Sampling of the Wiener Process for Remote Estimation Over a Channel With Random Delay

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Abstract—In this paper, we consider a problem of sampling a Wiener process, with samples forwarded to a remote estimator over a channel that is modeled as a queue. The estimator reconstructs an estimate of the *real-time* signal value from causally received samples. We study the optimal *online* sampling strategy that minimizes the mean square estimation error subject to a sampling rate constraint. We prove that the optimal sampling strategy is a threshold policy, and find the optimal threshold. This threshold is determined by how much the Wiener process varies during the random service time and the maximum allowed sampling rate. Further, if the sampling times are independent of the observed Wiener process, the above sampling problem for minimizing the estimation error is equivalent to a sampling problem for minimizing the age of information. This reveals an interesting connection between the age of information and remote estimation error. Our comparisons show that the estimation error achieved by the optimal sampling policy can be much smaller than those of age-optimal sampling, zero-wait sampling, and periodic sampling.

Index Terms—Sampling, remote estimation, age of information, Wiener process, queuing system.

I. INTRODUCTION

IN MANY real-time control and cyber-physical systems (e.g., airplane/vehicular control, sensor networks, smart grid, stock trading, robotics, etc.), timely updates about the system status are critical for state estimation and decision making. For example, real-time knowledge about the location, orientation, speed, and acceleration of motor vehicles is imperative for autonomous driving, and fresh information about stock price, financial news, and interest-rate movements is of
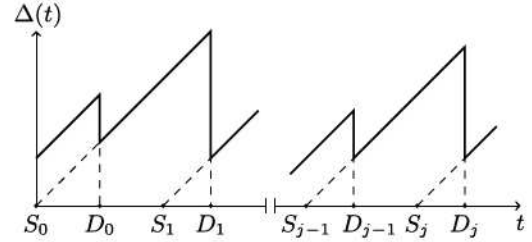
Fig. 1. Evolution of the age of information $\Delta(t)$ over time.

paramount importance for stock trading. In [2], [3], the age of information was introduced to measure the timeliness of status samples about a remote source. Suppose that the i-th status sample is generated at the source at time S_i ($0 \leq S_1 \leq S_2 \leq \ldots$) and is delivered to the destination at time D_i. At time t, the freshest sample available at the destination was generated at time $U(t) = \max\{S_i : D_i \leq t\}$. The *age of information*, or simply the *age*, is a function of time t that is defined as

$$\Delta(t) = t - U(t) = t - \max\{S_i : D_i \leq t\}, \quad (1)$$

which is the time difference between the generation time $U(t)$ of the freshest received sample and the current time t. Hence, a small age $\Delta(t)$ implies that there exists a fresh status sample at the destination. As plotted in Fig. 1, the age increases linearly over time and is reset to a smaller value once a new sample is received. Hence, the age $\Delta(t)$ exhibits a sawtooth pattern. Recently, the age of information concept has received significant attention, because of the rapid growth of real-time applications. A number of status update policies have been developed to keep the age $\Delta(t)$ small, subject to constraints on limited network resources, e.g., [3]–[17].

In practice, the state of many systems is in the form of a time-varying signal W_t, such as the location of a vehicle, the wind speed of a hurricane, and the price chart of a stock. These signals may change slowly at some time and vary more dynamically later. Hence, the time difference between the source and destination, described by the age $\Delta(t) = t - U(t)$, cannot fully characterize the amount of change $W_t - W_{U(t)}$ in the signal value. This motivated us to go beyond the age of information concept and investigate *timely updates of signal samples*.

Let us consider a status update system with two terminals (see Fig. 2): An observer taking samples from a continuous-time signal W_t which is modeled as a Wiener process, and an
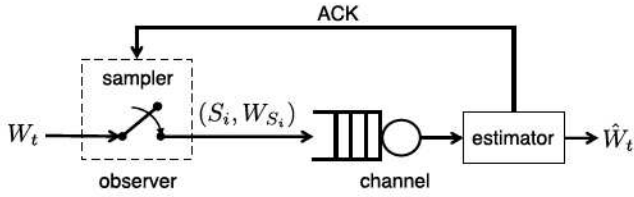
Fig. 2.   System model.

estimator, whose goal is to provide the best-guess $\hat{W}_t$ for the real-time signal value W_t at all time t.[1] The two terminals are connected by a channel that transmits time-stamped samples of the form (S_i, W_{S_i}) according to a first-in, first-out (FIFO) order, where S_i is the sampling time of the i-th sample and W_{S_i} is the value of the i-th sample. The samples are stored in a queue while they wait to be served by the channel. We assume that the samples experience *i.i.d.* random transmission times over the channel, which may be caused by fading, interference, collisions, retransmissions, and etc. As such, the channel is modeled as a FIFO queue with *i.i.d.* service time Y_i satisfying $\mathbb{E}[Y_i^2] < \infty$, where $Y_i \geq 0$ is the transmission time of sample i. This queueing model is helpful to understand the robustness of remote estimation and control systems under occasionally slow service. For example, a UAV flying by a WiFi access point may run into a communication outage caused by interference from the access point. The resulting delay in packet reception may affect the stability of UAV flight control and navigation [19].

Let G_i be the service starting time of sample i such that $S_i \leq G_i$. The delivery time of sample i is $D_i = G_i + Y_i$. The initial value $W_0 = 0$ is known by the estimator for free, which is represented by $S_0 = D_0 = 0$. At any time t, the estimator forms an estimate $\hat{W}_t$ using the samples received up to time t. Similar to [20], we assume that the estimator neglects the implied knowledge when no sample was delivered. The quality of remote estimation is evaluated via the time-average mean-square error (MSE) between W_t and $\hat{W}_t$:

$$\mathrm{mse} = \limsup_{T \to \infty} \frac{1}{T} \mathbb{E}\left[\int_0^T (W_t - \hat{W}_t)^2 dt \right]. \qquad (2)$$

The sampler is subject to a sampling rate constraint

$$\liminf_{n \to \infty} \frac{1}{n} \mathbb{E}[S_n] \geq \frac{1}{f_{\max}}, \qquad (3)$$

where $f_{\max}$ is the maximum allowed sampling rate. In practice, the sampling rate constraint (3) is imposed when there is a need to reduce the cost (e.g., energy consumption) for the transmission, storage, and processing of the samples.

Our goal is to find an optimal *online* sampling strategy that minimizes the MSE in (2) by choosing the sampling times S_i *causally* subject to the sampling rate constraint (3). The contributions of this paper are summarized as follows:

- We formulate the optimal sampling problem as a constrained continuous-time Markov decision problem with

a continuous state space, and solve it exactly. We prove that the optimal online sampling strategy for the Wiener process is a threshold policy,[2] and find the optimal threshold. Let Y be a random variable with the same distribution as Y_i. The optimal threshold is determined by $f_{\max}$ and W_Y, where W_Y is a random variable that has the same distribution as the amount of signal variation $(W_{t+Y} - W_t)$ that occurs during the random service time Y. The random variable W_Y indicates a tight *coupling*, in the optimal sampling policy, between the source process W_t and the service time Y.

- Our threshold-based optimal sampling policy has an important difference from the previous threshold-based sampling policies studied in, e.g., [21]–[38]: We have proven that it is better to not take any new sample when the server is busy. Consequently, the threshold should be *disabled* when the server is busy and *reactivated* once the server becomes available again. This is one of the reasons that sampling policies that ignore the state of the server, such as periodic sampling, can have a large estimation error.

- We show, perhaps surprisingly, even in the absence of a sampling rate constraint (i.e., $f_{\max} = \infty$), the optimal sampling strategy is *not* zero-wait sampling in which a new sample is generated once the previous sample is delivered; rather, it is optimal to wait for a certain amount of time after the previous sample is delivered, and then take the next sample.

- Our study reveals a relationship between the age of information and the estimation error of Wiener process: If the sampling times S_i are independent of the observed Wiener process (i.e., the sampling times S_i are chosen without using any information about the Wiener process), the MSE in (2) is exactly equal to the time-average expectation of the age of information $\limsup_{T \to \infty} \frac{1}{T} \mathbb{E}[\int_0^T \Delta(t) dt]$. Hence, the sampling problem for minimizing the MSE is equivalent to a sampling problem for minimizing the age, where the second problem was solved recently in [9]–[11]. If the sampling times S_i are chosen based on causal knowledge of the Wiener process, the age-optimal sampling policy (i.e., the sampling policy that minimizes the time-average expected age of information) no longer minimizes the MSE: Specifically, in the age-optimal sampling policy, a new sample is taken only when the *age of information* $\Delta(t)$, or equivalently the *expected estimation error* $\mathbb{E}[(W_t - \hat{W}_t)^2]$, is no smaller than a threshold; while in the MSE-optimal sampling policy, a new sample is taken only when *the instantaneous estimation error* $|W_t - \hat{W}_t|$ is no smaller than a threshold. The asymptotics of the MSE-optimal and age-optimal sampling policies at long/short service time or low/high sampling rates are also studied.

- Our theoretical and numerical comparisons show that the MSE of the optimal sampling policy can be much

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[1]This paper focuses on a Wiener process signal model, which has some nice properties that are used in our analysis. An important future direction is to study more general signal models. A recent result along this direction was reported in [18].

[2]A sampling policy is said to be a *threshold* policy if a new sample is taken when a threshold condition is satisfied. Examples of threshold policies can be found in Section III-B.

smaller than those of age-optimal sampling, periodic sampling, and the zero-wait sampling policy described in (9) below. In particular, periodic sampling is far from optimal when the sampling rate is sufficiently low or sufficiently high; age-optimal sampling is far from optimal when the sampling rate is sufficiently low; periodic sampling, age-optimal sampling, and zero-wait sampling policies are all far from optimal if the service time distribution is heavy-tailed.

The rest of this paper is organized as follows. In Section II, we discuss some related work. In Section III, we describe the system model and the formulation of the optimal sampling problem. In Section IV, we present the solution to this problem and compare it with some other sampling policies. In Section V, we describe the proof of this optimal solution. Some simulation results are provided in Section VI.

II. RELATED WORK

Lossy source coding and the rate-distortion function of the Wiener process was studied in, e.g., [39], [40], where the rate-distortion function represents the optimal tradeoff between the source coding rate and the distortion (i.e., MSE) for recovering of the Wiener process. The goal of these studies is to reconstruct the realization of Wiener process during a past time interval with a small distortion, which can be regarded as an *offline* signal reconstruction problem. This differs from our *online* signal tracking problem, where the real-time value of the Wiener process is estimated at the destination from causally received samples.

This paper is related to recent studies on the age of information, e.g., [3]–[17]. As mentioned above in Section I, a connection between the age of information and the estimation error of Wiener process is characterized in this paper. The estimation error of Wiener process was also mentioned in [4], [5] as an illustration of the age of information, where age-based sampling was not studied and the condition that the sampling times are independent of the Wiener process was used implicitly. Recently, a relationship between a nonlinear function of the age of information and the estimation error of the Ornstein-Uhlenbeck (OU) process was found in a follow-up study of the current paper [18].

This paper can also be considered as a contribution to the rich literature on remote estimation, e.g., [21]–[38], by including a queueing model. In [21], Åström and Bernhardsson showed that a threshold-based sampling method, in which a new sample is taken once the amount of signal variation since the previous sample has reached a threshold, can achieve a smaller estimation error than the traditional periodic sampling method with the same sampling rate. Such a threshold-based sampler and a Kalman-like estimator have been proven to be jointly optimal for minimizing the remote estimation error of several discrete-time signal processes in [22]–[27]. The sampling and remote estimation of continuous-time signal processes were considered in [28], [29], where it was shown that a threshold-based sampling policy is optimal for minimizing the estimation error of the Wiener process, and the optimal threshold was found. In [22]–[29], it was assumed that

the samples are transmitted from the sampler to the estimator over a perfect channel that is error and noise free. There are some recent studies that used explicit channel models. In [30]–[32], Gao et. al. considered the optimal transmission scheduling and remote estimation of an *i.i.d.* discrete-time source process X_t over an additive noise channel. Because of the noise, the transmitter needs to encode its message before transmission. In [30], [31], it was shown that, for a class of symmetric probability distributions on the source symbol X_t, if the transmission scheduling policy is threshold-based, i.e., a new coded packet is sent if $|X_t|$ is no smaller than a threshold, then the optimal encoder and decoder are piecewise affine. In [32], it was shown that if (i) the encoder and decoder (i.e., estimator) are piecewise affine, and (ii) the transmission scheduler satisfies some technical assumption, the optimal transmission scheduling policy is threshold-based. Some extensions of this research were reported in [33]–[35]. In [36]–[38], Chakravorty and Mahajan considered optimal transmission scheduling and remote estimation over a few channel models, where it was proved that a threshold-based transmission policy and a Kalman-like estimator are jointly optimal for minimizing the remote estimation error.

The closest study to this paper are [28], [29], where the optimal sampler of the Wiener process was designed in the absence of queueing and random service time (i.e., $Y_i = 0$). As we will see later, the queueing model affects the structure of the optimal sampler. Specifically, the sampler should disable the threshold when there is a packet in service and reactivate the threshold after all previous packets are delivered. A novel proof procedure is developed in the current paper to find the optimal sampler design.

III. SYSTEM MODEL AND PROBLEM FORMULATION

A. MMSE Estimation Policy

At time t, the information available to the estimator contains two part: (i) $M_t = \{(S_i, W_{S_i}, D_i) : D_i \leq t\}$, which contains the sampling time S_i, sample value W_{S_i}, and delivery time D_i of the samples delivered by time t and (ii) the facts that no sample has been received after the latest sample delivery time $\max\{D_i : D_i \leq t\}$. Similar with [20], we assume that the estimator neglects the implied knowledge when no sample was delivered. In this case, the minimum mean-square error (MMSE) estimation policy [41] is given by (see Appendix A for its derivation)

$$\hat{W}_t = \mathbb{E}[W_t | M_t]$$
$$= W_{S_i}, \text{ if } t \in [D_i, D_{i+1}), \ i = 0, 1, 2, \ldots, \quad (4)$$

which is illustrated in Fig. 3(b).

B. Sampling Policies

Let $I_t \in \{0, 1\}$ denote the idle/busy state of the server at time t. As shown in Fig. 2, the server state I_t is known by the sampler through acknowledgements (ACKs). We assume that once a sample is delivered to the estimator, an ACK is fed back to the sampler with zero delay. Hence, the information that is available to the sampler at time t can be expressed as $\{W_s, I_s : 0 \leq s \leq t\}$.
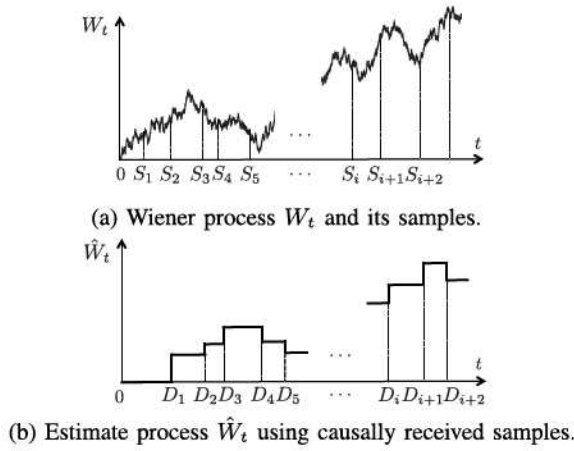
(a) Wiener process W_t and its samples.

(b) Estimate process $\hat{W}_t$ using causally received samples.

Fig. 3. Illustration of the MMSE estimation policy (4).

(a) If $|W_{S_i+Y_i} - W_{S_i}| \geq \sqrt{\beta}$, sample $i+1$ is taken at time $S_{i+1} = S_i + Y_i$.

(b) If $|W_{S_i+Y_i} - W_{S_i}| < \sqrt{\beta}$, sample $i+1$ is taken at time t that satisfies $t \geq S_i + Y_i$ and $|W_t - W_{S_i}| = \sqrt{\beta}$.

Fig. 4. Illustration of the threshold-based sampling policy (12), where no sample is taken during $[S_i, S_i + Y_i)$.

In online sampling policies, each sampling time S_i is chosen causally using the information available at the sampler. To characterize this statement precisely, we define

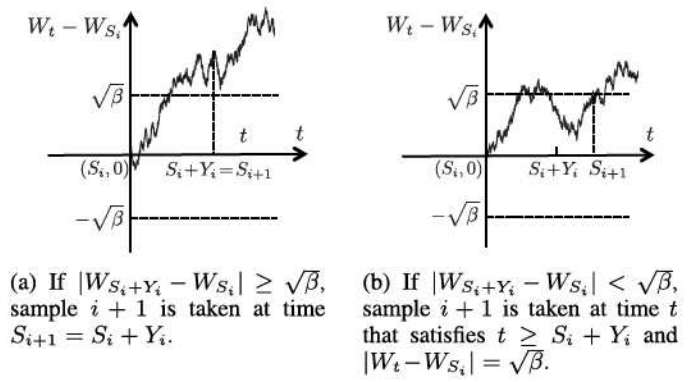$$\mathcal{N}_t = \sigma(W_s, I_s : 0 \leq s \leq t), \quad \mathcal{N}_t^+ = \cap_{s>t} \mathcal{N}_s, \quad (5)$$

where $\sigma(X_1, X_2, \ldots, X_n)$ represents the σ-field generated by the random variables $X_1, X_2, \ldots, X_n$. Then, $\{\mathcal{N}_t^+, t \geq 0\}$ is a *filtration* (i.e., a non-decreasing and right-continuous family of σ-fields) of the information available at the sampler. Each sampling time S_i is a *stopping time* with respect to the filtration $\{\mathcal{N}_t^+, t \geq 0\}$, i.e.,

$$\{S_i \leq t\} \in \mathcal{N}_t^+, \quad \forall t \geq 0. \quad (6)$$

Let $\pi = (S_1, S_2, \ldots)$ denote a sampling policy where $S_1 \leq S_2 \leq \cdots$ form an increasing sequence of sampling times. Let Π denote a set of *online* (also called *causal*) sampling policies satisfying the following two conditions: *(i)* Each sampling policy $\pi \in \Pi$ satisfies (6) for all $i = 0, 1, \ldots$ *(ii)* The inter-sampling times $\{T_i = S_{i+1} - S_i, i = 0, 1, \ldots\}$ form a *regenerative process* [42, Section 6.1]: There exist an increasing sequence $0 \leq k_1 < k_2 < \ldots$ of almost surely finite random integers such that the post-k_j process $\{T_{k_j+i}, i = 0, 1, \ldots\}$ has the same distribution as the post-k_1 process $\{T_{k_1+i}, i = 0, 1, \ldots\}$ and is independent of the pre-k_j process $\{T_i, i = 0, 1, \ldots, k_j - 1\}$; in addition, $\mathbb{E}[k_{j+1} - k_j] < \infty$, $\mathbb{E}[S_{k_1}^2] < \infty$, and $0 < \mathbb{E}[(S_{k_{j+1}} - S_{k_j})^2] < \infty$ for $j = 1, 2, \ldots$ By Condition (ii), we can obtain that, almost surely,

$$\lim_{i \to \infty} S_i = \infty, \quad \lim_{i \to \infty} D_i = \infty. \quad (7)$$

We analyze the MSE in (2), but operationally a nicer criterion is $\limsup_{n \to \infty} \mathbb{E}[\int_0^{D_n} (W_t - \hat{W}_t)^2 dt]/\mathbb{E}[D_n]$. These two criteria are associated to two definitions of "average cost per unit time" used in the literature of infinite-horizon undiscounted semi-Markov decision problems [43]–[47]. They are equivalent, if $\{T_1, T_2, \ldots\}$ is a regenerative process, or more generally, if $\{T_1, T_2, \ldots\}$ has only one ergodic class [43]–[45]. If no condition is imposed, however, these two criteria are different.

Some examples of the sampling policies in Π are:
1. *Periodic sampling* [48], [49]:
   The inter-sampling times are constant, such that for some $\beta \geq 0$,

   $$S_{i+1} = S_i + \beta. \quad (8)$$

2. *Zero-wait sampling* [3], [6], [9], [10]: A new sample is generated once the previous sample is delivered, i.e.,

   $$S_{i+1} = S_i + Y_i. \quad (9)$$

3. *Threshold policy on expected estimation error* [6], [9], [10]: The sampling times are given by

   $$S_{i+1} = \inf\{t \geq S_i + Y_i : t - S_i \geq \beta\} \quad (10)$$
   $$= \inf\left\{t \geq S_i + Y_i : \mathbb{E}[(W_t - \hat{W}_t)^2] \geq \beta\right\}, \quad (11)$$

   where $S_i + Y_i = D_i$ and, according to (4), $\hat{W}_t = W_{S_i}$.

4. *Threshold policy on instantaneous estimation error*: The sampling times are given by

   $$S_{i+1} = \inf\left\{t \geq S_i + Y_i : |W_t - \hat{W}_t| \geq \sqrt{\beta}\right\}, \quad (12)$$

   where $\hat{W}_t = W_{S_i}$. The sampling policy in (12) can be understood as follows: As illustrated in Fig. 4, if $|W_{S_i+Y_i} - W_{S_i}| \geq \sqrt{\beta}$, sample $i+1$ is generated at the time $S_{i+1} = S_i + Y_i$ when sample i is delivered; otherwise, if $|W_{S_i+Y_i} - W_{S_i}| < \sqrt{\beta}$, sample $i+1$ is generated at the earliest time t such that $t \geq S_i + Y_i$ and $|W_t - W_{S_i}|$ reaches the threshold $\sqrt{\beta}$. It is worthwhile to emphasize that even if there exists time $t \in [S_i, S_i + Y_i)$ such that $|W_t - W_{S_i}| \geq \sqrt{\beta}$, no sample is taken at such time t, as depicted in both cases of Fig. 4. In other words, the threshold-based control is *disabled* during $[S_i, S_i + Y_i)$ and is *reactivated* at time $S_i + Y_i$.

A sampling policy $\pi \in \Pi$ is said to be *signal-ignorant* (*signal-aware*), if π is (not) independent of the Wiener process $\{W_t, t \geq 0\}$. The sampling policies (8), (9), and (11) are signal-ignorant, and the sampling policy (12) is signal-aware.

C. Optimal Sampling Problem

We assume that the source process $\{W_t, t \geq 0\}$ and the service times $\{Y_i, i = 1, 2, \ldots\}$ are *mutually independent* and do not change according to the sampling policy. In addition, we assume that the Y_i's are *i.i.d.* with $\mathbb{E}[Y_i^2] < \infty$. The optimal sampling problem for minimizing the MSE subject to a sampling rate constraint is formulated as

$$\mathsf{mse}_{\text{opt}} \triangleq \inf_{\pi \in \Pi} \limsup_{T \to \infty} \frac{1}{T} \mathbb{E}\left[\int_0^T (W_t - \hat{W}_t)^2 dt\right] \quad (13)$$

$$\text{s.t.} \quad \liminf_{n \to \infty} \frac{1}{n}\mathbb{E}[S_n] \geq \frac{1}{f_{\max}}, \quad (14)$$

where $\mathsf{mse}_{\text{opt}}$ denotes the optimal value of (13). Later on in the paper, the unconstrained problem with $f_{\max} = \infty$ will also be studied.

IV. OPTIMAL SAMPLING POLICIES

A. Signal-Aware Sampling

Problem (13) is a constrained continuous-time Markov decision problem with a continuous state space. Such problems are often lack of closed-form or analytical solutions, however we were able to solve (13) exactly:

Theorem 1. *If the service times Y_i's are i.i.d. with $\mathbb{E}[Y_i^2] < \infty$, then there exists $\beta \geq 0$ such that the sampling policy (12) is an optimal solution of (13), and the optimal β is determined by solving[3]*

$$\mathbb{E}[\max(\beta, W_Y^2)] = \max\left(\frac{1}{f_{\max}}, \frac{\mathbb{E}[\max(\beta^2, W_Y^4)]}{2\beta}\right), \quad (15)$$

where Y is a random variable with the same distribution as Y_i. The optimal value of (13) is then given by

$$\mathsf{mse}_{\text{opt}} = \frac{\mathbb{E}[\max(\beta^2, W_Y^4)]}{6\mathbb{E}[\max(\beta, W_Y^2)]} + \mathbb{E}[Y]. \quad (16)$$

Proof: See Section V. □

According to Theorem 1, in the optimal signal-aware sampling policy, the $(i+1)$-th sample is taken at the earliest time t satisfying two conditions: (*i*) The i-th sample has already been delivered by time t, i.e., $t \geq D_i = S_i + Y_i$, and (*ii*) the instantaneous estimation error $|W_t - \hat{W}_t|$ at time t is no smaller than a threshold $\sqrt{\beta}$. In addition, the threshold $\sqrt{\beta}$ is determined by the maximum allowed sampling rate $f_{\max}$ and W_Y, where W_Y is a random variable that has the same distribution with the amount of signal variation $(W_{t+Y} - W_t)$ during the random service time Y for all starting time t. This indicates a tight *coupling* between the source process W_t and the service time Y, in the optimal sampling policy.

Equation (15) can be solved by using the bisection method with a low computational complexity. Hence, Problem (13) does not suffer from the curse of dimensionality encountered in most Markov decision problems with continuous state spaces. We note that the sampling policy in (12) and (15) is quite general in the sense that it is optimal for any service time

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[3]If $\beta \to 0$, the last terms in (15) and (22) are determined by L'Hospital's rule.

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distribution satisfying $\mathbb{E}[Y^2] < \infty$. The optimal signal-aware sampling policy in (12) and (15) is also called the "MSE-optimal" sampling policy in the sequel.

B. Signal-Ignorant Sampling and the Age of Information

Let $\Pi_{\text{signal-ignorant}} \subset \Pi$ denote the set of signal-ignorant sampling policies, defined as

$$\Pi_{\text{signal-ignorant}} = \{\pi \in \Pi : \pi \text{ is independent of } \{W_t, t \geq 0\}\}. \quad (17)$$

In these policies, the sampling decisions depend only on the service time $\{Y_i, i = 1, 2, \ldots\}$ but not the source process $\{W_t, t \geq 0\}$. For each $\pi \in \Pi_{\text{signal-ignorant}}$, the objective function in (13) can be rewritten as (see Appendix B for the proof)

$$\limsup_{T \to \infty} \frac{1}{T}\mathbb{E}\left[\int_0^T (W_t - \hat{W}_t)^2 dt\right]$$
$$= \limsup_{T \to \infty} \frac{1}{T}\mathbb{E}\left[\int_0^T \Delta(t)dt\right], \quad (18)$$

where $\Delta(t)$ is the *age of information* defined in (1). In FIFO queueing systems, $D_i \leq D_{i+1}$ holds for all i. Hence, the age $\Delta(t)$ can be equivalently expressed as

$$\Delta(t) = t - S_i, \quad t \in [D_i, D_{i+1}), \quad i = 0, 1, 2, \ldots \quad (19)$$

If the set of feasible policies is restricted from Π to $\Pi_{\text{signal-ignorant}}$, (13) reduces to the following sampling problem for minimizing the time-average expectation of the age of information [9]–[11]:

$$\mathsf{mse}_{\text{age-opt}} \triangleq \inf_{\pi \in \Pi_{\text{signal-ignorant}}} \limsup_{T \to \infty} \frac{1}{T}\mathbb{E}\left[\int_0^T \Delta(t)dt\right] \quad (20)$$

$$\text{s.t.} \quad \liminf_{n \to \infty} \frac{1}{n}\mathbb{E}[S_n] \geq \frac{1}{f_{\max}},$$

where $\mathsf{mse}_{\text{age-opt}}$ denotes the optimal value of (20). Because $\Pi_{\text{signal-ignorant}} \subset \Pi$,

$$\mathsf{mse}_{\text{opt}} \leq \mathsf{mse}_{\text{age-opt}}. \quad (21)$$

Note that problem (20) is simpler than (13) because the sampler does not use knowledge of W_t to make decisions. To solve (13), stronger techniques than those in [9]–[11] are developed in Section V.

Theorem 2. *[11] If the service times Y_i's are i.i.d. with $\mathbb{E}[Y_i^2] < \infty$, then there exists $\beta \geq 0$ such that the sampling policy (11) is an optimal solution of (20), and the optimal β is determined by solving*

$$\mathbb{E}[\max(\beta, Y)] = \max\left(\frac{1}{f_{\max}}, \frac{\mathbb{E}[\max(\beta^2, Y^2)]}{2\beta}\right), \quad (22)$$

where Y is a random variable with the same distribution as Y_i. The optimal value of (20) is then given by

$$\mathsf{mse}_{\text{age-opt}} \triangleq \frac{\mathbb{E}[\max(\beta^2, Y^2)]}{2\mathbb{E}[\max(\beta, Y)]} + \mathbb{E}[Y]. \quad (23)$$

Theorem 2 was proven in [9], [10] under an extra condition that the time difference $S_{i+1} - D_i$ is upper bounded by a

constant $M > 0$. In [11], Theorem 2 was established without requiring this extra condition.

One can obtain some interesting observations by comparing Theorem 1 and Theorem 2: In the optimal signal-ignorant sampling policy presented in Theorem 2, the $(i+1)$-th sample is taken at the earliest time t satisfying two conditions: *(i)* The i-th sample has already been delivered by time t, i.e., $t \geq D_i = S_i + Y_i$, and *(ii)* the expected estimation error $\mathbb{E}[(W_t - \hat{W}_t)]^2$ at time t, which, by (10) and (19), is equal to the age $\Delta(t)$, is no smaller than a threshold β. The first condition is the same with that in Theorem 1, but the second condition is quite different: Because the sampler has no knowledge about the Wiener process (except for its distribution), it can only use expected estimation error to make decisions. Further, the threshold β in Theorem 2 is determined by the maximum allowed sampling rate $f_{\max}$ and the random service time Y, which is also different from the case in Theorem 1. The optimal signal-ignorant sampling policy in (11) and (22) is also referred to as the "age-optimal" sampling policy.

In the following, the asymptotics of the MSE-optimal and age-optimal sampling policies at low/high service time or low/high sampling frequencies are studied.

C. Short Service Time or Low Sampling Rate

Let

$$Y_i = \alpha X_i \qquad (24)$$

represent the scaling of the service time Y_i with α, where $\alpha \geq 0$ and the X_i's are *i.i.d.* positive random variables. If $\alpha \to 0$ or $f_{\max} \to 0$, we can obtain from (15) that (see Appendix C for the proof)

$$\beta = \frac{1}{f_{\max}} + o\left(\frac{1}{f_{\max}}\right), \qquad (25)$$

where $f(x) = o(g(x))$ as $x \to a$ means that $\lim_{x \to a} f(x)/g(x) = 0$. In this case, the MSE-optimal sampling policy in (12) and (15) becomes

$$S_{i+1} = \inf\left\{t \geq S_i : |W_t - W_{S_i}| \geq \sqrt{\frac{1}{f_{\max}}}\right\}, \qquad (26)$$

and as shown in Appendix C, the optimal value of (13) becomes

$$\mathsf{mse}_{\mathsf{opt}} = \frac{1}{6 f_{\max}} + o\left(\frac{1}{f_{\max}}\right). \qquad (27)$$

The sampling policy (26) was also obtained in [29] for the case that $Y_i = 0$ for all i.

Similarly, if $\alpha \to 0$ or $f_{\max} \to 0$, the age-optimal sampling policy in (11) and (22) becomes periodic sampling (8) with $\beta = 1/f_{\max} + o(1/f_{\max})$, and the optimal value of (20) is $\mathsf{mse}_{\mathsf{age-opt}} = 1/(2 f_{\max}) + o(1/f_{\max})$. Therefore,

$$\lim_{\alpha \to 0} \frac{\mathsf{mse}_{\mathsf{opt}}}{\mathsf{mse}_{\mathsf{age-opt}}} = \lim_{f_{\max} \to 0} \frac{\mathsf{mse}_{\mathsf{opt}}}{\mathsf{mse}_{\mathsf{age-opt}}} = \frac{1}{3}. \qquad (28)$$

D. Long Service Time or Unbounded Sampling Rate

If $\alpha \to \infty$ or $f_{\max} \to \infty$, as shown in Appendix D, the MSE-optimal sampling policy for solving (13) is given by (12) where β is determined by solving

$$2\beta\mathbb{E}[\max(\beta, W_Y^2)] = \mathbb{E}[\max(\beta^2, W_Y^4)]. \qquad (29)$$

Similarly, if $\alpha \to \infty$ or $f_{\max} \to \infty$, the age-optimal sampling policy for solving (20) is given by (11) where β is determined by solving

$$2\beta\mathbb{E}[\max(\beta, Y)] = \mathbb{E}[\max(\beta^2, Y^2)]. \qquad (30)$$

In these limits, the ratio between $\mathsf{mse}_{\mathsf{opt}}$ and $\mathsf{mse}_{\mathsf{age-opt}}$ depends on the distribution of Y.

When the sampling rate is unbounded, i.e., $f_{\max} = \infty$, one logically reasonable policy is the zero-wait sampling policy in (9) [3], [6], [9], [10]. This zero-wait sampling policy achieves the maximum throughput and the minimum queueing delay. However, this zero-wait sampling policy *almost never* minimizes the MSE in (13) and *does not always* minimize the age of information in (20), as stated in the following two theorems:

We note that the zero-wait sampling policy can be expressed as (12) with $\beta = 0$. By checking when $\beta = 0$ is satisfied in (12) and (15), one can obtain

Theorem 3. *Suppose that $f_{\max} = \infty$. Then, the zero-wait sampling policy* (9) *is an optimal solution to* (13) *if and only if the service time Y is equal to zero with probability one.*

Proof: See Appendix E. $\square$

Hence, as long as the service time Y has a small probability to be positive, the zero-waiting sampling policy is not an optimal solution to (13). Similarly, the optimality of zero-wait sampling policy for solving (20) is characterized as

Theorem 4. *[10] Suppose that $f_{\max} = \infty$. Then, the zero-wait sampling policy* (9) *is an optimal solution to* (20) *if and only if*

$$\mathbb{E}[Y^2] \leq 2 \operatorname{ess\,inf} Y \, \mathbb{E}[Y], \qquad (31)$$

where $\operatorname{ess\,inf} Y = \sup\{y \in [0, \infty) : \Pr[Y < y] = 0\}$ can be considered as the minimum possible value of Y.

Proof: See Appendix E. $\square$

V. PROOF OF THEOREM 1

We prove Theorem 1 in four steps: First, we show that no sample should be generated when the server is busy, which simplifies the optimal online sampling problem. Second, we study the Lagrangian dual problem of the simplified problem, and decompose the Lagrangian dual problem into a series of *mutually independent* per-sample control problems. Each of these per-sample control problems is a continuous-time Markov decision problem. Further, we utilize optimal stopping theory [50] to solve the per-sample control problems. Finally, we show that the Lagrangian duality gap of our Markov decision problem is zero. By this, Problem (13) is solved. The details are as follows.

A. Simplification of Problem (13)

The following lemma is useful for simplifying (13).

Lemma 1. *Suppose that π is a feasible policy for Problem (13), in which at least one sample is taken when the server is busy processing an earlier generated sample. Then, there exists another feasible policy π_1 for Problem (13) that has a smaller estimation error than policy π. Hence, it is suboptimal in Problem (13) to take a new sample before the previous sample is delivered.*

Proof: Lemma 1 is proven by using the strong Markov property of the Wiener process and the orthogonality principle of MMSE estimation. The details are provided in Appendix F. □

By Lemma 1, we only need to consider a sub-class of sampling policies $\Pi_1 \subset \Pi$ such that each sample is generated and submitted to the server after the previous sample is delivered, i.e.,

$$\Pi_1 = \{\pi \in \Pi : S_i \geq D_{i-1} \text{ for all } i\}. \tag{32}$$

This completely eliminates the waiting time wasted in the queue, and hence the queue is always kept empty. The *information* that is available for determining S_i includes the history of signal values $(W_t, t \in [0, S_i])$ and the service times $(Y_1, \ldots, Y_{i-1})$ of previous samples.[4] To characterize this statement precisely, let us define the σ-fields $\mathcal{F}_t = \sigma(W_s : s \in [0, t])$ and $\mathcal{F}_t^+ = \cap_{s>t} \mathcal{F}_s$. Then, $\{\mathcal{F}_t^+, t \geq 0\}$ is the filtration (i.e., a non-decreasing and right-continuous family of σ-fields) of the Wiener process W_t. Given the service times $(Y_1, \ldots, Y_{i-1})$ of previous samples, S_i is a *stopping time* with respect to the filtration $\{\mathcal{F}_t^+, t \geq 0\}$ of the Wiener process W_t, that is

$$[\{S_i \leq t\}|Y_1, \ldots, Y_{i-1}] \in \mathcal{F}_t^+, \ \forall \, t \geq 0. \tag{33}$$

Then, the policy space Π_1 can be alternatively expressed as

$$\Pi_1 = \{S_i : [\{S_i \leq t\}|Y_1, \ldots, Y_{i-1}] \in \mathcal{F}_t^+, \ \forall \, t \geq 0,$$
$$S_i \geq D_{i-1} \text{ for all } i,$$
$$T_i = S_{i+1} - S_i \text{ is a regenerative process}\}. \tag{34}$$

Recall that any policy in Π satisfies "$T_i = S_{i+1} - S_i$ is a regenerative process".

Let $Z_i = S_{i+1} - D_i \geq 0$ represent the *waiting time* between the delivery time D_i of sample i and the generation time S_{i+1} of sample $i + 1$. Then, $S_i = Z_0 + \sum_{j=1}^{i-1}(Y_j + Z_j)$ and $D_i = \sum_{j=0}^{i-1}(Z_j + Y_{j+1})$. If $(Y_1, Y_2, \ldots)$ is given, $(S_0, S_1, \ldots)$ is uniquely determined by $(Z_0, Z_1, \ldots)$. Hence, one can also use $\pi = (Z_0, Z_1, \ldots)$ to represent a sampling policy.

Because T_i is a regenerative process, using the renewal theory [51] and [42, Section 6.1], one can show that in

[4]Note that the generation times $(S_1, \ldots, S_{i-1})$ of previous samples are also included in this information.

Problem (13), $\frac{1}{n}\mathbb{E}[S_n]$ is a convergent sequence and

$$\limsup_{T \to \infty} \frac{1}{T} \mathbb{E}\left[\int_0^T (W_t - \hat{W}_t)^2 dt\right]$$
$$= \lim_{n \to \infty} \frac{\mathbb{E}\left[\int_0^{D_n}(W_t - \hat{W}_t)^2 dt\right]}{\mathbb{E}[D_n]}$$
$$= \lim_{n \to \infty} \frac{\sum_{i=0}^{n-1} \mathbb{E}\left[\int_{D_i}^{D_{i+1}}(W_t - W_{S_i})^2 dt\right]}{\sum_{i=0}^{n-1} \mathbb{E}[Y_i + Z_i]},$$

where in the last step we have used $\mathbb{E}[D_n] = \mathbb{E}[\sum_{i=0}^{n-1}(Z_i + Y_{i+1})] = \mathbb{E}[\sum_{i=0}^{n-1}(Y_i + Z_i)]$. Hence, (13) can be rewritten as the following Markov decision problem:

$$\mathsf{mse}_{\mathrm{opt}} \triangleq \inf_{\pi \in \Pi_1} \lim_{n \to \infty} \frac{\sum_{i=0}^{n-1} \mathbb{E}\left[\int_{D_i}^{D_{i+1}}(W_t - W_{S_i})^2 dt\right]}{\sum_{i=0}^{n-1} \mathbb{E}[Y_i + Z_i]} \tag{35}$$

$$\text{s.t. } \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{E}[Y_i + Z_i] \geq \frac{1}{f_{\max}}, \tag{36}$$

where $\mathsf{mse}_{\mathrm{opt}}$ is the optimal value of (35).

In order to solve (35), let us consider the following Markov decision problem with a parameter $c \geq 0$:

$$p(c) \triangleq \inf_{\pi \in \Pi_1} \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{E}\left[\int_{D_i}^{D_{i+1}}(W_t - W_{S_i})^2 dt - c(Y_i + Z_i)\right] \tag{37}$$

$$\text{s.t. } \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{E}[Y_i + Z_i] \geq \frac{1}{f_{\max}}, \tag{}$$

where $p(c)$ is the optimum value of (37). Similar to Dinkelbach's method [52] for nonlinear fractional programming, we can obtain the following lemma for our Markov decision problem:

Lemma 2. *The following assertions are true:*

(a). $\mathsf{mse}_{\mathrm{opt}} \gtreqless c$ *if and only if* $p(c) \gtreqless 0$.

(b). *If $p(c) = 0$, the solutions to (35) and (37) are identical.*

Proof: See Appendix G. □

Hence, the solution to (35) can be obtained by solving (37) and seeking a $\mathsf{mse}_{\mathrm{opt}} \geq 0$ such that

$$p(\mathsf{mse}_{\mathrm{opt}}) = 0. \tag{38}$$

B. Lagrangian Dual Problem of (37) when $c = \mathsf{mse}_{\mathrm{opt}}$

Although (37) is a continuous-time Markov decision problem with a continuous state space, rather than a convex optimization problem, it is possible to use the Lagrangian dual approach to solve (37) and show that it admits no duality gap. When $c = \mathsf{mse}_{\mathrm{opt}}$, define the following Lagrangian

$$L(\pi; \lambda)$$
$$= \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{E}\left[\int_{D_i}^{D_{i+1}}(W_t - W_{S_i})^2 dt - (\mathsf{mse}_{\mathrm{opt}} + \lambda)(Y_i + Z_i)\right]$$
$$+ \frac{\lambda}{f_{\max}}, \tag{39}$$

where $\lambda \geq 0$ is the dual variable. Let

$$e(\lambda) \triangleq \inf_{\pi \in \Pi_1} L(\pi; \lambda). \tag{40}$$

Then, the Lagrangian dual problem of (37) is defined by

$$d \triangleq \max_{\lambda \geq 0} e(\lambda), \tag{41}$$

where d is the optimum value of (41). Weak duality [53], [54] implies that $d \leq p(\text{mse}_{\text{opt}})$. In Section V-D, we will establish strong duality, i.e., $d = p(\text{mse}_{\text{opt}})$.

In the sequel, we solve (40). Using the stopping times and martingale theory of the Wiener process, we can obtain the following lemma:

Lemma 3. *Let $\tau \geq 0$ be a stopping time of the Wiener process W_t with $\mathbb{E}[\tau^2] < \infty$, then*

$$\mathbb{E}\left[\int_0^\tau W_t^2 \, dt\right] = \frac{1}{6}\mathbb{E}\left[W_\tau^4\right]. \tag{42}$$

Proof: See Appendix H. □

By using Lemma 3 and the sufficient statistics of (40), we can show that for every $i = 1, 2, \ldots$,

$$\mathbb{E}\left[\int_{D_i}^{D_{i+1}} (W_t - W_{S_i})^2 dt\right]$$
$$= \frac{1}{6}\mathbb{E}\left[(W_{S_i+Y_i+Z_i} - W_{S_i})^4\right] + \mathbb{E}[Y_i + Z_i]\mathbb{E}[Y_i], \tag{43}$$

which is proven in Appendix I.

For any $s \geq 0$, define the σ-fields $\mathcal{F}_t^s = \sigma(W_{s+v} - W_s : v \in [0, t])$ and $\mathcal{F}_t^{s+} = \cap_{v>t}\mathcal{F}_v^s$, as well as the filtration $\{\mathcal{F}_t^{s+}, t \geq 0\}$ of the time-shifted Wiener process $\{W_{s+t} - W_s, t \in [0, \infty)\}$. Define $\mathfrak{M}_s$ as the set of square-integrable stopping times of $\{W_{s+t} - W_s, t \in [0, \infty)\}$, i.e.,

$$\mathfrak{M}_s = \{\tau \geq 0 : \{\tau \leq t\} \in \mathcal{F}_t^{s+}, \mathbb{E}\left[\tau^2\right] < \infty\}.$$

By substituting (43) into (40) and using again the sufficient statistics of (40), we can obtain

Theorem 5. *An optimal solution $(Z_0, Z_1, \ldots)$ to (40) satisfies*

$$Z_i = \arg\inf_{\tau \in \mathfrak{M}_{S_i+Y_i}} \mathbb{E}\left[\frac{1}{2}(W_{S_i+Y_i+\tau} - W_{S_i})^4\right.$$
$$\left. -\beta(Y_i + \tau)\Big| W_{S_i+Y_i} - W_{S_i}, Y_i\right], \tag{44}$$

where β is given by

$$\beta = 3(\text{mse}_{\text{opt}} + \lambda - \mathbb{E}[Y]) \geq 0. \tag{45}$$

Proof: See Appendix J. □

Note that because the Y_i's are *i.i.d.* and the strong Markov property of the Wiener process, the Z_i's as solutions of (44) are also *i.i.d.*

C. Per-Sample Optimal Stopping Solution to (44)

We use optimal stopping theory [50] to solve (44). Let us first pose (44) in the language of optimal stopping. A continuous-time two-dimensional Markov chain X_t on a probability space $(\mathbb{R}^2, \mathcal{F}, \mathbb{P})$ is defined as follows: Given the initial state $X_0 = x = (s, b)$, the state X_t at time t is

$$X_t = (s + t, b + W_t), \tag{46}$$

where $\{W_t, t \geq 0\}$ is a standard Wiener process. Define $\mathbb{P}_x(A) = \mathbb{P}(A|X_0 = x)$ and $\mathbb{E}_x Z = \mathbb{E}(Z|X_0 = x)$, respectively, as the conditional probability of event A and the conditional expectation of random variable Z for given initial state $X_0 = x$. Define the σ-fields $\mathcal{F}_t^X = \sigma(X_v : v \in [0, t])$ and $\mathcal{F}_t^{X+} = \cap_{v>t}\mathcal{F}_v^X$, as well as the filtration $\{\mathcal{F}_t^{X+}, t \geq 0\}$ of the Markov chain X_t. A random variable $\tau : \mathbb{R}^2 \to [0, \infty)$ is said to be a *stopping time* of X_t if $\{\tau \leq t\} \in \mathcal{F}_t^{X+}$ for all $t \geq 0$. Let $\mathfrak{M}$ be the set of square-integrable stopping times of X_t, i.e.,

$$\mathfrak{M} = \{\tau \geq 0 : \{\tau \leq t\} \in \mathcal{F}_t^{X+}, \mathbb{E}\left[\tau^2\right] < \infty\}.$$

Our goal is to solve the following optimal stopping problem:

$$\sup_{\tau \in \mathfrak{M}} \mathbb{E}_x g(X_\tau), \tag{47}$$

where $X_0 = x$ is the initial state of the Markov chain X_t, the function $g : \mathbb{R}^2 \to \mathbb{R}$ is defined as

$$g(s, b) = \beta s - \frac{1}{2}b^4 \tag{48}$$

with parameter $\beta \geq 0$. Notice that (44) is a special case of (47) where the initial state is $x = (Y_i, W_{S_i+Y_i} - W_{S_i})$, and W_t is replaced by the time-shifted Wiener process $W_{S_i+Y_i+t} - W_{S_i}$.

Theorem 6. *For all $x = (s, b) \in \mathbb{R}^2$ and $\beta \geq 0$, an optimal stopping time for solving (47) is*

$$\tau^* = \inf\left\{t \geq 0 : |b + W_t| \geq \sqrt{\beta}\right\}. \tag{49}$$

In order to prove Theorem 6, let us define the function

$$u(x) = \mathbb{E}_x g(X_{\tau^*}) \tag{50}$$

and establish some properties of $u(x)$.

Lemma 4. *$u(x) \geq g(x)$ for all $x \in \mathbb{R}^2$, and*

$$u(s, b) = \begin{cases} \beta s - \frac{1}{2}b^4, & \text{if } b^2 \geq \beta; \\ \beta s + \frac{1}{2}\beta^2 - \beta b^2, & \text{if } b^2 < \beta. \end{cases} \tag{51}$$

Proof: See Appendix K. □

A function $f(x)$ is said to be *excessive* for the process X_t if [50]

$$\mathbb{E}_x f(X_t) \leq f(x), \quad \text{for all } t \geq 0, \ x \in \mathbb{R}^2. \tag{52}$$

By using the Itô-Tanaka-Meyer formula [55, Theorem 7.14 and Corollary 7.35] in stochastic calculus, we can obtain

Lemma 5. *The function $u(x)$ is excessive for the process X_t.*

Proof: See Appendix L. □

Now, we are ready to prove Theorem 6. *Proof:* [Proof of Theorem 6] In Lemma 4 and Lemma 5, we have shown that

$u(x) = \mathbb{E}_x g(X_{\tau^*})$ is an excessive function and $u(x) \geq g(x)$. In addition, it is known that $\mathbb{P}_x(\tau^* < \infty) = 1$ for all $x \in \mathbb{R}^2$ [56, Theorem 8.5.3]. These conditions, together with the Corollary to Theorem 1 in [50, Section 3.3.1], imply that τ^* is an optimal stopping time of (47). This completes the proof. □

A consequence of Theorem 6 is

Corollary 1. *An optimal solution to (44) is*

$$Z_i = \inf\left\{t \geq 0 : \left|W_{S_i+Y_i+t} - W_{S_i}\right| \geq \sqrt{\beta}\right\}. \quad (53)$$

In addition, this solution satisfies

$$\mathbb{E}[Y_i + Z_i] = \mathbb{E}[\max(\beta, W_Y^2)], \quad (54)$$

$$\mathbb{E}[(W_{S_i+Y_i+Z_i} - W_{S_i})^4] = \mathbb{E}[\max(\beta^2, W_Y^4)]. \quad (55)$$

Proof: See Appendix M. □

D. Zero Duality Gap Between (37) and (41)

Strong duality is established in the following theorem:

Theorem 7. *If $c = \mathsf{mse}_{\mathrm{opt}}$, the following assertions are true:*

(a). *The duality gap between (37) and (41) is zero, i.e., $d = p(\mathsf{mse}_{\mathrm{opt}})$.*

(b). *A common optimal solution to (13), (35), and (37) is given by (12) and (15). The optimal value of (13) is given by (16).*

Proof: [Proof Sketch of Theorem 7] We use [53, Prop. 6.2.5] to find a *geometric multiplier* [53, Definition 6.1.1] for Problem (37). This tells us that the duality gap between (37) and (41) must be zero, because otherwise there is no geometric multiplier [53, Prop. 6.2.3(b)].[5] This result holds not only for convex optimization problem, but also for general non-convex optimization and Markov decision problems like (37). See Appendix N for the details. □

Hence, Theorem 1 follows from Theorem 7.

VI. NUMERICAL RESULTS

In this section, we evaluate the estimation performance achieved by the following four sampling policies:

1. *Periodic sampling:* The policy in (8) with $\beta = f_{\max}$.
2. *Zero-wait sampling [3], [6], [9], [10]:* The sampling policy in (9), which is feasible when $f_{\max} \geq 1/\mathbb{E}[Y_i]$.
3. *Age-optimal sampling [9], [10]:* The sampling policy in (11) and (22), which is the optimal solution to (20).
4. *MSE-optimal sampling:* The sampling policy in (12) and (15), which is the optimal solution to (13).

Let $\mathsf{mse}_{\mathrm{periodic}}$, $\mathsf{mse}_{\mathrm{zero\text{-}wait}}$, $\mathsf{mse}_{\mathrm{age\text{-}opt}}$, and $\mathsf{mse}_{\mathrm{opt}}$, be the MSEs of periodic sampling, zero-wait sampling, age-optimal sampling, MSE-optimal sampling, respectively. According to (21), as well as the facts that periodic sampling is feasible for (20) and zero-wait sampling is feasible for (20) when $f_{\max} \geq 1/\mathbb{E}[Y_i]$, we can obtain

$$\mathsf{mse}_{\mathrm{opt}} \leq \mathsf{mse}_{\mathrm{age\text{-}opt}} \leq \mathsf{mse}_{\mathrm{periodic}},$$

$$\mathsf{mse}_{\mathrm{opt}} \leq \mathsf{mse}_{\mathrm{age\text{-}opt}} \leq \mathsf{mse}_{\mathrm{zero\text{-}wait}}, \quad \text{when } f_{\max} \geq \frac{1}{\mathbb{E}[Y_i]},$$

[5] Note that geometric multiplier is different from the traditional Lagrangian multiplier.
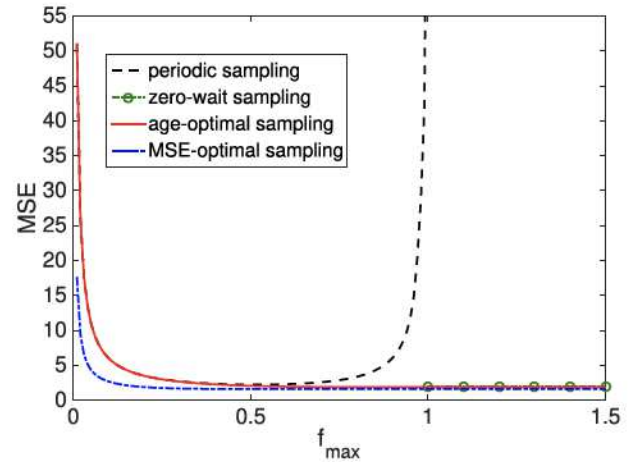
Fig. 5. MSE vs. $f_{\max}$ tradeoff for *i.i.d.* exponential service time and $\mathbb{E}[Y_i] = 1$, where zero-wait sampling is not feasible when $f_{\max} < 1/\mathbb{E}[Y_i]$ and hence is not plotted in that regime.
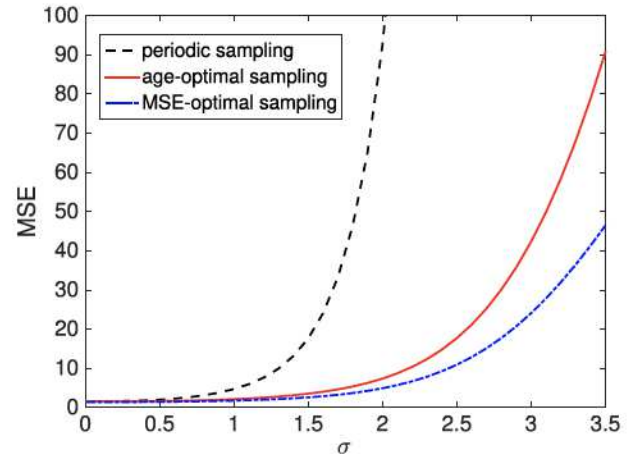
Fig. 6. MSE vs. the scale parameter σ of *i.i.d.* log-normal service time for $f_{\max} = 0.8$ and $\mathbb{E}[Y_i] = 1$, where zero-wait sampling is not feasible because $f_{\max} < 1/\mathbb{E}[Y_i]$.

which fit with our numerical results below.

Figure 5 depicts the tradeoff between MSE and $f_{\max}$ for *i.i.d.* exponential service time with mean $\mathbb{E}[Y_i] = 1/\mu = 1$. Hence, the maximum throughput of the queue is $\mu = 1$. In this setting, $\mathsf{mse}_{\mathrm{periodic}}$ is characterized by eq. (25) of [3], which was obtained using a D/M/1 queueing model [57]. For small values of $f_{\max}$, age-optimal sampling is similar with periodic sampling, and hence $\mathsf{mse}_{\mathrm{age\text{-}opt}}$ and $\mathsf{mse}_{\mathrm{periodic}}$ are of similar values. However, as $f_{\max}$ approaches the maximum throughput 1, $\mathsf{mse}_{\mathrm{periodic}}$ blows up to infinity. This is because the queue length in periodic sampling is large at high sampling frequencies, and the samples become stale during their long waiting times in the queue. On the other hand, $\mathsf{mse}_{\mathrm{opt}}$ and $\mathsf{mse}_{\mathrm{age\text{-}opt}}$ decrease with respect to $f_{\max}$. The reason is that the set of feasible policies satisfying the constraint in (13) and (20) becomes larger as $f_{\max}$ grows, and hence the optimal values of (13) and (20) are decreasing in $f_{\max}$. Moreover, the gap between $\mathsf{mse}_{\mathrm{opt}}$ and $\mathsf{mse}_{\mathrm{age\text{-}opt}}$ is large for small values of $f_{\max}$. The ratio $\mathsf{mse}_{\mathrm{opt}}/\mathsf{mse}_{\mathrm{age\text{-}opt}}$ tends to $1/3$ as $f_{\max} \to 0$, which is in accordance with (28). As we expected, $\mathsf{mse}_{\mathrm{zero\text{-}wait}}$ is larger than $\mathsf{mse}_{\mathrm{opt}}$ and $\mathsf{mse}_{\mathrm{age\text{-}opt}}$ when $f_{\max} \geq 1$. In summary, periodic sampling is far from
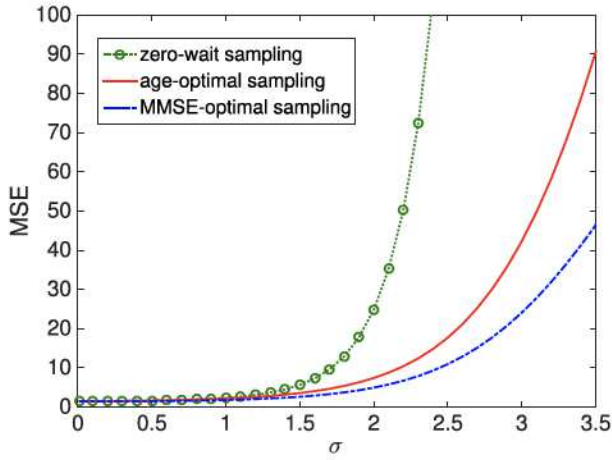
Fig. 7. MSE vs. the scale parameter σ of *i.i.d.* log-normal service time for $f_{max} = 1.5$ and $\mathbb{E}[Y_i] = 1$, where $\text{mse}_{\text{periodic}} = \infty$ due to queueing.

optimal if the sampling rate is too low or sufficiently high; age-optimal sampling is far from optimal if the sampling rate is too low.

Figure 6 and Figure 7 illustrate the MSE of *i.i.d.* log-normal service time for $f_{max} = 0.8$ and $f_{max} = 1.5$, respectively, where $Y_i = e^{\sigma X_i}/\mathbb{E}[e^{\sigma X_i}]$, $\sigma > 0$ is the scale parameter of log-normal distribution, and $(X_1, X_2, \ldots)$ are *i.i.d.* Gaussian random variables with zero mean and unit variance. Because $\mathbb{E}[Y_i] = 1$, the maximum throughput of the queue is 1. In Fig. 6, since $f_{max} < 1$, zero-wait sampling is not feasible and hence is not plotted. As the scale parameter σ grows, the tail of the log-normal distribution becomes heavier and heavier. We observe that $\text{mse}_{\text{periodic}}$ grows quickly with respect to σ, much faster than mse_{opt} and $\text{mse}_{\text{age-opt}}$. In addition, the gap between mse_{opt} and $\text{mse}_{\text{age-opt}}$ increases as σ grows. In Fig. 7, because $f_{max} > 1$, $\text{mse}_{\text{periodic}}$ is infinite and hence is not plotted. We can find that $\text{mse}_{\text{zero-wait}}$ grows quickly with respect to σ and is much larger than mse_{opt} and $\text{mse}_{\text{age-opt}}$. In summary, periodic sampling, age-optimal sampling, and zero-wait sampling policies are all far from optimal if the service times follow a heavy-tail distribution.

VII. CONCLUSION

In this paper, we have investigated optimal sampling of the Wiener process for remote estimation over a queue. The optimal sampling policy for minimizing the mean square estimation error subject to an average sampling rate constraint has been obtained in a semi-closed form. We prove that a threshold-based sampler is optimal and the optimal threshold is found exactly. Analytical and numerical comparisons with several important sampling policies, including age-optimal sampling, zero-wait sampling, and traditional periodic sampling, have been provided. The results in this paper generalize recent research on age of information by adding a signal-based control model, and generalize existing studies on remote estimation by adding a queueing model with random service times.

APPENDIX A
PROOF OF (4)

We use the calculus of variations to prove (4). Define $x \wedge y = \min\{x, y\}$. Let us consider a functional h of the estimate $\hat{W}_t$,

which is defined as

$$h(\hat{W}_t) = \mathbb{E}\left\{\int_{D_i \wedge T}^{D_{i+1} \wedge T} (\hat{W}_t - W_t)^2 dt \bigg| (S_j, W_{S_j}, D_j)_{j \leq i}\right\}$$

(56)

for any $T > 0$. By using Lemma 4 in [10], it is not hard to show that $h(\hat{W}_t)$ is a convex functional of the estimate $\hat{W}_t$. In the sequent, we will find the optimal estimate that solves

$$\min_{\hat{W}_t} h(\hat{W}_t).$$

(57)

Let f_t and g_t be two estimates, which are functions of the information available at the estimator $\{S_i, W_{S_i}, D_i : D_i \leq t\}$. Similar to the one-sided sub-gradient in finite dimensional space, the one-sided Gâteaux derivative of the functional h in the direction of g at a point f is given by

$$\delta h(f; g)$$
$$= \lim_{\epsilon \to 0^+} \frac{h(f_t + \epsilon g_t) - h(f_t)}{\epsilon}$$
$$= \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \mathbb{E}\left\{\int_{D_i \wedge T}^{D_{i+1} \wedge T} (f_t + \epsilon g_t - W_t)^2 - (f_t - W_t)^2 dt \bigg| (S_j, W_{S_j}, D_j)_{j \leq i}\right\}.$$
$$= \lim_{\epsilon \to 0^+} \mathbb{E}\left\{\int_{D_i \wedge T}^{D_{i+1} \wedge T} 2(f_t - W_t)g_t + \epsilon g_t^2 dt \bigg| (S_j, W_{S_j}, D_j)_{j \leq i}\right\}$$
$$= \mathbb{E}\left\{\int_{D_i \wedge T}^{D_{i+1} \wedge T} 2(f_t - W_t)g_t dt \bigg| (S_j, W_{S_j}, D_j)_{j \leq i}\right\}$$
$$= \mathbb{E}\left\{\int_{D_i \wedge T}^{D_{i+1} \wedge T} 2\left(f_t - \mathbb{E}\left[W_t | (S_j, W_{S_j}, D_j)_{j \leq i}\right]\right) g_t dt \bigg| (S_j, W_{S_j}, D_j)_{j \leq i}\right\},$$

(58)

where the last step follows from the iterated law of expectations. According to [61, p. 710], f_t is an optimal solution to (57) if and only if

$$\delta h(f; g) \geq 0, \quad \forall g.$$

By $\delta h(f; g) = -\delta h(f; -g)$, we get

$$\delta h(f; g) = 0, \quad \forall g.$$

(59)

Since g_t is arbitrary, by (58) and (59), the optimal solution to (57) is

$$f_t = \mathbb{E}[W_t | (S_j, W_{S_j}, D_j)_{j \leq i}],$$
$$= W_{S_i} + \mathbb{E}[W_t - W_{S_i} | (S_j, W_{S_j}, D_j)_{j \leq i}]$$
$$t \in [D_i \wedge T, D_{i+1} \wedge T).$$

(60)

Notice that under any online sampling policy π, $\{S_j, W_{S_j}, D_j, j \leq i\}$ are determined by the source $(W_t, t \in [0, S_i])$ and the service times $(Y_1, \ldots, Y_i)$. According to (i) the strong Markov property of the Wiener process [55, Theorem 2.16 and Remark 2.17] and (ii) the fact that the Y_i's are independent of the Wiener process W_t, we obtain that for any given realization

of $(S_j, W_{S_j}, D_j)_{j \le i}$, $\{W_t - W_{S_i}, t \ge S_i\}$ is a Wiener process. Hence,

$$\mathbb{E}[W_t - W_{S_i}|(S_j, W_{S_j}, D_j)_{j \le i}] = 0 \qquad (61)$$

for all $t \ge S_i$. Therefore, the optimal solution to (57) is

$$f_t = W_{S_i}, \quad \text{if } t \in [D_i \wedge T, D_{i+1} \wedge T), i = 1, 2, \dots \qquad (62)$$

Finally, we note that

$$\lim_{n \to \infty} \sum_{i=0}^{n} \mathbb{E}\left\{ \int_{D_i \wedge T}^{D_{i+1} \wedge T} (W_t - \hat{W}_t)^2 dt \right\}$$

$$= \lim_{n \to \infty} \mathbb{E}\left\{ \int_{0}^{D_n \wedge T} (W_t - \hat{W}_t)^2 dt \right\}$$

$$\overset{(a)}{=} \mathbb{E}\left\{ \lim_{n \to \infty} \int_{0}^{D_n \wedge T} (W_t - \hat{W}_t)^2 dt \right\}$$

$$\overset{(b)}{=} \mathbb{E}\left\{ \int_{0}^{T} (W_t - \hat{W}_t)^2 dt \right\} \qquad (63)$$

where in Step (a) we have used the monotonic convergence theorem, and Step (a) is due to $\lim_{n \to \infty} D_n = \infty$ almost surely, which was obtained in (7). Hence, the MMSE estimation problem can be formulated as

$$\min_{\hat{W}_t} \limsup_{T \to \infty} \frac{1}{T} \mathbb{E}\left\{ \int_{0}^{T} (W_t - \hat{W}_t)^2 dt \right\}$$

$$= \min_{\hat{W}_t} \limsup_{T \to \infty} \lim_{n \to \infty} \frac{1}{T} \sum_{i=0}^{n} \mathbb{E}\left\{ \int_{D_i \wedge T}^{D_{i+1} \wedge T} (W_t - \hat{W}_t)^2 dt \right\}. \qquad (64)$$

Recall that (62) is the solution of (56) for any $T > 0$. Let $T \to \infty$ in (62), we obtain that (4) is the MMSE estimator for solving (64). This completes the proof.

APPENDIX B
PROOF OF (18)

If π is independent of $\{W_t, t \in [0, \infty)\}$, the S_i's and D_i's are independent of $\{W_t, t \in [0, \infty)\}$. Define $x \wedge y = \min\{x, y\}$. For any $T > 0$, let us consider the term

$$\mathbb{E}\left\{ \int_{D_i \wedge T}^{D_{i+1} \wedge T} (W_t - \hat{W}_t)^2 dt \right\}$$

in the following two cases:

Case 1: If $D_i \wedge T \ge S_i$, we can obtain

$$\mathbb{E}\left\{ \int_{D_i \wedge T}^{D_{i+1} \wedge T} (W_t - \hat{W}_t)^2 dt \right\}$$

$$= \mathbb{E}\left\{ \int_{D_i \wedge T}^{D_{i+1} \wedge T} (W_t - \hat{W}_{S_i})^2 dt \right\}$$

$$\overset{(a)}{=} \mathbb{E}\left\{ \mathbb{E}\left\{ \int_{D_i \wedge T}^{D_{i+1} \wedge T} (W_t - W_{S_i})^2 dt \Big| S_i, D_i, D_{i+1} \right\} \right\}$$

$$\overset{(b)}{=} \mathbb{E}\left\{ \int_{D_i \wedge T}^{D_{i+1} \wedge T} \mathbb{E}\left\{ (W_t - W_{S_i})^2 | S_i, D_i, D_{i+1} \right\} dt \right\}$$

$$\overset{(c)}{=} \mathbb{E}\left\{ \int_{D_i \wedge T}^{D_{i+1} \wedge T} (t - S_i) dt \right\}$$

$$\overset{(d)}{=} \mathbb{E}\left\{ \int_{D_i \wedge T}^{D_{i+1} \wedge T} \Delta(t) dt \right\}, \qquad (65)$$

where Step (a) is due to the law of iterated expectations, Step (b) is due to Fubini's theorem, Step (c) is due to the strong Markov property of the Wiener process [55, Theorem 2.16] and the fact that S_i, D_i, D_{i+1} are independent of the Wiener process, and Step (d) is due to (19).

Case 2: If $D_i \wedge T < S_i$, then the fact $D_i \ge S_i$ implies that $T < S_i \le D_i \le D_{i+1}$. Hence, $D_i \wedge T = D_{i+1} \wedge T = T$ and

$$\mathbb{E}\left\{ \int_{D_i \wedge T}^{D_{i+1} \wedge T} (W_t - \hat{W}_t)^2 dt \right\} = \mathbb{E}\left\{ \int_{D_i \wedge T}^{D_{i+1} \wedge T} \Delta(t) dt \right\} = 0.$$

Therefore, (65) holds in both cases.

By using an argument similar to (63), we can obtain

$$\lim_{n \to \infty} \sum_{i=0}^{n} \mathbb{E}\left\{ \int_{D_i \wedge T}^{D_{i+1} \wedge T} \Delta(t) dt \right\} = \mathbb{E}\left\{ \int_{0}^{T} \Delta(t) dt \right\}. \qquad (66)$$

Combining (63)-(66), (18) is proven.

APPENDIX C
PROOFS OF (25) AND (27)

If $f_{\max} \to 0$, (15) tells us that

$$\mathbb{E}[\max(\beta, W_Y^2)] = \frac{1}{f_{\max}}, \qquad (67)$$

which implies

$$\beta \le \frac{1}{f_{\max}} \le \beta + \mathbb{E}[W_Y^2] = \beta + \mathbb{E}[Y]. \qquad (68)$$

Hence,

$$\frac{1}{f_{\max}} - \mathbb{E}[Y] \le \beta \le \frac{1}{f_{\max}}. \qquad (69)$$

If $f_{\max} \to 0$, (25) follows.

Because Y is independent of the Wiener process, using the law of iterated expectations and the Gaussian distribution of the Wiener process, we can obtain $\mathbb{E}[W_Y^4] = 3\mathbb{E}[Y^2]$ and $\mathbb{E}[W_Y^2] = 3\mathbb{E}[Y]$. Hence,

$$\beta \le \mathbb{E}[\max(\beta, W_Y^2)] \le \beta + \mathbb{E}[W_Y^2] = \beta + \mathbb{E}[Y],$$

$$\beta^2 \le \mathbb{E}[\max(\beta^2, W_Y^4)] \le \beta^2 + \mathbb{E}[W_Y^4] = \beta^2 + 3\mathbb{E}[Y^2].$$

Therefore,

$$\frac{\beta^2}{\beta + \mathbb{E}[Y]} \le \frac{\mathbb{E}[\max(\beta^2, W_Y^4)]}{\mathbb{E}[\max(\beta, W_Y^2)]} \le \frac{\beta^2 + 3\mathbb{E}[Y^2]}{\beta}. \qquad (70)$$

By combining (16), (25), and (70), (27) follows in the case of $f_{\max} \to 0$.

If $\alpha \to 0$, then $Y \to 0$ and $W_Y \to 0$ with probability one. Hence, $\mathbb{E}[\max(\beta, W_Y^2)] \to \beta$ and $\mathbb{E}[\max(\beta^2, W_Y^4)] \to \beta^2$. Substituting these into (15) and (70), yields

$$\lim_{\alpha \to 0} \beta = \frac{1}{f_{\max}}, \quad \lim_{\alpha \to 0} \left\{ \frac{\mathbb{E}[\max(\beta^2, W_Y^4)]}{6\mathbb{E}[\max(\beta, W_Y^2)]} + \mathbb{E}[Y] \right\} = \frac{1}{6 f_{\max}}.$$

By this, (25) and (27) are proven in the case of $\alpha \to 0$. This completes the proof.

APPENDIX D
PROOF OF (29)

If $f_{max} \to \infty$, the sampling rate constraint in (13) can be removed. By (15), the optimal β is determined by (29).

If $\alpha \to \infty$, let us consider the equation

$$\mathbb{E}[\max(\beta, W_Y^2)] = \frac{\mathbb{E}[\max(\beta^2, W_Y^4)]}{2\beta}. \tag{71}$$

If Y grows by α times, then β and $\mathbb{E}[\max(\beta, W_Y^2)]$ in (71) both should grow by α times, and $\mathbb{E}[\max(\beta^2, W_Y^4)]$ in (71) should grow by α^2 times. Hence, if $\alpha \to \infty$, it holds in (15) that

$$\frac{1}{f_{max}} \leq \frac{\mathbb{E}[\max(\beta^2, W_Y^4)]}{2\beta} \tag{72}$$

and the solution to (15) is given by (29). This completes the proof.

APPENDIX E
PROOFS OF THEOREMS 3 AND 4

Proof: [Proof of Theorem 3] The zero-wait policy can be expressed as (12) with $\beta = 0$. Because Y is independent of the Wiener process, using the law of iterated expectations and the Gaussian distribution of the Wiener process, we can obtain $\mathbb{E}[W_Y^4] = 3\mathbb{E}[Y^2]$. According to (29), $\beta = 0$ if and only if $\mathbb{E}[W_Y^4] = 3\mathbb{E}[Y^2] = 0$ which is equivalent to $Y = 0$ with probability one. This completes the proof. □

Proof: [Proof of Theorem 4] In the one direction, the zero-wait policy can be expressed as (11) with $\beta \leq \text{ess inf } Y$. If the zero-wait policy is optimal, then the solution to (30) must satisfy $\beta \leq \text{ess inf } Y$, which further implies $\beta \leq Y$ with probability one. From this, we can get

$$2\text{ess inf } Y\mathbb{E}[Y] \geq 2\beta\mathbb{E}[Y] = \mathbb{E}[Y^2], \tag{73}$$

By this, (31) follows.

In the other direction, if (31) holds, we will show that the zero-wait policy is age-optimal by considering the following two cases.

Case 1: $\mathbb{E}[Y] > 0$. By choosing

$$\beta = \frac{\mathbb{E}[Y^2]}{2\mathbb{E}[Y]}, \tag{74}$$

we can get $\beta \leq \text{ess inf } Y$ from (31) and hence

$$\beta \leq Y \tag{75}$$

with probability one. According to (74) and (75), such a β is the solution to (30). Hence, the zero-wait policy expressed by (11) with $\beta \leq \text{ess inf } Y$ is the age-optimal policy.

Case 2: $\mathbb{E}[Y] = 0$ and hence $Y = 0$ with probability one. In this case, $\beta = 0$ is the solution to (30). Hence, the zero-wait policy expressed by (11) with $\beta = 0$ is the age-optimal policy.

Combining these two cases, the proof is completed. □

APPENDIX F
PROOF OF LEMMA 1

Recall that S_i and G_i are the sampling time and the service starting time of sample i, respectively. Suppose that in the sampling policy π, sample i is generated when the server is busy sending another sample, and hence sample i needs to wait for some time before being submitted to the server, i.e., $S_i < G_i$. Let us consider a *virtual* sampling policy $\pi' = \{S_0, \ldots, S_{i-1}, G_i, S_{i+1}, \ldots\}$ such that the generation time of sample i is postponed from S_i to G_i. We call policy π' a virtual policy because it may happen that $G_i > S_{i+1}$. However, this will not affect our proof below. We will show that the MSE of the sampling policy π' is smaller than that of the sampling policy $\pi = \{S_0, \ldots, S_{i-1}, S_i, S_{i+1}, \ldots\}$.

Note that the Wiener process $\{W_t : t \in [0, \infty)\}$ does not change according to the sampling policy, and the sample delivery times $\{D_0, D_1, D_2, \ldots\}$ remain the same in policy π and policy π'. Hence, the only difference between policies π and π' is that *the generation time of sample i is postponed from S_i to G_i*. The MMSE estimator under policy π is given by (4) and the MMSE estimator under policy π' is given by

$$\hat{W}_t = \mathbb{E}[W_t | (S_j, W_{S_j}, D_j)_{j \leq i-1}, (G_i, W_{G_i}, D_i)]$$
$$= \begin{cases} 0, & t \in [0, D_1); \\ W_{G_i}, & t \in [D_i, D_{i+1}); \\ W_{S_j}, & t \in [D_j, D_{j+1}), \ j \neq i, j \geq 1. \end{cases} \tag{76}$$

Next, we consider a third virtual sampling policy π'' in which the samples (W_{G_i}, G_i) and (W_{S_i}, S_i) are both delivered to the estimator at time D_i. Clearly, the estimator under policy π'' has more information than those under policies π and π'. By following the arguments in Appendix A, one can show that the MMSE estimator under policy π'' is

$$\hat{W}_t = \mathbb{E}[W_t | (S_j, W_{S_j}, D_j)_{j \leq i}, (G_i, W_{G_i}, D_i)]$$
$$= \begin{cases} 0, & t \in [0, D_1); \\ W_{G_i}, & t \in [D_i, D_{i+1}); \\ W_{S_j}, & t \in [D_j, D_{j+1}), \ j \neq i, j \geq 1. \end{cases} \tag{77}$$

Notice that, because of the strong Markov property of Wiener process, the estimator under policy π'' uses the fresher sample W_{G_i}, instead of the stale sample W_{S_i}, to construct $\hat{W}_t$ during $[D_i, D_{i+1})$. Because the estimator under policy π'' has more information than that under policy π, one can imagine that policy π'' has a smaller estimation error than policy π, i.e., for any $T > 0$

$$\mathbb{E}\left\{ \int_{D_i \wedge T}^{D_{i+1} \wedge T} (W_t - W_{S_i})^2 dt \right\}$$
$$\geq \mathbb{E}\left\{ \int_{D_i \wedge T}^{D_{i+1} \wedge T} (W_t - W_{G_i})^2 dt \right\}. \tag{78}$$

To prove (78), we invoke the orthogonality principle of the MMSE estimator [41, Prop. V.C.2] under policy π'' and obtain

$$\mathbb{E}\left\{ \int_{D_i \wedge T}^{D_{i+1} \wedge T} 2(W_t - W_{G_i})(W_{G_i} - W_{S_i}) dt \right\} = 0, \tag{79}$$

where we have used the fact that W_{G_i} and W_{S_i} are available by the MMSE estimator under policy π''. Next, from (79),

we can get

$$\mathbb{E}\left\{\int_{D_i \wedge T}^{D_{i+1} \wedge T} (W_t - W_{S_i})^2 dt\right\}$$

$$=\mathbb{E}\left\{\int_{D_i \wedge T}^{D_{i+1} \wedge T} (W_t - W_{G_i})^2 + (W_{G_i} - W_{S_i})^2 \ dt\right\}$$

$$+\mathbb{E}\left\{\int_{D_i \wedge T}^{D_{i+1} \wedge T} 2(W_t - W_{G_i})(W_{G_i} - W_{S_i}) dt\right\}$$

$$=\mathbb{E}\left\{\int_{D_i \wedge T}^{D_{i+1} \wedge T} (W_t - W_{G_i})^2 + (W_{G_i} - W_{S_i})^2 \ dt\right\}$$

$$\geq\mathbb{E}\left\{\int_{D_i \wedge T}^{D_{i+1} \wedge T} (W_t - W_{G_i})^2 dt\right\}. \tag{80}$$

In other words, the estimation error of policy π'' is no greater than that of policy π. Furthermore, by comparing (76) and (77), we can see that the MMSE estimators under policies π'' and π' are exactly the same. Therefore, the estimation error of policy π' is no greater than that of policy π.

By repeating the above arguments for all samples i satisfying $S_i < G_i$, one can show that the sampling policy $\pi_1 = \{S_0, G_1, \ldots, G_{i-1}, G_i, G_{i+1}, \ldots\}$ is better than the sampling policy $\pi = \{S_0, S_1, \ldots, S_{i-1}, S_i, S_{i+1}, \ldots\}$. This completes the proof.

APPENDIX G
PROOF OF LEMMA 2

Part (a) is proven in two steps:

Step 1: We will prove that $\mathsf{mse}_{\mathrm{opt}} \leq c$ if and only if $p(c) \leq 0$.

If $\mathsf{mse}_{\mathrm{opt}} \leq c$, then there exists a policy $\pi = (Z_0, Z_1, \ldots) \in \Pi_1$ that is feasible for both (35) and (37), which satisfies

$$\lim_{n \to \infty} \frac{\sum_{i=0}^{n-1} \mathbb{E}\left[\int_{D_i}^{D_{i+1}} (W_t - W_{S_i})^2 dt\right]}{\sum_{i=0}^{n-1} \mathbb{E}[Y_i + Z_i]} \leq c. \tag{81}$$

Hence,

$$\lim_{n \to \infty} \frac{\frac{1}{n}\sum_{i=0}^{n-1} \mathbb{E}\left[\int_{D_i}^{D_{i+1}} (W_t - W_{S_i})^2 dt - c(Y_i + Z_i)\right]}{\frac{1}{n}\sum_{i=0}^{n-1} \mathbb{E}[Y_i + Z_i]} \leq 0. \tag{82}$$

Because the inter-sampling times $T_i = Y_i + Z_i$ are regenerative, $\mathbb{E}[k_{j+1} - k_j] < \infty$ and $0 < \mathbb{E}[(S_{k_{j+1}} - S_{k_j})^2] < \infty$ for all j, the renewal theory [51] tells us that the limit $\lim_{n \to \infty} \frac{1}{n}\sum_{i=0}^{n-1} \mathbb{E}[Y_i + Z_i]$ exists and is positive. By this, we get

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{E}\left[\int_{D_i}^{D_{i+1}} (W_t - W_{S_i})^2 dt - c(Y_i + Z_i)\right] \leq 0. \tag{83}$$

Therefore, $p(c) \leq 0$.

On the reverse direction, if $p(c) \leq 0$, then there exists a policy $\pi = (Z_0, Z_1, \ldots) \in \Pi_1$ that is feasible for both (35) and (37), which satisfies (83). Because the limit $\lim_{n \to \infty} \frac{1}{n}\sum_{i=0}^{n-1} \mathbb{E}[Y_i + Z_i]$ exists and is positive, from (83),

we can derive (82) and (81). Hence, $\mathsf{mse}_{\mathrm{opt}} \leq c$. By this, we have proven that $\mathsf{mse}_{\mathrm{opt}} \leq c$ if and only if $p(c) \leq 0$.

Step 2: We needs to prove that $\mathsf{mse}_{\mathrm{opt}} < c$ if and only if $p(c) < 0$. This statement can be proven by using the arguments in *Step 1*, in which "$\leq$" should be replaced by "$<$". Finally, from the statement of *Step 1*, it immediately follows that $\mathsf{mse}_{\mathrm{opt}} > c$ if and only if $p(c) > 0$. This completes the proof of part (a).

Part (b): We first show that each optimal solution to (35) is an optimal solution to (37). By the claim of part (a), $p(c) = 0$ is equivalent to $\mathsf{mse}_{\mathrm{opt}} = c$. Suppose that policy $\pi = (Z_0, Z_1, \ldots) \in \Pi_1$ is an optimal solution to (35). Then, $\mathsf{mse}_{\pi} = \mathsf{mse}_{\mathrm{opt}} = c$. Applying this in the arguments of (81)-(83), we can show that policy π satisfies

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{E}\left[\int_{D_i}^{D_{i+1}} (W_t - W_{S_i})^2 dt - c(Y_i + Z_i)\right] = 0.$$

This and $p(c) = 0$ imply that policy π is an optimal solution to (37).

Similarly, we can prove that each optimal solution to (37) is an optimal solution to (35). By this, part (b) is proven.

APPENDIX H
PROOF OF LEMMA 3

According to Theorem 2.51 of [55], $W_t^4 - 6\int_0^t W_s^2 \, ds$ is an martingale of the Wiener process $\{W_t, t \in [0, \infty)\}$. Because the minimum of two stopping times is a stopping time and constant times are stopping times [56], it follows that $t \wedge \tau$ is a bounded stopping time for every $t \in [0, \infty)$, where $x \wedge y = \min\{x, y\}$. Then, it follows from Theorem 8.5.1 of [56] that for every $t \in [0, \infty)$

$$\mathbb{E}\left[\int_0^{t \wedge \tau} W_s^2 \, ds\right] = \frac{1}{6}\mathbb{E}\left[W_{t \wedge \tau}^4\right]. \tag{84}$$

Notice that $\int_0^{t \wedge \tau} W_s^2 \, ds$ is positive and increasing with respect to t. By applying the monotone convergence theorem [56, Theorem 1.5.5], we can obtain

$$\lim_{t \to \infty} \mathbb{E}\left[\int_0^{t \wedge \tau} W_s^2 \, ds\right] = \mathbb{E}\left[\int_0^{\tau} W_s^2 ds\right]. \tag{85}$$

Hence, the limit $\lim_{t \to \infty} \mathbb{E}\left[W_{t \wedge \tau}^4\right]$ exists. The remaining task is to show that

$$\lim_{t \to \infty} \mathbb{E}\left[W_{t \wedge \tau}^4\right] = \mathbb{E}\left[W_{\tau}^4\right]. \tag{86}$$

Towards this goal, let us consider

$$\mathbb{E}\left[W_{\tau}^4\right]$$

$$=\mathbb{E}\left\{[W_{t \wedge \tau} - (W_{\tau} - W_{t \wedge \tau})]^4\right\}$$

$$=\mathbb{E}\left[W_{t \wedge \tau}^4\right] + 4\mathbb{E}\left[W_{t \wedge \tau}^3 (W_{\tau} - W_{t \wedge \tau})\right]$$

$$+ 6\mathbb{E}\left[W_{t \wedge \tau}^2 (W_{\tau} - W_{t \wedge \tau})^2\right]$$

$$+ 4\mathbb{E}\left[W_{t \wedge \tau}(W_{\tau} - W_{t \wedge \tau})^3\right] + \mathbb{E}\left[(W_{\tau} - W_{t \wedge \tau})^4\right]$$

$$=\mathbb{E}\left[W_{t \wedge \tau}^4\right] + 4\mathbb{E}\left[W_{t \wedge \tau}^3\right]\mathbb{E}[W_{\tau} - W_{t \wedge \tau}]$$

$$+ 6\mathbb{E}\left[W_{t\wedge\tau}^2\right]\mathbb{E}\left[(W_\tau - W_{t\wedge\tau})^2\right]$$
$$+ 4\mathbb{E}\left[W_{t\wedge\tau}\right]\mathbb{E}\left[(W_\tau - W_{t\wedge\tau})^3\right] + \mathbb{E}\left[(W_\tau - W_{t\wedge\tau})^4\right],$$

where in the last step we have used the strong Markov property of the Wiener process [55, Theorem 2.16]. By Wald's lemma for Wiener process [55, Theorem 2.44 and Theorem 2.48], $\mathbb{E}[W_\tau] = 0$ and $\mathbb{E}[W_\tau^2] = \mathbb{E}[\tau]$ for any stopping time τ with $\mathbb{E}[\tau] < \infty$. Hence,

$$\mathbb{E}[W_\tau - W_{t\wedge\tau}] = 0, \tag{87}$$
$$\mathbb{E}[W_{t\wedge\tau}] = 0, \tag{88}$$

which implies

$$\mathbb{E}\left[W_\tau^4\right] = \mathbb{E}\left[W_{t\wedge\tau}^4\right] + 6\mathbb{E}\left[W_{t\wedge\tau}^2\right]\mathbb{E}\left[(W_\tau - W_{t\wedge\tau})^2\right]$$
$$+ \mathbb{E}\left[(W_\tau - W_{t\wedge\tau})^4\right]$$
$$\geq \mathbb{E}\left[W_{t\wedge\tau}^4\right], \tag{89}$$

and hence

$$\mathbb{E}\left[W_\tau^4\right] \geq \lim_{t\to\infty}\mathbb{E}\left[W_{t\wedge\tau}^4\right]. \tag{90}$$

On the other hand, by Fatou's lemma [56, Theorem 1.5.4],

$$\mathbb{E}\left[W_\tau^4\right] = \mathbb{E}\left[\liminf_{t\to\infty} W_{t\wedge\tau}^4\right] \leq \lim_{t\to\infty}\mathbb{E}\left[W_{t\wedge\tau}^4\right]. \tag{91}$$

Combining (90) and (91), yields (86). This completes the proof.

APPENDIX I
PROOF OF (43)

The following lemma is needed in the proof of (43):

Lemma 6. *For any $\lambda \geq 0$, there exists an optimal solution $(Z_0, Z_1, \ldots)$ to (40) in which Z_i is independent of $(W_t, t \in [0, S_i])$ for all $i = 1, 2, \ldots$*

Proof: Because the Y_i's are *i.i.d.*, Z_i is independent of $Y_{i+1}, Y_{i+2}, \ldots$, and the strong Markov property of the Wiener process [55, Theorem 2.16], in the Lagrangian $L(\pi; \lambda)$ the term related to Z_i is

$$\mathbb{E}\left[\int_{S_i+Y_i}^{S_i+Y_i+Z_i+Y_{i+1}} (W_t - W_{S_i})^2 dt - (\text{mse}_{\text{opt}} + \lambda)(Y_i + Z_i)\right], \tag{92}$$

which is determined by the control decision Z_i and the recent information of the system $\mathcal{I}_i = (Y_i, (W_{S_i+t} - W_{S_i}, t \geq 0))$. According to [47, p. 252] and [62, Chapter 6], $\mathcal{I}_i$ is a *sufficient statistics* for determining Z_i in (40). Therefore, there exists an optimal policy $(Z_0, Z_1, \ldots)$ in which Z_i is determined based on only $\mathcal{I}_i$, which is independent of $(W_t : t \in [0, S_i])$. This completes the proof. $\square$

Proof: [Proof of (43)] By using (34) and Lemma 6, we obtain that for given Y_i and Y_{i+1}, Y_i and $Y_i + Z_i + Y_{i+1}$ are stopping times of the time-shifted Wiener process $\{W_{S_i+t} - W_{S_i}, t \geq 0\}$. Hence,

$$= \mathbb{E}\left\{\int_{D_i}^{D_{i+1}} (W_t - W_{S_i})^2 dt\right\}$$

$$= \mathbb{E}\left\{\int_{Y_i}^{Y_i+Z_i+Y_{i+1}} (W_{S_i+t} - W_{S_i})^2 dt\right\}$$
$$\overset{(a)}{=} \mathbb{E}\left\{\mathbb{E}\left\{\int_{Y_i}^{Y_i+Z_i+Y_{i+1}} (W_{S_i+t} - W_{S_i})^2 dt \Big| Y_i, Y_{i+1}\right\}\right\}$$
$$\overset{(b)}{=} \frac{1}{6}\mathbb{E}\left\{\mathbb{E}\left\{(W_{S_i+Y_i+Z_i+Y_{i+1}} - W_{S_i})^4 \Big| Y_i, Y_{i+1}\right\}\right\}$$
$$- \frac{1}{6}\mathbb{E}\left\{\mathbb{E}\left\{(W_{S_i+Y_i} - W_{S_i})^4 \Big| Y_i, Y_{i+1}\right\}\right\}$$
$$\overset{(c)}{=} \frac{1}{6}\mathbb{E}\left[(W_{S_i+Y_i+Z_i+Y_{i+1}} - W_{S_i})^4\right] - \frac{1}{6}\mathbb{E}\left[(W_{S_i+Y_i} - W_{S_i})^4\right], \tag{93}$$

where Step (a) and Step (c) are due to the law of iterated expectations, and Step (b) is due to Lemma 3. Because $S_{i+1} = S_i + Y_i + Z_i$, we have

$$\mathbb{E}\left[(W_{S_i+Y_i+Z_i+Y_{i+1}} - W_{S_i})^4\right]$$
$$= \mathbb{E}\left\{[(W_{S_i+Y_i+Z_i} - W_{S_i}) + (W_{S_{i+1}+Y_{i+1}} - W_{S_{i+1}})]^4\right\}$$
$$= \mathbb{E}\left[(W_{S_i+Y_i+Z_i} - W_{S_i})^4\right]$$
$$+ 4\mathbb{E}\left[(W_{S_i+Y_i+Z_i} - W_{S_i})^3(W_{S_{i+1}+Y_{i+1}} - W_{S_{i+1}})\right]$$
$$+ 6\mathbb{E}\left[(W_{S_i+Y_i+Z_i} - W_{S_i})^2(W_{S_{i+1}+Y_{i+1}} - W_{S_{i+1}})^2\right]$$
$$+ 4\mathbb{E}\left[(W_{S_i+Y_i+Z_i} - W_{S_i})(W_{S_{i+1}+Y_{i+1}} - W_{S_{i+1}})^3\right]$$
$$+ \mathbb{E}\left[(W_{S_{i+1}+Y_{i+1}} - W_{S_{i+1}})^4\right]$$
$$= \mathbb{E}\left[(W_{S_i+Y_i+Z_i} - W_{S_i})^4\right]$$
$$+ 4\mathbb{E}\left[(W_{S_i+Y_i+Z_i} - W_{S_i})^3\right]\mathbb{E}\left[(W_{S_{i+1}+Y_{i+1}} - W_{S_{i+1}})\right]$$
$$+ 6\mathbb{E}\left[(W_{S_i+Y_i+Z_i} - W_{S_i})^2\right]\mathbb{E}\left[(W_{S_{i+1}+Y_{i+1}} - W_{S_{i+1}})^2\right]$$
$$+ 4\mathbb{E}\left[(W_{S_i+Y_i+Z_i} - W_{S_i})\right]\mathbb{E}\left[(W_{S_{i+1}+Y_{i+1}} - W_{S_{i+1}})^3\right]$$
$$+ \mathbb{E}\left[(W_{S_{i+1}+Y_{i+1}} - W_{S_{i+1}})^4\right], \tag{94}$$

where in the last equation we have used the fact that Y_{i+1} is independent of Y_i and Z_i, and the strong Markov property of the Wiener process [55, Theorem 2.16]. By Wald's lemma for Wiener process [55, Theorem 2.44 and Theorem 2.48], $\mathbb{E}[W_\tau] = 0$ and $\mathbb{E}[W_\tau^2] = \mathbb{E}[\tau]$ for any stopping time τ with $\mathbb{E}[\tau] < \infty$. Hence,

$$\mathbb{E}\left[(W_{S_{i+1}+Y_{i+1}} - W_{S_{i+1}})\right] = 0, \tag{95}$$
$$\mathbb{E}\left[(W_{S_i+Y_i+Z_i} - W_{S_i})\right] = 0, \tag{96}$$
$$\mathbb{E}\left[(W_{S_i+Y_i+Z_i} - W_{S_i})^2\right]\mathbb{E}\left[(W_{S_{i+1}+Y_{i+1}} - W_{S_{i+1}})^2\right]$$
$$= \mathbb{E}[Y_i + Z_i]\mathbb{E}[Y_{i+1}]. \tag{97}$$

Therefore, we have

$$\mathbb{E}\left[(W_{S_i+Y_i+Z_i+Y_{i+1}} - W_{S_i})^4\right]$$
$$= \mathbb{E}\left[(W_{S_i+Y_i+Z_i} - W_{S_i})^4\right] + 6\mathbb{E}[Y_i + Z_i]\mathbb{E}[Y_{i+1}]$$
$$+ \mathbb{E}\left[(W_{S_{i+1}+Y_{i+1}} - W_{S_{i+1}})^4\right]. \tag{98}$$

Finally, because $(W_{S_i+t} - W_{S_i})$ and $(W_{S_{i+1}+t} - W_{S_{i+1}})$ are both Wiener processes, and the Y_i's are *i.i.d.*,

$$\mathbb{E}\left[(W_{S_i+Y_i} - W_{S_i})^4\right] = \mathbb{E}\left[(W_{S_{i+1}+Y_{i+1}} - W_{S_{i+1}})^4\right]. \quad (99)$$

Combining (93)-(99), yields (43). $\quad\square$

APPENDIX J
PROOF OF THEOREM 5

By (43), (92) can be rewritten as

$$\mathbb{E}\left[\int_{S_i+Y_i}^{S_i+Y_i+Z_i+Y_{i+1}} (W_t - W_{S_i})^2 dt - (\text{mse}_{\text{opt}} + \lambda)(Y_i + Z_i)\right]$$
$$= \mathbb{E}\left[\frac{1}{6}(W_{S_i+Y_i+Z_i} - W_{S_i})^4 - (\text{mse}_{\text{opt}} + \lambda - \mathbb{E}[Y])(Y_i + Z_i)\right]$$
$$= \mathbb{E}\left[\frac{1}{6}(W_{S_i+Y_i+Z_i} - W_{S_i})^4 - \frac{\beta}{3}(Y_i + Z_i)\right]$$
$$= \mathbb{E}\left[\frac{1}{6}[(W_{S_i+Y_i} - W_{S_i}) + (W_{S_i+Y_i+Z_i} - W_{S_i+Y_i})]^4\right.$$
$$\left. - \frac{\beta}{3}(Y_i + Z_i)\right]. \quad (100)$$

Because the Y_i's are *i.i.d.* and the strong Markov property of the Wiener process [55, Theorem 2.16], the expectation in (100) is determined by the control decision Z_i and the information $\mathcal{I}'_i = (W_{S_i+Y_i} - W_{S_i}, Y_i, (W_{S_i+Y_i+t} - W_{S_i+Y_i}, t \geq 0))$. According to [47, p. 252] and [62, Chapter 6], $\mathcal{I}'_i$ is a *sufficient statistics* for determining the waiting time Z_i in (40). Therefore, there exists an optimal policy $(Z_0, Z_1, \ldots)$ in which Z_i is determined based on only $\mathcal{I}'_i$. By this, (40) is decomposed into a sequence of per-sample control problems (44). Combining (35), (43), and Lemma 2, yields $\text{mse}_{\text{opt}} \geq \mathbb{E}[Y]$. Hence, $\beta \geq 0$.

We note that, because the Y_i's are *i.i.d.* and the strong Markov property of the Wiener process, the Z_i's in this optimal policy are *i.i.d.* Similarly, the $(W_{S_i+Y_i+Z_i} - W_{S_i})$'s in this optimal policy are *i.i.d.*

APPENDIX K
PROOF OF LEMMA 4

Case 1: If $b^2 \geq \beta$, then (49) tells us that

$$\tau^* = 0 \quad (101)$$

and

$$u(x) = \mathbb{E}[g(X_0)|X_0 = x] = g(x) = \beta s - \frac{1}{2}b^4. \quad (102)$$

Case 2: If $b^2 < \beta$, then $\tau^* > 0$ and $(b + W_{\tau^*})^2 = \beta$. Invoking Theorem 8.5.5 in [56], yields

$$\mathbb{E}_x \tau^* = -(\sqrt{\beta} - b)(-\sqrt{\beta} - b) = \beta - b^2. \quad (103)$$

Using this, we can obtain

$$u(x) = \mathbb{E}_x g(X_{\tau^*})$$
$$= \beta(s + \mathbb{E}_x \tau^*) - \frac{1}{2}\mathbb{E}_x\left[(b + W_{\tau^*})^4\right]$$
$$= \beta(s + \beta - b^2) - \frac{1}{2}\beta^2$$

$$= \beta s + \frac{1}{2}\beta^2 - b^2\beta. \quad (104)$$

Hence, in Case 2,

$$u(x) - g(x) = \frac{1}{2}\beta^2 - b^2\beta + \frac{1}{2}b^4 = \frac{1}{2}(b^2 - \beta)^2 \geq 0. \quad (105)$$

By combining these two cases, Lemma 4 is proven.

APPENDIX L
PROOF OF LEMMA 5

The function $u(s, b)$ is continuous differentiable in (s, b). In addition, $\frac{\partial^2}{\partial^2 b}u(s, b)$ is continuous everywhere but at $b = \pm\sqrt{\beta}$. By the Itô-Tanaka-Meyer formula [55, Theorem 7.14 and Corollary 7.35], we obtain that almost surely

$$u(s + t, b + W_t) - u(s, b)$$
$$= \int_0^t \frac{\partial}{\partial b}u(s + r, b + W_r)dW_r$$
$$+ \int_0^t \frac{\partial}{\partial s}u(s + r, b + W_r)dr$$
$$+ \frac{1}{2}\int_{-\infty}^{\infty} L^a(t)\frac{\partial^2}{\partial b^2}u(s + r, b + a)da, \quad (106)$$

where $L^a(t)$ is the local time that the Wiener process spends at the level a, i.e.,

$$L^a(t) = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon}\int_0^t 1_{\{|W_s-a|\leq\epsilon\}}ds, \quad (107)$$

and 1_A is the indicator function of event A. By the property of local times of the Wiener process [55, Theorem 6.18], we obtain that almost surely

$$u(s + t, b + W_t) - u(s, b)$$
$$= \int_0^t \frac{\partial}{\partial b}u(s + r, b + W_r)dW_r$$
$$+ \int_0^t \frac{\partial}{\partial s}u(s + r, b + W_r)dr$$
$$+ \frac{1}{2}\int_0^t \frac{\partial^2}{\partial b^2}u(s + r, b + W_r)dr. \quad (108)$$

Because

$$\frac{\partial}{\partial b}u(s, b) = \begin{cases} -2b^3, & \text{if } b^2 \geq \beta; \\ -2\beta b, & \text{if } b^2 < \beta, \end{cases} \quad (109)$$

we can obtain that for all $t \geq 0$ and all $x = (s, b) \in \mathbb{R}^2$

$$\mathbb{E}_x\left\{\int_0^t \left[\frac{\partial}{\partial b}u(s + r, b + W_r)\right]^2 dr\right\} < \infty. \quad (110)$$

This and Theorem 7.11 of [55] imply that $\int_0^t \frac{\partial}{\partial b}u(s + r, b + W_r)dW_r$ is a martingale and

$$\mathbb{E}_x\left[\int_0^t \frac{\partial}{\partial b}u(s + r, b + W_r)dW_r\right] = 0, \ \forall t \geq 0. \quad (111)$$

By combining (46), (108), and (111), we get

$$\mathbb{E}_x[u(X_t)] - u(x) = \mathbb{E}_x\left\{\int_0^t \left[\frac{\partial}{\partial s}u(X_r) + \frac{1}{2}\frac{\partial^2}{\partial b^2}u(X_r)\right]dr\right\}. \quad (112)$$

It is easy to compute that if $b^2 > \beta$,

$$\frac{\partial}{\partial s}u(s,b) + \frac{1}{2}\frac{\partial^2}{\partial b^2}u(s,b) = \beta - 3b^2 \leq 0; \quad (113)$$

and if $b^2 < \beta$,

$$\frac{\partial}{\partial s}u(s,b) + \frac{1}{2}\frac{\partial^2}{\partial b^2}u(s,b) = \beta - \beta = 0. \quad (114)$$

Hence,

$$\frac{\partial}{\partial s}u(s,b) + \frac{1}{2}\frac{\partial^2}{\partial b^2}u(s,b) \leq 0 \quad (115)$$

for all $(s,b) \in \mathbb{R}^2$ except for $b = \pm\sqrt{\beta}$. Since the Lebesgue measure of those r for which $b + W_r = \pm\sqrt{\beta}$ is zero, we get from (112) and (115) that $\mathbb{E}_x[u(X_t)] \leq u(x)$ for all $x \in \mathbb{R}^2$ and $t \geq 0$. This completes the proof.

APPENDIX M
PROOF OF COROLLARY 1

Because (49) is the optimal solution to (47), by choosing $s = Y_i$, $b = W_{S_i+Y_i} - W_{S_i}$, and using $W_{S_i+Y_i+t} - W_{S_i}$ to replace W_t, it is immediate that (53) is the optimal solution to (44).

The remaining task is to prove (54) and (55). According to (53) with $\beta \geq 0$, we have

$$W_{S_i+Y_i+Z_i} - W_{S_i}$$
$$= \begin{cases} W_{S_i+Y_i} - W_{S_i}, & \text{if } |W_{S_i+Y_i} - W_{S_i}| \geq \sqrt{\beta}; \\ \sqrt{\beta}, & \text{if } |W_{S_i+Y_i} - W_{S_i}| < \sqrt{\beta}. \end{cases} \quad (116)$$

Hence,

$$\mathbb{E}[(W_{S_i+Y_i+Z_i} - W_{S_i})^4] = \mathbb{E}[\max(\beta^2, (W_{S_i+Y_i} - W_{S_i})^4)]. \quad (117)$$

In addition, from (101) and (103) we know that if $|W_{S_i+Y_i} - W_{S_i}| \geq \sqrt{\beta}$, then (101) implies

$$\mathbb{E}[Z_i|Y_i] = 0; \quad (118)$$

otherwise, if $|W_{S_i+Y_i} - W_{S_i}| < \sqrt{\beta}$, then (103) implies

$$\mathbb{E}[Z_i|Y_i] = \beta - (W_{S_i+Y_i} - W_{S_i})^2. \quad (119)$$

By combining these two cases, we get

$$\mathbb{E}[Z_i|Y_i] = \max[\beta - (W_{S_i+Y_i} - W_{S_i})^2, 0]. \quad (120)$$

Using the law of iterated expectations, the strong Markov property of the Wiener process, and Wald's identity $\mathbb{E}[(W_{S_i+Y_i} - W_{S_i})^2] = \mathbb{E}[Y_i]$, yields

$$\mathbb{E}[Z_i + Y_i]$$
$$= \mathbb{E}[\mathbb{E}[Z_i|Y_i] + Y_i]$$
$$= \mathbb{E}[\max(\beta - (W_{S_i+Y_i} - W_{S_i})^2, 0) + Y_i]$$
$$= \mathbb{E}[\max(\beta - (W_{S_i+Y_i} - W_{S_i})^2, 0) + (W_{S_i+Y_i} - W_{S_i})^2]$$
$$= \mathbb{E}[\max(\beta, (W_{S_i+Y_i} - W_{S_i})^2)]. \quad (121)$$

Finally, because W_t and $W_{S_i+t} - W_{S_i}$ are of the same distribution, (54) and (55) follow from (121) and (117), respectively. This completes the proof.

APPENDIX N
PROOF OF THEOREM 7

According to [53, Prop. 6.2.5], if we can find $\pi^\star = (Z_0, Z_1, \ldots)$ and $\lambda^\star$ satisfying the following conditions:

$$\pi^\star \in \Pi_1, \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mathbb{E}[Y_i + Z_i] - \frac{1}{f_{\max}} \geq 0, \quad (122)$$

$$\lambda^\star \geq 0, \quad (123)$$

$$L(\pi^\star; \lambda^\star) = \inf_{\pi \in \Pi_1} L(\pi; \lambda^\star), \quad (124)$$

$$\lambda^\star \left\{\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mathbb{E}[Y_i + Z_i] - \frac{1}{f_{\max}}\right\} = 0, \quad (125)$$

then $\pi^\star$ is an optimal solution to the primal problem (37) and $\lambda^\star$ is a geometric multiplier [53] for the primal problem (37). Further, if we can find such $\pi^\star$ and $\lambda^\star$, then the duality gap between (37) and (41) must be zero, because otherwise there is no geometric multiplier [53, Prop. 6.2.3(b)]. We note that (122)-(125) are different from the Karush-Kuhn-Tucker (KKT) conditions because of (124).

The remaining task is to find $\pi^\star$ and $\lambda^\star$ that satisfies (122)-(125). According to Theorem 5 and Corollary 1, the solution $\pi^\star$ to (124) is given by (53) where $\beta = 3(\text{mse}_{\text{opt}}+\lambda^\star-\mathbb{E}[Y])$. In addition, as shown in the proof of Theorem 5, the Z_i's in policy $\pi^\star$ are *i.i.d.* Using (122), (123), and (125), the value of $\lambda^\star$ can be obtained by considering two cases: If $\lambda^\star > 0$, because the Z_i's are *i.i.d.*, we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mathbb{E}[Y_i + Z_i] = \mathbb{E}[Y_i + Z_i] = \frac{1}{f_{\max}}. \quad (126)$$

If $\lambda^\star = 0$, then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mathbb{E}[Y_i + Z_i] = \mathbb{E}[Y_i + Z_i] \geq \frac{1}{f_{\max}}. \quad (127)$$

Next, we use (126), (127), and $\beta = 3(\text{mse}_{\text{opt}} + \lambda^\star - \mathbb{E}[Y])$ to determine $\lambda^\star$. To compute mse_{opt}, we substitute policy $\pi^\star$ and (43) into (35), which yields

$$\text{mse}_{\text{opt}}$$
$$= \lim_{n\to\infty}\frac{\sum_{i=0}^{n-1}\mathbb{E}[(W_{S_i+Y_i+Z_i} - W_{S_i})^4 + (Y_i + Z_i)\mathbb{E}[Y]]}{6\sum_{i=0}^{n-1}\mathbb{E}[Y_i + Z_i]}$$
$$= \frac{\mathbb{E}[(W_{S_i+Y_i+Z_i} - W_{S_i})^4]}{6\mathbb{E}[Y_i + Z_i]} + \mathbb{E}[Y], \quad (128)$$

where in the last equation we have used that the Z_i's are *i.i.d.* and the $(W_{S_i+Y_i+Z_i} - W_{S_i})$'s are *i.i.d.*, which were shown in the proof of Theorem 5. Hence, the value of $\beta = 3(\text{mse}_{\text{opt}} + \lambda^\star - \mathbb{E}[Y])$ can be obtained by considering the following two cases:

Case 1: If $\lambda^\star > 0$, then (128) and (126) imply that

$$\mathbb{E}[Y_i + Z_i] = \frac{1}{f_{\max}}, \quad (129)$$

$$\beta > 3(\text{mse}_{\text{opt}} - \mathbb{E}[Y]) = \frac{\mathbb{E}[(W_{S_i+Y_i+Z_i} - W_{S_i})^4]}{2\mathbb{E}[Y_i + Z_i]}. \quad (130)$$

Case 2: If $\lambda^\star = 0$, then (128) and (127) imply that

$$\mathbb{E}[Y_i + Z_i] \geq \frac{1}{f_{\max}}, \tag{131}$$

$$\beta = 3(\mathsf{mse}_{\mathrm{opt}} - \mathbb{E}[Y]) = \frac{\mathbb{E}\left[(W_{S_i+Y_i+Z_i} - W_{S_i})^4\right]}{2\mathbb{E}[Y_i+Z_i]}. \tag{132}$$

Combining (129)-(132), yields that β is the root of

$$\mathbb{E}[Y_i + Z_i] = \max\left(\frac{1}{f_{\max}}, \frac{\mathbb{E}[(W_{S_i+Y_i+Z_i} - W_{S_i})^4]}{2\beta}\right). \tag{133}$$

Substituting (54) and (55) into (133), we obtain that β is the root of (15). Further, (53) can be rewritten as (12). Hence, if we choose $\pi^\star$ as the sampling policy in (12) and choose $\lambda^\star = \beta/3 - \mathsf{mse}_{\mathrm{opt}} + \mathbb{E}[Y]$ where β is the root of (15), then $\pi^\star$ and $\lambda^\star$ satisfies (122)-(125). By using the properties of geometric multiplier mentioned above, (12) and (15) is an optimal solution to the primal problem (37).

Because the problems (13), (35), and (37) are equivalent, (12) and (15) is also an optimal solution to (13) and (35).

The optimal objective value $\mathsf{mse}_{\mathrm{opt}}$ is given by (128). Substituting (54) and (55) into (128), (16) follows. This completes the proof.

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