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Scalable Robust Adaptive Control from the System Level Perspective

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***Abstract*— We will present a new general framework for robust and adaptive control that allows for distributed and scalable learning and control of large systems of interconnected linear subsystems. The control method is demonstrated for a linear time-invariant system with bounded parameter uncertainties, disturbances and noise. The presented scheme continuously collects measurements to reduce the uncertainty about the system parameters and adapts dynamic robust controllers online in a stable and performance-improving way. A key enabler for our approach is choosing a time-varying dynamic controller implementation, inspired by recent work on *System Level Synthesis* [1]. We leverage a new robustness result for this implementation to propose a general robust adaptive control algorithm. In particular, the algorithm allows us to impose communication and delay constraints on the controller implementation and is formulated as a sequence of robust optimization problems that can be solved in a distributed manner. The proposed control methodology performs particularly well when the interconnection between systems is sparse and the dynamics of local regions of subsystems depend only on a small number of parameters. As we will show on a five-dimensional exemplary chain-system, the algorithm can utilize system structure to efficiently learn and control the entire system while respecting communication and implementation constraints. Moreover, although current theoretical results require the assumption of small initial uncertainties to guarantee robustness, we will present simulations that show good closed-loop performance even in the case of large uncertainties, which suggests that this assumption is not critical for the presented technique and future work will focus on providing less conservative guarantees.**

I. INTRODUCTION

With the recent explosion of available computational resources and progress in the field of learning and estimation theory, there has been a resurging interest in robust adaptive control in the control and also machine learning community. In contrast to traditional work in [2] and [3], recent work has focused on analysis and development of adaptive control algorithms that merge learning and statistical theory techniques [4], [5], [6]. Although adaptive control algorithms are very useful for many systems of large-scale like communication networks, traffic networks or the power grid, there has not been a general theory of how to address the challenges in that setting. One of the major difficulties with deploying scalable adaptive algorithms in systems of that scale is, that the controller has to respect real-world implementation and communication constraints. Even in the non-adaptive case, incorporating these constraints into the control design is a challenging problem. Nevertheless, recent progress has been made by taking a new *System Level* approach [1], [7], [8], that allows to incorporate such constraints into optimal control problems in a tractable way. Aside from that, recent work [9], [4] has shown that the ideas in [8] can be used to provide robustness results that help to combine learning and control techniques with stability guarantees.

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In this work, we will leverage the system level approach to formulate a new general framework for robust adaptive control in large-scale systems. In particular, we will study the problem for linear systems with bounded uncertainty and disturbances. An appeal of this problem formulation is that in contrast to probabilistic guarantees as formulated in the results of [5], [6], we are able to provide worst-case safety guarantees that apply even in the presence of adversarial disturbances and small model non-linearities. Overall, the contribution of this paper is two-fold: We will derive robustness criteria similar to [8] for time-varying systems and controllers that provide a new general way to design stable adaptation in controllers. Secondly, we utilize these results to develop a robust and adaptive control scheme that can respect imposed communication and implementation constraints on the controller and allows for a distributed scalable implementation in large scale systems. Although our current stability proof is formulated for small initial uncertainties, in simulation we will show that the resulting control algorithm performs well even when the initial parameter uncertainties are large and the open loop system is unstable.

Due to lack of space, technical details and derivations are presented in an extended version of this paper found online.

II. A MOTIVATIONAL EXAMPLE: A 5-LINK CHAIN SYSTEM

We will begin by introducing the example which we use for our simulation results, to motivate the problem statement and the techniques presented in this work.

Consider the problem of controlling the following 5-link chain-system (1) with the state $x_t \in \mathbb{R}^5$, input $u_t \in \mathbb{R}^2$, disturbance $w_t \in \mathbb{R}^5$ and full state measurement $y_k = x_k$:

$$\begin{aligned} x_t &= Ax_{t-1} + Bu_{t-1} + w_{t-1} \\ y_t &= x_t \end{aligned} \tag{1}$$

Furthermore, assume that we do not have exact knowledge of A and B, but rather we do know that the system matrices

A and B are structured as

$$A = \begin{bmatrix} \alpha_2 & \alpha_3 & 0 & 0 & 0 \\ \alpha_1 & \alpha_2 & \alpha_3 & 0 & 0 \\ 0 & \alpha_1 & \alpha_2 & \alpha_3 & 0 \\ 0 & 0 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & 0 & 0 & \alpha_1 & \alpha_2 \end{bmatrix} B = \begin{bmatrix} \alpha_4 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \alpha_5 \end{bmatrix} \quad (2)$$

and that the parameters α lie within the bounds $0 \leq \alpha_2 \leq 1$, $0.1 \leq \alpha_1, \alpha_3 \leq 0.5$, $0.2 \leq \alpha_4 \leq 1$ and $-1 \leq \alpha_5 \leq -0.2$. In addition, assume that we know that the disturbance is bounded as $\|w\|_\infty \leq 0.5$.

We are interested in the problem of stable learning and control of this system under communication and computation constraints on the controller implementation. In particular, we will assume that (1) models a system of five interconnected, but otherwise separately acting scalar subsystems with state x_t^i and input u_t^i (where $u_t^i = 0$ for $i = 2, 3, 4$), where system i and j can communicate with eachother with a delay of $|i - j|$ time steps and each have limited computational power. Although this example is of small size, this problem setup captures the main difficulties that come with solving this type of robust adaptive control problem for large-scale systems, which will be the focus of the remainder of this paper.

III. Problem Statement

Consider a linear system of N subsystems with states, inputs and disturbances x^j, u^j, w^j that are interconnected w.r.t. the directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ i.e. $\mathcal{V} = \{1, \dots, N\}$ and $(j, i) \in \mathcal{E}$ implies that x_t^i influences x_{t+1}^j. Furthermore, define $\mathcal{N}(j)$ to be the set of subsystems that affect the subsystem j in the next time step i.e. $\mathcal{N}(j) = \{i \,|(j, i) \in \mathcal{E}\}$. The dynamics of the entire system can be written in the form

$$x_{t+1}^j = \sum_{i \in \mathcal{N}(j)} A^{j \leftarrow i} x_t^i + B^j u_t^j + w_t^j. \quad (3)$$

and we will refer to $x_t = \left[x_t^1, x_t^2, \dots, x_t^N\right]^T$ and $u_t = \left[u_t^1, u_t^2, \dots, u_t^N\right]^T$ as the global state and input of the system and accordingly, we will refer to A and B as the global system matrices, which are the corresponding compositions of the matrices $A^{j \leftarrow i}$ and B^j.

Remark 1. We allow for loops in the graph $\mathcal{G}$, which implies $j \in \mathcal{N}(j)$.

Similar to our introductory example in (Sec.II), we will assume that the matrices $A^{j \leftarrow i}$ and B^j are structured and have a low-dimensional representation of the form (4) w.r.t. some uncertain parameters $\alpha \in \mathbb{R}^p$ and known constant matrices $\mathcal{A}_s^{v \leftarrow u}$, $\mathcal{B}_s^u$.

$$A^{v \leftarrow u} = \sum_{s=1}^{p} \alpha_s \mathcal{A}_s^{v \leftarrow u} \qquad B^u = \sum_{s=1}^{p} \alpha_s \mathcal{B}_s^u \quad (4)$$

Furthermore, assume we are given the following information about the parameter α and the disturbance w^j in each subsystem:

$$\alpha \in \mathcal{P}_0 \qquad \left\|w_t^j\right\| \leq \eta \quad \forall t \geq 0 \quad (5)$$

Moreover we can setup the problem with any norm $\|.\|$, but for technical reasons, we will make the following assumption:

Assumption 1. The unit ball $\{x \,|\|x\| \leq 1\}$ of the norm is a polytope.

Remark 2. Common examples that satisfy (Ass.1) are $\|.\|_1$ and $\|.\|_\infty$.

A. Main Goal

Our objective will be to design causal controllers $u_k^j(x_k, x_{k-1}, \dots, x_0)$ that stabilize the global system despite the model uncertainties and allow for a scalable controller implementation when the total number of subsystems is very large. To make the second requirement more precise, we will break it down into the following three constraints:

Constraint 1 (Communication)**.** Every subsystem j can communicate with another subsystem i with a delay of $d^{j \leftarrow i}$ time-steps.

Constraint 2 (Localized Communication)**.** Every subsystem i only sends information to a local region of subsystems $\mathcal{S}(i)$.

Corresponding to (Const.2), let's define $\mathcal{R}(i)$ to be the set of subsystems from which subsystem i receives information:

Definition III.1. $\mathcal{R}(i) := \{j \,| i \in \mathcal{S}(j)\}$

Constraint 3 (Limited Computation)**.** Every subsystem j has limited computational resources.

IV. Outline of the Approach

We will briefly motivate our chosen control architecture and provide an overview of the results.

A. Ansatz: Time-Varying Controllers in SLS Implementation

The recent SLS approach [1] shows that any linear state feedback controller can be equivalently implemented in the form

$$u_t = \sum_{k=0}^{\infty} M(k+1)\hat{\delta}_{t-k} \quad \hat{\delta}_t = x_t - \sum_{k=1}^{\infty} R(k+1)\hat{\delta}_{t-k}$$

Inspired by this, we will make the ansatz (6),(7) for our control implementation,

$$u_t = \sum_{k=0}^{T-1} M_t(k+1)\hat{\delta}_{t-k} \quad (6)$$

$$\hat{\delta}_t = y_t - \sum_{k=1}^{T-1} R_t(k+1)\hat{\delta}_{t-k}. \quad (7)$$

where $M_t(i) \in \mathbb{R}^{p \times n}$, $R_t(i) \in \mathbb{R}^{n \times n}$, $\forall 1 \leq i \leq T$, $R(1) = I_n$ and we will refer to this as the *SLS implementation*. Our robust adaptive control scheme will propose algorithms that continuously use state observations to update the R_t and

M_t matrices in a stable and performance improving manner. More specifically, this happens in two steps: In the next section we will show how to use the system equation to continuously infer polytopes of feasible parameters α. Then, as later discussed in (Sec.VI) the polytopes are used to stably adapt the matrices R_t and M_t by utilizing a new robustness result.

V. Reducing Uncertainty through Polytopes of Consistent Parameters

Recall from (4), that $A^{v\leftarrow u}$ and B^u are structured as

$$A^{v\leftarrow u} = \sum_{s=1}^{p} \alpha_s \mathcal{A}_s^{v\leftarrow u} \qquad B^u = \sum_{s=1}^{p} \alpha_s \mathcal{B}_s^u \tag{8}$$

with $\alpha \in \mathcal{P}_0$. By plugging this form into the system equation (3) and recalling the constraint $\|w^j\| \leq \eta$, we see that for each pair of observations x_k, x_{k-1} and control action u_{k-1}, the true α has to be consistent with the inequality

$$\left\| x_k^j - \sum_{s=1}^{p} \alpha_s \hat{y}_{s,k-1}^j \right\| \leq \eta \tag{9}$$

$$\hat{y}_{s,k-1}^j = \sum_{i\in\mathcal{N}(j)} \mathcal{A}_s^{j\leftarrow i} x_{t-1}^i + \mathcal{B}_s^j u_{t-1}^j \tag{10}$$

Due to our assumption (Ass.1) on the norm $\|.\|$, condition (9) poses a polyhedral constraint on the system parameters α. We will define these inferred constraints from observations made in subsystem j at time t as $\mathcal{C}_t^j$:

$$\mathcal{C}_t^j = \left\{ \alpha \left| \left\| x_k^j - \sum_{s=1}^{p} \alpha_s \hat{y}_{s,k-1}^j \right\| \leq \eta \right. \right\} \tag{11}$$

By intersecting all constraints of the form (11), we can define $\mathcal{P}_t$ as the polytope of parameters consistent with the observations until time t:

$$\begin{aligned} &\mathcal{P}_t \\ &= \left\{ \alpha \in \mathcal{P}_0 | \, \forall j, \forall k \leq t : \left\| x_k^j - \sum_{s=1}^{p} \alpha_s \hat{y}_{s,k-1}^j \right\| \leq \eta \right\} && (12) \\ &= \mathcal{P}_0 \cap \bigcap_{j=1}^{N} \left(\mathcal{C}_1^j \cap \mathcal{C}_2^j \cap \cdots \cap \mathcal{C}_t^j \right) && (13) \end{aligned}$$

Correspondingly, define $\mathcal{M}_A^{j\leftarrow i}(\mathcal{P}_t)$ and $\mathcal{M}_B^j(\mathcal{P}_t)$ to be the set of consistent system matrices $A^{j\leftarrow i}$ and B^j at time t:

$$\mathcal{M}_A^{j\leftarrow i}(\mathcal{P}_t) = \left\{ \sum_{s=1}^{p} \alpha_s \mathcal{A}_s^{j\leftarrow i} \middle| \alpha \in \mathcal{P}_t \right\} \tag{14}$$

$$\mathcal{M}_B^j(\mathcal{P}_t) = \left\{ \sum_{s=1}^{p} \alpha_s \mathcal{B}_s^j \middle| \alpha \in \mathcal{P}_t \right\} \tag{15}$$

Furthermore, allowing every subsystem to share their observed constraints while respecting (Const.1) and (Const.2), we can define $\mathcal{P}_t^j$ as the polytope of consistent parameters for subsystem j at time t as:

$$\mathcal{P}_t^j = \mathcal{P}_{t-1}^j \bigcap_{i\in\mathcal{R}(j)} \hat{\mathcal{C}}_t^{j\leftarrow i} \qquad \hat{\mathcal{C}}_t^{j\leftarrow i} = \mathcal{C}_{t-d_{j\leftarrow i}}^i \tag{16}$$

where $\hat{\mathcal{C}}_t^{j\leftarrow i}$ denotes the constraints that j has obtained from system i at time t and $\mathcal{R}(j)$ is defined in (Def.III.1).

VI. A Scheme for Robust and Adaptive Control with SLS Implementations

We will combine the findings in (Sec.V) and a robustness result to propose a robust adaptive control scheme.

Recall the system dynamics from (Sec.III), with the structured uncertainties described by equations (4), (5) and our ansatz for the system-wide controller (6) and (7).

$$x_{t+1}^j = \sum_{i\in\mathcal{N}(j)} A^{j\leftarrow i} x_t^i + B^j u_t^j + w_t^j. \tag{17}$$

By enforcing additional sparsity constraints on R_t and M_t, we can represent the SLS implementation for subsystem j in the decomposed form:

$$u_t^j = \sum_i \sum_{k=0}^{T-1} \hat{M}_t^{j\leftarrow i}(k+1) \hat{\delta}_{t-k}^i \tag{18}$$

$$\hat{\delta}_t^j = x_t^j + v_t^j - \sum_i \sum_{k=1}^{T-1} \hat{R}_t^{j\leftarrow i}(k+1) \hat{\delta}_{t-k}^i \tag{19}$$

with $\hat{R}_t^{i\leftarrow i}(1) = I$ and $\hat{R}_t^{j\leftarrow i}(1) = 0$ for $i \neq j$. Furthermore, to allow for scalable implementation, we will enforce the following additional design constraints:

Constraint 4 (Distributed Computation)**.** $\hat{M}_t^{j\leftarrow i} := M_{t-d_{j\leftarrow i}}^{j\leftarrow i}$, $\hat{R}_t^{j\leftarrow i} := R_{t-d_{j\leftarrow i}}^{j\leftarrow i}$ where $M_t^{j\leftarrow i}$, $R_t^{j\leftarrow i}$ and $\hat{\delta}_t^i$ are computed locally in subsystem i and are broadcasted to the corresponding subsystem j with a delay of $d_{j\leftarrow i}$.

Constraint 5 (Localization and Communication Constraints)**.** For every subsystem i define a local region $\mathcal{L}(i) \subset \mathcal{S}(i)$ and enforce the constraints

$$\forall j \notin \mathcal{L}(i), k: \quad M_t^{j\leftarrow i}(k) = 0 \qquad R_t^{j\leftarrow i}(k) = 0, \tag{20}$$

$$\forall k < d_{j\leftarrow i}: \quad M_t^{j\leftarrow i}(k) = 0. \quad R_t^{j\leftarrow i}(k+1) = 0 \tag{21}$$

Under (Const.4) and (Const.5), the implementation (18), (19) can be verified to satisfy our previously discussed implementation constraints (Const.1) and (Const.2) for this problem setting.

With the abbreviations $\Delta_{k,t}^{j\leftarrow i}$,$\hat{w}_t^j$ and the function Δ_k^j

$$\Delta_{k,t}^{j\leftarrow i} := \Delta_k^j \left(A, B, R_t^{j\leftarrow i}, M_t^{j\leftarrow i} \right) := \ldots \tag{24}$$

$$\ldots R_t^{j\leftarrow i}(k+1) - \sum_{n\in\mathcal{N}(j)} A^{j\leftarrow n} R_t^{n\leftarrow i}(k) - B^j M_t^{j\leftarrow i}(k)$$

$$\hat{w}_t^j := v_t^j - \sum_i A^{j\leftarrow i} v_{t-1}^i + w_{t-1}^j. \tag{25}$$

and the following assumption, we can state the robustness result (Thm.VI.1):

Assumption 2. The communication speed between subsystems is faster than the propagation speed of disturbances, i.e.: $d_{j\leftarrow i} < 1 + d_{k\leftarrow i}, \quad \forall k \in \mathcal{N}(j)$

Remark 3. This is a common assumption in the distributed control community and known to not be very restrictive.

Theorem VI.1. Given (Ass.2), some λ, ρ and the local parameters m_1^i and m_2^i, assume that for all i, $R_t^{j\leftarrow i}$ and $M_t^{j\leftarrow i}$ satisfy the conditions (23), (22) and

$$\left\| \sum_{j\in\mathcal{L}(i)} \Delta_k^j(A,B,R_t^{j\leftarrow i},M_t^{j\leftarrow i}) \right\| \leq c_i\rho^{k-1} \tag{26}$$

$$\lambda_i = \sum_{k=1}^{T} c_i\rho^{k-1} \leq \lambda \tag{27}$$

for all times t, where $\bar{d}_i = \max_j d_{j\leftarrow i}$. Then the following inequality holds

$$\sum_j \left\|\hat{\delta}_t^j\right\| \leq \lambda \max_{1\leq k\leq T} \sum_j \left\|\hat{\delta}_{t-k}^j\right\| + \sum_{i=1}^{N}\left(m_1^i + \bar{d}_i m_2^i + \hat{w}_t^i\right).$$

Furthermore, the effective disturbance $\left\|\hat{\delta}_t^j\right\|$ is bounded as

$$\sum_j \left\|\hat{\delta}_t^j\right\| \leq \gamma_t \tag{28}$$

where γ_t is computed as

$$\gamma_t = \left(\sqrt[T]{\lambda}\right)^t \|x_0\| + \frac{1-\lambda^t}{1-\lambda}\left(\hat{\eta} + m_a\right) \qquad \text{if } \lambda < 1 \tag{29}$$

$$\gamma_t = \lambda^t \|x_0\| + \frac{1-\lambda^t}{1-\lambda}\left(\hat{\eta} + m_a\right) \qquad \text{if } \lambda \geq 1. \tag{30}$$

with

$$\sum_{i=1}^{N}\left(m_1^i + \bar{d}_i m_2^i + \hat{w}_t^i\right) \leq \hat{\eta} \tag{31}$$

and x_t, u_t are bounded if $\lambda < 1$

The value λ will be called a *robustness margin* of the closed loop if the conditions in (Thm.VI.1) are satisfied for it. We can derive from (Thm.VI.1) the distributed localized robust adaptive control (DLAR) algorithm (Algo.1). Every subsystem is constructing consistent polytopes $\mathcal{P}_t^j$ for their local parameters (Line 8-11) and optimizing for robust R_t^j and M_t^j that can satisfy the conditions of (Thm.VI.1). Moreover the algorithm finds such robust controllers in two steps. In each iteration, it first searches for R_t^j and M_t^j that achieve the smallest robustness margin λ_t^i (Line 12) and only if we find feasible controllers that guarantee a minimum desired level of robustness λ^*, the algorithm re-solves the optimization problem (32) in (Line 14) w.r.t. to a desired performance objective (35). The motivation behind this two-step procedure is clear: Optimizing for a performance objective is only reasonable if robust stability of the closed loop is possible. After the local controllers have been computed, the control actions and local constraints are broadcasted to the local region of subsystems (Line 18-19).

$$\min_{R_t^i, M_t^i, c_t^i, \lambda_t^i} \quad f(R_t^i, M_t^i, \lambda_t^i) \tag{32}$$

$$\text{s.t.} \forall A^{q\leftarrow p} \in \mathcal{M}_A^{q\leftarrow p}(\mathcal{E}\left(\mathcal{P}_t^i\right)), B^p \in \mathcal{M}_B^p(\mathcal{E}\left(\mathcal{P}_t^i\right)):$$
$$\text{holds } (26), (27), (22), (23), (21), (20)$$
$$R_t^{i\leftarrow i}(1) = I \text{ and } R_t^{j\leftarrow i}(1) = 0 \text{ for } i \neq j \tag{33}$$

$$f_\lambda(R, M, \lambda) = \lambda \tag{34}$$

$$f_{C^i,D^i}(R^i, M^i, \lambda) = \sum_{k=1}^{T}\left\|C^i R^i(k) + D^i M^i(k)\right\| \tag{35}$$

Algorithm 1: A DLAR Control Scheme with SLS

Input: $\mathcal{P}_0^i$, C^i, D^i, m_1, m_2, ρ, T, $\mathcal{A}_s^{j\leftarrow i}$, $\mathcal{B}_s^i$, λ^*

```
1  for subsystem j = 1 : N do
2  |   R_0^j, M_0^j ← solve (32) with P_0^j
3  |   δ̂_0^j ← x_0^j
4  |   apply u_0^j ← M_0^j(1) δ̂_0^j
5  end
6  for t = 1,2,... do
7  |   for subsystem i = 1 : N do
8  |   |   Ĉ_t^{i←j} ← C^j_{t-d_{i←j}} (16)
9  |   |   receive R̂_t^{i←j}, M̂_t^{i←j} ← R^{i←j}_{t-d_{i←j}}, M^{i←j}_{t-d_{i←j}}
10 |   |   compute C_t^i from (11)
11 |   |   update P_t^i ← P_{t-1}^i ∩ ⋂_{j∈R(i)} Ĉ_t^{i←j} (16)
12 |   |   λ_t^i, R_t^i, M_t^i ← solve (32) with f_λ (34)
13 |   |   if λ_t^i ≤ λ* then
14 |   |   |   R_t^i, M_t^i ← solve (32) with f_{C^i,D^i} (35)
   |   |   |     such that λ_t^i ≤ λ*
15 |   |   end
16 |   |   compute δ̂_t^i, u_t^i ← (19), (18)
17 |   |   apply u_t^i
18 |   |   broadcast R_t^{n←i}, M_t^{n←i} to all n ∈ L(i)
19 |   |   broadcast constraints C_t^i to all n ∈ S(i)
20 |   end
21 end
```

Moreover, we obtain the robustness results for the closed loop:

$$\forall 0 \leq h \leq \bar{d}_i - 1: \quad \sum_{j:d_{j\leftarrow i}\geq h+1} \left\| \sum_{k=d_{j\leftarrow i}+1}^{T} \Delta_k^j\left(A, B, R_t^{j\leftarrow i} - R_{t+h-d_{j\leftarrow i}}^{j\leftarrow i}, M_t^{j\leftarrow i} - M_{t+h-d_{j\leftarrow i}}^{j\leftarrow i}\right)\hat{\delta}_{t+h+1-k}^i \right\| \leq m_1^i \tag{22}$$

$$\forall 0 \leq h \leq \bar{d}_i - 1: \quad \sum_{j:d_{j\leftarrow i}=h+1} \left\| \sum_{k=h+2}^{T-1}\left(R_t^{j\leftarrow i} - R_{t-1}^{j\leftarrow i}\right)(k+1)\hat{\delta}_{t+h+1-k}^i \right\| \leq m_2^i \tag{23}$$

Proposition 1. $\lambda_t^i \leq \max\left\{\lambda_{t-1}^i, \lambda^*\right\}$

Proposition 2. $\max_i \lambda_{t'}^i$ is a robustness margin for the closed loop system for all $t \geq t'$ and the corresponding bounds of (Thm.VI.1) apply.

Proposition 3. If $\max_i \lambda_0^i < 1$ from (Line 2) of (Algo.1), then the closed loop system is stable for all time.

In addition, (21) in (Const.5) constrains the number of decision variables to the size of $\mathcal{L}(i)$ and with the corollary (Coro.VI.1),

Corollary VI.1 (Local Models)**.** The conditions (22), (23), (26) and (27) w.r.t. the subsytem i only depend on parameters $A^{v \leftarrow u}$ and B^u for which $u \in \mathcal{L}(i)$.

we can see that the complexity of the subproblem that every subsystems solves, grows only with the size of local regions $\mathcal{R}(i)$ and $\mathcal{L}(i)$. This addresses the implementation constraint (Const.3) and shows that (Algo.1) allows for a scalable implementation even for large number N of subsystems.

VII. Simulation

The algorithm (Algo.1) is applied to the exemplary control problem with implementation constraints discussed in (Sec.II). For this simulation we picked the true parameters $\alpha_1 = 0.3$, $\alpha_2 = 0.6$, $\alpha_3 = 0.2$ which produce an unstable open loop system ($\max_i |\lambda_i(A)| = 1.05$). (Fig.1) and (Fig.3) show simulation results of the presented adaptive controller scheme with $\rho = 0.7$, $T = 8$, $x_0 = [0, 3, 3, 3, 0]^T$, with respect to two different cases of initial available information. In (Fig.1) the controller has only knowledge of the initial entry-wise bounds on α described in (Sec.II), while in (Fig.3) the controller starts off with perfect knowledge of α. Furthermore, for the adaptive case, (Fig.2) summarizes the effective disturbances $\hat{\delta}_t^j$, the environment disturbances w_t^j and individually computed margins λ_t^j for every subsystem. In addition, the quantity μ_t in (Fig.2) computes the true robustness margin $\mu_t = \sum_k \|\Delta(k)\|_1$ of the closed loop adaptive controller w.r.t. to true plant[1]. Although the initial uncertainty provides a large robustness margin ($\max_i \lambda_0^i = 4$), the plot of μ_t in (Fig.2) shows that the controller learns enough by time-step $t = 20$ to render the closed loop stable.

Moreover, even though the controller in (Fig.3) has perfect knowledge of the parameters, its computed robustness margin is $\sum_k \|\Delta(k)\|_1 = 0.33$, which tells us that even in presence of full system knowledge, the communication constraints only allow for approximate localization of the disturbances. Putting this in relation to the simulation results in (Fig.1) shows us that the adaptive controller is performing quite well despite large initial uncertainty ($\lambda_0 >> 1$), communication/localization constraints and decentralized implementation. Although this observation is empirical at this point it shows that (Algo.1) is a promising approach even in the case of large parameter uncertainties.

[1]Note, that this information is not available to the controllers and is only displayed to show that the controller achieves robust stability.

VIII. Conclusion and Future Work

In this work, we derived a novel framework for adaptive and robust control of linear time-invariant systems. Using the new SLS framework [1] we derived time-varying robustness results which can be used as a new way to design stable adaptations in control systems. With this result we develop a robust adaptive control scheme for linear systems with state feedback under bounded parameter uncertainties, disturbances and noise. The resulting control system continuously infers polytopes of parameters that are consistent with the collected observations and use these sets to compute new robust controllers that improve control performance. In particular, inference of uncertainty sets is done efficiently, since structural properties of the system matrices are directly exploited and subsystems only need to model the dynamics in their local region. Moreover, we present how this approach can incorporate communication constraints and allows for a distributed and scalable control implementation. For the case of small initial uncertainties, a stability proof and worst-case bounds are provided for the closed loop. Finally, simulations with a chain-system empirically show that performance does not degrade too much even if we have large initial uncertainties in the parameters.

Future research will be focused on deriving performance bounds of this technique when dealing with a broader class of uncertainties and reducing the computational cost of the optimization procedures needed in the algorithm.

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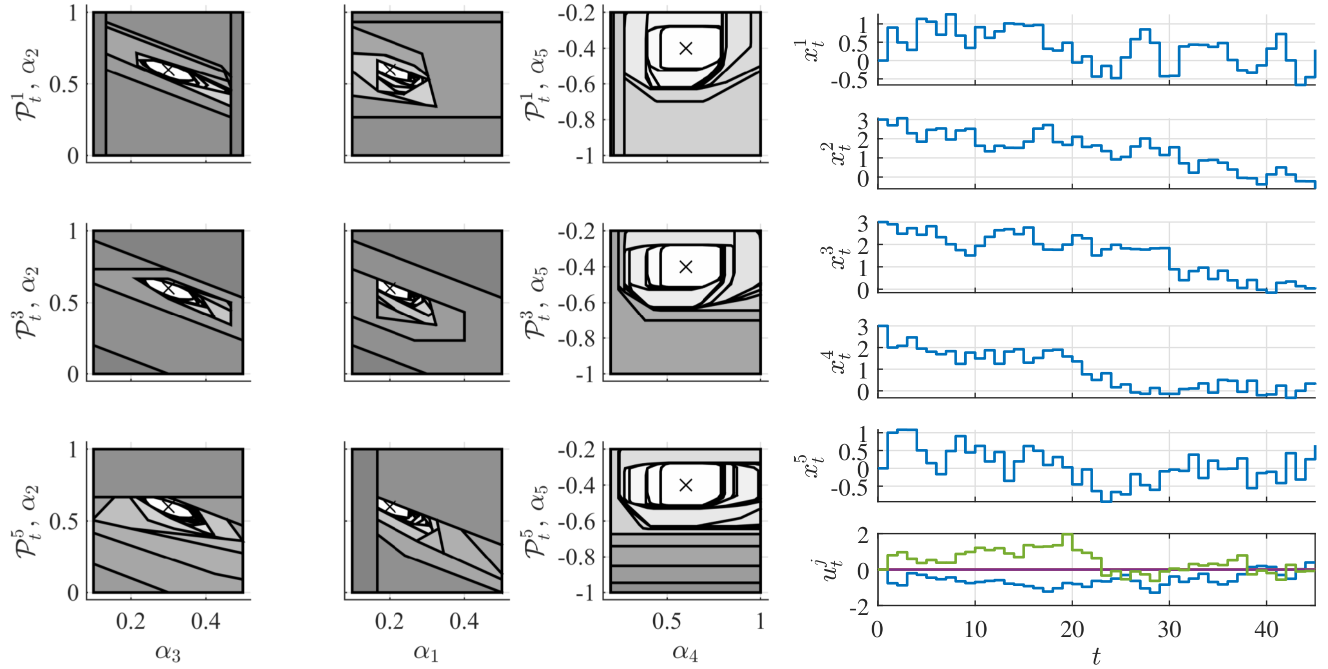
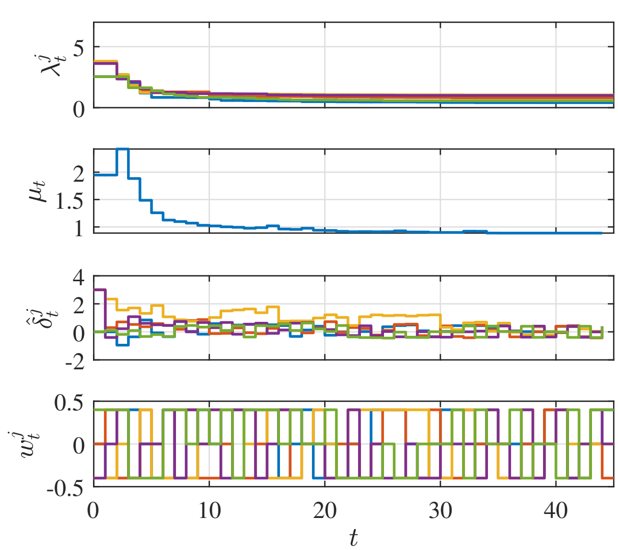


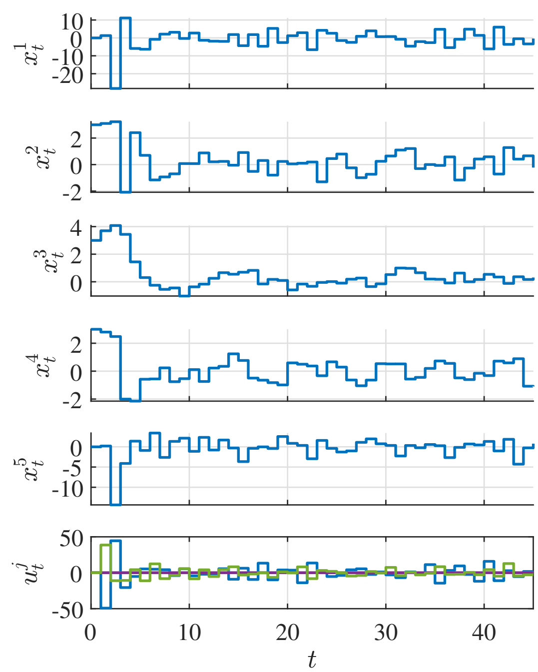
(a) (b)

Fig. 1: Left: Overlay of projections of $\mathcal{P}_t^j$ onto different coordinates for different time steps. Each row corresponds to a different subsystem x_1, x_3, x_5 (top to bottom). Shading indicates time of computation with shades lightening as simulation time passes. Right: state and input trajectories of closed loop simulation with (Algo.1) and uncertainties on α described in (Sec.II).



(a)

Fig. 2: Computed margins λ_t^j, effective disturbances $\hat{\delta}_t^j$ and disturbance w_t^j for every node x^j. $\mu_t = \sum_k \|\Delta_t(k)\|_1$ is computed with the real system and the controller at t.



(a)

Fig. 3: State and input trajectories for closed loop simulation of controller (Algo.1) with perfect parameter knowledge α.