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On the multiplicity of self-similar solutions of the semilinear heat equation

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ABSTRACT

In studies of superlinear parabolic equations

$$u_t = \Delta u + u^p, \quad x \in \mathbb{R}^N, \ t > 0,$$

where $p > 1$, backward self-similar solutions play an important role. These are solutions of the form $u(x,t) = (T-t)^{-1/(p-1)}w(y)$, where $y := x/\sqrt{T-t}$, T is a constant, and w is a solution of the equation $\Delta w - y \cdot \nabla w/2 - w/(p-1) + w^p = 0$. We consider (classical) positive radial solutions w of this equation. Denoting by p_S, p_{JL}, p_L the Sobolev, Joseph-Lundgren, and Lepin exponents, respectively, we show that for $p \in (p_S, p_{JL})$ there are only countably many solutions, and for $p \in (p_{JL}, p_L)$ there are only finitely many solutions. This result answers two basic open questions regarding the multiplicity of the solutions.

1. Introduction

In this paper, we consider positive radial self-similar solutions of the semilinear heat equation

$$u_t = \Delta u + u^p, \quad x \in \mathbb{R}^N, \ t > 0, \tag{1.1}$$

where $p > 1$ is a real number. A radial (backward) self-similar solution u is a solution of the form $u(x,t) = (T-t)^{-1/(p-1)}w(|y|)$, where $y := x/\sqrt{T-t}$, T is a constant, and w is a function in $C^1[0,\infty)$. Such a function u is a (regular) positive solution of (1.1) if w is a solution of the following problem

$$w_{rr} + \left(\frac{N-1}{r} - \frac{r}{2}\right) w_r - \frac{w}{p-1} + w^p = 0, \quad r > 0, \tag{1.2}$$

$$w_r(0) = 0, \quad w > 0. \tag{1.3}$$

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Self-similar solutions have an indispensable role in the theory of blowup of Eq. (1.1). They are examples of solutions exhibiting *type-I* blowup at time T, by which we mean that the rate of blowup is $(T-t)^{-1/(p-1)}$, the same as in the ordinary differential equation $u_t = u^p$. In fact, they often serve as canonical examples in the sense that general solutions with type-I blowup can be proved to approach in some way a self-similar solution as t approaches the blowup time (see, for example, [2,9,10,17,18], or the monograph [26] for results of this form). Moreover, self-similar solutions play an important role in the study of the asymptotic behavior of global solutions (see [8] or [26, the proof of Theorem 22.4], for example), in the construction of interesting solutions (like peaking or homoclinic solutions; see [8], [22] or [6], respectively), in the study of type-II blow-up (see [16, Proposition 1.8(ii)]), etc.

Eq. (1.2) has been scrutinized by a number of authors. To recall known results, we introduce several critical exponents (they are usually called the Sobolev, Joseph-Lundgren, and Lepin exponents, respectively):

$$p_S := \begin{cases} \frac{N+2}{N-2} & \text{if } N > 2, \\ \infty & \text{if } N \le 2, \end{cases}$$

$$p_{JL} := \begin{cases} 1 + 4\frac{N-4+2\sqrt{N-1}}{(N-2)(N-10)} & \text{if } N > 10, \\ \infty & \text{if } N \le 10, \end{cases}$$

$$p_L := \begin{cases} 1 + \frac{6}{N-10} & \text{if } N > 10, \\ \infty & \text{if } N \le 10. \end{cases}$$

Obviously, the constant $\kappa := (p-1)^{-1/(p-1)}$ is a solution of (1.2) for any $p > 1$. If $1 < p \le p_S$, κ is the only positive solution [9,11]. For $p_S < p < p_{JL}$, there exist at least countably (infinitely) many solutions; and for $p_{JL} \le p < p_L$ the existence of a (positive) finite number of nonconstant solutions has been established (see [3–5,8,14,22,29]). For $p > p_L$, κ is again the only positive solution. This was proved by [20] (the nonexistence was indicated by numerical experiments in the previous paper [23]). The same seems to be the case for $p = p_L$, as claimed in [21], however the proof given in [21] is not complete.

A natural and rather basic question, which, despite its importance on several levels, has been open until now is whether there may exist infinitely many solutions for some $p \in [p_{JL}, p_L)$, or uncountably many solutions for some $p \in (p_S, p_{JL})$. In particular, it is of significance to clarify whether there might be continua of solutions (1.2), (1.3) for some $p \in (p_S, p_L)$. For example, by ruling out the possibility that such continua exist, one could substantially simplify the proofs of some results on self-similar asymptotics of blowup solutions of (1.1), such as those in [17] or [15, Theorem 3.1]. Also, the problems of finiteness and countability of the set of the radial self-similar solutions are of great importance in our study of entire and ancient solutions of (1.1), which will appear in a forthcoming paper [24].

The goal of the present work is to address these problems. Our main results are stated in the following theorem.

Theorem 1.1. *Under the above notation, the following statements are valid.*

(i) For any $p \in (p_S, p_{JL})$ the set of solutions of (1.2), (1.3) *is infinite and countable. For $p = p_{JL}$ the set of solutions of* (1.2), (1.3) *is at most countable.*

(ii) For any $p \in (p_{JL}, p_L)$ the set of solutions of (1.2), (1.3) *is finite.*

Very likely, in the case $p = p_{JL}$ the set of solutions is finite, too, but our proof of the finiteness for $p \in (p_{JL}, p_L)$ does not cover the case $p = p_{JL}$. On the other hand, if there exist solutions for $p = p_L$, then our proof guarantees that the set of solutions is finite. Statement (i) can be made a little more precise. We will prove that the set formed by the values $w(0)$ of the solutions of (1.2), (1.3) is discrete.

Our method relies on two kinds of shooting techniques; one from the origin, considering a standard initial-value problem for (1.2) at $r = 0$, and another one where “initial conditions” are prescribed at $r = \infty$. In the proof of statement (i), we employ the analyticity of the nonlinearity $u \mapsto u^p$ in $(0,\infty)$. We prove that the solutions are isolated, hence there is at most countably many of them. Technical difficulties in this proof are caused by the fact that as $r \to \infty$ the nonconstant solutions of (1.2), (1.3) decay to 0, where we lose the analyticity. For the proof of statement (ii), we show that the solutions cannot accumulate at the singular solution. This involves a subtle analysis of how solutions of the initial value problems at $r = 0$ and $r = \infty$ behave near the singular solution.

The paper is organized as follows. In the next section, we introduce some notation and recall several technical results concerning solutions of (1.2). In Section 3, we use shooting arguments to show that the solutions of (1.2) are in one-to-one correspondence with the zeros of a real analytic function. This is key to showing that the solutions are isolated. In Section 4, we consider the set of solutions of (1.2) near a singular solution and complete the proof of Theorem 1.1(ii).

2. Notation and preliminaries

In the remainder of the paper, it is always assumed that $p > p_S$.

Although Eq. (1.2) has a singularity at $r = 0$, it is well known and easy to prove by an application of the Banach fixed point theorem to an integral operator (see, for example, [12]) that for each $\alpha > 0$ there is a unique local solution of (1.2) satisfying the initial conditions

$$w_r(0) = 0,\ \ w(0) = \alpha. \tag{2.1}$$

We denote this solution by $w(r,\alpha)$ and extend it to its maximal existence interval. If the solution changes sign, then the nonlinearity in (1.2) is interpreted as $w|w|^{p-1}$. Let

$$\mathcal{S} := \{\alpha > 0 : w(r,\alpha) > 0\ \ (r \in (0,\infty))\}. \tag{2.2}$$

Obviously, for each solution w of (1.2)–(1.3) one has $w = w(\cdot,\alpha)$ for some (unique) $\alpha \in \mathcal{S}$. We further denote

$$\phi_\infty(x) := L|x|^{-2/(p-1)}, \quad L := \left(\frac{2}{(p-1)^2}((N-2)p-N)\right)^{\frac{1}{p-1}}. \tag{2.3}$$

This is a singular solution of (1.2) (it is defined when $p(N-2) > N$). In fact, this is a unique solution of (1.2) with a singularity at $r = 0$ (see [21,25]).

We recall the following properties (as above, $\kappa = (p-1)^{-1/(p-1)}$):

Lemma 2.1. *The following statements are valid (for each $p > p_S$).*

(i) One has $\alpha \geq \kappa$ for all $\alpha \in \mathcal{S}$, and κ is isolated in $\mathcal{S}$.
(ii) For each $\alpha \in \mathcal{S} \setminus \{\kappa\}$ there exists a positive constant $\ell(\alpha)$ such that

$$w(r,\alpha) = \ell(\alpha) r^{-\frac{2}{p-1}}(1 - c(\alpha)r^{-2} + o(r^{-2})) \quad \text{as } r \to \infty, \tag{2.4}$$

where $c(\alpha) := (\ell(\alpha))^{p-1} - L^{p-1}$ (with L as in (2.3)). Moreover, one has

$$\frac{w_r(r,\alpha)}{w(r,\alpha)} = -\frac{2}{(p-1)} r^{-1} + 2c(\alpha)r^{-3} + o(r^{-3}) \quad \text{as } r \to \infty. \tag{2.5}$$

(iii) With $\ell(\alpha)$ as in statement (ii), the function $\alpha \mapsto \ell(\alpha)$ is one-to-one on $\mathcal{S} \setminus \{\kappa\}$.

Proof. Statement (i) is proved in Lemmas 2.2 and 2.3 of [20]. Statements (ii), (iii) for regular *bounded* solutions are proved in [17, Section 2]; the boundedness assumption can be removed due to [20, Lemma 2.1]. □

We need two additional properties of the function $\alpha \mapsto \ell(\alpha)$:

Lemma 2.2. *Suppose $\alpha_k \in \mathcal{S}$, $k = 1, 2, \dots$, and $\alpha_k \to \alpha_0 \in (\kappa, \infty]$ as $k \to \infty$. The following statements are valid.*

(i) If $\alpha_0 < \infty$, then $\alpha_0 \in \mathcal{S}$ and $\ell(\alpha_k) \to \ell(\alpha_0)$.
(ii) If $\alpha_0 = \infty$ and $p > p_{JL}$, then $\ell(\alpha_n) \to L$.

Proof. Statement (ii) is proved in [20, Lemma 2.7].

We prove statement (i). It is clearly sufficient to prove that the statement is valid if the sequence $\{\alpha_k\}_k$ is replaced by a subsequence (and then use this conclusion for any subsequence of $\{\alpha_k\}_k$ in place of the full sequence $\{\alpha_k\}_k$). We may in particular assume that $\alpha_j \neq \alpha_k$ if $j \neq k$ (for a constant sequence the statement is trivially true) and $\alpha_j > \kappa$ for any j.

The fact that $\alpha_0 \in \mathcal{S}$, that is, the solution $w(\cdot, \alpha_0)$ is positive, follows easily from the continuity of solutions with respect to initial data.

Consider now the function $v(r, \alpha) := w(r, \alpha) r^{2/(p-1)}$. It solves the equation

$$v_{rr} + \Big(\frac{N - 1 - 4/(p-1)}{r} - \frac{r}{2}\Big) v_r + \frac{1}{r^2}(v^p - L^{p-1} v) = 0, \quad r > 0. \tag{2.6}$$

If $\alpha, \bar{\alpha} \in \mathcal{S}$, $\alpha \neq \bar{\alpha}$ and $h(r) := v(r, \alpha) - v(r, \bar{\alpha})$, then h solves the equation

$$h_{rr} + \Big(\frac{N - 1 - 4/(p-1)}{r} - \frac{r}{2}\Big) h_r + \frac{1}{r^2}(p v_\theta^{p-1} - L^{p-1}) h = 0, \quad r > 0, \tag{2.7}$$

where $v_\theta = v_\theta(r)$ belongs to the interval with end points $v(r, \alpha)$ and $v(r, \bar{\alpha})$.

First we show that if $r > \sqrt{2N}$, then for any $k \neq j$ one has $v(r, \alpha_k) \neq v(r, \alpha_j)$. We go by contradiction. Assume $r_1 > \sqrt{2N}$, $k \neq j$ and $w(r_1, \alpha_k) = w(r_1, \alpha_j)$. Without loss of generality we may assume $w_r(r_1, \alpha_k) > w_r(r_1, \alpha_j)$. Set

$$\phi(r) := w(r, \alpha_k) - \kappa, \quad \phi^*(r) := w(r, \alpha_j) - \kappa,$$
$$r_2 := \sup\{\tilde{r} > r_1 : \phi^*(r) < \phi(r) \text{ in } (r_1, \tilde{r})\} \leq \infty.$$

Then [5, Proposition 2.3] implies $\phi(r), \phi^*(r) < 0$ for $r > \sqrt{2N}$, and we also have $\phi(r), \phi^*(r) \to -\kappa$ as $r \to \infty$. The arguments in the first paragraph of the proof of [5, Proposition 2.4] guarantee that [5, (16)] is true, that is,

$$\phi(r) > \phi^*(r)\Big(1 - C_1 \int_{r_1}^{r} s^{1-N} e^{s^2/4} \phi^*(s)^{-2}\, ds\Big) > \phi^*(r), \qquad r \in (r_1, r_2),$$

where $C_1 > 0$. This estimate shows that $r_2 = \infty$ and also that $\phi(r) > 0$ for r large enough, which is a contradiction.

Fix $R > \sqrt{2N}$; from now on we consider the solutions v on the interval $[R, \infty)$ only. Passing to a subsequence, we may assume that the sequence $\{v(R, \alpha_k)\}_{k \geq 1}$ is strictly monotone. Assume that it is decreasing (the other case is analogous), hence $v_k := v(\cdot, \alpha_k)$ satisfy $v_1 > v_2 > \cdots > v_0$ on $[R, \infty)$, $v_k \to v_0$ in $C_{loc}([R, \infty))$.

Set $h_k := v_k - v_{k+1}$, $k = 1, 2, \dots$. Then h_k is positive and it solves (2.7) with $v_{k+1} < v_\theta < v_k$. Moreover, $h_k \to 0$ in $C_{loc}([R, \infty))$ and $\sum_k h_k \leq v_1 - v_0 \leq C$, for some constant $C > 0$.

First assume

$$p v_0^{p-1} > L^{p-1} \quad \text{on } [R, \infty). \tag{2.8}$$

If $h_k'(r_0) \leq 0$ for some $r_0 \geq R$ then (2.7) guarantees $h_k''(r_0) < 0$, hence $h_k', h_k'' < 0$ for $r > r_0$ which contradicts the positivity of h. Consequently, $h_k' \geq 0$.

Set $h_k^\infty := \lim_{r\to\infty} h_k(r)$. Since

$$v_1(r) - v_0(r) = \sum_{k=1}^{\infty} h_k(r) \nearrow \sum_{k=1}^{\infty} h_k^\infty \quad \text{as} \quad r \to \infty, \tag{2.9}$$

we have $\sum_{k=1}^\infty h_k^\infty < \infty$. Fix $\varepsilon > 0$. Then there exists k_0 such that $\sum_{k=k_0}^\infty h_k^\infty < \varepsilon$, hence $v_j(r) - v_0(r) = \sum_{k=j}^\infty h_k(r) < \varepsilon$ for any $r \in [R,\infty)$ and $j \ge k_0$. This implies $\ell(\alpha_0) < \ell(\alpha_j) \le \ell(\alpha_0)+\varepsilon$, hence $\ell(\alpha_k) \to \ell(\alpha_0)$.

Next assume that (2.8) fails. Notice that there exist $c_1, c_2 > 0$ such that

$$\frac{N-1-4/(p-1)}{r} - \frac{r}{2} \le -c_1 r \quad \text{and} \quad p v_\theta^{p-1} - L^{p-1} \ge -c_2 \quad \text{for} \quad r \ge R.$$

Set $c_3 := c_2/c_1$. If $h_k'(r_0) < -c_3 h_k(r_0) r_0^{-3}$ for some $r_0 \ge R$ then (2.7) guarantees $h_k''(r_0) < 0$, hence $h_k'(r) < -c_3 h_k(r) r^{-3}$ and $h_k''(r) < 0$ for $r > r_0$ (since $r \mapsto -c_3 h_k(r) r^{-3}$ is increasing if $h_k' < 0$), which contradicts the positivity of h. Consequently, $h_k' \ge -c_3 h_k(r) r^{-3}$, or, equivalently, $(e^{-c_3/(2r^2)} h_k)' \ge 0$.

Set $h_k^\infty := \lim_{r\to\infty} h_k(r) = \lim_{r\to\infty} e^{-c_3/(2r^2)} h_k(r)$. Now

$$e^{-c_3/(2r^2)}(v_1(r) - v_0(r)) = \sum_{k=1}^{\infty} e^{-c_3/(2r^2)} h_k(r) \nearrow \sum_{k=1}^{\infty} h_k^\infty \quad \text{as} \quad r \to \infty,$$

and similar arguments as above show that $\ell(\alpha_k) \to \ell(\alpha_0)$ again. □

We will also need the following information on the behavior of the solutions $w(\cdot,\alpha)$ for large α. This result – in fact, a stronger version of it – is proved in [20, Lemma 2.5].

Lemma 2.3. *Assume that $p > p_{JL}$. Then, as $\alpha \nearrow \infty$, one has*

$$w(r,\alpha) \to \phi_\infty(r), \quad w_r(r,\alpha) \to \phi_\infty'(r), \tag{2.10}$$

uniformly for r in any compact subinterval of $(0,\infty)$.

In some comparison arguments below, we will employ radial eigenfunctions of the linearization of (1.2) at the singular solution ϕ_∞. Specifically, we consider the following eigenvalue problem:

$$\begin{aligned} \psi_{rr} + \left(\frac{N-1}{r} - \frac{r}{2}\right)\psi_r + \left(-\frac{1}{p-1} + \frac{pL^{p-1}}{r^2} + \lambda\right)\psi = 0, \quad r > 0 \\ \psi \in H^1_\omega(0,\infty). \end{aligned} \tag{2.11}$$

Here $H^1_\omega(0,\infty)$ is the usual weighted Sobolev space with the weight

$$\omega(r) := r^{N-1}\exp(-r^2/4). \tag{2.12}$$

The inclusion $\psi \in H^1_\omega(0,\infty)$ means that if $\tilde\psi$ equals ψ or ψ', then

$$\int_0^\infty \tilde\psi^2(r)\omega(r)\,dr < \infty.$$

This eigenvalue problem is well understood. The following lemma summarizes some basic known results (see [13,19]).

Lemma 2.4. *Assume that $p > p_{JL}$. The eigenvalues of (2.11) form a sequence explicitly given by*

$$\lambda_j = \frac{\beta}{2} + \frac{1}{p-1} + j, \quad j = 0,1,2,\dots, \tag{2.13}$$

where

$$\beta := \frac{-(N-2)+\sqrt{(N-2)^2-4pL^{p-1}}}{2} < 0. \tag{2.14}$$

For $j = 0, 1, 2, \dots,$ the eigenfunction corresponding to λ_j, which is unique up to scalar multiples, has exactly j zeros, all of them simple, and satisfies the following asymptotic relations with some positive constants k_j, $\tilde{k}_j$:

$$\psi_j(r) = k_j r^\beta + o(r^\beta) \quad \text{as } r \to 0,$$
$$\psi_j(r) = \tilde{k}_j r^{-\frac{2}{p-1}+2\lambda_j} + o(r^{-\frac{2}{p-1}+2\lambda_j}) \quad \text{as } r \to \infty.$$

If $p_{JL} < p < p_L$, then $\lambda_2 < 0$ and if $p = p_L$, then $\lambda_2 = 0$.

The following result, which is a part of analysis used in [13,19], will also be useful below. It can be easily derived from well-known properties of Kummer's equation (as shown in [13,19]). Consider the following equation (the same equation as in (2.11), but with $\lambda = 0$).

$$\psi_{rr} + \left(\frac{N-1}{r} - \frac{r}{2}\right)\psi_r + \left(-\frac{1}{p-1} + \frac{pL^{p-1}}{r^2}\right)\psi = 0, \quad r > 0 \tag{2.15}$$

Lemma 2.5. *Assume that $p > p_{JL}$. Eq. (2.15) has (linearly independent) solutions ψ_1, ψ_2 satisfying the following asymptotic relations with some positive constants κ_1, κ_2:*

$$\psi_1(r) = \kappa_1 r^\beta + o(r^\beta) \quad \text{as } r \to 0, \tag{2.16}$$
$$\psi_2(r) = \kappa_2 r^{\beta^-} + o(r^\beta) \quad \text{as } r \to 0. \tag{2.17}$$

Here β is as in (2.14) and

$$\beta^- := \frac{-(N-2)-\sqrt{(N-2)^2-4pL^{p-1}}}{2} < \beta < 0. \tag{2.18}$$

Problem (2.11) has $\lambda = 0$ as an eigenvalue precisely when ψ_1 also satisfies the following asymptotic relation with some positive constant $\tilde{\kappa}_1$

$$\psi_1(r) = \tilde{\kappa}_1 r^{-\frac{2}{p-1}} + o(r^{-\frac{2}{p-1}}) \quad \text{as } r \to \infty. \tag{2.19}$$

Obviously, if (2.19) holds, then ψ_1 is an eigenfunction corresponding to the eigenvalue $\lambda = 0$. We remark that ψ_2 cannot be an eigenfunction of (2.11), for (2.17), (2.18) imply that it is not in H^1_ω.

We conclude this section with a monotonicity property of the function $\alpha \mapsto w(\cdot, \alpha)$. It will be useful to note that on any interval $(0, r_0]$ where $w(\cdot, \alpha) > 0$, $w_\alpha(\cdot, \alpha)$ satisfies the linear equation

$$z_{rr} + \left(\frac{N-1}{r} - \frac{r}{2}\right)z_r + \left(-\frac{1}{p-1} + p(w(r,\alpha))^{p-1}\right)z = 0, \tag{2.20}$$

and it also satisfies the initial conditions $w_\alpha(0, \alpha) = 1$, $w_{\alpha r}(0, \alpha) = 0$.

Lemma 2.6. *Assume $p > p_{JL}$. There exist positive constants α^*, R, and C_1 such that for all $\alpha > \alpha^*$ and $r \in [0, R]$ one has*

$$w_\alpha(r, \alpha) > 0 \quad (r \in [0, R]) \tag{2.21}$$
$$\frac{w_\alpha(r,\alpha)}{w_\alpha(r_0,\alpha)} \le C_1 r^\beta \quad (r \in [0, R],\ r_0 \in [R/2, R]), \tag{2.22}$$

where β is as in (2.14).

Proof. All arguments needed for the proof of these estimates are essentially given in the proof of Lemma 2.8 of [20], although the estimates are not formulated there explicitly for the same functions. Nonetheless, the arguments are easy to use, or adapt, in our case. We include the following details for the convenience of the reader.

Let $r_1(\alpha) \in (0, \infty]$ be the first zero of $w(\cdot, \alpha)$. We first use a Sturmian comparison to show that the first zero of $\phi_\infty - w(\cdot, \alpha)$ is less than $r_1(\alpha)$. (Similar Sturm comparison arguments are used at several other places in this proof.) We go by contradiction. Assume $w := w(\cdot, \alpha) < \phi_\infty$ in $(0, r_1(\alpha))$. Multiplying (1.2) with $\omega\phi_\infty$ and integrating by parts over $(0, r_1(\alpha))$ we obtain

$$
\begin{aligned}
0 &= \int_0^{r_1(\alpha)} ((\omega w')' - \frac{\omega}{p-1} w + \omega w^p)\phi_\infty \, dr \\
&= \int_0^{r_1(\alpha)} ((\omega \phi_\infty')' - \frac{\omega}{p-1} \phi_\infty + \omega w^{p-1}\phi_\infty) w \, dr \\
&\quad + [\omega w' \phi_\infty]_0^{r_1(\alpha)} - [\omega w \phi_\infty']_0^{r_1(\alpha)} \\
&< \int_0^{r_1(\alpha)} ((\omega \phi_\infty')' - \frac{\omega}{p-1} \phi_\infty + \omega \phi_\infty^p) w \, dr = 0
\end{aligned}
$$

which is a contradiction (cf. [20, p. 2919, lines 1–4]).

Take an eigenfunction ψ_j of (2.11) corresponding to a positive eigenvalue λ_j and let $r_1 > 0$ be its first zero. We will assume that $\psi_j > 0$ in $(0, r_1)$ (replace ψ_j by $-\psi_j$ if necessary). Set $R = r_1/2$. Considering the linear equation for $\phi_\infty - w(\cdot, \alpha)$ and using a Sturmian comparison with ψ_j, it is shown in the proof of Lemma 2.8 of [20] that the first zero of $\phi_\infty - w(\cdot, \alpha)$ is greater than r_1, that is,

$$
w(r, \alpha) < \phi_\infty(r) \quad (r \in [0, 2R]). \tag{2.23}
$$

Now consider the linear equation (2.20) satisfied by $w_\alpha(\cdot, \alpha)$. Due to (2.23), the zero order coefficient in this equation is smaller on $(0, 2R]$ than the zero order coefficient in the equation for ψ_j, see (2.11). Therefore, a similar Sturmian comparison of $w_\alpha(\cdot, \alpha)$ with ψ_j shows that $w_\alpha(\cdot, \alpha) \neq 0$ in $[0, r_1]$. Since $w_\alpha(0, \alpha) = 1$, we have $w_\alpha(\cdot, \alpha) > 0$ in $[0, R]$, proving (2.21).

We now prove (2.22). Given an arbitrary $r_0 \in [R/2, R]$, set $\tilde{\psi}(r, \alpha) := w_\alpha(r, \alpha)/w_\alpha(r_0, \alpha)$. We claim that for all large enough α one has

$$
\tilde{\psi}(\cdot, \alpha) < C_2 \psi_j \text{ on } [R/2, R], \tag{2.24}
$$

where C_2 is a constant independent of α and r_0. To prove this, we use the Harnack inequality for $\tilde{\psi}(r, \alpha)$—a positive solution of (2.20). Note that the coefficients of (2.20) are bounded in $[R/4, 2R]$ uniformly in α. This follows from (2.23). Since $\tilde{\psi}(r_0, \alpha) = 1$, the Harnack inequality yields a uniform upper bound on $\tilde{\psi}(\cdot, \alpha)$ in $[R/2, R]$. Property (2.24) follows from this and the positivity of ψ_j on the interval $(0, r_1) \supset [R/2, R]$.

To complete the proof of the lemma, assume for a contradiction that given any $C_1 > 0$ and α^*, one can find $\alpha > \alpha^*$ and $r_0 \in [R/2, R]$ violating the estimate in (2.22). This, in conjunction with the asymptotics of ψ_j given in Lemma 2.4, implies that there exists α (and a corresponding $r_0 \in [R/2, R]$) such that estimate (2.24) holds and at the same time $\tilde{\psi}(\cdot, \alpha) > C_2 \psi_j$ somewhere in $(0, R/2)$. Consequently, as $\tilde{\psi}(r, \alpha)$ stays bounded as $r \to 0$ while $\psi_j(r) \to \infty$, there exist two points $r_1 < r_2$ in $(0, R/2)$ such that the function $\hat{\psi} := \tilde{\psi}(\cdot, \alpha) - C_2 \psi_j$ is positive in (r_1, r_2) and vanishes at r_1 and r_2. Eqs. (2.20) and (2.11) with $\lambda = \lambda_j > 0$ yield an inequality satisfied by $\hat{\psi}$ in $(0, R/2)$:

$$
\begin{aligned}
&\hat{\psi}_{rr} + \left(\frac{N-1}{r} - \frac{r}{2}\right)\hat{\psi}_r + \left(-\frac{1}{p-1} + \frac{pL^{p-1}}{r^2}\right)\hat{\psi} \\
&\quad = \left(\frac{pL^{p-1}}{r^2} - p(w(r, \alpha))^{p-1}\right)\tilde{\psi}(r, \alpha) + C_2 \lambda_j \psi_j(r) \geq 0.
\end{aligned}
$$

Using this inequality, Eq. (2.11) with $\lambda = \lambda_j > 0$, and a Sturmian comparison argument, we obtain that ψ_j has a zero in (r_1, r_2), a contradiction. This contradiction proves that (2.22) holds for all sufficiently large α. $\square$

3. Shooting techniques and the proof of Theorem 1.1(i)

In this section, we employ two kinds of shooting arguments. The first one is a standard shooting technique for (1.2), (2.1) (shooting from $r = 0$). The second one is a kind of shooting from $r = \infty$, which becomes a more standard shooting technique after Eq. (1.2) is transformed suitably. Shooting arguments of such sort were already used in [14], cf. also [22].

3.1. Shooting from $r = 0$

We return to the initial-value problem (1.2), (2.1). As noted above, a local solution can be found in a standard way by applying the Banach fixed point theorem to a suitable integral operator. Since the nonlinearity $w \mapsto w^p$ is analytic in intervals not containing 0, the local solution depends analytically on α. Away from $r = 0$, there are no singularities and standard theory of ordinary differential equations applies. We thus obtain the following regularity property of the function $w(r, \alpha)$.

Lemma 3.1. *Given any $\alpha_0 \in \mathcal{S}$ and $r_0 \in (0, \infty)$, there is $\epsilon > 0$ with the following property. The solution $w(\cdot, \alpha)$ is (defined and) positive on $[0, 2r_0]$ for any $\alpha \in (\alpha_0 - \epsilon, \alpha_0 + \epsilon)$, and the function w is analytic on $(0, 2r_0) \times (\alpha_0 - \epsilon, \alpha_0 + \epsilon)$.*

Clearly, if $\alpha_0 \in \mathcal{S}$, then the function $w_\alpha(\cdot, \alpha_0)$ solves on $(0, \infty)$ the linear equation

$$z_{rr} + \left(\frac{N-1}{r} - \frac{r}{2}\right) z_r + \left(-\frac{1}{p-1} + p(w(r, \alpha_0))^{p-1}\right) z = 0, \tag{3.1}$$

and satisfies the initial conditions $w_\alpha(0, \alpha_0) = 1$, $w_{\alpha r}(0, \alpha_0) = 0$. In particular, $w_\alpha(\cdot, \alpha_0)$ is a nontrivial solution of (3.1) and as such it has only simple zeros.

3.2. Shooting from $r = \infty$

By Lemma 2.1, if $u = w(\cdot, \alpha)$ for some $\alpha \in \mathcal{S} \setminus \{\kappa\}$, then, as $r \to \infty$, one has

$$u(r) = \ell r^{-\frac{2}{p-1}}(1 + o(r^{-1})), \qquad \frac{u'(r)}{u(r)} = -\frac{2}{(p-1)} r^{-1} + o(r^{-2}), \tag{3.2}$$

for a suitable constant $\ell = \ell(\alpha)$. The same is of course true, with $\ell = L$, if $u = \phi_\infty$. Conditions (3.2) can be viewed as a kind of “initial conditions” at $r = \infty$. We show that Eq. (1.2) with these conditions is well posed and has analytic solutions. This is true in spite of the fact, pointed to us by one of the referees of this paper, that the two conditions in (3.2) are not independent. In fact, as noted by the referee, the solutions of (1.2) satisfying the first condition in (3.2) automatically satisfy the second one since the first condition alone is sufficient to derive identity (3.8) below. We prove the following.

Lemma 3.2. *Given any $\ell_0 \in \{L\} \cup \{\ell(\alpha) : \alpha \in \mathcal{S} \setminus \{\kappa\}\}$ and $r_0 \in (0, \infty)$, there is $\theta > 0$ and an analytic function $u : (r_0/2, \infty) \times (\ell_0 - \theta, \ell_0 + \theta) \to (0, \infty)$ with the following properties.*

(i) For any $\ell \in (\ell_0 - \theta, \ell_0 + \theta)$, the function $u(\cdot, \ell)$ is a positive solution of (1.2) on $[r_0/2, \infty)$ satisfying (3.2), and it is the only solution (up to extensions and restrictions) of (1.2) satisfying (3.2).

(ii) The function $u_\ell(\cdot, \ell_0)$ can be extended to $(0, \infty)$, where it satisfies the linear equation

$$z_{rr} + \left(\frac{N-1}{r} - \frac{r}{2}\right) z_r + \left(-\frac{1}{p-1} + p\phi^{p-1}(r)\right) z = 0, \tag{3.3}$$

with $\phi = w(\cdot, \alpha_0)$ if $\ell_0 = \ell(\alpha_0)$ for some $\alpha_0 \in \mathcal{S} \setminus \{\kappa\}$, and $\phi = \phi_\infty$ if $\ell_0 = L$. Moreover, $r^{2/(p-1)} u_\ell(r, \ell_0) \to 1$ as $r \to \infty$.

We prepare the proof of this lemma by transforming the problem to one on a bounded interval. First, setting $v(r) := w(r) r^{2/(p-1)}$, we transform equations (1.2) to (2.6). Next, we set $y(\rho) = v(r)$, $\rho = 1/r$. A simple computation shows that w is a solution of (1.2) on (r_0, ∞) for some $r_0 > 0$ if and only if y is a solution of the following equation on $(0, 1/r_0)$:

$$y_{\rho\rho} + \frac{\gamma}{\rho} y_\rho + \frac{1}{2\rho^3} y_\rho + \frac{1}{\rho^2}(y^p - L^{p-1} y) = 0, \tag{3.4}$$

with $\gamma := 3 - N + 4/(p-1)$. Moreover, if conditions (3.2) are satisfied by $u = w$, then, as $\rho \searrow 0$, one has $y(\rho) \to \ell$ and

$$y'(\rho) = -v'(r) r^2 = -r^{\frac{2}{p-1}} w(r) r^2 \left(\frac{2}{p-1} r^{-1} + \frac{w'(r)}{w(r)}\right) \to 0 \quad \text{as } r = \frac{1}{\rho} \to \infty.$$

So y extends to a C^1 function on $[0, 1/r_0)$ with

$$y(0) = \ell, \quad y'(0) = 0. \tag{3.5}$$

Conversely, if y is C^1 on $[0, 1/r_0)$ and conditions (3.5) hold, then w is easily shown to satisfy (3.2).

To show that problem (3.4), (3.5) is well posed, we write it in an integral form. Define a function H on $[0, \infty)$ by

$$H(0) = 0, \quad H(\rho) = \rho^\gamma e^{-\rho^{-2}/4} \text{ if } \rho > 0, \tag{3.6}$$

so that $H'(\rho) = \gamma H(\rho)/\rho + H(\rho)/(2\rho^3)$. Notice that $H'(\rho) > 0$ for all sufficiently small $\rho > 0$. Eq. (3.4) is equivalent to the following equation

$$(H(\rho) y'(\rho))' + \rho^{-2} H(\rho)(y^p(\rho) - L^{p-1} y(\rho)) = 0. \tag{3.7}$$

Assuming y satisfies (3.5), we integrate (3.7) to obtain

$$\begin{aligned} y'(\rho) &= -\frac{1}{H(\rho)} \int_0^\rho \eta^{-2} H(\eta)(y^p(\eta) - L^{p-1} y(\eta))\, d\eta \\ &= -\frac{2}{H(\rho)} \int_0^\rho ((\eta H(\eta))' - (\gamma+1) H(\eta))(y^p(\eta) - L^{p-1} y(\eta))\, d\eta. \end{aligned}$$

After an integration by parts this becomes

$$\begin{aligned} y'(\rho) = &-2\rho(y^p(\rho) - L^{p-1} y(\rho)) \\ &+ \frac{2(\gamma+1)}{H(\rho)} \int_0^\rho H(\eta)(y^p(\eta) - L^{p-1} y(\eta))\, d\eta \\ &+ \frac{2}{H(\rho)} \int_0^\rho (\eta H(\eta))(p y^{p-1}(\eta) - L^{p-1}) y'(\eta)\, d\eta. \end{aligned} \tag{3.8}$$

Conversely, noting that $H(\eta)/H(\rho) < 1$ if $0 < \eta < \rho < \delta$ and $\delta > 0$ is sufficiently small, one shows easily that if $y \in C^1[0, \delta]$, $y(0) = \ell$ and (3.8) holds, then $y'(0) = 0$ and (3.4) is satisfied.

We can now set up a suitable fixed point argument. We work in the Banach space $X := C([0, \delta], \mathbb{R}^2)$ with a usual norm, say $\|U\| = \|y\|_{L^\infty(0,\delta)} + \|z\|_{L^\infty(0,\delta)}$ for $U = (y, z) \in X$. Let $U_0 \in X$ stand for the constant function $(\ell_0, 0)$. Fix any $\epsilon \in (0, \ell_0/2)$ and let B stand for the open ball ($\bar{B}$ for the closed ball) in X with center U_0 and radius ϵ. Note that the choice of ϵ guarantees that for any $(y, z) \in \bar{B}$ one has $y \geq \ell_0/2$. For any ℓ sufficiently close to ℓ_0, we consider the map $\Psi^\ell : \bar{B} \to X$ defined by $\Psi^\ell(y, z) = (\tilde{y}, \tilde{z})$, where, for $\rho \in [0, \delta]$,

$$\begin{aligned}\tilde{y}(\rho) &= \ell + \int_0^\rho z(\eta)\,d\eta,\\ \tilde{z}(\rho) &= -2\rho(y^p(\rho) - L^{p-1}y(\rho)) + \frac{2(\gamma+1)}{H(\rho)}\int_0^\rho H(\eta)(y^p(\eta) - L^{p-1}y(\eta))\,d\eta \\ &\quad + \frac{2}{H(\rho)}\int_0^\rho \eta H(\eta)(py^{p-1}(\eta) - L^{p-1})z(\eta)\,d\eta.\end{aligned} \tag{3.9}$$

Clearly, y is a $C^1[0, \delta]$-solution of (3.4), (3.5) if and only if (y, y') is a fixed point of the map Ψ^ℓ.

Lemma 3.3. *If δ and θ are sufficiently small positive numbers, then the map Ψ^ℓ defined above is for each $\ell \in (\ell_0 - \theta, \ell_0 + \theta)$ a 1/2-contraction on $\bar{B}$. Denoting its unique fixed point by U^ℓ, the map $\ell \to U^\ell$ is an analytic X-valued map on $(\ell_0 - \theta, \ell_0 + \theta)$.*

Before proving this lemma, we use it to complete the proof of Lemma 3.2.

Proof of Lemma 3.2. Lemma 3.3 and the notes preceding it yield a positive solution of (2.6), (3.5) on some interval $[r_1, \infty)$ and also imply the uniqueness of the solution and its analytic dependence on ℓ. Of course, as the equation has no singularity in $(0, \infty)$, we can combine these results with standard results from ordinary differential equations to prove the existence of an analytic function u on $(r_0/2, \infty) \times (\ell_0 - \theta, \ell_0 + \theta)$ (with θ possibly smaller than in Lemma 3.3) such that statement (i) of Lemma 3.2 holds.

Having proved that given any $r_0 > 0$ the function $u(r, \ell)$ is defined for $r \in [r_0/2, \infty)$ if ℓ is close enough to ℓ_0, we see that $u_\ell(r, \ell_0)$ is defined for any $r \in (0, \infty)$. Differentiating the fixed point equation (3.9) with respect to ℓ (using the smooth dependence of the fixed point on ℓ) and reversing the transformations relating y and w, we obtain that $r^{2/(p-1)}u_\ell(r, \ell_0) \to 1$ as $r \to \infty$. The regularity of the function u allows us to differentiate equation (1.2), with $w = u(\cdot, \ell)$, with respect to ℓ to obtain the equation for $u_\ell(\cdot, \ell_0)$. This yields Eq. (3.3) with $\phi = u(\cdot, \ell_0)$. The uniqueness property of the solution u implies that $u(\cdot, \ell_0) = w(\cdot, \alpha_0)$ if $\ell_0 = \ell(\alpha_0)$ for some $\alpha_0 \in \mathcal{S} \setminus \{\kappa\}$, and $u(\cdot, \ell_0) = \phi_\infty$ if $\ell_0 = L$. This completes the proof of Lemma 3.2. □

Remark 3.4. Clearly, we can differentiate equation (1.2) with $w = u(\cdot, \ell)$ further to find equations for higher derivatives of $u(\cdot, \ell)$ with respect to ℓ. For example, $u_{\ell\ell}(\cdot, L)$ is a solution of the following nonhomogeneous equation on $(0, \infty)$:

$$z_{rr} + \left(\frac{N-1}{r} - \frac{r}{2}\right) z_r + \left(-\frac{1}{p-1} + p\phi_\infty^{p-1}(r)\right) z = -p(p-1)\phi_\infty^{p-2}(r)u_\ell^2(r, L). \tag{3.10}$$

Note that the function $r^{2/(p-1)}u_{\ell\ell}(r, L) = y_{\ell\ell}(1/r, L)$ stays bounded as $r \to \infty$. This observation will be useful in the next section.

Proof of Lemma 3.3. As noted above, $\epsilon < \ell_0/2$ guarantees that for any $(y, z) \in \bar{B}$ one has $y \geq \ell_0/2$. It follows that the maps

$$(y, z) \mapsto y^p, \quad (y, z) \mapsto y^{p-1}z \tag{3.11}$$

are analytic $C[0, \delta]$-valued maps on B. Note also that the map sending $u \in C[0, \delta]$ to the function $\int_0^\rho H(\eta)/H(\rho)u(\eta)\,d\eta$ is a bounded linear operator on $C[0, \delta]$. It follows that the map $(\ell, U) \mapsto \Psi^\ell(U)$ is an analytic X-valued map on $(\ell_0 - \theta, \ell_0 + \theta) \times B$ (the smallness of θ, δ is not needed here).

Choose $\delta > 0$ so small that $H' > 0$ on $(0, \delta)$. Clearly, the maps (3.11) are globally Lipschitz on $\bar{B}$. This and the relation $H(\eta)/H(\rho) < 1$ for $0 < \eta < \rho \le \delta$ imply that, possibly after making $\delta > 0$ smaller, $\Psi^\ell : \bar{B} \to X$ is a 1/2-contraction (for any ℓ).

We now show that if θ is sufficiently small and δ is made yet smaller, if needed, then for each $\ell \in (\ell_0 - \theta, \ell_0 + \theta)$ one has $\Psi^\ell(\bar{B}) \subset \bar{B}$, that is, Ψ^ℓ is a 1/2-contraction on $\bar{B}$.

To that aim, for any $U \in \bar{B}$ we estimate

$$\begin{aligned} \|\Psi^\ell(U) - U_0\| &= \|\Psi^\ell(U_0) - U_0\| + \|\Psi^\ell(U) - \Psi^\ell(U_0)\| \\ &\le \|\Psi^\ell(U_0) - U_0\| + \frac{1}{2}\|U - U_0\| \\ &\le \|\Psi^\ell(U_0) - U_0\| + \frac{\epsilon}{2}. \end{aligned} \tag{3.12}$$

Now

$$\Psi^\ell(U_0)(\rho) - U_0 = \left(\ell - \ell_0, C_0\big(-2\rho + 2(\gamma+1)\int_0^\rho \frac{H(\eta)}{H(\rho)}\, d\eta\big)\right),$$

where $C_0 = \ell_0^p - L^{p-1}\ell_0$. Clearly,

$$\|\Psi^\ell(U_0) - U_0\| \le |\ell - \ell_0| + C\delta \le \theta + C\delta, \tag{3.13}$$

where C is determined by C_0 and γ (and is independent of θ and δ). Taking $0 < \theta < \epsilon/4$ and making $\delta > 0$ smaller, if necessary, so that $C\delta < \epsilon/4$, we obtain from (3.13), (3.12) that $\|\Psi^\ell(U) - U_0\| < \epsilon$ – that is, $\Psi^\ell(U) \in \bar{B}$ – for any $U \in \bar{B}$.

The uniform contraction theorem implies the existence of a unique fixed point U^ℓ of Ψ^ℓ, and it also gives the analyticity of the map $\ell \to U^\ell : (\ell_0 - \theta, \ell_0 + \theta) \to X$. □

Although not needed below, we add a remark on the dependence of the solutions on p. Clearly, when dealing with solutions bounded below by a positive constant, one can view p as a parameter, with the nonlinearity w^p depending analytically on p. Therefore the uniform contraction arguments employed in the shooting from 0 and ∞ imply that the solutions $w(\cdot, \alpha)$, $u(\cdot, \ell)$ given by Lemmas 3.1, 3.2 depend analytically on p, too.

3.3. The discreteness of the set $\mathcal{S}$

We now show that the set $\mathcal{S}$ is discrete, hence at most countable. This will prove statement (i) of Theorem 1.1.

We go by contradiction. Suppose that $\mathcal{S}$ contains an element which is not isolated in $\mathcal{S}$. Set

$$\alpha_0 := \inf\{\alpha \in \mathcal{S} : \alpha \text{ is an accumulation point of } \mathcal{S}\}. \tag{3.14}$$

Clearly, α_0 itself is an accumulation point of $\mathcal{S}$. By the continuity of the solutions $w(\cdot, \alpha)$ with respect to α, one has $\alpha_0 \in \mathcal{S}$. By Lemma 2.1(i), $\alpha_0 > \kappa$. Set $\ell_0 := \ell(\alpha_0)$ (cp. Lemma 2.1(ii)).

Choose $\epsilon > 0$ and $\theta > 0$ such that the function $w(\cdot, \alpha)$ is positive on $(0, 2)$ for all $\alpha \in (\alpha_0 - \epsilon, \alpha_0 + \epsilon)$ and the function $u(\cdot, \ell)$ is (defined and is) positive on $(1, \infty)$ for all $\ell \in (\ell_0 - \theta, \ell_0 + \theta)$ (see Lemmas 3.1, 3.2). Recalling from Section 3.1 and Lemma 3.2 that the functions $w_\alpha(\cdot, \alpha_0)$, $u_\ell(\cdot, \ell_0)$ are nontrivial solutions of the linear equation (3.1), we pick $r_0 \in (1, 2)$ such that neither of these functions vanishes at r_0. Then, making $\epsilon > 0$ and $\theta > 0$ smaller if necessary, we may assume that

$$\begin{aligned} w_\alpha(r_0, \alpha) \neq 0 \quad (\alpha \in (\alpha_0 - \epsilon, \alpha_0 + \epsilon)), \\ u_\ell(r_0, \ell) \neq 0 \quad (\ell \in (\ell_0 - \theta, \ell_0 + \theta)). \end{aligned} \tag{3.15}$$

Now, Lemma 2.2 guarantees that, possibly after making $\epsilon > 0$ yet smaller, one has $\ell(\alpha) \in (\ell_0 - \theta, \ell_0 + \theta)$ for any $\alpha \in (\alpha_0 - \epsilon, \alpha_0 + \epsilon) \cap \mathcal{S}$. For any such α, Lemmas 2.1(iii) and 3.2 imply that $w(\cdot, \alpha) \equiv u(\cdot, \ell(a))$; in particular,

$$(w(r_0, \alpha), w_r(r_0, \alpha)) = (u(r_0, \ell(\alpha)), u_r(r_0, \ell(\alpha))) \quad (\alpha \in (\alpha_0 - \epsilon, \alpha_0 + \epsilon) \cap \mathcal{S}). \tag{3.16}$$

Consider the following two analytic curves

$$J_1 := \{(w(r_0, \alpha), w_r(r_0, \alpha)) : \alpha \in (\alpha_0 - \epsilon, \alpha_0 + \epsilon)\},$$
$$J_2 := \{(u(r_0, \ell), u_r(r_0, \ell)) : \ell \in (\ell_0 - \theta, \ell_0 + \theta)\}.$$

In view of (3.15), they can be reparameterized by the first component, namely,

$$J_1 := \{(\zeta, F(\zeta)) : \zeta \in I_1\},$$
$$J_2 := \{(\zeta, G(\zeta)) : \zeta \in I_2\},$$

where I_1 is the open interval with the end points $w(r_0, \alpha_0 \pm \epsilon)$, I_2 is the open interval with the end points $u(r_0, \ell_0 \pm \theta)$, and F and G are analytic functions: $F(\zeta) = w_r(r_0, \hat{\alpha}(\zeta))$, where $\hat{\alpha}$ is the inverse to $\alpha \mapsto w(r_0, \alpha)$; and similarly for G. Since $\ell_0 = \ell(\alpha_0)$, relation (3.16) implies that $w(r_0, \alpha_0) = u(r_0, \ell_0) =: \zeta_0 \in I_1 \cap I_2$ and $F(\zeta_0) - G(\zeta_0) = 0$. Further, using (3.16) in conjunction with the fact that α_0 is an accumulation point of $\mathcal{S}$, we obtain that ζ_0 is an accumulation point of the set of zeros of the function $F - G$. By the analyticity, $F - G$ vanishes identically on a neighborhood of ζ_0. From this and the relation $w(\cdot, \alpha) \equiv u(\cdot, \ell(\alpha))$, we conclude that for α in a neighborhood of α_0 the solution $w(\cdot, \alpha)$ is positive on $(0, \infty)$, that is, $\alpha \in \mathcal{S}$. This is a contradiction to the definition of α_0 (cp. (3.14)). With this contradiction, the discreteness of $\mathcal{S}$ and statement (i) of Theorem 1.1 are proved.

4. Solutions near ϕ_∞ and the proof of Theorem 1.1(ii)

In this section we assume that $p_{JL} < p \leq p_L$.

Our goal is to show that there is a constant $\alpha^* > 0$ such that

$$\mathcal{S} \cap (\alpha^*, \infty) = \emptyset. \tag{4.1}$$

In conjunction with statement (i) of Lemma 2.1 and the discreteness of the set $\mathcal{S}$ proved in the previous section, (4.1) implies that the set $\mathcal{S}$ is finite. Thus, once we prove (4.1), the proof of statement (ii) of Theorem 1.1 will be complete.

To prove (4.1), we initially employ the functions $w(r, \alpha)$, $u(r, \ell)$ in a very similar manner as in Section 3.3, taking ℓ close to the constant L from the singular solution (cp. (2.3)).

First we choose $\alpha^* > 0$ and $R > 0$ such that

$$w_\alpha(r, \alpha) > 0, \; w(r, \alpha) > 0 \quad (r \in [0, R], \; \alpha > \alpha^*),$$

and (2.22) holds for some constant $C > 0$ (cp. Lemmas 2.6, 2.3). Next we choose $\theta > 0$ such that the function $u(\cdot, \ell)$ is (defined and) positive on $(R/2, \infty)$ for all $\ell \in (L - \theta, L + \theta)$. Pick $r_0 \in (R/2, R)$ such that $u_\ell(r_0, L) \neq 0$. Making $\theta > 0$ smaller if necessary, we have

$$u_\ell(r_0, \ell) \neq 0 \quad (\ell \in (L - \theta, L + \theta)). \tag{4.2}$$

Further, by Lemma 2.2(ii), we have, possibly after making α^* larger, that $\ell(\alpha) \in (L - \theta, L + \theta)$ for any $\alpha \in (\alpha^*, \infty) \cap \mathcal{S}$. For any such α, Lemmas 2.1(ii) and 3.2 imply that $w(\cdot, \alpha) \equiv u(\cdot, \ell(a))$; in particular,

$$(w(r_0, \alpha), w_r(r_0, \alpha)) = (u(r_0, \ell(\alpha)), u_r(r_0, \ell(\alpha))) \quad (\alpha \in (\alpha^*, \infty) \cap \mathcal{S}). \tag{4.3}$$

Consider the following two analytic curves

$$J_1 := \{(w(r_0,\alpha), w_r(r_0,\alpha)) : \alpha \in (\alpha^*, \infty)\},$$
$$J_2 := \{(u(r_0,\ell), u_r(r_0,\ell)) : \ell \in (L-\theta, L+\theta)\}.$$

In view of the relations $w_\alpha(r_0,\alpha) > 0$ and (4.2), using also the fact that $w(r_0,\alpha) \to \phi_\infty(r_0) =: \zeta_0$ as $\alpha \nearrow \infty$ (cp. Lemma 2.6), we reparameterize the curves J_1, J_2 as follows:

$$J_1 := \{(\zeta, F(\zeta)) : \zeta \in (w(r_0,\alpha^*), \zeta_0)\}, \tag{4.4}$$
$$J_2 := \{(\zeta, G(\zeta)) : \zeta \in I\}. \tag{4.5}$$

Here I is the open interval with the end points $u(r_0, L \pm \theta)$, and F and G are analytic functions: $F(\zeta) = w_r(r_0, \hat{\alpha}(\zeta))$, where $\hat{\alpha}$ is the inverse to $\alpha \mapsto w(r_0,\alpha)$; and, similarly, $G(\zeta) = u_r(r_0, \hat{\ell}(\zeta))$, where $\hat{\ell}$ is the inverse to $\ell \mapsto u(r_0,\ell)$.

Since $u(\cdot, L) = \phi_\infty$, we have $\zeta_0 \in I$ and $G(\zeta_0) = \phi'_\infty(r_0)$. Also, from the fact that $w_r(r_0,\alpha) \to \phi'_\infty(r_0)$ as $\alpha \to \infty$ (cp. (2.10)), we infer that $\lim_{\zeta \nearrow \zeta_0} F(\zeta) = \phi'_\infty(r_0)$. Thus, we may define $F(\zeta_0) := \phi'_\infty(r_0)$ and F becomes a continuous function on $(w(r_0,\alpha^*), \zeta_0]$.

If F were analytic on the interval $(w(r_0,\alpha^*), \zeta_0]$, we could use simple analyticity arguments, similar to those in Section 3.3, to conclude the proof of (4.1). However, it turns out that in some cases F is not even of class C^2 at ζ_0, and we thus need a different reasoning.

We will prove the following statements.

Proposition 4.1. *Let F and G be as above. Then the function F is of class C^1 on $(w(r_0,\alpha^*), \zeta_0]$ and the following statements hold:*

(i) $\lambda = 0$ is an eigenvalue of problem (2.11) if and only if $F'(\zeta_0) = G'(\zeta_0)$.

(ii) If $\lambda = 0$ is an eigenvalue of problem (2.11), then $\lim_{\zeta \to \zeta_0} F''(\zeta)$ exists and is distinct from $G''(\zeta_0)$. More specifically, the following statements are valid (with β as in (2.14)):

(a) If

$$N - 1 + 3\beta - \frac{2(p-2)}{(p-1)} \le -1, \tag{4.6}$$

then

$$F''(\zeta) \to -\infty \quad \text{as } \zeta \nearrow \zeta_0. \tag{4.7}$$

(b) If (4.6) is not true, then $F''(\zeta)$ has a finite limit as $\zeta \to \zeta_0$ and

$$\lim_{\zeta \to \zeta_0} F''(\zeta) \ne G''(\zeta_0). \tag{4.8}$$

Remark 4.2.

(i) It may be instructive – and will be useful below – to list the exponents $p > p_{JL}$ for which $\lambda = 0$ is an eigenvalue of problem (2.11). These can be computed from (2.13), (2.14): assuming $N > 10$, for $j \ge 2$ we have $\lambda_j = 0$ if and only if $p = p_j$, where

$$p_j := 1 + \frac{4j-2}{N(j-1) - 2j^2 - 2j + 2}. \tag{4.9}$$

As already mentioned in Lemma 2.4, $\lambda_2 = 0$ for $p = p_L$ (in other words, $p_2 = p_L$), and $\lambda_2 < 0$ for any $p_{JL} < p < p_L$. To have $p_j > p_{JL}$ for some $j \ge 3$, N has to be sufficiently large. Specifically, $p_j > p_{JL}$ if and only if $N > (2j-1)^2 + 1$. Thus, for example, if $N \le 26$, then $\lambda = 0$ is not an eigenvalue of problem (2.11) for any $p \in (p_{JL}, p_L)$; if $26 < N \le 50$, it is an eigenvalue for exactly one $p \in (p_{JL}, p_L)$, namely $p = p_3$; and so on.

(ii) Assume $p \in (p_{JL}, p_L)$. As noted in the previous remark, the assumption of statement (ii) of Proposition 4.1 ($\lambda = 0$ being an eigenvalue of (2.11)) is void if $N \le 26$. Also, if $26 < N \le 50$ and the assumption is satisfied, then necessarily $p = p_3$ (and $j = 3$). In this case, condition (4.6) is automatically satisfied. This follows from the relations (4.9) and $\beta = -2/(p-1) - 6$ (cp. (2.13)). However, for larger dimensions, (4.6) is not always satisfied. For example, in the case of $p = p_3$ (when $\lambda_3 = 0$), (4.6) is not satisfied if $N > 56$.

Before proving Proposition 4.1 , we show how it implies (4.1).

Proof of (4.1). Recall that the function G is analytic in a neighborhood of ζ_0. Proposition 4.1 implies that either $F'(\zeta_0) \ne G'(\zeta_0)$ or there exists $\zeta_1 < \zeta_0$ such that $F''(\zeta) \ne G''(\zeta)$ for all $\zeta \in (\zeta_1, \zeta_0)$. In either case, ζ_0 is clearly not an accumulation point of the set of zeros of the function $F - G$. This is equivalent to (4.1). □

Remark 4.3. There is a strong indication (see Remark 4.4) that whenever $\lambda = 0$ is an eigenvalue of problem (2.11), then there is an integer $k \ge 2$ such that

$$|F^{(k)}(\zeta)| \to \infty \quad \text{as } \zeta \nearrow \zeta_0. \tag{4.10}$$

If confirmed, this could be used – instead of statement (ii)(b) of Proposition 4.1 – as an alternative proof of (4.1) (the arguments would be similar as with $k = 2$ in the case (ii)(a)).

The rest of the section devoted to the proof of Proposition 4.1. We carry out the proof in several steps. In some cases, we do the computations in greater generality than needed for the proof, as these may be of some interest and do not require much extra work.

STEP 1: Computation of the derivatives $F^{(k)}(\zeta)$, $G^{(k)}(\zeta)$.

Recall that, assuming α is sufficiently large, we have $F(\zeta) = w_r(r_0, \hat\alpha(\zeta))$, where $\hat\alpha$ is the inverse to $\alpha \mapsto w(r_0, \alpha)$. Therefore, we have

$$F'(\zeta) = \frac{w_{r\alpha}(r_0, \alpha)}{w_\alpha(r_0, \alpha)}, \quad \text{with } \alpha = \hat\alpha(\zeta). \tag{4.11}$$

Similarly, for any integer $k > 1$, if $F^{(k)}(\zeta) =: g(\alpha)$ for $\alpha = \hat\alpha(\zeta)$, then

$$F^{(k+1)}(\zeta) = \frac{1}{w_\alpha(r_0, \alpha)} \partial_\alpha g(\alpha).$$

Hence, by induction, for $k = 1, 2, \dots$ we have

$$F^{(k)}(\zeta) = (\hat\partial_\alpha)^k w_r(r_0, \alpha), \quad \text{with } \alpha = \hat\alpha(\zeta), \tag{4.12}$$

where $\hat\partial_\alpha$ is a differential operator given by

$$\hat\partial_\alpha := \frac{1}{w_\alpha(r_0, \alpha)} \partial_\alpha.$$

Set

$$\psi(r, \alpha) := \hat\partial_\alpha w(r, \alpha) = \frac{w_\alpha(r, \alpha)}{w_\alpha(r_0, \alpha)}. \tag{4.13}$$

Note that $\psi(\cdot, \alpha)$ is a solution of the following problem with a homogeneous differential equation:

$$z_{rr} + \left(\frac{N-1}{r} - \frac{r}{2}\right) z_r + \left(-\frac{1}{p-1} + p(w(r,\alpha))^{p-1}\right) z = 0, \quad r \in (0, R], \tag{4.14}$$

$$z(r_0) = 1, \quad z(r) \text{ is bounded as } r \searrow 0. \tag{4.15}$$

Similarly, for $k = 2, 3, \dots$, $\hat{\partial}_\alpha^k w(r, \alpha)$ is a solution of the following problem with a nonhomogeneous differential equation:

$$z_{rr} + \left(\frac{N-1}{r} - \frac{r}{2}\right) z_r + \left(-\frac{1}{p-1} + p(w(r,\alpha))^{p-1}\right) z = f_k(r,\alpha), \quad r \in (0, R], \tag{4.16}$$

$$z(r_0) = 0, \quad z(r) \text{ is bounded as } r \searrow 0, \tag{4.17}$$

where

$$f_k(r,\alpha) = p(w(r,\alpha))^{p-1} \hat{\partial}_\alpha^k w(r,\alpha) - \hat{\partial}_\alpha^k w^p(r,\alpha). \tag{4.18}$$

We remark that (4.18) is just a compact way of writing the right-hand side. The k-derivative actually cancels out in (4.18) so f_k depends on lower derivatives only. Obviously, for each fixed α, the function $f_k(r,\alpha)$ is bounded as $r \to 0$.

For the function G and $k = 1, 2 \dots$, we have similarly as for F in (4.11), (4.12),

$$G^{(k)}(\zeta) = (\tilde{\partial}_\ell)^k u_r(r_0, \ell), \quad \text{with } \ell = \hat{\ell}(\zeta), \tag{4.19}$$

where

$$\tilde{\partial}_\ell := \frac{1}{u_\ell(r_0,\ell)} \partial_\ell. \tag{4.20}$$

STEP 2: Relation of $F'(\zeta_0)$ to $G'(\zeta_0)$ and the proof of statement (i) of Proposition 4.1.

We find the (left) derivative $F'(\zeta_0)$ using the definition of F and the L'Hospital rule:

$$\begin{aligned} \lim_{\zeta \nearrow \zeta_0} \frac{F(\zeta) - F(\zeta_0)}{\zeta - \zeta_0} &= \lim_{\alpha \to \infty} \frac{w_r(r_0,\alpha) - \phi'_\infty(r_0)}{w(r_0,\alpha) - \phi_\infty(r_0)} \\ &= \lim_{\alpha \to \infty} \frac{w_{\alpha r}(r_0,\alpha)}{w_\alpha(r_0,\alpha)} = \lim_{\alpha \to \infty} \psi_r(r_0,\alpha), \end{aligned} \tag{4.21}$$

where ψ is as in (4.13). Using Lemma 2.3, the uniform bound (2.22), and regularity properties of solutions of linear differential equations, one shows easily that, as $\alpha \to \infty$,

$$\begin{aligned} \psi(\cdot,\alpha) &\to \psi_\infty, \\ \psi_r(\cdot,\alpha) &\to \psi'_\infty, \end{aligned} \tag{4.22}$$

with the convergence in $L^\infty_{loc}(0,\infty)$, where ψ_∞ is a positive solution of Eq. (2.15) satisfying the following relations for some positive constant C_1:

$$\psi_\infty(r_0) = 1, \qquad \psi_\infty(r) \le C_1 r^\beta \quad (r \in [0, r_0]) \tag{4.23}$$

(β is as in (2.14)). The solution ψ_∞ is uniquely determined. In fact, it follows from (4.23) that

$$\psi_\infty \equiv c_0 \psi_1 \ \text{ with } c_0 := \frac{1}{\psi_1(r_0)}, \tag{4.24}$$

where ψ_1 is as in Lemma 2.5.

Thus, the limit in (4.21) exists and we have $F'(\zeta_0) = \psi'_\infty(r_0)$. Now, by (4.12), (4.13), we also have

$$\lim_{\zeta \nearrow \zeta_0} F'(\zeta) = \lim_{\alpha \to \infty} \psi_r(r_0,\alpha) = \psi'_\infty(r_0), \tag{4.25}$$

showing that F is of class C^1 on $(w(r_0,\alpha^*), \zeta_0]$.

The derivative $G'(\zeta_0)$ is obtained directly from (4.19), (4.20) using the relations $u(r_0, L) = \phi_\infty(r_0) = \zeta_0$:

$$G'(\zeta_0) = \frac{u_{r\ell}(r_0,L)}{u_\ell(r_0,L)}.$$

By Lemma 3.2, $\hat{\psi}(r) := u_\ell(r, L)/u_\ell(r_0, L)$ is a solution of (2.15) satisfying $\hat{\psi}(r_0) = 1$ and

$$r^{2/(p-1)}\hat{\psi}(r) \to \frac{1}{u_\ell(r_0, L)} \quad \text{as } r \to \infty. \tag{4.26}$$

We now complete the proof of statement (i) of Proposition 4.1.The relation $F'(\zeta_0) = G'(\zeta_0)$ is equivalent to $\psi'_\infty(r_0) = \hat{\psi}'(r_0)$. Since also $\psi_\infty(r_0) = 1 = \hat{\psi}(r_0)$ and ψ_∞, $\hat{\psi}$ are solutions of (2.15), the relation $F'(\zeta_0) = G'(\zeta_0)$ is actually equivalent to the identity $\hat{\psi} \equiv \psi_\infty$. In view of (4.24) and (4.26), the identity means that the solution ψ_1 in Lemma 2.5 satisfies (2.19). By Lemma 2.5, this is equivalent to $\lambda = 0$ being an eigenvalue of (2.11). Statement (i) is proved.

STEP 3: Variation of constants and an integral formula for $F^{(k)}(\zeta)$.

We find a tangible formula for the functions $F^{(k)}(\zeta)$, $k = 2, 3, \ldots$. For a while, we will consider $\alpha > \alpha^*$ fixed and write ψ for $\psi(\cdot, \alpha)$, f_k for $f_k(\cdot, \alpha)$. Remember that ψ is a solution of (4.14), (4.15). Let φ be the solution of (4.14) with

$$\varphi(r_0) = 0, \quad \varphi'(r_0) = -\frac{1}{\omega(r_0)}, \tag{4.27}$$

where ω is defined in (2.12).

Obviously, ψ, φ are linearly independent. We claim that for some constant $c \neq 0$ one has

$$\varphi(r) = cr^{-(N-2)} + o(r^{-(N-2)}) \quad \text{as } r \to 0. \tag{4.28}$$

In fact, any solution φ linearly independent from ψ has this property. One way to see this is by using the Frobenius method. Observe that multiplying Eq. (4.14) by r^2, we obtain an equation with analytic coefficients (near $r = 0$) and a regular singular point at $r = 0$. We look for solutions in the form of a convergent Frobenius series

$$z(r) = r^\vartheta \sum_{j=0}^{\infty} c_j r^j, \tag{4.29}$$

where c_j are real coefficients, $c_0 \neq 0$, and ϑ is a root of the indicial equation

$$\vartheta(\vartheta - 1) + (N - 1)\vartheta = 0.$$

The larger root $\vartheta = 0$ always yields solutions of the form (4.29); such solutions are bounded near 0 and they are all scalar multiples of ψ. Now, since the smaller root, $\vartheta := -(N - 2)$, is also an integer, the linearly independent solution φ is either given by (4.29) (with $\vartheta = -(N - 2)$), or by the formula

$$\varphi(r) = C\psi(r) \log r + z(r) \tag{4.30}$$

where z is as in (4.29) with $c_0 \neq 0$ and $C \in \mathbb{R}$ (possibly $C = 0$), see [28, Theorem 4.5], for example. In either case, (4.28) holds.

We use the linearly independent solutions ψ, φ in the variation of constants formula. The homogeneous equation (4.14) can be written as

$$(\omega(r)z_r)_r + \omega(r)\left(-\frac{1}{p-1} + pw^{p-1}(r, \alpha)\right)z = 0.$$

The Wronskian of the solutions ψ, φ, that is, the function

$$W(r) := \psi'(r)\varphi(r) - \psi(r)\varphi'(r),$$

satisfies $(\omega(r)W(r))' = 0$ for all $r > 0$ (as long as $w(\cdot, \alpha)$ stays positive) and $\omega(r_0)W(r_0) = 1$ (cp. (4.15), (4.27)). So $W(r) = 1/\omega(r)$ for all $r > 0$. A standard variation of constants formula (easily verified by direct differentiation) yields the general solution of (4.16):

$$z(r) = \left(c_1 - \int_r^{r_0} \omega(s) f_k(s)\varphi(s)\, ds\right)\psi(r) + \left(c_2 + \int_r^{r_0} \omega(s) f_k(s)\psi(s)\, ds\right)\varphi(r). \tag{4.31}$$

Here $c_1, c_2 \in \mathbb{R}$ are arbitrary parameters. For (4.31) to give a solution with $z(r_0) = 0$, it is necessary and sufficient that $c_1 = 0$. For this solution to also be bounded as $r \to 0$, it is necessary that

$$c_2 + \int_0^{r_0} \omega(s) f_k(s)\psi(s)\, ds = 0.$$

This follows from the boundedness of ψ, f_k, and formulas (4.28), (2.12). Thus, we get

$$z(r) = -\psi(r)\int_r^{r_0} \omega(s) f_k(s)\varphi(s)\, ds - \varphi(r)\int_0^{r} \omega(s) f_k(s)\psi(s)\, ds, \tag{4.32}$$

showing in particular that the solution of (4.16), (4.17) is unique. Using (4.28), (2.12), one verifies easily that the function z given by (4.32) is bounded near $r = 0$, so it is the unique solution of (4.16), (4.17).

Differentiating (4.32) and using (4.27), we obtain

$$z'(r_0) = \frac{1}{\omega(r_0)}\int_0^{r_0} \omega(s) f_k(s)\psi(s)\, ds. \tag{4.33}$$

We now summarize the above computations, bringing back the α-variable. Using (4.12), (4.16), (4.17), and substituting from (2.12), we obtain that for $k = 2, 3, \ldots$

$$F^{(k)}(\zeta) = \frac{1}{\omega(r_0)}\int_0^{r_0} s^{N-1}e^{-s^2/4} f_k(s,\alpha)\psi(s,\alpha)\, ds, \quad \text{with } \alpha = \hat{\alpha}(\zeta), \tag{4.34}$$

where

$$f_k(r,\alpha) = pw^{p-1}(r,\alpha)\hat{\partial}_\alpha^k w(r,\alpha) - \hat{\partial}_\alpha^k w^p(r,\alpha). \tag{4.35}$$

STEP 4: Estimates of $F''(\zeta)$ as $\zeta \nearrow \zeta_0$ and the proof of statement (ii)(a) of Proposition 4.1.

For $k = 2$, formulas (4.34), (4.35), (4.13) give

$$F''(\zeta) = -\frac{p(p-1)}{\omega(r_0)}\int_0^{r_0} s^{N-1}e^{-s^2/4}(w(s,\alpha))^{p-2}\psi^3(s,\alpha)\, ds \quad (\alpha = \hat{\alpha}(\zeta)). \tag{4.36}$$

Recall that $\psi^3(\cdot,\alpha) > 0$ on $[0,R) \supset [0,r_0]$ for all $\alpha > \alpha^*$.

We have $\alpha \to \infty$ as $\zeta \to \zeta_0$. Also, for any $s \in (0, r_0]$,

$$\begin{aligned}\lim_{\alpha\to\infty}(w(s,\alpha))^{p-2}\psi^3(s,\alpha) &= (\phi_\infty(s))^{p-2}(\psi_\infty(s))^3 \\ &= L^{p-2}s^{-2(p-2)/(p-1)}(\psi_\infty(s))^3\end{aligned} \tag{4.37}$$

(see Step 1). Now, by (4.24), ψ_∞ is a nonzero scalar multiple of the solution ψ_1 in Lemma 2.5. Moreover, being the limit of $\psi(\cdot,\alpha)$, ψ_∞ is nonnegative (hence positive) in $(0, r_0]$. Therefore, by (2.16), there is a positive constant c_1 such that

$$c_1 s^{3\beta} \le (\psi_\infty(s))^3 \le c_1^{-1}s^{3\beta} \quad (s \in (0, r_0)). \tag{4.38}$$

Assume now that (4.6) is true. It follows from (4.36)–(4.38) and Fatou's lemma that for some positive constant c_2 one has

$$\limsup_{\zeta\nearrow\zeta_0} F''(\zeta) \le -c_2\int_0^{r_0} s^{N-1+3\beta-2(p-2)/(p-1)}\, ds. \tag{4.39}$$

By (4.6), the integral is infinite, hence (4.8) holds. With this we have completed the proof statement (ii)(a) of Proposition 4.1.

Returning to (4.36), we now find the limit of $F''(\zeta)$ assuming (4.6) is not true, that is,

$$\gamma := N - 1 + 3\beta - \frac{2(p-2)}{(p-1)} > -1. \tag{4.40}$$

Using the relations (2.22) and $w(r,\alpha) < \phi_\infty(r)$, we find an upper bound on the integrand in (4.36) in the form cs^γ, where $c > 0$ is a constant. By (4.40), this is an integrable function, hence (4.37) and the Lebesgue dominated convergence theorem yield the finite limit

$$\lim_{\zeta \nearrow \zeta_0} F''(\zeta) = -\frac{p(p-1)L^{p-2}}{\omega(r_0)} \int_0^{r_0} s^{N-1-2(p-2)/(p-1)} e^{-s^2/4} (\psi_\infty(s))^3 \, ds. \tag{4.41}$$

STEP 5: Fredholm alternative and the proof of statement (ii)(b) of Proposition 4.1.

We assume that $\lambda = 0$ is an eigenvalue of (2.11) (that is, $F'(\zeta_0) = G'(\zeta_0)$) and (4.40) holds. Thus the limit $\lim_{\zeta \nearrow \zeta_0} F''(\zeta)$ is given by (4.41) and it is finite. We claim that for the relation

$$\lim_{\zeta \nearrow \zeta_0} F''(\zeta) = G''(\zeta_0) \tag{4.42}$$

to hold it is necessary that

$$\int_0^\infty s^{N-1-2(p-2)/(p-1)} e^{-s^2/4} \psi_\infty^3(s) \, ds = 0. \tag{4.43}$$

We first prove this claim and then verify that (4.43) does not hold. This will prove statement (ii)(b) and complete the proof of Proposition 4.1.

To prove the claim, assume (4.42). According to (4.12), the finite limit in (4.42) is also the limit of $(\hat{\partial}_\alpha)^2 w_r(r_0,\alpha)$ as $\alpha \to \infty$. Therefore, using the condition $(\hat{\partial}_\alpha)^2 w(r_0,\alpha) = 0$ (cp. (4.17)) and taking the limit in Eq. (4.16) with $k = 2$, we infer that, as $\alpha \to \infty$,

$$z_2(\cdot,\alpha) := (\hat{\partial}_\alpha)^2 w(\cdot,\alpha) \to z_2^\infty \quad \text{in } C^1_{loc}(0,\infty)$$

where z_2^∞ is the solution of the initial value problem

$$z_{rr} + \left(\frac{N-1}{r} - \frac{r}{2}\right) z_r + \left(-\frac{1}{p-1} + p\phi_\infty^{p-1}(r)\right) z = f_2^\infty(r), \tag{4.44}$$

$$z(r_0) = 0, \quad z'(r_0) = G''(\zeta_0), \tag{4.45}$$

with

$$f_2^\infty(r) := -p(p-1)\phi_\infty^{p-2}(r)\psi_\infty^2(r) = -p(p-1)L^{p-2} r^{-\frac{2(p-2)}{p-1}} \psi_\infty^2(r) \tag{4.46}$$

(ψ_∞ is as in (4.22)). We can also take the limit in the variation of constants formula for $z_2(\cdot,\alpha)$, namely, formula (4.32) with $k = 2$, $f_k = f_k(\cdot,\alpha)$, $\psi = \psi(\cdot,\alpha)$, and $\varphi = \varphi(\cdot,\alpha)$—the solution of the linear equation (4.14) with the initial conditions (4.27). This gives

$$z_2^\infty(r) = -\psi_\infty(r) \int_r^{r_0} \omega(s) f_2^\infty(s) \varphi_\infty(s) \, ds - \varphi_\infty(r) \int_0^r \omega(s) f_2^\infty(s) \psi_\infty(s) \, ds, \tag{4.47}$$

where φ_∞ is the solution of (2.15) with $\varphi_\infty(r_0) = 0$, $\varphi'_\infty(r_0) = -1/\omega(r_0)$. Note that ψ_∞, φ_∞ are linearly independent solutions of the homogeneous equation (2.15) and (4.47) is a version of the variation of constant formula for the solution z_2^∞; it is valid for all $r > 0$.

We next use the relations $F'(\zeta_0) = G'(\zeta_0)$ and (4.42) to show that the function $r^{2/(p-1)} z_2^\infty(r)$ is bounded as $r \to \infty$. By (4.19), (4.20),

$$G''(\zeta_0) = (\tilde{\partial}_\ell)^2 u_r(r_0,\ell)\,\big|_{\ell=L}. \tag{4.48}$$

Just like z_2^∞, the function

$$\tilde{z}_2(r) := (\tilde{\partial}_\ell)^2 u(r,\ell)\,\big|_{\ell=L} = \frac{1}{u_\ell(r_0,L)}\partial_\ell\left(\frac{u_\ell(r,\ell)}{u_\ell(r_0,\ell)}\right)\big|_{\ell=L} \tag{4.49}$$

is a solution of a nonhomogeneous linear equation, namely, Eq. (3.10) with the function $u_\ell(r,L)$ on the right-hand side replaced by the function

$$\tilde{\partial}_\ell u(r,\ell)\,\big|_{\ell=L} = \frac{u_\ell(r,L)}{u_\ell(r_0,L)}.$$

From Step 2 we know that this function is identical to ψ_∞. Thus z_2^∞ and $\tilde{z}_2$ both solve equation (4.44). Also, since $\tilde{\partial}_\ell u(r_0,\ell) = 1$ for all $\ell \approx L$, we have $\tilde{z}_2(r_0) = 0$. Thus, (4.48) and (4.45) imply that $\tilde{z}_2 \equiv z_2^\infty$. The boundedness of the function $r^{2/(p-1)} z_2^\infty(r)$ as $r \to \infty$ is now a consequence of (4.49), Remark 3.4 and the fact that the function $r^{2/(p-1)} u_\ell(r,L)$ is bounded as $r \to \infty$ (cp. Lemma 3.2).

We now prove (4.43), making use of (4.47). (Alternatively, one could invoke the Fredholm alternative for the nonhomogeneous equation (4.44) after estimating the solution z_2^∞ and its derivative near $r = 0$.) We need some information on the asymptotics of the function $\varphi_\infty(r)$ as $r \to \infty$. Recall that the asymptotics of ψ_∞ is the same as the asymptotics of the function ψ_1 given in (2.19).

Similarly as for the functions ψ, φ, the Wronskian of the functions φ_∞, ψ_∞ satisfies the following identity

$$\psi'_\infty(r)\varphi_\infty(r) - \psi_\infty(r)\varphi'_\infty(r) = \frac{1}{\omega(r)} \quad (r > 0).$$

Therefore, for large enough r we have ($\psi_\infty(r) \neq 0$ and)

$$\frac{d}{dr}\frac{\varphi_\infty(r)}{\psi_\infty(r)} = -\frac{1}{\omega(r)\psi_\infty^2(r)}. \tag{4.50}$$

Hence, for any $R > 0$ there is a constant c such that

$$\varphi_\infty(r) = \psi_\infty(r)\left(c - \int_R^r \frac{1}{\omega(s)\psi_\infty^2(s)}\,ds\right).$$

Using this, the asymptotics of ψ_∞, and expression (4.46), one shows via a simple computation that the function $\omega f_2^\infty \varphi_\infty$ is integrable on (r_0,∞). Also, the growth of φ_∞ and the boundedness of the function $r^{2/(p-1)} z_2^\infty(r)$ as $r \to \infty$ imply that the coefficient of $\varphi_\infty(r)$ in (4.47) approaches 0 as $r \to \infty$, that is, (4.43) holds. This proves our claim.

It remains to prove that (4.43) does not hold. Recall, that we assume that $\lambda = 0$ is an eigenvalue of (2.11), or, in other words, that for some $j \geq 2$ we have

$$\lambda_j = \frac{\beta}{2} + \frac{1}{p-1} + j = 0 \tag{4.51}$$

(cp. Lemma 2.4). Also recall that ψ_∞ is an eigenfunction associated with λ_j. As shown in [13,19], up to a scalar multiple, $\psi_\infty(r) = r^\beta M_j(r^2/4)$, where

$$M_j(z) := M\Big(-j, \beta + \frac{N}{2}, z\Big) \tag{4.52}$$

is the standard Kummer function. The assumption $\lambda_j = 0$ yields formula (4.9) for $p = p_j$ and $b := \beta + \frac{N}{2}$ can be expressed as

$$b = (N - 4(j-1)^2)/(4j-2). \tag{4.53}$$

Note that (4.40) implies that $b - j - 1 > 0$.

Using (4.52), (4.53) in the integral in (4.43) and making the substitution $z = s^2/4$, we see that (4.43) is equivalent to the following relation

$$\int_0^\infty z^{b-j-2} e^{-z} M_j^3(z)\, dz = 0. \tag{4.54}$$

Since j is an integer, the Kummer function M_j can be expressed in terms of a generalized Laguerre polynomial of degree j:

$$M(-j, b, z) = \frac{\Gamma(j+1)\Gamma(b)}{\Gamma(j+b)} L(j, b-1, z)$$

(Γ stands for the standard Gamma function). Thus we have the following relation equivalent to (4.43):

$$\int_0^\infty z^{b-j-2} e^{-z} L^3(j, b-1, z)\, dz = 0. \tag{4.55}$$

Since $B := b-j-1$ is positive, Proposition A.1 in the Appendix shows that the integral in (4.55) is positive. Thus (4.43) does not hold, and the proof of Proposition 4.1 is complete.

Remark 4.4. Returning to Remark 4.3, we comment on the validity of relations (4.10), which could be used instead of Step 5 and the Appendix in the proof of (4.1). By (4.18) and the chain rule, the function $s^{N-1} f_k(s,\alpha)\psi(s,\alpha)$ in (4.34) contains in particular the term

$$p(p-1)\dots(p-k+1)s^{N-1}(w(s,\alpha))^{p-k}\psi^{k+1}(s,\alpha)$$

whose limit as $\alpha \nearrow \infty$ is

$$L^{p-k} p(p-1)\dots(p-k+1)s^{N-1-\frac{2(p-k)}{p-1}}(\psi_\infty(s))^{k+1}$$

(cp. (4.37)). This term has the singularity of

$$s^{N-1-\frac{2(p-k)}{(p-1)}+(k+1)\beta}$$

at $s = 0$. Since $\beta + \frac{2}{p-1} = -2j$ for some $j \geq 2$ (cp. (4.51)), this singularity is not integrable near 0 if k is large enough. This makes it reasonable to expect that (4.10) holds for some $k \geq 2$. However, to make this into a proof, one would need to account for all the other terms in $s^{N-1} f_k(s,\alpha)\psi(s,\alpha)$ obtained from (4.18). It is difficult to keep track of possible cancellations of the singularities of these terms in the limit as $\alpha \to \infty$.

Appendix. Integrals with Laguerre polynomials

Proposition A.1. *Let*

$$Q_j(B) := \frac{1}{\Gamma(B)} \int_0^\infty x^{B-1} e^{-x} L^3(j, B+j, x)\, dx, \tag{A.1}$$

where $B > 0$, $j \geq 2$ and

$$L(j, \alpha, x) := \sum_{i=0}^{j} (-1)^i \binom{j+\alpha}{j-i} \frac{x^i}{i!} \tag{A.2}$$

is the generalized Laguerre polynomial. Then Q_j is a polynomial in B with positive coefficients; in particular, $Q(B) > 0$ for any $B > 0$.

Positivity of similar integrals involving Laguerre polynomials has been established in a number of combinatorics papers (see for example [7,27] and references therein). However, in these papers special relations between the exponent of x and the second argument of L are needed, and we were not able to make use of the integrals or techniques in these papers for proving Proposition A.1. Our proof is completely independent.

Proof of Proposition A.1. Given integers n_1, n_2, set

$$T_{n_1}^{n_2}(B) := \prod_{n=n_1}^{n_2} (B+n) \qquad (T_{n_1}^{n_2}(B) = 1 \text{ if } n_1 > n_2).$$

Let $0 \le i, m, n \le j$. The recurrence relation for the Laguerre polynomials and the orthogonality of these polynomials (see [1, Chapter 22], for example) give

$$L(j, B+j, x) = \sum_{m=0}^{j} \binom{2j-i-m}{j-m} L(m, B+i-1, x), \tag{A.3}$$

and

$$\int_0^\infty x^{B+i-1} e^{-x} L(m, B+i-1, x) L(n, B+i-1, x)\, dx = \frac{\Gamma(B+m+i)}{m!}\delta_{nm} = \frac{\Gamma(B)}{m!} T_0^{m+i-1}(B)\delta_{nm}. \tag{A.4}$$

Using (A.2), (A.3), and (A.4), we obtain

$$Q_j(B) = \frac{1}{\Gamma(B)} \sum_{i=0}^{j} (-1)^i \binom{B+2j}{j-i} \frac{1}{i!} \int_0^\infty x^{B+i-1} e^{-x} L^2(j, B+j, x)\, dx$$
$$= \frac{1}{j!} \sum_{i=0}^{j} (-1)^i \binom{j}{i} T_{j+i+1}^{2j}(B) \sum_{m=0}^{j} \binom{2j-i-m}{j-m}^2 \frac{1}{m!} T_0^{m+i-1}(B),$$

hence

$$Q_j(B) = \frac{1}{j!} T_0^{2j}(B) \sum_{m=0}^{j} \frac{1}{m!} S_{j,m}(B),$$

where

$$S_{j,m}(B) := \sum_{i=0}^{j} (-1)^i \binom{j}{i} \binom{2j-i-m}{j-m}^2 \frac{1}{T_{m+i}^{j+i}(B)}.$$

We show that the polynomial $T_0^{2j}(B) S_{j,m}(B)$ has positive coefficients for each $m = 0, 1, \dots, j$.

Given $j \ge 0$, $0 \le k_1 \le k_2$ and $B > k_2 - j$, set

$$S(j, k_1, k_2, B) := \sum_{i=0}^{j} (-1)^i \binom{j}{i} \binom{j+k_1-i}{k_1}^2 \frac{1}{T_{j-k_2+i}^{j+i}(B)}.$$

Notice that $S(j, j-m, j-m, B) = S_{j,m}(B)$.

Using the identities

$$\binom{j+1}{i} = \binom{j}{i} + \binom{j}{i-1}, \quad T_{j+1-k+i}^{j+1+i}(B) = T_{j-k+i}^{j+i}(B+1)$$

and the induction in j, one easily obtains

$$S(j, 0, k_2, B) = \frac{(j+k_2)!}{k_2!} \frac{1}{T_{j-k_2}^{2j}(B)}, \qquad j, k_2 \ge 0,\ B > k_2 - j. \tag{A.5}$$

Next assume $j \ge 2$, $0 < k_1 \le k_2$, and $B > k_2 - j$. Using the identity

$$(j+k_1-i)^2 = (j+k_1)^2 - i(2j+2k_1-1) + i(i-1)$$

and denoting

$$T_i := T_{j-k_2+i}^{j+i}(B) = T_{j-1-k_2+i-1}^{j-1+i-1}(B+2) = T_{j-2-k_2+i-2}^{j-2+i-2}(B+4)$$

we obtain

$$\begin{aligned}
&S(j,k_1,k_2,B)\\
&= \frac{1}{k_1^2}\sum_{i=0}^{j}(-1)^i\binom{j}{i}\binom{j+k_1-1-i}{k_1-1}^2\frac{[(j+k_1)^2-i(2j+2k_1-1)+i(i-1)]}{T_i}\\
&= \frac{(j+k_1)^2}{k_1^2}\sum_{i=0}^{j}(-1)^i\binom{j}{i}\binom{j+k_1-1-i}{k_1-1}^2\frac{1}{T_i}\\
&\quad+\frac{(2j+2k_1-1)j}{k_1^2}\sum_{i=1}^{j}(-1)^{i-1}\binom{j-1}{i-1}\binom{j-1+k_1-1-(i-1)}{k_1-1}^2\frac{1}{T_i}\\
&\quad+\frac{j(j-1)}{k_1^2}\sum_{i=2}^{j}(-1)^{i-2}\binom{j-2}{i-2}\binom{j-2+k_1-1-(i-2)}{k_1-1}^2\frac{1}{T_i}\\
&= \frac{(j+k_1)^2}{k_1^2}S(j,k_1-1,k_2,B)+\frac{(2j+2k_1-1)j}{k_1^2}S(j-1,k_1-1,k_2,B+2)\\
&\qquad+\frac{j(j-1)}{k_1^2}S(j-2,k_1-1,k_2,B+4).
\end{aligned}$$

Repeating this argument finitely many times, we obtain

$$S(j,k_1,k_2,B)=\sum_{(\tilde{j},\tilde{k}_1)\in A}c_{\tilde{j},\tilde{k}_1}S(\tilde{j},\tilde{k}_1,k_2,B+2(j-\tilde{j})), \tag{A.6}$$

where

$$A=\{(\tilde{j},\tilde{k}_1): 0\le\tilde{j}\le j,\ 0\le\tilde{k}_1\le k,\ \text{either } \tilde{k}_1=0 \text{ or } \tilde{j}\le 1\},$$

$c_{\tilde{j},\tilde{k}_1}\ge 0$ and $\sum_A c_{\tilde{j},\tilde{k}_1}>0$.

Fix $j\ge 2$, $0\le m\le j$, $k_1=k_2=j-m$, $B>0$, and let $(\tilde{j},\tilde{k}_1)\in A$. Then (A.5) and the definition of $S(j,k_1,k_2,B)$ imply

$$S(\tilde{j},\tilde{k}_1,k_2,B+2(j-\tilde{j}))=\begin{cases}\dfrac{(\tilde{j}+k_2)!}{k_2!}\,\dfrac{1}{T^{2j}_{j-k_2+(j-\tilde{j})}(B)} & \text{if } \tilde{k}_1=0,\\[2ex] \dfrac{1}{T^{2j}_{2j-k_2}(B)} & \text{if } \tilde{j}=0,\\[2ex] \dfrac{(\tilde{k}_1^2+2\tilde{k}_1)(B+2j)+k_2+1}{T^{2j}_{2j-1-k_2}(B)} & \text{if } \tilde{j}=1.\end{cases}$$

This and (A.6) imply the desired conclusion. □

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