THE BRAUER GROUP OF $\mathscr{M}_{1,1}$ OVER ALGEBRAICALLY CLOSED FIELDS OF CHARACTERISTIC 2

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Abstract. We prove that the Brauer group of the moduli stack of elliptic curves $\mathscr{M}_{1,1,k}$ over an algebraically closed field k of characteristic 2 is isomorphic to $\mathbb{Z}/(2)$. We also compute the Brauer group of $\mathscr{M}_{1,1,k}$ where k is a finite field of characteristic 2.

1. Introduction

Let $\mathscr{M}_{1,1,\mathbb{Z}}$ denote the moduli stack of elliptic curves over $\mathbb{Z}$. For any scheme S, we denote by $\mathscr{M}_{1,1,S} := S \times_{\mathbb{Z}} \mathscr{M}_{1,1,\mathbb{Z}}$ the restriction of $\mathscr{M}_{1,1,\mathbb{Z}}$ to the category of schemes over S.

Antieau and Meier [AM16, 11.2] computed the Brauer group $\operatorname{Br} \mathscr{M}_{1,1,S}$ for various base schemes S, and in particular proved that for any algebraically closed field k of characteristic not 2 the Brauer group $\operatorname{Br} \mathscr{M}_{1,1,k}$ is trivial. The purpose of this note is to compute $\operatorname{Br} \mathscr{M}_{1,1,k}$ in the characteristic 2 case. This then completes the calculation of $\operatorname{Br} \mathscr{M}_{1,1,k}$ over algebraically closed fields k. We summarize the result in the following theorem.

Theorem 1.1 ([AM16, 11.2] in $\operatorname{char} k \neq 2$)**.** *Let k be an algebraically closed field. Then $\operatorname{Br} \mathscr{M}_{1,1,k}$ is 0 unless $\operatorname{char} k = 2$, in which case $\operatorname{Br} \mathscr{M}_{1,1,k} = \mathbb{Z}/(2)$.*

To prove the theorem, we calculate the cohomology groups $\mathrm{H}^2_{\text{ét}}(\mathscr{M}_{1,1,k}, \mu_n)$ for varying n. There are essentially two ways to approach this calculation: (1) using the coarse moduli space; (2) using a presentation of $\mathscr{M}_{1,1,k}$ as a quotient stack. In this paper we give a new proof of the Antieau-Meier result using approach (1), and calculate in characteristic 2 using approach (2).

We also compute the Brauer group of $\mathscr{M}_{1,1,k}$ where k is a finite field of characteristic 2:

Theorem 1.2. *Let k be a finite field of characteristic 2. Then*

$$\operatorname{Br} \mathscr{M}_{1,1,k} = \begin{cases} \mathbb{Z}/(12) \oplus \mathbb{Z}/(2) & \text{if } x^2+x+1 \text{ has a root in } k \\ \mathbb{Z}/(24) & \text{otherwise.} \end{cases}$$

An outline of the paper is as follows.

In Section 2 we state definitions and recall general facts about the Brauer group of algebraic stacks.

In Section 3 we record some general remarks regarding $\operatorname{Br} \mathscr{M}_{1,1,S}$. We show that if S is a quasi-compact scheme admitting an ample line bundle and if at least one prime is invertible on S, then $\operatorname{Br} \mathscr{M}_{1,1,S} \simeq \operatorname{Br}' \mathscr{M}_{1,1,S}$. The restriction of $\mathscr{M}_{1,1,\mathbb{Z}}$ to the dense open substack of elliptic curves E/S with j-invariant $j(E) \in \Gamma(S, \mathcal{O}_S)$ for which $j(E)$ and $j(E) - 1728$ are invertible is a trivial $\mathbb{Z}/(2)$-gerbe over the coarse space $\mathbb{A}^1_{\mathbb{Z}} \setminus \{0, 1728\}$, and we use this fact

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to conclude that $\operatorname{Br}\mathscr{M}_{1,1,k}$ is a subgroup of $\mathbb{Z}/(2)\oplus\mathbb{Z}/(2)$ for an algebraically closed field k of arbitrary characteristic.

In Section 4 we give a second proof of Antieau and Meier's result above (that $\operatorname{Br}\mathscr{M}_{1,1,k}=0$ if $k=\overline{k}$ and $\operatorname{char} k\neq 2$). Using a dévissage argument, we study the relationship between the cohomology of μ_n on the stack $\mathscr{M}_{1,1,k}$ and on $\mathbb{A}^1_k$, in terms of the stabilizer groups of elliptic curves with j-invariant $0,1728\in\mathbb{A}^1_k$. This may be of independent interest for computing the Brauer groups of other separated Deligne-Mumford stacks whose coarse moduli space is a smooth curve over an algebraically closed field with vanishing Picard group.

In Section 5 we prove Theorem 1.1 and Theorem 1.2. Antieau and Meier suggest in [AM16, 11.3] that the characteristic 2 case can be settled using the $\mathrm{GL}_2(\mathbb{Z}/(3))$-cover $Y(3)\to\mathscr{M}_{1,1,k}$, where $Y(3)$ denotes the moduli stack of elliptic curves with full level 3 structure, and indeed we use this presentation $\mathscr{M}_{1,1,k}\simeq[Y(3)/\mathrm{GL}_2(\mathbb{Z}/(3))]$ as a global quotient stack to show that its Brauer group is in fact nonzero. We use the "Hesse presentation" of $Y(3)$ as in [FO10]; it is shown in Appendix A that this presentation coincides with the usual Weierstrass presentation as in [KM85]. The cohomological descent spectral sequence associated to the covering $Y(3)\to\mathscr{M}_{1,1,k}$ reduces our task to a computation of the first group cohomology of a 6-dimensional representation of $\mathrm{GL}_2(\mathbb{Z}/(3))$ over $\mathbb{F}_2$.

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2. The Brauer group of algebraic stacks

Let $(X,\mathcal{O}_X)$ be a locally ringed site [Gir71, V, §4], [Sta18, 04EU]. For any quasi-coherent $\mathcal{O}_X$-module $\mathscr{E}$, we set $\mathrm{GL}(\mathscr{E}):=\underline{\mathrm{Aut}}_{\mathcal{O}_X\text{-mod}}(\mathscr{E})$ and let $\mathrm{PGL}(\mathscr{E})$ be the sheaf quotient of $\mathrm{GL}(\mathscr{E})$ by $\mathbb{G}_{m,X}$ via the diagonal embedding. We denote $\mathrm{GL}_n(\mathcal{O}_X):=\mathrm{GL}(\mathcal{O}_X^{\oplus n})$ and $\mathrm{PGL}_n(\mathcal{O}_X):=\mathrm{PGL}(\mathcal{O}_X^{\oplus n})$. A basic fact about these groups is the Skolem-Noether theorem, which states that the morphism

$$\mathrm{PGL}_n(\mathcal{O}_X)\to\underline{\mathrm{Aut}}_{\mathcal{O}_X\text{-alg}}(\mathrm{Mat}_{n\times n}(\mathcal{O}_X))$$

is an isomorphism (see [Gir71, V.4.1]).

Definition 2.1 (Azumaya algebras). [Gro68a, §2], [Gir71, V, §4] Let $(X,\mathcal{O}_X)$ be a locally ringed site. An *Azumaya $\mathcal{O}_X$-algebra* is a quasi-coherent (non-commutative, unital) $\mathcal{O}_X$-algebra $\mathscr{A}$ such that there exists a covering $\{X_i\to X\}_{i\in I}$, positive integers n_i, and $\mathcal{O}_{X_i}$-algebra isomorphisms $\mathscr{A}|_{X_i}\simeq\mathrm{Mat}_{n_i\times n_i}(\mathcal{O}_{X_i})$.

Two Azumaya algebras $\mathscr{A}_1$ and $\mathscr{A}_2$ are *Morita equivalent* if there exist finite type locally free $\mathcal{O}_X$-modules $\mathcal{E}_1$ and $\mathcal{E}_2$, everywhere of positive rank, and an isomorphism

$$\mathscr{A}_1\otimes_{\mathcal{O}_X}\underline{\mathrm{End}}_{\mathcal{O}_X\text{-mod}}(\mathcal{E}_1)\simeq\mathscr{A}_2\otimes_{\mathcal{O}_X}\underline{\mathrm{End}}_{\mathcal{O}_X\text{-mod}}(\mathcal{E}_2)$$

of $\mathcal{O}_X$-algebras. Under tensor product of Azumaya algebras, Morita equivalence classes of Azumaya algebras form an abelian group $\operatorname{Br}X$ called the *(Azumaya) Brauer group* of X in which $[\mathscr{A}]^{-1}=[\mathscr{A}^{\mathrm{op}}]$ and the identity element is the class of trivial Azumaya algebras $[\underline{\mathrm{End}}_{\mathcal{O}_X\text{-mod}}(\mathcal{E})]$.

Definition 2.2 (Gerbe of trivializations). [Gir71, IV, §4.2], [Ols16, 12.3.5] There is a natural way to associate, to every Azumaya $\mathcal{O}_X$-algebra $\mathscr{A}$, a $\mathbb{G}_{m,X}$-gerbe $\mathcal{G}_{\mathscr{A}}$ called the *gerbe of trivializations of* $\mathscr{A}$. An object of $\mathcal{G}_{\mathscr{A}}$ is a triple

$$(U, \mathcal{E}, \sigma)$$

consisting of an object $U \in X$, a finite type locally free $\mathcal{O}_U$-module $\mathcal{E}$ (necessarily everywhere positive rank), and an isomorphism $\sigma : \underline{\mathrm{End}}_{\mathcal{O}_U\text{-mod}}(\mathcal{E}) \to \mathscr{A}|_U$ of $\mathcal{O}_U$-algebras. A morphism

$$(f, f^\sharp) : (U_1, \mathcal{E}_1, \sigma_1) \to (U_2, \mathcal{E}_2, \sigma_2)$$

consists of a morphism $f \in \mathrm{Mor}_X(U_1, U_2)$ and an isomorphism $f^\sharp : f^*\mathcal{E}_2 \to \mathcal{E}_1$ of $\mathcal{O}_{U_1}$-modules such that $\sigma_2 = \sigma_1 \circ \rho_{f^\sharp}$ where $\rho_{f^\sharp}$ denotes conjugation by $f^\sharp$. For any object $(U, \mathcal{E}, \sigma) \in \mathcal{G}_{\mathscr{A}}$, there is a canonical injection

$$\iota_{(U,\mathcal{E},\sigma)} : \mathbb{G}_{m,U} \to \underline{\mathrm{Aut}}_{(U,\mathcal{E},\sigma)}$$

of sheaves on X/U, sending $u \mapsto (\mathrm{id}_U, u)$; this is in fact an isomorphism, since if $(\mathrm{id}_U, f^\sharp) \in \mathrm{Aut}_{\mathcal{G}_{\mathscr{A}}(U)}((U, \mathcal{E}, \sigma))$ then $f^\sharp \in Z(\mathrm{End}_{\mathcal{O}_U\text{-mod}}(\mathcal{E}))$, which coincides with $\mathcal{O}_U$ since $Z(\mathrm{Mat}_{n\times n}(A)) = A$ for any commutative, unital ring A.

By the Skolem-Noether theorem, any two local trivializations of $\mathscr{A}$ are locally related by an automorphism of the trivializing vector bundle $\mathcal{E}$, i.e. any two objects of $\mathcal{G}_{\mathscr{A}}$ are locally isomorphic. Furthermore, according to the definition, an Azumaya algebra is locally trivial, i.e. for any $U \in X$ there exists a covering $\{U_i \to U\}$ such that the fiber category $\mathcal{G}_{\mathscr{A}}(U_i)$ is nonempty. These considerations show that $\mathcal{G}_{\mathscr{A}}$ is a $\mathbb{G}_{m,X}$-gerbe.

The assignment $\mathscr{A} \mapsto \mathcal{G}_{\mathscr{A}}$ induces a group homomorphism

(2.2.1) $$\alpha'_X : \operatorname{Br} X \to \mathrm{H}^2(X, \mathbb{G}_{m,X})$$

which is injective since a $\mathbb{G}_{m,X}$-gerbe $\mathcal{G}$ is trivial if and only if $\mathcal{G}(X)$ is nonempty.

For a morphism

$$(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

of locally ringed sites, the diagram

(2.2.2) $$\begin{array}{ccc} \operatorname{Br} X & \xrightarrow{\alpha'_X} & \mathrm{H}^2(X, \mathbb{G}_{m,X}) \\ {\scriptstyle f^*}\big\uparrow & & \big\uparrow{\scriptstyle f^*} \\ \operatorname{Br} Y & \xrightarrow[\alpha'_Y]{} & \mathrm{H}^2(Y, \mathbb{G}_{m,Y}) \end{array}$$

is commutative.

Lemma 2.3. *Let $\mathscr{X}$ be a $\mathbb{G}_{m,X}$-gerbe over a locally ringed site X. The class $[\mathscr{X}] \in \mathrm{H}^2(X, \mathbb{G}_{m,X})$ is in the image of α'_X if and only if $\mathscr{X}$ admits a 1-twisted finite locally free sheaf of everywhere positive rank.*

The usual proof (c.f. [dJ03, 2.14], [Lie08, 3.1.2.1], [Ols16, 12.3.11]) of Lemma 2.3 applies more generally to the case of $\mathbb{G}_m$-gerbes over an arbitrary locally ringed site.

We will only consider locally ringed sites $(X, \mathcal{O}_X)$ whose underlying site X is quasi-compact [Sta18, 090G]. For such X, the Brauer group $\operatorname{Br} X$ is a torsion group.

Definition 2.4. The torsion subgroup of $\mathrm{H}^2(X, \mathbb{G}_{m,X})$, denoted $\mathrm{Br}' X$, is called the *cohomological Brauer group* and the restriction

$$\alpha_X : \mathrm{Br}\, X \to \mathrm{Br}' X \tag{2.4.1}$$

of α'_X to $\mathrm{Br}' X$ is called the *Brauer map.*

We will consider algebraic stacks using the étale topology except in Section 5 (the case of characteristic 2) in which we will require the flat topology.

Surjectivity of the Brauer map may be checked on a finite flat surjective covering (c.f. [Gab78, II, Lemma 4], [dJ03, 2.15], [Lie08, 3.1.3.5]):

Proposition 2.5. *Let $f : X \to Y$ be a finitely presented, finite, flat, surjective morphism of algebraic stacks. A class $\beta \in \mathrm{H}^2(Y, \mathbb{G}_{m,Y})$ is in the image of α'_Y if and only if its pullback $f^*\beta \in \mathrm{H}^2(X, \mathbb{G}_{m,X})$ is in the image of α'_X.*

Proof. Let $\mathscr{Y}$ be the $\mathbb{G}_{m,Y}$-gerbe corresponding to β. Set $\mathscr{X} := X \times_Y \mathscr{Y}$ and let $F : \mathscr{X} \to \mathscr{Y}$ be the induced morphism of algebraic stacks. If $\mathscr{X}$ is in the image of α'_X, then there exists a 1-twisted finite locally free $\mathcal{O}_{\mathscr{X}}$-module $\mathscr{E}$ of everywhere positive rank. The pushforward $F_*\mathscr{E}$ is a 1-twisted, finite locally free $\mathcal{O}_{\mathscr{Y}}$-module of everywhere positive rank. Hence $\mathscr{Y}$ is in the image of α'_Y.

The other direction follows from commutativity of the diagram (2.2.2). □

Corollary 2.6. *Let $f : X \to Y$ be a finitely presented, finite, flat, surjective morphism of algebraic stacks. If α_X is an isomorphism, then α_Y is an isomorphism.*

Corollary 2.7. *Let X be a smooth separated generically tame Deligne-Mumford stack over a field k with quasi-projective coarse moduli space. Then the Brauer map α_X is surjective.*

Proof. By Kresch-Vistoli [KV04, 2.1,2.2], such X has a finite flat surjection $Z \to X$ where Z is a quasi-projective k-scheme. By Gabber's theorem (see [dJ03, 1.1]), the Brauer map is surjective for Z. Thus the Brauer map is surjective for X by Proposition 2.5. □

Remark 2.8. If $\mathrm{char}\, k \neq 2$, the stack $\mathscr{M}_{1,1,k}$ is generically tame and so Corollary 2.7 implies surjectivity of the Brauer map $\alpha_{\mathscr{M}_{1,1,k}}$. For the case $\mathrm{char}\, k = 2$, see Lemma 3.1.

3. Preliminary observations

The purpose of this section is to prove Lemma 3.4 below. Let us start, however, with a few preliminary observations about the stack $\mathscr{M}_{1,1}$ and its Brauer group.

The stack $\mathscr{M}_{1,1,\mathbb{Z}}$ is a Deligne-Mumford stack smooth and separated over $\mathbb{Z}$ [Ols16, 13.1.2]; hence if S is a regular Noetherian scheme then $\mathscr{M}_{1,1,S}$ is a regular Noetherian stack. For any locally Noetherian scheme S, the morphism

$$\pi : \mathscr{M}_{1,1,S} \to \mathbb{A}^1_S$$

sending an elliptic curve to its j-invariant identifies $\mathbb{A}^1_S$ with the coarse moduli space of $\mathscr{M}_{1,1,S}$ [FO10, 4.4].

In general, if $\mathscr{X}$ is a separated Deligne-Mumford stack and $\pi : \mathscr{X} \to X$ is its coarse moduli space, then π is initial among maps from $\mathscr{X}$ to an algebraic space, so the map $X(\mathcal{G}) \to \mathscr{X}(\mathcal{G})$ is an isomorphism for any group scheme $\mathcal{G}$; moreover if $U \to X$ is an etale morphism, then π_U :

$\mathscr{X} \times_X U \to U$ is a coarse moduli space. Applying these observations to $\mathcal{G} = \mathbb{G}_a, \mathbb{G}_m, \mu_n$ implies that the canonical maps $\mathcal{O}_X \to \pi_*\mathcal{O}_{\mathscr{X}}$, $\mathbb{G}_{m,X} \to \pi_*\mathbb{G}_{m,\mathscr{X}}$, $\mu_{n,X} \to \pi_*\mu_{n,\mathscr{X}}$ are isomorphisms; thus we will omit subscripts and denote $\mu_n, \mathbb{G}_m$ for the corresponding sheaves on either $\mathscr{M}_{1,1,S}$ or $\mathbb{A}^1_S$.

Lemma 3.1. *Let S be a quasi-compact scheme admitting an ample line bundle, and suppose that at least one prime p is invertible in S. Then the Brauer map $\alpha_{\mathscr{M}_{1,1,S}} : \operatorname{Br}\mathscr{M}_{1,1,S} \to \operatorname{Br}'\mathscr{M}_{1,1,S}$ is an isomorphism.*

Proof. By [KM85, 4.7.2], for $N \geq 3$ the moduli stack of full level N structures is representable by an affine $\mathbb{Z}[\frac{1}{N}]$-scheme $Y(N)$. Set $Y(N)_S := Y(N) \times_{\mathbb{Z}[\frac{1}{N}]} S$; the projection $Y(N)_S \to S$ is an affine morphism, hence $Y(N)_S$ is quasi-compact and admits an ample line bundle, hence the Brauer map $\alpha_{Y(N)_S}$ is surjective by Gabber's theorem (see [dJ03]), and, since the map $Y(N)_S \to \mathscr{M}_{1,1,S}$ is finite locally free, we have by Corollary 2.6 that $\alpha_{\mathscr{M}_{1,1,S}}$ is surjective. □

Lemma 3.2. *Let $U := \operatorname{Spec}\mathbb{Z}[t, (t(t-1728))^{-1}] \subset \mathbb{A}^1_{\mathbb{Z}}$ and let $\mathscr{M}^\circ_{1,1,\mathbb{Z}} := U \times_{\mathbb{A}^1_{\mathbb{Z}}} \mathscr{M}_{1,1,\mathbb{Z}}$. Then the restriction $\pi^\circ : \mathscr{M}^\circ_{1,1,\mathbb{Z}} \to U$ of π to U is a trivial $\mathbb{Z}/(2)$-gerbe, i.e. $\mathscr{M}^\circ_{1,1,\mathbb{Z}} \simeq \mathrm{B}(\mathbb{Z}/(2))_U$.*

Proof. Let S be a scheme and let E_1, E_2 be two elliptic curves over S. If $j(E_1) = j(E_2) \in \Gamma(S, \mathcal{O}_S)$ and $j(E_i), j(E_i) - 1728$ are units of $\Gamma(S, \mathcal{O}_S)$, then by [Del75, 5.3] one can find a finite étale cover $S' \to S$ such that there is an isomorphism $S' \times_S E_1 \simeq S' \times_S E_2$ of elliptic curves over S'. For any connected scheme S and an elliptic curve E/S for which $j(E)$ and $j(E) - 1728$ are invertible, we have $\operatorname{Aut}(E/S) \simeq \mathbb{Z}/(2)$ by [KM85, (8.4.2)]. It suffices now to show that there is an elliptic curve E_U over U with j-invariant t. For this we may take the elliptic curve E_U defined by the Weierstrass equation

$$Y^2Z + XYZ = X^3 - \tfrac{36}{t-1728}XZ^2 - \tfrac{1}{t-1728}Z^3$$

which satisfies $\Delta(E_U) = \frac{t^2}{(t-1728)^3}$ and $j(E_U) = t$ (see [Sil09, Proposition III.1.4(c)]). □

Lemma 3.3. *Let k be an algebraically closed field and let U be a smooth curve over k. If $\operatorname{Pic}(U) = 0$, then $\operatorname{Br}'\mathrm{B}(\mathbb{Z}/(2))_U \simeq (\mathbb{G}_m(U))/(2)$.*

Proof. The cohomological descent spectral sequence associated to the cover $U \to \mathrm{B}(\mathbb{Z}/(2))_U$ is of the form

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(\mathbb{Z}/(2), \mathrm{H}^q_{\text{ét}}(U, \mathbb{G}_m)) \implies \mathrm{H}^{p+q}_{\text{ét}}(\mathrm{B}(\mathbb{Z}/(2))_U, \mathbb{G}_m) \tag{3.3.1}$$

with differentials $\mathrm{E}_2^{p,q} \to \mathrm{E}_2^{p+2,q-1}$. We have by [Mil80, III.2.22 (d)] that $\mathrm{H}^q_{\text{ét}}(U, \mathbb{G}_m) = 0$ for all $q \geq 2$. Moreover, we have $\mathrm{H}^1_{\text{ét}}(U, \mathbb{G}_m) = \operatorname{Pic}(U) = 0$ by assumption. Thus the only row of the E_2-page of (3.3.1) containing nonzero entries is $q = 0$, which gives an isomorphism

$$\mathrm{H}^2_{\text{ét}}(\mathrm{B}(\mathbb{Z}/(2))_U, \mathbb{G}_m) \simeq \mathrm{H}^2(\mathbb{Z}/(2), \mathrm{H}^0_{\text{ét}}(U, \mathbb{G}_m)) \simeq (\mathbb{G}_m(U))/(2)$$

of abelian groups. □

Lemma 3.4. *Let k be an algebraically closed field. If $\operatorname{char} k \neq 2, 3$, then $\operatorname{Br}'\mathscr{M}_{1,1,k}$ is a subgroup of $\mathbb{Z}/(2) \oplus \mathbb{Z}/(2)$. If $\operatorname{char} k$ is 2 or 3, then $\operatorname{Br}'\mathscr{M}_{1,1,k}$ is a subgroup of $\mathbb{Z}/(2)$.*

Proof. We have that $\mathscr{M}_{1,1,k}$ is regular Noetherian and that $\mathscr{M}^\circ_{1,1,k} := \mathscr{M}^\circ_{1,1,\mathbb{Z}} \times_{\mathbb{Z}} k$ is a dense open substack; thus by [AM16, 2.5(iv)] the map

$$\operatorname{Br}'\mathscr{M}_{1,1,k} \to \operatorname{Br}'\mathscr{M}^\circ_{1,1,k}$$

induced by restriction is an injection. Here Lemma 3.2 implies $\mathrm{Br}' \mathscr{M}^{\circ}_{1,1,k} = \mathrm{Br}' \mathrm{B}(\mathbb{Z}/(2))_U$ for $U = \operatorname{Spec} k[t, (t(t-1728))^{-1}]$, and Lemma 3.3 implies $\mathrm{Br}' \mathrm{B}(\mathbb{Z}/(2))_U$ is $\mathbb{Z}/(2) \oplus \mathbb{Z}/(2)$ if $\operatorname{char} k \neq 2, 3$ and $\mathbb{Z}/(2)$ otherwise (here we use that $k^\times = (k^\times)^2$ since k is algebraically closed). ☐

4. The case $\operatorname{char} k$ is not 2

Antieau and Meier [AM16] compute the Brauer group $\mathrm{Br}\,\mathscr{M}_{1,1,S}$ for various base schemes S, including algebraically closed fields k of odd characteristic [AM16, 11.2] (the case $\operatorname{char} k \neq 2$ in Theorem 1.1). In this section we give a proof via a dévissage argument, using the fact that the coarse moduli space morphism $\pi : \mathscr{M} \to \mathbb{A}^1_k$ is a trivial $\mathbb{Z}/(2)$-gerbe away from $0, 1728 \in \mathbb{A}^1_k$ (see Lemma 3.2). Our proof is divided into two cases, depending on whether $\operatorname{char} k = 3$ or $\operatorname{char} k \neq 3$ (this will determine whether we puncture $\mathbb{A}^1_k$ at one or two points, respectively). We first fix notation and record some observations that apply to both cases.

4.1. We abbreviate $\mathscr{M} := \mathscr{M}_{1,1,k}$. By Lemma 3.1, the Brauer map $\alpha_{\mathscr{M}} : \mathrm{Br}\,\mathscr{M} \to \mathrm{Br}'\,\mathscr{M}$ is an isomorphism. By Lemma 3.4, the main task is to show that the 2-torsion in $\mathrm{Br}\,\mathscr{M}$ is 0.

For any integer $n \geq 1$, the étale Kummer sequence

$$1 \to \mu_{2^n} \to \mathbb{G}_m \stackrel{\times 2^n}{\to} \mathbb{G}_m \to 1$$

gives an exact sequence

$$0 \to (\operatorname{Pic} \mathscr{M})/(2^n) \to \mathrm{H}^2(\mathscr{M}, \mu_{2^n}) \to \mathrm{H}^2(\mathscr{M}, \mathbb{G}_m)[2^n] \to 0 \tag{4.1.1}$$

of abelian groups. Since we have $\operatorname{Pic} \mathscr{M} \simeq \mathbb{Z}/(12)$ by [FO10], we wish to compute $\mathrm{H}^2(\mathscr{M}, \mu_{2^n})$.

Set

$$U := \operatorname{Spec} k[t, (t(t-1728))^{-1}] = \mathbb{A}^1_k \setminus \{0, 1728\}$$

with inclusion $j : U \to \mathbb{A}^1_k$ and let $i : Z \to \mathbb{A}^1_k$ be the complement with reduced induced closed subscheme structure. (Thus, if $\operatorname{char} k$ is 2 or 3 then $Z \simeq \operatorname{Spec} k$, otherwise $Z \simeq \operatorname{Spec} k \amalg \operatorname{Spec} k$.) Set

$$\mathscr{M}^{\circ} := U \times_{\mathbb{A}^1_k} \mathscr{M}$$
$$\mathscr{M}_Z := Z \times_{\mathbb{A}^1_k} \mathscr{M}$$

with projections $\pi^{\circ} : \mathscr{M}^{\circ} \to U$ and $\pi_Z : \mathscr{M}_Z \to Z$. We have a commutative diagram

$$\begin{tikzcd} \mathscr{M}^{\circ} \arrow[r] \arrow[d, "\pi^{\circ}"'] & \mathscr{M} \arrow[d, "\pi"] & \mathscr{M}_Z \arrow[l] \arrow[d, "\pi_Z"] \\ U \arrow[r, "j"'] & \mathbb{A}^1_k & Z \arrow[l, "i"] \end{tikzcd} \tag{4.1.2}$$

with cartesian squares.

We have a distinguished triangle

$$j_! j^* \mathbf{R}\pi_* \mu_{2^n} \to \mathbf{R}\pi_* \mu_{2^n} \to i_* i^* \mathbf{R}\pi_* \mu_{2^n} \stackrel{+1}{\to} \tag{4.1.3}$$

in the derived category of bounded-below complexes of abelian sheaves on the étale site of $\mathbb{A}^1_k$, whose associated long exact sequence has the form

$$\begin{array}{c} \mathrm{H}^0(\mathbb{A}^1_k, j_!\mathbf{R}\pi^\circ_*\mu_{2^n}) \to \mathrm{H}^0(\mathscr{M}, \mu_{2^n}) \longrightarrow \mathrm{H}^0(Z, i^*\mathbf{R}\pi_*\mu_{2^n}) \\ \to \mathrm{H}^1(\mathbb{A}^1_k, j_!\mathbf{R}\pi^\circ_*\mu_{2^n}) \to \mathrm{H}^1(\mathscr{M}, \mu_{2^n}) \longrightarrow \mathrm{H}^1(Z, i^*\mathbf{R}\pi_*\mu_{2^n}) \\ \to \mathrm{H}^2(\mathbb{A}^1_k, j_!\mathbf{R}\pi^\circ_*\mu_{2^n}) \to \mathrm{H}^2(\mathscr{M}, \mu_{2^n}) \longrightarrow \mathrm{H}^2(Z, i^*\mathbf{R}\pi_*\mu_{2^n}) \end{array} \tag{4.1.4}$$

since $j^*\mathbf{R}\pi_*\mu_{2^n} \simeq \mathbf{R}\pi^\circ_*\mu_{2^n}$ and

$$\mathrm{H}^s(\mathbb{A}^1_k, \mathbf{R}\pi_*\mu_{2^n}) \simeq \mathrm{H}^s(\mathscr{M}, \mu_{2^n})$$
$$\mathrm{H}^s(\mathbb{A}^1_k, i_*i^*\mathbf{R}\pi_*\mu_{2^n}) \simeq \mathrm{H}^s(Z, i^*\mathbf{R}\pi_*\mu_{2^n})$$

for all s. We will first compute the groups $\mathrm{H}^s(\mathbb{A}^1_k, j_!j^*\mathbf{R}\pi_*\mu_{2^n})$ in the left column of (4.1.4).

Lemma 4.2. *Let k be an algebraically closed field, let $x_1, \ldots, x_r \in \mathbb{A}^1_k$ be r distinct k-points, set*

$$Z := \mathrm{Spec}\, k(x_1) \amalg \cdots \amalg \mathrm{Spec}\, k(x_r)$$

and let $U = \mathbb{A}^1_k \setminus Z$ be the complement with inclusion $j : U \to \mathbb{A}^1_k$. For any positive integer ℓ invertible in k, we have

$$\mathrm{H}^s(\mathbb{A}^1_k, j_!\mu_\ell) = \begin{cases} 0 & s \neq 1 \\ (\mu_\ell(k))^{\oplus(r-1)} & s = 1 \end{cases}.$$

Proof. Let $i : Z \to \mathbb{A}^1_k$ be the inclusion. We have a distinguished triangle

$$j_!\mu_\ell|_U \to \mu_\ell \to i_*i^*\mu_\ell \xrightarrow{+1}$$

in the derived category of bounded-below complexes of abelian sheaves on the big étale site of $\mathbb{A}^1_k$, which gives a long exact sequence

$$\begin{array}{c} \mathrm{H}^0(\mathbb{A}^1_k, j_!\mu_\ell|_U) \longrightarrow \mathrm{H}^0(\mathbb{A}^1_k, \mu_\ell) \longrightarrow \mathrm{H}^0(Z, \mu_\ell) \\ \to \mathrm{H}^1(\mathbb{A}^1_k, j_!\mu_\ell|_U) \longrightarrow \mathrm{H}^1(\mathbb{A}^1_k, \mu_\ell) \longrightarrow \mathrm{H}^1(Z, \mu_\ell) \\ \to \mathrm{H}^2(\mathbb{A}^1_k, j_!\mu_\ell|_U) \longrightarrow \mathrm{H}^2(\mathbb{A}^1_k, \mu_\ell) \longrightarrow \mathrm{H}^2(Z, \mu_\ell) \\ \to \mathrm{H}^3(\mathbb{A}^1_k, j_!\mu_\ell|_U) \longrightarrow \cdots \end{array}$$

in cohomology. The map $\mathrm{H}^0(\mathbb{A}^1_k, \mu_\ell) \to \mathrm{H}^0(Z, \mu_\ell)$ is identified with the diagonal map $\mu_\ell(k) \to (\mu_\ell(k))^{\oplus r}$. Since k is algebraically closed, the etale site of Z is trivial, hence $\mathrm{H}^s(Z, \mu_\ell) = 0$ for $s \geq 1$. By [Del77, Exp. 1, III, (3.6)] we have $\mathrm{H}^s(\mathbb{A}^1_k, \mu_\ell) = 0$ for $s \geq 2$. We have $\mathbb{G}_m(\mathbb{A}^1_k) \simeq \mathbb{G}_m(k)$ and the multiplication-by-ℓ map $\times\ell : \mathbb{G}_m(k) \to \mathbb{G}_m(k)$ is surjective; thus $\mathrm{H}^1(\mathbb{A}^1_k, \mu_\ell) = \mathrm{H}^1(\mathbb{A}^1_k, \mathbb{G}_m)[\ell] = (\mathrm{Pic}\, \mathbb{A}^1_k)[\ell] = 0$ by the Kummer sequence. $\square$

Lemma 4.3. *In the setup of Lemma 4.2, let n be any positive integer and let $\pi^\circ : \mathrm{B}(\mathbb{Z}/(n))_U \to U$ be the trivial $\mathbb{Z}/(n)$-gerbe over U. Then*

$$\mathrm{H}^s(\mathbb{A}^1_k, j_!\mathbf{R}\pi^\circ_*\mu_\ell) = \begin{cases} 0 & \text{if } s = 0\ , \\ (\mu_\ell(k))^{\oplus(r-1)} & \text{if } s = 1\ , \\ (\mu_{\gcd(n,\ell)}(k))^{\oplus(r-1)} & \text{if } s = 2\ . \end{cases}$$

Proof. We set

$$\mathcal{C} := j_!\mathbf{R}\pi^\circ_*\mu_\ell$$

for convenience. We will compute the groups $\mathrm{H}^s(\mathbb{A}^1_k, \mathcal{C})$ using the fact that the canonical truncations $\tau_{\leq s}\mathcal{C}$ satisfy

$$\mathrm{H}^s(\mathbb{A}^1_k, \tau_{\leq t}\mathcal{C}) \simeq \mathrm{H}^s(\mathbb{A}^1_k, \mathcal{C}) \tag{4.3.1}$$

for $s \leq t$. For any $s \in \mathbb{Z}$, the distinguished triangle

$$\tau_{\leq s-1}\mathcal{C} \to \tau_{\leq s}\mathcal{C} \to (h^s\mathcal{C})[-s] \xrightarrow{+1} \tag{4.3.2}$$

gives a long exact sequence

$$\begin{tikzcd} \mathrm{H}^0(\mathbb{A}^1_k, \tau_{\leq s-1}\mathcal{C}) \arrow[r] & \mathrm{H}^0(\mathbb{A}^1_k, \tau_{\leq s}\mathcal{C}) \arrow[r] & \mathrm{H}^{0-s}(\mathbb{A}^1_k, j_!\mathbf{R}^s\pi^\circ_*\mu_\ell) \arrow[dll, out=-5, in=175] \\ \mathrm{H}^1(\mathbb{A}^1_k, \tau_{\leq s-1}\mathcal{C}) \arrow[r] & \mathrm{H}^1(\mathbb{A}^1_k, \tau_{\leq s}\mathcal{C}) \arrow[r] & \mathrm{H}^{1-s}(\mathbb{A}^1_k, j_!\mathbf{R}^s\pi^\circ_*\mu_\ell) \arrow[dll, out=-5, in=175] \\ \mathrm{H}^2(\mathbb{A}^1_k, \tau_{\leq s-1}\mathcal{C}) \arrow[r] & \mathrm{H}^2(\mathbb{A}^1_k, \tau_{\leq s}\mathcal{C}) \arrow[r] & \mathrm{H}^{2-s}(\mathbb{A}^1_k, j_!\mathbf{R}^s\pi^\circ_*\mu_\ell) \end{tikzcd} \tag{4.3.3}$$

where

$$h^s\mathcal{C} \simeq j_!\mathbf{R}^s\pi^\circ_*\mu_\ell$$

since $j_!$ is exact.

Since $\pi^\circ : \mathrm{B}(\mathbb{Z}/(n))_U \to U$ is a trivial $\mathbb{Z}/(n)$-gerbe, by Lemma B.1 we have

$$\mathbf{R}^s\pi^\circ_*\mu_\ell \simeq \begin{cases} \mu_\ell & s = 0 \\ \mu_\ell[n] & s = 1, 3, 5, \dots \\ \mu_\ell/(n) & s = 2, 4, 6, \dots \end{cases} \tag{4.3.4}$$

where $\mu_\ell[n]$ and $\mu_\ell/(n)$ are defined by the exact sequence

$$1 \to \mu_\ell[n] \to \mu_\ell \xrightarrow{\times n} \mu_\ell \to \mu_\ell/(n) \to 1$$

of abelian sheaves. Since k is algebraically closed of characteristic prime to ℓ, the sheaves $\mu_\ell[n]$ and $\mu_\ell/(n)$ are both isomorphic to $\mu_{\gcd(n,\ell)}$, but for us the difference is important for reasons of functoriality (as ℓ is allowed to vary). More precisely, if ℓ_1 divides ℓ_2, then the inclusion $\mu_{\ell_1} \to \mu_{\ell_2}$ induces an inclusion

$$\mu_{\ell_1}[n] \to \mu_{\ell_2}[n]$$

whereas

$$\mu_{\ell_1}/(n) \to \mu_{\ell_2}/(n) \tag{4.3.5}$$

is not necessarily injective since an element $x \in \mu_{\ell_1}$ which is not an nth power of any $y_1 \in \mu_{\ell_1}$ may be an nth power of some $y_2 \in \mu_{\ell_2}$ (in particular, if $\ell_2 = n\ell_1$, then (4.3.5) is the zero morphism).

We have

$$\tau_{\leq 0}\mathcal{C} \simeq h^0\mathcal{C} \simeq j_!\mathbf{R}^0\pi^\circ_*\mu_\ell \simeq j_!\pi^\circ_*\mu_\ell \simeq j_!\mu_\ell$$

since π° is a coarse moduli space morphism and $\mathbf{R}^1\pi^\circ_*\mu_\ell \simeq \mu_{\gcd(n,\ell)}$ by (4.3.4). Applying Lemma 4.2 to the case $s = 1$ in (4.3.3) implies $\mathrm{H}^0(\mathbb{A}^1_k, \tau_{\leq 1}\mathcal{C}) = 0$ and gives isomorphisms $\mathrm{H}^1(\mathbb{A}^1_k, j_!\mu_\ell) \simeq \mathrm{H}^1(\mathbb{A}^1_k, \tau_{\leq 1}\mathcal{C})$ and $\mathrm{H}^2(\mathbb{A}^1_k, \tau_{\leq 1}\mathcal{C}) \simeq \mathrm{H}^1(\mathbb{A}^1_k, j_!\mu_{\gcd(n,\ell)})$.

Since $\mathbf{R}^2\pi^\circ_*\mu_\ell \simeq \mu_{\gcd(n,\ell)}$ by (4.3.4) and $\mathrm{H}^s(\mathbb{A}^1_k, j_!\mu_{\gcd(n,\ell)}) = 0$ for $s = -2, -1, 0$, the case $s = 2$ in (4.3.3) gives isomorphisms $\mathrm{H}^s(\mathbb{A}^1_k, \tau_{\leq 1}\mathcal{C}) \simeq \mathrm{H}^s(\mathbb{A}^1_k, \tau_{\leq 2}\mathcal{C})$ for $s = 0, 1, 2$, which implies the desired result. □

4.4 (Proof of Theorem 1.1 for $\mathrm{char}\, k = 3$). If $\mathrm{char}\, k = 3$, then Z consists of one point, so taking $r = 1$ in Lemma 4.3 implies

$$\mathrm{H}^s(\mathbb{A}^1_k, j_!\mathbf{R}\pi^\circ_*\mu_{2^n}) = 0 \tag{4.4.1}$$

for $s = 0, 1, 2$. Therefore, to compute $\mathrm{H}^2(\mathscr{M}, \mu_{2^n})$, it now remains to compute $\mathrm{H}^2(Z, i^*\mathbf{R}\pi_*\mu_{2^n})$ in (4.1.4). The stabilizer of any object of $\mathscr{M}$ of lying over $i : Z \to \mathbb{A}^1_k$ is the automorphism group of an elliptic curve with j-invariant 0, which is the semidirect product $\Gamma = \mathbb{Z}/(3) \rtimes \mathbb{Z}/(4)$ since k has characteristic 3. The underlying reduced stack $(\mathscr{M}_Z)_{\mathrm{red}}$ is the residual gerbe associated to the unique point of $|\mathscr{M}_Z|$ and is isomorphic to the classifying stack $\mathrm{B}\Gamma_k$. We have natural isomorphisms

$$\mathrm{H}^2(Z, i^*\mathbf{R}\pi_*\mu_{2^n}) \simeq i^*\mathbf{R}^2\pi_*\mu_{2^n} \overset{1}{\simeq} \mathrm{H}^2(\mathscr{M}_Z, \mu_{2^n}) \overset{2}{\simeq} \mathrm{H}^2(\mathrm{B}\Gamma_k, \mu_{2^n}) \overset{3}{\simeq} \mathrm{H}^2(\Gamma, \mu_{2^n}(k))$$

where isomorphism 1 follows from proper base change [Ols05, 1.3], isomorphism 2 is by invariance of étale site for nilpotent thickenings and the fact that 2^n is invertible on $\mathscr{M}_Z$, and isomorphism 3 is by the cohomological descent spectral sequence for the covering $\mathrm{Spec}\, k \to \mathrm{B}\Gamma_k$ (and the fact that $\mathrm{H}^i(\mathrm{Spec}\, k, \mu_{2^n}) = 0$ for $i > 0$ since k is algebraically closed). The Hochschild-Serre spectral sequence for the exact sequence

$$1 \to \mathbb{Z}/(3) \to \Gamma \to \mathbb{Z}/(4) \to 1$$

gives an isomorphism

$$\mathrm{H}^2(\Gamma, \mu_{2^n}(k)) \simeq \mathrm{H}^2(\mathbb{Z}/(4), \mu_{2^n}(k)) \simeq \mu_{2^n}(k)/(4)$$

where $\mathrm{H}^i(\mathbb{Z}/(3), \mu_{2^n}(k)) = 0$ for $i > 0$ since 3 is coprime to the order of $\mu_{2^n}(k)$. Since the first term in the last row of the diagram (4.1.4) is zero by (4.4.1), the above observations imply that we have natural inclusions

$$\mathrm{H}^2(\mathscr{M}, \mu_{2^n}) \to \mu_{2^n}(k)/(4)$$

compatible with the inclusions $\mu_{2^n} \subset \mu_{2^{n+1}}$ for all n. The inclusion $\mu_{2^n} \subset \mu_{2^{n+2}}$ induces the zero map $\mu_{2^n}(k)/(4) \to \mu_{2^{n+2}}(k)/(4)$, so $\mathrm{H}^2(\mathscr{M}, \mu_{2^n}) \to \mathrm{H}^2(\mathscr{M}, \mu_{2^{n+2}})$ is the zero map as well, hence

$$\varinjlim_{n\in\mathbb{N}} \mathrm{H}^2(\mathscr{M}, \mu_{2^n}) = 0$$

which by (4.1.1) gives $\mathrm{H}^2(\mathscr{M}, \mathbb{G}_m)[2^n] = 0$ for all n.

4.5 (Proof of Theorem 1.1, for char $k \neq 2, 3$). We describe the terms in (4.1.4). For the right column, we have

$$\mathrm{H}^s(Z, i^*\mathbf{R}\pi_*\mu_{2^n}) \simeq \mathrm{H}^s(\mathbb{Z}/(4), \mu_{2^n}(k)) \oplus \mathrm{H}^s(\mathbb{Z}/(6), \mu_{2^n}(k))$$

by [ACV03, A.0.7]. For the middle column, we have

$$\mathrm{H}^0(\mathscr{M}, \mu_{2^n}) \simeq \mathrm{H}^0(\mathbb{A}^1_k, \mu_{2^n}) \simeq \mu_{2^n}(k)$$

since $\mathbb{A}^1_k$ is the coarse moduli space of $\mathscr{M}$, and we have

$$\mathrm{H}^1(\mathscr{M}, \mu_{2^n}) \overset{1}{\simeq} \mathrm{H}^1(\mathscr{M}, \mathbb{G}_m)[2^n] \overset{2}{\simeq} (\mathbb{Z}/(12))[2^n] \overset{3}{\simeq} \mathbb{Z}/(4)$$

where isomorphism 1 follows since $k^\times = (k^\times)^{2^n}$, isomorphism 2 is by [Mum65], and isomorphism 3 holds for $n \gg 0$. For the left column, we have

$$\mathrm{H}^s(\tau_{\leq 1} j_! \mathbf{R}\pi^\circ_* \mu_{2^n}) = \begin{cases} 0 & s = 0 \\ \mu_{2^n} & s = 1 \\ \mu_2 & s = 2 \end{cases}$$

by Lemma 4.3.

To summarize, (4.1.4) simplifies to

(4.5.1)
$$\begin{tikzcd} 0 \arrow[r] & \mu_{2^n} \arrow[r] & \mu_{2^n} \oplus \mu_{2^n} \arrow[dll, hook] \\ \mu_{2^n} \arrow[r] & \mathbb{Z}/(4) \arrow[r] & \mu_4 \oplus \mu_2 \arrow[dll, hook] \\ \mu_2 \arrow[r] & \mathrm{H}^2(\mathscr{M}, \mu_{2^n}) \arrow[r] & \mu_{2^n}/(4) \oplus \mu_{2^n}/(6) \end{tikzcd}$$

for $n \gg 0$, and counting the number of elements in each group in (4.5.1) implies that the last morphism

$$\mathrm{H}^2(\mathscr{M}, \mu_{2^n}) \to \mu_{2^n}/(4) \oplus \mu_{2^n}/(6)$$

is injective. Furthermore, the inclusion

$$\mu_{2^n} \subset \mu_{2^{n+2}}$$

induces the zero map

$$\mu_{2^n}/(4) \oplus \mu_{2^n}/(6) \to \mu_{2^{n+2}}/(4) \oplus \mu_{2^{n+2}}/(6)$$

so the map $\mathrm{H}^2(\mathscr{M}, \mu_{2^n}) \to \mathrm{H}^2(\mathscr{M}, \mu_{2^{n+2}})$ is the zero map as well, hence

$$\varinjlim_{n \in \mathbb{N}} \mathrm{H}^2(\mathscr{M}, \mu_{2^n}) = 0$$

which by (4.1.1) gives $\mathrm{H}^2(\mathscr{M}, \mathbb{G}_m)[2^n] = 0$ for all n.

5. The case $\operatorname{char} k$ is 2

In this section we prove Theorem 1.1 (in case $\operatorname{char} k = 2$) and Theorem 1.2. For convenience, we denote $\mathrm{GL}_{n,p} := \mathrm{GL}_n(\mathbb{Z}/(p))$ and $\mathrm{SL}_{n,p} := \mathrm{SL}_n(\mathbb{Z}/(p))$. We denote by e the identity element of $\mathrm{GL}_{n,p}$.

5.1 (Hesse presentation of $\mathscr{M}_{1,1,k}$). By [FO10, 6.2] (and explained in more detail in A.6), there is a left action of $\mathrm{GL}_{2,3}$ on the $\mathbb{Z}[\frac{1}{3}]$-algebra

$$A_{\mathrm{H}} := \mathbb{Z}[\tfrac{1}{3}, \mu, \omega, \tfrac{1}{\mu^3-1}]/(\omega^2+\omega+1)$$

sending

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} * (\mu,\omega) &= (\mu,\omega^2) \\ \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} * (\mu,\omega) &= (\omega\mu,\omega) \\ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} * (\mu,\omega) &= (\tfrac{\mu+2}{\mu-1},\omega) \end{aligned} \tag{5.1.1}$$

for which the corresponding right action of $\mathrm{GL}_{2,3}$ on the $\mathbb{Z}[\frac{1}{3}]$-scheme

$$S_{\mathrm{H}} := \operatorname{Spec} A_{\mathrm{H}}$$

gives a presentation

$$\mathscr{M}_{1,1,\mathbb{Z}[\frac{1}{3}]} \simeq [S_{\mathrm{H}}/\mathrm{GL}_{2,3}] \tag{5.1.2}$$

of $\mathscr{M}_{1,1,\mathbb{Z}[\frac{1}{3}]}$ as a global quotient stack. The morphism

$$S_{\mathrm{H}} \to \mathscr{M}_{1,1,\mathbb{Z}[\frac{1}{3}]} \tag{5.1.3}$$

is given by the elliptic curve

$$X^3 + Y^3 + Z^3 = 3\mu XYZ$$

over S_{H}.

5.2 (Cohomological descent). Let k be an algebraically closed field of characteristic 2. The Brauer map $\alpha_{\mathscr{M}_{1,1,k}} : \operatorname{Br}\mathscr{M}_{1,1,k} \to \operatorname{Br}'\mathscr{M}_{1,1,k}$ is an isomorphism by Lemma 3.1. By Lemma 3.4, there is only 2-torsion in $\operatorname{Br}\mathscr{M}_{1,1,k}$. By Grothendieck's fppf-étale comparison theorem for smooth commutative group schemes [Gro68b, (11.7)], it suffices to compute the 2-torsion in $\mathrm{H}^2_{\mathrm{fppf}}(\mathscr{M}_{1,1,k}, \mathbb{G}_m)$. Since $\operatorname{Spec} k$ is a reduced scheme, we have

$$\mathrm{H}^1_{\mathrm{fppf}}(\mathscr{M}_{1,1,k}, \mathbb{G}_m) = \operatorname{Pic}(\mathscr{M}_{1,1,k}) = \mathbb{Z}/(12)$$

by [FO10, 1.1]. Thus, for any integer n, the fppf Kummer sequence

$$1 \to \mu_2 \to \mathbb{G}_m \xrightarrow{\times 2} \mathbb{G}_m \to 1 \tag{5.2.1}$$

gives an exact sequence

$$1 \to \mathbb{Z}/(2) \xrightarrow{\partial} \mathrm{H}^2_{\mathrm{fppf}}(\mathscr{M}_{1,1,k}, \mu_2) \to \mathrm{H}^2_{\mathrm{fppf}}(\mathscr{M}_{1,1,k}, \mathbb{G}_m)[2] \to 1 \tag{5.2.2}$$

of abelian groups. It remains to compute the middle term $\mathrm{H}^2_{\mathrm{fppf}}(\mathscr{M}_{1,1,k}, \mu_2)$.

The cohomological descent spectral sequence associated to the cover (5.1.3) is of the form

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(\mathrm{GL}_{2,3}, \mathrm{H}^q_{\mathrm{fppf}}(S_{\mathrm{H},k}, \mu_2)) \implies \mathrm{H}^{p+q}_{\mathrm{fppf}}(\mathscr{M}_{1,1,k}, \mu_2) \tag{5.2.3}$$

with differentials $\mathrm{E}_2^{p,q} \to \mathrm{E}_2^{p+2,q-1}$.

Let

$$\xi \in k$$

be a fixed primitive 3rd root of unity. By the Chinese Remainder Theorem, there is a k-algebra isomorphism

$$A_{\mathrm{H},k} = k[\mu, \omega, \tfrac{1}{\mu^3-1}]/(\omega^2+\omega+1) \to k[\nu_1, \tfrac{1}{\nu_1^3-1}] \times k[\nu_2, \tfrac{1}{\nu_2^3-1}] \tag{5.2.4}$$

sending $\mu \mapsto (\nu_1, \nu_2)$ and $\omega \mapsto (\xi, \xi^2)$. Since $S_{\mathrm{H},k}$ is a smooth curve over an algebraically closed field, we have by [Mil80, III.2.22 (d)] that $\mathrm{H}^q_{\text{ét}}(S_{\mathrm{H},k}, \mathbb{G}_m) = 0$ for all $q \geq 2$; since $S_{\mathrm{H},k}$ is a disjoint union of two copies of a distinguished affine open subset of $\mathbb{A}^1_k$, we have $\mathrm{H}^1_{\text{ét}}(S_{\mathrm{H},k}, \mathbb{G}_m) = \mathrm{Pic}(S_{\mathrm{H},k}) = 0$. By [Gro68b, (11.7)] we have $\mathrm{H}^q_{\mathrm{fppf}}(S_{\mathrm{H},k}, \mathbb{G}_m) = \mathrm{H}^q_{\text{ét}}(S_{\mathrm{H},k}, \mathbb{G}_m)$ for all $q \geq 0$; thus the fppf Kummer sequence implies $\mathrm{H}^q_{\mathrm{fppf}}(S_{\mathrm{H},k}, \mu_2) = 0$ for all $q \geq 2$. Furthermore, we have $\mathrm{H}^0_{\mathrm{fppf}}(S_{\mathrm{H},k}, \mu_2) = 0$ since $S_{\mathrm{H},k}$ is the product of two integral domains of characteristic 2. Thus the only nonzero terms on the E_2-page of (5.2.3) occur on the $q = 1$ row, so we have an isomorphism

$$\mathrm{H}^{p+1}_{\mathrm{fppf}}(\mathscr{M}_{1,1,k}, \mu_2) \simeq \mathrm{H}^p(\mathrm{GL}_{2,3}, \mathrm{H}^1_{\mathrm{fppf}}(S_{\mathrm{H},k}, \mu_2)) \tag{5.2.5}$$

for all $p \geq 0$. We are interested in the case $p = 1$.

5.3 (Description of the $\mathrm{GL}_{2,3}$-action on $\mathrm{H}^1_{\mathrm{fppf}}(S_{\mathrm{H},k}, \mu_2)$). We describe the abelian group

$$M := \mathrm{H}^1_{\mathrm{fppf}}(S_{\mathrm{H},k}, \mu_2)$$

and the left $\mathrm{GL}_{2,3}$-module structure it inherits from (5.1.1). Since $k[\mu, (\mu^3-1)^{-1}]$ is a principal localization of the polynomial ring $k[\mu]$ by a polynomial $\mu^3-1 = (\mu-1)(\mu-\xi)(\mu-\xi^2)$ splitting into three distinct irreducible factors, we have an isomorphism

$$(k[\mu, \tfrac{1}{\mu^3-1}])^\times \simeq k^\times \cdot (\mu-1)^{\mathbb{Z}} \cdot (\mu-\xi)^{\mathbb{Z}} \cdot (\mu-\xi^2)^{\mathbb{Z}} \tag{5.3.1}$$

of abelian groups. Thus (5.2.4) and the Kummer sequence (5.2.1) gives an isomorphism

$$M \simeq (\mathbb{Z}/(2))^{\oplus 6} \tag{5.3.2}$$

of abelian groups, with generators given by the classes of $\nu_i - \xi^j$ for $i = 1, 2$ and $j = 0, 1, 2$.

The isomorphism (5.2.4) is given by the map

$$s_1(\mu)\omega + s_0(\mu) \mapsto \big(s_1(\nu_1)\xi + s_0(\nu_1), s_1(\nu_2)\xi^2 + s_0(\nu_2)\big) \tag{5.3.3}$$

for $s_0, s_1 \in k[\mu, \frac{1}{\mu^3-1}]$. The inverse of (5.2.4) is given by the map

$$(f_1(\nu_1), f_2(\nu_2)) \mapsto f_1(\mu)\left(\frac{\omega}{\xi-\xi^2} + \frac{\xi}{\xi-1}\right) + f_2(\mu)\left(\frac{-\omega}{\xi-\xi^2} + \frac{-1}{\xi-1}\right) \tag{5.3.4}$$

where $f_i(\nu_i) \in k[\nu_i, \frac{1}{\nu_i^3-1}]$. (Note that, if we set $\mathrm{A}_1(t) := \frac{t}{\xi-\xi^2} + \frac{\xi}{\xi-1}$ and $\mathrm{A}_2(t) := \frac{-t}{\xi-\xi^2} + \frac{-1}{\xi-1}$, then $\mathrm{A}_i(\xi^j)$ is the Kronecker delta function.)

A computation with (5.1.1), (5.3.3), (5.3.4) shows that the action of $\mathrm{GL}_{2,3}$ on the right hand side of (5.2.4) is given by

$$\begin{aligned} \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} * (f_1(\nu_1), f_2(\nu_2)) &= (f_2(\nu_1), f_1(\nu_2)) \\ \begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix} * (f_1(\nu_1), f_2(\nu_2)) &= (f_1(\xi\nu_1), f_2(\xi^2\nu_2)) \\ \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} * (f_1(\nu_1), f_2(\nu_2)) &= (f_1(\tfrac{\nu_1+2}{\nu_1-1}), f_2(\tfrac{\nu_2+2}{\nu_2-1})) \end{aligned} \tag{5.3.5}$$

for $f_i(\nu_i) \in k[\nu_i, \frac{1}{\nu_i^3-1}]$. A computation with (5.3.5) (and using that $\operatorname{char} k = 2$) shows that the action of $\mathrm{GL}_{2,3}$ on (5.3.2) is given by (5.3.6), where every element is considered up to multiplication by $k^\times$.

$$\begin{array}{r|c|c|c|c|c|c} & \nu_1 - 1 & \nu_1 - \xi & \nu_1 - \xi^2 & \nu_2 - 1 & \nu_2 - \xi & \nu_2 - \xi^2 \\ \mathsf{M}_1 := \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} & \nu_2 - 1 & \nu_2 - \xi & \nu_2 - \xi^2 & \nu_1 - 1 & \nu_1 - \xi & \nu_1 - \xi^2 \\ \mathsf{M}_2 := \begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix} & \nu_1 - \xi^2 & \nu_1 - 1 & \nu_1 - \xi & \nu_2 - \xi & \nu_2 - \xi^2 & \nu_2 - 1 \\ \mathsf{i} := \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} & \dfrac{1}{\nu_1 - 1} & \dfrac{\nu_1 - \xi^2}{\nu_1 - 1} & \dfrac{\nu_1 - \xi}{\nu_1 - 1} & \dfrac{1}{\nu_2 - 1} & \dfrac{\nu_2 - \xi^2}{\nu_2 - 1} & \dfrac{\nu_2 - \xi}{\nu_2 - 1} \end{array} \tag{5.3.6}$$

5.4. We compute $\mathrm{H}^1(\mathrm{GL}_{2,3}, M)$. (In Appendix C we provide MAGMA code that can be used to verify this computation.) We have a filtration of groups

$$\mathrm{Q}_8 \trianglelefteq \mathrm{SL}_{2,3} \trianglelefteq \mathrm{GL}_{2,3} \tag{5.4.1}$$

where each is a normal subgroup of the next. Here Q_8 denotes the quaternion group

$$\mathrm{Q}_8 = \{\pm\mathsf{e}, \pm\mathsf{i}, \pm\mathsf{j}, \pm\mathsf{k} \ : \ \mathsf{ijk} = \mathsf{i}^2 = \mathsf{j}^2 = \mathsf{k}^2 = -\mathsf{e}\}$$

and is identified with the subgroup of $\mathrm{GL}_{2,3}$ as follows:

$$\mathsf{i} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} \qquad \mathsf{j} = \begin{bmatrix}-1 & -1\\ -1 & 1\end{bmatrix} \qquad \mathsf{k} = \begin{bmatrix}1 & -1\\ -1 & -1\end{bmatrix}$$

The quotient $\mathrm{GL}_{2,3} / \mathrm{SL}_{2,3}$ is cyclic of order 2 and is generated by M_1 in (5.3.6). The quotient $\mathrm{SL}_{2,3} / \mathrm{Q}_8$ is cyclic of order 3 and is generated by M_2 in (5.3.6). For $i = 1, 2$, let $\langle \mathsf{M}_i \rangle$ denote the subgroup of $\mathrm{GL}_{2,3}$ generated by M_i. We note that $\mathrm{SL}_{2,3}$ is generated by i and M_2.

Let

$$F : (\mathbb{Z}[\mathrm{GL}_{2,3}]\text{-Mod}) \to (\mathbb{Z}[\mathrm{SL}_{2,3}]\text{-Mod})$$

be the forgetful functor. An inspection of (5.3.6) implies that $F(M)$ is the direct sum $N_1 \oplus N_2$ where N_i is the $\mathrm{SL}_{2,3}$-submodule of $F(M)$ generated by the classes of $\nu_i - 1, \nu_i - \xi, \nu_i - \xi^2$, and moreover M_1 switches the summands N_1 and N_2. Under the adjunction

$$\mathrm{Hom}_{\mathrm{SL}_{2,3}}(F(M), N_1) \simeq \mathrm{Hom}_{\mathrm{GL}_{2,3}}(M, \mathrm{Ind}_{\mathrm{SL}_{2,3}}^{\mathrm{GL}_{2,3}}(N_1))$$

the projection map $F(M) \simeq N_1 \oplus N_2 \to N_1$ onto the first factor corresponds to a morphism

$$M \to \mathrm{Ind}_{\mathrm{SL}_{2,3}}^{\mathrm{GL}_{2,3}}(N_1) \tag{5.4.2}$$

of $\mathrm{GL}_{2,3}$-modules. Given $m \in M$, write $m = n_1 + n_2$ for $n_i \in N_i$; then the image of m under (5.4.2) is the function $\varphi_m \in \mathrm{Hom}_{\mathbb{Z}[\mathrm{SL}_{2,3}]}(\mathbb{Z}[\mathrm{GL}_{2,3}], N_1)$ such that $\varphi_m([\mathsf{e}]) = n_1$ and $\varphi_m([\mathsf{M}_1]) = \mathsf{M}_1 \cdot n_2$; thus (5.4.2) is an isomorphism.

A computation using (5.3.6) and the identities

$$\begin{aligned} \mathsf{k} &= \mathsf{M}_2^{-1} \cdot \mathsf{i} \cdot \mathsf{M}_2 \\ \mathsf{i} &= \mathsf{M}_2^{-1} \cdot \mathsf{j} \cdot \mathsf{M}_2 \\ \mathsf{j} &= \mathsf{M}_2^{-1} \cdot \mathsf{k} \cdot \mathsf{M}_2 \end{aligned} \tag{5.4.3}$$

shows that the action of an element $\mathsf{g} \in \mathrm{SL}_{2,3}$ on N_1 is by left multiplication by the matrix T_{g} as in (5.4.4), with elements of N_1 being viewed as vertical vectors. We note $T_{-\mathsf{e}} = T_{\mathsf{i}}^2 = T_{\mathsf{j}}^2 = T_{\mathsf{k}}^2 = \mathrm{id}_{N_1}$, i.e. $-\mathsf{e}$ acts trivially on N_1.

$$\begin{array}{c|cccc} \mathsf{g} & \mathsf{M}_2 & \mathsf{i} & \mathsf{j} & \mathsf{k} \\ T_{\mathsf{g}} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array} \tag{5.4.4}$$

Since M is an induced module, the restriction map

$$\mathrm{H}^1(\mathrm{GL}_{2,3}, M) \to \mathrm{H}^1(\mathrm{SL}_{2,3}, N_1) \tag{5.4.5}$$

is an isomorphism so we reduce to computing $\mathrm{H}^1(\mathrm{SL}_{2,3}, N_1)$.

The Hochschild-Serre spectral sequence for the inclusion $\mathrm{Q}_8 \trianglelefteq \mathrm{SL}_{2,3}$ degenerates on the E_2 page since the order of the quotient group $\langle \mathsf{M}_2 \rangle$ is coprime to the order of N_1. In particular the restriction map

$$\mathrm{H}^1(\mathrm{SL}_{2,3}, N_1) \to \mathrm{H}^0(\langle \mathsf{M}_2 \rangle, \mathrm{H}^1(\mathrm{Q}_8, N_1)) \tag{5.4.6}$$

is an isomorphism.

Let $\mathrm{C}^i(\mathrm{Q}_8, N_1) := \mathrm{Fun}((\mathrm{Q}_8)^i, N_1)$ denote the group of inhomogeneous i-cochains. By Remark 5.5, the group $\mathrm{SL}_{2,3}$ has a natural left action on $\mathrm{C}^i(\mathrm{Q}_8, N_1)$ (by entrywise conjugation on the source $(\mathrm{Q}_8)^i$ and by its usual action on N_1) such that the differentials in the inhomogeneous cochain complex

$$\mathrm{C}^0(\mathrm{Q}_8, N_1) \xrightarrow{d_0} \mathrm{C}^1(\mathrm{Q}_8, N_1) \xrightarrow{d_1} \mathrm{C}^2(\mathrm{Q}_8, N_1) \to \cdots$$

are $\mathrm{SL}_{2,3}$-linear. Since the order of the subgroup $\langle \mathsf{M}_2 \rangle$ is coprime to the orders of $\mathrm{C}^i(\mathrm{Q}_8, N_1)$, we have that $\mathrm{H}^0(\langle \mathsf{M}_2 \rangle, \mathrm{H}^1(\mathrm{Q}_8, N_1)) \simeq (\mathrm{H}^1(\mathrm{Q}_8, N_1))^{\mathsf{M}_2}$ is isomorphic to the middle cohomology of the sequence

$$(\mathrm{C}^0(\mathrm{Q}_8, N_1))^{\mathsf{M}_2} \xrightarrow{(d_0)^{\mathsf{M}_2}} (\mathrm{C}^1(\mathrm{Q}_8, N_1))^{\mathsf{M}_2} \xrightarrow{(d_1)^{\mathsf{M}_2}} (\mathrm{C}^2(\mathrm{Q}_8, N_1))^{\mathsf{M}_2}$$

i.e. cohomology commutes with taking M_2-invariants.

We now describe $\ker((d_1)^{\mathsf{M}_2})$ and $\mathrm{im}((d_0)^{\mathsf{M}_2})$.

An element $f \in (\mathrm{C}^1(\mathrm{Q}_8, N_1))^{\mathsf{M}_2}$ is a function $f : \mathrm{Q}_8 \to N_1$ satisfying

$$f(\mathsf{g}) = \mathsf{M}_2 \cdot f(\mathsf{M}_2^{-1} \mathsf{g} \mathsf{M}_2) \tag{5.4.7}$$

for all $\mathsf{g} \in \mathrm{Q}_8$. We have that $f \in \ker d_1$ if

$$f(\mathsf{g}_1 \cdot \mathsf{g}_2) = \mathsf{g}_1 \cdot f(\mathsf{g}_2) + f(\mathsf{g}_1) \tag{5.4.8}$$

for all $\mathsf{g}_1, \mathsf{g}_2 \in \mathrm{Q}_8$.

Suppose $f \in \ker((d_1)^{\mathsf{M}_2}) = (\ker d_1) \cap (\mathrm{C}^1(\mathrm{Q}_8, N_1))^{\mathsf{M}_2}$; taking $(\mathsf{g}_1, \mathsf{g}_2) = (\mathsf{e}, \mathsf{e})$ in (5.4.8) implies $f(\mathsf{e}) = 0$; taking $\mathsf{g} = -\mathsf{e}$ in (5.4.7) implies that

$$f(-\mathsf{e}) = (s, s, s)$$

for some $s \in \mathbb{Z}/(2)$; taking $(\mathsf{g}_1, \mathsf{g}_2) = (-\mathsf{e}, -\mathsf{e})$ in (5.4.8) and using the fact that $-\mathsf{e}$ acts trivially on N_1 implies that $2f(-\mathsf{e}) = 0$, which imposes no condition on s. We note that

$$\mathsf{g} \cdot f(-\mathsf{e}) = f(-\mathsf{e})$$

for any $\mathsf{g} \in \mathrm{SL}_{2,3}$.

Setting $\mathsf{g} = \mathsf{i}, \mathsf{j}, \mathsf{k}$ in (5.4.7) and using (5.4.3) gives

$$\begin{aligned} f(\mathsf{i}) &= \mathsf{M}_2 \cdot f(\mathsf{k}) \\ f(\mathsf{j}) &= \mathsf{M}_2 \cdot f(\mathsf{i}) \\ f(\mathsf{k}) &= \mathsf{M}_2 \cdot f(\mathsf{j}) \end{aligned} \tag{5.4.9}$$

respectively; thus we have

$$\begin{aligned} f(\mathsf{i}) &= (s_1, s_2, s_3) \\ f(\mathsf{j}) &= (s_2, s_3, s_1) \\ f(\mathsf{k}) &= (s_3, s_1, s_2) \end{aligned}$$

for some $s_1, s_2, s_3 \in \mathbb{Z}/(2)$.

Setting either $\mathsf{g}_1 = -\mathsf{e}$ or $\mathsf{g}_2 = -\mathsf{e}$ in (5.4.8) implies

$$f(-\mathsf{g}) = f(\mathsf{g}) + f(-\mathsf{e}) \tag{5.4.10}$$

for any $\mathsf{g} \in \mathrm{Q}_8$.

Setting $(\mathsf{g}_1, \mathsf{g}_2) = (\pm\mathsf{i}, \pm\mathsf{j}), (\pm\mathsf{j}, \pm\mathsf{k}), (\pm\mathsf{k}, \pm\mathsf{i})$ in (5.4.8) (where the signs can vary independently of each other) all impose the condition

$$s_2 = 0 \tag{5.4.11}$$

for s, s_2 (check the case $(\mathsf{g}_1, \mathsf{g}_2) = (\mathsf{i}, \mathsf{j})$, then use (5.4.10) to show that changing the signs don't give new relations, then use (5.4.9) to show that one can permute using left multiplication by M_2).

Setting $(\mathsf{g}_1, \mathsf{g}_2) = (\pm\mathsf{j}, \pm\mathsf{i}), (\pm\mathsf{k}, \pm\mathsf{j}), (\pm\mathsf{i}, \pm\mathsf{k})$ in (5.4.8) (where the signs can vary independently of each other) all impose the condition

$$s = s_3 \tag{5.4.12}$$

for s_3 (check the case $(\mathsf{g}_1, \mathsf{g}_2) = (\mathsf{j}, \mathsf{i})$, then use (5.4.10) to show that changing the signs don't give new relations, then use (5.4.9) to show that one can permute using left multiplication by M_2).

Setting $(\mathsf{g}_1, \mathsf{g}_2) = (\pm\mathsf{g}, \pm\mathsf{g})$ for $\mathsf{g} = \mathsf{i}, \mathsf{j}, \mathsf{k}$ (where the signs can vary independently of each other) all impose the condition

$$s = s_2 + s_3 \tag{5.4.13}$$

on s, s_2, s_3 (check the case $\mathsf{g} = \mathsf{i}$, then use (5.4.10) to show that changing the signs don't give new relations, then use (5.4.9) to show that one can permute using left multiplication by M_2), but (5.4.13) is implied by (5.4.11) and (5.4.12).

These are the only relations satisfied by the s, s_1, s_2, s_3. Thus we have

$$\ker((d_1)^{\mathsf{M}_2}) \simeq \mathbb{Z}/(2) \oplus \mathbb{Z}/(2)$$

since there are no relations on $s, s_1 \in \mathbb{Z}/(2)$.

An element of $(\mathrm{C}^0(\mathrm{Q}_8, N_1))^{\mathsf{M}_2}$ corresponds to an element $(t,t,t) \in N_1$; since every element of $\mathrm{SL}_{2,3}$ fixes elements of this form (see (5.4.4)), the image of (t,t,t) under $(d_0)^{\mathsf{M}_2}$ corresponds to the function $f : \mathrm{Q}_8 \to N_1$ sending every element to $(0,0,0)$, in other words

$$\mathrm{im}((d_1)^{\mathsf{M}_2}) = 0$$

which implies

$$\mathrm{H}^0(\langle \mathsf{M}_2 \rangle, \mathrm{H}^1(\mathrm{Q}_8, N_1)) \simeq \mathbb{Z}/(2) \oplus \mathbb{Z}/(2) \tag{5.4.14}$$

and so

$$\mathrm{Br}\, \mathscr{M}_{1,1,k} = \mathrm{H}^2_{\mathrm{fppf}}(\mathscr{M}_{1,1,k}, \mathbb{G}_m)[2] = \mathbb{Z}/(2) \tag{5.4.15}$$

by combining (5.4.14) with (5.4.6), (5.4.5), (5.2.5), and (5.2.2). □

Remark 5.5 (The inhomogeneous cochain complex admits a left G-action). Let G be a group, let $H \trianglelefteq G$ be a normal subgroup, and let M be a left G-module. Set $P_i := \mathbb{Z}[H^{i+1}]$; we denote by $[\mathsf{h}_0, \dots, \mathsf{h}_i]$ the canonical $\mathbb{Z}$-basis of P_i. We view P_i as a left H-module via the diagonal action $\mathsf{h} \cdot [\mathsf{h}_0, \dots, \mathsf{h}_i] = [\mathsf{h}\mathsf{h}_0, \dots, \mathsf{h}\mathsf{h}_i]$; then P_i is a free left $\mathbb{Z}[H]$-module with basis consisting of elements of the form $[\mathsf{e}, \mathsf{h}_1, \dots, \mathsf{h}_i]$. Applying the functor $\mathrm{Hom}_H(-, M)$ to the bar resolution

$$\cdots \to P_2 \to P_1 \to P_0 \to \mathbb{Z} \to 0$$

gives the usual homogeneous cochain complex

$$\mathrm{Hom}_{\mathbb{Z}[H]}(P_0, M) \xrightarrow{\delta_0} \mathrm{Hom}_{\mathbb{Z}[H]}(P_1, M) \xrightarrow{\delta_1} \mathrm{Hom}_{\mathbb{Z}[H]}(P_2, M) \to \cdots$$

whose cohomology gives $\mathrm{H}^i(H, M)$.

We note that there is a natural left G-action on $\mathrm{Hom}_{\mathbb{Z}[H]}(P_i, M)$ for which the differential $\delta_i : \mathrm{Hom}_{\mathbb{Z}[H]}(P_i, M) \to \mathrm{Hom}_{\mathbb{Z}[H]}(P_{i+1}, M)$ is G-linear. Namely, the action of $\mathsf{g} \in G$ on $\varphi_i \in \mathrm{Hom}_{\mathbb{Z}[H]}(P_i, M)$ is described by

$$(\mathsf{g}\varphi_i)([\mathsf{h}_0, \dots, \mathsf{h}_i]) := \mathsf{g} \cdot (\varphi_i([\mathsf{g}^{-1}\mathsf{h}_0\mathsf{g}, \dots, \mathsf{g}^{-1}\mathsf{h}_i\mathsf{g}]))$$

for all $\mathsf{h}_0, \dots, \mathsf{h}_i \in H$. Let

$$\mathrm{C}^i(H, M) := \mathrm{Fun}(H^i, M)$$

denote the abelian group of functions $H^i \to M$. Via the usual abelian group isomorphism

$$\mathrm{Hom}_{\mathbb{Z}[H]}(P_i, M) \simeq \mathrm{C}^i(H, M)$$

sending $\varphi_i \mapsto \{(\mathsf{h}_1, \dots, \mathsf{h}_i) \mapsto \varphi_i(\mathsf{e}, \mathsf{h}_1, \mathsf{h}_1\mathsf{h}_2, \dots, \mathsf{h}_1 \cdots \mathsf{h}_i)\}$, the abelian group $\mathrm{C}^i(H, M)$ inherits a left action of G described by

$$(\mathsf{g}f_i)(\mathsf{h}_1, \dots, \mathsf{h}_i) = \mathsf{g} \cdot (f_i(\mathsf{g}^{-1}\mathsf{h}_1\mathsf{g}, \dots, \mathsf{g}^{-1}\mathsf{h}_i\mathsf{g})) \tag{5.5.1}$$

for $\mathsf{g} \in G$ and $f_i \in \mathrm{C}^i(H, M)$. The inhomogeneous cochain complex

$$\mathrm{C}^0(H, M) \xrightarrow{d_0} \mathrm{C}^1(H, M) \xrightarrow{d_1} \mathrm{C}^2(H, M) \to \cdots$$

is G-linear as well.

For $f_0 \in \mathrm{C}^0(H, M)$, we have $(d_0 f_0)(\mathsf{h}_1) = \mathsf{h}_1 \cdot f_0(\mathsf{e}) - f_0(\mathsf{e})$.

For $f_1 \in \mathrm{C}^1(H, M)$, we have $(d_1 f_1)(\mathsf{h}_1, \mathsf{h}_2) = \mathsf{h}_1 \cdot f_1(\mathsf{h}_2) - f_1(\mathsf{h}_1\mathsf{h}_2) + f_1(\mathsf{h}_1)$.

Let $\Sigma := G/H$ be the quotient; then there is an induced left action of Σ on the cohomology $\mathrm{h}^i(\mathrm{C}^\bullet(H, M))$. In case $G \to \Sigma$ has a section, in which case G is the semi-direct product $G \simeq H \rtimes \Sigma$, then this Σ-action coincides with the one obtained by restricting the G-action on $\mathrm{C}^\bullet(H, M)$ to Σ.

Remark 5.6. The arguments used in 5.3 and 5.4 are similar to those of Mathew and Stojanoska [MS16, Appendix B], who show $\mathrm{H}^1(\mathrm{GL}_{2,3}, (TMF(3)_0)^\times) = \mathbb{Z}/(12)$ where $\mathrm{GL}_{2,3}$ acts on

$$TMF(3)_0 = \mathbb{Z}[\tfrac{1}{3}, \zeta, t, \tfrac{1}{t}, \tfrac{1}{1-\zeta t}, \tfrac{1}{1+\zeta^2 t}]/(\zeta^2 + \zeta + 1) \tag{5.6.1}$$

as in [Sto14, §4.3].

Note 5.7 (Explicit description of inhomogeneous 1-cocycles). We describe the 1-cocycles $\mathrm{GL}_{2,3} \to M$ obtained via the compositions (5.4.6) and (5.4.5). By our computation in 5.4, the 1-cocycles

$$f_{\mathrm{Q}_8} : \mathrm{Q}_8 \to N_1$$

are of the form

$$\begin{aligned} \mathsf{e} &\mapsto (0,0,0) & \qquad -\mathsf{e} &\mapsto (s,s,s) \\ \mathsf{i} &\mapsto (s_1,0,s) & -\mathsf{i} &\mapsto (s_1+s,s,0) \\ \mathsf{j} &\mapsto (0,s,s_1) & -\mathsf{j} &\mapsto (s,0,s_1+s) \\ \mathsf{k} &\mapsto (s,s_1,0) & -\mathsf{k} &\mapsto (0,s_1+s,s) \end{aligned}$$

for some $s, s_1 \in \mathbb{Z}/(2)$. Suppose

$$f_{\mathrm{SL}_{2,3}} : \mathrm{SL}_{2,3} \to N_1$$

is a 1-cocycle such that $f_{\mathrm{SL}_{2,3}}$ is fixed by the action of M_2 (see (5.5.1)) and which satisfies $f_{\mathrm{SL}_{2,3}}(\mathsf{g}) = f_{\mathrm{Q}_8}(\mathsf{g})$ for $\mathsf{g} \in \mathrm{Q}_8$. We have

$$\mathsf{M}_2 \cdot f_{\mathrm{SL}_{2,3}}(\mathsf{M}_2^{-1} \cdot \mathsf{g} \cdot \mathsf{M}_2) = f_{\mathrm{SL}_{2,3}}(\mathsf{g})$$

for all $\mathsf{g} \in \mathrm{SL}_{2,3}$; taking $\mathsf{g} = \mathsf{M}_2$ gives $\mathsf{M}_2 \cdot f_{\mathrm{SL}_{2,3}}(\mathsf{M}_2) = f_{\mathrm{SL}_{2,3}}(\mathsf{M}_2)$. Taking $\mathsf{g}_1 = \mathsf{g}_2 = \mathsf{M}_2$ in the 1-cocycle condition (5.4.8) then gives $f_{\mathrm{SL}_{2,3}}(\mathsf{M}_2) = 0$. Thus we have

$$f_{\mathrm{SL}_{2,3}}(\mathsf{g} \cdot \mathsf{M}_2) = f_{\mathrm{SL}_{2,3}}(\mathsf{g}) \tag{5.7.1}$$

for any $\mathsf{g} \in \mathrm{SL}_{2,3}$, again by (5.4.8).

By Shapiro's lemma (5.4.5), there is a 1-cocycle

$$f_{\mathrm{GL}_{2,3}} : \mathrm{GL}_{2,3} \to \mathrm{Ind}_{\mathrm{SL}_{2,3}}^{\mathrm{GL}_{2,3}}(N_1)$$

such that precomposing with the inclusion $\mathrm{SL}_{2,3} \subset \mathrm{GL}_{2,3}$ and postcomposing with the projection $\mathrm{Ind}_{\mathrm{SL}_{2,3}}^{\mathrm{GL}_{2,3}}(N_1) \to N_1$ gives $f_{\mathrm{SL}_{2,3}}$. After altering $f_{\mathrm{GL}_{2,3}}$ by a 1-coboundary, we may assume by Note 5.8 that $f_{\mathrm{GL}_{2,3}}$ is given by the formula (5.8.1), namely

$$f_{\mathrm{GL}_{2,3}}(\mathsf{g} \cdot \mathsf{M}_1^i)([\mathsf{M}_1^j]) := f_{\mathrm{SL}_{2,3}}(\mathsf{M}_1^j \cdot \mathsf{g} \cdot \mathsf{M}_1^{-j}) \tag{5.7.2}$$

for any $i, j \in \{0, 1\}$ and $\mathsf{g} \in \mathrm{SL}_{2,3}$. Any element $\mathsf{g} \in \mathrm{GL}_{2,3}$ may be expressed in the form

$$\mathsf{h} \cdot \mathsf{M}_2^{i_2} \cdot \mathsf{M}_1^{i_1}$$

where $i_1 \in \{0,1\}$ and $i_2 \in \{0,1,2\}$ and $\mathsf{h} \in \mathrm{Q}_8$. We have formulas

$$\begin{aligned} \mathsf{M}_1 \cdot \mathsf{M}_2^{-1} \cdot \mathsf{M}_1 &= \mathsf{M}_2^{-1} \\ \mathsf{M}_1 \cdot \mathsf{i} \cdot \mathsf{M}_1^{-1} &= -\mathsf{i} \\ \mathsf{M}_1 \cdot \mathsf{j} \cdot \mathsf{M}_1^{-1} &= -\mathsf{k} \\ \mathsf{M}_1 \cdot \mathsf{k} \cdot \mathsf{M}_1^{-1} &= -\mathsf{j} \end{aligned} \tag{5.7.3}$$

and so

$$\begin{aligned} f_{\mathrm{GL}_{2,3}}(\mathsf{h} \cdot \mathsf{M}_2^{i_2} \cdot \mathsf{M}_1^{i_1})([\mathsf{M}_1^j]) &\overset{1}{=} f_{\mathrm{SL}_{2,3}}(\mathsf{M}_1^j \cdot \mathsf{h} \cdot \mathsf{M}_2^{i_2} \cdot \mathsf{M}_1^{-j}) \\ &= f_{\mathrm{SL}_{2,3}}((\mathsf{M}_1^j \cdot \mathsf{h} \cdot \mathsf{M}_1^{-j}) \cdot (\mathsf{M}_1^j \cdot \mathsf{M}_2^{i_2} \cdot \mathsf{M}_1^{-j})) \\ &\overset{2}{=} f_{\mathrm{SL}_{2,3}}(\mathsf{M}_1^j \cdot \mathsf{h} \cdot \mathsf{M}_1^{-j}) \\ &\overset{3}{=} f_{\mathrm{Q}_8}(\mathsf{M}_1^j \cdot \mathsf{h} \cdot \mathsf{M}_1^{-j}) \end{aligned}$$

where equality 1 is by (5.7.2) and equality 2 is by (5.7.1) and (5.7.3) and equality 3 is since $\mathsf{M}_1^j \cdot \mathsf{h} \cdot \mathsf{M}_1^{-j} \in \mathrm{Q}_8$ (see (5.7.3)). This is summarized in (5.7.4) below.

$$\begin{aligned} f_{\mathrm{GL}_{2,3}}(\mathsf{e}) &= (f_{\mathrm{Q}_8}(\mathsf{e}), f_{\mathrm{Q}_8}(\mathsf{e})) = ((0,0,0),(0,0,0)) \\ f_{\mathrm{GL}_{2,3}}(\mathsf{i}) &= (f_{\mathrm{Q}_8}(\mathsf{i}), f_{\mathrm{Q}_8}(-\mathsf{i})) = ((s_1,0,s),(s_1+s,s,0)) \\ f_{\mathrm{GL}_{2,3}}(\mathsf{j}) &= (f_{\mathrm{Q}_8}(\mathsf{j}), f_{\mathrm{Q}_8}(-\mathsf{k})) = ((0,s,s_1),(0,s_1+s,s)) \\ f_{\mathrm{GL}_{2,3}}(\mathsf{k}) &= (f_{\mathrm{Q}_8}(\mathsf{k}), f_{\mathrm{Q}_8}(-\mathsf{j})) = ((s,s_1,0),(s,0,s_1+s)) \end{aligned} \tag{5.7.4}$$

Note 5.8 (The Shapiro isomorphism and inhomogeneous 1-cocycles). [1] Let G be a group, let $H \subseteq G$ be a normal subgroup of finite index such that the projection $G \to G/H$ has a section $G/H \to G$ whose image corresponds to a subgroup Σ of G. Let N be a left H-module and let $\mathrm{Ind}_H^G N := \mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], N)$ denote the associated induced left G-module. We recall that the left G-action on $\mathrm{Ind}_H^G N$ sends $\varphi \mapsto \mathsf{g}\varphi$ where $(\mathsf{g}\varphi)(x) = \varphi(x\mathsf{g})$.

We describe the inverse of the Shapiro isomorphism $\mathrm{H}^1(G, \mathrm{Ind}_H^G N) \to \mathrm{H}^1(H, N)$ in terms of inhomogeneous cochains. Suppose given a function

$$f : H \to N$$

which satisfies

$$f(\mathsf{h}_1\mathsf{h}_2) = \mathsf{h}_1 \cdot f(\mathsf{h}_2) + f(\mathsf{h}_1)$$

for all $\mathsf{h}_1, \mathsf{h}_2 \in H$. We construct a 1-cocycle

$$s : G \to \mathrm{Ind}_H^G(N)$$

which restricts to f, i.e. satisfies $s(\mathsf{h})(1 \cdot [\mathsf{e}]) = f(\mathsf{h})$ for all $\mathsf{h} \in H$. Note that every element of $\mathsf{g} \in G$ may be written uniquely in the form

$$\mathsf{g} = \mathsf{h}\sigma$$

for $\mathsf{h} \in H$ and $\sigma \in \Sigma$, hence the collection $\{[\sigma]\}_{\sigma \in \Sigma}$ forms a basis for $\mathbb{Z}[G]$ as a left $\mathbb{Z}[H]$-module. We set

$$s(\mathsf{h}\sigma)([\xi]) := f(\xi\mathsf{h}\xi^{-1}) \tag{5.8.1}$$

[1] Ehud Meir's MathOverflow post [Mei16] was helpful in working out the details of this section.

for $\mathsf{h} \in H$ and $\sigma, \xi \in \Sigma$ and extend $\mathbb{Z}[H]$-linearly. Given $\mathsf{g}_1, \mathsf{g}_2 \in G$ where $\mathsf{g}_i = \mathsf{h}_i\sigma_i$ with $\mathsf{h}_i \in H$ and $\sigma_i \in \Sigma$, for any $\xi \in \Sigma$ we have

$$\begin{aligned} s(\mathsf{g}_1\mathsf{g}_2)([\xi]) &= s(\mathsf{h}_1\sigma_1\mathsf{h}_2\sigma_2)([\xi]) \\ &= s(\mathsf{h}_1(\sigma_1\mathsf{h}_2\sigma_1^{-1})\sigma_1\sigma_2)([\xi]) \\ &= f(\xi\mathsf{h}_1(\sigma_1\mathsf{h}_2\sigma_1^{-1})\xi^{-1}) \end{aligned}$$

and

$$\begin{aligned} (\mathsf{g}_1 \cdot s(\mathsf{g}_2))([\xi]) &= s(\mathsf{h}_2\sigma_2)([\xi\mathsf{h}_1\sigma_1]) \\ &= s(\mathsf{h}_2\sigma_2)([(\xi\mathsf{h}_1\xi^{-1})\xi\sigma_1]) \\ &= (\xi\mathsf{h}_1\xi^{-1}) \cdot s(\mathsf{h}_2\sigma_2)([\xi\sigma_1]) \\ &= (\xi\mathsf{h}_1\xi^{-1}) \cdot f((\xi\sigma_1)\mathsf{h}_2(\xi\sigma_1)^{-1}) \end{aligned}$$

and

$$s(\mathsf{g}_1)([\xi]) = s(\mathsf{h}_1\sigma_1)([\xi]) = f(\xi\mathsf{h}_1\xi^{-1})$$

which implies

$$s(\mathsf{g}_1\mathsf{g}_2) = \mathsf{g}_1 \cdot s(\mathsf{g}_2) + s(\mathsf{g}_1)$$

by $\mathbb{Z}[H]$-linearity and since f is a 1-cocycle; hence s is a 1-cocycle. □

5.9 (Proof of Theorem 1.2). Let k^{sep} be a fixed separable closure of k and let $\mathrm{G}_k := \mathrm{Gal}(k^{\mathrm{sep}}/k) \simeq \widehat{\mathbb{Z}}$ be the absolute Galois group. Set $\mathscr{M} := \mathscr{M}_{1,1,k}$ and $\mathscr{M}^{\mathrm{sep}} := \mathscr{M}_{1,1,k^{\mathrm{sep}}}$. We have $\operatorname{Br}\mathscr{M} = \operatorname{Br}'\mathscr{M}$ by Lemma 3.1. The Leray spectral sequence for the map $\mathscr{M} \to \operatorname{Spec} k$ is of the form

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(\mathrm{G}_k, \mathrm{H}^q_{\text{ét}}(\mathscr{M}^{\mathrm{sep}}, \mathbb{G}_m)) \implies \mathrm{H}^{p+q}_{\text{ét}}(\mathscr{M}, \mathbb{G}_m)$$

with differentials $\mathrm{E}_2^{p,q} \to \mathrm{E}_2^{p+2,q-1}$. Here we have $\Gamma(\mathscr{M}^{\mathrm{sep}}, \mathbb{G}_m) = \Gamma(\mathbb{A}^1_{k^{\mathrm{sep}}}, \mathbb{G}_m) = (k^{\mathrm{sep}})^\times$ since $\mathscr{M}^{\mathrm{sep}} \to \mathbb{A}^1_{k^{\mathrm{sep}}}$ is the coarse moduli space map. Since k is a finite field, we have that $\mathrm{H}^0_{\text{ét}}(\mathscr{M}^{\mathrm{sep}}, \mathbb{G}_m)$ is a torsion group. Moreover $\mathrm{H}^1_{\text{ét}}(\mathscr{M}^{\mathrm{sep}}, \mathbb{G}_m) \simeq \operatorname{Pic}(\mathscr{M}^{\mathrm{sep}}) \simeq \mathbb{Z}/(12)$ is a torsion group by [FO10]. Thus by e.g. [Fu11, 4.3.7] or [GS06, 6.1.3] we have $\mathrm{E}_2^{p,q} = 0$ for $(p,q) \in \mathbb{Z}_{\geq 2} \times \{0,1\}$. This means there is an exact sequence

$$0 \to \mathrm{E}_2^{1,1} \to \mathrm{H}^2_{\text{ét}}(\mathscr{M}, \mathbb{G}_m) \to \mathrm{E}_2^{0,2} \to 0 \tag{5.9.1}$$

of abelian groups.

By [FO10], we have that $\operatorname{Pic}(\mathscr{M}^{\mathrm{sep}}) \simeq \mathbb{Z}/(12)$ is generated by the class of the Hodge bundle; since G_k acts trivially on invariant differentials of elliptic curves $E \to S$ where S is a k-scheme, the action of G_k on $\operatorname{Pic}(\mathscr{M}^{\mathrm{sep}})$ is trivial. Hence we have

$$\mathrm{E}_2^{1,1} = \mathrm{H}^1(\mathrm{G}_k, \mathrm{H}^1_{\text{ét}}(\mathscr{M}^{\mathrm{sep}}, \mathbb{G}_m)) \stackrel{1}{=} \operatorname{Hom}_{\mathrm{cont}}(\mathrm{G}_k, \operatorname{Pic}(\mathscr{M}^{\mathrm{sep}})) \stackrel{2}{=} \mathbb{Z}/(12)$$

where equality 1 is by [Fu11, 4.3.7] and equality 2 is since $\mathrm{G}_k \simeq \widehat{\mathbb{Z}}$. We have

$$\mathrm{E}_2^{0,2} = \mathrm{H}^0(\mathrm{G}_k, \mathrm{H}^2_{\text{ét}}(\mathscr{M}^{\mathrm{sep}}, \mathbb{G}_m)) \stackrel{1}{=} (\mathbb{Z}/(2))^{\mathrm{G}_k} \stackrel{2}{=} \mathbb{Z}/(2)$$

where equality 1 is by the computation for an algebraically closed field (Theorem 1.1) and also the fact that $\mathrm{H}^2_{\text{ét}}(\mathscr{M}^{\mathrm{sep}}, \mathbb{G}_m)$ is a torsion group (see [AM16, Proposition 2.5 (iii)]) and equality 2 is because any group action on the group of order 2 is necessarily trivial. Thus (5.9.1) reduces to a natural extension

$$0 \to \mathbb{Z}/(12) \to \operatorname{Br} \mathscr{M} \to \mathbb{Z}/(2) \to 0 \tag{5.9.2}$$

and it remains to see whether (5.9.2) is split. It suffices to compute the size of $(\operatorname{Br} \mathscr{M})[2]$, since $(\operatorname{Br} \mathscr{M})[2]$ has 4 or 2 elements depending on whether (5.9.2) is split or not, respectively.

As in 5.2, the fppf Kummer sequence

$$1 \to \mu_2 \to \mathbb{G}_m \xrightarrow{\times 2} \mathbb{G}_m \to 1 \tag{5.9.3}$$

gives an exact sequence

$$1 \to \mathbb{Z}/(2) \xrightarrow{\partial} \mathrm{H}^2_{\text{fppf}}(\mathscr{M}, \mu_2) \to (\operatorname{Br} \mathscr{M})[2] \to 1 \tag{5.9.4}$$

of abelian groups. We compute $\mathrm{H}^2_{\text{fppf}}(\mathscr{M}, \mu_2)$ using the Leray spectral sequence which is of the form

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(\mathrm{G}_k, \mathrm{H}^q_{\text{fppf}}(\mathscr{M}^{\text{sep}}, \mu_2)) \implies \mathrm{H}^{p+q}_{\text{fppf}}(\mathscr{M}, \mu_2)$$

with differentials $\mathrm{E}_2^{p,q} \to \mathrm{E}_2^{p+2,q-1}$. We have

$$\mathrm{H}^p_{\text{fppf}}(\mathscr{M}^{\text{sep}}, \mu_2) = \begin{cases} 0 & \text{if } p = 0 \\ \mathbb{Z}/(2) & \text{if } p = 1 \\ \mathbb{Z}/(2) \oplus \mathbb{Z}/(2) & \text{if } p = 2 \end{cases}$$

from the fppf Kummer sequence on $\mathscr{M}^{\text{sep}}$, where the $p = 0$ case follows since we are in characteristic 2 and $\Gamma(\mathscr{M}^{\text{sep}}, \mathbb{G}_m) = \Gamma(\mathbb{A}^1_{k^{\text{sep}}}, \mathbb{G}_m) = (k^{\text{sep}})^\times$, the $p = 1$ case is since the multiplication-by-2 map on $\Gamma(\mathscr{M}^{\text{sep}}, \mathbb{G}_m) = (k^{\text{sep}})^\times$ is an isomorphism, and the $p = 2$ case is by the computation in the algebraically closed case (combine (5.2.5), (5.4.5), (5.4.6), (5.4.14)).

Since k has characteristic 2, the 2-cohomological dimension of k satisfies $\operatorname{cd}_2(k) \le 1$ by e.g. [GS06, 6.1.9]; hence $\mathrm{E}_2^{p,q} = 0$ for $p \ge 2$ and any q. Hence there is an exact sequence

$$0 \to \mathrm{H}^1(\mathrm{G}_k, \mathrm{H}^1_{\text{fppf}}(\mathscr{M}^{\text{sep}}, \mu_2)) \to \mathrm{H}^2_{\text{fppf}}(\mathscr{M}, \mu_2) \to \mathrm{H}^0(\mathrm{G}_k, \mathrm{H}^2_{\text{fppf}}(\mathscr{M}^{\text{sep}}, \mu_2)) \to 0 \tag{5.9.5}$$

of abelian groups. As above, the G_k-action on $\mathrm{H}^1_{\text{fppf}}(\mathscr{M}^{\text{sep}}, \mu_2)$ is necessarily trivial so we have an isomorphism $\mathrm{H}^1(\mathrm{G}_k, \mathrm{H}^1_{\text{fppf}}(\mathscr{M}^{\text{sep}}, \mu_2)) \simeq \operatorname{Hom}_{\text{cont}}(\mathrm{G}_k, \mathbb{Z}/(2)) \simeq \mathbb{Z}/(2)$.

To describe $\mathrm{H}^0(\mathrm{G}_k, \mathrm{H}^2_{\text{fppf}}(\mathscr{M}^{\text{sep}}, \mu_2))$, we describe the G_k-action on $\mathrm{H}^2_{\text{fppf}}(\mathscr{M}^{\text{sep}}, \mu_2)$. Let

$$\xi \in k^{\text{sep}}$$

be a fixed root of $x^2 + x + 1$ (i.e. a primitive 3rd root of unity).

If $\xi \in k$, then G_k acts trivially on $\mathrm{H}^2_{\text{fppf}}(\mathscr{M}^{\text{sep}}, \mu_2)$; hence $\mathrm{H}^0(\mathrm{G}_k, \mathrm{H}^2_{\text{fppf}}(\mathscr{M}^{\text{sep}}, \mu_2))$ has 4 elements, hence $\mathrm{H}^2_{\text{fppf}}(\mathscr{M}, \mu_2)$ has 8 elements by (5.9.5), hence $(\operatorname{Br} \mathscr{M})[2]$ has 4 elements by (5.9.4), hence $\operatorname{Br} \mathscr{M} \simeq \mathbb{Z}/(2) \oplus \mathbb{Z}/(12)$.

Suppose $\xi \notin k$. The k-algebra map

$$k[\mu, \omega, \tfrac{1}{\mu^3 - 1}]/(\omega^2 + \omega + 1) \to k^{\text{sep}}[\nu_1, \tfrac{1}{\nu_1^3 - 1}] \times k^{\text{sep}}[\nu_2, \tfrac{1}{\nu_2^3 - 1}]$$

sending $\mu \mapsto (\nu_1, \nu_2)$ and $\omega \mapsto (\xi, \xi^2)$ induces an isomorphism

$$k[\mu, \omega, \tfrac{1}{\mu^3 - 1}]/(\omega^2 + \omega + 1) \otimes_k k^{\text{sep}} \to k^{\text{sep}}[\nu_1, \tfrac{1}{\nu_1^3 - 1}] \times k^{\text{sep}}[\nu_2, \tfrac{1}{\nu_2^3 - 1}] \tag{5.9.6}$$

of k^{sep}-algebras. The inverse to (5.9.6) sends

$$(f_1(\nu_1), f_2(\nu_2)) \mapsto f_1(\mu)\left(\omega \otimes \frac{1}{\xi - \xi^2} + 1 \otimes \frac{\xi}{\xi - 1}\right) + f_2(\mu)\left((-\omega) \otimes \frac{1}{\xi - \xi^2} + (-1) \otimes \frac{1}{\xi - 1}\right)$$

for $f_i(\nu_i) \in k[\nu_i, \frac{1}{\nu_i^3 - 1}]$.

Let

$$\lambda \in \mathrm{G}_k$$

be an automorphism of k^{sep} such that $\lambda(\xi) = \xi^2$. Then the k-algebra automorphism of $k^{\text{sep}}[\nu_1, \frac{1}{\nu_1^3-1}] \times k^{\text{sep}}[\nu_2, \frac{1}{\nu_2^3-1}]$ induced by (5.9.6) sends $(\nu_1, 0) \mapsto (0, \nu_2)$ and $(0, \nu_2) \mapsto (\nu_1, 0)$ and $(\xi, 0) \mapsto (0, \xi^2)$ and $(0, \xi) \mapsto (\xi^2, 0)$. We see that the action of λ on M (see (5.3.2)) is given by (5.9.7).

$$\begin{array}{c|c|c|c|c|c|c} & \nu_1 - 1 & \nu_1 - \xi & \nu_1 - \xi^2 & \nu_2 - 1 & \nu_2 - \xi & \nu_2 - \xi^2 \\ \lambda & \nu_2 - 1 & \nu_2 - \xi^2 & \nu_2 - \xi & \nu_1 - 1 & \nu_1 - \xi^2 & \nu_1 - \xi \end{array} \tag{5.9.7}$$

A computation with (5.9.7) and (5.3.6) shows that

$$\lambda \mathsf{g} \lambda^{-1} \cdot m = \mathsf{g} \cdot m \tag{5.9.8}$$

for any $m \in M$ and $\mathsf{g} \in \mathrm{GL}_{2,3}$.

Let $f_{\mathrm{GL}_{2,3}} : \mathrm{GL}_{2,3} \to M$ be an inhomogeneous 1-cocycle as in Note 5.7. Multiplying the 1-cocycle condition (5.4.8) on the left by λ gives

$$\begin{aligned} \lambda \cdot f_{\mathrm{GL}_{2,3}}(\mathsf{g}_1 \cdot \mathsf{g}_2) &= \lambda \mathsf{g}_1 \cdot f_{\mathrm{GL}_{2,3}}(\mathsf{g}_2) + \lambda \cdot f_{\mathrm{GL}_{2,3}}(\mathsf{g}_1) \\ &\overset{1}{=} \mathsf{g}_1 \cdot (\lambda \cdot f_{\mathrm{GL}_{2,3}}(\mathsf{g}_2)) + \lambda \cdot f_{\mathrm{GL}_{2,3}}(\mathsf{g}_1) \end{aligned}$$

where equality 1 follows from (5.9.8). Hence the function $\lambda \cdot f_{\mathrm{GL}_{2,3}} : \mathrm{GL}_{2,3} \to M$ sending $\mathsf{g} \mapsto \lambda \cdot f_{\mathrm{GL}_{2,3}}(\mathsf{g})$ is a 1-cocycle as well. Using (5.9.7) and (5.7.4), we have that

$$\begin{aligned} (\lambda \cdot f_{\mathrm{GL}_{2,3}})(\mathsf{e}) &= ((0,0,0),(0,0,0)) \\ (\lambda \cdot f_{\mathrm{GL}_{2,3}})(\mathsf{i}) &= ((s_1 + s, 0, s),(s_1, s, 0)) \\ (\lambda \cdot f_{\mathrm{GL}_{2,3}})(\mathsf{j}) &= ((0, s, s_1 + s),(0, s_1, s)) \\ (\lambda \cdot f_{\mathrm{GL}_{2,3}})(\mathsf{k}) &= ((s, s_1 + s, 0),(s, 0, s_1)) \end{aligned} \tag{5.9.9}$$

and so

$$\begin{aligned} f_{\mathrm{GL}_{2,3}}(\mathsf{e}) - (\lambda \cdot f_{\mathrm{GL}_{2,3}})(\mathsf{e}) &= ((0,0,0),(0,0,0)) \\ f_{\mathrm{GL}_{2,3}}(\mathsf{i}) - (\lambda \cdot f_{\mathrm{GL}_{2,3}})(\mathsf{i}) &= ((s,0,0),(s,0,0)) \\ f_{\mathrm{GL}_{2,3}}(\mathsf{j}) - (\lambda \cdot f_{\mathrm{GL}_{2,3}})(\mathsf{j}) &= ((0,0,s),(0,s,0)) \\ f_{\mathrm{GL}_{2,3}}(\mathsf{k}) - (\lambda \cdot f_{\mathrm{GL}_{2,3}})(\mathsf{k}) &= ((0,s,0),(0,0,s)) \end{aligned} \tag{5.9.10}$$

for the same $s, s_1 \in \mathbb{Z}/(2)$ as in (5.7.4).

Suppose $f_{\mathrm{GL}_{2,3}}$ and $\lambda \cdot f_{\mathrm{GL}_{2,3}}$ differ by a 1-coboundary, in other words there exists an element

$$m := ((m_1^1, m_2^1, m_3^1),(m_1^2, m_2^2, m_3^2)) \in M$$

such that

$$f_{\mathrm{GL}_{2,3}}(\mathsf{g}) - (\lambda \cdot f_{\mathrm{GL}_{2,3}})(\mathsf{g}) = \mathsf{g} \cdot m - m \tag{5.9.11}$$

for all $\mathsf{g} \in \mathrm{GL}_{2,3}$. By (5.9.10), taking $\mathsf{g} = \mathsf{M}_2$ in (5.9.11) gives $m^i := m_1^i = m_2^i = m_3^i$ for $i = 1, 2$; then taking $\mathsf{g} = \mathsf{M}_1$ gives $m^1 = m^2$; then taking $\mathsf{g} = \mathsf{i}$ gives $m = 0$. We see that $f_{\mathrm{GL}_{2,3}}$ and $\lambda \cdot f_{\mathrm{GL}_{2,3}}$ differ by a 1-coboundary if and only if $s = 0$.

Hence we have that $\mathrm{H}^0(\mathrm{G}_k, \mathrm{H}^2_{\mathrm{fppf}}(\mathscr{M}^{\mathrm{sep}}, \mu_2)) \simeq \mathbb{Z}/(2)$, hence $\mathrm{H}^2_{\mathrm{fppf}}(\mathscr{M}, \mu_2)$ has 4 elements by (5.9.5), hence $(\mathrm{Br}\,\mathscr{M})[2]$ has 2 elements by (5.9.4), hence $\mathrm{Br}\,\mathscr{M} \simeq \mathbb{Z}/(24)$. $\square$

Appendix A. The Weierstrass and Hesse presentations of $[\Gamma(3)]$

The purpose of this section is to prove Proposition A.4 below, which we could not find proved in the literature. For completeness of exposition, we first recall the definition of a full level N structure on an elliptic curve E/S.

A.1 (Full level N structure). [KM85, Ch. 3] Let N be a positive integer. We define $[\Gamma(N)]$ to be the category of pairs

$$(E/S, \xi)$$

where

$$E/S = (f : E \to S, e : S \to E)$$

is an elliptic curve and

$$\xi : (\mathbb{Z}/(N))^2_S \to E$$

is a morphism of S-group schemes inducing an isomorphism $(\mathbb{Z}/(N))^2_S \simeq E[N]$. A morphism

$$(E_1/S_1, \xi_1) \to (E_2/S_2, \xi_2)$$

is a pair

$$(\alpha : E_1 \to E_2\ ,\ \beta : S_1 \to S_2)$$

of morphisms of schemes such that the diagram

(A.1.1)
$$\begin{tikzcd}
& E_1 \arrow[rr, "\alpha"] \arrow[dd, "f_1"] & & E_2 \arrow[dd, "f_2"] \\
(\mathbb{Z}/(N))^2_{S_1} \arrow[ur, "\xi_1"] \arrow[dr] \arrow[rr, "\mathrm{id}\times\beta"] & & (\mathbb{Z}/(N))^2_{S_2} \arrow[ur, "\xi_2"] \arrow[dr] & \\
& S_1 \arrow[rr, "\beta"'] & & S_2
\end{tikzcd}$$

commutes, where the morphism $\mathrm{id}\times\beta$ is the one induced by the identity on $(\mathbb{Z}/(3))^2_{\mathbb{Z}}$ and β, and such that α induces an isomorphism of S_1-group schemes $E_1 \simeq S_1 \times_{\beta, S_2} E_2$.

There is a functor

$$[\Gamma(N)] \to \mathscr{M}_{1,1,\mathbb{Z}}$$

sending $(E/S, \xi) \mapsto E/S$ on objects and $(\alpha, \beta) \mapsto (\alpha, \beta)$ on morphisms. If E/S admits a full level N structure, then N is invertible on S by [KM85, 2.3.2], hence the above functor factors through $\mathscr{M}_{1,1,\mathbb{Z}[\frac{1}{N}]}$. If $N \geq 3$, then for any scheme S the fiber category $[\Gamma(N)](S)$ is equivalent to a set by [KM85, 2.7.2], so $[\Gamma(N)]$ is fibered in sets over the category of schemes.

A.2 (The $\mathrm{GL}_2(\mathbb{Z}/(N))$-action on $[\Gamma(N)]$). Fix a scheme S. For any element

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$$

in $\mathrm{GL}_2(\mathbb{Z}/(N))$, let

$$\varphi_\sigma : (\mathbb{Z}/(N))^2_S \to (\mathbb{Z}/(N))^2_S$$

be the S-group scheme automorphism of $(\mathbb{Z}/(N))^2_S$ corresponding to the abelian group homomorphism $(\mathbb{Z}/(N))^2 \to (\mathbb{Z}/(N))^2$ defined by

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \sigma_{11}x_1 + \sigma_{12}x_2 \\ \sigma_{21}x_1 + \sigma_{22}x_2 \end{bmatrix}$$

for $x_1, x_2 \in \mathbb{Z}/(N)$, i.e. acting by multiplication on the left on $(\mathbb{Z}/(N))^2$ viewed as vertical vectors. We have

$$\varphi_{\sigma_1}\varphi_{\sigma_2} = \varphi_{\sigma_1\sigma_2}$$

for $\sigma_1, \sigma_2 \in \mathrm{GL}_2(\mathbb{Z}/(N))$.

Fix an object $(E/S, \xi) \in [\Gamma(N)](E/S)$; then $(E/S, \xi \circ \varphi_\sigma)$ is another object of $[\Gamma(N)](E/S)$, i.e. corresponds to another full level N structure on E/S. This implies that there is a natural action of $\mathrm{GL}_2(\mathbb{Z}/(N))$ on each fiber category $[\Gamma(N)](E/S)$; the action is a right action since it is defined by precomposition.

Theorem A.3. [KM85, 4.7.2] *If $N \geq 3$, the category $[\Gamma(N)]$ is representable by a smooth affine curve $Y(N)$ over $\mathbb{Z}[\frac{1}{N}]$.*

We are primarily interested in the case $N = 3$. The 3-torsion points of an elliptic curve correspond to its inflection points (also "flex points"). In [KM85, (2.2.11)] it is shown that $Y(3) \simeq \mathrm{Spec}\, A_{\mathrm{W}}$ where

$$A_{\mathrm{W}} := \mathbb{Z}[\tfrac{1}{3}, B, C, \tfrac{1}{C}, \tfrac{1}{a_3}, \tfrac{1}{a_1^3 - 27a_3}]/(B^3 - (B+C)^3)$$

and the universal elliptic curve over A_{W} with full level 3 structure is the pair

$$\text{(A.3.1)} \qquad \begin{cases} E_{\mathrm{W}} := \mathrm{Proj}\, A_{\mathrm{W}}[X, Y, Z]/(Y^2Z + a_1XYZ + a_3YZ^2 = X^3) \\ [0:0:1], [C : B+C : 1] \end{cases}$$

where

$$\text{(A.3.2)} \qquad a_1 = 3C - 1$$

$$\text{(A.3.3)} \qquad a_3 = -3C^2 - B - 3BC\,.$$

The formulas (A.3.2) and (A.3.3) are obtained by imposing the condition that the line $Y = X + BZ$ is a flex tangent to E_{W} at $[C : B+C : 1]$. The ring A_{W} is isomorphic to $TMF(3)_0$ (5.6.1), with mutually inverse ring isomorphisms $TMF(3)_0 \to A_{\mathrm{W}}$ and $A_{\mathrm{W}} \to TMF(3)_0$ given by $(\zeta, t) \mapsto (\frac{B+C}{C}, \frac{1}{3C})$ and $(B, C) \mapsto (\frac{1}{3(\zeta-1)t}, \frac{1}{3t})$ respectively.

In this paper, however, we use the "Hesse presentation" of $Y(3)$ as in [FO10, 5.1]. The following is claimed without proof in the Introduction to [DR73] and [Har11, 5.2.30].

Proposition A.4. *There is an isomorphism* $Y(3) \simeq \operatorname{Spec} A_{\mathrm{H}}$ *where*

$$A_{\mathrm{H}} := \mathbb{Z}[\tfrac{1}{3}, \mu, \omega, \tfrac{1}{\mu^3-1}]/(\omega^2+\omega+1)$$

and the universal elliptic curve over A_{H} *with full level 3 structure is the pair*

$$\begin{cases} E_{\mathrm{H}} := \operatorname{Proj} A_{\mathrm{H}}[X,Y,Z]/(X^3+Y^3+Z^3 = 3\mu XYZ) \\ [-1:0:1], [1:-\omega:0] \end{cases} \tag{A.4.1}$$

with identity section $[1:-1:0]$.

The explicit $\mathbb{Z}[\frac{1}{3}]$-algebra isomorphisms $A_{\mathrm{H}} \to A_{\mathrm{W}}$ and $A_{\mathrm{W}} \to A_{\mathrm{H}}$ are given in (A.8.7) and (A.8.8) respectively.

A.5. By [Sma01, §4], the group law of an elliptic curve $E = \operatorname{Proj} A[X,Y,Z]/(X^3+Y^3+Z^3 = 3\mu XYZ)$ in Hessian form over a ring A is as follows. If $P = [x:y:z]$, then $2P = [x':y':z']$ where

$$\begin{aligned} x' &= y(z^3-x^3) \\ y' &= x(y^3-z^3) \\ z' &= z(x^3-y^3) \end{aligned}$$

and if $P_i = [x_i : y_i : z_i]$ are points of E_{H} for $i = 1,2,3$ satisfying $P_1 + P_2 = P_3$, then

$$\begin{aligned} x_3 &= x_2 y_1^2 z_2 - x_1 y_2^2 z_1 \\ y_3 &= x_1^2 y_2 z_2 - x_2^2 y_1 z_1 \\ z_3 &= x_2 y_2 z_1^2 - x_1 y_1 z_2^2 \end{aligned}$$

which only makes sense if $P_1 \neq P_2$.

Using the above formulas, we may check that the full level 3 structure $\xi_{\mathrm{H}} : (\mathbb{Z}/(3))^2_{A_{\mathrm{H}}} \to E_{\mathrm{H}}$ is given by the table (A.5.1).

$$\xi_{\mathrm{H}}\left(\begin{bmatrix} (0,0) & (1,0) & (2,0) \\ (0,1) & (1,1) & (2,1) \\ (0,2) & (1,2) & (2,2) \end{bmatrix}\right) = \begin{bmatrix} [1:-1:0] & [-1:0:1] & [0:1:-1] \\ [1:-\omega:0] & [-\omega:0:1] & [0:1:-\omega] \\ [1:-\omega^2:0] & [-\omega^2:0:1] & [0:1:-\omega^2] \end{bmatrix} \tag{A.5.1}$$

The Hesse presentation (A.4.1) is sometimes easier to work with than the Weierstrass presentation (A.3.1) since the equation of the universal elliptic curve is symmetric in X, Y, Z, which means that there is also considerable symmetry in the 3-torsion points (A.5.1).

A.6. We describe the $\mathrm{GL}_2(\mathbb{Z}/(3))$-action on $E_{\mathrm{H}}/A_{\mathrm{H}}$. Set $S_{\mathrm{H}} := \operatorname{Spec} A_{\mathrm{H}}$. The functor $[\Gamma(3)]$ being representable by S_{H} means explicitly that for any $\mathbb{Z}[\frac{1}{3}]$-scheme T and object $(E/T, \xi) \in ([\Gamma(3)])(T)$, there exists a unique pair (α, β) of morphisms of schemes $\alpha : E \to E_{\mathrm{H}}$ and $\beta : T \to S_{\mathrm{H}}$ such that the diagram

$$\begin{tikzcd}
 & E \arrow[rr, "\alpha"] \arrow[dd, "f_T" near end] & & E_{\mathrm{H}} \arrow[dd, "f_{S_{\mathrm{H}}}"] \\
(\mathbb{Z}/(3))^2_T \arrow[ur, "\xi"] \arrow[dr] \arrow[rr, "\mathrm{id} \times \beta" near end] & & (\mathbb{Z}/(3))^2_{S_{\mathrm{H}}} \arrow[ur, "\xi_{\mathrm{H}}"] \arrow[dr] & \\
 & T \arrow[rr, "\beta"'] & & S_{\mathrm{H}}
\end{tikzcd}$$

commutes and induces an isomorphism of T-group schemes $E \simeq T \times_{\beta, S_{\mathrm{H}}} E_{\mathrm{H}}$ as in (A.1.1).

As in A.2, for every $\sigma \in \mathrm{GL}_2(\mathbb{Z}/(3))$, let φ_σ be the S_{H}-automorphism of $(\mathbb{Z}/(3))^2_{S_{\mathrm{H}}}$ induced by σ; then precomposition $\xi_{\mathrm{H}}\varphi_\sigma$ defines another full level 3 structure on $E_{\mathrm{H}}/S_{\mathrm{H}}$. Taking $T = S_{\mathrm{H}}$ and $\xi = \xi_{\mathrm{H}}\varphi_\sigma$ above, there is a unique pair $(\alpha_\sigma, \beta_\sigma)$ of morphisms of schemes $\alpha_\sigma : E_{\mathrm{H}} \to E_{\mathrm{H}}$ and $\beta_\sigma : S_{\mathrm{H}} \to S_{\mathrm{H}}$ such that the diagram

$$\begin{tikzcd}
 & E_{\mathrm{H}} \arrow[rr, "\alpha_\sigma"] \arrow[dd, "f_{S_{\mathrm{H}}}" near end] & & E_{\mathrm{H}} \arrow[dd, "f_{S_{\mathrm{H}}}"] \\
(\mathbb{Z}/(3))^2_{S_{\mathrm{H}}} \arrow[ur, "\xi_{\mathrm{H}}\varphi_\sigma"] \arrow[dr] \arrow[rr, "\mathrm{id} \times \beta_\sigma" near end] & & (\mathbb{Z}/(3))^2_{S_{\mathrm{H}}} \arrow[ur, "\xi_{\mathrm{H}}"] \arrow[dr] & \\
 & S_{\mathrm{H}} \arrow[rr, "\beta_\sigma"'] & & S_{\mathrm{H}}
\end{tikzcd}$$

commutes and induces an isomorphism of S_{H}-group schemes $E_{\mathrm{H}} \simeq S_{\mathrm{H}} \times_{\beta_\sigma, S_{\mathrm{H}}} E_{\mathrm{H}}$. Given two elements $\sigma_1, \sigma_2 \in \mathrm{GL}_2(\mathbb{Z}/(3))$, we have a commutative diagram

$$\begin{tikzcd}
 & E_{\mathrm{H}} \arrow[rr, "\alpha_{\sigma_1}"] \arrow[dd, "f_{S_{\mathrm{H}}}" near end] & & E_{\mathrm{H}} \arrow[rr, "\alpha_{\sigma_2}"] \arrow[dd, "f_{S_{\mathrm{H}}}" near end] & & E_{\mathrm{H}} \arrow[dd, "f_{S_{\mathrm{H}}}"] \\
(\mathbb{Z}/(3))^2_{S_{\mathrm{H}}} \arrow[ur, "\xi_{\mathrm{H}}\varphi_{\sigma_1}\varphi_{\sigma_2}"] \arrow[dr] \arrow[rr] & & (\mathbb{Z}/(3))^2_{S_{\mathrm{H}}} \arrow[ur, "\xi_{\mathrm{H}}\varphi_{\sigma_2}"] \arrow[dr] \arrow[rr] & & (\mathbb{Z}/(3))^2_{S_{\mathrm{H}}} \arrow[ur, "\xi_{\mathrm{H}}"] \arrow[dr] & \\
 & S_{\mathrm{H}} \arrow[rr, "\beta_{\sigma_1}"'] & & S_{\mathrm{H}} \arrow[rr, "\beta_{\sigma_2}"'] & & S_{\mathrm{H}}
\end{tikzcd}$$

which implies

$$\beta_{\sigma_2}\beta_{\sigma_1} = \beta_{\sigma_1\sigma_2}$$

since $\varphi_{\sigma_1\sigma_2} = \varphi_{\sigma_1}\varphi_{\sigma_2}$ (see A.2). Thus the assignment

$$\sigma \mapsto \beta_\sigma \tag{A.6.1}$$

defines a right action of $\mathrm{GL}_2(\mathbb{Z}/(3))$ on the scheme S_{H}.

In terms of the generators

$$\mathsf{M}_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ \mathsf{M}_2 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \ \mathsf{i} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

of $\mathrm{GL}_2(\mathbb{Z}/(3))$, the action of $\mathrm{GL}_2(\mathbb{Z}/(3))$ on $E_{\mathrm{H}}/A_{\mathrm{H}}$ is as follows. (We refer to (A.5.1) for the additive structure on $E_{\mathrm{H}}[3]$.)

(1) For $\sigma = \mathsf{M}_1$, the new level 3 structure $\xi_{\mathrm{H}}\varphi_{\mathsf{M}_1}$ is

$$\big[[-1:0:1] \quad [1:-\omega:0]\big] \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \big[[-1:0:1] \quad [1:-\omega^2:0]\big]$$

and the scheme morphisms $\alpha_{\mathsf{M}_1} : E_{\mathrm{H}} \to E_{\mathrm{H}}$ and $\beta_{\mathsf{M}_1} : S_{\mathrm{H}} \to S_{\mathrm{H}}$ correspond to the ring homomorphisms sending

$$\begin{cases} (X,Y,Z) \hookleftarrow (X,Y,Z) \\ (\mu,\omega^2) \hookleftarrow (\mu,\omega) \end{cases}$$

respectively.

(2) For $\sigma = \mathsf{M}_2$, the new level 3 structure $\xi_{\mathrm{H}}\varphi_{\mathsf{M}_2}$ is

$$\big[[-1:0:1] \quad [1:-\omega:0]\big] \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \big[[-\omega^2:0:1] \quad [1:-\omega:0]\big]$$

and the scheme morphisms $\alpha_{\mathsf{M}_2} : E_{\mathrm{H}} \to E_{\mathrm{H}}$ and $\beta_{\mathsf{M}_2} : S_{\mathrm{H}} \to S_{\mathrm{H}}$ correspond to the ring homomorphisms sending

$$\begin{cases} (X,Y,\omega^2 Z) \hookleftarrow (X,Y,Z) \\ (\omega\mu,\omega) \hookleftarrow (\mu,\omega) \end{cases}$$

respectively.

(3) For $\sigma = \mathsf{i}$, the new level 3 structure $\xi_{\mathrm{H}}\varphi_{\mathsf{i}}$ is

$$\big[[-1:0:1] \quad [1:-\omega:0]\big] \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \big[[1:-\omega:0] \quad [0:1:-1]\big]$$

and the scheme morphisms $\alpha_{\mathsf{i}} : E_{\mathrm{H}} \to E_{\mathrm{H}}$ and $\beta_{\mathsf{i}} : S_{\mathrm{H}} \to S_{\mathrm{H}}$ correspond to the ring homomorphisms sending

$$\begin{cases} (\omega X + \omega^2 Y + Z, \omega^2 X + \omega Y + Z, X+Y+Z) \hookleftarrow (X,Y,Z) \\ (\frac{\mu+2}{\mu-1},\omega) \hookleftarrow (\mu,\omega) \end{cases}$$

respectively.

Remark A.7. According to our convention, the action of $\mathrm{GL}_2(\mathbb{Z}/(3))$ on the fiber category $[\Gamma(3)](E_{\mathrm{H}}/\operatorname{Spec} A_{\mathrm{H}})$ is by precomposition, hence the action of $\mathrm{GL}_2(\mathbb{Z}/(3))$ on pairs of points on the right hand side of (A.5.1) is a *right* action; thus the induced action of $\mathrm{GL}_2(\mathbb{Z}/(3))$ on the scheme $\operatorname{Spec} A_{\mathrm{H}}$ is a *right* action (as described in (A.6.1)) and the corresponding action of $\mathrm{GL}_2(\mathbb{Z}/(3))$ on the coordinate ring A_{H} is a *left* action.

A.8 (Proof of Proposition A.4). In fact, it turns out that the identities

$$a_1^3 - 27a_3 = (3C + 9B - 1)^3 \tag{A.8.1}$$

$$a_3 = B(6C + 9B - 1) \tag{A.8.2}$$

hold in A_{W} which yields a simpler description

$$A_{\mathrm{W}} \simeq \mathbb{Z}[\tfrac{1}{3}, B, C, \tfrac{1}{C}, \tfrac{1}{3C+9B-1}, \tfrac{1}{6C+9B-1}]/(C^2 + 3CB + 3B^2)$$

of A_{W}. (For (A.8.1), write out $a_1^3 - 27a_3$ in terms of B, C and notice that it is of the form $9C + 27B - 1$ plus higher order terms; then check that the naive guess works. To see (A.8.2), substitute $C^2 = -3CB - 3B^2$ into (A.3.3).)

We follow the argument of [AD09, 2.1]; see also [Con96, §1.4.1, §1.4.2]. Working "generically", we will assume that a_1 is a unit to obtain the coordinate change formula (A.8.9), then observe that it applies also to the case when a_1 is not a unit. Starting with

$$Y_1 Z_1 (Y_1 + a_1 X_1 + a_3 Z_1) = X_1^3 \tag{A.8.3}$$

we define X_2, Y_2, Z_2 by the system

$$\begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} = \begin{bmatrix} u^2 & & \\ & u^3 & \\ & & 1 \end{bmatrix} \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix}$$

where $u = a_1/3$ and substitute into (A.8.3) to get

$$Y_2 Z_2 (Y_2 + 3X_2 + \tfrac{27a_3}{a_1^3} Z_2) = X_2^3 \, . \tag{A.8.4}$$

We define X_3, Y_3, Z_3 by the system

$$\begin{bmatrix} 1 & 1 & \\ 1 & & \frac{27a_3}{a_1^3} \\ 1 & & \end{bmatrix} \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = \begin{bmatrix} \omega & \omega^2 & \\ \omega^2 & \omega & \\ & & 1 \end{bmatrix} \begin{bmatrix} X_3 \\ Y_3 \\ Z_3 \end{bmatrix}$$

where $\omega = \frac{C+B}{B}$ [2] and substitute into (A.8.4) to get

$$(\omega X_3 + \omega^2 Y_3 - Z_3)(\omega^2 X_3 + \omega Y_3 - Z_3)(-X_3 - Y_3 + Z_3) = \tfrac{27a_3}{a_1^3} Z_3^3$$

or equivalently

$$X_3^3 + Y_3^3 + \tfrac{27a_3 - a_1^3}{a_1^3} Z_3^3 = -3X_3 Y_3 Z_3 \, . \tag{A.8.5}$$

We know that the coefficient of Z_3^3 in (A.8.5) is a cube (A.8.1) so we normalize by defining X_4, Y_4, Z_4 by the system

$$\begin{bmatrix} X_3 \\ Y_3 \\ Z_3 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & \frac{-a_1}{3C+9B-1} \end{bmatrix} \begin{bmatrix} X_4 \\ Y_4 \\ Z_4 \end{bmatrix}$$

and substitute into (A.8.5) to get

$$X_4^3 + Y_4^3 + Z_4^3 = 3\tfrac{a_1}{3C+9B-1} X_4 Y_4 Z_4 \, . \tag{A.8.6}$$

[2]Since 3 is invertible, if x is a root of the polynomial $T^2 + 3T + 3$ then $x + 1$ is a root of the polynomial $T^2 + T + 1$, thus it is natural to take $\frac{C+B}{B}$ as our ω.

To summarize the above, there is a ring homomorphism $\varphi_{21} : A_{\mathrm{H}} \to A_{\mathrm{W}}$ sending

$$\begin{aligned} \mu &\mapsto \frac{3C-1}{3C+9B-1} \\ \omega &\mapsto \frac{C+B}{B} \end{aligned} \tag{A.8.7}$$

and solving for B, C in terms of μ, ω implies that the inverse $\varphi_{12} : A_{\mathrm{W}} \to A_{\mathrm{H}}$ sends

$$\begin{aligned} B &\mapsto \frac{\mu-1}{3(\omega+2)(\mu-\omega)} \\ C &\mapsto \frac{(\omega-1)(\mu-1)}{3(\omega+2)(\mu-\omega)} \end{aligned} \tag{A.8.8}$$

where $\omega+2$ is a unit of A_{H} since $(\omega+2)(\omega-1) = -3$ and $\mu-\omega$ is a unit of A_{H} since $\mu^3 - 1 = (\mu-1)(\mu-\omega)(\mu-\omega^2)$. We may check that the product

$$\begin{bmatrix} u^2 & & \\ & u^3 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & \\ 1 & & \frac{27a_3}{a_1^3} \\ 1 & & \end{bmatrix}^{-1} \begin{bmatrix} \omega & \omega^2 & \\ \omega^2 & \omega & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & \frac{-a_1}{3C+9B-1} \end{bmatrix}$$

is "projectively equivalent" to the matrix

$$X := \begin{bmatrix} 0 & 0 & \frac{-3}{3C+9B-1} \\ \omega & \omega^2 & \frac{3u}{3C+9B-1} \\ \frac{\omega^2}{a_3} & \frac{\omega}{a_3} & \frac{3u}{a_3(3C+9B-1)} \end{bmatrix} \tag{A.8.9}$$

whose determinant is a unit of A_{W}. Given a section $[s_X : s_Y : s_Z]$ of (A.8.3), the corresponding section of (A.8.6) is $X^{-1} \cdot [s_X : s_Y : s_Z]^{\mathrm{T}}$ where

$$X^{-1} = \begin{bmatrix} \frac{-a_1}{3} & \frac{B}{C} & \frac{-9CB-18B^2-C}{3} \\ \frac{-a_1}{3} & \frac{-B}{C+3B} & \frac{-9CB-9B^2+C+3B}{3} \\ \frac{-3C-9B+1}{3} & 0 & 0 \end{bmatrix} .$$

The above implies that the sections

$$[0 : 1 : 0] \ , \ [0 : 0 : 1] \ , \ [C : B + C : 1]$$

of (A.8.3) (i.e. the identity section and ordered basis for the 3-torsion) correspond to the sections

$$[1 : -\omega : 0] \ , \ [1 : -\omega^2 : 0] \ , \ [-1 : 0 : 1] \tag{A.8.10}$$

of (A.8.6). We may apply an automorphism of the pair $(A_{\mathrm{H}}, E_{\mathrm{H}}/A_{\mathrm{H}}) \in \mathscr{M}_{1,1,\mathbb{Z}}$ of the form A.6(2) (for Y instead of Z) to (A.8.10) to get

$$[1 : -1 : 0] \ , \ [1 : -\omega : 0] \ , \ [-1 : 0 : 1] \tag{A.8.11}$$

and using the fact that there is a simply transitive action of $\mathrm{GL}_2(\mathbb{Z}/(3))$ on the set of ordered bases of the 3-torsion in $E_{\mathrm{H}}/A_{\mathrm{H}}$, we may switch the second and third sections of (A.8.11) to obtain

$$[1 : -1 : 0] \ , \ [-1 : 0 : 1] \ , \ [1 : -\omega : 0] \tag{A.8.12}$$

as desired. $\square$

Remark A.9. For (A.8.1), see also Stojanoska's derivation [Sto14, §4.1].

Remark A.10. There are coordinate change formulas in [Sma01, §3] transforming a Weierstrass equation into Hesse normal form, but there it is assumed that the base ring is a finite field $\mathbb{F}_q$ where $q \equiv 2 \pmod 3$, in order to take cube roots of $a_1^3 - 27a_3$, but from this description it is not clear that the cube root is an algebraic function. As shown in (A.8.1), it turns out that in fact $a_1^3 - 27a_3$ is a cube in the ring A_{W}. One suspects that this is the case after tracing through the proof of [AD09, 2.1] and arriving at the equation $x^3 + y^3 + \frac{27a_3 - a_1^3}{a_1^3} z^3 = 3xyz$, in which case we know that $\frac{27a_3 - a_1^3}{a_1^3}$ is a cube by Lemma A.11.

Lemma A.11. *Let k be a field of characteristic not 3, and let*

$$x^3 + y^3 + \beta = 3xy \tag{A.11.1}$$

be a curve in $\mathbb{A}_k^2$. Suppose that

$$ax + by + c = 0 \tag{A.11.2}$$

is the tangent line to a flex point of E and suppose that $a^3 \neq b^3$. Then β is a cube in k.

Proof. If $a = 0$, then $b \neq 0$ and substituting $y = -\frac{c}{b}$ into (A.11.1) and rearranging gives $x^3 + \frac{3c}{b}x - (\frac{c}{b})^3 + \beta = 0$ which by assumption is of the form $(x + \ell)^3$ for some $\ell \in k$. Comparing coefficients, we have $\ell = 0$ and so $\beta = (\frac{c}{b})^3$.

By symmetry we may assume that $a, b \neq 0$. By scaling (A.11.2), we may assume that $b = -1$. Substituting $y = ax + c$ into E gives

$$(a^3 + 1)x^3 + 3(a)(ac - 1)x^2 + 3(c)(ac - 1)x + (c^3 + \beta)$$

and dividing by the leading coefficient gives

$$x^3 + 3\left(\frac{a(ac-1)}{a^3+1}\right)x^2 + 3\left(\frac{c(ac-1)}{a^3+1}\right)x + \left(\frac{c^3+\beta}{a^3+1}\right)$$

and comparing this to

$$x^3 + 3\ell x^2 + 3\ell^2 x + \ell^3$$

gives either $ac - 1 = 0$ in which case $c^3 + \beta = 0$ as well (so that $\beta = (-1/a)^3 = (-c)^3$), otherwise if $ac - 1 \neq 0$ then

$$\frac{c}{a} = a\left(\frac{ac-1}{a^3+1}\right)$$

which implies $c = -a^2$ so that the original equation of the tangent line is $y = ax - a^2$. Substituting this back into E gives $\beta = (-a)^3$. ∎

Appendix B. Higher direct images of sheaves on classifying stacks of discrete groups

The material in this section is standard and we claim no originality.

For a category $\mathcal{C}$, we denote by $\mathrm{PSh}(\mathcal{C})$ (resp. $\mathrm{PAb}(\mathcal{C})$) the category of presheaves (resp. abelian presheaves) on $\mathcal{C}$. If $\mathcal{C}$ is a site, we denote by $\mathrm{Sh}(\mathcal{C})$ (resp. $\mathrm{Ab}(\mathcal{C})$) the category of sheaves (resp. abelian sheaves) on $\mathcal{C}$.

Let $\mathcal{C}$ be a site, let G be a finite (discrete) group, let $\mathrm{B}G_{\mathcal{C}}$ be the classifying stack associated to G over $\mathcal{C}$. Let

$$\pi : \mathrm{B}G_{\mathcal{C}} \to \mathcal{C}$$

be the projection and let

$$\varphi : \mathcal{C} \to \mathrm{B}G_{\mathcal{C}}$$

be the canonical section of π. We view any fibered category $p : \mathcal{F} \to \mathcal{C}$ as a site via the Grothendieck topology inherited from $\mathcal{C}$ via p.

Lemma B.1. *In the setup above, for any abelian sheaf $\mathscr{F} \in \mathrm{Ab}(\mathrm{B}G_{\mathcal{C}})$ the higher pushforward $\mathbf{R}^i\pi_*\mathscr{F}$ is naturally isomorphic to the sheaf associated to the presheaf whose value on an object $U \in \mathcal{C}$ is $\mathrm{H}^i(G, \Gamma(U, \varphi^*\mathscr{F}))$.*

Proof. Let $\mathrm{P}G_{\mathcal{C}}$ denote the category whose objects are the objects of $\mathcal{C}$ and where a morphism $X_1 \to X_2$ in $\mathrm{P}G_{\mathcal{C}}$ is a pair (φ, g) where $\varphi \in \mathrm{Mor}_{\mathcal{C}}(X_1, X_2)$ and $g \in G$. (In other words, there is an equivalence of categories $\mathrm{P}G_{\mathcal{C}} \simeq \mathcal{C} \times [*/G]$ where $[*/G]$ is the category with one object $*$ and where $\mathrm{Hom}_{[*/G]}(*, *)$ is isomorphic to G.) The fibered category $\mathrm{P}G_{\mathcal{C}}$ is a (separated) prestack whose associated stack is $\mathrm{B}G_{\mathcal{C}}$, and the inclusion $\mathrm{P}G_{\mathcal{C}} \to \mathrm{B}G_{\mathcal{C}}$ induces an equivalence of topoi $\mathrm{Sh}(\mathrm{P}G_{\mathcal{C}}) \simeq \mathrm{Sh}(\mathrm{B}G_{\mathcal{C}})$. Hence in the statement of the lemma we may replace $\mathrm{B}G_{\mathcal{C}}$ by $\mathrm{P}G_{\mathcal{C}}$ where by abuse of notation we also denote

$$\pi : \mathrm{P}G_{\mathcal{C}} \to \mathcal{C}$$

the projection morphism. Since sheafification is an exact functor, the diagram

$$\begin{tikzcd} \mathrm{PAb}(\mathrm{P}G_{\mathcal{C}}) \arrow[r, "\pi_*^{\mathrm{pre}}"] \arrow[d, "\mathrm{sh}"'] & \mathrm{PAb}(\mathcal{C}) \arrow[d, "\mathrm{sh}"] \\ \mathrm{Ab}(\mathrm{P}G_{\mathcal{C}}) \arrow[r, "\pi_*"'] & \mathrm{Ab}(\mathcal{C}) \end{tikzcd}$$

is (2-)commutative. For the same reason, we have a natural isomorphism

$$(\mathbf{R}\pi_*^{\mathrm{pre}}(\mathscr{F}))^{\mathrm{sh}} \simeq \mathbf{R}\pi_*(\mathscr{F}^{\mathrm{sh}}) \tag{B.1.1}$$

in $\mathrm{D}^+(\mathrm{Ab}(\mathcal{C}))$ for any abelian presheaf $\mathscr{F} \in \mathrm{PAb}(\mathrm{P}G_{\mathcal{C}})$. Presheaves on $\mathrm{P}G_{\mathcal{C}}$ correspond to presheaves $\mathscr{F}$ on $\mathcal{C}$ equipped with a G-action, and under this identification $\pi_*^{\mathrm{pre}}(\mathscr{F}) = \mathscr{F}^G$ where $\Gamma(U, \mathscr{F}^G) := (\Gamma(U, \mathscr{F}))^G$ for all $U \in \mathcal{C}$. Let $\mathscr{F} \in \mathrm{Ab}(\mathrm{P}G_{\mathcal{C}})$ be an abelian sheaf, and let

$$\mathscr{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \mathcal{I}^2 \to \cdots$$

be a resolution of $\mathscr{F}$ by injective abelian presheaves $\mathcal{I}^i \in \mathrm{PAb}(\mathrm{P}G_{\mathcal{C}})$. Then $\mathbf{R}\pi_*^{\mathrm{pre}}(\mathscr{F})$ is isomorphic to

$$(\mathcal{I}^\bullet)^G = \{(\mathcal{I}^0)^G \to (\mathcal{I}^1)^G \to (\mathcal{I}^2)^G \to \cdots\} \tag{B.1.2}$$

in $\mathrm{D}^+(\mathrm{PAb}(\mathcal{C}))$, and $\Gamma(U, \mathbf{R}\pi_*^{\mathrm{pre}}(\mathscr{F}))$ is isomorphic to

$$\Gamma(U, (\mathcal{I}^\bullet)^G) = \{(\Gamma(U, \mathcal{I}^0))^G \to (\Gamma(U, \mathcal{I}^1))^G \to (\Gamma(U, \mathcal{I}^2))^G \to \cdots\} \tag{B.1.3}$$

in $\mathrm{D}^+(\mathrm{PAb}(\mathcal{C}))$. Furthermore $\Gamma(U, \mathcal{I}^i) \simeq (i_U)^*\mathcal{I}^i$ is an injective G-module for all i by Lemma B.2, thus we have an isomorphism

$$h^i(\Gamma(U, (\mathcal{I}^\bullet)^G)) \simeq \mathrm{H}^i(G, \Gamma(U, \mathscr{F}))$$

of abelian groups. $\square$

Lemma B.2. *Let $\mathcal{C}$ be a category, let $U \in \mathcal{C}$ be an object, let $\mathcal{A}_{\mathcal{C},U}$ denote the full subcategory of $\mathcal{C}$ containing exactly U, and let $i_U : \mathcal{A}_{\mathcal{C},U} \to \mathcal{C}$ denote the inclusion. The inverse image functor $(i_U)^* : \mathrm{PAb}(\mathcal{C}) \to \mathrm{PAb}(\mathcal{A}_{\mathcal{C},U})$ preserves injectives.*

Proof. The functor $(i_U)^* : \mathrm{PAb}(\mathrm{P}G_{\mathcal{C}}) \to \mathrm{PAb}(\mathcal{A}_{\mathcal{C},U})$ has an exact left adjoint, namely the "extension by zero" functor $i_{U,!} : \mathrm{PAb}(\mathcal{A}_{\mathcal{C},U}) \to \mathrm{PAb}(\mathrm{P}G_{\mathcal{C}})$ which sends $M \in \mathrm{PAb}(\mathcal{A}_{\mathcal{C},U})$ to the abelian presheaf $i_{U,\dagger}(M)$ where $\Gamma(V, i_{U,\dagger}(M)) = M$ if $V = U$ and 0 otherwise (with the only nontrivial restriction morphisms being those corresponding to the endomorphisms of U). $\square$

Appendix C. Computation using Magma

We compute $\mathrm{H}^1(\mathrm{GL}_2(\mathbb{Z}/(3)), M)$ in 5.4 using Magma [BCP97]. Here `G` is defined as the subgroup of $\mathrm{GL}_2(\mathbb{Z}/(3))$ generated by the matrices in (5.3.6), but the specified matrices constitute a generating set so in fact `G` $= \mathrm{GL}_2(\mathbb{Z}/(3))$. The group `G` acts on the abelian group $M = (\mathbb{Z}/(2))^{\oplus 6}$ by the three specified elements of $\mathrm{Mat}_{6\times 6}(\mathbb{Z})$, where each $\mathbf{x} \in M$ is viewed as a horizontal vector and each 6×6 matrix $\mathbf{A}$ acts on M by right multiplication $\mathbf{x} \mapsto \mathbf{x} \cdot \mathbf{A}$. The last line computes $\mathrm{H}^1(G, (\mathbb{Z}/(2))^{\oplus 6})$.

```
G := MatrixGroup< 2 , FiniteField(3) |
  [ 1,0 , -1,1 ] , [ 0,-1 , 1,0] , [ 1,0 , 0,-1 ]
>;
mats := [
  Matrix(Integers() , 6 , 6 , [
    0, 0, 1, 0, 0, 0 ,
    1, 0, 0, 0, 0, 0 ,
    0, 1, 0, 0, 0, 0 ,
    0, 0, 0, 0, 1, 0 ,
    0, 0, 0, 0, 0, 1 ,
    0, 0, 0, 1, 0, 0 ]) ,
  Matrix(Integers() , 6 , 6 , [
    1, 0, 0, 0, 0, 0 ,
    1, 0, 1, 0, 0, 0 ,
    1, 1, 0, 0, 0, 0 ,
    0, 0, 0, 1, 0, 0 ,
    0, 0, 0, 1, 0, 1 ,
    0, 0, 0, 1, 1, 0 ]) ,
  Matrix(Integers() , 6 , 6 , [
    0, 0, 0, 1, 0, 0 ,
    0, 0, 0, 0, 1, 0 ,
    0, 0, 0, 0, 0, 1 ,
    1, 0, 0, 0, 0, 0 ,
    0, 1, 0, 0, 0, 0 ,
    0, 0, 1, 0, 0, 0 ])
];
CM := CohomologyModule(G,[2,2,2,2,2,2],mats);
CohomologyGroup(CM,1);
```

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