Journal of Hyperbolic Differential Equations
Vol. 16, No. 4 (2019) 663–700

DOI: 10.1142/S021989161950022X

L^p time asymptotic decay for general hyperbolic–parabolic balance laws with applications

Yanni Zeng
Department of Mathematics
University of Alabama at Birmingham
1402 10th Avenue South
Birmingham, AL 35294-1241, USA
ynzeng@uab.edu

Received 15 June 2018
Accepted 30 October 2019
Published 14 February 2020

Communicated by Tai-Ping Liu

Abstract. We study the time asymptotic decay of solutions for a general system of hyperbolic–parabolic balance laws in one space dimension. The system has a physical viscosity matrix and a lower-order term for relaxation, damping or chemical reaction. The viscosity matrix and the Jacobian matrix of the lower-order term are rank deficient. For Cauchy problem around a constant equilibrium state, existence of solution global in time has been established recently under a set of reasonable assumptions. In this paper, we obtain optimal L^p decay rates for $p \geq 2$. Our result is general and applies to models such as Keller–Segel equations with logarithmic chemotactic sensitivity and logistic growth, and gas flows with translational and vibrational non-equilibrium. Our result also recovers or improves the existing results in literature on the special cases of hyperbolic–parabolic conservation laws and hyperbolic balance laws, respectively.

Keywords: Hyperbolic–parabolic; balance laws; time-decay rate; Keller–Segel model; chemotaxis; logistic growth; thermal non-equilibrium; Cauchy problem.

Mathematics Subject Classification 2010: 35L65, 35B25, 35K15, 35K59, 35L60

1. Introduction

Consider a general class of partial differential equations in the form

$$w_t + \sum_{j=1}^{m} f_j(w)_{x_j} = \sum_{j,k=1}^{m} [B_{jk}(w)w_{x_k}]_{x_j} + r(w), \quad m \geq 1, \tag{1.1}$$

where $w, f_j, r \in \mathbb{R}^n$ and $B_{jk} \in \mathbb{R}^{n\times n}$. The unknown function $w = w(x,t)$ depends on the space variable $x = (x_1, \ldots, x_m)^t \in \mathbb{R}^m$ and the time variable $t \in \mathbb{R}^+$. The equation describes a variety of phenomena from continuum mechanics, with w being physical densities such as mass density, momentum density, energy density, etc. As given functions of w, f_j are flux functions, and r represents external forces, relaxation, chemical reactions and so forth. The matrices B_{jk} are known as viscosity matrices and are functions of w. They describe viscosity, heat conduction, species diffusion, etc. Several examples of (1.1), such as Navier–Stokes equations for compressible flows and the system for polyatomic gas flows in translational and vibrational non-equilibrium, can be derived from the Boltzmann equation by Chapman–Enskog expansion. A common feature of these equations is that B_{jk} and r' (the Jacobian matrix of r) are rank deficient. We refer (1.1) as hyperbolic–parabolic balance laws, which describe the balance of physical quantities.

A special case of (1.1) is hyperbolic–parabolic conservation laws, where $r = 0$

$$w_t + \sum_{j=1}^{m} f_j(w)_{x_j} = \sum_{j,k=1}^{m} [B_{jk}(w)w_{x_k}]_{x_j}, \quad m \geq 1. \tag{1.2}$$

Among examples are Navier–Stokes equations and the full system of magneto-hydrodynamics. Another special case of (1.1) is hyperbolic balance laws, with $B_{jk} = 0$

$$w_t + \sum_{j=1}^{m} f_j(w)_{x_j} = r(w), \quad m \geq 1. \tag{1.3}$$

Important examples include Euler equations with damping and polyatomic gas flows in thermal non-equilibrium. For the most general form (1.1), with nontrivial B_{jk} and r, we have polyatomic gas flows in both translational and vibrational non-equilibrium as an important example. We also have Keller–Segel equations with logistic growth in chemotaxis as an interesting application.

For (1.1), we consider the Cauchy problem with initial condition:

$$w(x, 0) = w_0(x), \tag{1.4}$$

where w_0 is assumed to be a small perturbation of a constant equilibrium state $\bar{w}$, $r(\bar{w}) = 0$. The author has proposed a set of structural conditions for (1.1), which leads to the existence of solution for (1.1), (1.4) global in time if w_0 is near $\bar{w}$ [15]. The result applies to all space dimensions $m \geq 1$. The same set of structural conditions also give rise to the L^p $(p \geq 2)$ convergence rates of w to $\bar{w}$. The conclusion has been proved for multispace dimensions $(m \geq 2)$ in [16]. The purpose of this paper is to prove the parallel result on decay rates for one space dimension, $m = 1$. The approach for $m = 1$ needs to be different: The Green's function decays slower in one space dimension than in multispace dimensions. The approach for $m \geq 2$ no longer applies in several technical areas, where integrals with respect to time now become divergent. Thus for $m = 1$, we need to make use of details from spectral analysis, which on the other hand is not available in a straightforward manner for $m \geq 2$.

In this paper, we also discuss applications. We recover the known results for the special cases of hyperbolic–parabolic conservation laws and hyperbolic balance laws. We also apply the general results to the Keller–Segel chemotaxis model and to the polyatomic gas flows in both translational and vibrational non-equilibrium.

Now, we formulate the decay results in one space dimension. Letting $m = 1$ in (1.1) we have

$$w_t + f(w)_x = [B(w)w_x]_x + r(w), \quad x \in \mathbb{R}, \ t \in \mathbb{R}^+. \tag{1.5}$$

We state the structural conditions for (1.5), which are the one-dimensional version of the set proposed in [15]. Let $\mathbb{O}$ be a small neighborhood of a constant equilibrium state $\bar{w}$, and $\mathbb{E}$ be the equilibrium manifold in $\mathbb{O}$

$$\mathbb{E} = \{w \in \mathbb{O} \,|\, r(w) = 0\}. \tag{1.6}$$

Assume that $f(w)$, $B(w)$ and $r(w)$ are smooth in $\mathbb{O}$. In the following we use f' to denote the Jacobian matrix of f with respect to w, etc.

Assumption 1.1. (1) There exists a strictly convex entropy function η, which is a scalar function of w in $\mathbb{O}$, such that $\eta'' f'$ is symmetric in $\mathbb{O}$, $\eta'' B$ is symmetric, semi-positive definite in $\mathbb{O}$, and $\eta'' r'$ is symmetric, semi-negative definite on $\mathbb{E}$. Here, η'' is the Hessian of η with respect to w.

(2) Equation (1.5) has n_1 conservation laws. That is, there is a partition $n = n_1 + n_2$, $n_1, n_2 \geq 0$, such that

$$r(w) = \begin{pmatrix} 0_{n_1 \times 1} \\ r_2(w) \end{pmatrix}, \quad w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \tag{1.7}$$

with $w_1 \in \mathbb{R}^{n_1}$, $r_2, w_2 \in \mathbb{R}^{n_2}$, and $(r_2)_{w_2} \in \mathbb{R}^{n_2 \times n_2}$ is nonsingular (if $n_2 > 0$). Here, $(r_2)_{w_2}$ denotes the Jacobian matrix of r_2 with respect to w_2, etc.

(3) There is a diffeomorphism $\varphi \to w$ from an open set $\tilde{\mathbb{O}} \subset \mathbb{R}^n$ to $\mathbb{O}$ and a constant orthogonal matrix $P \in \mathbb{R}^{n \times n}$ such that

$$P^t B(w(\varphi)) w_\varphi(\varphi) P = \begin{pmatrix} 0_{n_3 \times n_3} & 0_{n_3 \times n_4} \\ 0_{n_4 \times n_3} & B^*(\varphi) \end{pmatrix}. \tag{1.8}$$

Here, $n_3, n_4 \geq 0$ are constant integers such that $n_3 + n_4 = n$, and $B^* \in \mathbb{R}^{n_4 \times n_4}$ is nonsingular (if $n_4 > 0$) for $\varphi \in \tilde{\mathbb{O}}$.

(4) [12] Let $\mathbb{N}_1$ be the null space of $B(\bar{w})$ and $\mathbb{N}_2$ be the null space of $r'(\bar{w})$. Then $\mathbb{N}_1 \cap \mathbb{N}_2$ contains no eigenvectors of $f'(\bar{w})$.

Remark 1.2. The partitions $n = n_1 + n_2$ and $n = n_3 + n_4$ in Assumption 1.1 are independent. Here, n_1 is the number of conservation laws in (1.5), including viscous and inviscid ones. On the other hand, n_3 is the number of equations without a viscosity term, or being "hyperbolic type", including both conservation laws and non-conservation laws. The locations of the conservation laws and rate equations are also independent to the locations of the "hyperbolic" equations and "parabolic" equations. Matrix P in condition (3) of Assumption 1.1 is usually a permutation to showcase such independence. In Sec. 6, we use Keller–Segel model with logistic growth and the system for polyatomic gas in translational and vibrational non-equilibrium to illustrate the two partitions by different choices of dissipation parameters.

We introduce the following notations to abbreviate the norms of Sobolev spaces with respect to x:

$$\|\cdot\|_s = \|\cdot\|_{H^s(\mathbb{R})}, \quad \|\cdot\| = \|\cdot\|_{L^2(\mathbb{R})}. \tag{1.9}$$

With φ and P given in condition (3) of Assumption 1.1, we define

$$\tilde{w} = \begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2 \end{pmatrix} \equiv P^t \varphi(w), \tag{1.10}$$

where $\tilde{w}_1 \in \mathbb{R}^{n_3}$ and $\tilde{w}_2 \in \mathbb{R}^{n_4}$. The following theorem is a special case for $m = 1$ of [15, Theorem 1.7].

Theorem 1.3 ([15]). *Let $\bar{w}$ be a constant equilibrium state, Assumption* 1.1 *be satisfied, $s \geq 2$ be an integer, and $w_0 - \bar{w} \in H^s(\mathbb{R})$. Then there exists a constant $\varepsilon > 0$ such that if $\|w_0 - \bar{w}\|_s \leq \varepsilon$, the Cauchy problem* (1.5), (1.4) *has a unique global solution w. The solution satisfies $w - \bar{w} \in C([0,\infty); H^s(\mathbb{R}))$, $D_x w \in L^2([0,\infty); H^{s-1}(\mathbb{R}))$, $D_x \tilde{w}_2(w) \in L^2([0,\infty); H^s(\mathbb{R}))$, $r(w) \in L^2([0,\infty); H^s(\mathbb{R}))$, and*

$$\begin{aligned} \sup_{t\geq 0} \|w - \bar{w}\|_s^2(t) + \int_0^\infty [\|D_x w\|_{s-1}^2 + \|D_x \tilde{w}_2(w)\|_s^2 + \|r_2(w)\|_s^2](t)dt \\ \leq C\|w_0 - \bar{w}\|_s^2, \end{aligned} \tag{1.11}$$

where $C > 0$ is a constant.

Our main result is the following L^2 decay estimates of w to $\bar{w}$.

Theorem 1.4. *Let $\bar{w}$ be a constant equilibrium state of* (1.5), *and Assumption* 1.1 *be true. Let $s \geq 4$ be an integer, and $w_0 - \bar{w} \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$. Then there exists a constant $\varepsilon > 0$ such that if $\delta_0 \equiv \|w_0 - \bar{w}\|_s + \|w_0 - \bar{w}\|_{L^1} \leq \varepsilon$, the solution of* (1.5), (1.4) *given in Theorem* 1.3 *has the following estimates for $t \geq 0$:*

$$\|D_x^l (w - \bar{w})\|(t) \leq C\delta_0 (t+1)^{-\frac{1}{4}-\frac{l}{2}}, \quad 0 \leq l \leq s-2, \tag{1.12}$$

$$\|D_x^l r_2(w)\|(t) \leq C\delta_0 (t+1)^{-\frac{3}{4}-\frac{l}{2}}, \quad 0 \leq l \leq s-4. \tag{1.13}$$

Here, $C > 0$ in (1.12) *and* (1.13) *is a constant.*

Recall Gagliardo–Nirenberg inequality [11]: There is a constant $C > 0$ such that for $g \in H^k(\mathbb{R})$,

$$\|D_x^l g\|_{L^p} \leq C\|D_x^k g\|^\theta \|g\|^{1-\theta}, \tag{1.14}$$

where $0 \leq l \leq k$, $p \in [2, \infty]$, and $\theta = \frac{l+1/2-1/p}{k} \leq 1$. Applying (1.14) to $g = w - \bar{w}$ with $k = s-2$, and to $g = r_2(w)$ with $k = s-4$, respectively, we have the following corollary of Theorem 1.4.

Corollary 1.5. *Under the assumptions of Theorem* 1.4, *the solution of* (1.5), (1.4) *has the following* L^p *estimates with* $p \geq 2$: *For* $t \geq 0$,

$$\|D_x^l(w - \bar{w})\|_{L^p}(t) \leq C\delta_0(t+1)^{-\frac{1}{2}+\frac{1}{2p}-\frac{l}{2}}, \quad 0 \leq l \leq s - \frac{5}{2} + \frac{1}{p}, \tag{1.15}$$

$$\|D_x^l r_2(w)\|_{L^p}(t) \leq C\delta_0(t+1)^{-1+\frac{1}{2p}-\frac{l}{2}}, \quad 0 \leq l \leq s - \frac{9}{2} + \frac{1}{p}. \tag{1.16}$$

Here, δ_0 *is as defined in Theorem* 1.4, *and* $C > 0$ *is a constant.*

Remark 1.6. For the special case of hyperbolic balance laws, i.e. when $B = 0$, we only need $s \geq 3$, and (1.15) is true for $0 \leq l \leq s - 3/2 + 1/p$ (rather than $0 \leq l \leq s - 5/2 + 1/p$). Similarly, (1.16) is true for $0 \leq l \leq s - 5/2 + 1/p$. This is due to the lack of second derivatives in the equation. See Sec. 5 for details.

Remark 1.7. Results for multispace dimensions parallel to Theorem 1.4 and Corollary 1.5 are given in [16]. One may further show that for $m \geq 2$, a time asymptotic solution to (1.1), (1.4) is the solution of the corresponding linear equation, linearized around the constant equilibrium state $\bar{w}$, with the same initial condition (1.4) [17]. However, such a conclusion is not true for one space dimension.

The plan of the paper is as follows. In Sec. 2, we give some preliminaries. In Sec. 3, we carry out the spectral analysis of the linear system, which leads to decay estimates of its solution. In Sec. 4, we perform weighted energy estimate. In Sec. 5, we prove Theorem 1.4. Finally, in Sec. 6, we discuss applications, to hyperbolic–parabolic conservation laws, to hyperbolic balance laws, to a chemotaxis model of Keller–Segel type with logistic growth, and to polyatomic gas flows in translational and vibrational non-equilibrium.

Throughout this paper, we use C to denote a universal positive constant. We also use the bar accent for the value of a variable taken at the constant equilibrium state $\bar{w}$, e.g. $\bar{\varphi} = \varphi(\bar{w})$, etc.

2. Preliminaries

In this section, we assume that condition (2) of Assumption 1.1 holds. As evidenced by (1.11), $r_2(w)$ represents the part of solution with faster decay rate, and can be used to isolate the leading term in the solution. For this, we introduce a new variable ψ using the notations in (1.7):

$$\psi = \psi(w) = \begin{pmatrix} w_1 \\ r_2(w) \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}(w), \tag{2.1}$$

where $\psi_1 = w_1 \in \mathbb{R}^{n_1}$ and $\psi_2 = r_2 \in \mathbb{R}^{n_2}$. Under condition (2) of Assumption 1.1, ψ is a diffeomorphism, with the Jacobian matrices

$$\psi_w = \begin{pmatrix} I_{n_1\times n_1} & 0_{n_1\times n_2} \\ (r_2)_{w_1} & (r_2)_{w_2} \end{pmatrix}, \quad w_\psi = \psi_w^{-1} = \begin{pmatrix} I_{n_1\times n_1} & 0_{n_1\times n_2} \\ -(r_2)_{w_2}^{-1}(r_2)_{w_1} & (r_2)_{w_2}^{-1} \end{pmatrix}. \tag{2.2}$$

Next, we linearize (1.5) around the constant equilibrium state $\bar{w}$ using the new variable ψ. Let

$$\tilde{\psi} = \psi - \bar{\psi} = \begin{pmatrix} w_1 - \bar{w}_1 \\ r_2(w) \end{pmatrix} \equiv \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix} \tag{2.3}$$

be the perturbation. Multiplying (1.5) from the left by ψ_w, we have

$$\tilde{\psi}_t + \psi_w f' w_\psi \tilde{\psi}_x = \psi_w (B w_\psi \tilde{\psi}_x)_x + \psi_w r. \tag{2.4}$$

Multiplying the equation by a constant matrix $\tilde{A}_0$ to be defined in (2.6), we write (2.4) or (1.5) as

$$\tilde{A}_0 \tilde{\psi}_t + \tilde{A} \tilde{\psi}_x = \tilde{B} \tilde{\psi}_{xx} + \tilde{L} \tilde{\psi} + \tilde{R}, \tag{2.5}$$

where

$$\begin{aligned} \tilde{A}_0 &= (w_\psi^t \eta'' w_\psi)(\bar{w}), \\ \tilde{A} &= (w_\psi^t \eta'' f' w_\psi)(\bar{w}), \\ \tilde{B} &= (w_\psi^t \eta'' B w_\psi)(\bar{w}), \\ \tilde{L} &= (w_\psi^t \eta'' r' w_\psi)(\bar{w}), \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} \tilde{R} &= R_1 + R_2 + R_3, \\ R_1 &= \tilde{A}_0 [(\psi_w f' w_\psi)(\bar{w}) - (\psi_w f' w_\psi)(w)] \psi_x, \\ R_2 &= \tilde{A}_0 \{\psi_w(w)[B(w) w_x]_x - (\psi_w B w_\psi)(\bar{w}) \psi_{xx}\}, \\ R_3 &= \tilde{A}_0 [(\psi_w r)(w) - (\psi_w r' w_\psi)(\bar{w}) \tilde{\psi}]. \end{aligned} \tag{2.7}$$

Without the nonlinear source $\tilde{R}$, (2.5) would be a linear system of $\tilde{\psi}$ with constant coefficients. It is straightforward to verify the following lemma for properties of those coefficient matrices.

Lemma 2.1. *Under conditions* (1), (2) *and* (4) *of Assumption* 1.1, *we have the following*:

(1) $\tilde{A}_0$, $\tilde{A}$, $\tilde{B}$ *and* $\tilde{L}$ *are real, symmetric.* $\tilde{A}_0$ *is positive definite,* $\tilde{B}$ *is semi-positive definite, and* $\tilde{L}$ *is semi-negative definite.*
(2) *If* $\zeta \in \mathbb{R}^n \backslash \{0\}$ *and* $\tilde{B}\zeta = \tilde{L}\zeta = 0$, *then* $\lambda \tilde{A}_0 \zeta + \tilde{A}\zeta \neq 0$ *for any* $\lambda \in \mathbb{R}$.

When (1) of Lemma 2.1 holds, there are several equivalent forms of (2), see [12, Theorem 1.1]. Among them there is the existence of a so-called compensating matrix. Here, we state the following lemma as a consequence of Lemma 2.1 applying [12, Theorem 1.1].

Lemma 2.2. *Let conditions* (1), (2), *and* (4) *of Assumption* 1.1 *be true. Then we have the following*:

(1) *There exists a matrix* $K \in \mathbb{R}^{n\times n}$, *called a compensating matrix, such that* $K\tilde{A}_0$ *is real, skew symmetric, and*

$$S = \frac{1}{2}(K\tilde{A} + \tilde{A}K^t) + \tilde{B} - \tilde{L} \tag{2.8}$$

is real, symmetric and positive definite.

(2) *Let* $\lambda(i\xi)$ *be the value of* λ *such that*

$$\lambda\tilde{A}_0\zeta + (i\xi\tilde{A} + \xi^2\tilde{B} - \tilde{L})\zeta = 0$$

has a nontrivial solution $\zeta \in \mathbb{R}^n$, *where* $\xi \in \mathbb{R}$. *Then there exists a constant* $c > 0$ *such that the real part of* $\lambda(i\xi)$ *has the estimate*

$$\Re\lambda(i\xi) \le -c\rho(|\xi|), \quad \xi \in \mathbb{R}, \tag{2.9}$$

where

$$\rho(r) = \frac{r^2}{1+r^2}, \quad r \ge 0.$$

To study a linear system with constant coefficients we use Fourier transform. We use the hat accent to denote the Fourier transform of a function in x:

$$\begin{aligned} \hat{\tilde{\psi}}(\xi,t) &= \int_{\mathbb{R}} \tilde{\psi}(x,t)e^{-ix\xi}\,dx, \\ \tilde{\psi}(x,t) &= \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\tilde{\psi}}(\xi,t)e^{ix\xi}\,d\xi. \end{aligned} \tag{2.10}$$

Taking Fourier transform of (2.5) we have

$$\tilde{A}_0\hat{\tilde{\psi}}_t + (i\xi\tilde{A} + \xi^2\tilde{B} - \tilde{L})\hat{\tilde{\psi}} = \hat{\tilde{R}}. \tag{2.11}$$

To simplify $\tilde{R}$ defined in (2.7) we write

$$R_1 = \tilde{A}_0\begin{pmatrix} R_{11} \\ R_{12} \end{pmatrix}, \quad R_2 = \tilde{A}_0\begin{pmatrix} R_{21} \\ R_{22} \end{pmatrix}, \quad R_3 = \tilde{A}_0\begin{pmatrix} R_{31} \\ R_{32} \end{pmatrix}, \tag{2.12}$$

where $R_{k1} \in \mathbb{R}^{n_1}$, and $R_{k2} \in \mathbb{R}^{n_2}$, $k = 1,2,3$. If we write

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad f_1 \in \mathbb{R}^{n_1}, \quad f_2 \in \mathbb{R}^{n_2},$$

then from (2.7), (2.2) and (2.3) we have

$$\begin{pmatrix} R_{11} \\ R_{12} \end{pmatrix} = [(\psi_w f' w_\psi)(\bar{w}) - (\psi_w f' w_\psi)(w)]\psi_x, \tag{2.13}$$

$$R_{11} = \tilde{f}_{1x}, \quad \tilde{f}_1 = -[f_1(w) - f_1(\bar{w}) - (f_1' w_\psi)(\bar{w})\tilde{\psi}] = O(1)|\tilde{\psi}|^2. \tag{2.14}$$

Similarly,

$$\begin{pmatrix} R_{21} \\ R_{22} \end{pmatrix} = \psi_w(w)[B(w)w_x]_x - (\psi_w B w_\psi)(\bar{w})\psi_{xx}, \tag{2.15}$$

$$R_{21} = b_{1x}, \quad b_1 = \begin{pmatrix} I_{n_1\times n_1} & 0_{n_1\times n_2} \end{pmatrix} [B(w)w_x - (Bw_\psi)(\bar{w})\psi_x]. \tag{2.16}$$

With (1.7) we also have

$$R_{31} = 0, \quad R_{32} = [(r_2)_{w_2}(w) - (r_2)_{w_2}(\bar{w})]r_2(w) = O(1)|w - \bar{w}|^2. \tag{2.17}$$

From (2.7), (2.12), (2.14), (2.16) and (2.17) we have

$$\tilde{R} = \tilde{A}_0 \begin{pmatrix} \tilde{f}_{1x} + b_{1x} \\ 0_{n_2\times 1} \end{pmatrix} + \tilde{A}_0 \begin{pmatrix} 0_{n_1\times 1} \\ R_{12} + R_{22} + R_{32} \end{pmatrix}. \tag{2.18}$$

We cite [14, Lemma 3.4].

Lemma 2.3 ([14]). *If* $(\eta'' r')(\bar{w})$ *is symmetric then*

$$(\eta'' w_\psi)(\bar{w}) = \begin{pmatrix} \eta_{w_1w_1} - \eta_{w_2w_1}(r_2)_{w_2}^{-1}(r_2)_{w_1} & \eta_{w_2w_1}(r_2)_{w_2}^{-1} \\ 0_{n_2\times n_1} & \eta_{w_2w_2}(r_2)_{w_2}^{-1} \end{pmatrix} (\bar{w}). \tag{2.19}$$

Taking transpose of (2.19) and using (1.7) and (2.2), we simplify $\tilde{L}$ in (2.6). Under the assumption of Lemma 2.3 we have

$$\tilde{L} = \mathrm{diag}(0_{n_1\times n_1}, (\eta_{w_2w_2}(r_2)_{w_2}^{-1})(\bar{w})). \tag{2.20}$$

For the treatment of r in (1.5) in Sec. 4, we need the following crucial estimate obtained in [14].

Lemma 2.4 ([14]). *Under conditions* (1) *and* (2) *of Assumption* 1.1, *in a small neighborhood of* $\bar{w}$ *we have*

$$\eta_{w_1w_2} - \eta_{w_2w_2}(r_2)_{w_2}^{-1}(r_2)_{w_1} = O(1)|r_2(w)|. \tag{2.21}$$

For the treatment of the viscosity term in (1.5) we need the diffeomorphism φ defined in condition (3) of Assumption 1.1. Here, we cite the following result from [15].

Lemma 2.5 ([15]). *Conditions* (1) *and* (3) *in Assumption* 1.1 *imply that* $P^t w_\varphi^t \eta'' P$ *is block-upper triangular, and* $P^t w_\varphi^t \eta'' B w_\varphi P$ *is block diagonal in the partition* $n = n_3 + n_4$ *as follows*:

$$P^t w_\varphi^t \eta'' P = \begin{pmatrix} \tilde{\eta}_1 & 0_{n_3\times n_4} \\ \tilde{\eta}_3 & \tilde{\eta}_4 \end{pmatrix}, \tag{2.22}$$

$$P^t w_\varphi^t \eta'' B w_\varphi P = \mathrm{diag}(0_{n_3\times n_3}, \tilde{\eta}_4 B^*), \tag{2.23}$$

where $\tilde{\eta}_4 B^*$ *is symmetric, positive definite for all* $w \in \mathbb{O}$.

In Secs. 4 and 5, we need some tools from analysis, which are summarized as Lemma 2.6 as follows. It can be verified by using Gagliardo–Nirenberg inequality, for instance, see [14].

Lemma 2.6. (1) (*Sobolev-type inequality*) *For* $w \in H^1(\mathbb{R})$,

$$\|w\|_{L^\infty} \le \sqrt{2}\|w\|^{\frac{1}{2}}\|w_x\|^{\frac{1}{2}}. \tag{2.24}$$

(2) *Let* g *be a given smooth function of* w *in* $\mathbb{O}$. *If* $w - \bar{w} \in H^s(\mathbb{R})$, $s \ge 1$, *and* $\|w - \bar{w}\|_1$ *is bounded, then*

$$\|D_x^l g\| \le C\|D_x^l w\|, \quad 1 \le l \le s, \tag{2.25}$$

where $C > 0$ *is a constant depending only on* s *and the bound of* $\|w - \bar{w}\|_1$.

(3) *If* $D_x g, \tilde{g} \in H^{l-1}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *then*

$$\|D_x^l(g\tilde{g}) - gD_x^l\tilde{g}\| \le C(\|D_x g\|_{L^\infty}\|D_x^{l-1}\tilde{g}\| + \|D_x^l g\|\|\tilde{g}\|_{L^\infty}), \tag{2.26}$$

where $C > 0$ *is a constant depending only on* l.

3. Spectral Analysis

In this section, we assume conditions (1), (2) and (4) of Assumption 1.1, and carry out the spectral analysis of the linear system. Based upon it we derive estimates for the linearized system. We write (2.11), which is equivalent to (1.5), as

$$\hat{\tilde{\psi}}_t = E\hat{\tilde{\psi}} + \tilde{A}_0^{-1}\hat{\tilde{R}}, \tag{3.1}$$

where

$$E = -\tilde{A}_0^{-1}(i\xi\tilde{A} + \xi^2\tilde{B} - \tilde{L}) = E(i\xi). \tag{3.2}$$

The solution of (3.1) is

$$\hat{\tilde{\psi}}(\xi, t) = e^{Et}\hat{\tilde{\psi}}(\xi, 0) + \int_0^t e^{E(t-\tau)}\tilde{A}_0^{-1}\hat{\tilde{R}}(\xi, \tau)\, d\tau. \tag{3.3}$$

To estimate e^{Et} we first have the eigen-decomposition of E. Since $\tilde{A}_0$ is real, symmetric and positive definite, we may choose an $\tilde{A}_0^{\frac{1}{2}}$ that is real, symmetric and positive definite. We write

$$E = (\tilde{A}_0^{\frac{1}{2}})^{-1}\tilde{E}\tilde{A}_0^{\frac{1}{2}}, \tag{3.4}$$

where

$$\tilde{E} = (\tilde{A}_0^{\frac{1}{2}})^{-1}(\tilde{L} - i\xi\tilde{A} - \xi^2\tilde{B})(\tilde{A}_0^{\frac{1}{2}})^{-1} = \tilde{E}(i\xi). \tag{3.5}$$

Note that $\tilde{E}(z)$ is holomorphic in $z \in \mathbb{C}$. From Kato's perturbation theory [2], the number $\tilde{n}$ of distinct eigenvalues of $\tilde{E}(z)$ is constant if z is not one of the exceptional points, of which there are only a finite number in each compact set of $\mathbb{C}$. In each simply connected domain $\mathbb{D}$ containing no exceptional points, the eigenvalues of $\tilde{E}(z)$ can be expressed as $\tilde{n}$ holomorphic functions $\lambda_j(z)$, $j = 1, \ldots, \tilde{n}$, with the eigenvalues $\lambda_j(z)$ having constant multiplicities m_j. The $\lambda_j(z)$ are branches of one or several analytic functions in $\mathbb{C}$, which have only algebraic singularities, and which are everywhere continuous in $\mathbb{C}$. An exceptional point z_0 is either a branch point of some of the $\lambda_j(z)$, or a regular point for all of them; in the latter case, the values of some of the different $\lambda_j(z)$ coincide at $z = z_0$. Hence, there is always the splitting of eigenvalues at (and only at) an exceptional point.

The eigenprojections $P_j(z)$ and the eigennilpotents $D_j(z)$ for the eigenvalue $\lambda_j(z)$ of $\tilde{E}(z)$ are also holomorphic in each simply connected domain $\mathbb{D}$ containing no exceptional points, being branches of one or several analytic functions with only algebraic singularities. The analytic functions $P_j(z)$ and $\lambda_j(z)$ have common branch points of the same order, but $P_j(z)$ always has a pole at a branch point, see Theorem 3.1, while $\lambda_j(z)$ is always continuous there. $P_j(z)$ and $D_j(z)$ may have a pole even at an exceptional point, even when $\lambda_j(z)$ is holomorphic there.

From (3.5) we write

$$\tilde{E}(i\xi) = (i\xi)^2 \tilde{E}_\infty\left(\frac{1}{i\xi}\right),$$

$$\tilde{E}_\infty\left(\frac{1}{i\xi}\right) = (\tilde{A}_0^{\frac{1}{2}})^{-1}\left[\tilde{B} - \frac{1}{i\xi}\tilde{A} + \frac{1}{(i\xi)^2}\tilde{L}\right](\tilde{A}_0^{\frac{1}{2}})^{-1}.$$

With the same argument, $\tilde{E}_\infty(\tilde{z})$ has a finite number of exceptional points on $|\tilde{z}| \le 1$. Thus, $\tilde{E}(z)$ has a finite number of exceptional points on $|z| \ge 1$, hence on the whole complex plane $\mathbb{C}$.

From (3.5) and Lemma 2.1, $\tilde{E}(z)$ is real, symmetric for $z \in \mathbb{R}$. Thus, it has spectral decomposition

$$\tilde{E}(z) = \sum_{j=1}^{\tilde{n}} \lambda_j(z) P_j(z), \quad z \in \mathbb{R},$$

where $P_j(z)$ are real, symmetric, semi-positive definite for $z \in \mathbb{R}$. That is, $D_j(z) = 0$, $1 \le j \le \tilde{n}$, for $z \in \mathbb{R}$. By analytic continuation, $D_j(z) = 0$, $1 \le j \le \tilde{n}$, for all $z \in \mathbb{C}$. Therefore, for all non-exceptional points $z \in \mathbb{C}$, $\tilde{E}(z)$ has the spectral decomposition

$$\tilde{E}(z) = \sum_{j=1}^{\tilde{n}} \lambda_j(z) P_j(z), \quad z \in \mathbb{C}, \tag{3.6}$$

where if $z \in \mathbb{R}$, $P_j(z)$ are orthogonal projections and their Euclidean norms $|P_j(z)| = 1$.

Theorem 3.1 (Butler's Theorem). *If z_0 is a branch point of $\lambda_j(z)$ (hence also of $P_j(z)$) of order $p - 1 \ge 1$, then $P_j(z)$ has a pole there. That is, the Laurent expansion of $P_j(z)$ in powers of $(z - z_0)^{1/p}$ necessarily contains negative powers. In particular, $|P_j(z)| \to \infty$ as $z \to z_0$.*

A proof of Butler's Theorem can be found in [2]. Here, we note that according to the theorem, any real z is not a branch point of $\lambda_j(z)$. Consequently, all $\lambda_j(z)$

are holomorphic for real z. The $P_j(z)$ are single-valued for real z. Since they do not have a pole, $P_j(z)$ are holomorphic for real z as well. In particular, $\lambda_j(z)$ and $P_j(z)$ are holomorphic at $z = 0$.

From (3.4) and (3.6) we have

$$E(z) = (\tilde{A}_0^{\frac{1}{2}})^{-1}\tilde{E}(z)\tilde{A}_0^{\frac{1}{2}} = \sum_{j=1}^{\tilde{n}} \lambda_j(z)\tilde{P}_j(z), \tag{3.7}$$

where

$$\tilde{P}_j(z) = (\tilde{A}_0^{\frac{1}{2}})^{-1}P_j(z)\tilde{A}_0^{\frac{1}{2}} \tag{3.8}$$

are eigenprojections of $E(z)$ corresponding to the eigenvalues $\lambda_j(z)$. From (3.8) and the discussion above, we conclude that the eigenvalues $\lambda_j(z)$ and eigenprojections $\tilde{P}_j(z)$ of $E(z)$ are holomorphic at $z = 0$.

Taking Taylor expansions at $z = 0$ and by (3.2) and (3.7) we have

$$\begin{aligned} E(z) &= \tilde{A}_0^{-1}(\tilde{L} - z\tilde{A} + z^2\tilde{B}) \\ &= \sum_{j=1}^{\tilde{n}} [\lambda_j(0) + \lambda_j'(0)z + \cdots](\tilde{P}_{j0} + z\tilde{P}_{j1} + \cdots), \end{aligned} \tag{3.9}$$

where $\tilde{P}_{j0}$, $\tilde{P}_{j1}, \ldots$ are constant matrices. In particular,

$$\tilde{P}_{j0} = \tilde{P}_j(0).$$

Comparing the constant terms on the both sides of (3.9) we have

$$\tilde{A}_0^{-1}\tilde{L} = \sum_{j=1}^{\tilde{n}} \lambda_j(0)\tilde{P}_{j0}. \tag{3.10}$$

Noting

$$\tilde{P}_j(z)\tilde{P}_k(z) = \delta_{jk}\tilde{P}_j(z), \quad \sum_{j=1}^{\tilde{n}} \tilde{P}_j(z) = I,$$

by Taylor expansions we have

$$(\tilde{P}_{j0} + z\tilde{P}_{j1} + \cdots)(\tilde{P}_{k0} + z\tilde{P}_{k1} + \cdots) = \delta_{jk}(\tilde{P}_{j0} + z\tilde{P}_{j1} + \cdots),$$

$$\sum_{j=1}^{\tilde{n}} (\tilde{P}_{j0} + z\tilde{P}_{j1} + \cdots) = I.$$

Now, we compare the constant terms to have

$$\tilde{P}_{j0}\tilde{P}_{k0} = \delta_{jk}\tilde{P}_{j0}, \quad \sum_{j=1}^{\tilde{n}} \tilde{P}_{j0} = I. \tag{3.11}$$

Therefore, by combining those $\tilde{P}_{j0}$ with the same value of $\lambda_j(0)$ in (3.10), we obtained the spectral decomposition of $\tilde{A}_0^{-1}\tilde{L}$. From Lemma 2.1, $(\tilde{A}_0^{\frac{1}{2}})^{-1}\tilde{L}(\tilde{A}_0^{\frac{1}{2}})^{-1}$ is real, symmetric and semi-negative definite, hence its eigenvalues are either zero or negative. Noting

$$\tilde{A}_0^{-1}\tilde{L} = (\tilde{A}_0^{\frac{1}{2}})^{-1}[(\tilde{A}_0^{\frac{1}{2}})^{-1}\tilde{L}(\tilde{A}_0^{\frac{1}{2}})^{-1}]\tilde{A}_0^{\frac{1}{2}},$$

we conclude that $\lambda_j(0)$ in (3.10) are either zero or negative.

Now, we label λ_j such that

$$\lambda_1(0) = \cdots = \lambda_{\tilde{n}_1}(0) = 0, \quad \lambda_j(0) < 0 \quad \text{for } \tilde{n}_1 < j \le \tilde{n}. \tag{3.12}$$

Then $\sum_{j=1}^{\tilde{n}_1} \tilde{P}_{j0}$ is the eigenprojection of $\tilde{A}_0^{-1}\tilde{L}$ corresponding to the eigenvalue zero. From (2.6), (2.2) and (1.7),

$$\tilde{A}_0^{-1}\tilde{L} = (w_\psi^{-1} r' w_\psi)(\bar{w}) = \operatorname{diag}(0_{n_1\times n_1}, (r_2)_{w_2}(\bar{w})). \tag{3.13}$$

It is clear that the eigenprojection of $\tilde{A}_0^{-1}\tilde{L}$ corresponding to the eigenvalue zero is

$$P_0 \equiv \sum_{j=1}^{\tilde{n}_1} \tilde{P}_{j0} = \operatorname{diag}(I_{n_1\times n_1}, 0_{n_2\times n_2}). \tag{3.14}$$

Note that by (3.11)

$$\tilde{P}_{j0}P_0 = P_0\tilde{P}_{j0} = \tilde{P}_{j0}, \quad 1 \le j \le \tilde{n}_1. \tag{3.15}$$

The following lemma is from [3]. We modify the proof to fit our assumptions.

Lemma 3.2 ([3]). *Under conditions* (1), (2) *and* (4) *of Assumption* 1.1, *there exist positive constants C and c such that*

$$|e^{E(i\xi)t}| \le Ce^{-\frac{c\xi^2 t}{1+\xi^2}}, \quad \xi \in \mathbb{R}, \ t \ge 0. \tag{3.16}$$

Proof. For definiteness we use Euclidean norm as the matrix norm. Consider the linear system

$$\hat{u}_t = E(i\xi)\hat{u}, \tag{3.17}$$

where the solution is $\hat{u} = e^{E(i\xi)t}\hat{u}(\xi, 0)$. From (3.2), this is equivalent to

$$\tilde{A}_0\hat{u}_t = (-i\xi\tilde{A} - \xi^2\tilde{B} + \tilde{L})\hat{u}. \tag{3.18}$$

Multiplying (3.18) by $\hat{u}^*$, the conjugate transpose of $\hat{u}$, from the left, and taking the real part, we have

$$\frac{1}{2}(\hat{u}^*\tilde{A}_0\hat{u})_t = -\xi^2\hat{u}^*\tilde{B}\hat{u} + \hat{u}^*\tilde{L}\hat{u}, \tag{3.19}$$

where we have applied (1) of Lemma 2.1.

From Lemma 2.2, there is a compensating matrix K, such that $K\tilde{A}_0$ is real, skew symmetric, and

$$S = \frac{1}{2}(K\tilde{A} + \tilde{A}K^t) + \tilde{B} - \tilde{L}$$

is real, symmetric and positive definite. We multiply (3.18) by $-i\xi\hat{u}^*K$ from the left, and take the real part. This gives us

$$\begin{aligned}\left(-\frac{i}{2}\xi\hat{u}^*K\tilde{A}_0\hat{u}\right)_t &+ \frac{1}{2}\xi^2\hat{u}^*(K\tilde{A}+\tilde{A}K^t)\hat{u}\\ &= \Re\{i\xi^3\hat{u}^*K\tilde{B}\hat{u} - i\xi\hat{u}^*K\tilde{L}\hat{u}\}\\ &\leq |\xi|^3|\hat{u}||K\tilde{B}^{\frac{1}{2}}||\tilde{B}^{\frac{1}{2}}\hat{u}| + |\xi||\hat{u}||K(-\tilde{L})^{\frac{1}{2}}||(-\tilde{L})^{-\frac{1}{2}}\hat{u}|\\ &\leq \frac{\tilde{c}}{2}\xi^2|\hat{u}|^2 + C\xi^4|\tilde{B}^{\frac{1}{2}}\hat{u}|^2 + C|(-\tilde{L})^{-\frac{1}{2}}\hat{u}|^2, \end{aligned} \tag{3.20}$$

where $\tilde{B}^{\frac{1}{2}}$ and $(-\tilde{L})^{\frac{1}{2}}$ are real, symmetric, and semi-positive definite, and $\tilde{c} > 0$ is the smallest eigenvalue of S. Adding $\xi^2\hat{u}^*(\tilde{B}-\tilde{L})\hat{u}$ to both sides of (3.20) we have

$$\left(-\frac{i}{2}\xi\hat{u}^*K\tilde{A}_0\hat{u}\right)_t + \frac{1}{2}\xi^2\hat{u}^*S\hat{u} \leq \xi^2\hat{u}^*(\tilde{B}-\tilde{L})\hat{u} + C_1\xi^4\hat{u}^*\tilde{B}\hat{u} + C_1\hat{u}^*(-\tilde{L})\hat{u} \tag{3.21}$$

for some constant $C_1 > 1$.

Next, we multiply (3.19) by $(1+\xi^2)$, and (3.21) by $\alpha > 0$ to be determined. Summing up the results gives us

$$\begin{aligned}(1+\xi^2)(E_\alpha)_t &+ \frac{\alpha}{2}\xi^2\hat{u}^*S\hat{u} + (1+\xi^2)(\xi^2\hat{u}^*\tilde{B}\hat{u} - \hat{u}^*\tilde{L}\hat{u})\\ &\leq \alpha C_1(1+\xi^2)(\xi^2\hat{u}^*\tilde{B}\hat{u} - \hat{u}^*\tilde{L}\hat{u}),\end{aligned} \tag{3.22}$$

where

$$E_\alpha = \frac{1}{2}\left(\hat{u}^*\tilde{A}_0\hat{u} - \frac{i\xi\alpha}{1+\xi^2}\hat{u}^*K\tilde{A}_0\hat{u}\right).$$

Taking $\alpha \leq 1/C_1$ in (3.22), we have

$$(E_\alpha)_t + \frac{\alpha}{2}\frac{\xi^2}{1+\xi^2}\hat{u}^*S\hat{u} \leq 0. \tag{3.23}$$

Now, we show that E_α is equivalent to $|\hat{u}|^2$ for appropriately chosen α. Since

$$\left|i\frac{\xi}{1+\xi^2}\hat{u}^*K\tilde{A}_0\hat{u}\right| \leq \frac{|\xi|}{1+\xi^2}|\hat{u}|^2|K||\tilde{A}_0| \leq C_2|\hat{u}|^2$$

for some constant $C_2 > 0$, we have

$$\frac{1}{2}(\lambda_m - \alpha C_2)|\hat{u}|^2 \leq E_\alpha \leq \frac{1}{2}(\lambda_M + \alpha C_2)|\hat{u}|^2,$$

where $\lambda_m > 0$ and $\lambda_M > 0$ are the smallest and the largest eigenvalues of $\tilde{A}_0$, respectively. Taking $\alpha = \min\{1/C_1, \lambda_m/(2C_2)\}$ we conclude that E_α is equivalent to $|\hat{u}|^2$. Noting S is positive definite, (3.23) implies

$$(E_\alpha)_t + 2c\frac{\xi^2}{1+\xi^2}E_\alpha \le 0 \tag{3.24}$$

for some constant $c > 0$. Solving (3.24) we obtain

$$E_\alpha(t) \le e^{-\frac{2c\xi^2 t}{1+\xi^2}}E_\alpha(0),$$

which implies

$$|\hat{u}(\xi,t)|^2 \le Ce^{-\frac{2c\xi^2 t}{1+\xi^2}}|\hat{u}(\xi,0)|^2.$$

Noting $\hat{u}(\xi,t) = e^{E(i\xi)t}\hat{u}(\xi,0)$ we arrive at (3.16). □

Lemma 3.3. *Let $h(x) \in \mathbb{R}^n$. Under conditions* (1), (2) *and* (4) *of Assumption* 1.1, *if $h \in L^1(\mathbb{R})$ and $D_x^k h \in L^2(\mathbb{R})$, then*

$$\|e^{E(i\xi)t}(i\xi)^k\hat{h}(\xi)\| \le C(t+1)^{-\frac{1}{4}-\frac{k}{2}}\|h\|_{L^1} + Ce^{-ct}\|D_x^k h\|, \tag{3.25}$$

where C and c are positive constants. If in addition, h takes the form

$$h = \begin{pmatrix} 0_{n_1\times 1} \\ h_2 \end{pmatrix}, \quad h_2 \in \mathbb{R}^{n_2}, \tag{3.26}$$

then

$$\|e^{E(i\xi)t}(i\xi)^k\hat{h}(\xi)\| \le C(t+1)^{-\frac{3}{4}-\frac{k}{2}}\|h\|_{L^1} + Ce^{-ct}\|D_x^k h\|. \tag{3.27}$$

Proof. Applying (3.16) we have

$$\begin{aligned}
\|e^{E(i\xi)t}(i\xi)^k\hat{h}(\xi)\|^2 &= \int_{\mathbb{R}} |e^{E(i\xi)t}(i\xi)^k\hat{h}(\xi)|^2 d\xi \\
&\le C\int_{\{|\xi|\le\varepsilon\}\cup\{|\xi|\ge\varepsilon\}} e^{-\frac{2c\xi^2 t}{1+\xi^2}}|\xi|^{2k}|\hat{h}(\xi)|^2 d\xi \\
&\le C\int_{|\xi|\le\varepsilon} e^{-c\xi^2 t}|\xi|^{2k}|\hat{h}(\xi)|^2 d\xi \\
&\quad + C\int_{|\xi|\ge\varepsilon} e^{-\frac{2c\varepsilon^2 t}{1+\varepsilon^2}}|(i\xi)^k\hat{h}(\xi)|^2 d\xi \\
&\le C\left[\|\hat{h}\|_{L^\infty}^2\int_{|\xi|\le\varepsilon} e^{-c\xi^2 t}|\xi|^{2k}d\xi + e^{-\frac{2c\varepsilon^2 t}{1+\varepsilon^2}}\|(i\xi)^k\hat{h}(\xi)\|^2\right] \\
&\le C[(t+1)^{-\frac{1}{2}-k}\|h\|_{L^1}^2 + e^{-\frac{2c\varepsilon^2 t}{1+\varepsilon^2}}\|D_x^k h\|^2],
\end{aligned} \tag{3.28}$$

where $0 < \varepsilon \le 1$ is a small constant. Taking square root on both sides and resetting the constant c we obtain (3.25).

If h further satisfies (3.26), we refine the integral over $|\xi| \le \varepsilon$ to obtain (3.27) as follows. From (3.7),

$$e^{E(z)t} = \sum_{j=1}^{\tilde{n}} e^{\lambda_j(z)t} \tilde{P}_j(z),$$

where $\lambda_j(z)$ and $\tilde{P}_j(z)$ are holomorphic at $z = 0$. Thus for $|\xi| \le \varepsilon$ with a small ε, taking Taylor expansions of $\tilde{P}_j(i\xi)$, $1 \le j \le \tilde{n}_1$, and applying (3.14), (3.15) and (3.26), we have

$$\begin{aligned} e^{E(i\xi)t}(i\xi)^k \hat{h}(\xi) &= \sum_{j=1}^{\tilde{n}_1} e^{\lambda_j(i\xi)t}[\tilde{P}_{j0} + O(|\xi|)](i\xi)^k \hat{h}(\xi) \\ &\quad + \sum_{j=\tilde{n}_1+1}^{\tilde{n}} e^{\lambda_j(i\xi)t} \tilde{P}_j(i\xi)(i\xi)^k \hat{h}(\xi) \\ &= \sum_{j=1}^{\tilde{n}_1} e^{\lambda_j(i\xi)t} O(|\xi|)(i\xi)^k \hat{h}(\xi) + \sum_{j=\tilde{n}_1+1}^{\tilde{n}} e^{\lambda_j(i\xi)t} \tilde{P}_j(i\xi)(i\xi)^k \hat{h}(\xi). \end{aligned}$$

This implies

$$|e^{E(i\xi)t}(i\xi)^k \hat{h}(\xi)| \le C \left[\sum_{j=1}^{\tilde{n}_1} e^{\Re\{\lambda_j(i\xi)\}t} |\xi|^{k+1} |\hat{h}(\xi)| + \sum_{j=\tilde{n}_1+1}^{\tilde{n}} e^{\Re\{\lambda_j(i\xi)\}t} |\xi|^k |\hat{h}(\xi)| \right]. \tag{3.29}$$

From Lemma 2.2, noting $\lambda_j(i\xi)$ is an eigenvalue of E defined in (3.2), we have

$$\Re\{\lambda_j(i\xi)\} \le -\frac{\bar{c}\xi^2}{1+\xi^2}, \quad \xi \in \mathbb{R}, \tag{3.30}$$

where $\bar{c} > 0$ is a constant. From (3.12) we also have

$$\Re\{\lambda_j(i\xi)\} \le \frac{1}{2}\lambda_j(0) \le -\bar{c}, \quad \tilde{n}_1 + 1 \le j \le \tilde{n}, \tag{3.31}$$

for small ξ and by resetting $\bar{c} > 0$ if needed. Using (3.29)–(3.31), we refine the integral over $|\xi| \le \varepsilon$ as

$$\begin{aligned} \int_{|\xi|\le\varepsilon} |e^{E(i\xi)t}(i\xi)^k \hat{h}(\xi)| d\xi &\le C \int_{|\xi|\le\varepsilon} (e^{-\bar{c}\xi^2 t}|\xi|^{2k+2} + e^{-2\bar{c}t})|\hat{h}(\xi)|^2 d\xi \\ &\le C(t+1)^{-\frac{3}{2}-k} \|\hat{h}\|_{L^\infty}^2 \le C(t+1)^{-\frac{3}{2}-k} \|h\|_{L^1}^2. \end{aligned} \tag{3.32}$$

Replacing the corresponding integral in (3.28) by (3.32) we obtain (3.27). □

4. Weighted Energy Estimate

In this section, we carry out weighted energy estimate to derive decay rates for the nonlinear system. The non-optimal decay rates for higher derivatives obtained in this section help us to obtain the optimal ones given in Theorem 1.4. Similar methodology has been used for higher space dimensions [16]. However, the analysis in [16] does not apply to one space dimension that we are considering. This is because in one space dimension, the solution decays to the constant equilibrium state at a slower rate. The new idea in this section is that we assume optimal decay rates for lower derivatives, and perform weighted energy estimate to obtain rough estimates on higher derivatives of the solution. In Sec. 5, we use these estimates to obtain optimal ones for lower derivatives. In other words, the *a priori* estimate in this section is not closed independently (unlike the case of higher space dimensions). Instead, it is closed together with the analysis in Sec. 5. Similar ideas have been employed to resolve other challenging problems [5, 9, 10].

We introduce the following notation for $t \geq 0$ and $0 \leq l \leq s$:

$$N_l^2(t) = \sup_{0\leq\tau\leq t}\left[(\tau+1)^{\frac{1}{2}}\sum_{k=0}^{l}(\tau+1)^k\|D_x^k(w-\bar{w})\|^2(\tau)\right]. \tag{4.1}$$

Theorem 4.1. *Let* $\bar{w}$ *be a constant equilibrium state of* (1.5), *and Assumption* 1.1 *be true. Let* $s \geq 2$ *and* $w_0 - \bar{w} \in H^s(\mathbb{R})$. *Then there exists a constant* $\varepsilon > 0$ *such that if* $\|w_0 - \bar{w}\|_s \leq \varepsilon$ *and* $N_2(t) \leq \varepsilon$, *the solution of* (1.5), (1.4) *given in Theorem* 1.3 *has the following estimates*:

$$\|D_x^l(w-\bar{w})\|(t) \leq C\|w_0-\bar{w}\|_s(t+1)^{-\frac{l}{2}}, \quad t\geq 0,\ 0\leq l\leq s, \tag{4.2}$$

$$\begin{aligned}&\int_0^\infty\sum_{l=0}^{s-1}(t+1)^l\|D_x^{l+1}w\|_{s-l-1}^2(t)dt\\&\qquad+\int_0^\infty\sum_{l=0}^{s}(t+1)^l(\|D_x^{l+1}\tilde{w}_2\|_{s-l}^2+\|D_x^l r_2(w)\|_{s-l}^2)(t)dt\\&\quad\leq C\|w_0-\bar{w}\|_s^2,\end{aligned} \tag{4.3}$$

where $C > 0$ *is a constant.*

Proof. For $t \geq 0$ we define

$$\begin{aligned}M^2(t) = {}&\sum_{l=0}^{s}\sup_{0\leq\tau\leq t}[(\tau+1)^l\|D_x^l(w-\bar{w})\|_{s-l}^2(\tau)]\\&+\int_0^t\sum_{l=0}^{s-1}(\tau+1)^l\|D_x^{l+1}w\|_{s-l-1}^2(\tau)d\tau\\&+\int_0^t\sum_{l=0}^{s}(\tau+1)^l(\|D_x^{l+1}\tilde{w}_2\|_{s-l}^2+\|D_x^l r_2(w)\|_{s-l}^2)(\tau)d\tau.\end{aligned} \tag{4.4}$$

Our goal is to prove

$$M^2(t) \leq C\|w_0-\bar{w}\|_s^2, \tag{4.5}$$

where $C > 0$ is a constant. Equations (4.2) and (4.3) are then direct consequence of (4.4) and (4.5). In what follows, we assume that $M(t)$ and $N_2(t)$ are small.

First, we use $N_2(t)$ to express some L^∞-norms needed in this section. From (2.24), (4.1), (1.10) and (2.25) we have

$$\begin{aligned}
&\|w-\bar{w}\|_{L^\infty} \le C\|w-\bar{w}\|^{\frac{1}{2}}\|w_x\|^{\frac{1}{2}} \le CN_2(t)(t+1)^{-\frac{1}{2}},\\
&\|w_x\|_{L^\infty} \le C\|w_x\|^{\frac{1}{2}}\|w_{xx}\|^{\frac{1}{2}} \le CN_2(t)(t+1)^{-1},\\
&\|D_x^2\tilde{w}_2\|_{L^\infty} \le C\|D_x^2 w\|^{\frac{1}{2}}\|D_x^3\tilde{w}_2\|^{\frac{1}{2}} \le CN_2^{\frac{1}{2}}(t)(t+1)^{-\frac{5}{8}}\|D_x^3\tilde{w}_2\|^{\frac{1}{2}}.
\end{aligned} \tag{4.6}$$

To estimate $\|r_2(w)\|_{L^\infty}$, we note $r_2(w) = \tilde{\psi}_2(w)$ by (2.3). Thus taking the lower half of (2.4) and using (2.2) we have

$$r_{2t} - \overline{(r_2)}_{w_2} r_2 = R, \tag{4.7}$$

where $\overline{(r_2)}_{w_2} = (r_2)_{w_2}(\bar{w})$ and

$$R = \left((r_2)_{w_1}(r_2)_{w_2}\right)\{-f(w)_x + [B(w)w_x]_x\} + [(r_2)_{w_2}(w) - \overline{(r_2)}_{w_2}]r_2(w). \tag{4.8}$$

Solving the linear system (4.7) with respect to t gives us

$$r_2(w(x,t)) = e^{t\overline{(r_2)}_{w_2}} r_2(w_0(x)) + \int_0^t e^{(t-\tau)\overline{(r_2)}_{w_2}} R(x,\tau)d\tau. \tag{4.9}$$

From Lemma 2.1 and (2.20), $(\eta_{w_2w_2}(r_2)_{w_2}^{-1})(\bar{w})$ is real, symmetric and semi-negative definite. Since it is nonsingular by Assumption 1.1, it is in fact negative definite. This implies $[\eta^{\frac{1}{2}}_{w_2w_2}(r_2)^{-1}_{w_2}(\eta^{\frac{1}{2}}_{w_2w_2})^{-1}](\bar{w})$ is real, symmetric and negative definite, hence the eigenvalues of $\overline{(r_2)}_{w_2}$ are all negative. Therefore, there is a constant $c > 0$ such that

$$\|r_2(w)\|_{L^\infty}(t) \le Ce^{-ct}\|r_2(w_0)\|_{L^\infty} + C\int_0^t e^{-c(t-\tau)}\|R(\cdot,\tau)\|_{L^\infty}d\tau. \tag{4.10}$$

From condition (3) of Assumption 1.1 and (1.10) we have

$$D_x^l[B(w)w_x] = PD_x^l[P^tB(w)w_\varphi P\tilde{w}_x] = PD_x^l\begin{pmatrix} 0_{n_3\times 1}\\ B^*(w)\tilde{w}_{2x}\end{pmatrix}, \quad l \ge 0. \tag{4.11}$$

From (4.8) and (4.11), and applying triangle inequality, we have

$$\|R\|_{L^\infty} \le C(\|w_x\|_{L^\infty} + \|D_x^2\tilde{w}_2\|_{L^\infty} + \|w-\bar{w}\|_{L^\infty}\|r_2(w)\|_{L^\infty}). \tag{4.12}$$

Substituting (4.6) into (4.12), we integrate both sides to arrive at

$$\begin{aligned}
&\int_0^t e^{-c(t-\tau)}\|R\|_{L^\infty}(\tau)d\tau\\
&\quad \le CN_2(t)(t+1)^{-1} + CN_2^{\frac{1}{2}}(t)\int_0^t e^{-c(t-\tau)}(\tau+1)^{-\frac{9}{8}}[(\tau+1)^2\|D_x^3\tilde{w}_2\|^2(\tau)]^{\frac{1}{4}}d\tau\\
&\qquad + CN_2(t)(t+1)^{-\frac{3}{2}}\sup_{0\le\tau\le t}[(\tau+1)\|r_2(w)\|_{L^\infty}(\tau)].
\end{aligned}$$

Applying Hölder's inequality and (4.4), the second term on the right-hand side is bounded by $CN_2^{\frac{1}{2}}(t)M^{\frac{1}{2}}(t)(t+1)^{-\frac{9}{8}}$. Substituting the result into (4.10) gives us

$$\begin{aligned}\|r_2(w)\|_{L^\infty}(t) &\le Ce^{-ct}\|r_2(w_0)\|_{L^\infty} + C[N_2(t)+M(t)](t+1)^{-1} \\ &\quad + CN_2(t)(t+1)^{-\frac{3}{2}} \sup_{0\le\tau\le t}[(\tau+1)\|r_2(w)\|_{L^\infty}(\tau)],\end{aligned}$$

which implies

$$\sup_{0\le\tau\le t}[(\tau+1)\|r_2(w)\|_{L^\infty}(\tau)] \le C[\|r_2(w_0)\|_{L^\infty} + M(t) + N_2(t)]$$

for small $N_2(t)$. Applying (2.24) and (2.25) to the right-hand side, we have

$$\|r_2(w)\|_{L^\infty}(t) \le C[\|w_0-\bar{w}\|_{s-1} + M(t) + N_2(t)](t+1)^{-1}. \tag{4.13}$$

Now, we start the weighed energy estimate. Applying D_x^l to (1.5) and multiplying the result by $D_x^l w^t \eta''(w)$, we have

$$\begin{aligned}&D_x^l w^t\eta''(w)D_x^l w_t + D_x^l w^t\eta''(w)D_x^l[f'(w)w_x] \\ &\quad = D_x^l w^t\eta''(w)D_x^{l+1}[B(w)w_x] + D_x^l w^t\eta''(w)D_x^l r(w).\end{aligned}$$

We replace the time variable by τ, multiply the equation by the weight function $(\tau+1)^k$, and integrate the result over $\mathbb{R}\times[0,t]$. After integration by parts and noting the symmetry of $\eta''f'$, for $1\le k\le l\le s$, we have

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}}(t+1)^k[D_x^l w^t\eta''(w)D_x^l w](x,t)dx \\ &\quad = \frac{1}{2}\int_{\mathbb{R}}[D_x^l w^t\eta''(w)D_x^l w](x,0)dx + \sum_{j=1}^{5} I_j,\end{aligned} \tag{4.14}$$

where

$$\begin{aligned}I_1 &= \frac{1}{2}\int_0^t\int_{\mathbb{R}}(\tau+1)^k[D_x^l w^t\eta''(w)_t D_x^l w](x,\tau)dxd\tau \\ &\quad + \frac{k}{2}\int_0^t\int_{\mathbb{R}}(\tau+1)^{k-1}[D_x^l w^t\eta''(w)D_x^l w](x,\tau)dxd\tau, \\ I_2 &= \int_0^t\int_{\mathbb{R}}(\tau+1)^k\frac{1}{2}\{D_x^l w^t[\eta''(w)f'(w)]_x D_x^l w\}(x,\tau)dxd\tau, \\ I_3 &= -\int_0^t\int_{\mathbb{R}}(\tau+1)^k\{D_x^l w^t\eta''(w)[D_x^l(f'(w)w_x) - f'(w)D_x^{l+1}w]\}(x,\tau)dxd\tau, \\ I_4 &= -\int_0^t\int_{\mathbb{R}}(\tau+1)^k\{[D_x^{l+1}w^t\eta''(w) + D_x^l w^t\eta''(w)_x]D_x^l[B(w)w_x]\}(x,\tau)dxd\tau, \\ I_5 &= \int_0^t\int_{\mathbb{R}}(\tau+1)^k[D_x^l w^t\eta''(w)D_x^l r(w)](x,\tau)dxd\tau.\end{aligned} \tag{4.15}$$

From (1.5) and applying (4.6), (4.11) and (4.13) we have

$$\begin{aligned}\|w_t\|_{L^\infty} &\le C(\|w_x\|_{L^\infty} + \|[B(w)w_x]_x\|_{L^\infty} + \|r(w)\|_{L^\infty})\\ &\le C(\|w_x\|_{L^\infty} + \|D_x^2\tilde{w}_2\|_{L^\infty} + \|r_2(w)\|_{L^\infty})\\ &\le C[\|w_0 - \bar{w}\|_{s-1} + M(t) + N_2(t)](t+1)^{-1}\\ &\quad + CN_2^{\frac{1}{2}}(t)(t+1)^{-\frac{5}{8}}\|D_x^3\tilde{w}_2\|^{\frac{1}{2}}. \end{aligned} \tag{4.16}$$

Substituting (4.16) into (4.15) and noting $1 \le k \le l \le s$, we have

$$\begin{aligned} I_1 &\le C\int_0^t (\tau+1)^k\|w_t\|_{L^\infty}(\tau)\|D_x^l w\|^2(\tau)d\tau + C\int_0^t (\tau+1)^{k-1}\|D_x^l w\|^2(\tau)d\tau\\ &\le C[\|w_0 - \bar{w}\|_{s-1} + M(t) + N_2(t)]M^2(t) + C\int_0^t (\tau+1)^{k-1}\|D_x^l w\|^2(\tau)d\tau. \end{aligned} \tag{4.17}$$

Similarly, from (4.15) and (4.6),

$$I_2 \le C\int_0^t (\tau+1)^k(\|w_x\|_{L^\infty}\|D_x^l w\|^2)(\tau)\,d\tau \le CN_2(t)M^2(t). \tag{4.18}$$

For I_3 we apply Cauchy–Schwarz inequality, (2.26), (2.25) and (4.6) to have

$$\begin{aligned} I_3 &\le C\int_0^t (\tau+1)^k\|D_x^l w\|(\tau)\|D_x^l[f'(w)w_x] - f'(w)D_x^{l+1}w\|(\tau)d\tau\\ &\le C\int_0^t (\tau+1)^k(\|D_x^l w\|^2\|w_x\|_{L^\infty})(\tau)d\tau \le CN_2(t)M^2(t). \end{aligned} \tag{4.19}$$

To estimate I_4 we need the diffeomorphism φ defined in condition (3) of Assumption 1.1. With $\tilde{w}$ defined in (1.10) we have

$$D_x^{l+1}w^t\eta''(w) = D_x^{l+1}\tilde{w}^tP^tw_\varphi^t\eta''(w) + [D_x^l(w_\varphi\varphi_x) - w_\varphi D_x^{l+1}\varphi]^t\eta''(w).$$

Applying (1.10), (2.22) and (4.11) gives us

$$\begin{aligned} D_x^{l+1}w^t\eta''(w)D_x^l[B(w)w_x] &= D_x^{l+1}\tilde{w}_2^t\tilde{\eta}_4D_x^l[B^*(w)\tilde{w}_{2x}]\\ &\quad + [D_x^l(w_\varphi\varphi_x) - w_\varphi D_x^{l+1}\varphi]^t\eta''(w)D_x^l[B(w)w_x]. \end{aligned}$$

Substituting the equation into (4.15) and applying (2.25), (2.26) and (4.11), we have

$$\begin{aligned} I_4 &= -\int_0^t\int_{\mathbb{R}} (\tau+1)^k(D_x^{l+1}\tilde{w}_2^t\tilde{\eta}_4B^*(w)D_x^{l+1}\tilde{w}_2)(x,\tau)dxd\tau\\ &\quad + O(1)\int_0^t (\tau+1)^k(\|D_x^{l+1}\tilde{w}_2\|\|w_x\|_{L^\infty}\|D_x^l w\| + \|w_x\|_{L^\infty}^2\|D_x^l w\|^2)(\tau)d\tau. \end{aligned}$$

Noting that $\tilde{\eta}_4 B^*(w)$ is symmetric, positive definite by Lemma 2.5, we conclude that there is a constant $c > 0$ such that

$$I_4 \le -c\int_0^t (\tau+1)^k \|D_x^{l+1}\tilde{w}_2\|^2(\tau)d\tau + CN_2(t)M^2(t), \tag{4.20}$$

where we have applied (4.4) and (4.6), noting $1 \le k \le l \le s$.

To estimate I_5, by (1.7) and the key estimate (2.21) we write

$$\begin{aligned} D_x^l w^t \eta''(w) D_x^l r(w) &= D_x^l r^t(w)\eta''(w)D_x^l w \\ &= D_x^l r_2^t(w)(\eta_{w_1w_2}D_x^l w_1 + \eta_{w_2w_2}D_x^l w_2) \\ &= D_x^l r_2^t(w)\eta_{w_2w_2}[(r_2)_{w_2}^{-1}(r_2)_{w_1}D_x^l w_1 + D_x^l w_2] \\ &\quad + O(1)|D_x^l r_2(w)||r_2(w)||D_x^l w_1|. \end{aligned}$$

Noting $l \ge 1$ and $D_x w_2 = (w_2)_\psi \psi_x = -(r_2)_{w_2}^{-1}(r_2)_{w_1}w_{1x} + (r_2)_{w_2}^{-1}r_2(w)_x$, by (2.1) and (2.2), we further have

$$\begin{aligned} D_x^l w^t \eta''(w) D_x^l r(w) = {} & D_x^l r_2^t(w)\eta_{w_2w_2}D_x^{l-1}[(r_2)_{w_2}^{-1}(r_2)(w)_x] \\ & + O(1)|D_x^l r_2(w)||(r_2)_{w_2}^{-1}(r_2)_{w_1}D_x^l w_1 \\ & - D_x^{l-1}[(r_2)_{w_2}^{-1}(r_2)_{w_1}w_{1x}]| \\ & + O(1)|D_x^l r_2(w)||r_2(w)||D_x^l w_1|. \end{aligned}$$

By linearizing at $\bar{w}$ of the leading term on the right-hand side, from (4.15), (2.25) and (2.26) we arrive at

$$\begin{aligned} I_5 = {} & \int_0^t\int_{\mathbb{R}} (\tau+1)^k\{D_x^l r_2^t(w)[\eta_{w_2w_2}(r_2)_{w_2}^{-1}](\bar{w})D_x^l r_2(w)\}(x,\tau)dxd\tau \\ & + O(1)\int_0^t (\tau+1)^k\|D_x^l r_2(w)\|^2(\tau)\|w-\bar{w}\|_{L^\infty}(\tau)d\tau \\ & + O(1)\int_0^t (\tau+1)^k\|D_x^l r_2(w)\|(\tau)(\|w_x\|_{L^\infty}\|D_x^{l-1}w\| \\ & + \|r_2(w)\|_{L^\infty}\|D_x^l w\|)(\tau)d\tau, \end{aligned}$$

where the term containing $\|D_x^{l-1}w\|$ disappears if $l = 1$. In the derivation of (4.10), we have made conclusion that $[\eta_{w_2w_2}(r_2)_{w_2}^{-1}](\bar{w})$ is real, symmetric, and negative definite. Thus, there exists a constant $c > 0$ such that

$$I_5 \le -c\int_0^t (\tau+1)^k\|D_x^l r_2(w)\|^2(\tau)d\tau + C[\|w_0-\bar{w}\|_{s-1} + M(t) + N_2(t)]M^2(t), \tag{4.21}$$

again, with the help of (4.4), (4.6) and (4.13).

Combining (4.14) and (4.17)–(4.21) and noting η'' is symmetric, positive definite, for $1 \le k \le l \le s$ we obtain

$$\begin{aligned}(t+1)^k\|D_x^l w\|^2(t) &+ \int_0^t (\tau+1)^k(\|D_x^{l+1}\tilde{w}_2\|^2 + \|D_x^l r_2(w)\|^2)(\tau)d\tau \\ &\le C\|D_x^l w_0\|^2 + C[\|w_0-\bar{w}\|_{s-1} + M(t) + N_2(t)]M^2(t) \\ &\quad + C\int_0^t (\tau+1)^{k-1}\|D_x^l w\|^2(\tau)d\tau, \end{aligned} \tag{4.22}$$

where $C > 0$ is a constant.

We still need an estimate of the integral on the right-hand side of (4.22). For this we use the variable $\tilde{\psi}$ instead. Apply D_x^l to (2.5) and multiply the result by $D_x^{l+1}\tilde{\psi}^t K$, where K is the compensating matrix. Then we have

$$D_x^{l+1}\tilde{\psi}^t K\tilde{A}_0 D_x^l\tilde{\psi}_t + D_x^{l+1}\tilde{\psi}^t K\tilde{A}D_x^{l+1}\tilde{\psi} = D_x^{l+1}\tilde{\psi}^t K(\tilde{B}D_x^{l+2}\tilde{\psi} + \tilde{L}D_x^l\tilde{\psi} + D_x^l\tilde{R}).$$

We replace t by τ, multiply the equation by $(\tau+1)^k$, and integrate the result over $\mathbb{R}\times[0,t]$. Noting $K\tilde{A}_0$ is real, skew symmetric by Lemma 2.2, and using S defined in (2.8), we have

$$\int_0^t\int_{\mathbb{R}}(\tau+1)^k(D_x^{l+1}\tilde{\psi}^t S D_x^{l+1}\tilde{\psi})(x,\tau)dxd\tau = \sum_{j=6}^{9} I_j, \tag{4.23}$$

where

$$\begin{aligned} I_6 &= \frac{1}{2}\int_0^t\int_{\mathbb{R}}(\tau+1)^k(D_x^l\tilde{\psi}^t K\tilde{A}_0 D_x^{l+1}\tilde{\psi})_t(x,\tau)dxd\tau, \\ I_7 &= \int_0^t\int_{\mathbb{R}}(\tau+1)^k(D_x^{l+1}\tilde{\psi}^t\tilde{B}D_x^{l+1}\tilde{\psi})(x,\tau)dxd\tau, \\ I_8 &= -\int_0^t\int_{\mathbb{R}}(\tau+1)^k(D_x^{l+1}\tilde{\psi}^t\tilde{L}D_x^{l+1}\tilde{\psi})(x,\tau)dxd\tau, \\ I_9 &= \int_0^t\int_{\mathbb{R}}(\tau+1)^k D_x^{l+1}\tilde{\psi}^t K(\tilde{B}D_x^{l+2}\tilde{\psi} + \tilde{L}D_x^l\tilde{\psi} + D_x^l\tilde{R})(x,\tau)dxd\tau. \end{aligned} \tag{4.24}$$

Let $1 \le k \le l \le s-1$. By integration by parts and Cauchy–Schwarz inequality we have

$$\begin{aligned} I_6 &\le C(t+1)^k(\|D_x^{l+1}\tilde{\psi}\|\|D_x^l\tilde{\psi}\|)(t) + C(\|D_x^{l+1}\tilde{\psi}\|\|D_x^l\tilde{\psi}\|)(0) \\ &\quad + \alpha\int_0^t(\tau+1)^k\|D_x^{l+1}\tilde{\psi}\|^2(\tau)d\tau + C_\alpha\int_0^t(\tau+1)^{k-2}\|D_x^l\tilde{\psi}\|^2(\tau)d\tau, \end{aligned} \tag{4.25}$$

where $\alpha > 0$ is a constant to be determined, and $C_\alpha > 0$ is a constant depending on the choice of α.

To estimate I_7 we need to convert the variable ψ associated with the kinetic term to φ with viscosities via the mappings $\psi \to w \to \varphi$ as follows: By (2.3) and (2.6),

$$\begin{aligned} D_x^{l+1}\tilde{\psi}^t\tilde{B}D_x^{l+1}\tilde{\psi} &= D_x^{l+1}\psi^t(w_\psi^t\eta''Bw_\psi)(\bar{w})D_x^{l+1}\psi \\ &= D_x^{l+1}\psi^t(w_\psi^t\eta''Bw_\psi)(w)D_x^{l+1}\psi + O(1)|D_x^{l+1}\psi|^2|w-\bar{w}| \\ &= D_x^l(\psi_\varphi\varphi_x)^t w_\psi^t\eta''Bw_\psi D_x^l(\psi_\varphi\varphi_x) + O(1)|D_x^{l+1}\psi|^2|w-\bar{w}| \\ &= D_x^{l+1}\varphi^t w_\varphi^t\eta''Bw_\varphi D_x^{l+1}\varphi + O(1)|D_x^l(\psi_\varphi\varphi_x) \\ &\quad - \psi_\varphi D_x^{l+1}\varphi|(|D_x^{l+1}\psi| + |D_x^{l+1}\varphi|) + O(1)|D_x^{l+1}\psi|^2|w-\bar{w}|. \end{aligned}$$

Applying (4.24), (1.10), (2.23), (2.25), (2.26) and (4.6), we arrive at

$$\begin{aligned} I_7 &\le \int_0^t\int_{\mathbb{R}}(\tau+1)^k(D_x^{l+1}\tilde{w}_2^t\tilde{\eta}_4B^*D_x^{l+1}\tilde{w}_2)(x,\tau)dxd\tau \\ &\quad + C\int_0^t(\tau+1)^k\|D_x^l(\psi_\varphi\varphi_x) - \psi_\varphi D_x^{l+1}\varphi\|(\tau)(\|D_x^{l+1}\psi\| + \|D_x^{l+1}\varphi\|)(\tau)d\tau \\ &\quad + C\int_0^t(\tau+1)^k(\|D_x^{l+1}\psi\|^2\|w-\bar{w}\|_{L^\infty})(\tau)d\tau \\ &\le C\int_0^t(\tau+1)^k\|D_x^{l+1}\tilde{w}_2\|^2(\tau)d\tau \\ &\quad + C\int_0^t(\tau+1)^k(\|w_x\|_{L^\infty}\|D_x^l w\|\|D_x^{l+1}w\|)(\tau)d\tau \\ &\quad + C\int_0^t(\tau+1)^k(\|D_x^{l+1}w\|^2\|w-\bar{w}\|_{L^\infty})(\tau)d\tau \\ &\le C\int_0^t(\tau+1)^k\|D_x^{l+1}\tilde{w}_2\|^2(\tau)d\tau + CN_2(t)M^2(t). \end{aligned} \tag{4.26}$$

To estimate I_8 we substitute (2.3) and (2.20) into (4.24) to have

$$\begin{aligned} I_8 &= -\int_0^t\int_{\mathbb{R}}(\tau+1)^k[D_x^{l+1}r_2(w)^t(\eta_{w_1w_2}(r_2)_{w_2}^{-1})(\bar{w})D_x^{l+1}r_2(w)](x,\tau)dxd\tau \\ &\le C\int_0^t(\tau+1)^k\|D_x^{l+1}r_2(w)\|^2(\tau)d\tau. \end{aligned} \tag{4.27}$$

For I_9 we simplify the integrand first. From (2.6) and (2.7) we have

$$\begin{aligned} &\tilde{B}D_x^{l+2}\tilde{\psi} + \tilde{L}D_x^l\tilde{\psi} + D_x^l\tilde{R} = D_x^lR_1 + D_x^l\tilde{R}_2 + D_x^l\tilde{R}_3, \\ &R_1 = \tilde{A}_0[(\psi_wf'w_\psi)(\bar{w}) - (\psi_wf'w_\psi)(w)]\psi_x, \\ &\tilde{R}_2 = R_2 + \tilde{B}\tilde{\psi}_{xx} = \tilde{A}_0\psi_w(w)[B(w)w_x]_x, \\ &\tilde{R}_3 = R_3 + \tilde{L}\tilde{\psi} = \tilde{A}_0(\psi_wr)(w). \end{aligned} \tag{4.28}$$

Therefore,

$$
\begin{aligned}
I_9 &\leq C\int_0^t\int_{\mathbb{R}}(\tau+1)^k(|D_x^{l+1}\tilde{\psi}||D_x^lR_1+D_x^l\tilde{R}_2+D_x^l\tilde{R}_3|)(x,\tau)dxd\tau\\
&\leq \alpha\int_0^t(\tau+1)^k\|D_x^{l+1}\tilde{\psi}\|^2(\tau)d\tau\\
&\quad+C_\alpha\int_0^t(\tau+1)^k(\|D_x^lR_1\|^2+\|D_x^l\tilde{R}_2\|^2+\|D_x^l\tilde{R}_3\|^2)(\tau)d\tau, \qquad (4.29)
\end{aligned}
$$

where $\alpha>0$ again is a constant to be determined, and $C_\alpha>0$ is a constant depending on α. Applying (2.25), (2.26), (4.11) and (4.28), it is clear that the second integral on the right-hand side of (4.29) is bounded by

$$
\begin{aligned}
C_\alpha\int_0^t&(\tau+1)^k(\|w_x\|_{L^\infty}^2\|D_x^lw\|^2+\|w-\bar{w}\|_{L^\infty}^2\|D_x^{l+1}w\|^2+\|w_x\|_{L^\infty}^2\|D_x^{l+1}w\|^2\\
&+\|D_x^lw\|^2\|D_x^2\tilde{w}_2\|_{L^\infty}^2+\|D_x^{l+2}\tilde{w}_2\|^2+\|w_x\|_{L^\infty}^2\|D_x^{l-1}r_2(w)\|^2\\
&+\|D_x^lw\|^2\|r_2(w)\|_{L^\infty}^2+\|D_x^lr_2(w)\|^2)(\tau)d\tau.
\end{aligned}
$$

Therefore, using (4.6) and (4.13), we have

$$
\begin{aligned}
I_9 &\leq \alpha\int_0^t(\tau+1)^k\|D_x^{l+1}\tilde{\psi}\|^2(\tau)d\tau\\
&\quad+C_\alpha\int_0^t(\tau+1)^k(\|D_x^{l+2}\tilde{w}_2\|^2+\|D_x^lr_2(w)\|^2)(\tau)d\tau\\
&\quad+C_a[\|w_0-\bar{w}\|_{s-1}^2+M^2(t)+N_2^2(t)]M^2(t). \qquad (4.30)
\end{aligned}
$$

Combining (4.23), (4.25)–(4.27) and (4.30), and noting S is symmetric and positive definite by Lemma 2.2, we conclude that there is a constant $c>0$ such that for $1\leq k\leq l\leq s-1$,

$$
\begin{aligned}
c\int_0^t&(\tau+1)^k\|D_x^{l+1}\tilde{\psi}\|^2(\tau)d\tau\\
&\leq 2\alpha\int_0^t(\tau+1)^k\|D_x^{l+1}\tilde{\psi}\|^2(\tau)d\tau+C_\alpha\int_0^t(\tau+1)^{k-2}\|D_x^l\tilde{\psi}\|^2(\tau)d\tau\\
&\quad+C_\alpha\int_0^t(\tau+1)^k(\|D_x^{l+1}\tilde{w}_2\|_1^2+\|D_x^lr_2(w)\|_1^2)(\tau)d\tau\\
&\quad+C(t+1)^k(\|D_x^{l+1}\tilde{\psi}\|^2+\|D_x^l\tilde{\psi}\|^2)(t)\\
&\quad+C_\alpha\|w-\bar{w}\|_s^2+C_\alpha[N_2(t)+M^2(t)]M^2(t).
\end{aligned}
$$

Now, we choose $\alpha < c/4$ to arrive at

$$\begin{aligned}\int_0^t (\tau+1)^k \|D_x^{l+1} w\|^2(\tau)d\tau \\ &\le C\int_0^t (\tau+1)^k \|D_x^{l+1}\tilde{\psi}\|^2(\tau)d\tau \\ &\le C\bigg\{\int_0^t (\tau+1)^{k-2}\|D_x^l w\|^2(\tau)d\tau + \int_0^t (\tau+1)^k(\|D_x^{l+1}\tilde{w}_2\|_1^2 \\ &\quad + \|D_x^l r_2(w)\|_1^2)(\tau)d\tau + (t+1)^k\|D_x^l w\|_1^2(t) + \|w_0-\bar{w}\|_s^2 \\ &\quad + [N_2(t)+M^2(t)]M^2(t)\bigg\},\end{aligned} \tag{4.31}$$

where $1 \le k \le l \le s-1$.

Recall the energy estimate from Theorem 1.3,

$$\begin{aligned}\sup_{0\le\tau\le t}\|w-\bar{w}\|_s^2(\tau) + \int_0^t (\|D_x w\|_{s-1}^2 + \|D_x\tilde{w}_2\|_s^2 + \|r_2(w)\|_s^2)(\tau)d\tau \\ \le C\|w_0-\bar{w}\|_s^2.\end{aligned} \tag{4.32}$$

Summing up (4.22) for $k \le l \le s$, we have for $1 \le k \le s$,

$$\begin{aligned}\sup_{0\le\tau\le t}[(\tau+1)^k\|D_x^k w\|_{s-k}^2(\tau)] + \int_0^t (\tau+1)^k[\|D_x^{k+1}\tilde{w}_2\|_{s-k}^2 + \|D_x^k r_2(w)\|_{s-k}^2](\tau)d\tau \\ \le C\left[\|w_0-\bar{w}\|_s^2 + M^3(t) + N_2(t)M^2(t) + \int_0^t (\tau+1)^{k-1}\|D_x^k w\|_{s-k}^2(\tau)d\tau\right].\end{aligned} \tag{4.33}$$

Also, summing up (4.31) for $k \le l \le s-1$, we have for $1 \le k \le s-1$,

$$\begin{aligned}\int_0^t (\tau+1)^k\|D_x^{k+1}w\|_{s-k-1}^2(\tau)d\tau \\ &\le C\bigg[\|w_0-\bar{w}\|_s^2 + M^4(t) + N_2(t)M^2(t) + (t+1)^k\|D_x^k w\|_{s-k}^2(t) \\ &\quad + \int_0^t (\tau+1)^k(\|D_x^{k+1}\tilde{w}_2\|_{s-k}^2 + \|D_x^k r_2(w)\|_{s-k}^2)(\tau)d\tau \\ &\quad + \int_0^t (\tau+1)^{k-2}\|D_x^k w\|_{s-k-1}^2(\tau)d\tau\bigg].\end{aligned} \tag{4.34}$$

By Induction, (4.32)–(4.34) imply

$$\begin{aligned}\sup_{0\le\tau\le t}[(\tau+1)^k\|D_x^k(w-\bar{w})\|_{s-k}^2(\tau)] \\ &+ \int_0^t (\tau+1)^k[\|D_x^{k+1}\tilde{w}_2\|_{s-k}^2 + \|D_x^k r_2(w)\|_{s-k}^2](\tau)d\tau \\ &\le C[\|w_0-\bar{w}\|_s^2 + M^3(t) + N_2(t)M^2(t)], \quad 0 \le k \le s,\end{aligned} \tag{4.35}$$

$$\int_0^t (\tau+1)^k \|D_x^{k+1} w\|_{s-k-1}^2(\tau)d\tau$$
$$\leq C[\|w_0-\bar{w}\|_s^2 + M^3(t) + N_2(t)M^2(t)], \quad 0 \leq k \leq s-1. \tag{4.36}$$

The induction goes as follows: (4.32) implies both (4.35) and (4.36) for $k=0$. Next, taking $k=1$ in (4.33) and $k=0$ in (4.36) give us (4.35) for $k=1$. Now taking $k=1$ in (4.34) and (4.35) and $k=0$ in (4.36) give us (4.36) for $k=1$. As it continues, we make the conclusion that (4.35) is true for $0 \leq k \leq s$ while (4.36) is true for $0 \leq k \leq s-1$. Summing up (4.35) and (4.36) we arrive at

$$M^2(t) \leq C[\|w_0-\bar{w}\|_s^2 + M^3(t) + N_2(t)M^2(t)],$$

which implies

$$[1 - CM(t) - CN_2(t)]M^2(t) \leq C\|w_0-\bar{w}\|_s^2.$$

Therefore, if $M(t)$ and $N_2(t)$ are small, we have

$$M^2(t) \leq C\|w_0-\bar{w}\|_s^2. \tag{4.37}$$

By a standard continuity argument, (4.37) is true if $\|w_0-\bar{w}\|_s$ and $N_2(t)$ are sufficiently small. □

5. Optimal Decay Rates

In this section, we finish the nonlinear analysis to prove our main result, Theorem 1.4, to obtain optimal decay rates of the solution w of (1.5), (1.4) to the constant equilibrium state $\bar{w}$. This is to perform *a priori* estimate via Duhamel's principle, using results derived in Secs. 3 and 4.

Recall N_k^2 defined in (4.1),

$$N_k^2(t) = \sup_{0\leq\tau\leq t}\left[(\tau+1)^{\frac{1}{2}}\sum_{l=0}^{k}(\tau+1)^l\|D_x^l(w-\bar{w})\|^2(\tau)\right], \tag{5.1}$$

where $t \geq 0$ and $0 \leq k \leq s$. By a standard continuity argument, to prove (1.12) in Theorem 1.4 under the smallness assumption on the initial data, we only need to prove the following proposition.

Proposition 5.1. *Under the hypotheses of Theorem* 1.4, *if $N_{s-2}(T)$ is bounded by a small positive constant, which is independent of $T>0$, then*

$$N_{s-2}(T) \leq C\delta_0, \tag{5.2}$$

where $\delta_0 \equiv \|w_0-\bar{w}\|_s + \|w_0-\bar{w}\|_{L^1}$, and $C>0$ is a constant independent of T.

Proof. From (5.1) and (4.2) (assuming $N_{s-2}(T)$ is small with $s\geq 4$), for $0\leq t\leq T$ we have

$$\|D_x^l(w-\bar{w})\|(t) \leq N_{s-2}(t)(t+1)^{-\frac{1}{4}-\frac{l}{2}}, \quad 0\leq l\leq s-2, \tag{5.3}$$

$$\|D_x^l w\|(t) \leq C\delta_0(t+1)^{-\frac{l}{2}}, \quad l=s-1, s. \tag{5.4}$$

We carry out our analysis with the variable $\tilde{\psi}$ defined in (2.3). Applying Plancherel theorem, (3.3), and triangle inequality, we have

$$\begin{aligned}\|D_x^l\tilde{\psi}\|(t) &= \|(i\xi)^l\hat{\tilde{\psi}}\|(t)\\ &\le \|(i\xi)^l e^{E(i\xi)t}\hat{\tilde{\psi}}(\xi,0)\| + \int_0^t \|(i\xi)^l e^{E(i\xi)(t-\tau)}\tilde{A}_0^{-1}\hat{\tilde{R}}(\xi,\tau)\|d\tau. \end{aligned} \tag{5.5}$$

From (3.25), (2.3) and (2.25) we have

$$\begin{aligned}\|(i\xi)^l e^{E(i\xi)t}\hat{\tilde{\psi}}(\xi,0)\| &\le C(t+1)^{-\frac{1}{4}-\frac{l}{2}}(\|\tilde{\psi}\|_{L^1} + \|D_x^l\tilde{\psi}\|)(0)\\ &\le C(t+1)^{-\frac{1}{4}-\frac{l}{2}}(\|w_0-\bar{w}\|_{L^1} + \|w_0-\bar{w}\|_l). \end{aligned} \tag{5.6}$$

Similarly, from (2.18), (3.25) and (3.26) we have

$$\int_0^t \|(i\xi)^l e^{E(i\xi)(t-\tau)}\tilde{A}_0^{-1}\hat{\tilde{R}}(\xi,\tau)\|d\tau = \sum_{j=10}^{12} I_j, \tag{5.7}$$

where

$$\begin{aligned}I_{10} &= O(1)\int_0^{\frac{t}{2}}(t-\tau+1)^{-\frac{1}{4}-\frac{l+1}{2}}\left(\|\tilde{f}_1\|_{L^1} + \|b_1\|_{L^1} + \sum_{k=1}^3\|R_{k2}\|_{L^1}\right)(\tau)d\tau,\\ I_{11} &= O(1)\int_{\frac{t}{2}}^t(t-\tau+1)^{-\frac{3}{4}}\left(\|D_x^l\tilde{f}_1\|_{L^1} + \|D_x^l b_1\|_{L^1} + \sum_{k=1}^3\|D_x^l R_{k2}\|_{L^1}\right)(\tau)d\tau,\\ I_{12} &= O(1)\int_0^t e^{-c(t-\tau)}\left(\|D_x^{l+1}\tilde{f}_1\| + \|D_x^{l+1}b_1\| + \sum_{k=1}^3\|D_x^l R_{k2}\|\right)(\tau)d\tau, \end{aligned} \tag{5.8}$$

with a positive constant c.

To estimate I_{10} we note that $\tilde{f}_1$, b_1 and R_{k2}, $1 \le k \le 3$, are defined in (2.13)–(2.17). Thus by (2.3), (2.25) and (5.3),

$$\begin{aligned}\|\tilde{f}_1\|_{L^1}(\tau) &\le C\|\tilde{\psi}\|^2(\tau) \le C\|w-\bar{w}\|^2(\tau) \le CN_{s-2}^2(\tau)(\tau+1)^{-\frac{1}{2}},\\ \|b_1\|_{L^1}(\tau) &\le C(\|w-\bar{w}\|\|\psi_x\|)(\tau) \le C(\|w-\bar{w}\|\|w_x\|)(\tau) \le CN_{s-2}^2(\tau)(\tau+1)^{-1},\\ \|R_{12}\|_{L^1}(\tau) &\le C(\|w-\bar{w}\|\|\psi_x\|)(\tau) \le CN_{s-2}^2(\tau)(\tau+1)^{-1},\\ \|R_{22}\|_{L^1}(\tau) &\le C(\|w_x\|\|\psi_x\| + \|w-\bar{w}\|\|\psi_{xx}\|)(\tau)\\ &\le C(\|w_x\|^2 + \|w-\bar{w}\|\|w_{xx}\|)(\tau)\\ &\le CN_{s-2}^2(\tau)(\tau+1)^{-\frac{3}{2}},\\ \|R_{32}\|_{L^1}(\tau) &\le C\|w-\bar{w}\|^2(\tau) \le CN_{s-2}^2(\tau)(\tau+1)^{-\frac{1}{2}}.\end{aligned}$$

Substituting these into (5.8) gives us

$$\begin{aligned} I_{10} &\le CN_{s-2}^2(t)\int_0^{\frac{t}{2}}(t-\tau+1)^{-\frac{3}{4}-\frac{l}{2}}(\tau+1)^{-\frac{1}{2}}d\tau \\ &\le CN_{s-2}^2(t)(t+1)^{-\frac{1}{4}-\frac{l}{2}}. \end{aligned} \tag{5.9}$$

To estimate I_{11} we consider $0 \le l \le s-2$. Again, by (2.14), (2.3), (2.25) and (5.3) we have

$$\begin{aligned} \|D_x^l\tilde{f}_1\|_{L^1}(\tau) &= \|-D_x^{l-1}[f_1'(w)w_\psi\psi_x] + (f_1'w_\psi)(\bar{w})D_x^l\psi\|_{L^1}(\tau) \\ &\le C(\|w-\bar{w}\|\|D_x^l\psi\| + \|w_x\|\|D_x^{l-1}\psi\| + \cdots + \|D_x^{l-1}w\|\|\psi_x\|)(\tau) \\ &\le C\sum_{k=0}^{l-1}(\|D_x^k(w-\bar{w})\|\|D_x^{l-k}(w-\bar{w})\|)(\tau) \\ &\le CN_{s-2}^2(\tau)(\tau+1)^{-\frac{1}{2}-\frac{l}{2}}, \end{aligned} \tag{5.10}$$

where we have considered $l \ge 1$ while the result is clearly true for $l = 0$. Similarly, from (2.13) and (2.15)–(2.17) we have

$$\begin{aligned} \|D_x^l b_1\|_{L^1}(\tau) &\le C\sum_{k=0}^{l}\|D_x^k(w-\bar{w})\|\|D_x^{l+1-k}(w-\bar{w})\|(\tau) \\ &\le C[N_{s-2}^2(\tau)(\tau+1)^{-1-\frac{l}{2}} + N_{s-2}(\tau)\delta_0(\tau+1)^{-\frac{3}{4}-\frac{l}{2}}], \\ \|D_x^l R_{12}\|_{L^1}(\tau) &\le C[N_{s-2}^2(\tau)(\tau+1)^{-1-\frac{l}{2}} + N_{s-2}(\tau)\delta_0(\tau+1)^{-\frac{3}{4}-\frac{l}{2}}], \\ \|D_x^l R_{22}\|_{L^1}(\tau) &\le C\left[\sum_{k=0}^{l+1}\|D_x^k(w-\bar{w})\|\|D_x^{l+2-k}w\|\right. \\ &\qquad \left. + \|w_x\|_{L^\infty}\sum_{k=1}^{l}\|D_x^k w\|\|D_x^{l+1-k}w\|\right](\tau) \\ &\le C[N_{s-2}^2(\tau)(\tau+1)^{-\frac{3}{2}-\frac{l}{2}} + N_{s-2}(\tau)\delta_0(\tau+1)^{-\frac{5}{4}-\frac{l}{2}}], \\ \|D_x^l R_{32}\|_{L^1}(\tau) &\le CN_{s-2}^2(\tau)(\tau+1)^{-\frac{1}{2}-\frac{l}{2}}, \end{aligned} \tag{5.11}$$

where we have applied (5.4) and (2.24)–(2.26) as well. Substituting (5.10) and (5.11) into (5.8) gives us

$$\begin{aligned} I_{11} &\le C[N_{s-2}^2(t) + N_{s-2}(t)\delta_0]\int_{\frac{t}{2}}^{t}(t-\tau+1)^{-\frac{3}{4}}(\tau+1)^{-\frac{1}{2}-\frac{l}{2}}d\tau \\ &\le C[N_{s-2}^2(t) + N_{s-2}(t)\delta_0](t+1)^{-\frac{1}{4}-\frac{l}{2}}. \end{aligned} \tag{5.12}$$

To estimate I_{12}, again we consider $0 \le l \le s-2$ and apply (2.24)–(2.26) to (2.13)–(2.17) to have

$$\begin{aligned}
\|D_x^{l+1}\tilde{f}_1\|(\tau) &\le \|-D_x^l[f_1'(w)w_\psi\psi_x] + (f_1'w_\psi)(w)D_x^{l+1}\psi\|(\tau) \\
&\quad + C(\|w-\bar{w}\|_{L^\infty}\|D_x^{l+1}\psi\|)(\tau) \\
&\le C(\|w_x\|_{L^\infty}\|D_x^l w\| + \|w-\bar{w}\|_{L^\infty}\|D_x^{l+1}w\|)(\tau) \\
&\le C[N_{s-2}^2(\tau)(\tau+1)^{-\frac{5}{4}-\frac{l}{2}} + N_{s-2}(\tau)\delta_0(\tau+1)^{-1-\frac{l}{2}}], \\
\|D_x^{l+1}b_1\|(\tau) &\le C(\|w_x\|_{L^\infty}\|D_x^{l+1}w\| + \|w-\bar{w}\|_{L^\infty}\|D_x^{l+2}w\|)(\tau) \\
&\le C[N_{s-2}^2(\tau)(\tau+1)^{-\frac{7}{4}-\frac{l}{2}} + N_{s-2}(\tau)\delta_0(\tau+1)^{-\frac{3}{2}-\frac{l}{2}}], \\
\|D_x^l R_{12}\|(\tau) &\le C[N_{s-2}^2(\tau)(\tau+1)^{-\frac{5}{4}-\frac{l}{2}} + N_{s-2}(\tau)\delta_0(\tau+1)^{-1-\frac{l}{2}}], \\
\|D_x^l R_{22}\|(\tau) &\le C(\|w_x\|_{L^\infty}\|D_x^{l+1}w\| + \|w-\bar{w}\|_{L^\infty}\|D_x^{l+2}w\| \\
&\quad + \|w_x\|_{L^\infty}^2\|D_x^l w\| + \|w_{xx}\|_{L^\infty}\|D_x^l w\|)(\tau) \\
&\le C[N_{s-2}^2(\tau)(\tau+1)^{-\frac{7}{4}-\frac{l}{2}} + N_{s-2}(\tau)\delta_0(\tau+1)^{-\frac{3}{2}-\frac{l}{2}}], \\
\|D_x^l R_{32}\|(\tau) &\le C(\|w-\bar{w}\|_{L^\infty}\|D_x^l w\| + \|w_x\|_{L^\infty}\|D_x^{l-1}w\|)(\tau) \\
&\le CN_{s-2}^2(\tau)(\tau+1)^{-\frac{3}{4}-\frac{l}{2}}.
\end{aligned}$$

Here, we have assumed $l \ge 1$ or $l \ge 2$ in the above derivation but the results are clearly true for $0 \le l \le s-2$. Substituting these estimates into (5.8) gives us

$$\begin{aligned}
I_{12} &\le C[N_{s-2}^2(t) + N_{s-2}(t)\delta_0]\int_0^t e^{-c(t-\tau)}(\tau+1)^{-\frac{3}{4}-\frac{l}{2}}d\tau \\
&\le C[N_{s-2}^2(t) + N_{s-2}(t)\delta_0](t+1)^{-\frac{3}{4}-\frac{l}{2}}. \qquad (5.13)
\end{aligned}$$

Combining (5.7), (5.9), (5.12) and (5.13) we have

$$\begin{aligned}
&\int_0^t \|(i\xi)^l e^{E(i\xi)(t-\tau)}\tilde{A}_0^{-1}\hat{\hat{R}}(\xi,\tau)\|d\tau \\
&\quad \le C[N_{s-2}^2(t) + N_{s-2}(t)\delta_0](t+1)^{-\frac{1}{4}-\frac{l}{2}}, \quad 0 \le l \le s-2. \qquad (5.14)
\end{aligned}$$

Substituting (5.6) and (5.14) into (5.5), for $0 \le l \le s-2$ we have

$$\|D_x^l(w-\bar{w})\|(t) \le C\|D_x^l\tilde{\psi}\|(t) \le C[\delta_0 + N_{s-2}^2(t)](t+1)^{-\frac{1}{4}-\frac{l}{2}}.$$

Equivalently, we have

$$(t+1)^{\frac{1}{2}+l}\|D_x^l(w-\bar{w})\|^2(t) \le C[\delta_0 + N_{s-2}^2(t)]^2.$$

Summing up for $0 \le l \le s-2$ and taking supremum for $0 \le t \le T$, we have

$$N_{s-2}^2(T) \le C[\delta_0 + N_{s-2}^2(T)]^2,$$

or

$$[1 - CN_{s-2}(T)]N_{s-2}(T) \le C\delta_0.$$

This gives us (5.2) provided $N_{s-2}(T) \le 1/(2C)$, which is independent of T. □

Next, we carry out the proof of (1.13) in Theorem 1.4. For this we recall (4.9), which implies the following L^2-counterpart to (4.10).

$$\|D_x^l r_2(w)\|(t) \le Ce^{-ct}\|D_x^l r_2(w_0)\| + C\int_0^t e^{-c(t-\tau)}\|D_x^l R(\cdot,\tau)\|d\tau, \tag{5.15}$$

where C and c are positive constants, and R is defined in (4.8).

Noting $r_2(\bar{w}) = 0$ and applying (2.25) we have

$$\|D_x^l r_2(w_0)\| \le C\|D_x^l(w_0 - \bar{w})\|, \quad 0 \le l \le s. \tag{5.16}$$

To estimate $\|D_x^l R\|$ we use (4.8), (2.24)–(2.26) and (4.11) to arrive at

$$\begin{aligned}\|D_x^l R\|(\tau) &\le \|D_x^l[(r_2)_w f(w)_x]\|(\tau) + \|D_x^l[(r_2)_w(Bw_x)_x]\|(\tau)\\ &\quad + \|D_x^l\{[(r_2)_{w_2}(w) - (r_2)_{w_2}(\bar{w})]r_2(w)\}\|(\tau)\\ &\le C[\|D_x^{l+1}w\| + (\|w_x\|_{L^\infty} + \|D_x^2\tilde{w}_2\|_{L^\infty} + \|w-\bar{w}\|_{L^\infty})\|D_x^l(w-\bar{w})\|\\ &\quad + \|w_x\|_{L^\infty}\|D_x^{l-1}(w-\bar{w})\| + \|D_x^{l+2}\tilde{w}_2\|](\tau),\end{aligned}$$

where the terms with $\|D_x^{l-1}(w-\bar{w})\|$ does not exist if $l = 0$. Applying (1.12) and taking $0 \le l \le s-4$ we further have

$$\|D_x^l R\|(\tau) \le C\delta_0[(\tau+1)^{-\frac{3}{4}-\frac{l}{2}} + (\tau+1)^{-\frac{1}{4}-\frac{l}{2}}\|D_x^2\tilde{w}_2\|_{L^\infty}(\tau)]. \tag{5.17}$$

Finally, we substitute (5.16) and (5.17) into (5.15), and apply Cauchy–Schwarz inequality and (4.3) to obtain (1.13),

$$\begin{aligned}\|D_x^l r_2(w)\|(t) &\le Ce^{-ct}\|w_0 - \bar{w}\|_{s-4} + C\delta_0\int_0^t e^{-c(t-\tau)}(\tau+1)^{-\frac{3}{4}-\frac{l}{2}}d\tau\\ &\quad + C\delta_0\int_0^t e^{-c(t-\tau)}(\tau+1)^{-\frac{3}{4}-\frac{l}{2}}[(\tau+1)^{\frac{1}{2}}\|D_x^2\tilde{w}_2\|_{s-1}(\tau)]d\tau\\ &\le C\delta_0(t+1)^{-\frac{3}{4}-\frac{l}{2}}, \quad 0 \le l \le s-4.\end{aligned} \tag{5.18}$$

This completes the proof of Theorem 1.4.

To finish this section, we justify Remark 1.6 on the special case of hyperbolic balance laws, i.e. $B = 0$ in (1.5). In this case, $b_1 = 0$ and $R_{22} = 0$ in (5.8). We may replace $N_{s-2}(T)$ by $N_{s-1}(T)$ in Proposition 5.1 and let $s \ge 3$. It is straightforward to verify that (5.12) and (5.13) are true for $0 \le l \le s-1$, with N_{s-2} replaced by N_{s-1}. As a consequence, (5.2) becomes $N_{s-1}(T) \le C\delta_0$, which gives (1.12) for $0 \le l \le s-1$, or (1.15) for $0 \le l \le s - 3/2 + 1/p$. Similarly, (5.17), hence (5.18), is true for $0 \le l \le s-2$. This implies (1.16) for $0 \le l \le s - 5/2 + 1/p$.

6. Applications

In this section, we discuss several applications. First, we consider the special case of hyperbolic–parabolic conservation laws (1.2),

$$\begin{aligned} w_t + f(w)_x &= [B(w)w_x]_x, \\ w(x,0) &= w_0(x). \end{aligned} \tag{6.1}$$

Assumption 1.1 is simplified to

Assumption 6.1. (1) There exists a strictly convex entropy function η of w such that in $\mathbb{O}$, $\eta'' f'$ is symmetric, and $\eta'' B$ is symmetric, semi-positive definite.

(2) There is a diffeomorphism $\varphi \to w$ from an open set $\tilde{\mathbb{O}} \subset \mathbb{R}^n$ to $\mathbb{O}$ such that

$$B(w(\varphi))w_\varphi(\varphi) = \operatorname{diag}(0_{n_3 \times n_3}, B^*(\varphi)),$$

where n_3 and $n_4 = n - n_3 > 0$ are constant integers, and $B^* \in \mathbb{R}^{n_4 \times n_4}$ is nonsingular in $\tilde{\mathbb{O}}$.

(3) The null space of $B(\bar{w})$ contains no eigenvectors of $f'(\bar{w})$.

We note that the constant orthogonal matrix P in Assumption 1.1 is not needed since there is no intertwining of dissipation from viscosity and kinetic terms. We also set $n_4 > 0$ since otherwise (6.1) would be a system of hyperbolic conservation laws, and condition (3) in Assumption 6.1 would not be satisfied. Theorem 1.4 and Corollary 1.5 are reduced to Theorem 6.2.

Theorem 6.2. *Let $\bar{w}$ be a constant state and Assumption* 6.1 *be true. Let $s \geq 4$ be an integer, and $w_0 - \bar{w} \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$. Then there exists a constant $\varepsilon > 0$ such that if $\delta_0 \equiv \|w_0 - \bar{w}\|_s + \|w_0 - \bar{w}\|_{L^1} \leq \varepsilon$, the Cauchy problem* (6.1) *has a unique solution for $t \geq 0$, satisfying*

$$\|D_x^l(w - \bar{w})\|_{L^p}(t) \leq C\delta_0(t+1)^{-\frac{1}{2}+\frac{1}{2p}-\frac{l}{2}} \tag{6.2}$$

for $0 \leq l \leq s - 5/2 + 1/p$ with $p \geq 2$. In particular, the decay rate in L^2 is $(t+1)^{-\frac{1}{4}-\frac{l}{2}}$ in (6.2).

With slightly simplified assumptions (condition (2) of Assumption 6.1), Theorem 6.2 recovers existing results in [3, 4].

The second application is to the special case of hyperbolic balance laws (1.3),

$$\begin{aligned} w_t + f(w)_x &= r(w), \\ w(x,0) &= w_0(x). \end{aligned} \tag{6.3}$$

In this case, Assumption 1.1 is simplified to

Assumption 6.3. (1) There exists a strictly convex entropy function η of w in $\mathbb{O}$, such that $\eta'' f'$ is symmetric in $\mathbb{O}$ and $\eta'' r'$ is symmetric, semi-negative definite on $\mathbb{E}$.

(2) Equation (6.3) has n_1 conservation laws, i.e. there is a partition $n = n_1 + n_2$, $n_1, n_2 > 0$, such that

$$r(w) = \begin{pmatrix} 0_{n_1\times 1} \\ r_2(w) \end{pmatrix}, \quad w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

with $w_1 \in \mathbb{R}^{n_1}$, $r_2, w_2 \in \mathbb{R}^{n_2}$, and $(r_2)_{w_2} \in \mathbb{R}^{n_2\times n_2}$ is nonsingular.

(3) The null space of $r'(\bar{w})$ contains no eigenvectors of $f'(\bar{w})$.

In Assumption 6.3, we set $n_1, n_2 > 0$. The case $n_1 = 0$ leads to better decay rates than those in (6.4) below while physical models often demand $n_1 > 0$. The case $n_2 = 0$ is precluded as otherwise the system would be one of hyperbolic conservation laws. Noting Remark 1.6, Theorem 1.4 and Corollary 1.5 are reduced to Theorem 6.4.

Theorem 6.4. *Let $\bar{w}$ be a constant equilibrium state of* (6.3) *and Assumption* 6.3 *be true. Let $s \geq 3$ be an integer, and $w_0 - \bar{w} \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$. Then there exists a constant $\varepsilon > 0$ such that if $\delta_0 \equiv \|w_0 - \bar{w}\|_s + \|w_0 - \bar{w}\|_{L^1} \leq \varepsilon$, the Cauchy problem* (6.3) *has a unique solution for $t \geq 0$, satisfying*

$$\|D_x^l(w - \bar{w})\|_{L^p}(t) \leq C\delta_0(t+1)^{-\frac{1}{2}+\frac{1}{2p}-\frac{l}{2}} \tag{6.4}$$

for $0 \leq l \leq s - 3/2 + 1/p$, and

$$\|D_x^l r_2(w)\|_{L^p}(t) \leq C\delta_0(t+1)^{-1+\frac{1}{2p}-\frac{l}{2}} \tag{6.5}$$

for $0 \leq l \leq s - 5/2 + 1/p$, where $p \geq 2$. In particular, the L^2 rates in (6.4) *and* (6.5) *are $(t+1)^{-\frac{1}{4}-\frac{l}{2}}$ and $(t+1)^{-\frac{3}{4}-\frac{l}{2}}$, respectively.*

Under Assumption 6.3, which is simpler and slightly weaker, Theorem 6.4 recovers existing results in [5]. A comparison of Assumption 6.3 and the set used in [5] is given in [14].

The third application is to Keller–Segel model with logistic growth. The following chemotaxis model was proposed by Keller and Segel [6] to describe the oriented movement of cells toward a chemical concentration gradient

$$\begin{cases} c_t = \varepsilon c_{xx} - \mu u c^m, \\ u_t = (Du_x - \chi u c^{-1} c_x)_x, \end{cases} \quad x \in \mathbb{R}, \ \ t > 0, \tag{6.6}$$

where the unknown functions $c(x,t)$ and $u(x,t)$ denote the chemical concentration and cell density, respectively. The constants $\varepsilon \geq 0$ and $D \geq 0$ are, respectively, diffusion coefficients of the chemical and cells. The constants $\mu > 0$ and $\chi > 0$ are the coefficients of density-dependent degradation rate and of chemotactic sensitivity, respectively, while $m \geq 0$ is the degradation rate.

In our discussion, we set $m = 1$, and the degradation term in (6.6) is $-\mu uc$. This implies that the chemical (oxygen) is consumed only when cells (bacteria) encounter the chemical. We also consider that cells undergo logistic growth. Therefore, our model reads

$$\begin{cases} c_t = \varepsilon c_{xx} - \mu uc, \\ u_t = (Du_x - \chi uc^{-1}c_x)_x + au\Big(1 - \dfrac{u}{K}\Big), \end{cases} \quad x \in \mathbb{R}, \ \ t > 0, \tag{6.7}$$

where the constants $a \geq 0$ and $K > 0$ are the natural growth rate and the typical carrying capacity.

The singularity in the chemotactic sensitivity in (6.6) or (6.7) can be removed by the inverse Hopf-Cole transformation [7],

$$v = (\ln c)_x. \tag{6.8}$$

Under the new variables v and u, we write (6.7) as

$$\begin{cases} v_t + (\mu u - \varepsilon v^2)_x = \varepsilon v_{xx}, \\ u_t + \chi(uv)_x = Du_{xx} + au\left(1 - \dfrac{u}{K}\right). \end{cases} \tag{6.9}$$

Using the positive parameters μ, χ and K we simplify (6.9) by rescaling

$$\tilde{t} = \mu Kt, \quad \tilde{x} = \sqrt{\frac{\mu K}{\chi}}x, \quad \tilde{u} = \frac{u}{K}, \quad \tilde{v} = \sqrt{\frac{\chi}{\mu K}}v. \tag{6.10}$$

This converts (6.9) into

$$\begin{cases} \tilde{v}_{\tilde{t}} + (\tilde{u} - \tilde{\varepsilon}\tilde{v}^2)_{\tilde{x}} = \tilde{\varepsilon}\tilde{v}_{\tilde{x}\tilde{x}}, \\ \tilde{u}_{\tilde{t}} + (\tilde{u}\tilde{v})_{\tilde{x}} = \tilde{D}\tilde{u}_{\tilde{x}\tilde{x}} + \tilde{a}\tilde{u}(1 - \tilde{u}), \end{cases} \tag{6.11}$$

where

$$\tilde{\varepsilon} = \frac{\varepsilon}{\chi}, \quad \tilde{D} = \frac{D}{\chi}, \quad \tilde{a} = \frac{a}{\mu K}. \tag{6.12}$$

Dropping the tilde accent we write (6.11) as

$$\begin{cases} v_t + (u - \varepsilon v^2)_x = \varepsilon v_{xx}, \\ u_t + (uv)_x = Du_{xx} + au(1 - u), \end{cases} \tag{6.13}$$

where $\varepsilon \geq 0$, $D \geq 0$ and $a \geq 0$ are constant parameters.

We consider Cauchy problem of (6.13) with initial data

$$(v, u)(x, 0) = (v_0, u_0)(x), \tag{6.14}$$

where (v_0, u_0) is a perturbation of a constant equilibrium state $(\bar{v}, \bar{u})$. Here to be equilibrium, $\bar{u} = 0$ or 1, and to be stable equilibrium $\bar{u} = 1$. Therefore, we take the constant equilibrium state as $(\bar{v}, 1)$, where $\bar{v}$ is a constant. Now, we take a neighborhood $\mathbb{O}$ of $(\bar{v}, 1)$. The equilibrium manifold is

$$\mathbb{E} = \{(v, 1)\} \cap \mathbb{O}.$$

Equation (6.13) is in the form of (1.5), with

$$w = \begin{pmatrix} v \\ u \end{pmatrix}, \quad f(w) = \begin{pmatrix} u - \varepsilon v^2 \\ uv \end{pmatrix}, \quad B = \begin{pmatrix} \varepsilon & 0 \\ 0 & D \end{pmatrix}, \quad r(w) = \begin{pmatrix} 0 \\ au(1-u) \end{pmatrix}.$$

In what follows, we verify that Assumption 1.1 is satisfied. Let

$$\eta = \frac{1}{2}v^2 + u \ln u - u$$

be the entropy function [8]. By direct calculation,

$$\eta'' = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{u} \end{pmatrix}, \quad f' = \begin{pmatrix} -2\varepsilon v & 1 \\ u & v \end{pmatrix}, \quad r' = \begin{pmatrix} 0 & 0 \\ 0 & a(1-2u) \end{pmatrix}.$$

Clearly, η is strictly convex in $\mathbb{O}$; $\eta'' f'$ is symmetric, and $\eta'' B$ is symmetric, semi-positive definite in $\mathbb{O}$ (for $\varepsilon \geq 0$ and $D \geq 0$); and $\eta'' r'$ is symmetric, semi-negative definite on $\mathbb{E}$.

If $a > 0$, (6.13) has one conservation law, and the partition $n = n_1 + n_2$ in condition (2) is $2 = 1 + 1$. In this case, $(r_2)_{w_2} = \frac{\partial}{\partial u}[au(1-u)] = a(1-2u) \neq 0$. On the other hand, if $a = 0$ then (6.13) has two conservation laws, and the partition is $2 = 2 + 0$.

The diffeomorphism φ in condition (3) is the identity. The partition $n = n_3 + n_4$ is $2 = 0 + 2$ if $\varepsilon > 0$ and $D > 0$ (two parabolic equations), $2 = 1 + 1$ if $\varepsilon > 0$ and $D = 0$, or $\varepsilon = 0$ and $D > 0$ (one parabolic equation), and $2 = 2 + 0$ if $\varepsilon = D = 0$ (two hyperbolic equations). The constant orthogonal matrix $P \in \mathbb{R}^{2\times 2}$ is the identity in all cases except when $\varepsilon > 0$ and $D = 0$, in which P is the permutation to interchange the two equations in (6.13).

Finally, condition (4) is satisfied if at least one of ε, D and a is positive (otherwise, (6.13) becomes a system of hyperbolic conservation laws). This is readily verified since $\mathbb{N}_1 \cap \mathbb{N}_2$ is either zero-dimensional, or one-dimensional, spanned by $(1, 0)^t$ or $(0, 1)^t$. Each of these subspaces of $\mathbb{R}^2$ does not contain eigenvectors of $f'(\bar{v}, 1)$.

We now conclude that Assumption 1.1 is satisfied by (6.13) as long as one of ε, D and a is positive. This breaks down to seven cases as follows:

Case 1. $\varepsilon > 0$, $D > 0$ and $a > 0$;
Case 2. $\varepsilon = 0$, $D > 0$ and $a > 0$;
Case 3. $\varepsilon > 0$, $D = 0$ and $a > 0$;
Case 4. $\varepsilon = D = 0$ and $a > 0$;
Case 5. $\varepsilon > 0$, $D > 0$ and $a = 0$;
Case 6. $\varepsilon = 0$, $D > 0$ and $a = 0$;
Case 7. $\varepsilon > 0$, $D = 0$ and $a = 0$.

Here, Case 4 fits (6.3) and Theorem 6.4 applies, while Cases 5–7 fit (6.1) and Theorem 6.2 applies. Therefore, we focus on Cases 1–3. In these cases, $r_2(w) = au(1-u)$, which is equivalent to $1-u$ since u is about one and $a>0$. Applying Theorems 1.3, 1.4 and Corollary 1.5 to (6.13), (6.14), we have the following.

Theorem 6.5. *Let $\bar{v}$ be a constant, $a>0$, $s\geq 2$ be an integer, and $(v_0-\bar{v},u_0-1)\in H^s(\mathbb{R})$. Then there exists a constant $\varepsilon>0$ such that if $\|(v_0-\bar{v},u_0-1)\|_s\leq\varepsilon$, the Cauchy problem* (6.13), (6.14) *has a unique global solution. The solution has an energy estimate as follows*:

Case 1. *If $\varepsilon>0$ and $D>0$ then*

$$\sup_{t\geq 0}\|(v-\bar{v},u-1)\|_s^2(t)+\int_0^\infty(\|v_x\|_s^2+\|u-1\|_{s+1}^2)(t)dt \leq C\|(v_0-\bar{v},u_o-1)\|_s^2.$$

Case 2. *If $\varepsilon=0$ and $D>0$ then*

$$\sup_{t\geq 0}\|(v-\bar{v},u-1)\|_s^2(t)+\int_0^\infty(\|v_x\|_{s-1}^2+\|u-1\|_{s+1}^2)(t)dt \leq C\|(v_0-\bar{v},u_o-1)\|_s^2.$$

Case 3. *If $\varepsilon>0$ and $D=0$ then*

$$\sup_{t\geq 0}\|(v-\bar{v},u-1)\|_s^2(t)+\int_0^\infty(\|v_x\|_s^2+\|u-1\|_s^2)(t)dt \leq C\|(v_0-\bar{v},u_o-1)\|_s^2.$$

Theorem 6.6. *Let $\bar{v}$ be a constant and $a>0$. Let $s\geq 4$ be an integer, and $(v_0-\bar{v},u_0-1)\in H^s(\mathbb{R})\cap L^1(\mathbb{R})$. Then there exists a constant $\varepsilon>0$ such that if $\delta_0\equiv\|(v_0-\bar{v},u_0-1)\|_s+\|(v_0-\bar{v},u_0-1)\|_{L^1}\leq\varepsilon$, the solution of* (6.13), (6.14) *given in Theorem* 6.5 *has the following L^p estimate with $p\geq 2$: For $t\geq 0$,*

$$\|D_x^l(v-\bar{v},u-1)\|_{L^p}(t)\leq C\delta_0(t+1)^{-\frac{1}{2}+\frac{1}{2p}-\frac{l}{2}},\quad 0\leq l\leq s-\frac{5}{2}+\frac{1}{p},$$

$$\|D_x^l(u-1)\|_{L^p}(t)\leq C\delta_0(t+1)^{-1+\frac{1}{2p}-\frac{l}{2}},\quad 0\leq l\leq s-\frac{9}{2}+\frac{1}{p}.$$

We comment that due to the specific form of (6.13), such as the reduced system being a scalar equation hence no wave interaction between different characteristic families, we may obtain results better than what the general theory offers. For instance, in a recent paper [18], we establish for cases one and two the global existence of solution to (6.13), (6.14) under the assumption $(v_0,u_0-1)\in H^2(\mathbb{R})$ and $u_0>0$ (initial density positive), without the smallness requirement on the H^2-norm. Asymptotic behavior of solution and decay rates are also obtained. In an upcoming paper [19], we further obtain optimal time decay rates with non-small initial data for the special case when the chemical is non-diffusive while v_0 has zero mass. The latter corresponds to the initial distribution of chemical being around a constant background state, $\lim_{x\to\pm\infty}c(x,0)=\bar{c}$, where $\bar{c}>0$ is a constant.

Our final application is to polyatomic gas flows in both translational and vibrational non-equilibrium [1, 13],

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + ((\rho u^2 + p)_x = (\mu u_x)_x, \\ (\rho E)_t + (\rho E u + pu)_x = (\mu u u_x + \kappa T_{1x} + \nu \rho e_{2x})_x, \\ (\rho e_2)_t + (\rho e_2 u)_x = (\nu \rho e_{2x})_x + \rho \dfrac{e_2^* - e_2}{\tau}, \end{cases} \tag{6.15}$$

where ρ, u and p are the density, velocity and pressure, respectively. The total energy is

$$E = e + \frac{1}{2}u^2, \quad e = e_1 + e_2,$$

where the internal energy e consists of two parts: e_2 is the non-equilibrium vibrational energy, and e_1 is the rest of the internal energy.

We comment that since the relaxation time scale of the translational mode is much smaller than that of vibrational mode, through Chapman–Enskog expansion we introduce dissipation mechanisms to compensate the translational mode, and single out the vibrational mode as the non-equilibrium mode. Therefore, we need two sets of thermal dynamic variables, one for the translational mode and for all other internal modes at the same pace of the translational mode, and the other for the vibrational mode. We use subscript "1" for the former and "2" for the latter. For instance, like e_1 and e_2 introduced above, we use T_1 for the common temperature of the translational mode and all other internal modes except the vibrational mode, and T_2 for the vibrational temperature. The two sets of variables obey different thermodynamic equations as follows:

$$T_1 ds_1 = de_1 + pdv, \quad T_2 ds_2 = de_2, \tag{6.16}$$

where $v = 1/\rho$ is the specific volume, and s_1 and s_2 are the equilibrium and vibrational entropies, respectively.

The dissipation mechanisms due to the translational mode are realized by the viscosity coefficient μ, thermal conductivity κ, and self-diffusion coefficient ν. The first three equations in (6.15) are conservation of mass, momentum and energy. The last equation is the relaxation of vibrational energy to its local equilibrium state e_2^* at the time scale τ. Both e_2^* and τ are known functions of the thermal dynamic variables in set one. In view of (6.16), (6.15) is a system of four equations for four unknowns: the velocity, two thermodynamic variables in set one, and one in set two.

It has been shown in [15] that (6.15) satisfies Assumption 1.1 under physical assumptions. For this we introduce the following notations based on the relation among thermodynamic variables,

$$p = p(v, e_1) = \tilde{p}(v, T_1), \quad T_1 = T_1(v, e_1), \quad e_2 = \omega(T_2).$$

A state is an equilibrium state if and only if $T_2 = T_1$. Thus, the equilibrium manifold is characterized as

$$\mathbb{E} = \{T_2 = T_1\} \cap \mathbb{O},$$

and e_2 satisfies

$$e_2^* = \omega(T_1), \quad \bar{e}_2 = \omega(\bar{T}_1).$$

Here, we recall that the bar accent is for the constant equilibrium state $\bar{w}$. Without loss of generality, we set $\bar{u} = 0$, hence

$$\bar{w} = (\bar{\rho}, 0, \bar{\rho}\bar{e}, \bar{\rho}\bar{e}_2)^t. \tag{6.17}$$

The physical assumptions to be imposed are

$$\begin{aligned} \tilde{p}_v &= \frac{\partial}{\partial v}\tilde{p}(v, T_1) < 0, \quad T_{1e_1} = \frac{\partial}{\partial e_1}T_1(v, e_1) > 0, \\ p_{e_1} &= \frac{\partial}{\partial e_1}p(v, e_1) \neq 0, \quad \omega'(T) > 0. \end{aligned} \tag{6.18}$$

We cite [15, Propositions 4.1 and 4.2].

Proposition 6.7. *Let* (6.18) *be true, and the dissipation parameters in* (6.15) *satisfy*

$$\kappa > 0, \quad \nu \geq 0, \quad \mu \geq 0. \tag{6.19}$$

Then (6.15) *satisfies Assumption* 1.1 *in a small neighborhood* $\mathbb{O}$ *of* $\bar{w}$.

We comment that the entropy function η in condition (1) of Assumption 1.1 is $-\rho s$, where $s = s_1 + s_2$ is the physical entropy. The partition $n = n_1 + n_2$ in condition (2) is $4 = 3+1$. The diffeomorphism φ in condition (3) is $\varphi(w) = (\rho, u, T_1, e_2)^t$. The partition $n = n_3 + n_4$ is $4 = 1 + 3$ if $\nu > 0$ and $\mu > 0$; $4 = 2 + 2$ if $\nu > 0$ and $\mu = 0$, or $\nu = 0$ and $\mu > 0$; and $4 = 3+1$ if $\nu = \mu = 0$. The matrix P is the identity except when $\nu = 0$. In the latter, P is a permutation to move the rate equation above the momentum equation.

We consider the Cauchy problem of (6.15) with prescribed initial data

$$\begin{aligned} w(x, 0) &\equiv (\rho, \rho u, \rho E, \rho e_2)^t(x, 0) \\ &= \left(\rho_0, \rho_0 u_0, \rho_0\left(e_0 + \frac{1}{2}u_0^2\right), \rho_0 e_{20}\right)^t (x) \equiv w_0(x). \end{aligned} \tag{6.20}$$

Applying Theorem 1.4, Corollary 1.5 and Proposition 6.7 to (6.15), (6.20) we have the following theorem.

Theorem 6.8. *Let* $\bar{\rho}, \bar{e}_1 > 0$ *be constants,* $\bar{T}_1 = T_1(1/\bar{\rho}, \bar{e}_1)$, $\bar{e}_2 = \omega(\bar{T}_1)$ *and* $\bar{e} = \bar{e}_1 + \bar{e}_2$. *Let* (6.18) *and* (6.19) *be true,* $s \geq 4$ *be an integer, and* $w_0 - \bar{w} \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$ *for* w_0 *and* $\bar{w}$ *in* (6.20) *and* (6.17), *respectively. Then there exists a constant* $\varepsilon > 0$ *such that if* $\delta_0 \equiv \|w_0 - \bar{w}\|_s + \|w_0 - \bar{w}\|_{L^1} \leq \varepsilon$, *the Cauchy problem*

(6.15), (6.20) *has a unique solution, with* $(\rho-\bar\rho, \rho u, \rho E-\bar\rho\bar e, \rho e_2-\bar\rho\bar e_2)\in C([0,\infty); H^s(\mathbb{R}))$. *The solution satisfies the following* L^p *decay properties with* $p\ge 2$: *For* $t\ge 0$,

$$\|D_x^l(\rho-\bar\rho, \rho u, \rho E-\bar\rho\bar e, \rho e_2-\bar\rho\bar e_2)\|_{L^p}(t)\le C\delta_0(t+1)^{-\frac{1}{2}+\frac{1}{2p}-\frac{l}{2}} \tag{6.21}$$

for $0\le l\le s-5/2+1/p$, *and*

$$\|D_x^l[\rho(e_2^*-e_2)/\tau]\|_{L^p}(t)\le C\delta_0(t+1)^{-1+\frac{1}{2p}-\frac{l}{2}} \tag{6.22}$$

for $0\le l\le s-9/2+1/p$. *In particular, the* L^2 *decay rates in* (6.21) *and* (6.22) *are* $(t+1)^{-\frac{1}{4}-\frac{l}{2}}$ *and* $(t+1)^{-\frac{3}{4}-\frac{l}{2}}$, *respectively.*

Acknowledgment

This work was partially supported by the Simons Foundation (#244905 to Yanni Zeng) and the National Science Foundation (DMS-1908195).

References

[1] J. F. Clarke and M. McChesney, *Dynamics of Relaxing Gases*, 2nd edn. (Butterworths, 1976).
[2] T. Kato, *Perturbation Theory for Linear Operators*, 2nd edn. (Springer, 1976).
[3] S. Kawashima, Systems of a hyperbolic–parabolic composite type, with applications to the equations of magnetohydrodynamics, Doctoral thesis, Kyoto University (1983).
[4] S. Kawashima, Large-time behavior of solutions to hyperbolic–parabolic systems of conservation laws and applications, *Proc. Roy. Soc. Edinburgh Sect. A* **106** (1987) 169–194.
[5] S. Kawashima and W.-A. Yong, Decay estimates for hyperbolic balance laws, *Z. Anal. Anwend.* **28** (2009) 1–33.
[6] E. F. Keller and L. A. Segel, Traveling bands of chemotactic bacteria: A theoretical analysis, *J. Theor. Biol.* **30** (1971) 235–248.
[7] H. A. Levine and B. D. Sleeman, A system of reaction diffusion equations arising in the theory of reinforced random walks, *SIAM J. Appl. Math.* **57** (1997) 683–730.
[8] D. Li, R. Pan and K. Zhao, Quantitative decay of a hybrid type chemotaxis model with large data, *Nonlinearity* **28** (2015) 2181–2210.
[9] T.-P. Liu and Y. Zeng, Compressible Navier–Stokes equations with zero heat conductivity, *J. Diff. Equ.* **153** (1999) 225–291.
[10] T.-P. Liu and Y. Zeng, Shock waves in conservation laws with physical viscosity, *Mem. Amer. Math. Soc.* **234**(1105) (2015) vi+168 pp.
[11] L. Nirenberg, On elliptic partial differential equations, *Ann. Scuola Norm. Sup. Pisa* **13** (1959) 115–162.
[12] Y. Shizuta and S. Kawashima, Systems of equations of hyperbolic–parabolic type with applications to the discrete Boltzmann equation, *Hokkaido Math. J.* **14** (1985) 249–275.
[13] W. Vincenti and C. Kruger Jr., *Introduction to Physical Gas Dynamics* (Krieger, 1986).
[14] Y. Zeng, Global existence theory for a general class of hyperbolic balance laws, *Bulletin, Inst. Math. Academia Sinica* **10** (2015) 143–170.

[15] Y. Zeng, Global existence theory for general hyperbolic–parabolic balance laws with application, *J. Hyperbolic Diff. Equ.* **14** (2017) 359–391.
[16] Y. Zeng, L^p decay for general hyperbolic–parabolic systems of balance laws, *Discrete Contin. Dyn. Syst.* **38** (2018) 363–396.
[17] Y. Zeng, Asymptotic behavior of solutions to general hyperbolic–parabolic systems of balance laws in multi space-dimensions, *Pure Appl. Math. Q.* **14** (2018) 161–192.
[18] Y. Zeng and K. Zhao, On the logarithmic Keller–Segel–Fisher/KPP system, *Discrete Contin. Dyn. Syst.* **39** (2019) 5365–5402.
[19] Y. Zeng and K. Zhao, Optimal decay rates for a chemotaxis model with logistic growth, logarithmic sensitivity and density-dependent production/consumption rate, *J. Diff. Eqs.* **268** (2020) 1379–1411.