LOWER TAIL OF THE KPZ EQUATION

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Abstract
We provide the first tight bounds on the lower tail probability of the one-point distribution of the Kardar–Parisi–Zhang (KPZ) equation with narrow wedge initial data. Our bounds hold for all sufficiently large times T and demonstrates a crossover between superexponential decay with exponent $\frac{5}{2}$ (and leading prefactor $\frac{4}{15\pi}T^{1/3}$) for tail depth greater than $T^{2/3}$, and exponent 3 (with leading prefactor at least $\frac{1}{12}$) for tail depth less than $T^{2/3}$.

Contents

1. Introduction

The $(1+1)$d *stochastic heat equation (SHE)* with multiplicative space-time white noise ξ is[1]

$$\partial_T \mathcal{Z}(T,X) = \frac{1}{2}\partial_X^2 \mathcal{Z}(T,X) + \mathcal{Z}(T,X)\xi(T,X), \tag{1.1}$$

where $T \geq 0$ and $X \in \mathbb{R}$. The SHE is ubiquitous, modeling the density of particles diffusing in space-time random environments (with random killing/branching as in [94], [116] or random drifts as in [12], [42]). Via the Feynman–Kac formula, it is the partition function for the continuum directed polymer model (see, e.g., [4], [38], [84]).

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[1]The solution theory for this stochastic PDE is classical (see, e.g., [39], [127], [139]), based on Itô stochastic integrals or martingale problems.

Taking logarithms formally[2] leads to the Kardar–Parisi–Zhang (KPZ) equation

$$\partial_T \mathcal{H}(T,X) = \frac{1}{2}\partial_X^2 \mathcal{H}(T,X) + \frac{1}{2}\big(\partial_X \mathcal{H}(T,X)\big)^2 + \xi(T,X),$$

which is a paradigm for random interface growth (see [91]) and a testing ground for the study of nonlinear *stochastic PDEs (SPDEs)* (see, e.g., [69]–[71], [74], [76]). The KPZ equation's spatial derivative formally solves the stochastic Burgers equation—a continuum model for turbulence (see [18], [68]), interacting particle systems, and driven lattice gases (see [137]).

The *Cole–Hopf solution to the KPZ equation* with *narrow wedge initial data* is given by

$$\mathcal{H}(T,X) := \log \mathcal{Z}(T,X), \quad \text{with } \mathcal{Z}(0,X) = \delta_{X=0}. \tag{1.2}$$

The well-definedness of $\log \mathcal{Z}$ for all $T > 0$ and $X \in \mathbb{R}$ relies on the almost-sure strict positivity of $\mathcal{Z}$ proved in [119] to hold for a wide class of initial data (including the delta function). This is the physically relevant notion of solution and it has been shown to arise quite generally from various regularization or discretization schemes for the equation and for noise (see, e.g., [17], [20], [45], [46], [69]–[71], [74], [76], [77]). The Cole–Hopf solution also coincides with the solutions constructed from regularity structures (see [74]), paracontrolled distributions (see [70]), and energy solution methods (see [71]).

The present work establishes tight bounds on the *lower tail* probability that $\mathcal{Z}(T,X)$ is close to zero, or equivalently that $\mathcal{H}(T,X)$ is very negative.[3] The first result in this direction was the aforementioned almost sure positivity of $\mathcal{Z}$ established in [119] via large deviation bounds and a comparison principle. Using Malliavin calculus, Mueller and Nualart [120] proved a quantitative upper bound on the decay of the lower tail probability. Working with the SHE on an interval with Dirichlet boundary conditions and constant initial data, they showed that for any $\delta > 0$

[2] Due to the nonlinearity of the KPZ equation and the roughness of the white noise, it is challenging to construct a solution theory for it directly. Smoothing the noise in space, Bertini and Cancrini [17] showed that the logarithm of the smoothed noise SHE solves the KPZ equation with the same smoothed noise, up to an Itô correction whose size diverges as the smoothing disappears. More recently, the techniques of regularity structures in [74] and [75], energy solutions in [69] and [72], paracontrolled distributions in [70], and [71], and renormalization group in [104] have been used to construct the solution theory of the KPZ equation directly. These solutions all agree with the Cole–Hopf solution.

[3] To avoid confusion, let us distinguish our present investigation from earlier work of [59], [60], and [61] which studied the stochastic Burgers equation (the spatial derivative of KPZ) but with a noise which is smooth in space and white in time. In that case, which has no direct relationship to our work, the tail of the local slope has $-\frac{7}{2}$ power-law (not exponential) decay. A proxy for the question we consider here, [6] studied the tail behavior of the invicid Burgers equation with white-noise initial data, showing cubic exponential decay. That result, however, also has no direct bearing on our present work.

there are constants $c_1, c_2 > 0$ so that $\mathbb{P}(\mathcal{H}(T, X) \leq -s) \leq c_1 \exp(-c_2 s^{\frac{3}{2}-\delta})$. Using Talagrand's concentration of measure methods, Moreno Flores [117] improved the exponent. In particular, he considered the full-line SHE with $\mathcal{Z}(0, X) = \delta_{X=0}$ initial data (this is the setting we address here) and he proved a similar bound to [120] but with the $3/2 - \delta$ exponent replaced by the Gaussian exponent 2. Quite recently, the use of Malliavin calculus [82] extended these sort of results to noises with more general covariance structure. There is some work in progress [95] which seeks to use stochastic analytic methods to prove a lower bound with exponent $\frac{5}{2}$ on this tail probability. As we prove here, the exponents accessed in earlier work are not optimal and, moreover, these previous results are (in a sense we now describe) not well adapted to study the long (or intermediate) time solution tail.

When time increases, the KPZ equation shows an overall decay at linear rate $-T/24$ with fluctuations which grows like $T^{1/3}$. The first author with Amir and Quastel [4] proved (see also [130] for a less rigorous treatment done in parallel, and [34], [57] for physics results) that when $\mathcal{Z}(0, X) = \delta_{X=0}$,

$$\lim_{T\to\infty} \mathbb{P}(\Upsilon_T \leq s) = F_{\mathrm{GUE}}(s), \quad \text{where } \Upsilon_T := \frac{\mathcal{H}(2T, 0) + \frac{T}{12}}{T^{\frac{1}{3}}}. \tag{1.3}$$

The $T^{1/3}$ scaling is a characteristic of models in the KPZ universality class, as is the limiting *Gaussian unitary ensemble (GUE)* Tracy–Widom distribution $F_{\mathrm{GUE}}(s)$ (see [39]). We consider $\mathcal{H}$ at time $2T$ to simplify some factors of 2 in formulas. Reinterpreting the tail bounds of [120] and [117] in terms of the lower tail of Υ_T, one sees that their effectiveness degrades as T grows (i.e., they do not reflect the centering or scaling associated with the long-time fluctuations).

While the distributional limit in (1.3) does not control the tails of Υ_T for finite T, it does suggest a natural conjecture. For s large, $F_{\mathrm{GUE}}(-s) \approx e^{-\frac{1}{12}s^3}$ (see Proposition 5.1 and [7], [128], [134]). Thus one might expect a similar lower tail bound for Υ_T, at least for large enough T. As we prove in Theorem 1.1, this is only half true. In fact, there are two types of decay regimes for the lower tail $\mathbb{P}(\Upsilon_T < -s)$: for $T^{2/3} \gg s \gg 0$, a cubic exponent controls the tail decay, whereas for $s \gg T^{2/3}$, the tail exponent becomes $\frac{5}{2}$ and the leading constant in the exponential is $\frac{4}{15\pi} T^{1/3}$ instead of $\frac{1}{12}$ in the first regime (in fact, in Theorem 1.1 we are only able to lower bound the prefactor to the cubic exponent by $\frac{1}{12}$).

The existence of these two regimes has been discussed extensively in the physics literature for many years. The KPZ equation is believed to be the unique heteroclinic orbit between the Edwards–Wilkinson (or weak coupling) and KPZ (or strong coupling) fixed points (see [111]). The cubic exponent is (as explained above) representative of the KPZ fixed point behavior. On the other hand, the Edwards–Wilkinson fixed point is described by a Gaussian process, suggesting naively a tail exponent of

2, not $\frac{5}{2}$. The $\frac{5}{2}$ exponent was first demonstrated in [97]–[99] (see also more recent works of [89], [113] and the discussion in the footnote of Section 2.3) by studying the short-time deep tail. Since the Gaussian Edwards–Wilkinson fixed point arises as a limit of the short-time shallow tail, there is no contradiction.

Studying the short-time deep tail is equivalent to putting a small constant in front of the noise. In that case, it is possible to reformulate the tail behavior in terms of a large deviation problem for the underlying space-time white noise and then to optimize over all possible instances of the noise which realize the desired one-point deviation. Although this approach only applies to short time, quite interestingly the behavior it predicts seems to remain valid for all times, provided one goes deep enough into the tail.

This *weak noise theory* (WNT) or *optimal fluctuation theory* generalizes Freidlin–Wentzell theory for stochastic differential equations. In the mathematics literature, it is only recently that this sort of approach for nonlinear SPDEs has begun to be put on a rigorous mathematical footing. Namely, [78] and [35] take the first step of this approach (computing the rate function for a given space-time trajectory) for certain SPDEs (not presently including KPZ) in a periodic setting. Besides adapting this to the KPZ equation on the line, there is significant work needed to extract close-form one-point tail behavior by optimizing over all possible space-time trajectories.

While the cubic shallow tail exponent should be quite universal (i.e., for all KPZ class models), the $\frac{5}{2}$ exponent's universality is much less clear. For instance, for some discrete KPZ class models like last passage percolation with bounded entries, the very deep tail will be controlled by the tail behavior of the underlying noise and it is unlikely to conform with this exponent. On the other hand, the $\frac{5}{2}$ exponent seems to show up when studying the total current of certain periodic space KPZ class particle systems (i.e., in terms of height functions, this corresponds with the spatial average height function). This behavior was first demonstrated in the physics literature for *totally asymmetric simple exclusion process (TASEP)* in [55] and then extended to *asymmetric simple exclusion process (ASEP)* in [54] and tested numerically for other models in [108]. Studying the tail behavior for other integrable models (as discussed in Section 2.6) and extending the weak noise theory to other systems should help to shed further light on the generality of this $\frac{5}{2}$ exponent.

Between the cubic and $\frac{5}{2}$ exponent regimes, there is a crossover in the tail of Υ_T at depth of order $T^{2/3}$, or for $\mathcal{H}(2T, 0)$ at a depth of order T. As T goes to infinity, this crossover corresponds with the large deviation rate function (generally denoted by $\Phi_-(z)$) for the KPZ equation which has speed T^2 (see Section 2.3). Recently there has been significant work focused on determining this rate function. The authors of [131] computed $\Phi_-(z)$ (see (2.5) for the formula) and showed that it crosses over between cubic and $\frac{5}{2}$ power-law behavior as $|z|$ moves from near zero to near infinity. That calculation involves a nonrigorous Wentzel–Kramers–Brillouin (WKB) approximation analysis of an integro-differential generalization of the Painlevé II equation which [4] related to the distribution of the KPZ equation. Making this WKB approximation rigorous seems to require solving an inverse scattering problem—see Section 2.1 for further discussion on this.

In Section 2.3 we outline another approach to computing $\Phi_-(z)$ via large deviations for the Airy point process. A month after posting the first version of this article, along with P. Le Doussal, A. Krajenbrink, and L.-C. Tsai, we filled in the details of this outline. In particular, we conjectured an electrostatic formula for this rate function of the empirical measure for the Airy point process and solved the resulting variation problem. This approach (which has not yet been made rigorous) yields the same formula for $\Phi_-(z)$ as in [131].

The first rigorous proof of the rate function $\Phi_-(z)$ came half a year after the first version of our article posted, and it is due to L.-C. Tsai [136]. Instead of proving a general Airy point process large deviation principle, he considered just the special test function (namely, $a \mapsto (a-z)_+$ discussed in Section 2.3) which arises in this application and uses the stochastic Airy operator representation for the point process to reduce the computation of $\Phi_-(z)$ into a large deviation result for the underlying Brownian motion driving that operator. This approach is delicate and relies on a special property of the test function which does not hold in general. Besides the three approaches mentioned so far (integro-differential equation, electrostatic Airy rate function, and stochastic Airy operator), there is another nonrigorous approach introduced recently in [100] which relies on computing cumulants of certain linear statistics of the Airy point process. All four approaches are discussed in [103].

Before stating our main result, it should be emphasized that the tail result which we prove, and the abovementioned large deviation rate function results are complementary in their nature. Our result demonstrates that for all time there are precisely two regimes of tail decay (and gives their behavior), and the large deviation results identify the long-time limit of the crossover mechanism between these two regimes.

We now state our main result.

THEOREM 1.1

Let Υ_T denote the centered and scaled KPZ solution with narrow wedge initial data as in (1.2). Fix $\epsilon, \delta \in (0, \frac{1}{3})$ and $T_0 > 0$. Then there exist $S = S(\epsilon, \delta, T_0)$, $C = C(T_0) > 0$, $K_1 = K_1(\epsilon, \delta, T_0) > 0$, and $K_2 = K_2(T_0) > 0$ such that for all $s \geq S$ and $T \geq T_0$,

$$\mathbb{P}(\Upsilon_T \leq -s) \leq e^{-\frac{4(1-C\epsilon)}{15\pi} T^{\frac{1}{3}} s^{\frac{5}{2}}} + e^{-K_1 s^{3-\delta} - \epsilon T^{1/3} s} + e^{-\frac{(1-C\epsilon)}{12} s^3} \tag{1.4}$$

and

$$\mathbb{P}(\Upsilon_T \le -s) \ge e^{-\frac{4(1+C\epsilon)}{15\pi} T^{\frac{1}{3}} s^{\frac{5}{2}}} + e^{-K_2 s^3}. \tag{1.5}$$

We prove this in Section 3. Note that the right-hand side of (1.4) is a sum of three terms. The first dominates the other two when $s \gg T^{\frac{2}{3}}$. In the region $T^{\frac{2}{3}} \gg s \gg 0$, the second and third terms dominate, and when $T \to \infty$, the third dominates the second and recovers the $\frac{1}{12}s^3$ tail behavior of the GUE Tracy–Widom distribution. There is a similar interplay between the two terms in (1.5), although in this lower bound we do not recover the $\frac{1}{12}$ constant as $T \to \infty$.[4]

In a follow-up work [40], we have extended the upper bound (1.4) to general initial data. We do not yet have a matching lower bound, and we expect that the coefficients in the exponents will depend on the initial data. That work relies on Theorem 1.1 as an input and also uses the Brownian Gibbs property for the KPZ line ensemble from [43].

We now briefly explain the three steps in our proof, although to simplify the exposition we will leave off the ϵ and δ's which are present in the statement and proof.

Step 1: The first step in our proof is to reduce the study of the lower tail asymptotics for the KPZ equation to the large-parameter (u in (1.6)) asymptotics of the SHE Laplace transform. This is the content of the proof of Theorem 1.1, and the desired SHE Laplace transform asymptotics are then recorded as Proposition 3.1. The fundamental identity which allows us to prove these asymptotics in Proposition 3.1 is the one-point formula (see, e.g., [4], [34], [57], [130]). Recently, [26] reformulated that result as an identity between the Laplace transform of the SHE and the expectation of a specific multiplicative functional of the Airy point process (see Proposition 1.2). Armed with this, we reduce Proposition 3.1 to Proposition 4.2 which studies Airy point process asymptotics and whose proof is the main technical feat here.

Step 2: The proof of Proposition 4.2 relies on three results (Theorems 1.4, 1.5, and 1.6) about large deviations of the number of Airy points in large intervals and their rigidity around typical locations. Theorems 1.4 and 1.5 respectively probe the lower and upper large deviation tails for the fluctuations of the number of Airy points in a large interval $[-s, \infty)$. The mean number of points grows[5] (see Proposition 1.3) like $\frac{2}{3\pi}s^{\frac{3}{2}}$ and these theorems probe the probability of finding a different constant than $\frac{2}{3\pi}$. On the lower tail, Theorem 1.4 shows that the exponential decay power law has exponent 3, while Theorem 1.5 shows that the corresponding upper tail exponent is $\frac{3}{2}$. To our knowledge, such large deviation results are new for the Airy point process (see

[4]We expect this is just a limitation of our result and would follow from a finer analysis.
[5]The variance grows like $\log(s)$ and the fluctuations satisfy a central limit theorem in this scale; see [132].

Sections 1.2 and 2.3 for further discussion). Theorem 1.6 controls the maximum (over the entire Airy point process) deviation of points outside bands around their typical locations. We do not expect this result to be nearly as tight as Theorems 1.4 and 1.5, but it will suffice for our purposes. Using these three theorems we can establish control over the probabilities of various scenarios for the Airy point process and hence establish precise upper and lower bounds on the expectation value needed to prove Proposition 4.2.

Step 3: The proofs of Theorems 1.4, 1.5, and 1.6 are each rather different. The first two rely on the determinantal structure of the Airy point process (see Section 4.1), while the third uses its relation to the stochastic Airy operator (see Section 4.3). The proof of Theorem 1.4 is technically the most challenging. Via Markov's inequality, it reduces to a bound on the cumulant generating function for the number of Airy points in the interval $[-s, \infty)$, when the parameter v of the generating function is of order $s^{\frac{3}{2}}$ (see Section 1.3). Theorem 1.7 relates (via standard determinantal methods) this generating function $F(x; v)$ to the Ablowitz–Segur solution of the Painlevé II equation, and then proves the needed decay bound on the generating function using a delicate analysis of an asymptotic formula (given in recent work in [30] in terms of oscillatory Jacobi elliptic functions) for this solution to Painlevé II. The proof of Theorem 1.5 is considerably simpler. It uses the fact that the number of Airy points in an interval equals (in law) the sum of independent Bernoulli random variables (with parameters related to the eigenvalues of the Airy kernel projected onto the interval). The theorem follows by combining Bennett's concentration inequality on such sums, along with estimates on mean and variance given in Proposition 1.3. Theorem 1.6 uses the identity in law (see Proposition 4.4) between the Airy point process and the spectrum of the stochastic Airy operator. The typical locations of the Airy points are given by the zeros of the Airy function, and the estimate on uniform deviations from bands around those typical locations can be reduced (through operator manipulations such as those used in [128], [138]) to an exponential tail estimate (proved in Lemma 4.7) of the maximum oscillation of Brownian motion.[6]

The rest of this Introduction records the main results (summarized above) which go into our proof of Theorem 1.1. Section 1.1 provides the key identity relating the Laplace transform of the SHE and the expectation of a multiplicative functional of the Airy point process. Section 1.2 records the Airy point process large deviation and rigidity estimates upon which we rely. Section 1.3 records the precise asymptotics of the Ablowitz–Segur solution of the Painlevé II equation needed in the proof of Theorem 1.4.

[6]The Brownian motion is the driving noise for the stochastic Airy operator.

1.1. Laplace transform formula

The starting point for our study is the exact formula characterizing the one-point distribution of the SHE with delta initial data. This was simultaneously and independently computed in [4], [34], [130], and [57] (rigorous proof provided in [4]). That formula can, by straightforward manipulations, be reformulated (Proposition 1.2) in terms of the expectation of a multiplicative functional of the Airy point process (Section 4). This was done in [26, Theorem 2.2], and the resulting formula offers a major benefit since it enables one to bring to bear on the KPZ equation the vast range of tools and understanding developed for the Airy point process. In fact, prior to our present work, it was not clear how to prove directly that the formula in [4], [34], [130], and [57] defines a probability distribution.[7] Proposition 1.2 makes such a result immediate, and the lower tail decay becomes more tractable.

Proposition 1.2 is a special limit case of a general matching between stochastic vertex models and Macdonald measures in [23, Corollary 4.4]. In special cases, the Macdonald measures reduce to determinantal Schur measures and hence are analyzable in the spirit of our work here (see [11], [24], [27] or Sections 2.5 and 2.6 for further discussion).

PROPOSITION 1.2 ([26, Theorem 2.2])
Let $\mathcal{Z}(T,X)$ be the unique solution to the SHE (1.1) with $\mathcal{Z}(0,X)=\delta_{X=0}$. Denote the ordered points of the Airy point process (Section 4) by $\mathbf{a}_1>\mathbf{a}_2>\cdots$. Then for any $T,u>0$, we have[8]

$$\mathbb{E}_{\mathrm{SHE}}\Big[\exp\Big(-u\mathcal{Z}(2T,0)\exp\Big(\frac{T}{12}\Big)\Big)\Big]=\mathbb{E}_{\mathrm{Airy}}\Big[\prod_{k=1}^{\infty}\frac{1}{1+u\exp(T^{\frac{1}{3}}\mathbf{a}_k)}\Big]. \tag{1.6}$$

Setting $u=\exp(T^{\frac{1}{3}}s)$ and rewriting the above result in terms of Υ_T from (1.3), we find that

$$\mathbb{E}_{\mathrm{SHE}}\big[\exp(-\exp\big((T^{\frac{1}{3}}(\Upsilon_T+s)\big))\big]=\mathbb{E}_{\mathrm{Airy}}\Big[\prod_{k=1}^{\infty}\frac{1}{1+\exp(T^{\frac{1}{3}}(s+\mathbf{a}_k))}\Big]. \tag{1.7}$$

Let G be a Gumbel random variable. Then the function $\exp(-\exp(x))$ is equal to $\mathbb{P}(G<-x)$. Armed with this, one can now see that the left-hand side of (1.7) is equal to $\mathbb{P}(\Upsilon_T+T^{-\frac{1}{3}}G<-s)$, where Υ_T and G are independent. Due to the rapid decay of the lower tail probability of a Gumbel distribution,[9] when s is large and T is greater than some $T_0>0$, $\mathbb{P}(\Upsilon_T+T^{-\frac{1}{3}}G<-s)$ is approximately $\mathbb{P}(\Upsilon_T<-s)$ which is

[7] The hard part is to prove that the lower tail probability decays to zero.
[8] A similar result holds for any X up to multiplying $\mathcal{Z}$ by a Gaussian factor; see [4, Proposition 1.4].
[9] The lower tail probability of a Gumbel distribution has double exponential decay.

exactly the tail we are looking to control. Now consider the right-hand side of (1.7). If $s + \mathbf{a}_k \gg 0$, then the corresponding term in the product will be exponentially small, whereas if $s + \mathbf{a}_k \ll 0$, then the term will be very close to 1. Thus, the tail decay on the left-hand side is linked with the number of exponentially small terms (and their exponential factors) on the right-hand side.

Typically, the Airy point process is close to the zeros of the Airy function (Proposition 4.5), and hence $\mathbf{a}_k \sim -(\frac{3\pi}{2}k)^{\frac{2}{3}}$ (Proposition 4.6). Plugging in this estimate readily yields decay like $\exp(-\frac{4}{15\pi}T^{\frac{1}{3}}s^{\frac{5}{2}})$. The Airy points may, however, differ from these typical locations. For instance, $\mathbf{a}_1$ (which is GUE Tracy–Widom distributed) may dip below $-s$ in which case the product in the expectation on the right-hand side of (1.7) becomes very close to 1. The probability of such a drastic dip behaves like $\exp(-\frac{1}{12}s^3)$. Of course, there are many other scenarios in which the Airy points deviate from their typical locations in less drastic ways, and the contributions of those to the overall expectation need to be controlled and ultimately contribute to the other terms in our bounds in Theorem 1.1. We give a brief overview of this in Section 1.2. Proposition 3.1 (which follows directly from Proposition 4.2) contains precise statements of the bounds that we prove on the behavior of the right-hand side of (1.7).

1.2. Rigidity bounds for Airy point process

The Airy point process $\mathbf{a}_1 > \mathbf{a}_2 > \cdots$ (see Section 4) is a determinantal point process on the real line introduced by Tracy and Widom [134] as the scaling limit of the edge of the spectrum of the GUE. There they found that $F(s) := \mathbb{P}(\mathbf{a}_1 < s)$ can be written in terms of the Hastings–McLeod (HM) solution of Painlevé II:

$$F(s) = \exp\Big(-\int_x^\infty (y-x)u_{\mathrm{HM}}^2(y)\,dy\Big), \tag{1.8}$$

where $u_{\mathrm{HM}}(y)$ is the solution of the Painlevé II equation (introduced in [124], [125]; see also the review [67]) with specific boundary behavior as $x \to \infty$:

$$\begin{aligned} u''_{\mathrm{HM}} &= xu_{\mathrm{HM}} + 2u_{\mathrm{HM}}^3, \quad (') = \frac{d}{dx}, \\ u_{\mathrm{HM}}(x) &= \frac{x^{-\frac{1}{4}}}{2\sqrt{\pi}}e^{-\frac{2}{3}x^{\frac{3}{2}}}\big(1+o(1)\big), \quad \text{as } x \to \infty. \end{aligned} \tag{1.9}$$

This solution was introduced by Hastings and McLeod in [81], where they determined an asymptotic formula for $u_{\mathrm{HM}}(x)$ as $x \to -\infty$ (this is called solving the *connection problem*). Plugging this into (1.8) allowed [134] to demonstrate that $F(-s)$ decays like $\exp(-\frac{s^3}{12})$. Later, using the nonlinear steepest descent technique pioneered by Deift and Zhou [52], the authors of [50], [53], and [7] determined smaller-order terms in the asymptotic expansion of $F(-s)$. Similar results have been established for other determinantal point processes related to KPZ class models (see, e.g., [8], [10]).

In order to make rigorous the heuristic described in the last section, we need to establish sufficiently precise control over the deviations of a large number of Airy points from their typical locations. Controlling deviations of eigenvalues from their typical locations is a central theme in some works on random matrix universality (see, e.g., [63], [64] and subsequent works). We require very precise upper and lower bounds on large deviations which do not seem to be present in the existing literature. In fact, we must ultimately rely upon the Ablowitz–Segur solution of Painlevé II to establish suitably precise bounds.

Our rigidity bounds are established in terms of counting Airy points in intervals. Define

$$\chi^{\mathrm{Ai}} : \mathcal{B}(\mathbb{R}) \to \mathbb{Z}_{\geq 0}, \qquad \chi^{\mathrm{Ai}}(B) := \#\{k : \mathbf{a}_k \in B\}, \quad \forall B \in \mathcal{B}(\mathbb{R}),$$

where $\mathcal{B}(\mathbb{R})$ denotes the Borel σ-algebra of $\mathbb{R}$. The cumulants of $\chi^{\mathrm{Ai}}(B)$ have been studied in [132] when the Borel set B is a semi-infinite interval of the form $[-s, \infty)$ or a finite interval of the form $[-ks, -(k-1)s)$. Following [132, Theorem 1],[10] we can record the following formulas for the expectation and the variance of the random variable $\chi^{\mathrm{Ai}}(B)$.

PROPOSITION 1.3
Define intervals $\mathfrak{B}_k(s) := [-ks, -(k-1)s)$ *for* $k \in \mathbb{Z}_{>1}$ *and* $\mathfrak{B}_1(s) := [-s, \infty)$*. For any* $s > 0$*,*

$$\mathbb{E}_{\mathrm{Airy}}\big[\chi^{\mathrm{Ai}}\big(\mathfrak{B}_1(s)\big)\big] = \frac{2}{3\pi} s^{\frac{3}{2}} + \mathfrak{D}(s),$$

$$\mathrm{Var}_{\mathrm{Airy}}\big(\chi^{\mathrm{Ai}}\big(\mathfrak{B}_1(s)\big)\big) = \frac{3}{4\pi^2} \log(s) + \mathfrak{E}(s),$$

where $\sup_{s\geq 0} |\mathfrak{D}(s)| < \infty$*, and* $\sup_{s\geq 0} |\mathfrak{E}(s)| < \infty$*. For any* $k \in \mathbb{Z}_{>1}$*, there exists* $s_0 = s_0(k) > 0$ *and* $C = C(k) > 0$ *such that for all* $s > s_0$*,*

$$\mathrm{Var}_{\mathrm{Airy}}\big(\chi^{\mathrm{Ai}}\big(\mathfrak{B}_k(s)\big)\big) \leq C \log(s).$$

[10] As pointed out by Hägg [73, footnote, p. 16], there is a mistake in the calculation leading up to [132, Theorem 1] which erroneously produced the constant $\frac{11}{12\pi^2}$ instead of the correct constant $\frac{3}{4\pi^2}$ for the variance of $\chi^{\mathrm{Ai}}(\mathfrak{B}_1(s))$ recorded in Proposition 1.3. In fact, for our purposes, the exact value of this constant is unnecessary. For the variance of $\chi^{\mathrm{Ai}}(\mathfrak{B}_1(s))$ with $k \in \mathbb{Z}_{>1}$, the constant should also be modified from that in [132, Theorem 1]. Since [73] does not provide the fixed value in this case (and to avoid redoing the calculation), we can easily argue that there must be some constant $c(k) > 0$ such that $\mathrm{Var}_{\mathrm{Airy}}(\chi^{\mathrm{Ai}}(\mathfrak{B}_k(s))) \leq c(k)\log(s)$. To see this, define $X = \chi^{\mathrm{Ai}}(\mathfrak{B}_1(ks))$, $X = \chi^{\mathrm{Ai}}(\mathfrak{B}_1((k-1)s))$, and $Z = \chi^{\mathrm{Ai}}(\mathfrak{B}_k(s))$. Since $X = Y + Z$, $\mathrm{Var}_{\mathrm{Airy}}(X) = \mathrm{Var}_{\mathrm{Airy}}(Y) + \mathrm{Var}_{\mathrm{Airy}}(Z) + \mathrm{Cov}_{\mathrm{Airy}}(Y, Z)$. Using the bounds for the variance of X and Y, and the inequality $|\mathrm{Cov}(Y, Z)| \leq \sqrt{\mathrm{Var}(Y)\,\mathrm{Var}(Z)}$, we arrive at the desired type of bound for $\mathrm{Var}(Z)$. (We thank Thomas Claeys for pointing us to the work of [73].)

These estimates can be used to prove a central limit theorem for linear statistics (including the number of particles in large intervals) for the Airy point process (see, e.g., [132]). By studying higher-order cumulants, [56, Theorem 5.2] derives a moderate deviation result for $\chi^{\mathrm{Ai}}(B_n)$ where $B_n := [-n, n]$. However, their result does not probe far enough into the tails of the distribution (it is still effectively Gaussian) to be of use in our desired application.

The theorems which we now state effectively show that the deviations of $\chi^{\mathrm{Ai}}([-s,\infty))$ have the same exponential order (up to some small correct terms) tail behavior as the deviations of $\mathbf{a}_1$. In other terms, the probability of having far too few or far too many points in a large interval is similar to the probability of having the first point far to the left or to the right.

THEOREM 1.4

For any $\delta \in (0, \frac{2}{5})$, there exist $s_0 = s_0(\delta) > 0$ and $K = K(\delta) > 0$ such that for all $s \geq s_0$ and $c > 0$,

$$\mathbb{P}\big(\chi^{\mathrm{Ai}}\big([-s,\infty)\big) - \mathbb{E}[\chi^{\mathrm{Ai}}\big([-s,\infty)\big)] \leq -cs^{\frac{3}{2}}\big) \leq \exp\big(-cs^{3-\delta}(1 - Ks^{-\frac{2\delta}{11}})\big). \quad (1.10)$$

THEOREM 1.5

Recall $\mathfrak{B}_k(s)$ from Proposition 1.3. Fix any $k \in \mathbb{Z}_{\geq 1}$, $c > 0$, and $\epsilon \in (0,1)$. Then there exists $s = s_0(k,\epsilon)$ such that for all $s \geq s_0$,

$$\begin{aligned}\mathbb{P}\big(\chi^{\mathrm{Ai}}\big(\mathfrak{B}_k(s)\big) - \mathbb{E}\big[\chi^{\mathrm{Ai}}\big(\mathfrak{B}_k(s)\big)\big] \geq cs^{\frac{3}{2}}\big) \\ \leq \exp\big(-cs^{\frac{3}{2}}\big(\log(cs^{\frac{3}{2}}) - (1+\epsilon)\log\big(\log(s)\big)\big)\big).\end{aligned}$$

Theorems 1.4 and 1.5 are, respectively, proved in Sections 4.5 and 4.6. The proof of Theorem 1.4 is based on a connection between the cumulant generating function of $\chi^{\mathrm{Ai}}([-s,\infty))$ and the Ablowitz–Segur solution of Painlevé II (see Section 1.3). The proof of Theorem 1.5 is simpler, relying on estimates in Proposition 1.3 along with Bennett's concentration inequality.

In addition to controlling the number of Airy points in large intervals, we require some uniform bound on the distance between the points and their typical locations. Let $\lambda_1 < \lambda_2 < \cdots$ denote the eigenvalues of the Airy operator (see Section 4.3). As shown in Proposition 4.6, $\lambda_n \approx (\frac{3\pi}{2}n)^{\frac{2}{3}}$. The following result follows directly from combining Proposition 4.5 with $\beta = 2$, and Proposition 4.4. Proposition 4.5 is a similar bound for the Airy$_\beta$ point process, and its proof relies on studying the spectrum of the stochastic Airy operator. To control the deviations of that random operator's spectrum, we prove a result (Lemma 4.7) which precisely controls the oscillations of Brownian motion. We do not claim that the next rigidity result is optimal and it may be possible to prove similar (or better) results about uniform rigidity of Airy points via other methods (see, e.g., [32, Theorem 3.1]).

THEOREM 1.6
For $\epsilon \in (0,1)$*, let* C_ϵ^{Ai} *be the smallest real number such that for all* $k \geq 1$*,*

$$(1-\epsilon)\boldsymbol{\lambda}_k - C_\epsilon^{\mathrm{Ai}} \leq -\mathbf{a}_k \leq (1+\epsilon)\boldsymbol{\lambda}_k + C_\epsilon^{\mathrm{Ai}}. \tag{1.11}$$

Then for all $\epsilon, \delta \in (0,1)$ *there exist* $s_0 = s_0(\epsilon,\delta)$ *and* $\kappa = \kappa(\epsilon,\delta)$ *such that for* $s \geq s_0$*,*

$$\mathbb{P}(C_\epsilon^{\mathrm{Ai}} \geq s) \leq \kappa \exp(-\kappa s^{1-\delta}). \tag{1.12}$$

1.3. Asymptotics of the Ablowitz–Segur solution of Painlevé II
The proof of Theorem 1.4 relies on Markov's inequality which shows that for any $v > 0$,

$$\begin{aligned}\mathbb{P}\big(\chi^{\mathrm{Ai}}\big([-s,\infty)\big) - \mathbb{E}[\chi^{\mathrm{Ai}}\big([-s,\infty)\big)] \leq -cs^{\frac{3}{2}}\big) \\ \leq e^{-cvs^{\frac{3}{2}} + v\mathbb{E}[\chi^{\mathrm{Ai}}([-s,\infty))]} F(-s;v),\end{aligned} \tag{1.13}$$

where

$$F(x;v) := \mathbb{E}[\exp\big(-v\chi^{\mathrm{Ai}}\big([x,\infty)\big)\big)].$$

In (1.13) we choose $v = s^{\frac{3}{2}-\delta}$. In order to extract asymptotics of $F(x;v)$ (see Theorem 1.7), we rely on a connection to the Ablowitz–Segur (AS) solution of the Painlevé II equation.

The Ablowitz–Segur (AS) solution $u_{\mathrm{AS}}(\cdot;\gamma)$ of the Painlevé II equation is a one-parameter family of solutions to (1.9) characterized by the following boundary condition:

$$u_{\mathrm{AS}}(x;\gamma) = \sqrt{\gamma}\frac{x^{-\frac{1}{4}}}{2\sqrt{\pi}} e^{-\frac{2}{3}x^{\frac{3}{2}}}\big(1+o(1)\big) \quad \text{as } x \to \infty.$$

(Here $o(1)$ means any function which goes to zero as $x \to \infty$.) For fixed $\gamma \in (0,1)$, [1] and [2] solved the connection problem (behavior as $x \to -\infty$). The case $\gamma = 1$ is the Hastings–McLeod solution analyzed in [81], and the case in which $\gamma > 1$ was subsequently studied in [90].

THEOREM 1.7
For K^{Ai} *the Airy point process correlation kernel (see Section 4) and* $\gamma = 1 - e^{-v}$*,*

$$F(x;v) = \det(I - \gamma K^{\mathrm{Ai}})_{L^2([x,\infty))} = \exp\Big(-\int_x^\infty (y-x)u_{\mathrm{AS}}^2(y;\gamma)\,dy\Big). \tag{1.14}$$

Fix any $\delta \in (0, \frac{2}{5})$, *and set* $v = s^{\frac{3}{2}-\delta}$. *Then, as* s *goes to* ∞,

$$\log F(-s; v) \leq -\frac{2}{3\pi} v s^{\frac{3}{2}} + \mathcal{O}(s^{3-\frac{13\delta}{11}}). \tag{1.15}$$

The first part of this result, (1.14), contains two equalities. The first follows from general theory relating multiplicative functions of determinant point processes to Fredholm determinants (see [5, Section 3.4] for background on Fredholm determinants). For a determinantal point process X with state space $\mathcal{X}$ and correlation kernel K^X, and a function $\phi : \mathcal{X} \to \mathbb{C}$,

$$\mathbb{E}\Big[\prod_{x \in X} \phi(x)\Big] = \det\big(1 - (1-\phi)K^X\big)_{L^2(\mathcal{X})}. \tag{1.16}$$

This identity requires $(1-\phi)K^X$ to be trace-class (see [22] for more details). The second equality in (1.14) relies on the integrable structure of the Airy kernel (see [134, Section 1.C]).

Proving the second part of the theorem, namely, (1.15), requires a close analysis of the AS solution to Painlevé II, as is provided in Section 6.

The AS solution has received some attention recently in [21] and [31] due to the fact that γK^{Ai} represents the kernel for a *thinned* version of the Airy point process—each particle is removed with probability $1-\gamma$. This thinning represents one way to achieve a crossover between the GUE Tracy–Widom distribution and more classical extreme-value statistics. The study of positive temperature free fermions in Section 2.2 represents another such mechanism.

Ablowitz and Segur [1], [2] solved the connection problem for the AS solution for $\gamma \in (0, 1)$ fixed. For our application, γ (or, equivalently, v) fixed would only yield an exponent of $s^{\frac{3}{2}-\delta}$ in Theorem 1.4 (not the desired $s^{3-\delta}$). Recently, utilizing Riemann–Hilbert steepest descent, Bothner [30] computed the asymptotic form of the AS solution $u_{\mathrm{AS}}(x; \gamma)$ as $x \to -\infty$ for a more general range of γ. The formulas are written in terms of Jacobi elliptic theta functions and take different forms depending on the values of γ. In particular, setting $\tau := -\frac{1}{(-x)^{3/2}} \log(1-\gamma)$, [30] computes asymptotic formulas in three different ranges of parameters: (a) $\tau \in (0, (-x)^{-\delta}]$; (b) $\tau \in (0, \frac{2}{3}\sqrt{2} - \eta]$; (c) $\tau \in (\frac{2}{3}\sqrt{2} - \aleph \frac{\log(-x)^{\frac{3}{2}}}{(-x)^{\frac{3}{2}}}, \infty)$. Here $\delta, \eta > 0$ are arbitrary small numbers and $\aleph \in (-\infty, \frac{7}{6}]$. For $\tau \in (0, \frac{2}{3}\sqrt{2})$ the resulting asymptotic form of $u_{\mathrm{AS}}(x; \gamma)$ as $x \to -\infty$ is pseudoperiodic, thus making it rather challenging to compute the integral in the exponential in (1.14) (as necessary to recover asymptotics for $F(x; v)$). As τ approaches 0 and $\frac{2}{3}\sqrt{2}$ the oscillations die out, although due to different mechanisms in each case.

In [30] and [29], Bothner managed to translate his asymptotic result for u_{AS} into a corresponding result for F only in the (c) region.[11] For region (a), [30] demonstrated a simplified form of $u_{\mathrm{AS}}(x;\gamma(x))$ for $\tau\in(0,(-x)^{-\delta})$ for any fixed $\delta>0$. However, this simplified form still retains its oscillatory nature which is one of the difficulties in getting a full expansion for $F(-s;1-e^{-s^{3/2-\eta}})$. Recently, Bothner and Buckingham [31] showed that for any $0<\epsilon<\frac{1}{2}$, there exist constants $s_0=s_0(\epsilon)$ and $c_j'=c_j'(\epsilon)$ for $j=1,2$ so that for $s\geq s_0$ and $0\leq v=-\log(1-\gamma)<s^{\frac{1}{2}-\epsilon}$,

$$\log F(-s;v)=-\frac{2v}{3\pi}s^{\frac{3}{2}}+\frac{v^2}{4\pi^2}\log(8s^{\frac{3}{2}})+\log\Big(G\Big(1+\frac{\mathbf{i}v}{2\pi}\Big)G\Big(1-\frac{\mathbf{i}v}{2\pi}\Big)\Big)+r(s,v). \tag{1.17}$$

Here $G(x)$ is the Barnes G-function and $|r(s,v)|\leq c_1'\frac{v^3}{s^{\frac{3}{2}}}+c_2'\frac{v}{s}$ for all $s\geq s_0$, $0\leq v\leq s^{\frac{1}{2}-\epsilon}$.

Since (1.17) gives the full expansion of $\log F(-s;s^{\frac{3}{2}-\delta})$ only when $\delta>\frac{2}{3}$, plugging it into the right-hand side of (1.13) only yields a leading term (in the upper bound of the lower tail probability of $\chi^{\mathrm{Ai}}([-s,\infty)))$ like $\exp(-cs^{2-})$. However, Theorem 1.4 asks that the upper bound be like $\exp(-cs^{3-})$. In Section 6 we demonstrate how we can work with δ close to zero. Presently, we cannot justify a full expansion of $F(-s;v)$ in Theorem 1.7 like that of (1.17). However, the weaker result in Theorem 1.7 suffices for our present needs.

Outline

The organization of this article is the following. Section 2 includes a brief discussion of how our results and methods connect to other problems and may be extended in other directions. Section 3 reduces the proof of our main result (Theorem 1.1) to a result (Proposition 3.1) for a cumulant generating function. Proposition 3.1 is subsequently proved in Section 4.2 by reducing it to a result (Proposition 4.2) about the Airy point process. The rest of Section 4 develops and proves various properties about the Airy point process, including the key rigidity estimates stated in the Introduction as Theorems 1.4, 1.5, and 1.6. Proposition 4.2 is proved in Section 5. Finally, Section 6 contains a discussion on asymptotics of the Ablowitz–Segur solution of Painlevé II and a proof of Theorem 1.7, stated earlier in the Introduction.

[11] The article [30] achieved this for $\tau>\frac{2}{3}\sqrt{2}$ based on the lack of oscillations in u_{AS} for such τ, and [29] provided an extension to the full region (c) (and slightly beyond).

2. Connections and extensions

We discuss various applications and extensions of our results and methods. Section 2.1 describes the relationship between our analysis and an inverse-scattering problem generalizing the Painlevé II equation. Section 2.2 explains how our results relate to the lower tail decay for positive temperature free fermions. Section 2.3 discusses extending our analysis to study the KPZ equation large deviation rate function, as well as relates our work to recent physics literature. Sections 2.4, 2.5, and 2.6 touch upon extensions of our methods and results to (respectively) the KPZ equation upper tail decay and general initial data, half-space geometry, and certain discretizations of the KPZ equation like ASEP or the stochastic six-vertex model.

2.1. An integro-differential generalization of Painlevé II

Using the explicit form of the Airy kernel and the fact (1.16) that expectations of multiplicative functions of determinant point processes can be written as Fredholm determinants, we can rewrite the equality in Proposition 1.2 (actually (1.7)) as

$$\mathbb{E}_{\mathrm{SHE}}\big[\exp(-\exp\big(\big(T^{\frac{1}{3}}(\Upsilon_T+s)\big)\big)\big] = \det(I-K)_{L^2(s,\infty)} =: Q(s), \tag{2.1}$$

where K is the Airy kernel deformed by a Fermi factor:

$$K(x,x') = \int_{-\infty}^{\infty} dr\,\sigma(r)\,\mathrm{Ai}(x+r)\,\mathrm{Ai}(x'+r), \quad \text{with } \sigma(r) = \frac{1}{1+e^{-T^{\frac{1}{3}}r}}. \tag{2.2}$$

It was proved in [4, Section 5.2] (following [135]) that for any choice of $\sigma(r)$ (which is smooth except at a finite number of points at which it has bounded jumps, and which approaches 0 at $-\infty$ and 1 at $+\infty$ exponentially fast), the resulting $Q(s)$ satisfies

$$\frac{d^2}{ds^2}\log Q(s) = \int_{-\infty}^{\infty} dr\,\sigma'(r) q_r^2(s),$$
$$Q(s) = \exp\Big(-\int_s^{\infty} dx\,(x-s)\int_{-\infty}^{\infty} dr\,\sigma'(r) q_r^2(x)\Big),$$

where $q_r(s)$ solves the following integro-differential generalization of Painlevé II:

$$\frac{d^2}{ds^2}q_r(s) = \Big(s+r+2\int_{-\infty}^{\infty} dr'\sigma'(r')q_{r'}^2(s)\Big)q_r(s),$$
$$\text{with } q_r(s)\sim \mathrm{Ai}(r+s) \text{ as } s\to+\infty. \tag{2.3}$$

If $\sigma(r) = \mathbf{1}_{r\geq 0}$, then the above equation recovers the Hastings–McLeod solution to Painlevé II. The derivation of the above result in [4, Section 5.2] came from an attempt to directly study the lower tail for the KPZ equation. Due to the complexity of this equation, [4] was unable to even show that the lower tail decays to zero and resorted to a more indirect route via the results of [119]. Sasorov, Meerson, and Prolhac [131] managed to extract asymptotics from this equation via a WKB approximation along with a self-consistency ansatz for the form of the solution to a Schrödinger equation in which the potential depends upon the solution. It would be valuable to make this approach rigorous, and below we mention possible ways to start. Before doing so, let us note that we may reverse the direction of inference and try to use our methods for studying the KPZ tail (via the Airy point process) to deduce results for the solution to (2.3).

The connection problem for (2.3) asks how the Airy behavior as $s \to \infty$ propagates through as $s \to -\infty$. This problem also falls under the realm of inverse scattering on the line (see [14], [51]) for the Airy operator. For the Hastings–McLeod solution of the Painlevé II equation, this problem has been resolved to a great level of detail using the steepest descent method for an associated 2×2 Riemann–Hilbert problem (see, e.g., [7], [31], [50], [52], [53], [67]).

For a general choice of $\sigma(r)$, the kernel K may be rewritten as

$$K(x,x') = \int_{-\infty}^{\infty} dr\sigma'(r) \frac{\mathrm{Ai}(x+r)\,\mathrm{Ai}'(x'+r) - \mathrm{Ai}'(x+r)\,\mathrm{Ai}(x'+r)}{x-x'}$$

and hence takes the form of an *integrable integral operator*. As shown in [85], the associated $Q(s)$ can be written in terms of an operator-valued Riemann–Hilbert problem.[12] The analysis of such problems is considerably more involved than in the finite-dimensional (namely, 2×2) matrix setting (see [86], [87] for some recent advances in this direction).

The approach developed in these pages may offer an alternative to studying the operator-valued Riemann–Hilbert problem. In our analysis there is nothing particularly special about the choice of $\sigma(r)$ (which translates into the choice of multiplicative functional). For another $\sigma(r)$ we could just as well similarly derive asymptotics for $Q(s)$. Turning this into a solution to the connection problem in (2.3) may still be a challenge. Should this work, the study of the operator-valued Riemann–Hilbert problem would be reduced to the study of the 2×2 matrix problem associated with the Hastings–McLeod and Ablowitz–Segur solutions. We do not pursue this idea further in the present text and leave it for future investigation.

[12] When $\sigma'(r)$ is a sum of N delta functions, the resulting Riemann–Hilbert problem is $(2N \times 2N)$-dimensional.

2.2. *Positive temperature free fermions*

Positive temperature free fermions and the related Moshe–Neuberger–Shapiro [118] matrix model[13] have recently been studied in [47] and [109] (and earlier in [88] in grand-canonical form). These ensembles are indexed by an inverse temperature β. When $\beta \to \infty$ this recovers the GUE. The authors of [47], [88], and [109] consider taking the number of fermions (or matrix dimension) $N \to \infty$. When β is fixed, the distribution of the rightmost fermion converges to the GUE Tracy–Widom distribution (see [109, Theorem 2(a)]); when β tends to 0 sufficiently fast relative to N going to infinity, the rightmost fermion converges to a Gumbel distribution; and when β tends to 0 and N tend to infinity in a critical manner, there is a crossover between the GUE Tracy–Widom and Gumbel distributions. The limit of the correlation kernel for a fermion point process at the edge converges under this critical scaling to the Fermi factor deformation of the Airy kernel given in (2.2). Hence, the Fredholm determinant (2.1) gives the probability that the rightmost limiting fermion is to the left of s, and Proposition 3.1 provides the lower tail probability decay of that distribution.

2.3. *Large deviation rate function*

Theorem 1.1 shows that there is a crossover between two types of tail decay which occurs when s is of order $T^{2/3}$. This can be understood in terms of large deviations. For $z \leq 0$, let

$$\Phi_-(z) = -\lim_{T\to\infty} T^{-2} \log\Big(\mathbb{P}\Big(\mathcal{H}(2T,0) + \frac{T}{12} \leq zT\Big)\Big). \tag{2.4}$$

The existence of the above limit is not a priori clear.[14] In terms of Φ_-, Theorem 1.1 suggests that $\Phi_-(z) \approx \frac{1}{12}(-z)^3$ for z near 0 and $\Phi_-(z) \approx \frac{4}{15\pi}(-z)^{\frac{5}{2}}$ for z near $-\infty$.

In the physics literature, the crossover between the exponents $\frac{5}{2}$ to 3 seems to have been first predicted via weak noise theory by Kolokolov and Korshunov [97] in the context of directed polymers, and quite recently by Meerson, Katsav, and Vilenkin [113] in the context of the KPZ equation. Weak noise theory (WNT), sometimes also called *optimal fluctuation theory*, studies the large deviations of the noise necessary to produce a given space-time trajectory of the KPZ equation (or more general systems).

[13] This is a one-parameter ($b \geq 0$) unitarily invariant measure on $n \times n$ Hermitian matrices H with density (relative to the Lebesgue measure on algebraically independent entries of H) given proportional to

$$e^{-(2b+1)\operatorname{Tr}(H^2)} \int_{\mathrm{U}(n)} dU e^{2b \operatorname{Tr}(UHU^\dagger H)},$$

where the integral is with respect to the Haar measure on the unitarty group $\mathrm{U}(n)$. When $b = 0$, the measure reduces to the GUE.

[14] Basu, Ganguly, and Sly [13] have an approach to proving the existence of such rate functions for first and directed last passage percolation. Whether this approach lifts to positive temperature models like KPZ remains to be seen.

It is a valid method only under "weak coupling" or when there is an exceedingly small parameter in front of the noise term. In many instances, this approach is only valid for short times (when the noise is, through rescaling, effectively weak). However, for the KPZ equation it seems that it remains valid for longer times, if one probes deep enough into the tail. WNT has a long and rich history within physics dating back to the 1960s in condensed matter physics (see, e.g., [79], [110], [140]) and was introduced into the study of the noisy Burgers equation by Fogedby [66] in the late 1990s. It also goes under names such as the *instanton method* in turbulence, *macroscopic fluctuation theory* in lattice gases (see [19]), and *WKB method* in reaction-diffusion systems (see [114] for a more extensive history). Within mathematics, the WNT for diffusions goes under the name *Freidlin–Wentzell theory*. For field-valued/infinite-dimensional diffusion processes (see [33]) and for certain nonlinear SPDEs (see [35], [78]), it has recently received some rigorous treatment. WNT alone does not provide the $\frac{5}{2}$ exponent or associated prefactor. Once the large deviations for the sample path (e.g., evolution of the KPZ equation) is determined, one still needs to solve a Hamiltonian variational problem to figure out the most likely trajectory among all those which achieve a given one-point large deviation. In the physics literature, [97], [99], and [113] worked through this calculation for KPZ with flat initial data and predicted the $\frac{5}{2}$ exponent along with a prefactor of $\frac{8}{15\pi}$. The authors of [89] worked with parabolic initial data (which interpolates between flat and narrow wedge) and predicted that the prefactor becomes $\frac{4}{15\pi}$ in the narrow wedge limit. These short-time predictions have been confirmed through exact formulas in [106] and [101].

Weak noise theory does not provide any explanation for the crossover of the exponents of $\Phi_-(z)$ from $\frac{5}{2}$ to 3 as z goes from 0 to $-\infty$. Recently, this crossover has been studied via analysis of the integro-differential equation discussed in Section 2.1. Le Doussal, Majumdar, and Schehr [107] performed a rough (nonrigorous) analysis of the equation and predicted the existence of a large deviation principle (LDP) with speed T^2 and cubic behavior for small z. However, their analysis missed the behavior of $\Phi_-(z)$ for $z \ll 0$ and hence did not predict that the $\frac{5}{2}$ exponent remains for a long time. Via nonrigorous WKB approximation analysis, Sasorov, Meerson, and Prolhac [131] predicted not only that the $\frac{5}{2}$ to 3 crossover should hold for all times sufficiently large, but also predicted a formula for the large deviation rate function $\Phi_-(z)$ from (2.4). The [131] prediction

$$\Phi_-(z) = \frac{4}{15\pi^6}(1-\pi^2 z)^{\frac{5}{2}} - \frac{4}{15\pi^6} + \frac{2}{3\pi^4}z - \frac{1}{2\pi^2}z^2 \tag{2.5}$$

indeed recovers the desired small and large z asymptotics. Hartmann, Le Doussal, Majumdar, Rosso, and Schehr [80] have performed simulations which numerically confirm the $\frac{5}{2}$ exponent for short and moderate values of time. The cubic exponent is harder to access numerically.

We now explain how our present work could be extended to prove a formula for $\Phi_-(z)$. The core challenge is that there is no proved large deviation theory for the empirical density of the Airy point process (such as was done for the GUE point process in [15]; see also [105] and the references therein). Since there are infinitely many points in the Airy point process, one cannot naively apply the Coulomb-gas/electrostatics approach to formulate a large deviation principle. Indeed, we have derived what should be the rate function in [41], although only in a physics way. The proof of this has not yet been made rigorous. We leave this challenge to future work.

In light of (1.6) and the argument used to prove Theorem 1.1, $\Phi_-(z)$ should be given by

$$\Phi_-(z) = \lim_{T\to\infty} \frac{1}{T^2} \log \mathbb{E}\Big[\exp\Big(-\sum_{i=1}^{\infty} \varphi_{T,-zT^{2/3}}(\mathbf{a}_i)\Big)\Big],$$

where the $\mathbf{a}_i$'s are the Airy point process, and $\varphi_{t,s}(a) := \log(1 + \exp(T^{1/3}(a + s)))$. For large T, $\varphi_{T,-zT^{2/3}}(T^{2/3}a) \approx T(a-z)_+$ (where $(\cdot)_+ := \max(\cdot, 0)$). Letting $\mu_T(\cdot) = T^{-1}\sum_{i\geq 1} \delta_{\mathbf{a}_i T^{-2/3}}(\cdot)$ denote the scaled empirical Airy point process measure,

$$\Phi_-(z) = \lim_{T\to\infty} \frac{1}{T^2} \log \mathbb{E}\Big[\exp\Big(-T^2 \int_{\mathbb{R}} \mathrm{d}a\, \mu_T(a)(a-z)_+\Big)\Big].$$

Now assume that, for a suitable class of functions μ, the empirical measure μ_T satisfies $\mathbb{P}(\mu_T \approx \mu) \approx \exp(-T^2 I(\mu))$ for a rate functional I. Then we would expect that

$$\Phi_-(z) = \min_{\mu}\Big(\int_{\mathbb{R}} \mathrm{d}a\, \mu(a)(a-z)_+ + I(\mu)\Big), \tag{2.6}$$

where the minimum is over the class of functions upon which I is finite.

Assuming (2.6), we can derive upper bounds on Φ_-. For instance, $I(\mu)$ should be minimized and equal to zero for the limiting density[15] of the Airy point process $\mu_*(a) = \pi^{-1}\sqrt{-a}\,\mathbf{1}_{a\leq 0}$. Plugging this choice into (2.6) and evaluating the integral gives $\Phi_-(z) \leq \frac{4}{15\pi}(-z)^{\frac{5}{2}}$. On the other hand, consider the limiting density of the Airy point process conditioned on $\mathbf{a}_1 \leq zT^{2/3}$ (after the scaling discussed above). Since that density will be supported strictly on $(-\infty, z]$, the integral in (2.6) will be zero. For that density, $I(\mu) = \frac{(-z)^3}{12}$, as can be determined by the known large deviations for $\mathbf{a}_1$ in Proposition 5.1. Thus, we find that $\Phi_-(z) \leq \frac{(-z)^3}{12}$.

A month after initially posting this paper, we (along with Le Doussal, Krajenbrink, Tsai in [41]) derived (based on a nonrigorous limit of the GUE LDP from [15];

[15] This can be calculated, for instance, by taking the trace of the Airy kernel.

see also [48], [49]) an electrostatic formula for $I(\mu)$.[16] Assuming the validity of this, we were able to solve the variational problem and confirm the formula for $\Phi_-(z)$ in (2.5). Proving the Airy large deviation principle remains a challenge for future work.

Half a year after initially posting the present article, Tsai [136] found a proof of the formula for $\Phi_-(z)$ in (2.5) based on the stochastic Airy operator representation for the point process. Essentially, the large deviation problem is transferred onto a large deviation problem for the driving noise for that operator which, being Gaussian, is readily studied via standard theory. This is obviously an oversimplification and there are some real subtleties which presently restrict the class of test functions for which this approach can be applied. Presently the result of [136] does not control the finite T tail behavior, although it is believable that the method could be extended to recover bounds such as proved here in Theorem 1.1. In certain applications, it is important to have uniform tail bounds for all times.

2.4. *Upper tail and general initial data*

Unlike for the lower tail, the upper tail probability $\mathbb{P}(\Upsilon_T > s)$ can be studied via Fredholm determinants (see [44, Proposition 10]). The large deviation rate should be T (instead of T^2 for the lower tail) and it is predicted in [107] and [131] that the rate function is $\frac{4}{3}s^{\frac{3}{2}}$. In our followup work [40] we have provided some analysis of this upper tail behavior.

The first author and Hammond [43] introduced a method (based on the KPZ line ensemble Gibbs property) to extend tail probabilities from narrow wedge initial data to general initial data. In [43, Theorem 13], the inputs came from [4] and [117] and were far from optimal. In our follow-up work [40], we employed the bounds from Theorem 1.1 to extend the upper bound (1.4) to general initial data. A matching lower bound is presently not accessible.

2.5. *Half-space KPZ*

The $(1+1)$-dimensional SHE $\mathcal{Z}^{\mathrm{hs}}(T,X)$ in the half-space $\mathbb{R}_+$ with delta initial data at the origin is uniquely defined (see [45]) by the SPDE in (1.1) and the Robin boundary condition (parameterized by $A \in \mathbb{R}$) which is formally given as $\partial_X \mathcal{Z}^{\mathrm{hs}}(T,X)|_{X=0} = A\mathcal{Z}^{\mathrm{hs}}(T,0)$, for all $T \geq 0$. The above half-space SHE/KPZ equation has been recently studied in [45] and [126] where it arises as the scaling limit of a corresponding ASEP. In the spirit of Proposition 1.2, [11] (see [126, Corollary 1.3]) computed a Laplace transform formula for the half-space SHE in terms of the (Pfaffian) GOE

[16] The conjectured formula is that $I(\mu)$ is finite only when $\int da(\mu(a) - \mu_*(a)) = 0$ and otherwise

$$I(\mu) = -\int \log|a_1 - a_2| \prod_{i=1}^{2} da_i\,(\mu(a_i) - \mu_*(a_i)) + \frac{4}{3}\int_0^\infty da|a|^{3/2}\mu(a).$$

point process. Using this, [136] proved the half-space KPZ large deviation rate function. Recently, [96] proved tights bounds on the lower tail of the half-space KPZ equation by extending our methods. The half-space problem has also received attention in the physics literature (see [102], [115]).

2.6. Other integrable probabilistic systems

Integrable probabilistic systems in the KPZ universality class (see [39]) fall into two classes: determinantal (i.e., free fermion) or nondeterminantal. For determinantal models like the longest increasing subsequence, polynuclear growth model, directed last passage percolation with geometric (or exponential) weights, and TASEP, various authors (see, e.g., [8], [9]) have obtained optimal lower tail[17] estimates via analysis of 2×2 Riemann–Hilbert problems related to Painlevé equations. Coulomb gas methods or loop equation also provide means to extract lower tail estimates in these contexts (see, e.g., [28], [36], [37], [48]). So far, our present work on the KPZ equation provides the only lower tail bounds for nondeterminantal models.

Besides studying one-point lower tail decay and large deviations, there is much interest in understanding the large deviations of the entire space-time trajectory. For TASEP, a recent attempt at this has been made in [121]. The rate is still N^2, although the rate function is only bounded above and below in [121]. The stochastic six-vertex model [25] is a discrete-time analogue of (T)ASEP. Significant efforts have been made (summarized, for instance, in [129]) to study large deviations and surface tensions for the six-vertex model. Until recently, the only rigorous results (i.e., large deviations for limit shapes) are for determinantal models such as uniform Aztec diamond or rhombus tilings (see, e.g., [92], [93]). Aggarwal [3] has now proved an arctic circle for the square ice model. This is, however, only the boundary of the limit shape and the method there does not seem able to capture the internal shape.

Using the methods considered here, we should be able to access tail/large deviation-type results for a few other nondeterminantal models. The starting point for our result is the identity in Proposition 1.2 which matches the SHE Laplace transform with a multiplicative function of the Airy point process. Similar formulas exist for ASEP (see [27, Theorem 1.1]), the stochastic six-vertex model (see [23, Corollary 4.4]), and q-TASEP (see [122, Proposition 6.1]). The methods of this article should extend to these other models, even though will likely involve some new analysis (such as of q-Laplace transforms and the associated variants of the Painlevé equations which may arise for these different models).

[17] For TASEP, the lower tail corresponds to the upper tail for the current of particles to pass the origin.

3. Proof of the main result

Recall Υ_T from (1.3). The random variable $\exp(-\exp(T^{\frac{1}{3}}(\Upsilon_T + s)))$ is equal to the conditional probability $\mathbb{P}(G \le -T^{1/3}(\Upsilon_T + s)|\Upsilon_T)$, where G is a Gumbel random variable independent of Υ_T. Thus, the expected value of $\exp(-\exp(T^{\frac{1}{3}}(\Upsilon_T + s)))$ is equal to the probability $\mathbb{P}(\Upsilon_T + T^{-1/3}G \le -s)$, which is approximately equal to $\mathbb{1}(\Upsilon_T \le -s)$ for large enough s and for all T greater than some $T_0 > 0$. Motivated by this heuristic, we prove Theorem 1.1 by estimating the Laplace transform formula $\mathbb{E}[\exp(-\exp(T^{\frac{1}{3}}(\Upsilon_T + s)))]$. We first state in Proposition 3.1 matching upper and lower bounds on the Laplace transform formula. Then, using Proposition 3.1, we finish the proof of Theorem 1.1 in Section 3.1.

PROPOSITION 3.1
Fix $\epsilon, \delta \in (0, \frac{1}{3})$ and $T_0 > 0$. Then there exist $s_0 = s_0(\epsilon, \delta, T_0)$, $C = C(T_0) > 0$, $K_1 = K_1(\epsilon, \delta, T_0) > 0$, and $K_2 = K_2(T_0) > 0$ such that for all $s \ge s_0$, one has

$$\mathbb{E}\big[\exp\big(-\exp\big(T^{\frac{1}{3}}(\Upsilon_T + s)\big)\big)\big] \le e^{-\frac{4(1-C\epsilon)}{15\pi}T^{\frac{1}{3}}s^{\frac{5}{2}}} + e^{-K_1 s^{3-\delta} - \epsilon T^{1/3}s} + e^{-\frac{(1-C\epsilon)}{12}s^3} \tag{3.1}$$

and

$$\mathbb{E}\big[\exp\big(-\exp\big(T^{\frac{1}{3}}(\Upsilon_T + s)\big)\big)\big] \ge e^{-\frac{4(1+C\epsilon)}{15\pi}T^{\frac{1}{3}}s^{\frac{5}{2}}} + e^{-K_2 s^3}. \tag{3.2}$$

We postpone the proof of Proposition 3.1 to Section 4.2.

3.1. Proof of Theorem 1.1

We show that (3.1) (resp., (3.2)) implies (1.4) (resp., (1.5)) of Theorem 1.1.

Let us first show that (3.1)⇒(1.4). Observe that using Markov's inequality

$$\begin{aligned}\mathbb{P}(\Upsilon_T \le -s) &= \mathbb{P}\big(\exp\big(-\exp\big(T^{\frac{1}{3}}(\Upsilon_T + s)\big)\big) \ge e^{-1}\big)\\ &\le e\mathbb{E}\big[\exp\big(-\exp\big(T^{\frac{1}{3}}(\Upsilon_T + s)\big)\big)\big].\end{aligned}$$

The inequality (3.1) bounds the right-hand side and yields (1.4).

Now we show that (3.2)⇒(1.5). Take $\bar{s} := (1-\epsilon)^{-1}s$. Observe that

$$\begin{aligned}\mathfrak{R} &:= \mathbb{E}\big[\exp\big(-\exp\big(T^{\frac{1}{3}}(\Upsilon_T + \bar{s})\big)\big)\big]\\ &\le \mathbb{E}\big[\mathbb{1}\{\Upsilon_T \le -s\} + \mathbb{1}\{\Upsilon_T > -s\}\exp\big(-\exp(\epsilon \bar{s} T^{\frac{1}{3}})\big)\big],\end{aligned}$$

where $\mathbb{1}\{A\}$ is an indicator function. The above inequality implies that

$$\mathbb{P}(\Upsilon_T \le -s) \ge \mathfrak{R} - \exp\big(-\exp(\epsilon \bar{s} T^{\frac{1}{3}})\big). \tag{3.3}$$

It follows from (3.2) that

$$\mathfrak{R} \ge \exp\Big(-(1 + C\epsilon + C'\epsilon)\frac{4}{15\pi} T^{\frac{1}{3}} s^{\frac{5}{2}}\Big) + \exp(-K_2 s^3) \tag{3.4}$$

for all $s \ge S = S(\epsilon, \delta)$. Here, the $C'\epsilon$ terms appear because $\bar{s}^{\frac{5}{2}} \le s^{\frac{5}{2}}(1 + C'\epsilon)$ for some $C' > 0$.

Now, we notice that there exists $S' = S'(\epsilon, T_0)$ such that for all $s \ge S'$,

$$\begin{aligned} \exp(\epsilon \bar{s} T^{\frac{1}{3}}) &\ge T^{\frac{1}{3}} \frac{4 s^{\frac{5}{2}}}{15\pi} - \log \epsilon \qquad \text{and} \\ \exp\big(-\exp(\epsilon \bar{s} T^{\frac{1}{3}})\big) &\le \epsilon \exp\Big(-\frac{4}{15\pi} T^{\frac{1}{3}} s^{\frac{5}{2}}\Big). \end{aligned} \tag{3.5}$$

Plugging the lower bound (3.4) on $\mathfrak{R}$ and the upper bound (3.5) on $\exp(-\exp(\epsilon \bar{s} T^{\frac{1}{3}}))$ into the right-hand side of (3.3) yields, for all $s \ge \max\{S, S'\}$,

$$\mathbb{P}(\Upsilon_T \le -s) \ge (1-\epsilon)\exp\Big(-\big(1 + (C + C')\epsilon\big)\frac{4}{15\pi} T^{\frac{1}{3}} s^{\frac{5}{2}}\Big) + \exp(-K_2 s^3).$$

The multiplicative factor $(1 - \epsilon)$ can be absorbed into the exponential factor $(1 + (C + C')\epsilon))$ on the right-hand side above; and rewriting it as $(1 + C\epsilon)$ for a slightly modified constant C yields the right-hand side of (1.5), thus completing the proof of Theorem 1.1. □

4. Airy point process

To prove Proposition 3.1, we use Proposition 1.2 which connects the SHE and the Airy point process. In this section we recall or prove various properties about the Airy point process. Section 4.1 reviews its determinantal structure. Section 4.2 contains a proof of Proposition 3.1. Section 4.3 relates the Airy point process to the stochastic Airy operator and derives properties about the typical point locations and deviations from there. Section 4.4 contains a heuristic explanation for certain terms in our tail bound. Finally, Sections 4.5, 4.6, and 4.7 provide proofs of, respectively, Theorem 1.4, Theorem 1.5, and Proposition 4.5.

4.1. *Determinantal point process definition*

The *Airy point process* (written here as χ^{Ai} or $\mathbf{a}_1 > \mathbf{a}_2 > \cdots$) is a *simple determinantal point process* (see [5, Section 4.2]). Let us briefly review these terms. Denote the Borel σ-algebra of the real line $\mathbb{R}$ by $\mathscr{B}(\mathbb{R})$, and let μ be a sigma-finite measure over $\mathbb{R}$. A *point process* is a probability distribution on locally finite configurations of the real points, or in other words, a nonnegative integer-valued random measure χ on the measure space $M = (\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$. A point process χ is called *simple* if $\mu(\{\exists x : \chi(x) \neq 0\}) = 0$. For any $k \geq 1$, the k-point *correlation function* of χ with respect to the measure μ is the locally integrable function $\rho_k : \mathbb{R}^k \to [0, \infty)$ such that for any mutually disjoint families of the Borel sets $B_1, \dots, B_k \in \mathcal{B}(\mathbb{R})$,

$$\mathbb{E}_\nu\Big[\prod_{i=1}^{k} \chi(B_i)\Big] = \int_{B_1\times\cdots\times B_k} \rho_k(x_1, \dots, x_k)\, d\mu(x_1)\cdots d\mu(x_k).$$

A simple point process χ is *determinantal* if there exists $K^\chi : \mathbb{R}^2 \to \mathbb{C}$ such that for all $k \geq 1$, $\rho_k(x_1, \dots, x_k) = \det[K^\chi(x_i, x_j)]_{1\leq i,j\leq k}$. We refer to K^χ as the *correlation kernel* of χ.

The Airy point process correlation kernel K^{Ai} relative to Lebesgue measure μ on $\mathbb{R}$ is[18]

$$K^{\mathrm{Ai}}(x, y) = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}(y)\mathrm{Ai}'(x)}{x - y} = \int_0^\infty \mathrm{Ai}(x + r)\mathrm{Ai}(y + r)\, dr. \tag{4.1}$$

We will write χ^{Ai} to denote the Airy point process (random) measure. We may also write $\chi^{\mathrm{Ai}} = \sum_{i=1}^{\infty} \delta_{\mathbf{a}_i}$ for random points $\mathbf{a}_1 > \mathbf{a}_2 > \cdots$.[19] We will use both of these notational conventions.

An integral operator $\mathfrak{K} : L^2(M) \to L^2(M)$ with kernel $K : \mathbb{R}^2 \to \mathbb{C}$ written as

$$(\mathfrak{K} f)(x) = \int K(x, y) f(y)\, d\mu(y), \quad \text{for } f \in L^2(M)$$

is *locally admissible* if for any compact set $D \subset \mathbb{R}$, the operator $\mathfrak{K}_D = \mathbb{1}_D \mathfrak{K} \mathbb{1}_D$, having kernel $K_D(x, y) = \mathbb{1}_D(x) K(x, y) \mathbb{1}_D(y)$, has the following representation:

$$\begin{aligned} (\mathfrak{K}_D f)(x) &= \sum_{k=1}^{n} \lambda_k \phi_k(x) \langle \phi_k, f\rangle_{L^2(M)}, \\ K_D(x, y) &= \sum_{k=1}^{n} \lambda_k \phi_k(x) \overline{\phi_k(y)}, \end{aligned} \tag{4.2}$$

where n may be finite or infinite, $\{\phi_k\}_k \in L^2(M)$ are orthonormal eigenfunctions, and the eigenvalues $(\lambda_k^D)_{k=1}^n$ of K_D are positive and satisfy $\sum_{k=1}^{n} \lambda_n^D < \infty$. We call $\mathfrak{K}$ *good* if for all compact D and all $1 \leq k \leq n$, $\lambda_k^D \in (0, 1]$. For a determinantal point process with locally admissible and good correlation kernel, for any compact

[18] Recall the *Airy function* $\mathrm{Ai}(x) := \frac{1}{\pi}\int_0^\infty \cos(tx + t^3/3)\, dt$.

[19] This follows from a calculation like in Proposition 1.3 which shows that, almost surely, there are infinitely many particles in χ^{Ai} but only finitely many to the right of any given point.

set $D \subset \mathbb{R}$, $\chi(D)$ equals in distribution the sum of n (same n as in (4.2)) independent Bernoulli random variables with the respective probabilities of equality to 1 given by the $\lambda_1^D, \dots, \lambda_n^D$ (see [5, Section 4.2]).

LEMMA 4.1
The kernel (4.1) of the Airy point process K^{Ai} is locally admissible and good.

We use this result in proving Theorem 1.5 (see [5, Proposition 4.2.30] for a proof).

4.2. *Proof of Proposition 3.1*

As above, let $\mathbf{a_1} > \mathbf{a_2} > \cdots$ denote the Airy point process. Denote

$$\mathcal{I}_s(x) := \frac{1}{1+\exp(T^{\frac{1}{3}}(s+x))} \qquad \text{and} \\ \mathcal{J}_s(x) := \log\big(1+\exp\big(T^{\frac{1}{3}}(s+x)\big)\big) \tag{4.3}$$

so that for any $x \in \mathbb{R}$, we have $\mathcal{I}_s(x) = \exp(-\mathcal{J}_s(x))$. Proposition 1.2 connects $\mathbb{E}_{\mathrm{Airy}}[\prod_{k=1}^{\infty} \mathcal{I}_s(\mathbf{a}_k)]$ with the Laplace transform of the SHE. We now state upper and lower bounds on this expectation and then subsequently complete the proof of Proposition 3.1.

PROPOSITION 4.2
Fix any $\epsilon, \delta \in (0, \frac{1}{3})$ and $T_0 > 0$. Then there exist $s_0 = s_0(\epsilon, \delta, T_0)$, an absolute constant $C > 0$, $K_1 = K_1(\epsilon, \delta, T_0) > 0$, and $K_2 = K_2(T_0) > 0$ such that for all $s \geq s_0$ and $T \geq T_0$,

$$\mathbb{E}_{\mathrm{Airy}}\Big[\prod_{k=1}^{\infty} \mathcal{I}_s(\mathbf{a}_k)\Big] \leq e^{-\frac{4(1-C\epsilon)}{15\pi} T^{\frac{1}{3}} s^{\frac{5}{2}}} + e^{-K_1 s^{3-\delta} - \epsilon T^{1/3} s} + e^{-\frac{(1-C\epsilon)}{12} s^3} \tag{4.4}$$

and

$$\mathbb{E}_{\mathrm{Airy}}\Big[\prod_{k=1}^{\infty} \mathcal{I}_s(\mathbf{a}_k)\Big] \geq e^{-\frac{4(1+C\epsilon)}{15\pi} T^{\frac{1}{3}} s^{\frac{5}{2}}} + e^{-K_2 s^3}. \tag{4.5}$$

Proof of Proposition 3.1
Using (1.7), (3.1)–(3.2) follows from (4.4)–(4.5). ☐

4.3. *Stochastic Airy operator*

As observed in [62] and proved in [128], the Airy point process equals in distribution the negated spectrum of the *stochastic Airy operator*. This yields a way to compute the typical locations of the points and establish a uniform bound (see Proposition 4.5) on the deviations from those locations. This bound is used in the proof of (4.5) of Proposition 4.2. It is not, however, tight enough to suffice for all of our needs, hence our need for Theorems 1.4 and 1.5.

Definition 4.3 (Stochastic Airy operator)
Let $D = D(\mathbb{R}^+)$ be the space of the generalized functions, that is, the continuous dual of the space C_0^∞ of all smooth compactly supported test functions endowed with the topology of compact convergence. For any function f, we denote its kth derivative by the symbol $f^{(k)}$ and define its action on any test function $\phi \in C_0^\infty$ by

$$\prec \phi, f^{(k)}(x) \succ := (-1)^k \int f(x)\phi^{(k)}(x)\, dx.$$

Define the space of functions $H^1_{\text{loc}} = H^1_{\text{loc}}(\mathbb{R})$, where for any $f \in H^1_{\text{loc}}$ and any compact set $I \subset \mathbb{R}$, we have $f^{(1)}\mathbb{1}_I \in L^2(\mathbb{R})$. The $\beta > 0$ stochastic Airy operator $\mathcal{H}_\beta$ is a linear map

$$\mathcal{H}_\beta : H^1_{\text{loc}} \to D \quad \text{with } \mathcal{H}_\beta f = -f^{(2)} + xf + \frac{2}{\sqrt{\beta}} f B'.$$

Here, B is a standard Brownian motion and B' is its derivative which belongs to the space D.[20] The nonrandom part of $\mathcal{H}_\beta$ is the *Airy operator* $\mathcal{A} = -\partial_x^2 + x$. Define the Hilbert space

$$L^* := \{ f : f(0) = 0, \|f\|_* < \infty \} \quad \text{where } \|f\|_*^2 = \int_0^\infty \big((f')^2 + (1+x)f^2\big)\, dx.$$

A pair $(f, \mathbf{\Lambda}) \in L^* \times \mathbb{R}$ is an eigenfunction/value pair for $\mathcal{H}_\beta$ if $\mathcal{H}_\beta f = \mathbf{\Lambda} f$ (likewise for $\mathcal{A}$).

PROPOSITION 4.4 ([128, Theorem 1.1])
Let $\mathbf{a} = (\mathbf{a}_1 > \mathbf{a}_2 > \cdots)$ *denote the Airy point process, and let* $\mathbf{\Lambda} = (\mathbf{\Lambda}_1 < \mathbf{\Lambda}_2 < \cdots)$ *denote the eigenvalues of* $\mathcal{H}_2$. *Then* $\mathbf{a}$ *and* $-\mathbf{\Lambda}$ *are equal in distribution.*

Results obtained in [128] and [138] show that the spectrum of $\mathcal{H}_\beta$ lies within a uniform random band around the spectrum of the Airy operator $\mathcal{A}$. The following is a strengthening of such a result wherein the tail decay of the band width (here C_ϵ) is controlled. This result is proved for generic β, although we need it later only for $\beta = 2$.

[20] To see that $fB' \in D$, observe that $\int_0^y fB'\, dx = -\int_0^y Bf'\, dx + f(y)B_y - f(0)B_0$ by integration by parts. One can now see that the latter is a continuous function. Thus, its derivative fB' belongs to the space D.

PROPOSITION 4.5

Denote the eigenvalues of the Airy operator $\mathcal{A}$ *by* $(\boldsymbol{\lambda}_1 < \boldsymbol{\lambda}_2 < \cdots)$, *and denote the eigenvalues of* $\mathcal{H}_\beta$ *by* $(\boldsymbol{\Lambda}_1^\beta, \boldsymbol{\Lambda}_2^\beta, \ldots)$. *For any* $\epsilon \in (0,1)$, *we define the random variable* C_ϵ *as the minimal real number such that for all* $k \geq 1$,

$$(1-\epsilon)\boldsymbol{\lambda}_k - C_\epsilon \leq \boldsymbol{\Lambda}_k^\beta \leq (1+\epsilon)\boldsymbol{\lambda}_k + C_\epsilon.$$

Then for all $\epsilon, \delta \in (0,1)$ *there exist* $s_0 = s_0(\epsilon, \delta)$, *and* $\kappa = \kappa(\epsilon, \delta)$ *such that for* $s \geq s_0$,

$$\mathbb{P}\Big(C_\epsilon \geq \frac{s}{\sqrt{\beta}}\Big) \leq \exp(-\kappa s^{1-\delta}). \tag{4.6}$$

Notice that (4.6) demonstrates a concentration inequality for the supremum of the deviations of the eigenvalues of $\mathcal{H}_\beta$ around their typical locations. We defer the proof of this proposition until Section 4.7.

Finally, we state a result on the position of the eigenvalues of the Airy operator $\mathcal{A}$. For the Airy operator, λ_k coincides with the kth zero of the Airy function. Classical works (see, e.g., [112], [133]) have addressed this question for more general operators $-\partial_x^2 + V(x)$ for $V(x)$ satisfying certain regularity conditions. Those works are not formulated in the style of theorem and proof, so we also mention that Proposition 4.6 is a special case of the result proved in [83, Theorem 3.3] for a general class of $V(x)$.

PROPOSITION 4.6

Denote the eigenvalues of the Airy operator $\mathcal{A}$ *by* $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_1 < \boldsymbol{\lambda}_2 < \cdots)$. *Then for any* $n \geq 1$, λ_n *satisfies*

$$\begin{aligned} \frac{1}{\pi}\int_0^{\lambda_n} \sqrt{(\lambda_n - x)}\,dx = n - \frac{1}{4} + \mathcal{R}(n), \qquad \textit{or} \\ \lambda_n = \Big(\frac{3\pi}{2}\Big(n - \frac{1}{4} + \mathcal{R}(n)\Big)\Big)^{\frac{2}{3}}, \end{aligned} \tag{4.7}$$

where $|\mathcal{R}(n)| \leq K/n$ *for some large constant* K.

4.4. Heuristics for Proposition 4.2

There are two main contributions to $\mathbb{E}_{\text{Airy}}[\prod_{k=1}^\infty \mathcal{I}_s(\mathbf{a}_k)]$—typical and atypical values of $\mathbf{a}$. Owing to Proposition 4.5, the typical values of $\mathbf{a}$ are close to the negatives of the Airy operator eigenvalues, whose locations are estimated in Proposition 4.6.

The asymptotic formula in (4.7) leads (as we now show) to the $\exp(-\frac{4}{15\pi}T^{1/3}s^{\frac{5}{2}})$ term in (4.4) and (4.5).[21] Replacing $\mathbf{a}_k$ by $-\lambda_k$ yields

[21]The ϵ error factor comes from various approximation errors and the fact that the replacement is only true with high probability.

$$\log\Big(\prod_{k=1}^{\infty} \mathcal{I}_s(\mathbf{a}_k)\Big) \approx \sum_{k=1}^{\infty} \mathcal{J}_s(-\lambda_k) = -\sum_{k=1}^{\infty} \log\big(1+\exp\big(T^{\frac{1}{3}}(s-\lambda_k)\big)\big).$$

When $s \gg \lambda_k$ and T is bounded away from 0, $\log(1+\exp(T^{\frac{1}{3}}(s-\lambda_k))) \approx T^{\frac{1}{3}}(s-\lambda_k)$. By Proposition 4.6, $\lambda_k \approx (3\pi k/2)^{\frac{2}{3}}$, hence

$$\begin{aligned}
&\sum_{\{k:\lambda_k<s\}} \log\big(1+\exp\big(T^{\frac{1}{3}}(s-\lambda_k)\big)\big)\\
&\quad\approx T^{\frac{1}{3}} \sum_{k<\frac{2}{3\pi}s^{\frac{3}{2}}} \Big(s-\Big(\frac{3\pi k}{2}\Big)^{\frac{2}{3}}\Big)\\
&\quad\approx T^{\frac{1}{3}}\Big(\frac{2}{3\pi}s^{\frac{5}{2}} - \frac{3}{5}\cdot\Big(\frac{3\pi}{2}\Big)^{\frac{2}{3}}\cdot\Big(\frac{2}{3\pi}s^{\frac{3}{2}}\Big)^{\frac{5}{3}}\Big) = \frac{4}{15\pi}T^{\frac{1}{3}}s^{\frac{5}{2}}.
\end{aligned} \tag{4.8}$$

To obtain the last approximation, we replace the sum $\sum_{k<x} k^{\frac{2}{3}}$ by the integral $\int_0^x z^{\frac{2}{3}}\,dz$ which is equal to $\frac{3}{5}\cdot x^{\frac{5}{3}}$. Thus (4.8) accounts for the first term in (4.4) and (4.5).

To complete the above heuristic, we must show that the sum of $\mathcal{J}_s(-\lambda_k)$ over all $\lambda_k > s$ can be ignored. For all $\lambda_k > s$, one has $0 \le \mathcal{J}_s(-\lambda_k) \le \exp(T^{\frac{1}{3}}(s-\lambda_k))$. Using this,

$$\begin{aligned}
0 \le \sum_{\{k:\lambda_k>s\}} \mathcal{J}_s(-\lambda_k) &\le \sum_{k\ge\frac{2}{3\pi}s^{\frac{3}{2}}} \exp\Big(T^{\frac{1}{3}}\Big(s-\Big(\frac{3\pi k}{2}\Big)^{\frac{2}{3}}\Big)\Big)\\
&\le \int_{\frac{2}{3\pi}s^{\frac{3}{2}}}^{\infty} \exp\Big(T^{\frac{1}{3}}\Big(s-\Big(\frac{3\pi z}{2}\Big)^{\frac{2}{3}}\Big)\Big)\,dz.
\end{aligned}$$

The final integrand is less than 1 inside $[\frac{2}{3\pi}s^{\frac{3}{2}}, \infty]$, and thanks to the inequality (see Lemma 5.6)

$$s-\Big(\frac{3\pi z}{2}\Big)^{\frac{2}{3}} \le -\Big(\frac{3\pi(z-\frac{2}{3\pi}s^{\frac{3}{2}})}{2}\Big)^{\frac{1}{3}} \quad \text{for all } z \ge \Big(\frac{2}{3\pi}\Big)s^{\frac{3}{2}} + \sqrt{\frac{2}{3\pi}}s^{\frac{3}{4}},$$

we obtain the following bound:

$$\int_{\frac{2}{3\pi}s^{\frac{3}{2}}}^{\infty} \exp\Big(T^{\frac{1}{3}}\Big(s-\Big(\frac{3\pi z}{2}\Big)^{\frac{2}{3}}\Big)\Big)\,dz \le \sqrt{\frac{2}{3\pi}}s^{\frac{3}{4}} + \int_0^{\infty} \exp\Big(-T^{\frac{1}{3}}\Big(\frac{3\pi z}{2}\Big)^{1/3}\Big)\,dz.$$

The final integral evaluates to a constant times $(T/2)^{-\frac{1}{3}}\int_0^{\infty} z^2\exp(-z)\,dz = (T/2)^{-\frac{1}{3}}\Gamma(3)$. Thus, when T is bounded away from 0, the contribution of the eigenvalues which are greater than s is of the order $\mathcal{O}(s^{\frac{3}{4}})$, which is certainly less than $s^{\frac{5}{2}}$ for enough large s.

The other terms in the bounds (4.4) and (4.5) come from the atypical deviations of the Airy points from their typical locations. For instance, if $\mathbf{a}_1$ is very negative, then this will clearly affect the validity of the above heuristic. The proof of Proposition 4.2 boils down to controlling these atypical deviations and measuring their effect on the multiplicative functional in question.

Before we prove Proposition 4.2, we give proofs of Theorems 1.4 and 1.5 and Proposition 4.5 which provide important control over the atypical deviations of the Airy point process.

4.5. *Proof of Theorem 1.4*

Let us denote $A := \{\chi^{\mathrm{Ai}}([-s,\infty)) - \mathbb{E}[\chi^{\mathrm{Ai}}([-s,\infty))] \le -cs^{\frac{3}{2}}\}$. Using Markov's inequality, we find that for any $\lambda > 0$,

$$\mathbb{P}(A) \le \exp\bigl(-\lambda c s^{\frac{3}{2}} + \lambda \mathbb{E}\bigl[\chi^{\mathrm{Ai}}\bigl([-s,\infty)\bigr)\bigr]\bigr)\mathbb{E}\bigl[\exp\bigl(-\lambda\chi^{\mathrm{Ai}}\bigl([-s,\infty)\bigr)\bigr)\bigr].$$

Set $\lambda = s^{\frac{3}{2}-\delta}$. Owing to Proposition 1.3 and Theorem 1.7,

$$\mathbb{E}\bigl[\chi^{\mathrm{Ai}}\bigl([-s,\infty)\bigr)\bigr] = \frac{2}{3\pi}s^{\frac{3}{2}} + \mathfrak{O}(s),$$

$$\mathbb{E}\bigl[\exp\bigl(-\lambda\chi^{\mathrm{Ai}}\bigl([-s,\infty)\bigr)\bigr)\bigr] = F(-s;\lambda) \le \exp\Bigl(-\frac{2\lambda}{3\pi}s^{\frac{3}{2}} + Ks^{3-\frac{13\delta}{11}}\Bigr),$$

where $K = K(\delta)$ is a large constant and s is large enough. Thus

$$\mathbb{P}(A) \le \exp\bigl(-cs^{3-\delta} + Ks^{3-\frac{13\delta}{11}} + \mathfrak{O}(s)\bigr).$$

Since $|\mathfrak{O}(s)|$ is uniformly bounded for all $s > 0$, the theorem is proved. □

4.6. *Proof of Theorem 1.5*

Fix any $k \in \mathbb{Z}_{\ge 0}$. By Lemma 4.1, the kernel of the Airy point process is locally admissible and good. Thus (as discussed before Lemma 4.1) for any compact set D, $\chi^{\mathrm{Ai}}(D) \overset{d}{=} \sum_i^\infty X_i$ where the X_i's are independent Bernoulli random variables satisfying $\mathbb{P}(X_i = 1) = 1 - \mathbb{P}(X_i = 0) = \lambda_i^D$. Here the λ_i^D's are the eigenvalues of the operator $\mathbb{1}_D K^{\mathrm{Ai}}\mathbb{1}_D$. Choose a sequence of compact sets D_n increasing to the interval $\mathfrak{B}_k$. By Bennett's concentration inequality in [16],

$$\mathbb{P}\bigl(\chi^{\mathrm{Ai}}(D_n) - \mathbb{E}\bigl[\chi^{\mathrm{Ai}}(D_n)\bigr] \ge cs^{\frac{3}{2}}\bigr) \le \exp\Bigl(-\sigma_n^2 h\Bigl(\frac{cs^{\frac{3}{2}}}{\sigma_n^2}\Bigr)\Bigr), \tag{4.9}$$

where $h(u) := (1+u)\log(1+u) - u$. By the dominated convergence theorem, as $n \to \infty$, $\mu_n := \mathbb{E}[\chi^{\mathrm{Ai}}(D_n)] \to \mathbb{E}[\chi^{\mathrm{Ai}}(\mathfrak{B}_k)]$ and $\sigma_n^2 := \mathrm{Var}(\chi^{\mathrm{Ai}}(D_n)) \to \mathrm{Var}(\chi^{\mathrm{Ai}}(\mathfrak{B}_k))$. By Proposition 1.3, for s large enough,

$$\mathrm{Var}\big(\chi^{\mathrm{Ai}}(\mathfrak{B}_k)\big) \le C \log s$$

for some constant $C > 0$. Therefore, for any given $\epsilon > 0$, there exist $S_0 = S_0(\epsilon)$ and $N_0 = N_0(\epsilon)$ such that for all $s \ge S_0$ and $n \ge N_0$,

$$\sigma_n^2 \le C(1+\epsilon)\log s. \tag{4.10}$$

Since $h(u) \ge u(\log u - 1)$, we find that $\sigma_n^2 h(cs^{\frac{3}{2}}/\sigma_n^2) \ge cs^{\frac{3}{2}}(\log(cs^{\frac{3}{2}}) - \log\sigma_n^2 - 1)$. Plugging the upper bound (4.10) on σ_n^2 into this inequality and exponentiating yields

$$\exp\big(-\sigma_n^2 h(cs^{\frac{3}{2}}/\sigma_n^2)\big) \le \exp\big(-cs^{\frac{3}{2}}\big(\log(cs^{\frac{3}{2}}) - (1+\epsilon)\log\log s\big)\big) \tag{4.11}$$

for all $n \ge N_0$ and s sufficiently large. Now, Fatou's lemma shows that

$$\begin{aligned}
&\mathbb{P}\big(\chi^{\mathrm{Ai}}(\mathfrak{B}_k) - \mathbb{E}\big[\chi^{\mathrm{Ai}}(\mathfrak{B}_k)\big] \ge cs^3\big) \\
&\quad \le \liminf_{n\to\infty} \mathbb{P}\big(\chi^{\mathrm{Ai}}(D_n) - \mathbb{E}\big[\chi^{\mathrm{Ai}}(D_n)\big] \ge cs^3\big).
\end{aligned} \tag{4.12}$$

Owing to (4.9) and (4.11), we find that

$$\begin{aligned}
\text{RHS of (4.12)} &\le \limsup_{n\to\infty} \exp\big(-\sigma_n^2 h(cs^{\frac{3}{2}}/\sigma_n^2)\big) \\
&\le \exp\big(-cs^{\frac{3}{2}}\big(\log(cs^{\frac{3}{2}}) - (1+\epsilon)\log\log s\big)\big).
\end{aligned}$$

□

4.7. Proof of Proposition 4.5

We start with a lemma about the tails of the distribution of Brownian motion oscillations.

LEMMA 4.7

Let B_x be a Brownian motion on $[0,\infty)$, and define

$$Z := \sup_{x>0} \sup_{y\in[0,1)} \frac{|B_{x+y} - B_x|}{6\sqrt{\log(3+x)}}. \tag{4.13}$$

Then, letting $\bar{B}_x = \int_x^{x+1} B_y\,dy$ and $\bar{B}'_x = \frac{d}{dx}\bar{B}_x (= B_{x+1} - B_x)$, we have that (1) $\max\{|\bar{B}'_x|, |\bar{B}_x - B_x|\} \le 6Z\sqrt{\log(3+x)}$, and (2) there exist $K_1, K_2, s_0 > 0$ such that for all $s > s_0$,

$$\mathbb{P}(Z \ge s) \le K_1 e^{-K_2 s^2}. \tag{4.14}$$

Proof

The proof of (1) follows from the following inequalities:

$$
\begin{aligned}
|\bar{B}'_x| = |B_{x+1} - B_x| &\le 6\sqrt{\log(3+x)} \sup_{y\in[0,1)} \frac{|B_{x+y} - B_x|}{6\sqrt{\log(3+x)}} \\
&\le 6Z\sqrt{\log(3+x)},
\end{aligned}
$$

$$
|\bar{B}_x - B_x| \le \int_0^1 |B_{x+y} - B_x|\,dy \le \sup_{y\in[0,1)} |B_{x+y} - B_x| \le 6Z\sqrt{\log(3+x)}.
$$

Turning to (2), for any $y \in [0,1)$,

$$
\begin{aligned}
|B_{x+y} - B_x| &\le |B_{x+y} - B_{\lceil x\rceil}| + |B_{\lceil x\rceil} - B_{\lfloor x\rfloor}| + |B_x - B_{\lfloor x\rfloor}| \\
&\le 2 \sup_{y\in[0,1]} |B_{\lceil x\rceil + y} - B_{\lceil x\rceil}| + 2 \sup_{y\in[0,1)} |B_{\lfloor x\rfloor + y} - B_{\lfloor x\rfloor}|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\sup_{y\in[0,1)} \frac{|B_{x+y} - B_x|}{\sqrt{\log(3+x)}} \le{}& 2 \sup_{y\in[0,1]} \frac{|B_{\lceil x\rceil + y} - B_{\lceil x\rceil}|}{\sqrt{\log(3+x)}} \\
&+ 2 \sup_{y\in[0,1]} \frac{|B_{\lfloor x\rfloor + y} - B_{\lfloor x\rfloor}|}{\sqrt{\log(3+x)}}.
\end{aligned}
\tag{4.15}
$$

To study Z, we must take the sup over all positive real x of the above bound. However, at the cost of replacing $3+x$ by $2+x$ in the denominator, using (4.15) we can bound $Z \le 4W$ where

$$
W := \max_{n\in\mathbb{Z}_{\ge 1}} \frac{W_n}{6\sqrt{\log(2+n)}}, \quad \text{where } W_n := \sum_{y\in[0,1)} |B_{n+y} - B_n|.
$$

The $\{W_n\}_{n\in\mathbb{Z}_{\ge 1}}$ are independent and identically distributed, and an application of the reflection principle shows that

$$
\mathbb{P}(W_n \ge a) \le 2\mathbb{P}\big(|B_{n+1} - B_n| \ge a/2\big) \le \frac{2}{a} e^{-a^2/8}.
\tag{4.16}
$$

The union bound shows that

$$
\begin{aligned}
\mathbb{P}(Z \ge s) \le \mathbb{P}(4W \ge s) &= \mathbb{P}\Big(\bigcup_{n=0}^{\infty} \frac{W_n}{6\sqrt{\log(2+n)}} \ge \frac{s}{4}\Big) \\
&\le \sum_{n=0}^{\infty} \mathbb{P}\Big(W_n \ge \frac{3}{2} s\sqrt{\log(2+n)}\Big).
\end{aligned}
$$

Combining this with (4.16) yields the desired decay bound as long as s is large enough. ∎

Proof of Proposition 4.5
We will make use of the following convention. For any two operators $A, B : H^1_{\mathrm{loc}} \to D$, we write $A \le B$ if for all $f \in L^*$, $\prec f, Af \succ \le \prec f, Bf \succ$. If $A \le B$, then $\lambda_k^A \le \lambda_k^B$, where λ_k^A and λ_k^B are kth lowest eigenvalues of the operators A and B, respectively.

In our proof we bound $\mathcal{H}_\beta$ above/below by the Airy operator plus/minus an error with well-controlled tails. This requires establishing a random operator bound on B'. Decomposing the Brownian motion $B_x = \bar{B}_x + (B_x - \bar{B}_x)$ ($\bar{B}_x$ is defined in Lemma 4.7), we find that for $f \in C_0^\infty$,

$$\prec f, B' f \succ = \int_0^\infty f^2 \bar{B}'_x \, dx + \int_0^\infty f(x) f'(x)(\bar{B}_x - B_x)\, dx. \tag{4.17}$$

CLAIM
Fix $\epsilon, \delta \in (0,1)$. Let $K_1 = K_1(\delta) \ge 1$ (resp., $K_2 = K_2(\delta) \ge 1$) be a constant such that $\sqrt{\log(3+x)} \le x^\delta$ (resp., $\log(3+x) \le x^\delta$) for all $x \ge K_1$ (resp., $x \ge K_2$). Define

$$\mathcal{U}_\epsilon := \max\Big\{ Z\Big(\Big(\frac{Z}{\epsilon}\Big)^{\frac{\delta}{(1-\delta)}} + K_1^\delta\Big), Z^2\Big(\Big(\frac{Z}{\epsilon}\Big)^{\frac{2\delta}{(1-\delta)}} + K_2^\delta\Big)\Big\}.$$

Then

$$-10\epsilon \mathcal{A} - 6\Big(1 + \frac{1}{2}\epsilon^{-1}\Big)\mathcal{U}_\epsilon \le B' \le 10\epsilon\mathcal{A} + 6\Big(1 + \frac{1}{2}\epsilon^{-1}\Big)\mathcal{U}_\epsilon \tag{4.18}$$

Proof of Claim
Recall that $\bar{B}'_x = B_{x+1} - B_x$. From Lemma 4.7, $|\bar{B}'_x| \le 6Z\sqrt{\log(3+x)}$ (see (4.13) for Z). Thus, we will start by establishing the following bound, valid for all $x \ge 0$:

$$\begin{aligned} Z\sqrt{\log(3+x)} \le \max\big\{ & Z\big((Z/\epsilon)^{\frac{\delta}{(1-\delta)}} + K_1^\delta\big) + \epsilon x, \\ & Z\sqrt{(Z/\epsilon)^{\frac{2\delta}{(1-\delta)}} + K_2^\delta + \epsilon^2 x}\big\}. \end{aligned} \tag{4.19}$$

We explain the derivation of the first bound by $Z((Z/\epsilon)^{\frac{\delta}{(1-\delta)}} + K_1^\delta) + \epsilon x$, as the second bound follows a similar type of argument. Let $z_0 := \max\{(Z/\epsilon)^{\frac{1}{(1-\delta)}}, K_1\}$. For $x < z_0$,

$$\begin{aligned} Z\sqrt{\log(3+x)} \le Z\sqrt{\log(3+z_0)} \le Z \cdot z_0^\delta \le Z\big((Z/\epsilon)^{\frac{\delta}{(1-\delta)}} + K_1^\delta\big) \\ \le Z\big((Z/\epsilon)^{\frac{\delta}{(1-\delta)}} + K_1^\delta\big) + \epsilon x. \end{aligned}$$

The second inequality uses $\sqrt{\log(3+z_0)} \le z_0^\delta$, and the third uses $\max\{a,b\} \le a+b$ for $a, b \ge 0$. For $x \ge z_0$,

$$Z\sqrt{\log(3+x)} \le Z\big(1+\sqrt{\log(3+x)}\big) \le Z + \epsilon x^{1-\delta}\sqrt{\log(3+x)}$$
$$\le Z + \epsilon x \le Z\big((Z/\epsilon)^{\frac{\delta}{(1-\delta)}} + K_1^{\delta}\big) + \epsilon x.$$

The second inequality uses $Z \le \epsilon x^{1-\delta}$ (as $(Z/\epsilon)^{\frac{1}{1-\delta}} \le x$), the third uses $\sqrt{\log(3+x)} \le x^{\delta}$ (since $x \ge K_1$), and the fourth uses $(Z/\epsilon)^{\frac{\delta}{(1-\delta)}} + K_1^{\delta} \ge 1$.

Combining (4.19) with the definition of $\mathcal{U}_\epsilon$, we see that for all $x \ge 0$,

$$Z\sqrt{\log(3+x)} \le \max\{\mathcal{U}_\epsilon + \epsilon x, \sqrt{\mathcal{U}_\epsilon + \epsilon^2 x}\}.$$

This along with Lemma 4.7 establishes that for all $x \ge 0$,

$$\begin{aligned} |\bar{B}_x'| &\le 6Z\sqrt{\log(3+x)} \le 6(\mathcal{U}_\epsilon + \epsilon x), \\ |\bar{B}_x - B_x| &\le 6Z\sqrt{\log(3+x)} \le 6\sqrt{\mathcal{U}_\epsilon + \epsilon^2 x}. \end{aligned} \tag{4.20}$$

Using the formula for $\prec f, B'f \succ$ in (4.17), along with the inequality $|f'(x)f(x)(\bar{B}_x - B_x)| \le 3\epsilon(f'(x))^2 + (12\epsilon)^{-1} f(x)^2|\bar{B}_x - B_x|^2$ (which follows by applying $ab \le \frac{1}{2}(a^2+b^2)$) we have that

$$\big|\prec f, B'f\succ\big| \le \int_0^\infty f^2(x)\big(\epsilon + |\bar{B}_x'|\big)\,dx + 3\epsilon\int_0^\infty \big(f'(x)\big)^2$$
$$+ (12\epsilon)^{-1}\int_0^\infty \big(f(x)\big)^2 |\bar{B}_x - B_x|^2\,dx.$$

Plugging the bounds from (4.20) into the above expression yields

$$\big|\prec f, B'f\succ\big| \le 6\mathcal{U}_\epsilon\|f\|^2 + 7\epsilon\langle f, \mathcal{A}f\rangle + 3\epsilon\int_0^\infty \big(f'(x)\big)^2\,dx$$
$$+ 3\epsilon^{-1}\int_0^\infty f^2(\mathcal{U}_\epsilon + \epsilon^2 x)\,dx$$
$$\le 6\big(1 + (2\epsilon)^{-1}\big)\mathcal{U}_\epsilon\|f\|^2 + 10\epsilon\langle f, \mathcal{A}f\rangle,$$

which implies (4.18), as claimed. □

Now, we turn to complete the proof of Proposition 4.5. In (4.18), we have shown that, for any $f \in L^*$,

$$-\frac{20}{\sqrt{\beta}}\prec f, \mathcal{A}f\succ - \frac{12}{\sqrt{\beta}}\mathcal{U}_\epsilon\|f\|^2$$
$$\le \prec f, \frac{2}{\sqrt{\beta}}B'f\succ \le \frac{20}{\sqrt{\beta}}\prec f, \mathcal{A}f\succ + \frac{12}{\sqrt{\beta}}\mathcal{U}_\epsilon\|f\|^2.$$

Adding $\prec f, \mathcal{A} f \succ$ on each side of the above inequalities and combining those with the definition that $\mathcal{H}_\beta = \mathcal{A} + \frac{2}{\sqrt{\beta}} B'$ yields

$$\mathcal{A}\Big(1 - \frac{20}{\sqrt{\beta}}\epsilon\Big) - \frac{12}{\sqrt{\beta}}\Big(1 + \frac{1}{2\epsilon}\Big)\mathcal{U}_\epsilon \leq \mathcal{H}_\beta \leq \mathcal{A}\Big(1 + \frac{20}{\sqrt{\beta}}\epsilon\Big) + \frac{12}{\sqrt{\beta}}\Big(1 + \frac{1}{2\epsilon}\Big)\mathcal{U}_\epsilon,$$

which shows that

$$\Big(1 - \frac{20}{\sqrt{\beta}}\epsilon\Big)\lambda_k - \frac{12}{\sqrt{\beta}}\Big(1 + \frac{1}{2\epsilon}\Big)\mathcal{U}_\epsilon \leq \mathbf{\Lambda}_k^\beta \leq \Big(1 - \frac{20}{\sqrt{\beta}}\epsilon\Big)\lambda_k - \frac{12}{\sqrt{\beta}}\Big(1 + \frac{1}{2\epsilon}\Big)\mathcal{U}_\epsilon$$

for all $k \in \mathbb{N}$. Replacing $\epsilon \mapsto \frac{\sqrt{\beta}}{20}\epsilon$ and using the tail bound (4.14) on $\mathcal{U}_\epsilon$ yields Proposition 4.5. ☐

5. Proof of Proposition 4.2

We prove the upper bound (4.4) in Section 5.1 and the lower bound (4.5) in Section 5.2. Before giving these proofs, we recall the behavior of the tail of $\mathbf{a}_1$ (the GUE Tracy–Widom distribution). Numerous works (see, e.g., [7], [28], [58], [128]) have focused on finding the exact tails of $\mathbf{a}_1$ and the following proposition follows from these (e.g., [128, Theorem 1.3]).

PROPOSITION 5.1
Let $\mathbf{a}_1$ denote the top particle in the Airy point process (which follows the GUE Tracy–Widom distribution). Then ($o(1)$ goes to zero as s goes to infinity)

$$\mathbb{P}(\mathbf{a}_1 < -s) = \exp\Big(-\frac{1}{12}\big(s^3 + o(1)\big)\Big). \tag{5.1}$$

5.1. *Proof of the upper bound (4.4)*

Recall $\mathcal{I}_s(\cdot)$ and $\mathcal{J}_s(\cdot)$ from (4.3), related by $\mathcal{I}_s(\cdot) = \exp(\mathcal{J}_s(\cdot))$. Thus, in order to obtain an upper bound on $\mathbb{E}[\prod_{k=1}^\infty \mathcal{I}_s(\mathbf{a}_k)]$, we derive a lower bound on $\sum_{k=1}^\infty \mathcal{J}_s(\mathbf{a}_k)$ by comparing the Airy point process with the corresponding eigenvalues λ_k of the Airy operator (see Section 4.3). Let us denote $\mathcal{D}_k := (-\lambda_k - \mathbf{a}_k)_+ = \max\{-\lambda_k - \mathbf{a}_k, 0\}$.

LEMMA 5.2
Fix some $\epsilon \in (0, 1/3)$. Denote $\theta_0 = \lceil 2s^{\frac{3}{2}}/3\pi \rceil$. There exist $S_0 = S_0(\epsilon) > 0$ and a constant $R > 0$ such that for all $s \geq S_0$,

$$\sum_{k=1}^\infty \mathcal{J}_s(\mathbf{a}_k) \geq T^{\frac{1}{3}}\Big(\frac{4s^{\frac{5}{2}}}{15\pi}(1 - 8\epsilon) - \sum_{k=1}^{\theta_0} \mathcal{D}_k - R\Big). \tag{5.2}$$

Proof
Using monotonicity of $\mathcal{J}_s(\cdot)$ and the inequality (1.11), we obtain the following:

$$\sum_{k=1}^{\infty} \mathcal{J}_s(\mathbf{a}_k) = \sum_{k=1}^{\infty} \mathcal{J}_s\big(-\lambda_k - (-\lambda_k - \mathbf{a}_k)_+ + (-\lambda_k - \mathbf{a}_k)_-\big) \geq \sum_{k=1}^{\infty} \mathcal{J}_s(-\lambda_k - \mathcal{D}_k). \tag{5.3}$$

We divide the sum on the right-hand side of (5.3) into three ranges: $[1,\theta_1]$, (θ_1,θ_2), and $[\theta_2,\infty)$, where θ_1 and θ_2 are defined as (recall $\mathcal{R}(n)$ from Proposition 4.6)

$$\mathcal{K} := \sup_{n\geq 1}\big\{\big|n\mathcal{R}(n)\big|\big\}, \qquad \theta_1 := \lceil 4\mathcal{K}\rceil, \qquad \theta_2 := \Big\lceil \frac{2s^{3/2}}{3\pi} + \frac{1}{2}\Big\rceil.$$

Note that as θ_1 does not depend on s, but θ_2 does, we choose s large enough so that $\theta_1 < \theta_2$.

CLAIM
We have

$$\sum_{k=1}^{\theta_1} \mathcal{J}_s(-\lambda_k - \mathcal{D}_k) \geq T^{\frac{1}{3}}\Big(\theta_1\Big(s - \Big(\frac{3\pi(4\mathcal{K}+1)}{2}\Big)^{\frac{2}{3}}\Big) - \sum_{k=1}^{\theta_1}\mathcal{D}_k\Big). \tag{5.4}$$

Proof of Claim
Since $\log(1+\exp(a)) \geq a$ for any $a \in \mathbb{R}$, $\mathcal{J}_s(\cdot) \geq T^{\frac{1}{3}}(s+\cdot)$. Using this and the monotonicity of λ_k in k, we find that

$$\sum_{k=1}^{\theta_1} \mathcal{J}_s(-\lambda_k - \mathcal{D}_k) \geq \sum_{k=1}^{\theta_1} T^{\frac{1}{3}}(s - \lambda_k - \mathcal{D}_k) \geq T^{\frac{1}{3}}\Big(\theta_1(s-\lambda_{\theta_1}) - \sum_{k=1}^{\theta_1}\mathcal{D}_k\Big).$$

From (4.7), $\lambda_{\theta_1} \leq (3\pi(\theta_1 - \frac{1}{4} + \mathcal{K}/\theta_1)/2)^{\frac{2}{3}} \leq (3\pi(4\mathcal{K}+1)/2)^{\frac{2}{3}}$; hence (5.4) follows immediately. □

CLAIM
We have

$$\sum_{k=\theta_1+1}^{\theta_2-1} \mathcal{J}_s(-\lambda_k - \mathcal{D}_k) \geq T^{\frac{1}{3}}\Big(\frac{4s^{\frac{5}{2}}}{15\pi}(1-3\epsilon) - (\theta_1+1)s - \sum_{k=\theta_1+1}^{\theta_2-1}\mathcal{D}_k\Big). \tag{5.5}$$

Proof of Claim

We assume that $s \geq (3\pi\epsilon^{-1}/4)^{\frac{2}{3}}(1+\epsilon)$. Observe that

$$\sum_{k=\theta_1+1}^{\theta_2-1} \mathcal{J}_s(-\lambda_k - \mathcal{D}_k) \geq T^{\frac{1}{3}} \sum_{k=\theta_1+1}^{\theta_2-1} \Big(\Big(s - \Big(\frac{3\pi k}{2}\Big)^{\frac{2}{3}}\Big) - \sum_{k=\theta_1+1}^{\theta_2-1} \mathcal{D}_k\Big). \tag{5.6}$$

This uses $\log(1+\exp(a)) \geq a$ for all $a \in \mathbb{R}$ and $\lambda_k \leq (3\pi k/2)^{\frac{2}{3}}$ for all $k > \theta_1$. Now we bound

$$\begin{aligned}\sum_{k=\theta_1+1}^{\theta_2-1} \Big(s - \Big(\frac{3\pi k}{2}\Big)^{\frac{2}{3}}\Big) \geq \sum_{k=\theta_1+1}^{\theta_2-1} \Big(s - \Big(\frac{3\pi k}{2}\Big)^{\frac{2}{3}}\Big) &\geq \int_{\theta_1+1}^{\theta_2-1} \Big(s - \Big(\frac{3\pi z}{2}\Big)^{\frac{2}{3}}\Big)\,dz \\ &\geq \int_{0}^{\theta_2-1} \Big(s - \Big(\frac{3\pi z}{2}\Big)^{\frac{2}{3}}\Big)\,dz - (\theta_1+1)s \\ &= (\theta_2-1)\Big(s - \frac{3\cdot(3\pi)^{\frac{2}{3}}}{5\cdot 2^{\frac{2}{3}}}(\theta_2-1)^{\frac{2}{3}}\Big) - (\theta_1+1)s.\end{aligned}$$

Noting that $(1-\epsilon)\frac{2s^{\frac{3}{2}}}{3\pi} \leq \theta_2 - 1 \leq \frac{2s^{\frac{3}{2}}}{3\pi} + 1$, we may bound the above expression such that combining with (5.6) we arrive at the claimed inequality (5.5). □

Plugging the bounds (5.4), (5.5), and $\sum_{k=\theta_2}^{\infty} \mathcal{J}_s(-\lambda_k - \mathcal{D}_k) \geq 0$ into (5.3) yields

$$\sum_{k=1}^{\infty} \mathcal{J}_s(\mathbf{a}_k) \geq \frac{T^{\frac{1}{3}}}{2^{\frac{1}{3}}}\Big(\frac{4s^{\frac{5}{2}}}{15\pi}(1-3\epsilon) - s - \sum_{k=1}^{\theta_2-1} \mathcal{D}_k - \theta_1\Big(\frac{3\pi(\mathcal{K}+1)}{2}\Big)^{3/2}\Big). \tag{5.7}$$

To finally arrive at the desired inequality in (5.2), we use two more bounds. Since we may assume that $s \leq \frac{4\epsilon s^{\frac{5}{2}}}{3\pi}$ for all $s \geq S_0 := (3\pi\epsilon^{-1}/4)^{\frac{2}{3}}(1+\epsilon)$, we can replace $-s$ by $-\frac{4\epsilon s^{\frac{5}{2}}}{3\pi}$ in the right-hand side of (5.7). Finally, for all $\epsilon < 1$, $\theta_1(3\pi(\mathcal{K}+1)/2)^{3/2}$ can be bounded above by a large constant R (independent of s and ϵ). Incorporating these bounds into (5.7) yields (5.2). □

Proof of (4.4) in Proposition 4.2

Multiplying (5.2) by -1 and exponentiating yields

$$\prod_{k=1}^{\infty} \mathcal{I}_s(\mathbf{a}_k) \leq \exp\Big(-T^{\frac{1}{3}}\Big(\frac{4s^{\frac{5}{2}}}{15\pi}(1-8\epsilon) - \sum_{k=1}^{\theta_0} \mathcal{D}_k - R\Big)\Big).$$

Recalling $\theta_0 = \lceil 2s^{\frac{3}{2}}/3\pi \rceil$ and defining $\mathcal{S}_{\theta_0} := \sum_{k=1}^{\theta_0} \mathcal{D}_k$, we have that

$$\mathbb{1}\{\mathcal{S}_{\theta_0} < \epsilon s\theta_0\} \prod_{k=1}^{\infty} \mathcal{I}_s(\mathbf{a}_k) \le \exp\Big(-T^{\frac{1}{3}} \frac{4s^{\frac{5}{2}}}{15\pi}(1-11\epsilon)\Big).$$

If $S_{\theta_0} \ge \epsilon s\theta_0$, then there exists at least one $k \in [1, \theta_0] \cap \mathbb{Z}$ such that $\mathcal{D}_k$ is greater than ϵs. Thus, $\{\mathcal{S}_{\theta_0} \ge \epsilon s\theta_0\} \subset \bigcup_{k=1}^{\theta_0}\{\mathcal{D}_k \ge \epsilon s\}$. Summarizing the discussion above, we have that

$$\begin{aligned}
&\mathbb{E}\Big[\prod_{k=1}^{\infty} \mathcal{I}_s(\mathbf{a}_k)\Big] \\
&\quad = \mathbb{E}\Big[\mathbb{1}\{\mathcal{S}_{\theta_0} < \epsilon s\theta_0\} \prod_{k=1}^{\infty} \mathcal{I}_s(\mathbf{a}_k)\Big] + \mathbb{E}\Big[\mathbb{1}\{\mathcal{S}_{\theta_0} \ge \epsilon s\theta_0\} \prod_{k=1}^{\infty} \mathcal{I}_s(\mathbf{a}_k)\Big] \\
&\quad \le \exp\Big(-T^{\frac{1}{3}} \frac{4s^{\frac{5}{2}}}{15\pi}(1-11\epsilon)\Big) + \mathbb{E}\Big[\mathbb{1}\Big\{\bigcup_{k=1}^{\theta_0}\{\mathcal{D}_k \ge \epsilon s\}\Big\} \prod_{k=1}^{\infty} \mathcal{I}_s(\mathbf{a}_k)\Big].
\end{aligned} \tag{5.8}$$

We may bound indicator functions

$$\mathbb{1}\Big\{\bigcup_{k=1}^{\theta_0}\{\mathcal{D}_k \ge \epsilon s\}\Big\} \le \mathbb{1}\Big\{\bigcup_{k=1}^{\theta_0}\{\mathcal{D}_k \ge \epsilon s\} \cap \{\mathbf{a}_1 \ge -(1-\epsilon)s\}\Big\} + \mathbb{1}\{\mathbf{a}_1 \le -(1-\epsilon)s\}.$$

Since $\mathcal{I}_s(\mathbf{a}_k) \le 1$ for all $k \in \mathbb{Z}_{>0}$, when $\mathbf{a}_1 \ge -(1-\epsilon)s$,

$$\prod_{k=1}^{\infty} \mathcal{I}_s(\mathbf{a}_k) \le \frac{1}{1+\exp(T^{\frac{1}{3}}(s+\mathbf{a}_1))} \le \exp(-\epsilon s T^{\frac{1}{3}}).$$

Combining these observations and taking expectations implies that

$$\begin{aligned}
&\mathbb{E}\Big[\mathbb{1}\Big\{\bigcup_{k=1}^{\theta_0}\{\mathcal{D}_k \ge \epsilon s\}\Big\} \prod_{k=1}^{\infty} \mathcal{I}_s(\mathbf{a}_k)\Big] \\
&\qquad \le \exp(-\epsilon s T^{\frac{1}{3}})\mathbb{P}\Big(\bigcup_{k=1}^{\theta_0}\{\mathcal{D}_k \ge \epsilon s\}\Big) + \mathbb{P}\big(\mathbf{a}_1 \le -(1-\epsilon)s\big).
\end{aligned} \tag{5.9}$$

By Proposition 5.1, there exists $C > 0$ such that for s large enough $\mathbb{P}(\mathbf{a}_1 \le -(1-\epsilon)s) \le \exp(-\frac{s^3}{12}(1-C\epsilon))$. Combining (5.8), (5.9), and (5.10) in Lemma 5.3, we find (4.4). ☐

LEMMA 5.3
Fix $\epsilon, \delta \in (0, 1/3)$. There exist $S_0 = S_0(\eta, \delta) > 0$ and $K_1 = K_1(\eta, \delta) > 0$ such that the following holds for all $s \ge S_0$. Divide the interval $[-s, 0]$ into $\lceil 2\epsilon^{-1}\rceil$ segments $\mathcal{Q}_i :=$

$[-j\epsilon s/2, -(j-1)\epsilon s/2)$ for $j = 1, \dots, \lceil 2\epsilon^{-1}\rceil$. Denote the right and left endpoints of $\mathcal{Q}_j$ by q_j and p_j. Define $k_j := \inf\{k : -\lambda_k \geq q_j\}$ ($\lambda_1 < \lambda_2 < \cdots$ are the Airy operator eigenvalues). Then (recalling $\theta_0 = \lceil 2s^{\frac{3}{2}}/3\pi\rceil$)

$$\mathbb{P}(\mathbf{a}_{k_j} \leq p_j) \leq \exp(-K_1 s^{3-\delta}) \quad \forall j \in \{1, \dots, \lceil 2\eta^{-1}\rceil\} \tag{5.10}$$

$$\mathbb{P}\Big(\bigcup_{k=1}^{\theta_0} \{\mathcal{D}_k \geq \epsilon s\}\Big) \leq \exp(-K_1 s^{3-\delta}).$$

Proof
We prove the first line of (5.10). For $1 \leq j \leq \lceil 2\epsilon^{-1}\rceil$, when $\mathbf{a}_{k_j} \leq p_j = -2^{-1}(j\epsilon s)$,

$$\chi^{\mathrm{Ai}}\big([-2^{-1}(j\epsilon s), \infty)\big) \leq k_j \leq \#\big\{k : -\lambda_k \geq -2^{-1}(j-1)\epsilon s\big\}. \tag{5.11}$$

Owing to Propositions 1.3 and 4.6, we have

$$\begin{aligned} \#\{k : -\lambda_k \leq -x\} &=: \frac{2}{3\pi}x^3 + C_1(x) \\ \mathbb{E}\big[\chi^{\mathrm{Ai}}\big([-x, \infty)\big)\big] &=: \frac{2}{3\pi}x^3 + C_2(x), \end{aligned} \tag{5.12}$$

where $\sup_{x\geq 0}\{|C_1(x)|, |C_2(x)|\} < \infty$. Combining (5.11) and (5.12) shows that when $\mathbf{a}_{k_j} \leq p_j$,

$$\begin{aligned} &\chi^{\mathrm{Ai}}\big([-2^{-1}(j\epsilon s), \infty)\big) - \mathbb{E}\big[\chi^{\mathrm{Ai}}\big([-2^{-1}(j\epsilon s), \infty)\big)\big] \\ &\quad \leq \#\big\{k : -\lambda_k \geq -2^{-1}(j-1)\epsilon s\big\} - \#\{k : -\lambda_k \geq -2^{-1} j\epsilon s\} \\ &\qquad + C_1(2^{-1}j\epsilon s) - C_2(2^{-1}j\epsilon s) \\ &\quad \leq \frac{(\epsilon s)^{\frac{3}{2}}}{3\sqrt{2}\pi}\big((j-1)^{\frac{3}{2}} - j^{\frac{3}{2}}\big) + C_1(2^{-1}j\epsilon s) - C_2(2^{-1}j\epsilon s) \\ &\quad \leq -M\sqrt{j}(\epsilon s)^{\frac{3}{2}} + C_1(2^{-1}j\epsilon s) - C_2(2^{-1}j\epsilon s) \end{aligned}$$

for some $M > 0$. Therefore,

$$\begin{aligned} \mathbb{P}(\mathbf{a}_{k_j} \leq p_j) \leq \mathbb{P}\Big(&\chi^{\mathrm{Ai}}\big([p_j, \infty)\big) - \mathbb{E}\big[\chi^{\mathrm{Ai}}\big([p_j, \infty)\big)\big] \\ &\leq -M\sqrt{j}(\epsilon s)^{\frac{3}{2}} + 2\sup_{x\geq 0}\big\{\big|C_1(x)\big|, \big|C_2(x)\big|\big\}\Big). \end{aligned}$$

For large enough s, $-M\sqrt{j}(\epsilon s)^{\frac{3}{2}} + 2\sup_{x\geq 0}\{|C_1(x)|, |C_2(x)|\} \leq -\frac{M}{2}\sqrt{j}(\epsilon s)^{\frac{3}{2}}$ for all $j \in \{1, \dots, \lceil 2\epsilon^{-1}\rceil\}$. The first line of (5.10) follows by applying (1.10) of Theorem 1.4 which shows that there exist $S_0(\epsilon, \delta)$ and $K_1 = K_1(\epsilon, \delta)$ such that for all

$s \geq S_0$,

$$\mathbb{P}\Big(\chi^{\mathrm{Ai}}\big([-(j\epsilon s),\infty)\big) - \mathbb{E}[\chi^{\mathrm{Ai}}\big([-(j\epsilon s),\infty)\big)] \leq -\frac{M}{2}\sqrt{j}(\epsilon s)^{\frac{3}{2}}\Big) \leq \exp(-K_1 s^{3-\delta}).$$

Turning to the second line of (5.10), we assume (as allowed by (4.7)) that s is large enough so that $\lambda_{\theta_0} < s$. We claim then that

$$\bigcup_{k=1}^{\theta_0}\{\mathcal{D}_k \geq \epsilon s\} \subset \bigcup_{j=1}^{\lceil 2\epsilon^{-1}\rceil}\{\mathbf{a}_{k_j} \leq p_j\}. \tag{5.13}$$

To see this, consider any integer $1 \leq k \leq \theta_0$, and assume that $\mathcal{D}_k \geq \epsilon s$. Let j be such that $-\lambda_k \in \mathcal{Q}_{j-1}$. Since $\mathcal{Q}_{j-1}$ is to the right of $\mathcal{Q}_j = [p_j, q_j]$, it follows that $\mathbf{a}_k \leq -\lambda_k - \epsilon s$. Moreover, $\mathbf{a}_{k_j} \leq \mathbf{a}_k$ because $-\lambda_{k_j} < -\lambda_k$. Combining these yields

$$\mathbf{a}_{k_j} < \mathbf{a}_k \leq -\lambda_k - \epsilon s = (\lambda_{k_j} - \lambda_k) - \lambda_{k_j} - \epsilon s \leq -\lambda_{k_j} - \frac{\epsilon s}{2},$$

where the last inequality uses $0 \leq (\lambda_{k_j} - \lambda_k) \leq \frac{\epsilon s}{2}$ (as $\lambda_{k_j}, \lambda_k \in \mathcal{Q}_{j-1}$). Hence, the distance between $\mathbf{a}_{k_j}$ and λ_{k_j} is greater than or equal to $\epsilon s/2$. This shows that $\mathbf{a}_{k_j} \leq p_j$, and hence (5.13).

The first line of (5.10) along with (5.13) implies that

$$\mathbb{P}\Big(\bigcup_{k=1}^{\theta_0}\{\mathcal{D}_k \geq \epsilon s\}\Big) \leq \sum_{i=1}^{\theta_0}\mathbb{P}(\mathbf{a}_{k_i} \leq p_i) \leq \lceil 2\epsilon^{-1}\rceil \exp(-K_1 s^{3-\delta}).$$

As long as s is sufficiently large, the $\lceil 2\epsilon^{-1}\rceil$ prefactor can be absorbed into the exponent at the cost of slightly modifying K_1. □

5.2. Proof of the lower bound (4.5)

In order to obtain a lower bound on $\mathbb{E}[\prod_{k=1}^{\infty}\mathcal{I}_s(\mathbf{a}_k)]$, we derive an upper bound on $\sum_{k=1}^{\infty}\mathcal{J}_s(\mathbf{a}_k)$.

LEMMA 5.4

There exist $B > 0$ and S_0 such that for all $\epsilon \in (0, 1/3)$ and all $s \geq S_0$,

$$\sum_{k=1}^{\infty}\mathcal{J}_s(\mathbf{a}_k) \leq \mathcal{L}_{T,\epsilon}(s + C_\epsilon^{\mathrm{Ai}}), \tag{5.14}$$

where

$$\mathcal{L}_{T,\epsilon}(x) := T^{\frac{1}{3}}\Big(\frac{4x^{\frac{5}{2}}}{15\pi}(1+3\epsilon) + 2x + B\Big) + \frac{x^{\frac{3}{2}}}{3(1-\epsilon)^{\frac{3}{2}}} + \sqrt{\frac{2}{\pi}}\frac{x^{\frac{3}{4}}}{(1-\epsilon)^{\frac{3}{4}}} + \frac{4}{T\pi(1-\epsilon)^3}.$$

Proof

Using the monotonicity of $\mathcal{J}_s(\cdot)$ and the inequality (1.11), we obtain

$$\sum_{k=1}^{\infty} \mathcal{J}_s(\mathbf{a}_k) \le \sum_{k=1}^{\infty} \mathcal{J}_s\big(-(1-\epsilon)\lambda_k + C_\epsilon^{\mathrm{Ai}}\big) = (\widetilde{\mathbf{I}}) + (\widetilde{\mathbf{II}}) + (\widetilde{\mathbf{III}}), \tag{5.15}$$

where $(\widetilde{\mathbf{I}})$, $(\widetilde{\mathbf{II}})$, and $(\widetilde{\mathbf{III}})$ equal the sum of $\mathcal{J}_s(-(1-\epsilon)\lambda_k + C_\epsilon^{\mathrm{Ai}})$ over all integers k in the intervals $[1, \theta_1']$, (θ_1', θ_2') and $[\theta_2', \infty)$, respectively, and (similar to Section 5.1) θ_1' and θ_2' are

$$\theta_1' := \theta_1 = \Big\lceil 4 \sup_{n\in\mathbb{Z}_{>0}} n\big|\mathcal{R}(n)\big|\Big\rceil, \qquad \theta_2' := \Big\lceil \frac{2(s+C_\epsilon^{\mathrm{Ai}})^{\frac{3}{2}}}{3\pi(1-\epsilon)^{\frac{3}{2}}} + \frac{1}{2}\Big\rceil.$$

For any integer $1 \le k \le \theta_1'$, $\mathcal{J}_s(-(1-\epsilon)\lambda_k + C_\epsilon^{\mathrm{Ai}}) \le \mathcal{J}_s(-(1-\epsilon)\lambda_1 + C_\epsilon^{\mathrm{Ai}})$. Using this upper bound and the inequality $\log(1+\exp(a)) \le a + \pi/2$ for $a > 0$, we obtain

$$(\widetilde{\mathbf{I}}) \le \theta_1' \mathcal{J}_s\big(-(1-\epsilon)\lambda_1 + C_\epsilon^{\mathrm{Ai}}\big) \le \theta_1' T^{\frac{1}{3}}\big(s - (1-\epsilon)\lambda_1 + C_\epsilon^{\mathrm{Ai}}\big) + \frac{\pi\theta'}{2}. \tag{5.16}$$

CLAIM

We have

$$(\widetilde{\mathbf{II}}) \le T^{\frac{1}{3}}\Big(\frac{4(s+C_\epsilon^{\mathrm{Ai}})^{\frac{5}{2}}}{15\pi}(1+3\epsilon) + (2-\theta_1')(s+C_\epsilon^{\mathrm{Ai}}) - \frac{3(3\pi)^{2/3}(\theta_1')^{5/3}}{5\cdot 2^{2/3}}\Big) + \frac{\pi(\theta_2'-\theta_1')}{2}. \tag{5.17}$$

Proof of Claim

For integers $k \in (\theta_1', \infty)$, it follows from the definition of θ_1' that

$$\lambda_k \ge \Big(\frac{3\pi(k - \frac{1}{4} - |\mathcal{R}(k)|)}{2}\Big)^{\frac{2}{3}} \ge \Big(\frac{3\pi(k-\frac{1}{2})}{2}\Big)^{\frac{2}{3}}. \tag{5.18}$$

This and the monotonicity of $\mathcal{J}_s(\cdot)$ implies that

$$\mathcal{J}_s\big(-(1-\epsilon)\lambda_k + C_\epsilon^{\mathrm{Ai}}\big) \le \mathcal{J}_s\Big(-(1-\epsilon)\Big(\frac{3\pi(k-\frac{1}{2})}{2}\Big)^{\frac{2}{3}} + C_\epsilon\Big).$$

Leveraging this and using the inequality $\mathcal{J}_s(a) \leq a + \pi/2$ for any $a > 0$, we obtain

$$(\widetilde{\mathbf{II}}) \leq \sum_{k=\theta_1'+1}^{\theta_2'-1} \Big(T^{\frac{1}{3}} f_s(k) + \frac{\pi}{2} \Big),$$

$$\text{where } f_s(z) := s + C_\epsilon^{\mathrm{Ai}} - (1-\epsilon)\Big(\frac{3\pi(z-\frac{1}{2})}{2} \Big)^{\frac{2}{3}}. \tag{5.19}$$

Bounding the sum in (5.19) by the corresponding integral, we find that

$$(\widetilde{\mathbf{II}}) \leq T^{\frac{1}{3}} \int_{\theta_1'}^{\theta_2'} f_s(z)\, dz + \frac{\pi(\theta_2' - \theta_1')}{2}. \tag{5.20}$$

To bound $\int_{\theta_1'}^{\theta_2'} f_s(z)\, dz$, we observe that

$$\begin{aligned}
\int_{\frac{1}{2}}^{\theta_2'} f_s(z)\, dz &\leq (s + C_\epsilon^{\mathrm{Ai}}) \Big(\frac{2(s + C_\epsilon^{\mathrm{Ai}})^{\frac{3}{2}}}{3\pi(1-\epsilon)^{\frac{3}{2}}} + \frac{3}{2} \Big) \\
&\quad - (1-\epsilon)\frac{3}{5} \cdot \Big(\frac{3\pi}{2} \Big)^{\frac{2}{3}} \Big(\frac{2(s + C_\epsilon^{\mathrm{Ai}})^{\frac{3}{2}}}{3\pi(1-\epsilon)^{\frac{3}{2}}} \Big)^{\frac{5}{3}} \\
&= \frac{4(s + C_\epsilon^{\mathrm{Ai}})^{\frac{5}{2}}}{15\pi(1-\epsilon)^{\frac{3}{2}}} + \frac{3}{2}(s + C_\epsilon^{\mathrm{Ai}}) \\
&\leq \frac{4(s + C_\epsilon^{\mathrm{Ai}})^{\frac{5}{2}}}{15\pi}(1 + 3\epsilon) + \frac{3}{2}(s + C_\epsilon^{\mathrm{Ai}}), \\
\int_{\frac{1}{2}}^{\theta_1'} f(z)\, dz &\geq (s + C_\epsilon^{\mathrm{Ai}})\Big(\theta_1' - \frac{1}{2} \Big) - \int_{\frac{1}{2}}^{\theta_1'} \Big(\frac{3\pi(z-1)}{2} \Big)^{\frac{2}{3}} dz \\
&= (s + C_\epsilon^{\mathrm{Ai}})\Big(\theta_1' - \frac{1}{2} \Big) - \frac{3}{5} \cdot \Big(\frac{3\pi}{2} \Big)^{\frac{2}{3}} \cdot \Big(\theta_1' - \frac{1}{2} \Big)^{\frac{5}{3}}.
\end{aligned}$$

Combining these bounds with (5.20) yields the upper bound on $(\widetilde{\mathbf{II}})$ in (5.17). □

CLAIM
We have

$$(\widetilde{\mathbf{III}}) \leq \sqrt{\frac{2}{\pi}} \frac{(s + C_\epsilon^{\mathrm{Ai}})^{\frac{3}{4}}}{(1-\epsilon)^{\frac{3}{4}}} + \frac{4}{T\pi(1-\epsilon)^3}. \tag{5.21}$$

Proof of Claim
Using the inequality $\log(1+z) \leq z$ for all $z \geq 0$, we find that

$$\mathcal{J}_s\big(-(1-\epsilon)\lambda_k + C_\epsilon^{\mathrm{Ai}}\big) \le \exp\big(T^{\frac{1}{3}}\big(s-(1-\epsilon)\lambda_k + C_\epsilon^{\mathrm{Ai}}\big)\big). \tag{5.22}$$

Plugging the lower bound on λ_k from (5.18) into (5.22), we find (recalling $f_s(z)$ from (5.19)) that

$$(\widetilde{\mathbf{III}}) \le \sum_{k=\theta_2'}^{\infty} \exp\big(T^{\frac{1}{3}} f_s(k)\big). \tag{5.23}$$

Noting that $f_s(k) \le f_s(\theta_2') < 0$ for all $k > \theta_2'$, we find that for all $k > \theta_2' + \sqrt{3\theta_2'}$,

$$f_s(k) < (1-\epsilon)\Big(\frac{3\pi(\theta_2'-\frac{1}{2})}{2}\Big)^{\frac{2}{3}} - (1-\epsilon)\Big(\frac{3\pi(k-\frac{1}{2})}{2}\Big)^{\frac{2}{3}} \le -(1-\epsilon)\Big(\frac{3\pi(k-\theta_2')}{2}\Big)^{\frac{1}{3}}.$$

The first inequality uses $f_s(\theta_2') < 0$ and the second follows from Lemma 5.6 (we assume that s is large enough so that $\theta_2' - \frac{1}{2} > 27$). Utilizing this estimate yields

$$\begin{aligned}
(\widetilde{\mathbf{III}}) &\le \sum_{k=\theta_2'}^{k=\theta_2'+\sqrt{3\theta_2'}} \exp\big(T^{\frac{1}{3}} f_s(k)\big) + \sum_{k>\theta_2'+\sqrt{3\theta_2'}} \exp\big(T^{\frac{1}{3}} f_s(k)\big) \\
&\le \sqrt{3\theta_2'} + \sum_{k=\theta_2'+\sqrt{3\theta_2'}}^{\infty} \exp\Big(-(1-\epsilon)T^{\frac{1}{3}}\Big(\frac{3\pi(k-\theta_2')}{2}\Big)^{\frac{1}{2}}\Big) \\
&\le \sqrt{3\theta_2'} + \int_0^{\infty} \exp\Big(-(1-\epsilon)T^{\frac{1}{3}}\Big(\frac{3\pi z}{2}\Big)^{\frac{1}{3}}\Big)\, dz \\
&= \sqrt{3\theta_2'} + \frac{4}{T\pi(1-\epsilon)^3} \le \sqrt{\frac{2}{\pi}}\, \frac{(s+C_\epsilon^{\mathrm{Ai}})^{\frac{3}{4}}}{(1-\epsilon)^{\frac{3}{4}}} + \frac{4}{T\pi(1-\epsilon)^3}.
\end{aligned} \tag{5.24}$$

The first inequality follows from (5.23); the second follows from the bound

$$\exp\big(T^{\frac{1}{3}} f_s(k)\big) \le \begin{cases} 1 & k \in [\theta_2', \theta_2' + \sqrt{3\theta_2'}], \\ \exp\big(-(1-\epsilon)T^{\frac{1}{3}}(\frac{3\pi(k-\theta_2')}{2})^{\frac{1}{3}}\big) & k \in [\theta_2' + \sqrt{3\theta_2'}, \infty); \end{cases}$$

the third uses that the sum is bounded by the integral; and the last inequality uses that for s large enough, $\sqrt{3\theta_2'} \le \sqrt{\frac{2}{\pi}} \frac{(s+C_\epsilon^{\mathrm{Ai}})^{\frac{3}{4}}}{(1-\epsilon)^{\frac{3}{4}}}$. This completes the proof of (5.21). □

Plugging the upper bounds of $(\widetilde{\mathbf{I}})$, $(\widetilde{\mathbf{II}})$, and $(\widetilde{\mathbf{III}})$ obtained in (5.16), (5.17), and (5.21), respectively, into (5.15), we arrive at (5.14). □

Proof of (4.5)

CLAIM

Fix any $\epsilon, \delta \in (0, 1/3)$ and $T_0 > 0$. Then there exist $\kappa = \kappa(\epsilon, \delta, T_0) > 0$ and $S_0 = S_0(\epsilon, \delta, T_0) > 0$ such that for all $s \geq S_0$ and $T > T_0$,

$$\mathbb{E}_{\mathrm{Airy}}\Big[\mathbb{1}(\mathbf{a}_1 \geq -s) \prod_{k=1}^{\infty} \mathcal{I}(\mathbf{a}_k)\Big] \\ \geq \big(1 - 2\exp(-\kappa s^{1-2\delta})\big) \exp\Big(-T^{\frac{1}{3}} \frac{4s^{\frac{5}{2}}}{15\pi}(1+9\epsilon)\Big). \tag{5.25}$$

Proof of Claim

Negating both sides of (5.14) and exponentiating yields $\prod_{k=1}^{\infty} \mathcal{I}(\mathbf{a}_k) \geq \exp(-\mathcal{L}_{T,\epsilon}(s + C_\epsilon^{\mathrm{Ai}}))$. Along with the monotonicity of $\mathcal{L}_{T,\epsilon}(\cdot)$, this implies that

$$\mathbb{E}_{\mathrm{Airy}}\Big[\mathbb{1}(\mathbf{a}_1 \geq -s) \prod_{k=1}^{\infty} \mathcal{I}(\mathbf{a}_k)\Big] \\ \geq \mathbb{P}(\mathbf{a}_1 \geq -s, C_\epsilon^{\mathrm{Ai}} < s^{1-\delta}) \exp\big(-\mathcal{L}_{T,\epsilon}(s + s^{1-\delta})\big). \tag{5.26}$$

Taking s large enough, we have the bounds

$$T^{\frac{1}{3}} \frac{4(s+s^{1-\delta})^{\frac{5}{2}}}{15\pi} \leq T^{\frac{1}{3}} \frac{4s^{\frac{5}{2}}}{15\pi}(1+5\epsilon), \qquad T^{\frac{1}{3}}\big(2(s+s^{1-\delta}) + B\big) \leq T^{\frac{1}{3}} \frac{4s^{\frac{5}{2}}}{15\pi}\epsilon,$$
$$\frac{(s+s^{1-\delta})^{\frac{3}{2}}}{3(1-\epsilon)^{\frac{3}{2}}} \leq T^{\frac{1}{3}} \frac{4s^{\frac{5}{2}}}{15\pi}\epsilon, \qquad \sqrt{\frac{2}{\pi}} \frac{(s+s^{1-\delta})^{\frac{3}{4}}}{(1-\epsilon)^{\frac{3}{4}}} \leq T^{\frac{1}{3}} \frac{4s^{\frac{5}{2}}}{15\pi}\epsilon,$$
$$\frac{4}{T\pi(1-\epsilon)^3} \leq T^{\frac{1}{3}} \frac{4s^{\frac{5}{2}}}{15\pi}\epsilon.$$

Using these bounds, we find that

$$\mathcal{L}_{T,\epsilon}(s + s^{1-\delta}) \leq T^{\frac{1}{3}} \cdot \frac{4s^{\frac{5}{2}}}{15\pi}(1+9\epsilon). \tag{5.27}$$

Thanks to (1.12) of Theorem 1.6, there exist $\kappa = \kappa(\epsilon, \delta)$ and $S_0' = S_0'(\epsilon, \delta)$ such that for all $s \geq S_0'$, $\mathbb{P}(C_\epsilon^{\mathrm{Ai}} < s^{1-\delta}) > 1 - \exp(-\kappa s^{1-2\delta})$. Moreover, using (5.1), we find that for large enough s, $\mathbb{P}(\mathbf{a}_1 \leq -s) \leq \kappa \exp(-\kappa s^{1-2\delta})$. This implies that for large enough s,

$$\mathbb{P}(\mathbf{a}_1 \geq -s, C_\epsilon^{\mathrm{Ai}} < s^{1-\delta}) \geq \mathbb{P}(\mathbf{a}_1 \geq -s) + \mathbb{P}(C_\epsilon^{\mathrm{Ai}} < s^{1-\delta}) - 1 \geq 1 - 2\exp(-\kappa s^{1-2\delta}).$$

Plugging this and (5.27) into (5.26) yields (5.25). ☐

CLAIM

Fix $\epsilon \in (0,1/3)$ *and* $T_0 > 0$. *Then there exist* $K = K(\epsilon, T_0) > 0$ *and* $S_0 = S_0(\epsilon, T_0) > 0$ *such that for all* $s \geq S_0$,

$$\mathbb{E}_{\mathrm{Airy}}\Big[\mathbb{1}(\mathbf{a}_1 < -s)\prod_{k=1}^{\infty} \mathcal{I}(\mathbf{a}_k)\Big] \geq \exp(-Ks^3). \tag{5.28}$$

Proof of Claim

We begin with a brief description of our proof technique. Let us denote $\theta_0' := \lceil s^{1+\delta}\rceil$. We consider a finite sequence of intervals

$$\mathfrak{I}_1 := [-s^2, -s), \mathfrak{I}_2 := [-2s^2, -s^2), \dots, \mathfrak{I}_{\theta_0'} := \big[-\theta_0' s^2, -(\theta_0' - 1)s^2\big).$$

The length of each of the intervals is s^2 and there are θ_0' intervals. For any integer $\ell \in (1, \theta_0'] \cap \mathbb{Z}$ (resp., $= 1$), note that $\sum_{\mathbf{a}_k \in \mathfrak{I}_\ell} \mathcal{J}_s(\mathbf{a}_k)$ is less than or equal to $\sum_{\mathbf{a}_k \in \mathfrak{I}_\ell} \mathcal{J}_s(-(\ell-1)s^2)$ (resp., $\sum_{\mathbf{a}_k \in \mathfrak{I}_1} \mathcal{J}_s(-s)$) with equality when all the $\mathbf{a}_k$'s in the interval $\mathfrak{I}_\ell$ coincide with the right endpoint $-(\ell-1)s^2$ (resp., $-s$). We show that with high probability the number of Airy points inside the interval $\mathfrak{I}_\ell$ cannot differ considerably from its expected value. Based on this, we argue that the probability of an abundant accumulation of the Airy points inside any of the intervals $\mathfrak{I}_1, \dots, \mathfrak{I}_{\theta_0'}$ is small in comparison to $\mathbb{P}(\mathbf{a}_1 \leq -s)$. Moreover, the contributions of those Airy points which fall into any of those intervals are bounded from above by the result of moving the points to the right endpoint of the interval. Finally, using the upper tail estimate of C_ϵ^{Ai} (see (1.12) of Theorem 1.6), we show that the $\mathbf{a}_k$'s which fall in the region $(-\infty, -\theta_0' s^2)$ hardly contribute to the product $\sum_{k=1}^{\infty} \mathcal{I}_s(\mathbf{a}_k)$.

Now we provide the details of the above sketch. First, we find an upper bound on $\sum_{\mathbf{a}_k \in \widetilde{\mathfrak{I}}} \mathcal{J}_s(\mathbf{a}_k)$, where $\widetilde{\mathfrak{I}} := \bigcup_{\ell=1}^{\theta_0'} \mathfrak{I}_\ell$. Recall that the number of $\mathbf{a}_k$'s in a Borel set D is given by $\chi^{\mathrm{Ai}}(D)$. By replacing all the $\mathbf{a}_k$'s inside the interval $\mathfrak{I}_k$ by the right endpoint of the interval, we obtain

$$\sum_{\mathbf{a}_k \in \mathfrak{I}_\ell} \mathcal{J}(\mathbf{a}_k) \leq \begin{cases} \chi^{\mathrm{Ai}}(\mathfrak{I}_\ell)\log(1+\exp(T^{\frac{1}{3}}(s-(\ell-1)s^2))) & \text{when } \ell > 1, \\ \chi^{\mathrm{Ai}}(\mathfrak{I}_1)\log(2) & \text{when } \ell = 1. \end{cases}$$

Next, using Theorem 1.5, we observe that for large enough s, $\chi^{\mathrm{Ai}}(\mathfrak{I}_l)$ is bounded above by $\mathbb{E}[\chi^{\mathrm{Ai}}(\mathfrak{I}_\ell)] + \epsilon s^3$ with probability greater than $1 - K_1\exp(-K_2 s^3 \log s)$. Owing to Proposition 1.3, there exists a constant M such that for large enough s,

$$\mathbb{E}\big[\chi^{\mathrm{Ai}}(\mathfrak{I}_\ell)\big] = \frac{2}{3\pi}\big(\ell^{\frac{3}{2}} - (\ell-1)^{\frac{3}{2}}\big)s^3 + \mathfrak{D}(\ell s^2) - \mathfrak{D}\big((\ell-1)s^2\big) \leq \frac{M\sqrt{\ell}s^3}{\pi}.$$

Consequently, with probability exceeding $1 - \theta_0' K_1 \exp(-K_2 s^3 \log s)$,

$$\sum_{\mathbf{a}_k \in \widetilde{\mathfrak{I}}} \mathcal{J}_s(\mathbf{a}_k) \leq \Big(\frac{Ms^3}{\pi} + \epsilon s^3\Big)\Big(\log 2 + \sum_{\ell=2}^{\theta_0'} \sqrt{\ell} \log\big(1 + \exp\big(T^{\frac{1}{3}}\big(s - (l-1)s^2\big)\big)\big)\Big).$$

Since $\log(1+x) \leq x$ for all $x > 0$ and $(\ell-1)s^2 - s \geq (l-1)s^2(1-\epsilon)$ for all $s \geq \epsilon^{-1}$, we conclude that there exists a constant C such that for large enough s, with probability exceeding $1 - \theta_0' K_1 \exp(-K_2 s^3 \log s)$,

$$\begin{aligned}\sum_{\mathbf{a}_k \in \widetilde{\mathfrak{I}}} \mathcal{J}_s(\mathbf{a}_k) &\leq s^3\Big(\frac{M}{\pi} + \epsilon\Big)\Big(\log 2 + \sum_{\ell=2}^{\theta_0'} \sqrt{\ell} \exp\big(-(\ell-1)(1-\epsilon)T^{\frac{1}{3}} s^2\big)\Big) \\ &\leq Cs^3. \end{aligned} \tag{5.29}$$

We now turn to bound the remaining sum $\sum_{\mathbf{a}_k < -\theta_0' s^2} \mathcal{J}_s(\mathbf{a}_k)$. For this, we consider the following decomposition:

$$\sum_{k:\mathbf{a}_k < -\theta_0' s^2} \mathcal{J}_s(\mathbf{a}_k) = (\mathbf{A}) + (\mathbf{B}),$$

$$(\mathbf{A}) := \sum_{k:\mathbf{a}_k < -\theta_0' s^2, \lambda_k \leq \theta_0' s^2} \mathcal{J}_s(\mathbf{a}_k),$$

$$(\mathbf{B}) := \sum_{k:\mathbf{a}_k < -\theta_0' s^2, \lambda_k > \theta_0' s^2} \mathcal{J}_s(\mathbf{a}_k).$$

Proposition 4.6 shows that $\#\{\lambda_k \leq \theta_0' s^2\} \leq Cs^{\frac{9}{2} + \frac{3\delta}{2}}$ for large enough s and some constant $C > 0$. This along with the bound $\log(1+a) \leq a$ for all $a > 0$ implies that

$$\mathcal{J}_s(\mathbf{a}_k) \leq \exp\big(T^{\frac{1}{3}}(s - \theta_0' s^2)\big) \leq \exp\big(-(1-\epsilon)T^{\frac{1}{3}} s^3\big)$$

when $\mathbf{a}_k \leq -\theta_0' s^2$ and $s \geq \epsilon^{-\frac{1}{2}}$. Thus, for large enough s,

$$(\mathbf{A}) \leq Cs^{\frac{9}{2} + \frac{3\delta}{2}} \exp\big(-(1-\epsilon)T^{\frac{1}{3}} s^3\big) \leq s^3. \tag{5.30}$$

Now we turn to bound $(\mathbf{B})$. Recall the inequality $\mathcal{J}_s(\mathbf{a}_k) \leq \mathcal{J}_s(-(1-\epsilon)\lambda_k + C_\epsilon^{\mathrm{Ai}})$ which we obtain by using the monotonicity of $\mathcal{J}_s$ and the inequality (1.11). We will now employ Theorem 1.6, but to avoid confusion in notation, let us temporarily rename the variables s and δ in the statement of Theorem 1.6 by $\tilde{s}$ and $\tilde{\delta}$. Then, taking $\tilde{s} = s^{3+\frac{\delta}{2}}$ and $\tilde{\delta} = \frac{\delta}{2(3+\delta/2)}$, the corollary implies that there exist $\kappa = \kappa(\epsilon, \delta) > 0$ and $S_0 = S_0(\epsilon, \delta) > 0$ such that for all $s \geq S_0$, $\mathbb{P}(C_\epsilon^{\mathrm{Ai}} < s^{3+\frac{\delta}{2}}) \geq 1 - \exp(-\kappa s^{3+\frac{\delta}{4}})$. Since $\theta_0' s^2 \approx s^{3+\delta}$, we have $s + s^{3+\frac{\delta}{2}} \leq (1-\epsilon)\theta_0' s^2$ for large enough s. Consequently, for large enough s,

$$\mathbb{P}\Big((\mathbf{B}) \leq \sum_{\lambda_k > \theta_0' s^2} \mathcal{J}_s\big((1-\epsilon)(\theta_0' s^2 - \lambda_k) - s\big)\Big) \geq 1 - \exp(-\kappa s^{3+\frac{\delta}{4}}). \tag{5.31}$$

Plugging the inequality (5.35) in Lemma 5.5 into (5.31) and using (5.30) along with the fact that $(\theta_0' s^2)^{\frac{3}{4}} \leq C s^3$ for some constant C, we find that for large enough s,

$$\mathbb{P}\big((\mathbf{A}) + (\mathbf{B}) \leq C s^3\big) \geq 1 - \exp(-\kappa s^{3+\frac{\delta}{4}}).$$

Combining this with the probability bound computed on the event in (5.29) implies that there exists a constant $C = C(\epsilon, \delta, T_0) > 0$ such that for s large enough,

$$\mathbb{P}(\mathcal{A}) \geq 1 - \theta_0' K_1 \exp(-K_2 s^3 \log s) - \exp(-\kappa s^{3+\frac{\delta}{4}}), \tag{5.32}$$

where $\mathcal{A} := \{\sum_{k=1}^{\infty} \mathcal{J}_s(\mathbf{a}_k) \leq C s^3\}$. Negating both sides above, exponentiating, then multiplying by $\mathbb{1}(\mathbf{a}_1 \leq -s)$ and taking expectations, we obtain

$$\mathbb{E}_{\mathrm{Airy}}\Big[\mathbb{1}(\mathbf{a}_1 \leq -s) \prod_{k=1}^{\infty} \mathcal{I}_s(\mathbf{a}_k)\Big] \geq \mathbb{P}\big(\{\mathbf{a}_1 \leq -s\} \cap \mathcal{A}\big) \exp(-C s^3). \tag{5.33}$$

It thus remains to estimate

$$\begin{aligned} \mathbb{P}\big(\{\mathbf{a}_1 \leq -s\} \cap \mathcal{A}\big) &\geq \mathbb{P}(\mathbf{a}_1 \leq -s) + \mathbb{P}(\mathcal{A}) - 1 \\ &\geq \exp(-s^3) - \theta_0' K_1 \exp(-K_2 s^3 \log s) - \exp(-\kappa s^{3+\frac{\delta}{4}}). \end{aligned} \tag{5.34}$$

The first inequality uses $\mathbb{P}(A \cap B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1$ for any events A and B. The second inequality uses the lower bound on $\mathbb{P}(\mathbf{a}_1 \leq -s)$ in (5.1) and the lower bound in (5.32). Combining (5.34) with (5.33) readily yields the claimed inequality (5.28) for some K and s large enough. □

Now we may complete the proof of (4.5) by combining (5.25) and (5.28) with

$$\mathbb{E}\Big[\prod_{k=1}^{\infty} \mathcal{I}_s(\mathbf{a}_k)\Big] = \mathbb{E}\Big[\mathbb{1}(\mathbf{a}_1 \geq -s) \prod_{k=1}^{\infty} \mathcal{I}_s(\mathbf{a}_k)\Big] + \mathbb{E}\Big[\mathbb{1}(\mathbf{a}_1 < -s) \prod_{k=1}^{\infty} \mathcal{I}_s(\mathbf{a}_k)\Big]. \qquad \square$$

LEMMA 5.5
As above, set $\theta_0' = \lceil s^{1+\delta} \rceil$. Then for all s such that $\theta_0' s^2 > 27$,

$$\sum_{\lambda_k > \theta_0' s^2} \mathcal{J}_s\big((1-\epsilon)(\theta_0' s^2 - \lambda_k) - s\big) \leq \sqrt{\frac{2}{\pi}} (\theta_0' s^2)^{\frac{3}{4}} \log 2 + \frac{4}{T\pi(1-\epsilon)^3}. \tag{5.35}$$

Proof
For s large enough, (4.7) implies that

$$\{k : \lambda_k > \theta_0' s^2\} \subseteq \Big\{k : k \geq \frac{2}{3\pi}(\theta_0' s^2)^{\frac{3}{2}}\Big\}.$$

From this, we may deduce the first inequality listed below

$$\begin{aligned}
&\sum_{\lambda_k > \theta_0' s^2} \mathcal{J}_s\big((1-\epsilon)(\theta_0' s^2 - \lambda_k) - s\big) \\
&\quad \leq \sum_{k \geq \frac{2}{3\pi}(\theta_0' s^2)^{\frac{3}{2}}} \mathcal{J}_s\big((1-\epsilon)(\theta_0' s^2 - \lambda_k) - s\big) \\
&\quad \leq \sqrt{\frac{2}{\pi}}(\theta_0' s^2)^{\frac{3}{4}} \log 2 \\
&\qquad + \sum_{k'=\sqrt{\frac{2}{\pi}}(\theta_0' s^2)^{\frac{3}{4}}}^{\infty} \exp\Big(-(1-\epsilon) T^{\frac{1}{3}} \Big(\frac{3\pi(k' - \frac{1}{2})}{2}\Big)^{1/3}\Big). \qquad (5.36)
\end{aligned}$$

To show the second inequality, let $\theta_0'' := \frac{2}{3\pi}(\theta_0' s^2)^{\frac{3}{2}}$, and let $\theta_0''' := \frac{2}{3\pi}(\theta_0' s^2)^{\frac{3}{4}} + \sqrt{\frac{2}{\pi}}(\theta_0' s^2)^{\frac{3}{4}}$. Using $\mathcal{J}_s(x) \leq \log 2$ and $\log(1+x) \leq x$ for $x \leq 0$, along with Lemma 5.6 (similarly to (5.24)),

$$\begin{aligned}
&\mathcal{J}_s\big((1-\epsilon)(\theta_0' s^2 - \lambda_k) - s\big) \\
&\quad \leq \begin{cases} \log 2 & k \in [\theta_0'', \theta_0'''] \cap \mathbb{Z}, \\ \exp(-(1-\epsilon) T^{\frac{1}{3}} (\frac{3\pi(k - \theta_0'' - \frac{1}{2})}{2})^{\frac{1}{3}}) & k \in (\theta_0''', \infty) \cap \mathbb{Z}. \end{cases}
\end{aligned}$$

Using this bound and substituting $k' = k - \theta_0''$, we obtain

$$\begin{aligned}
&\sum_{k \geq \frac{2}{3\pi}(\theta_0' s^2)^{\frac{3}{2}}} \mathcal{J}_s\big((1-\epsilon)(\theta_0' s^2 - \lambda_k) - s\big) \\
&\quad \leq (\theta_0''' - \theta_0'') \log 2 + \sum_{k' > \theta_0''' - \theta_0''} \exp\Big(-(1-\epsilon) T^{\frac{1}{3}} \Big(\frac{3\pi(k' - \frac{1}{2})}{2}\Big)^{\frac{1}{3}}\Big),
\end{aligned}$$

which implies the second inequality in (5.36). Bounding the sum by a corresponding integral and evaluating yields the bound in (5.35). □

LEMMA 5.6
Fix $a > 27$. Then we have $(a+x)^{\frac{2}{3}} \geq a^{\frac{2}{3}} + x^{\frac{1}{3}}$ for all $x \geq \sqrt{3a}$.

Proof
Observe that for all $x \geq \sqrt{3a}$ and $a > 27$, one can write $x < x^2$, and using $3a^{\frac{2}{3}} \leq a$, one has $3a^{\frac{4}{3}}x^{\frac{1}{3}} \leq ax$ and $3a^{\frac{2}{3}}x^{\frac{2}{3}} \leq ax^{\frac{2}{3}} \leq ax$. Combining these inequalities yields

$$(a+x)^2 = a^2 + x^2 + 2ax \geq a^2 + x + 3a^{\frac{4}{3}}x^{\frac{1}{3}} + 3a^{\frac{2}{3}}x^{\frac{2}{3}} = (a^{\frac{2}{3}} + x^{\frac{1}{3}})^3. \qquad \square$$

6. Ablowitz–Segur solution of Painlevé II

Recall (see Section 1.3) the Ablowitz–Segur solution $u_{\mathrm{AS}}(x;\gamma)$ of Painlevé II. We restate [30, Theorem 1.10] which provides the asymptotic form of $u_{\mathrm{AS}}(x;\gamma)$ as $x \to -\infty$. Lemmas 6.3 and 6.4 result from the analysis of this form. We will combine those two lemmas in Section 6.2 to yield a proof of Theorem 1.7.

6.1. *Asymptotic form for the Ablowitz–Segur solution of Painlevé II*

In order to restate [30, Theorem 1.10], we introduce a few special functions. For a real variable $\aleph \in (0,\infty)$, define $\kappa = \kappa(\aleph) \in (0,1)$ implicitly as

$$\aleph = \frac{2}{3}\sqrt{\frac{2}{1+\kappa^2}}\Big(E(\kappa') - \frac{2\kappa^2}{1+\kappa^2}K(\kappa')\Big),$$

where $\kappa' := \sqrt{1-\kappa^2}$, and K and E are standard complete elliptic integrals

$$K(\kappa) := \int_0^1 \frac{d\xi}{\sqrt{(1-\xi^2)(1-\kappa^2\xi^2)}} \qquad \text{and} \qquad E(\kappa) := \int_0^1 \sqrt{\frac{1-\xi^2\kappa^2}{1-\xi^2}}\, d\xi.$$

It follows from [30, Proposition 3.2] that κ is uniquely defined for all $\aleph \in (0, \frac{2}{3}\sqrt{2})$.

Further, define (using $\kappa = \kappa(\aleph)$)

$$V = V(\aleph) := -\frac{2}{3\pi}\sqrt{\frac{2}{1+\kappa^2}}\Big(E(\kappa) - \frac{1-\kappa^2}{1+\kappa^2}K(\kappa)\Big) \qquad \text{and}$$

$$\tau = \tau(\aleph) := 2\mathbf{i}\frac{K(\kappa)}{K(\kappa')}.$$

Define the Jacobi theta and elliptic functions (with $q = e^{\mathbf{i}\pi\tau}$ and $z \in \mathbb{C}$)

$$\theta_2(z,q) = 2\sum_{m=0}^{\infty} q^{(m+\frac{1}{2})^2}\cos\big((2m+1)\pi z\big),$$

$$\theta_3(z,q) = 1 + 2\sum_{m=1}^{\infty} q^{m^2}\cos(2\pi m z),$$

$$\mathrm{cd}\Big(2zK\Big(\frac{1-\kappa}{1+\kappa}\Big), \frac{1-\kappa}{1+\kappa}\Big) = \frac{\theta_3(0,q)\theta_2(z,q)}{\theta_2(0,q)\theta_3(z,q)}, \quad z \in \mathbb{C}\setminus \bigcup_{m,n\in\mathbb{Z}}\Big\{\frac{1}{2} + \frac{\tau}{2} + m + \tau n\Big\}.$$

The asymptotic formula below follows immediately from [30, Theorem 1.10].[22] In order to state it, let us define a bit more notation which will also be used in the subsequent lemmas.

Fix, throughout what follows, some $\delta \in (0, \frac{2}{5})$. For $s > 0$, define $\gamma = \gamma(s) := 1 - \exp(-s^{\frac{3}{2}-\delta})$, $v = v(s) := -\log(1-\gamma(s)) = s^{\frac{3}{2}-\delta}$, and an interval $\Psi_s := [-s, -\frac{3^{2/3}}{2} s^{1-\frac{2\delta}{3}})$. For $x \in (-\infty, 0)$ and $s > 0$, define $\aleph = \aleph(x,s) := \frac{v(s)}{(-x)^{3/2}}$. For $x \in \Psi_s$, it follows that $\aleph(x,s) \in (0, \frac{2}{3}\sqrt{2})$ and hence $\kappa = \kappa(\aleph(x,s))$ is a function[23] of $s > 0$ and $x \in \Psi_s$. In fact, in Proposition 6.1 and Lemmas 6.3 and 6.4 we will generally deal with functions of s and x, although we will often suppress the explicit dependence (mainly, to keep equations from growing too lengthy).

PROPOSITION 6.1

For any fixed[24] $\zeta \in (0, \frac{\sqrt{2}}{3})$ *there exist* $s_0 = s_0(\zeta) > 0$, $c_0 = c_0(\zeta) > 0$ *such that for all* $s \geq s_0$ *(using* $\kappa = \kappa(\aleph(x,s))$, $V = V(\aleph(x,s))$, $\gamma = \gamma(s)$, $v = v(s)$ *as above, and* $J_1 = J_1(x,s)$*),*

$$u_{\mathrm{AS}}(x;\gamma) = -\sqrt{-\frac{x}{2}}\frac{1-\kappa}{\sqrt{1+\kappa^2}}\mathrm{cd}\Big(2(-x)^{3/2} V K\Big(\frac{1-\kappa}{1+\kappa}\Big), \frac{1-\kappa}{1+\kappa}\Big) + J_1, \tag{6.1}$$

where[25]

$$|J_1| \leq c_0(-x)^{-\frac{1}{10}}, \quad \textit{for all } x \in \Psi_s \textit{ which satisfy } (-x)^{\frac{3}{2}}\Big(\frac{2\sqrt{2}}{3} - \zeta\Big) \geq v. \tag{6.2}$$

Continuing with the notation introduced above, the following result controls the small $\aleph$ behavior of $\kappa(\aleph)$ and $V(\aleph)$. The bound (6.3) follows immediately from equation (3.5) in [30, Proposition 3.2], and the bound (6.4) follows immediately from equation (3.9) in [30, Corollary 3.3].

[22]Since our notation is slightly different from that of [30], let us match it here. The parameter ε in [30] is equal to 1 in our case. Bothner [30] has parameters s and δ which do not correspond to our notation. Let us denote them as s and δ. Then in terms of our notation, $\mathsf{s} = \gamma$ and $\delta = \zeta$. In contrast to [30], we treat v as being parameterized by an underlying s, whereas Bothner treated v as a free variable in its own right. Finally, since we only utilize equation (1.26) from [30], we do not need to make use of his constants v_1, f_1, or c_1. His v_1 translates into our s_1 constant, and his c_0 is the same as ours.

[23]This follows from [30, Proposition 3.2] as noted above.

[24]The result, in fact, holds for $\zeta \in (0, \frac{2\sqrt{2}}{3})$. The more restrictive range $(0, \frac{\sqrt{2}}{3})$ in [30, Theorem 1.10] is only needed for equation (1.27) therein, not (1.26). However, since we only need this result for small ζ we do not provide the explanation for this wider range's validity.

[25]The condition assumed on x in (6.2) is equivalent to $\aleph \leq \frac{2\sqrt{2}}{3} - \zeta$.

PROPOSITION 6.2

Define $\mathfrak{Q}_1(\aleph)$ *and* $\mathfrak{Q}_2(\aleph)$ *via*

$$\kappa(\aleph) = 1 - 2\sqrt{\frac{\aleph}{\pi}} + \frac{2\aleph}{\pi} - \frac{29}{8}\Big(\frac{\aleph}{\pi}\Big)^{3/2} + \mathfrak{Q}_1(\aleph), \tag{6.3}$$

$$V(\aleph) = -\frac{2}{3\pi} - \frac{\aleph}{2\pi^2}\log(\aleph) + \frac{\aleph}{2\pi^2}\big(1 + \log(16\pi)\big) + \mathfrak{Q}_2(\aleph). \tag{6.4}$$

Then there exist $\aleph_0 \in (0, \frac{2}{3}\sqrt{2})$ *and* $C = C(\aleph_0)$ *such that for all* $\aleph \le \aleph_0$, $|\mathfrak{Q}_1(\aleph)| \le C\aleph^2$ *and* $|\mathfrak{Q}_2(\aleph)| \le C\aleph^2$.

Combining Propositions 6.1 and 6.2, we may simplify the asymptotic formula of $u_{\mathrm{AS}}(x;\gamma)$.

LEMMA 6.3

Recall the notation from Proposition 6.1 (namely, $\aleph = \aleph(x,s)$, $\kappa = \kappa(\aleph(x,s))$, $V = V(\aleph(x,s))$, $\gamma = \gamma(s)$, $v = v(s)$, *and* Ψ_s*). Define* $\phi = \phi(x,s)$ *via the relation*

$$\pi(-x)^{\frac{3}{2}}V = -\frac{2}{3}(-x)^{\frac{3}{2}} + \frac{v}{2\pi}\log\big(8(-x)^{\frac{3}{2}}\big) + \phi. \tag{6.5}$$

Fix any $\eta_0 \in (0, \frac{2}{5})$*. Then there exist* $s_0 = s_0(\eta_0) > 0$, $C = C(\eta_0) > 0$, *and* $C' = C'(\eta_0) > 0$ *such that for all pairs* (x,s) *which satisfy* $s \ge s_0$, $x \in \Psi_s$, *and* $\aleph = (-x)^{-\eta}$ *for some* $\eta \in (\eta_0, \frac{2}{5})$, *we have that (with the notation* $J_2 = J_2(x,s)$ *and* $J_3 = J_3(x,s)$*)*

$$u_{\mathrm{AS}}(x;\gamma) = (-x)^{-\frac{1}{4}}\sqrt{\frac{v}{\pi}}\cos\big(\pi(-x)^{\frac{3}{2}}V\big) + J_2, \tag{6.6}$$

$$\phi = \frac{v}{2\pi}\big(1 - \log(v/2\pi) + J_3\big), \tag{6.7}$$

where $|J_2| \le C(-x)^{\frac{1}{2}-\frac{3\eta}{2}}$ *and* $|J_3| \le C'(-x)^{-\eta}$.

Proof

It follows from [123, (22.11.4)] that

$$\mathrm{cd}(z,\kappa) = \frac{2\pi}{K(\kappa)\kappa}\sum_{n=0}^{\infty}(-1)^n\frac{\tilde{q}^{n+\frac{1}{2}}}{1-\tilde{q}^{2n+1}}\cos\big((2n+1)\zeta\big), \tag{6.8}$$

where $\zeta := \frac{\pi z}{2K(\kappa)}$ and $\tilde{q} := \exp(-\pi K(\kappa')/K(\kappa))$.

We claim that there exist $0 \le \kappa_0 < 1$ and $C_1 = C_1(\kappa_0) > 0$ such that for all $\kappa \le \kappa_0$,

$$\cos\big(\pi z/2K(\kappa)\big) - C_1\kappa^2 \le \mathrm{cd}(z,\kappa) \le \cos\big(\pi z/2K(\kappa)\big) + C_1\kappa^2. \tag{6.9}$$

Owing to [65] and [123, (19.5.5), (19.5.8)], there exist $0 \le \kappa_0 < 1$ and $0 < C_2 = C_2(\kappa_0) < C_3 = C_3(\kappa_0)$ such that for all $\kappa \le \kappa_0$,

$$C_2\kappa^4 + \frac{\kappa^2}{16} \le \tilde{q} \le \frac{\kappa^2}{16} + C_3\kappa^4, \qquad \frac{\pi}{2} + C_2\kappa^2 \le K(\kappa) \le \frac{\pi}{2} + C_3\kappa^2. \tag{6.10}$$

When $\kappa \le \kappa_0$, substituting (6.10) into (6.8) yields

$$\big|\mathrm{cd}(z,k) - \cos\big(\pi z/2K(\kappa)\big)\big| \le \sum_{n=1}^{\infty} \frac{(C_3\kappa)^{2n}}{1 - \frac{\kappa^2}{16} - C_2\kappa^4} + C_2\kappa^2.$$

For small enough κ, the right-hand side above is bounded by $C_1\kappa^2$ (for $C_1 = C_1(\kappa_0)$), which proves (6.9).

Return to the proof of the lemma, and define $\tilde{\kappa} := \frac{1-\kappa}{1+\kappa}$ (recall $\kappa = \kappa(\aleph)$ as above). If we further define $\mathfrak{H}_1 = \mathfrak{H}_1(\aleph)$ by the relation $\tilde{\kappa} = \sqrt{\aleph/\pi} + \mathfrak{H}_1$, then (6.3) implies that there exists a constant $C_4 > 0$ such that for small enough $\aleph > 0$, $|\mathfrak{H}_1| \le C_4\aleph$. Thus, as $\aleph$ goes to zero, so too does $\tilde{\kappa}$.

Now, recall that we have assumed that $\aleph$ satisfies $\aleph = (-x)^{\eta}$ for some $\eta \in (\eta_0, \frac{2}{5})$. This implies that $\aleph \le (-x)^{\eta_0}$ and hence, as $-x$ goes to ∞, $\aleph$ and $\tilde{\kappa}$ both go to zero. Combining this deduction with (6.9), we conclude that there exist $x_1 = x_1(\eta_0) > 0$ and $C_5 = C_5(\eta_0) > 0$ such that (with the notation $\mathfrak{H}_2 = \mathfrak{H}_2(x,s)$ defined by the relation below)

$$\mathrm{cd}\big(2(-x)^{3/2}VK(\tilde{\kappa}),\tilde{\kappa}\big) = \cos\big(\pi(-x)^{\frac{3}{2}}V\big) + \mathfrak{H}_2, \tag{6.11}$$

where $|\mathfrak{H}_2| \le C_5(-x)^{-\eta}$ for all $(-x) \ge x_1$.

Using (6.11) along with the expansion for κ provided by (6.3), we see that there exist $x_2 = x_2(\eta_0) > 0$ and $C_6 = C_6(\eta_0)$ such that (with the notation $\mathfrak{H}_3 = \mathfrak{H}_3(x,s)$)

$$\sqrt{-\frac{x}{2}}\frac{1-\kappa}{\sqrt{1+\kappa^2}}\mathrm{cd}\big(2(-x)^{\frac{3}{2}}VK(\tilde{\kappa}),\tilde{\kappa}\big) = \sqrt{\frac{v}{\pi(-x)^{\frac{1}{2}}}}\cos\big(\pi(-x)^{\frac{3}{2}}V\big) + \mathfrak{H}_3, \tag{6.12}$$

where $|\mathfrak{H}_3| \le C_6(-x)^{\frac{1}{2}-\frac{3\eta}{2}}$ for all $(-x) \ge x_2$.

We may now apply (6.1) and combine that with the deduction above in (6.12). The first result requires that $s \ge s_0$, $x \in \Psi_s$, and $(-x)^{\frac{3}{2}}(\frac{2\sqrt{2}}{3} - \zeta) \ge v$, and the second requires that $(-x) \ge x_2$. This second condition can be ensured by possibly increasing the value of s_0. In applying (6.1), we may use the inequality $|J_1| \le c_0(-x)^{-\frac{1}{10}} \le c_0(-x)^{\frac{1}{2}-\frac{3\eta}{2}}$ (thanks to (6.2) and $\frac{1}{2} - \frac{3\eta}{2} > -\frac{1}{10}$). Combining this bound with the bound on $\mathfrak{H}_3$ in (6.12), we see that $J_2 := J_1 + \mathfrak{H}_3$ satisfies the desired bound $|J_2| \le C(-x)^{\frac{1}{2}-\frac{3\eta}{2}}$ for some constant C. This proves the bound on the error J_2 in (6.6).

Now we turn to prove (6.7). Owing to (6.4), there exist $s_0' = s_0'(\eta_0) > 0$ and $C_7 = C_7(\eta_0) > 0$ such that for all $s \geq s_0'$ and $x \in \Psi_s$ satisfying $\aleph(x,s) = (-x)^{-\eta}$ for some $\eta \in (\eta_0, \frac{2}{5})$, one has

$$\pi(-x)^{\frac{3}{2}} V = -\frac{2}{3}(-x)^{\frac{3}{2}} + \frac{v}{2\pi}\log\big(8(-x)^{\frac{3}{2}}\big) - \frac{v}{\pi}\log(v/2\pi) + \frac{v}{2\pi} + vJ_3, \quad (6.13)$$

where $J_3 = J_3(\aleph) := \mathfrak{Q}_2(\aleph)/v$. By substituting (6.13) into (6.5), we arrive at the desired error bound on J_3 in (6.7). □

The next lemma highlights the critical oscillatory cancellation which enables us to prove Theorem 1.7 (done in Section 6.2). Let us introduce a shorthand notation (the first equality is the definition and the second follows from (6.5)):

$$\psi(x,s) := 2\pi(-x)^{\frac{3}{2}} V\big(\aleph(x,s)\big) = -\frac{4}{3}(-x)^{\frac{3}{2}} + \frac{v}{\pi}\log\big(8(-x)^{\frac{3}{2}}\big) + 2\phi(x,s).$$

LEMMA 6.4

Recall that we have fixed $\delta \in (0, \frac{2}{5})$ throughout this section. For $\theta \in (0,\delta)$, there exist $s_0 = s_0(\delta,\theta) > 0$ and $C = C(\delta,\theta) > 0$ such that for all $s \geq s_0$,

$$\Big|\int_{-s}^{-s^{1-\frac{2}{3}\theta}} \frac{(x+s)}{(-x)^{\frac{1}{2}}}\cos\big(\psi(x,s)\big)\,dx\Big| \leq C s^{\frac{3}{2}} s^{-(\delta-\theta)}. \quad (6.14)$$

Proof

We will apply Lemma 6.3 to provide an asymptotic expansion for $\psi(x,s)$ (via $V(\aleph(x,s))$). To do this, fix $\eta_0 = \delta - \theta$, which is in $(0, \frac{2}{5})$ since $\delta \in (0, \frac{2}{5})$ and $\theta \in (0,\delta)$. We must verify two conditions to apply the lemma: for large enough s, and for x in the domain of integration $[-s, -s^{1-\frac{2}{3}\theta}]$, (1) $x \in \Psi_s = [-s, -\frac{3^{2/3}}{2}s^{1-\frac{2\delta}{3}})$, and (2) $\aleph(x,s) = \frac{v(s)}{(-x)^{\frac{3}{2}}} = \frac{s^{\frac{3}{2}-\delta}}{(-x)^{\frac{3}{2}}}$ equals $(-x)^{-\eta}$ for some $\eta \in (\eta_0, \frac{2}{5})$. Condition (1) is immediate. Condition (2) follows by considering the two endpoints $x = -s$ and $x = -s^{1-\frac{2}{3}\theta}$. In the first case, we find that $\eta = \delta$ and in the second case, $\eta = (\delta-\theta)/(1-\frac{2}{3}\theta)$, which is bounded below by $\delta - \theta$. Clearly, for intermediate x, η ranges between these two extremes which are contained in the interval $[\delta-\theta, \frac{2}{5})$, as desired. Thus, conditions (1) and (2) are both confirmed. By applying Lemma 6.3, there exist $s_0 = s_0(\delta,\theta) > 0$ and $C' = C'(\delta,\theta) > 0$ such that for all $s \geq s_0$,

$$\big|J_3(x,s)\big| = \Big|\frac{2\pi}{v}\phi(x,s) - 1 + \log(v/2\pi)\Big| \leq C'(-x)^{-(\delta-\theta)} \quad (6.15)$$

for any $x \in [-s, -s^{1-\frac{2}{3}\theta})$ (here we use J_3 as in (6.7) and the fact that $\eta_0 = \delta - \theta$).

Given this control over the expansion for ϕ (and hence ψ), we now turn to estimating the integral in (6.14). In order to capture the scale of oscillations and hence bound their net effect, it will be necessary for us to divide the domain of integration $[-s, -s^{1-\frac{2}{3}\theta}]$ into a disjoint (except for endpoints) union of consecutive closed intervals $\mathcal{I}_1, \mathcal{I}_2, \dots, \mathcal{I}_K$. Here K and the intervals are chosen by the following prescription. Denote $\mathcal{I}_j = [a_j, b_j]$, and let $b_1 = -s^{1-\frac{2}{3}\theta}$. Inductively in $j \geq 1$, let

$$a_j = b_j - \pi(-b_j)^{-\frac{1}{2}} \qquad \text{and} \qquad b_{j+1} = a_j.$$

Let K denote the minimal integer k such that $b_k < -s$. Finally, reassign $b_K = -s$. This produces the desired intervals. Note that each interval has length of order between $s^{-\frac{1}{2}}$ and $s^{\frac{1}{3}\theta-\frac{1}{2}}$, and the total number of intervals K is of order $s^{\frac{3}{2}}$. These intervals are chosen so as to contain roughly one period of oscillation. This enables us to control the sum of oscillatory effects.

For any $1 \leq j < K$, we may parameterize the interval $\mathcal{I}_j$ via the function $\mathcal{I}_j(t) = b - \pi(-b)^{-\frac{1}{2}}t$, as t ranges over $[0, 1]$. Let us fix some j and for the moment drop the subscripts on a_j, b_j, and $\mathcal{I}_j(t)$. We claim the following bounds. There exist $s_1 = s_1(\delta, \theta) > 0$ and $C = C(\delta, \theta) > 0$ such that for all $s \geq s_1$ (note, the error terms J_4, J_5, J_6, J_7, J_8, J_9 below are functions of t, b, and s),

$$\frac{4}{3}\big(-\mathcal{I}(t)\big)^{\frac{3}{2}} = \frac{4}{3}(-b)^{\frac{3}{2}} + 2\pi t + J_4 \quad \text{where } |J_4| = C(-b)^{-\frac{3}{2}},$$

$$\phi\big(\mathcal{I}(t), s\big) = \phi(b, s) + J_5 \quad \text{where } |J_5| = C(-b)^{-(\delta-\theta)},$$

$$\frac{v}{\pi}\log\big(8\big(-\mathcal{I}(t)\big)^{\frac{3}{2}}\big) = \frac{v}{\pi}\log\big(8(-b)^{\frac{3}{2}}\big) + \frac{3vt}{2}(-b)^{-\frac{3}{2}} + J_6$$
$$\text{where } |J_6| \leq C(-b)^{-\frac{3}{2}-(\delta-\theta)/(1-\frac{2}{3}\theta)}.$$

The first bound above follows from Taylor's expansion of $(1+x)^{\frac{3}{2}}$; the second bound follows directly from (6.15); and the third bound needs a bit more argument. Combining Taylor's expansion of $(1+x)^{\frac{3}{2}}$ with that of $\log(1+x)$ yields the first two terms and an error of order $v(-b)^{-3}$. Since $b \in [-s, -s^{1-\frac{2}{3}\theta}]$, it follows that we may upper bound $s \leq (-b)^{(1-\frac{2}{3}\theta)^{-1}}$, and hence $v \leq (-b)^{(\frac{3}{2}-\delta)(1-\frac{2}{3}\theta)^{-1}}$. This enables us to reexpress the order of $v(-b)^{-3}$ entirely in terms of b, as claimed.

Observe that of J_4, J_5, and J_6, the largest error term is J_5. This is because $\delta - \theta \leq \frac{2}{5}$, whereas all other exponents have negative powers exceeding $\frac{3}{2}$. Thus, combining the three bounds with the definition of ψ, we find that for all $s \geq s_1$,

$$\psi\big(\mathcal{I}(t),s\big) = \underbrace{\psi(b,s) + 2\pi t + \frac{3vt}{2}(-b)^{-\frac{3}{2}}}_{=:\psi_t(b,s)} + J_7$$

$$\text{where } |J_7| \le 3C(-b)^{-(\delta-\theta)}. \tag{6.16}$$

We may now use (6.16) to bound the integral on the left-hand side of (6.14) over the interval $\mathcal{I}$ (i.e., $\mathcal{I}_j$ for any $1 \le j < K$). As we show below, there exist $s_2 = s_2(\delta,\theta) > 0$ and $C = C(\delta,\theta)$ such that for all $s \ge s_2$ and all $\mathcal{I} = \mathcal{I}_j$ for $1 \le j < K$,

$$\left|\int_{\mathcal{I}} \frac{(x+s)}{(-x)^{\frac{1}{2}}} \cos\big(\psi(x,s)\big)\,dx\right| \le \frac{C|b+s|}{|b|}\left(\frac{v|b|^{-\frac{3}{2}}}{\xi(b,s)} + |b|^{-2(\delta-\theta)}\right) + C|b|^{-\frac{1}{2}-\frac{7}{11}}, \tag{6.17}$$

where $\xi(b,s) = 2\pi + \frac{3}{2}v|b|^{-\frac{3}{2}}$.

To show this, observe first that by parameterizing the interval $\mathcal{I}$ via $\mathcal{I}(t)$ for $t \in [0,1]$ we have

$$\int_{\mathcal{I}} \frac{(x+s)}{(-x)^{\frac{1}{2}}} \cos\big(\psi(x,s)\big)\,dx = \frac{\pi}{(-b)^{\frac{1}{2}}}\int_0^1 \left(\frac{b+s}{(-b)^{\frac{1}{2}}} + J_8\right)\cos\big(\psi\big(\mathcal{I}(t),s\big)\big)\,dt, \tag{6.18}$$

where the error term $J_8 = J_8(t,b,s)$ comes from Taylor's expansion and can be bounded uniformly in t and for all intervals $\mathcal{I}$ by

$$|J_8| \le C(-b)^{(1-\frac{2}{3}\theta)^{-1}-2} \le C(-b)^{-\frac{7}{11}}$$

for some constant $C > 0$. The second bound comes by taking the worst value of $\theta \in (0, \frac{2}{5})$.

Using (6.16), $\cos(\psi(\mathcal{I}(t),s)) = \cos(\psi_t(b,s) + J_7)$. Expanding the sum in the cosine yields

$$\cos\big(\psi\big(\mathcal{I}(t),s\big)\big) = \cos\big(\psi_t(b,s)\big)\cos(J_7) - \sin\big(\psi_t(b,s)\big)\sin(J_7). \tag{6.19}$$

The bound on J_7 in (6.16) implies that for some $C = C(\delta,\theta) > 0$,

$$\max\big\{\big|\cos(J_7) - 1\big|, \big|\sin(J_7)\big|\big\} \le C(-b)^{-2(\delta-\theta)}.$$

Substituting this into (6.19) yields

$$\cos\big(\psi\big(\mathcal{I}(t),s\big)\big) = \cos\big(\psi_t(b,s)\big) + J_9 \quad \text{where } |J_9| \le C(-b)^{-2(\delta-\theta)}.$$

We may finally substitute this back into (6.18) and evaluate the main contribution to the integral as well as the error terms. This yields (recall $\xi(b,s)$ defined below (6.17) and note that the value of the constant C may change between lines)

$$\begin{aligned}
&\text{LHS of (6.17)}\\
&\quad\le \frac{\pi|b+s|}{(-b)}\Big(\Big|\int_0^1 \cos\big(\psi_t(b,s)\big)\,dt\Big| + C|b|^{-2(\delta-\theta)}\Big) + C|b|^{-\frac{1}{2}-\frac{7}{11}}\\
&\quad= \frac{\pi|b+s|}{(-b)}\Big(\frac{|\sin(\psi(b,s)) - \sin(\psi(b,s)+\frac{3}{2}v|b|^{-\frac{3}{2}})|}{\xi(b,s)} + C|b|^{-2(\delta-\theta)}\Big)\\
&\qquad + C|b|^{-\frac{1}{2}-\frac{7}{11}}\\
&\quad\le \frac{\pi|b+s|}{|b|}\Big(\frac{|\frac{3}{2}v|b|^{-\frac{3}{2}}|}{\xi(b,s)} + C|b|^{-2(\delta-\theta)}\Big) + C|b|^{-\frac{1}{2}-\frac{7}{11}},
\end{aligned}$$

where in the third line we have used the inequality $|\sin(x)-\sin(x+y)|\le|y|$ which holds for all y. Redefining C to absorb all needed constants yields the right-hand side of (6.17), as desired.

Now we turn to the final step of the proof where we sum the contributions over all the intervals $\mathcal{I}_1,\dots,\mathcal{I}_K$. Summing (6.17) over $1\le j<K$ yields

$$\begin{aligned}
&\Big|\sum_{j=1}^{K-1}\int_{\mathcal{I}_j}\frac{x+s}{(-x)^{\frac{1}{2}}}\cos\big(\psi(x,s)\big)\,dx\Big|\\
&\qquad\le \sum_{j=1}^{K-1}\Big(\frac{C|b_j+s|}{|b_j|}\Big(\frac{v|b_j|^{-\frac{3}{2}}}{\xi(b_j,s)} + |b_j|^{-2(\delta-\theta)}\Big) + C|b_j|^{-\frac{1}{2}-\frac{7}{11}}\Big). \qquad (6.20)
\end{aligned}$$

We may use the bound $\sum_{j=1}^{K-1}\frac{\pi(b_j+s)}{(-b_j)}\le 2\int_{-s}^{-s^{1-\frac{2}{3}\theta}}(-x)^{-\frac{1}{2}}(x+s)\,dx\le\frac{8}{3}s^{\frac{3}{2}}$ to see that (the constant C may change from the left-hand side to the right-hand side below)

$$\begin{aligned}
&\sum_{j=1}^{K-1}\frac{C|b_j+s|}{|b_j|}\Big(\frac{v|b_j|^{-\frac{3}{2}}}{\xi(b_j,s)} + |b_j|^{-2(\delta-\theta)}\Big)\\
&\qquad\le Cs^{\frac{3}{2}}\max_{1\le j\le K-1}\Big(\frac{v|b_j|^{-\frac{3}{2}}}{\xi(b_j,s)} + |b_j|^{-2(\delta-\theta)}\Big)\\
&\qquad\le Cs^{\frac{3}{2}}s^{-(\delta-\theta)}. \qquad (6.21)
\end{aligned}$$

The second inequality comes from noting that $v=v(s)=s^{\frac{3}{2}-\delta}$, the maximal value of $|b_j|$ to a negative power is realized when $j=1$, in which case $|b_1|=s^{1-\frac{2}{3}\theta}$, and $-(\delta-\theta)\geq -2(\delta-\theta)(1-\frac{2}{3}\theta)$ for $0\leq\theta<\delta<\frac{2}{5}$. As for the other term in the summation on the right-hand side of (6.20), we may bound it via an integral as

$$\sum_{j=1}^{K-1} C|b_j|^{-\frac{1}{2}-\frac{7}{11}} \leq C\int_{-s}^{-s^{1-\frac{2}{3}\theta}} (-x)^{-\frac{7}{11}}\,dx \leq Cs^{1-\frac{7}{11}}.$$

Combining this bound with (6.21) shows that

$$\text{LHS (6.20)} \leq Cs^{\frac{3}{2}}s^{-(\delta-\theta)}.$$

Here we have used that $s^{1-\frac{7}{11}} = s^{\frac{3}{2}}s^{-\frac{1}{2}-\frac{7}{11}}$ and that clearly $s^{-\frac{1}{2}-\frac{7}{11}} \leq s^{-(\delta-\theta)}$. Finally, we must deal with the summand when $j=K$. However, this is clearly bounded by a constant. Thus, we arrive at the desired bound (6.14) and complete the proof of the lemma. □

6.2. Proof of Theorem 1.7

Recall from (1.14) that (for $v=v(s)=s^{\frac{3}{2}-\delta}$ with $\delta\in(0,\frac{2}{5})$ fixed)

$$\log F(-s;v) = -\int_{-s}^{\infty}(x+s)u_{\mathrm{AS}}^2(x;\gamma)\,dx, \qquad \text{with } \gamma = 1-e^{-v}.$$

We seek to prove that

$$\int_{-s}^{\infty}(x+s)u_{\mathrm{AS}}^2(x;\gamma)\,dx \geq \frac{2}{3\pi}vs^{\frac{3}{2}} + \mathcal{O}(s^{3-\frac{12\delta}{11}}).$$

Most of the contribution to the left-hand side integral comes from x near $-s$. With this in mind, for $\theta\in(0,\delta)$, divide the integral into two parts:

$$\textbf{(a)} := \int_{-s^{1-\frac{2}{3}\theta}}^{\infty}(x+s)u_{\mathrm{AS}}^2(x;\gamma)\,dx \qquad \text{and}$$

$$\textbf{(b)} := \int_{-s}^{-s^{1-\frac{2}{3}\theta}}(x+s)u_{\mathrm{AS}}^2(x;\gamma)\,dx.$$

We use the obvious lower bound **(a)** ≥ 0. For **(b)** we use the asymptotic expansion for u_{AS} given in Lemma 6.3. The assumptions of this lemma were previously verified for this range of x in the beginning of the proof of Lemma 6.4, so we do not repeat them here. We may now use the expansion provided in (6.6) for u_{AS} to show (using $\eta_0=\delta-\theta$ in the lemma) that there exist $s_0=s_0(\delta,\theta)>0$ and $C=C(\delta,\theta)>0$ such that for all $s\geq s_0$,

$$\textbf{(b)} := \int_{-s}^{-s^{1-\frac{2}{3}\theta}}(x+s)\Big((-x)^{-\frac{1}{4}}\sqrt{\frac{v}{\pi}}\cos\big(\pi(-x)^{\frac{3}{2}}V\big)+J_2\Big)^2\,dx,$$

where $|J_2| \le C(-x)^{\frac{1}{2}-\frac{3}{2}(\delta-\theta)}$. Squaring the expansion term and using the identity that $2\cos(y)^2 = 1+\cos(2y)$, we can write

$$(\mathbf{b}) := \underbrace{\frac{v}{2\pi}\int_{-s}^{-s^{1-\frac{2}{3}\theta}} \frac{x+s}{(-x)^{\frac{1}{2}}}\,dx}_{(\mathbf{b_1})} + \underbrace{\frac{v}{2\pi}\int_{-s}^{-s^{1-\frac{2}{3}\theta}} \frac{x+s}{(-x)^{\frac{1}{2}}}\cos\bigl(2\pi(-x)^{\frac{3}{2}}V\bigr)\,dx}_{(\mathbf{b_2})}$$
$$+ \underbrace{\int_{-s}^{-s^{1-\frac{2}{3}\theta}} (x+s)\Bigl(2(-x)^{-\frac{1}{4}}\sqrt{\frac{v}{\pi}}\cos\bigl(\pi(-x)^{\frac{3}{2}}V\bigr)J_2 + J_2^2\Bigr)\,dx}_{(\mathbf{b_3})}.$$

By extending the domain of integration to $[-s,-1]$, we may lower bound

$$(\mathbf{b_1}) = \frac{v}{2\pi}\int_{-s}^{-1} \frac{x+s}{(-x)^{\frac{1}{2}}}\,dx - \frac{v}{2\pi}\int_{-s^{1-\frac{2}{3}\theta}}^{-1} \frac{x+s}{(-x)^{\frac{1}{2}}}\,dx \ge \frac{2}{3\pi}vs^{\frac{3}{2}} + J_{10},$$

where the term J_{10} comes from bounding the second integral along with an order vs residual from evaluating the first integral. It can be bounded (for some constant $C>0$) by

$$|J_{10}| \le Cs^{3-\delta-\frac{1}{3}\theta}.$$

The term $\frac{2}{3\pi}vs^{\frac{3}{2}}$ will constitute the main contribution.

The oscillatory integral in $(\mathbf{b_2})$ is lower bounded by applying Lemma 6.4 which shows that $|(\mathbf{b_2})| \ge Cvs^{\frac{3}{2}}s^{-(\delta-\theta)}$ for some constant $C>0$. We may bound

$$\begin{aligned}\bigl|(\mathbf{b_3})\bigr| &\le \int_{-s}^{-s^{1-\frac{2}{3}\theta}} (x+s)\Bigl(2(-x)^{-\frac{1}{4}}\sqrt{\frac{v}{\pi}}|J_2| + |J_2|^2\Bigr)\,dx \\ &\le s^{\frac{3}{2}}C\bigl(s^{\frac{3}{2}-\frac{3}{2}(\delta-\theta)-\frac{1}{2}\delta} + s^{\frac{3}{2}-3(\delta-\theta)}\bigr),\end{aligned}$$

where the second inequality follows from our bound on $|J_2|$ along with extending the integral from $-s$ to zero and evaluating.

Combining the above bounds we find that

$$(\mathbf{b}) \ge \frac{2}{3\pi}vs^{\frac{3}{2}} + J_{11},$$

where there is a constant $C>0$ such that

$$|J_{11}| \le Cs^{3-\delta}\bigl(s^{-\frac{1}{3}\theta} + s^{-\delta+\theta} + s^{-\delta+\frac{3}{2}\theta} + s^{-2\delta+3\theta}\bigr).$$

We are given $\delta \in (0,\frac{2}{5})$ but we are free to choose $\theta \in (0,\delta)$ so as to minimize $|J_{11}|$. Choosing $\theta = \frac{6}{11}\delta$ results in the best (i.e., lowest) upper bound of $|J_{11}| \le s^{3-\frac{13}{11}\delta}$. This is small compared to $\frac{2}{3\pi}vs^{\frac{3}{2}}$ (which is of order $s^{3-\delta}$); the proof is complete. □

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