

Hyperbolic Groups That Are Not Commensurably Co-Hopfian

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Sela proved that every torsion-free one-ended hyperbolic group is co-Hopfian. We prove that there exist torsion-free one-ended hyperbolic groups that are not commensurably co-Hopfian. In particular, we show that the fundamental group of every simple surface amalgam is not commensurably co-Hopfian.

1 Introduction

Hyperbolic groups, introduced by Gromov in his seminal essay [8], form a broad family of finitely presented groups satisfying a coarse notion of negative curvature. The large-scale geometry reflected in the definition has a variety of algebraic implications; for example, hyperbolic groups satisfy the Tits alternative and each of Dehn's decision problems—the word, conjugacy, and isomorphism problems [6, 23]—is solvable. A major development in the understanding of (torsion free) hyperbolic groups was the body of work developed by Rips and Sela, each independently and in partnership. The techniques they developed involved studying the action of hyperbolic groups on $\mathbb{R}$ -trees and then developing a JSJ decomposition—a canonical splitting of a hyperbolic group along two-ended subgroups.

A group is *co-Hopfian* if it is not isomorphic to any of its proper subgroups. Sela [24], building on work of Rips–Sela [21], proved that a torsion-free hyperbolic group is co-Hopfian if and only if it is freely indecomposable. Thus, by the “easy” direction of

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Stallings' Theorem [26], a torsion-free one-ended hyperbolic group is co-Hopfian. In his thesis, Moiola [17, Theorem 1.0.8] generalized this statement to prove every one-ended hyperbolic group is co-Hopfian. A group Γ is *commensurably co-Hopfian* if no finite-index subgroup of Γ is isomorphic to an infinite-index subgroup of Γ . As we explain below, this variation of coHopficity has its own merits as, unlike the classical notion, it is a commensurability invariant. Strebel [29] proved that the infinite-index subgroups of a Poincaré duality group have strictly smaller cohomological dimension than the group. Thus, Poincaré duality groups are commensurably co-Hopfian. In particular, fundamental groups of closed hyperbolic manifolds are commensurably co-Hopfian. Following the result of Sela, it was natural to ask if one ended hyperbolic groups are commensurably co-Hopfian.

In this paper, we exhibit one-ended hyperbolic groups that are not commensurably co-Hopfian, answering a question of Whyte on Bestvina's Problem list [2, (Whyte, Q. 1.12)] and also asked by Kapovich [12, Section 5].

Theorem 1.1. There exist one-ended hyperbolic groups that are not commensurably co-Hopfian.

Our proof of Theorem 1.1 is topological. A *simple surface amalgam* is the union of a finite collection of at least three surfaces with negative Euler characteristic and precisely one boundary component each and which have their boundary components identified. These spaces have been studied in [7, 14, 27, 28]. Such spaces are one-ended and Gromov hyperbolic. Hyperbolicity is an immediate consequence of the Bestvina-Feighn combination theorem [1], since free groups are amalgamated along malnormal subgroups. Moreover, by assigning each surface a suitable Riemannian metric, every simple surface amalgam admits a metric with a CAT(-1) universal cover (see [4, Theorem II.5.4] or Section 4 in [27]). We exhibit in Section 2 a simple surface amalgam X and two finite covers $X_1 \rightarrow X$ and $X_2 \rightarrow X$ so that the space X_1 π_1 -injectively embeds in the space X_2 and such that $\pi_1(X_1)$ embeds as an infinite-index subgroup of $\pi_1(X_2)$. See Figure 1. The construction given in Section 2 does not immediately extend to simple surface amalgams in which the subsurfaces have different Euler characteristics. Nonetheless, we prove the following in Section 3.

Theorem 1.2. The fundamental group of any simple surface amalgam is not commensurably co-Hopfian.

Bowditch [3] proved that if G is a one-ended hyperbolic group that is not Fuchsian, then there is a canonical graph of groups decomposition of G , called the

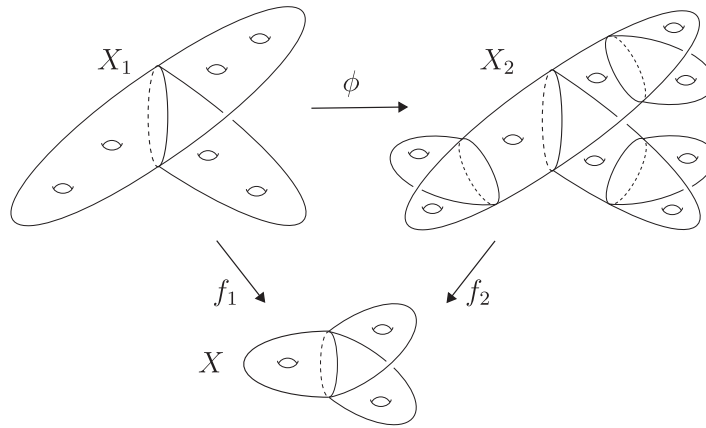


Fig. 1. Two finite covers of a simple surface amalgam X . The space X_1 embeds as a retract of the space X_2 , such that $\pi_1(X_1)$ embeds as an infinite-index subgroup of $\pi_1(X_2)$. Thus, the group $\pi_1(X)$ contains a finite-index subgroup isomorphic to $\pi_1(X_1)$ and an infinite-index subgroup isomorphic to $\pi_1(X_1)$.

JSJ decomposition of G , with edge groups that are two-ended and vertex groups of three types: two-ended; maximally hanging Fuchsian; and quasi-convex rigid vertex groups not of the first two types. For background and definition, see [9, 22, 25]. We conjecture that for a one-ended hyperbolic group the commensurably co-Hopfian property is related to the existence of maximal hanging Fuchsian vertex groups in the JSJ decomposition of the group over two-ended subgroups.

Conjecture 1.3. Let Γ be a one-ended hyperbolic group that is not Fuchsian. If Γ is not commensurably co-Hopfian, then its JSJ decomposition contains a maximal hanging Fuchsian vertex group. Moreover, if the JSJ decomposition of Γ only contains maximal hanging Fuchsian vertex groups and 2-ended vertex groups, then Γ is not commensurably co-Hopfian.

In this paper, the embeddings constructed are quasi-isometric embeddings and are surely not representative. Indeed, highly distorted subgroups may be counterexamples to Conjecture 1.3, so a quasi-convexity assumption may be required.

In Section 4, we present two examples of one-ended hyperbolic groups whose JSJ decomposition contains both maximal hanging Fuchsian and rigid vertex groups and so that one group is commensurably co-Hopfian and the other is not. We summarize our examples and open problems in Section 1.1.

Quasi-isometrically co-Hopfian

The quasi-isometrically co-Hopfian condition is a related coarse notion for metric spaces. A metric space X is *quasi-isometrically co-Hopfian* if every quasi-isometric embedding of X into itself is a quasi-isometry. This property holds for certain coarse PD(n) spaces [13][11,Section 3] and has been studied for certain Gromov hyperbolic spaces [16] and non-uniform lattices in rank-one semisimple Lie groups [11]. The infinite-index embeddings we give in Section 2 and Theorem 3.1 are retractions and therefore are quasi-isometric embeddings that are not quasi-isometries. These are the only known examples, as far as we know, of one-ended hyperbolic groups that are not quasi-isometrically co-Hopfian.

Motivation of the terminology

Kapovich [12] uses the term *weakly* co-Hopfian instead of commensurably co-Hopfian. We abandon this terminology, since the property is not weaker than the co-Hopfian property: every one-ended hyperbolic group is co-Hopfian by the theorem of Sela, but not every one-ended hyperbolic group is commensurably co-Hopfian as shown here. However, in general, the commensurably co-Hopfian property defined in this paper is not a stronger condition than the co-Hopfian property. For example, the integers $\mathbb{Z}$ are commensurably co-Hopfian, but not co-Hopfian. The adjective *commensurably* is justified by the fact that being commensurably co-Hopfian is an abstract commensurability invariant, which follows from the lemma below. The co-Hopfian property, on the other hand, is not an abstract commensurability invariant by work of Cornuier [5,Appendix A].

Lemma 1.4. If $H \leq G$ is a finite-index subgroup, then H is commensurably co-Hopfian if and only if G is commensurably co-Hopfian.

Proof. If G is commensurably co-Hopfian, then H is commensurably co-Hopfian. Indeed, otherwise, there exists a finite-index subgroup $H' \leq H \leq G$ with an infinite-index embedding $\varphi : H' \rightarrow H \leq G$, contradicting the commensurably co-Hopfianity of G .

Conversely, suppose G is not commensurably co-Hopfian. Then, there exists a finite-index subgroup $G' \leq G$ and an embedding $\varphi : G' \rightarrow G$ so that $\varphi(G')$ is an infinite-index subgroup of G . The intersection $\varphi^{-1}(H) \cap H$ is a finite-index subgroup of H , since the preimage $\varphi^{-1}(H)$ is a finite-index subgroup of G . Therefore, φ restricts to an infinite-index embedding $\varphi^{-1}(H) \cap H \rightarrow H$. Thus, H is not commensurably co-Hopfian. ■

The following example, explained to the authors by Cornulier, shows that the commensurably co-Hopfian property is not a quasi-isometry invariant.

Example 1.5. (Cornulier) Let Γ be an arithmetic lattice in $\mathrm{SL}_2(\mathbb{Q}_p) \times \mathrm{SL}_2(\mathbb{Q}_q)$ for suitable primes p, q . The group Γ acts geometrically on the product of the associated Bruhat–Tits buildings: the product of two trees. Thus, the group Γ is quasi-isometric to $\mathbb{F}_2 \times \mathbb{F}_2$, the product of two free groups, which is not commensurably co-Hopfian.

The commensurable co-Hopfity of Γ can be deduced from Margulis’ Super-rigidity Theorem. Indeed, suppose $\Gamma' \leq \Gamma$ is a finite-index subgroup and $\phi : \Gamma' \rightarrow \Gamma$ is an infinite-index embedding. Then superrigidity (see [15, Prop. VII.5.3, p225], or alternatively [18, Appendix C]), implies that ϕ extends to a continuous homomorphism $\Phi : \mathrm{SL}_2(\mathbb{Q}_p) \times \mathrm{SL}_2(\mathbb{Q}_q) \rightarrow \mathrm{SL}_2(\mathbb{Q}_p) \times \mathrm{SL}_2(\mathbb{Q}_q)$. It then follows from the representation theory of p -adic Lie groups that Φ is an automorphism, contradicting the fact that ϕ embeds Γ' as an infinite-index subgroup.

We also note that the following question of Bestvina remains open.

Question 1.6. Does there exist a one-ended hyperbolic group that contains isomorphic finite-index subgroups of different index?

1.1 Summary

The table below summarizes the results in this paper and related open problems.

G hyperbolic	JSJ decomposition has only 2-ended and maximal hanging Fuchsian vertex groups	JSJ decomposition has 2-ended, maximal hanging Fuchsian, and rigid vertex groups	JSJ decomposition has only 2-ended and rigid vertex groups	G is one-ended and hyperbolic with trivial JSJ decomposition
Commensurably co-Hopfian	Open problem	Example 4.4	Example 5.1	Poincaré duality groups [29]
Not commensurably co-Hopfian	Section 2; Theorem 3.1	Example 4.1	Open problem	Open problem

Section 2 demonstrates the main example of the paper, and Section 3 generalizes the main example to prove that all simple surface amalgams have fundamental groups that are not commensurably co-Hopfian. Section 4 gives examples of one-ended hyperbolic groups with mixed JSJ decompositions such that some groups are commensurably co-Hopfian and others are not. Section 5 gives an example of commensurably

co-Hopfian one-ended hyperbolic groups that have non-trivial JSJ decomposition and vertex groups that are only two-ended and rigid.

2 The Main Example

A *simple surface amalgam* X is the union of a finite set of surfaces $\Sigma_1, \dots, \Sigma_k$ with $k \geq 3$ and $\chi(\Sigma_i) < 0$ such that $\partial \Sigma_i \cong S^1$ and all boundary components are identified to a single copy of the circle S^1 by a homeomorphism.

The following lemma determines the finite covers of a surface with boundary.

Lemma 2.1. [20, Lemma 3.2] Let Σ be an oriented surface with positive genus. Fix a positive integer d . For each boundary component of Σ , pick a collection of degrees summing to d . Then a d -sheeted covering $\Sigma' \rightarrow \Sigma$ exists with the prescribed degree coverings in the preimage of each boundary component of Σ if and only if the total number of boundary components of Σ' has the same parity as $d\chi(\Sigma)$.

We will repeatedly use Lemma 2.1 to construct finite covers of a surface amalgam. Indeed, given a simple surface amalgam X constructed from surfaces $\Sigma_1, \dots, \Sigma_k$, we can specify covers of each Σ_i and then glue them together along boundary components provided the covering degrees of identified boundary components over the amalgamating curve in X match.

Proof. of Theorem 1.1 Let X be a simple surface amalgam with subsurfaces $\Sigma_1, \Sigma_2, \Sigma_3$, where Σ_i is a surface of genus one with a single boundary component. Demonstrating that $\pi_1(X)$ is not commensurably co-Hopfian follows from considering Figure 1.

We first construct a degree-3 cover $f_1 : X_1 \rightarrow X$. By Lemma 2.1, there exists a degree-3 cover $\Sigma'_i \rightarrow \Sigma_i$ so that Σ'_i has a single boundary component for $i \in \{1, 2, 3\}$. By an elementary Euler characteristic computation, the surface Σ'_i has genus two. The boundary components of each Σ'_i for $i \in \{1, 2, 3\}$ can be identified to each other by a homeomorphism to construct a 3-sheeted cover $f_1 : X_1 \rightarrow X$.

We now build a degree-4 cover $f_2 : X_2 \rightarrow X$. By Lemma 2.1, there exists a degree-2 cover $\Sigma''_i \rightarrow \Sigma_i$ so that Σ''_i has two boundary components for $i \in \{1, 2, 3\}$. Again, by an elementary Euler characteristic computation, the surface Σ''_i has genus one. By identifying a single boundary component from each Σ''_i and attaching copies of Σ_j, Σ_k to the other boundary component of Σ''_i we obtain the 4-sheeted covering $f_2 : X_2 \rightarrow X$.

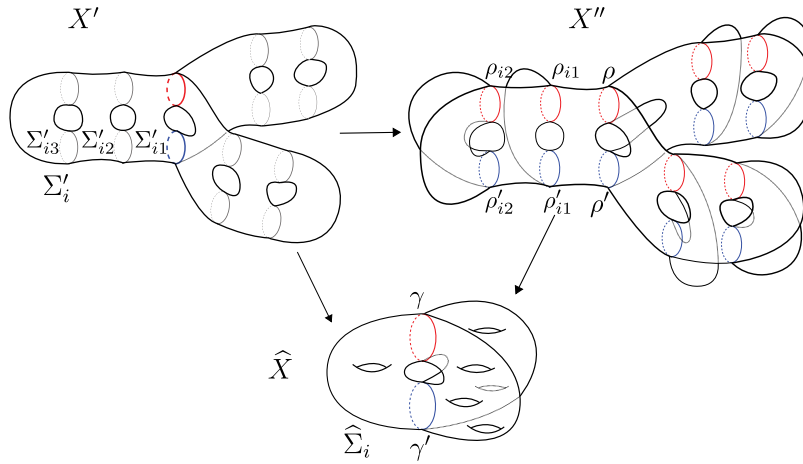


Fig. 2. The spaces X' and X'' finitely cover the space $\widehat{X}$, and the space X' embeds π_1 -injectively in the space X'' , such that $\pi_1(X')$ embeds as an infinite-index subgroup of $\pi_1(X'')$. Thus, the fundamental group of $\widehat{X}$ is not commensurably co-Hopfian since it contains a finite-index subgroup and an infinite-index subgroup isomorphic to $\pi_1(X')$.

There is a π_1 -injective proper embedding $\phi : X_1 \rightarrow X_2$ as shown in Figure 1 that yields an embedding of $\pi_1(X_1)$ in $\pi_1(X_2)$ as an infinite-index subgroup. Therefore, $\pi_1(X)$ contains a finite-index subgroup that is isomorphic to an infinite-index subgroup. ■

3 Simple Surface Amalgams Are Not Commensurably Co-Hopfian

Theorem 3.1. If G is the fundamental group of a simple surface amalgam, then G is not commensurably co-Hopfian.

Proof. Let G be the fundamental group of a simple surface amalgam X with k subsurfaces $\Sigma_1, \dots, \Sigma_k$. We construct a degree-2 cover $\widehat{X} \rightarrow X$ by an application of Lemma 2.1. (This step allows us to resolve any parity issues in the future application of Lemma 2.1.) Let $\widehat{X}$ be the union of k surfaces $\widehat{\Sigma}_1, \dots, \widehat{\Sigma}_k$, where $\chi(\widehat{\Sigma}_i) = 2\chi(\Sigma_i)$. Let $\widehat{\chi}_i = \chi(\widehat{\Sigma}_i)$. The surface $\widehat{\Sigma}_i$ has two boundary components γ_i and γ'_i , and $\widehat{X}$ is obtained by identifying the curves $\{\gamma_i \mid 1 \leq i \leq k\}$ to a single curve γ and the curves $\{\gamma'_i \mid 1 \leq i \leq k\}$ to a single curve γ' .

As in Section 2, we will construct two finite covers X' and X'' of the space $\widehat{X}$ so that the space X' embeds π_1 -injectively in the space X'' , such that $\pi_1(X')$ embeds as an infinite-index subgroup of $\pi_1(X'')$. See Figure 2. This construction relies on the next claim.

Claim 3.2. There exists a set of positive integers $\{D, d, d_i \mid 1 \leq i \leq k\}$ so that

$$\begin{aligned}(d + d_1)\widehat{\chi}_1 + 2d_1\widehat{\chi}_k + d_1\widehat{\chi}_1 &= D\widehat{\chi}_1 \\(d + d_2)\widehat{\chi}_2 + 2d_2\widehat{\chi}_1 + d_2\widehat{\chi}_2 &= D\widehat{\chi}_2 \\&\vdots \\(d + d_k)\widehat{\chi}_k + 2d_k\widehat{\chi}_{k-1} + d_k\widehat{\chi}_k &= D\widehat{\chi}_k.\end{aligned}$$

■

Proof of Claim. Rewrite the i -th equation in the following form (with indices mod k):

$$2d_i(\widehat{\chi}_{i-1} + \widehat{\chi}_i) = (D - d)\widehat{\chi}_i.$$

Choose positive integers D and d such that $D > d$ and $(D - d)$ is divisible by $2\text{lcm}\{(|\widehat{\chi}_{i-1} + \widehat{\chi}_i|) \mid 1 \leq i \leq k\}$. Then, we obtain positive integers

$$d_i := \frac{(D - d)\widehat{\chi}_i}{2(\widehat{\chi}_{i-1} + \widehat{\chi}_i)} \geq 1.$$

■

Let $D, d, d_1, \dots, d_k$ be positive integers satisfying the equations given by Claim 3.2. There exists a degree- D cover $X' \rightarrow \widehat{X}$ constructed as follows. Let X' be the union of k surfaces $\Sigma'_1, \dots, \Sigma'_k$, so that $\chi(\Sigma'_i) = D\widehat{\chi}_i = 2D\chi_i$, the surface Σ'_i has two boundary components ρ_i and ρ'_i , and X' is obtained by identifying the curves $\{\rho_i \mid 1 \leq i \leq k\}$ to a single curve ρ and the curves $\{\rho'_i \mid 1 \leq i \leq k\}$ to a single curve ρ' . By Lemma 2.1, there exists a degree- D covering map $\Sigma'_i \rightarrow \widehat{\Sigma}_i$ that restricts to a degree- D cover on each of the boundary components $\rho \rightarrow \gamma$ and $\rho' \rightarrow \gamma'$. Thus, these maps glue to yield a degree- D cover $X' \rightarrow \widehat{X}$.

To build the space X'' , for $i \in \{1, \dots, k\}$, we will partition each surface $\Sigma'_i \subset X'$ into three subsurfaces, $\Sigma'_{i1}, \Sigma'_{i2}, \Sigma'_{i3}$, and attach additional subsurfaces to the boundary curves of Σ'_{i2} as follows. The construction is illustrated in Figure 2. In particular, the construction ensures that the space X' embeds in X'' π_1 -injectively, and $\pi_1(X')$ embeds as an infinite-index subgroup of $\pi_1(X'')$. Let Σ'_{i1} be the subsurface of Σ'_i with Euler characteristic $(d + d_i)\widehat{\chi}_i$ and four boundary components, two of which are the curves ρ_i and ρ'_i ; call the other boundary curves ρ_{i1} and ρ'_{i1} . Let Σ'_{i2} be the subsurface with Euler characteristic $2d_i\widehat{\chi}_{i-1}$ (subscript mod k) and four boundary components, two of

which are ρ_{i1} and ρ'_{i1} ; call the other boundary curves ρ_{i2} and ρ'_{i2} . Finally, let Σ'_{i3} be the subsurface with Euler characteristic $d_i \widehat{\chi}_i$ and two boundary curves, ρ_{i2} and ρ'_{i2} . Claim 3.2 implies that we indeed have the decomposition $\Sigma'_i \cong \Sigma'_{i1} \cup \Sigma'_{i2} \cup \Sigma'_{i3}$, since the Euler characteristics of Σ'_{i1} , Σ'_{i2} , and Σ'_{i3} sum to $D\widehat{\chi}_i$. For $i \in \{1, \dots, k\}$ attach $k-2$ surfaces $\{\Sigma_j^i \mid j \in \{1, \dots, k\}, j \neq i, i-1\}$ with two boundary components and Euler characteristics $\chi(\Sigma_j^i) = d_i \widehat{\chi}_j$ to the curves $\{\rho_{i1}, \rho'_{i1}\}$. Similarly, attach $k-2$ surfaces $\{\Sigma_j^i \mid j \in \{1, \dots, k\}, j \neq i, i-1\}$ with two boundary components and Euler characteristics $\chi(\Sigma_j^i) = d_i \widehat{\chi}_j$ to the pair of curves $\{\rho_{i2}, \rho'_{i2}\}$.

We now prove there exists a degree- $(d + 2 \sum_{i=1}^k d_i)$ covering map $X'' \rightarrow \widehat{X}$. We describe the cover on the branching curves of X'' , and then we use Lemma 2.1 to show the cover extends to all of X'' . As above, suppose the curves $\{\rho_i\}_{i=1}^k$ and $\{\rho'_i\}_{i=1}^k$ are glued together to form the curves ρ and ρ' in X'' , respectively. Then, ρ and ρ' cover the curves γ and γ' by degree d . For all $i \in \{1, \dots, k\}$, the curves ρ_{i1} and ρ_{i2} cover the curve γ by degree d_i , and the curves ρ'_{i1} and ρ'_{i2} cover the curve γ' by degree d_i . By Lemma 2.1, there exists a degree $(d + d_i)$ cover $\Sigma'_{i1} \rightarrow \Sigma'_i$, a degree $2d_i$ cover $\Sigma'_{i2} \rightarrow \Sigma'_{i-1}$, and a degree d_i cover $\Sigma'_{i3} \rightarrow \Sigma'_i$. By Lemma 2.1, there are degree d_i covers $\Sigma_j^i \rightarrow \Sigma'_j$ and $\Sigma_j^i \rightarrow \Sigma'_j$. Since these covering maps agree on their intersection, there exists a finite cover $X'' \rightarrow \widehat{X}$.

4 Examples With Mixed JSJ Decomposition

Example 4.1. (Not commensurably co-Hopfian.) We adapt the proof in Section 2 to exhibit a one-ended hyperbolic group G whose JSJ decomposition contains both maximal hanging Fuchsian vertex groups and rigid vertex groups and so that G is not commensurably co-Hopfian. An illustration of this example appears in Figure 3 and Figure 4. Let X_0 be a simple surface amalgam with subsurfaces $\Sigma_1, \Sigma_2, \Sigma_3$, where Σ_i is a surface of genus one with a single boundary component. Let a_i be an essential simple closed curve on Σ_i that is not homotopic to the boundary. There exists a homeomorphism $\phi_{ij} : \Sigma_i \rightarrow \Sigma_j$ so that $\phi_{ij}(a_i) = a_j$ for all $i, j \in \{1, 2, 3\}$.

Let H be a torsion-free one-ended hyperbolic group that does not split over a virtually cyclic subgroup, and let X_H be a finite cell complex with $\pi_1(X_H) \cong H$. Suppose there exists an infinite-order element $h \in H$ represented by a closed curve a_h on X_H so that there exists a degree-2 cover $X'_H \rightarrow X_H$ in which a_h lifts to a single closed curve on X'_H . (For a concrete example, let $H \cong \pi_1(S) \rtimes \langle t \rangle$ be the fundamental group of a closed fibered hyperbolic 3-manifold with fiber a closed surface S , and let $h = t$.) For $i \in \{1, 2, 3\}$, let $\phi_i : X_H \rightarrow X_{H_i}$ be a homeomorphism, and let $a_{hi} = \phi(a_h)$. For $i \in \{1, 2, 3\}$ let A_i be an annulus. Glue one boundary component of A_i to the curve a_i and the other

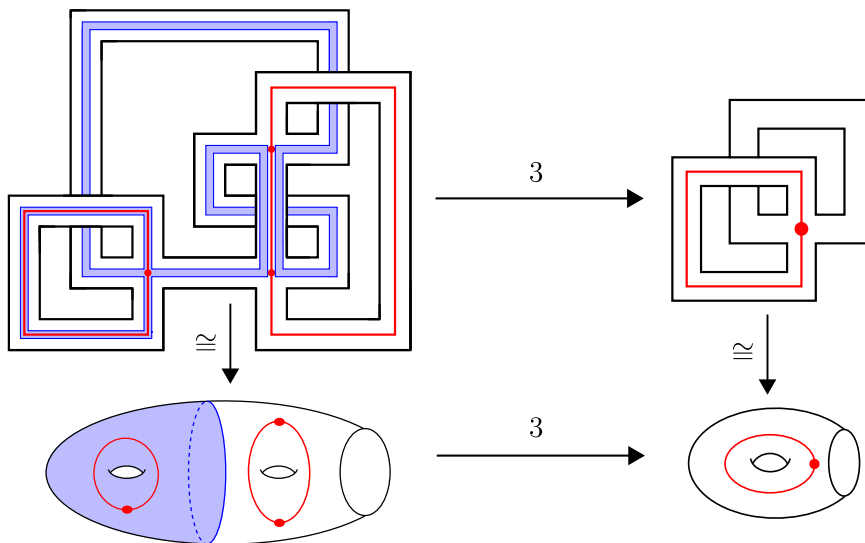


Fig. 3. A degree-3 cover of a surface Σ of genus one with one boundary component by a surface of genus two and one boundary component. The red curve on Σ has two pre-images that lie in subsurfaces separated by the blue curve. The existence of such a cover is evident from the “fat-graph” representation of the surfaces, drawn on the top row. The vertical homeomorphisms are color preserving.

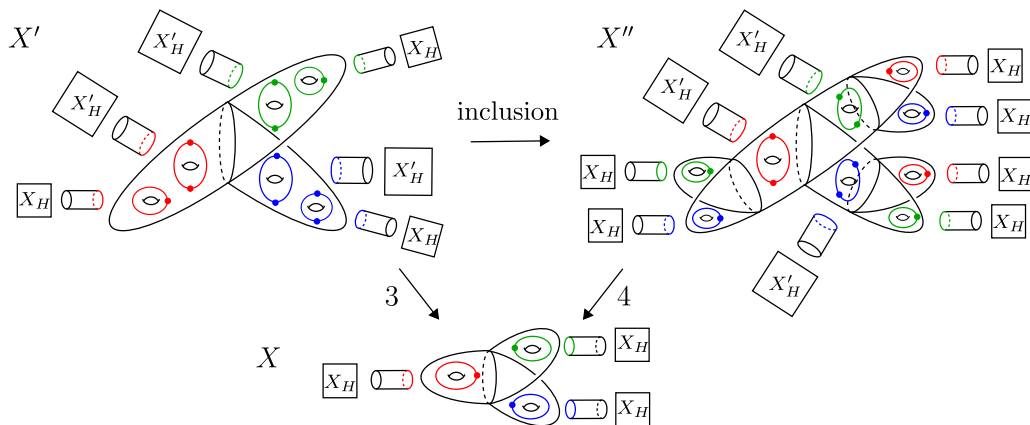


Fig. 4. The squares represent cell complexes X_H and 2-fold covers X'_H of X_H . The spaces X' and X'' are finite covers of the space X , and the space X' embeds π_1 -injectively in X'' , such that $\pi_1 X'$ embeds as an infinite index subgroup of $\pi_1 X''$. Thus, the group $\pi_1(X)$ contains a finite-index subgroup isomorphic to $\pi_1(X')$ and an infinite-index subgroup isomorphic to $\pi_1(X')$.

boundary component of the annulus to the curve a_{hi} by homeomorphisms. Let X be the resulting complex, and let G be the fundamental group of X . The JSJ decomposition of G over 2-ended vertex groups contains three maximal hanging Fuchsian vertex groups, $\pi_1(\Sigma_i)$, and three rigid vertex groups, $\pi_1(X_{Hi})$.

Claim 4.2. The group G is not commensurably co-Hopfian.

Proof. We first construct a degree-3 cover $X' \rightarrow X$. As shown in Figure 3, for $i \in \{1, 2, 3\}$ there exists a degree-3 cover $\Sigma'_i \rightarrow \Sigma_i$ so that Σ'_i has one boundary component and genus two and so that the preimage of the curve a_i has two components a'_i and a''_i , where a'_i covers a_i by degree one and a''_i covers a_i by degree two. Moreover, there exists a closed curve γ_i (shown in blue in Figure 3) that separates Σ'_i into two subsurfaces; one subsurface has boundary γ_i and contains the curve a'_i , and the other subsurface has two boundary components and contains the curve a''_i . Thus, as in Section 2, the boundary components of Σ'_i can be glued together to form a degree three cover of the simple surface amalgam $X'_0 \rightarrow X_0$. By assumption on the group H , the degree-3 cover of the simple surface amalgam extends to a degree-3 cover of X obtained by taking copies of X_H and copies of the degree-two cover X'_H and attaching them along annuli to lifts of the curves a_{hi} on X'_0 . See Figure 4.

The degree-4 cover $X'' \rightarrow X$ is constructed in analogy to the construction in Section 2. The space X'' contains the space X' as a subspace that induces a π_1 -injective embedding of the fundamental group as an infinite-index subgroup, and for $i \in \{1, 2, 3\}$ to each of the curves $\gamma_i \subset X'$ defined in the paragraph above a copy of $\Sigma_i \cup X_{Hi}$ is glued along the boundary component of Σ_i . As above, the space X'' forms a degree-4 cover of X . Then X' embeds π_1 -injectively in X'' such that $\pi_1(X')$ embeds as an infinite-index subgroup of $\pi_1(X'')$. So, the group $G = \pi_1(X)$ contains a finite-index subgroup isomorphic to $\pi_1(X')$ and an infinite-index subgroup isomorphic to $\pi_1(X'')$. Thus, G is not commensurably co-Hopfian. ■

Examples 4.4 and 5.1 make use of the notion of an *acylindrical submanifold*. Let M be a Riemannian manifold and $N \subseteq M$ a locally convex submanifold. Let A denote the annulus. The submanifold N is said to be *acylindrical* if any π_1 -injective map $(A, \partial A) \rightarrow (M, N)$ is relatively homotopic to a map $(A, \partial A) \rightarrow (N, N)$. Equivalently, the subgroup $\pi_1(N) \leq \pi_1(M)$ is *malnormal* in the sense that $\pi_1(N) \cap \pi_1(N)^g = \{1\}$ for all $g \in \pi_1(M) - \pi_1(N)$. In particular, if M is a closed hyperbolic manifold and N is a simple closed geodesic, then N is acylindrical in M .

We also require the following lemma; we include a proof as we are unaware of a reference.

Lemma 4.3. Let Σ and Σ' be compact surfaces with boundary with negative Euler characteristic. If $f : (\Sigma, \partial\Sigma) \rightarrow (\Sigma', \partial\Sigma')$ is a π_1 -injective map, then f is homotopic to a finite-degree covering map of Σ' .

Proof. It suffices to prove that $f_* : \pi_1(\Sigma) \rightarrow \pi_1(\Sigma')$ embeds $\pi_1(\Sigma)$ as a finite-index subgroup of $\pi_1(\Sigma')$. Let $D\Sigma$ and $D\Sigma'$ be the doubles of Σ and Σ' along their boundary components. The spaces $D\Sigma$ and $D\Sigma'$ are closed surfaces of genus greater than 1, and the map f determines a π_1 -injective map $F : D\Sigma \rightarrow D\Sigma'$. This last fact follows since Σ and Σ' have negative Euler characteristic so the subgroups corresponding to the boundary components give malnormal families. Thus, the lift $\tilde{f} : \tilde{\Sigma} \rightarrow \tilde{\Sigma}'$ sends each boundary component to a unique boundary component. If T and T' are the Bass–Serre trees corresponding to the splittings of $\pi_1(D\Sigma)$ and $\pi_1(D\Sigma')$ given by the doubling, then it follows from the previous observation that the F_* -equivariant map $T \rightarrow T'$ induced by the morphism between the graphs of groups is locally injective, and hence globally injective. Thus, $F_* : \pi_1(D\Sigma) \rightarrow \pi_1(D\Sigma')$ is injective.

The fundamental group of a closed surface $D\Sigma$ can only embed in the fundamental group of a closed surface $D\Sigma'$ as a finite-index subgroup, so the map F is homotopic to a finite-degree covering map F' , which induces that same map on the fundamental groups. The preimage $(F')^{-1}(\Sigma')$ may not be Σ , but it will be a subspace homeomorphic to Σ , so we can identify $(F')^{-1}(\Sigma')$ with Σ' , to obtain a covering map $f' : (\Sigma, \partial\Sigma) \rightarrow (\Sigma', \partial\Sigma')$ such that $f_* = f'_*$, so f' is homotopic to f . ■

Example 4.4. (Commensurably co-Hopfian.) Let M be a closed hyperbolic 3-manifold, and let γ be an embedded locally geodesic closed curve in M . Let Σ be a compact surface with positive genus and boundary $\partial\Sigma$ homeomorphic to S^1 . Identify γ with $\partial\Sigma$ via a homeomorphism to obtain a quotient space X . The fundamental group $G = \pi_1(X)$ is a one-ended hyperbolic group given by the amalgamation of the 3-manifold group and the free group $\pi_1(\Sigma)$ along the cyclic groups corresponding to γ and $\partial\Sigma$. This amalgamated free product corresponds to the canonical JSJ decomposition for $\pi_1(X)$.

Proposition 4.5. The group G is commensurably co-Hopfian.

Proof. Let $G' \leq G$ be a finite-index subgroup and $\varphi : G' \rightarrow G$ an injective homomorphism. Without loss of generality, assume that G' is a normal subgroup of G

and let $\pi : X' \rightarrow X$ denote the corresponding finite regular cover. Take the π -preimages of M and Σ to decompose X' as a collection of homeomorphic 3-manifolds $M'_1, \dots, M'_n$ and a collection of homeomorphic surfaces with boundary $\Sigma'_1, \dots, \Sigma'_m$ such that $M'_i \rightarrow M$ and $\Sigma'_j \rightarrow \Sigma$ are regular covers.

We first argue that the homomorphism $\varphi : G' \rightarrow G$ is induced by a map $\Phi : X' \rightarrow X$ such that the restriction of Φ to each 3-manifold M'_i is a covering map $\Phi_i : M'_i \rightarrow M$. Let T denote the Bass–Serre tree of the JSJ splitting of G . The subgroup $\varphi(G') \leq G$ acts on T . Since the group $\pi_1(M'_i)$ does not split over a virtually cyclic subgroup, the subgroup $\varphi(\pi_1(M'_i))$ stabilizes a vertex in T . Thus, there exists $g_i \in G$, such that $\varphi(\pi_1(M'_i))^{g_i} \leq \pi_1(M)$. As $\pi_1(M)$ is commensurably co-Hopfian, $\varphi(\pi_1(M'_i))^{g_i}$ is a finite-index subgroup of $\pi_1(M)$. By Mostow rigidity [19], the covering space corresponding the subgroup $\varphi(\pi_1(M'_i))^{g_i}$ is isometric to M'_i , so the homomorphism $\varphi^{g_i} : \pi_1(M'_i) \rightarrow \pi_1(M)$ is induced by a covering map $\Phi_i : M'_i \rightarrow M$. Since X and X' have contractible universal covers, there exists a continuous map $\Psi : X' \rightarrow X$ such that $\Psi_* = \varphi$. As φ^{g_i} induces the same map on the fundamental group as Φ_i and the space X is a classifying space, we can homotope Ψ to a map Φ so that it restricts to Φ_i on each $M'_i \subseteq X'$ (see [10, Proposition 1B.9.]). Thus, the resulting map Φ is as specified.

Suppose towards a contradiction that $\Phi : X' \rightarrow X$ is not homotopic to a covering map. Let $C_j \subset \Phi^{-1}(\gamma)$ be the set of curves in the full preimage of the amalgamating curve $\gamma \subset X$ that lie in the surface Σ'_j . After homotopy, we may assume that C_j is a set of disjoint curves and $\partial \Sigma'_j \subseteq C_j$. Moreover, since the curves $\gamma \subset M$ and $\partial \Sigma \subset \Sigma$ are acylindrical subspaces, applying a suitable homotopy removes parallel curves in the set C_j . Lemma 4.3 implies that if $C_j = \partial \Sigma'_j$, then Φ can be homotoped on Σ'_j , relative to $\partial \Sigma'_j$, to a covering map $\Sigma'_j \rightarrow \Sigma$. Since we assumed that Φ is not homotopic to a covering map we can say, without loss of generality, the set C_1 contains a curve that is not a component in $\partial \Sigma'_1$.

Let $\sigma_1, \dots, \sigma_\ell$ denote the closures of the components of $\Sigma'_1 - C_1$. Each σ_i is mapped by Φ into either Σ or M and we refer to the subsurfaces as either Σ -type or M -type, accordingly. We verify the following claims:

Claim 4.6. If σ_i is Σ -type and $\sigma_i \cap \sigma_j \neq \emptyset$, for $j \neq i$ then σ_j is M -type. ■

Proof. Suppose not. Then we would have a π_1 -injective map $f : (\sigma_i \cup \sigma_j, \partial(\sigma_i \cup \sigma_j)) \rightarrow (\Sigma, \partial \Sigma)$. By Lemma 4.3 the map f is homotopic to a covering map, which implies that the

curves $\sigma_1 \cap \sigma_2$ would have to be homotopic to a boundary curve in $\sigma_1 \cup \sigma_2$, contradicting the fact that σ_i and σ_j are not annuli. ■

Claim 4.7. If σ_i has a boundary component in $\partial \Sigma'_1$, then σ_i is Σ -type.

Proof. Suppose not. Then we have σ_i that intersects M'_j such that $\sigma_i \cup M'_j$ maps π_1 -injectively into M . But $\sigma_i \cup M'_j$ has cohomological dimension 3 while on the other hand the fundamental group will be infinite ended so it cannot induce a finite index embedding into $\pi_1 M$. By Strebel's theorem [29], such a map cannot exist. ■

Combining Claims 4.6 and 4.7 with the assumption that C'_1 contains a curve not in $\partial \Sigma'_1$ we can deduce that there exists at least some σ_j that is M -type. Moreover, for each Σ -type σ_i , since Φ induces a π_1 -injective map $(\sigma_i, \partial \sigma_i) \rightarrow (\Sigma, \partial \Sigma)$. Lemma 4.3 implies we can homotope, relative to its boundary curves, $\sigma_i \rightarrow \Sigma$ to a covering map.

Under the regular covering map $\pi : X' \rightarrow X$ corresponding to the finite-index subgroup $G' \leq G$, each boundary component in $\partial \Sigma'_j$ covers $\partial \Sigma$ with degree d for some $d \in \mathbb{N}$. Thus, the Euler characteristic satisfies

$$\chi(\Sigma'_j) = d|\partial \Sigma'_j| \cdot \chi(\Sigma).$$

The degrees of the covering maps from M'_i to M given by either π or Φ must coincide since they are determined by the ratio of the volumes. Similarly, if γ' is a curve in $\partial \Sigma'_i$, then since γ' and γ are geodesic curves in some M'_j and M , respectively, the degrees of the covering map $\gamma' \rightarrow \gamma$ given by π and Φ must also coincide since it is given by the ratios of their length. It follows that each component in $\partial \Sigma'_i$ covers γ with degree d .

Thus, we can deduce that

$$\begin{aligned} |\chi(\Sigma'_1)| &= \sum_i |\chi(\sigma_i)| > \sum_{\sigma_i \text{ is } \Sigma\text{-type}} |\chi(\sigma_i)| \\ &= \sum_{\sigma_i \text{ is } \Sigma\text{-type}} \deg(\Phi : \partial \sigma_i \rightarrow \gamma) |\chi(\Sigma)| > d|\partial \Sigma'_1| \cdot |\chi(\Sigma)|. \end{aligned}$$

(We let $\deg(\Phi : \partial \sigma_i \rightarrow \gamma)$ denote the sum of the degrees of the map restricted to each component in $\partial \sigma_i$.) The first inequality follows from discarding the M -type surfaces σ_i , and the second inequality follows from only counting the degrees of the curves in $\partial \Sigma'_1$. This contradicts the previous equality, and thus, Φ is homotopic to a covering map. Therefore, $\varphi(G')$ is a finite-index subgroup of G .

5 Commensurably Co-Hopfian Groups Without Hanging Fuchsian Subgroups in Their JSJ Decomposition

In this section we provide an example of a one-ended hyperbolic group with non-trivial JSJ decomposition and only rigid and two-ended vertex groups. The key point is that we choose the rigid vertex groups to be commensurably co-Hopfian.

Example 5.1. Let M and N be closed hyperbolic 3-manifolds. For simplicity we will assume that $\pi_1(M)$ and $\pi_1(N)$ are incommensurable. Let $\gamma \subseteq M$ and $\sigma \subseteq N$ be simple closed geodesics, and let A be an annulus. Let X be the space obtained from $M \sqcup A \sqcup N$ by gluing one boundary component of the annulus to γ and the other to σ .

Claim 5.2. $G = \pi_1 X$ is commensurably co-Hopfian.

Proof. The proof follows a similar strategy to Claim 4.5. Let $G' \leq G$ be a finite-index subgroup and $\varphi : G' \rightarrow G$ is an injective homomorphism. Assuming G' is a normal subgroup, let $X' \rightarrow X$ be the finite-sheeted, regular cover corresponding to G' . Considering the φ -preimages of M, N , and A , decompose X' as a graph of spaces with vertex spaces $M_{u_1}, \dots, M_{u_m}$ and $N_{v_1}, \dots, N_{v_n}$ and edge spaces $A_{e_1}, \dots, A_{e_a}$.

As $\pi_1(M)$ and $\pi_1(N)$ do not split over a virtually cyclic group, are commensurably co-Hopfian, and are incommensurable with each other, there exists $g_i, h_i \in G$ such that $\varphi(\pi_1(M_{u_i}))^{g_i}$ is a finite-index subgroup of $\pi_1(M)$ and $\varphi(\pi_1(N_{v_i}))^{h_i}$ is a finite-index subgroup of $\pi_1(N)$. By Mostow rigidity, there exist covering maps $\Phi_{u_i} : M_{u_i} \rightarrow M$ and $\Phi_{v_i} : N_{v_i} \rightarrow N$ that correspond to the embeddings $\varphi^{g_i} : \pi_1(M_{u_i}) \rightarrow \pi_1(M)$ and $\varphi^{h_i} : \pi_1(N_{v_i}) \rightarrow \pi_1(N)$.

The spaces X' and X have contractible universal covers, so there exists a continuous map $\Phi : X' \rightarrow X$ such that $\Phi_* = \varphi$. After homotopy, Φ restricts to Φ_{u_i} on M_{u_i} and Φ_{v_i} on N_{v_i} . The map Φ may be homotoped to a covering map since any annulus mapping (A, ∂) to either (M, γ) or (N, σ) can be homotoped into γ or σ , since γ and σ are acylindrical subspaces of M and N . Thus, $\varphi(G')$ is a finite-index subgroup of G . ■

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