SCATTERING BELOW THE GROUND STATE FOR THE $2d$ RADIAL NONLINEAR SCHRÖDINGER EQUATION

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ABSTRACT. We revisit the problem of scattering below the ground state threshold for the mass-supercritical focusing nonlinear Schrödinger equation in two space dimensions. We present a simple new proof that treats the case of radial initial data. The key ingredient is a localized virial/Morawetz estimate; the radial assumption aids in controlling the error terms resulting from the spatial localization.

1. INTRODUCTION

We consider the initial-value problem for the focusing nonlinear Schrödinger equation (NLS) in two space dimensions:

$$(1.1) \qquad \begin{cases} (i\partial_t + \Delta)u = -|u|^p u, \\ u(0) = u_0 \in H^1(\mathbb{R}^2), \end{cases}$$

where $u : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}$ and $2 < p < \infty$. This equation admits a global nonscattering solution of the form $u(t) = e^{it}Q(x)$, where Q is the *ground state* solution to the elliptic equation

$$-\Delta Q + Q - Q^{p+1} = 0$$

(see, e.g., [16]). In this note we will give a simple new proof of scattering for radial solutions to (1.1) with initial data "below the ground state threshold" (in the sense of (1.3) and (1.4) below). Such results have been established previously in related settings in [1, 3–6, 8, 10]. These results demonstrate that the ground state is the minimal nonscattering solution in an appropriate sense. Before stating the result precisely, let us introduce a few basic notions.

First, we recall that solutions to (1.1) conserve their *mass* and *energy*, defined by

$$M(u(t)) = \int_{\mathbb{R}^2} |u(t,x)|^2 \, dx,$$

$$E(u(t)) = \int_{\mathbb{R}^2} \tfrac{1}{2}|\nabla u(t,x)|^2 - \tfrac{1}{p+2}|u(t,x)|^{p+2} \, dx,$$

respectively.

Next, we observe that the class of solutions to (1.1) is invariant under the scaling

$$(1.2) \qquad\qquad u(t, x) \mapsto \lambda^{\frac{2}{p}} u(\lambda^2 t, \lambda x).$$

This defines a notion of *criticality* for (1.1). Specifically, if we define

$$s_c = 1 - \tfrac{2}{p},$$

then we find that the $\dot{H}^{s_c}$-norm of initial data is invariant under the rescaling (1.2). In particular (since we are working in two space dimensions) we always have $s_c < 1$, which means the equation is always *energy-subcritical*. The case $s_c = 0$ (i.e., $p = 2$) is called the *mass-critical* equation, since in this case the mass of solutions is left invariant under (1.2).

In this paper, we consider the mass-supercritical range $2 < p < \infty$. We will prove the following.

Theorem 1.1. *Let $2 < p < \infty$. Suppose $u_0 \in H^1(\mathbb{R}^2)$ is spherically-symmetric and satisfies*

$$(1.3) \qquad\qquad M(u_0)^2 E(u_0)^{p-2} < M(Q)^2 E(Q)^{p-2}$$

and

$$(1.4) \qquad\qquad \|u_0\|_{L^2}^2 \|\nabla u_0\|_{L^2}^{p-2} < \|Q\|_{L^2}^2 \|\nabla Q\|_{L^2}^{p-2}.$$

Then (1.1) admits a unique, global-in-time solution u with $u|_{t=0} = u_0$. Furthermore, the solution u scatters, that is, there exist $u_\pm \in H^1(\mathbb{R}^2)$ such that

$$\lim_{t \to \pm\infty} \|u(t) - e^{it\Delta} u_\pm\|_{H^1} = 0.$$

Remark 1.2. The powers of $M(u_0)$ and $E(u_0)$ are chosen so that the product scales like the critical $\dot{H}^{s_c}$-norm. Here $e^{it\Delta}$ denotes the Schrödinger group, so that $e^{it\Delta} u_\pm$ are solutions to the linear Schrödinger equation. The proof of Theorem 1.1 will also show that the solution u obeys global space-time bounds of the form

$$\|u\|_{L^{2p}_{t,x}(\mathbb{R}\times\mathbb{R}^2)} \leq C(M(u_0), E(u_0)).$$

The result of Theorem 1.1 was previously established in [1,3,8], extending the arguments of [6,10]. In fact, in [1,3,8] the same result is proven without the restriction to radial initial data. In these works the authors proceed via the concentration-compactness approach to induction on energy. The purpose of this note is to demonstrate a short and simple argument that suffices to handle the radial case; in particular, it avoids concentration-compactness entirely.[1] This extends our previous works [4,5] to the two-dimensional setting, which often presents new challenges due to issues with Morawetz estimates and weaker dispersive estimates. It is an interesting problem to find a simplified argument to handle the general (i.e., non-radial) case in two dimensions, as well as to consider the one-dimensional problem.

The strategy of proof will be to establish a virial/Morawetz estimate for solutions to (1.1) from which we may deduce scattering. The required coercivity in the virial/Morawetz estimate follows from the sub-threshold assumptions (1.3) and (1.4) (specifically through the use of the sharp Gagliardo–Nirenberg inequality). The radial assumption is used to get uniform control over error terms stemming

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[1]Shortly after the submission of the present work to arXiv, Guo and Shen submitted their work [9], which obtains the same result by similar methods (as well as some extensions to nonlinear Klein–Gordon equations).

from the spatial truncation, which is in turn needed to render the virial/Morawetz quantities finite. In particular, we utilize the radial Sobolev embedding estimate to deal with errors at large radii.

2. Preliminaries

We use the standard notation $A \lesssim B$ to denote $A \leq CB$ for some $C > 0$, with dependence on parameters indicated via subscripts. We also use the "big O" notation $\mathcal{O}$. When necessary we will write $C(A)$ to denote a positive constant depending on a parameter A.

We employ the standard Lebesgue and Sobolev spaces, including mixed space-time Lebesgue norms. We write r' for the Hölder dual of r, i.e., the solution to $\frac{1}{r} + \frac{1}{r'} = 1$.

We write $a\pm$ to denote $a \pm \varepsilon$ for sufficiently small $\varepsilon > 0$. When the exponent $L_x^{\infty-}$ appears, it should be interpreted as L_x^r for some r satisfying $\frac{1}{r} = 0+$.

We utilize the following radial Sobolev embedding estimate, which appears originally in [12] and follows from the fundamental theorem of calculus and Cauchy–Schwarz.

Lemma 2.1 (Radial Sobolev embedding, [12]). *If $f \in H^1(\mathbb{R}^2)$ is spherically-symmetric, then*

$$\||x|^{\frac{1}{2}} f\|_{L^\infty(\mathbb{R}^2)} \lesssim \|f\|_{H^1(\mathbb{R}^2)}.$$

2.1. **Linear estimates; local theory.** We recall the standard dispersive estimate

$$\|e^{it\Delta}\|_{L^{r'}(\mathbb{R}^2) \to L^r(\mathbb{R}^2)} \lesssim |t|^{-1+\frac{2}{r}}, \quad 2 \leq r \leq \infty,$$

which in turn yield the following Strichartz estimates [7, 11, 13]: for $t \in I \subset \mathbb{R}$ and $2 < q_j \leq \infty$ satisfying $\frac{1}{q_j} + \frac{1}{r_j} = \frac{1}{2}$ for $j = 1, 2$,

$$\|e^{it\Delta} f\|_{L_t^{q_1} L_x^{r_1}(I \times \mathbb{R}^2)} \lesssim \|f\|_{L^2(\mathbb{R}^2)},$$

$$\left\|\int_{\mathbb{R}} e^{-is\Delta} F(s)\, ds\right\|_{L^2(\mathbb{R}^2)} \lesssim \|F\|_{L_t^{q_1'} L_x^{r_1'}(\mathbb{R} \times \mathbb{R}^2)},$$

$$\left\|\int_0^t e^{i(t-s)\Delta} F(s)\, ds\right\|_{L_t^{q_1} L_x^{r_1}(I \times \mathbb{R}^2)} \lesssim \|F\|_{L_t^{q_2'} L_x^{r_2'}(I \times \mathbb{R}^2)}.$$

The endpoint case $(q, r) = (2, \infty)$ may also be included in the radial setting [14], although we will not need it here.

Local well-posedness for (1.1) follows from standard arguments using Strichartz estimates and Sobolev embedding. In particular, any $u_0 \in H^1$ leads to a local-in-time solution in $C_t H^1$, which may be extended to a global solution provided the H^1-norm remains uniformly bounded in time. See [2] for a textbook treatment.

2.2. **Variational analysis.** We recall that Q is the unique positive, radial, decaying solution to

(2.1) $$-\Delta Q + Q - Q^{p+1} = 0,$$

which may be constructed as an optimizer of the sharp Gagliardo–Nirenberg inequality

$$\|f\|_{L^{p+2}(\mathbb{R}^2)}^{p+2} \leq C_0 \|f\|_{L^2(\mathbb{R}^2)}^2 \|\nabla f\|_{L^2(\mathbb{R}^2)}^p.$$

For more details, see for example [2,16]. Multiplying (2.1) by Q and by $x \cdot \nabla Q$ and integrating leads to the Pohozaev identities

$$\|Q\|_{L^2}^2 = \tfrac{2}{p+2}\|Q\|_{L^{p+2}}^{p+2} \quad \text{and} \quad \|\nabla Q\|_{L^2}^2 = \tfrac{p}{p+2}\|Q\|_{L^{p+2}}^{p+2}.$$

In particular,

$$(2.2) \qquad \|Q\|_{L^2}^2 \|\nabla Q\|_{L^2}^{p-2} = \tfrac{p+2}{p}C_0^{-1}$$

and

$$(2.3) \qquad M(Q)^2 E(Q)^{p-2} = (\tfrac{p-2}{2p})^{p-2}(\tfrac{p+2}{p})^2 C_0^{-2}.$$

We will need the following two lemmas.

Lemma 2.2 (Coercivity I). *If*

$$M(u_0)^2 E(u_0)^{p-2} < (1-\delta)M(Q)^2 E(Q)^{p-2}$$

and

$$\|u_0\|_{L^2}^2 \|\nabla u_0\|_{L^2}^{p-2} \leq \|Q\|_{L^2}^2 \|\nabla Q\|_{L^2}^{p-2},$$

then there exists $\delta' > 0$ so that

$$(2.4) \qquad \|u(t)\|_{L^2}^2 \|\nabla u(t)\|_{L^2}^{p-2} < (1-\delta')\|Q\|_{L^2}^2 \|\nabla Q\|_{L^2}^{p-2}$$

for all $t \in I$, where $u : I \times \mathbb{R}^2 \to \mathbb{C}$ is the maximal-lifespan solution to (1.1). In particular, $I = \mathbb{R}$ and $u(t)$ is uniformly bounded in H^1.

Proof. By the sharp Gagliardo–Nirenberg inequality and conservation of mass/energy,

$$(1-\delta)M(Q)^2 E(Q)^{p-2} \geq M(u)^2 E(u)^{p-2}$$
$$\geq \left[\tfrac{1}{2}\|u\|_{L^2}^{\frac{4}{p-2}}\|\nabla u\|_{L^2}^2 - \tfrac{C_0}{p+2}\|u\|_{L^2}^{\frac{2p}{p-2}}\|\nabla u\|_{L^2}^p\right]^{p-2}$$

for all $t \in I$. Using (2.3), a short computation reveals that this is equivalent to

$$1-\delta \geq \frac{p}{p-2}\left(\frac{\|u(t)\|_{L^2}^{\frac{2}{p-2}}\|\nabla u(t)\|_{L^2}}{\|Q\|_{L^2}^{\frac{2}{p-2}}\|\nabla Q\|_{L^2}}\right)^2 - \frac{2}{p-2}\left(\frac{\|u(t)\|_{L^2}^{\frac{2}{p-2}}\|\nabla u(t)\|_{L^2}}{\|Q\|_{L^2}^{\frac{2}{p-2}}\|\nabla Q\|_{L^2}}\right)^p$$

for all $t \in I$. The desired bound now follows from a continuity argument and the fact that for $p > 2$ we have

$$\tfrac{p}{p-2}y^2 - \tfrac{2}{p-2}y^p \leq 1-\delta \implies |y-1| \geq \delta' \quad \text{for some} \quad \delta' > 0.$$

Global well-posedness and uniform H^1 bounds now follow from the conservation of L^2-norm and the H^1 blowup criterion (cf. Section 2.1). $\qquad\square$

Lemma 2.3 (Coercivity II). *If*

$$\|f\|_{L^2}^2 \|\nabla f\|_{L^2}^{p-2} \leq (1-\delta)\|Q\|_{L^2}^2 \|\nabla Q\|_{L^2}^{p-2}$$

for some $\delta > 0$, then there exists $\delta' > 0$ such that

$$\|\nabla f\|_{L^2}^2 - \tfrac{p}{p+2}\|f\|_{L^{p+2}}^{p+2} \geq \delta'\|f\|_{L^{p+2}}^{p+2}.$$

Proof. By the sharp Gagliardo–Nirenberg inequality and (2.2),

$$E(f) \geq \|\nabla f\|_{L^2}^2 [\tfrac{1}{2} - \tfrac{1}{p+2} C_0 \|f\|_{L^2}^2 \|\nabla f\|_{L^2}^{p-2}]$$
$$\geq \|\nabla f\|_{L^2}^2 [\tfrac{1}{2} - \tfrac{1}{p+2} C_0 (1-\delta) \|Q\|_{L^2}^2 \|\nabla Q\|_{L^2}^{p-2}]$$
$$= \|\nabla f\|_{L^2}^2 [\tfrac{p-2}{2p} + \tfrac{\delta}{p}].$$

Thus

$$\tfrac{1}{p} \|\nabla f\|_{L^2}^2 - \tfrac{1}{p+2} \|f\|_{L^{p+2}}^{p+2} = E(f) - \tfrac{p-2}{2p} \|\nabla f\|_{L^2}^2 \geq \tfrac{\delta}{p} \|\nabla f\|_{L^2}^2,$$

which implies the result after some rearranging. $\square$

3. Virial/Morawetz estimate

Let u_0 satisfy (1.3) and (1.4), and let u be the corresponding global-in-time solution to (1.1) guaranteed by Lemma 2.2. In particular, u is uniformly bounded in H^1 and obeys (2.4). Applying the scaling (1.2), we may assume

$$(3.1) \qquad\qquad M(u) = E(u) = E_0.$$

We will prove the following space-time estimate.

Proposition 3.1. *For any* $T > 0$,

$$\int_0^T \int |u(t,x)|^{p+2} \, dx \, dt \lesssim_{E_0} T^\alpha, \quad where \quad \alpha = \max\{\tfrac{1}{3}, \tfrac{2}{p+2}\}.$$

Proof. We let ϕ be a smooth radial function satisfying

$$\phi(x) = \begin{cases} 1, & 0 \leq |x| \leq 1, \\ 0, & |x| > 2. \end{cases}$$

We may write $\phi = \phi(r)$ where $r = |x|$. We use $'$ or ∂_r to denote radial derivatives.

We then set

$$\psi(x) = \tfrac{1}{|x|} \int_0^{|x|} \phi(\rho) \, d\rho.$$

In particular, $\psi(r) = \phi(r)$ for $r \leq 1$. We also have the following bound:

$$(3.2) \qquad\qquad |\psi(x)| \lesssim \min\{1, \tfrac{1}{|x|}\}.$$

Note that

$$(3.3) \qquad\qquad r\psi'(r) = \phi(r) - \psi(r).$$

In particular, $\psi'(r) = \phi'(r) = 0$ for $r \leq 1$, while we have

$$(3.4) \qquad\qquad |\psi'(x)| \lesssim \tfrac{1}{|x|^2} \quad for \quad |x| > 1.$$

Given $R \geq 1$, we define the Morawetz quantity

$$A(t) = \int \psi(\tfrac{x}{R}) x \cdot \text{Im}[\bar{u} \nabla u] \, dx$$

which satisfies

$$(3.5) \qquad\qquad |A(t)| \lesssim R E_0 \quad \text{uniformly over} \quad t \in \mathbb{R}.$$

Using (1.1), we compute

(3.6) $$\tfrac{dA}{dt} = \operatorname{Re} \int \psi(\tfrac{x}{R}) x_k \big[\bar{u} u_{jjk} - \bar{u}_{jj} u_k\big]\, dx$$

(3.7) $$+ \operatorname{Re} \int \psi(\tfrac{x}{R}) x_k \big[\bar{u}\partial_k(|u|^p u) - |u|^p \bar{u} u_k\big]\, dx,$$

where subscripts denote partial derivatives and repeated indices are summed.

For (3.6), we begin by observing that

$$\operatorname{Re}[\bar{u} u_{jjk} - \bar{u}_{jj} u_k] = \tfrac{1}{2}\partial_{jjk}|u|^2 - 2\operatorname{Re}\partial_j[\bar{u}_j u_k].$$

We will insert this identity into (3.6) and integrate by parts. Using (3.3) as well, this yields

$$(3.6) = -\tfrac{1}{2}\int \Delta[\psi(\tfrac{x}{R}) + \phi(\tfrac{x}{R})]|u|^2\, dx + 2\int \psi(\tfrac{x}{R})|\nabla u|^2 + \psi'(\tfrac{x}{R})\tfrac{|x|}{R}|\partial_r u|^2\, dx.$$

Now observe

$$\Delta[\psi(\tfrac{x}{R}) + \phi(\tfrac{x}{R})] = \tfrac{1}{R^2}\phi''(\tfrac{x}{R}) + \tfrac{1}{R|x|}\big[2\phi'(\tfrac{x}{R}) - \psi'(\tfrac{x}{R})\big].$$

Recalling (3.3) and (3.4), we deduce

$$(3.6) = 2\int \phi(\tfrac{x}{R})|\nabla u|^2 - \tfrac{1}{2R^2}\int \phi''(\tfrac{x}{R})|u|^2 - \int \tfrac{1}{2R|x|}[2\phi'(\tfrac{x}{R}) - \psi'(\tfrac{x}{R})]|u|^2\, dx$$

(3.8) $$= 2\int \phi(\tfrac{x}{R})|\nabla u|^2 + \mathcal{O}(\tfrac{1}{R^2}\|u\|_{L^2}^2).$$

We turn to (3.7). Noting that

$$\operatorname{Re}\{\bar{u}\partial_k(|u|^p u) - |u|^p \bar{u} u_k\} = \tfrac{p}{p+2}\partial_k(|u|^{p+2})$$

and

$$\partial_k[x_k \psi(\tfrac{x}{R})] = 2\psi(\tfrac{x}{R}) + \tfrac{x_k}{R}[\partial_k \psi](\tfrac{x}{R}),$$

we may integrate by parts and use (3.3) to write

(3.9) $$(3.7) = -\int [\psi(\tfrac{x}{R}) + \phi(\tfrac{x}{R})]\tfrac{p}{p+2}|u|^{p+2}\, dx.$$

We now collect (3.8) and (3.9) to obtain

(3.10) $$\tfrac{dA}{dt} \geq \int 2\phi(\tfrac{x}{R})|\nabla u|^2 - \tfrac{p}{p+2}[\psi(\tfrac{x}{R}) + \phi(\tfrac{x}{R})]|u|^{p+2}\, dx - \mathcal{O}(\tfrac{1}{R^2}).$$

Now, by construction (see, e.g., (3.2)) and radial Sobolev embedding (Lemma 2.1), we can estimate

(3.11)
$$\int |\psi(\tfrac{x}{R}) - \phi(\tfrac{x}{R})|\,|u|^{p+2}\, dx \lesssim \int_{|x|>R} \tfrac{R}{|x|}|u|^{p+2}\, dx$$
$$\lesssim R^{-\frac{p}{2}}\||x|^{\frac{1}{2}}u\|_{L^\infty}^p \|u\|_{L^2}^2 \lesssim R^{-\frac{p}{2}},$$

and so we may continue from (3.10) to get

(3.12) $$\tfrac{dA}{dt} \geq 2\int \phi(\tfrac{x}{R})\big[|\nabla u|^2 - \tfrac{p}{p+2}|u|^{p+2}\big]\, dx - \mathcal{O}(\tfrac{1}{R^\sigma}),$$

where

$$\sigma = \min\{2, \tfrac{p}{2}\}.$$

Next, let us establish a lower bound.

Lemma 3.2. *There exists $\eta > 0$ such that for R sufficiently large, we have*

$$2\int \phi(\tfrac{x}{R})[|\nabla u|^2 - \tfrac{p}{p+2}|u|^{p+2}]\,dx \geq \eta \int_{|x|\leq \frac{R}{2}} |u|^{p+2}\,dx - \mathcal{O}(\tfrac{1}{R^\sigma})$$

uniformly in time.

Proof of Lemma 3.2. Let us write $\phi(\tfrac{x}{R}) = \chi_R^2(x)$. We first wish to show

(3.13) $$\sup_t \|\chi_R u(t)\|_{L^2}^2 \|\chi_R u(t)\|_{\dot{H}^1}^{p-2} < (1 - \tfrac{1}{2}\delta')\|Q\|_{L^2}^2\|Q\|_{\dot{H}^1}^{p-2}$$

for R sufficiently large, where δ' is as in (2.4). Given that (2.4) holds and the cutoff only decreases the L^2-norm, we need only consider the $\dot{H}^1$-norm. For this we compute

$$\int \chi_R^2 |\nabla u|^2\,dx = \int |\nabla(\chi_R u)|^2 + \chi_R \Delta[\chi_R]|u|^2\,dx,$$

which shows

$$\|\chi_R u\|_{\dot{H}^1}^2 \leq \|u\|_{\dot{H}^1}^2 + \mathcal{O}(R^{-2}\|u\|_{L^2}^2),$$

and hence (3.13) holds for R large enough.

Using Lemma 2.3, (3.13) implies

$$\|\chi_R u(t)\|_{\dot{H}^1}^2 - \tfrac{p}{p+2}\|\chi_R u(t)\|_{L^{p+2}}^{p+2} \geq 10\eta \|\chi_R u(t)\|_{L^{p+2}}^{p+2}$$

for some $\eta > 0$, uniformly in t. Now, the argument just given above shows that we may replace

$$\|\chi_R u(t)\|_{\dot{H}^1}^2 \quad\text{with}\quad \int \phi(\tfrac{x}{R})|\nabla u|^2\,dx$$

with errors that are $\mathcal{O}(R^{-2})$ uniformly in time. Similarly, estimating as in (3.11), we may write

$$\int |\chi_R u|^{p+2} = \int \phi(\tfrac{x}{R})|u|^{p+2} + \mathcal{O}\left[\int_{|x|>\frac{R}{2}} |u|^{p+2}\,dx\right]$$

$$= \int \phi(\tfrac{x}{R})|u|^{p+2}\,dx + \mathcal{O}(R^{-\frac{p}{2}})$$

uniformly in t. This completes the proof. $\square$

With Lemma 3.2 in place, we may combine (3.12), the fundamental theorem of calculus, and (3.5) to deduce

$$\int_0^T \int_{|x|\leq \frac{R}{2}} |u(t,x)|^{p+2}\,dx\,dt \lesssim_{E_0} R + \tfrac{T}{R^\sigma}$$

uniformly in $T, R \geq 1$. As radial Sobolev embedding (Lemma 2.1) yields

$$\int_{|x|>\frac{R}{2}} |u(t,x)|^{p+2}\,dx \lesssim R^{-\frac{p}{2}}\||x|^{\frac{1}{2}}u\|_{L^\infty}^p\|u\|_{L^2}^2 \lesssim R^{-\frac{p}{2}}\|u\|_{H^1}^{p+1} \lesssim R^{-\frac{p}{2}}$$

uniformly in time, we deduce

$$\int_0^T \int |u(t,x)|^{p+2}\,dx\,dt \lesssim_{E_0} R + TR^{-\sigma}$$

uniformly in $T, R \geq 1$. Choosing $R = T^{\frac{1}{1+\sigma}}$ yields

$$\int_0^T \int |u(t,x)|^{p+2}\,dx\,dt \lesssim T^{\frac{1}{1+\sigma}}.$$

Observing that

$$\frac{1}{1+\sigma} = \frac{1}{1+\min\{2,\frac{p}{2}\}} = \max\{\tfrac{1}{3}, \tfrac{2}{p+2}\} = \alpha,$$

we complete the proof of the proposition. $\qquad\square$

4. SCATTERING

In this section we will use the virial/Morawetz estimate of Proposition 3.1 to establish scattering.

Proof of Theorem 1.1. For initial data u_0 as in Theorem 1.1, we are guaranteed a global-in-time solution $u(t)$ satisfying uniform H^1 bounds by Lemma 2.2. We rescale u so that (3.1) holds, and we have the virial/Morawetz estimate, Proposition 3.1.

Our argument is similar to the one appearing in [5], and in fact would be applicable in the settings considered previously in [4, 5].

Let $\varepsilon > 0$ be a small parameter to be chosen sufficiently small (depending on E_0) below. As Sobolev embedding and Strichartz yield

$$\|e^{it\Delta}u_0\|_{L^{2p}_{t,x}(\mathbb{R}\times\mathbb{R}^2)} \lesssim \|u_0\|_{H^1(\mathbb{R}^2)} \lesssim_{E_0} 1,$$

we may split $\mathbb{R}$ into $J = J(\varepsilon, E_0)$ intervals I_j such that

$$\tag{4.1} \|e^{it\Delta}u_0\|_{L^{2p}_{t,x}(I_j\times\mathbb{R}^2)} < \varepsilon$$

for each j. We let $T = T(\varepsilon)$ be a large parameter to be determined below. We will prove

$$\tag{4.2} \|u\|^{2p}_{L^{2p}_{t,x}(I_j\times\mathbb{R}^2)} \lesssim_{E_0} T$$

for each j. Then, summing over j yields the critical global space-time bound

$$\|u\|^{2p}_{L^{2p}_{t,x}(\mathbb{R}\times\mathbb{R}^2)} \lesssim_{E_0} T,$$

which in turn yields scattering by standard arguments (see, e.g., [15, Section 5.8]).

As Hölder's inequality and Sobolev embedding imply

$$\tag{4.3} \|u\|^{2p}_{L^{2p}_{t,x}(I\times\mathbb{R}^2)} \lesssim_{E_0} \langle I \rangle$$

for any interval $I \subset \mathbb{R}$, it suffices to consider j such that $|I_j| > 2T$.

Let us fix one such interval, say $I = (a, b)$ with $|I| > 2T$. We will show that there exists $t_1 \in (a, a+T)$ such that

$$\tag{4.4} \left\| \int_0^{t_1} e^{i(t-s)\Delta}(|u|^p u)(s)\,ds \right\|_{L^{2p}_{t,x}([t_1,\infty)\times\mathbb{R}^2)} \lesssim_{E_0} C(\varepsilon)T^{-\beta} + \varepsilon^{\frac{1}{2}}$$

for some $\beta > 0$.

Assuming (4.4) for the moment, let us complete the proof. We use the Duhamel formula to write

$$e^{i(t-t_1)\Delta}u(t_1) = e^{it\Delta}u_0 + i\int_0^{t_1} e^{i(t-s)\Delta}(|u|^p u)(s)\,ds.$$

Thus, choosing T sufficiently large depending on ε and recalling (4.1), we deduce

$$\|e^{i(t-t_1)\Delta}u(t_1)\|_{L^{2p}_{t,x}([t_1,b]\times\mathbb{R}^2)} \lesssim \varepsilon^{\frac{1}{2}}.$$

For ε small enough, this yields by a continuity argument the bound

$$\|u\|_{L^{2p}_{t,x}([t_1,b]\times\mathbb{R}^2)} \lesssim \varepsilon^{\frac{1}{2}}.$$

On the other hand, using $|t_1 - a| < T$ and (4.3), we get

$$\|u\|^{2p}_{L^{2p}_{t,x}([a,t_1]\times\mathbb{R}^2)} \lesssim T,$$

and hence (4.2) holds, as desired.

It remains to prove (4.4). By time-translation invariance, we may assume $a = 0$. We begin by applying Proposition 3.1, which yields

$$(4.5) \qquad \int_0^T \int |u(t,x)|^{p+2}\, dx\, dt \lesssim_{E_0} T^\alpha, \quad \text{where} \quad \alpha = \max\{\tfrac{1}{3}, \tfrac{2}{p+2}\}.$$

We claim that there exists $t_0 \in [\frac{T}{4}, \frac{T}{2}]$ and $\delta = \delta(\varepsilon, E_0) > 0$ such that

$$(4.6) \qquad \int_{t_0}^{t_0+\delta T^{1-\alpha}} \int |u(t,x)|^{p+2}\, dx\, dt < \varepsilon.$$

Indeed, as $[\frac{T}{4}, \frac{T}{2}]$ is covered by $\sim \delta^{-1} T^\alpha$ intervals of length $\delta T^{1-\alpha}$, the bound (4.5) shows that there must be some interval $[t_0, t_0 + \delta T^{1-\alpha}]$ obeying

$$\int_{t_0}^{t_0+\delta T^{1-\alpha}} \int |u(t,x)|^{p+2}\, dx\, dt \lesssim \delta C(E_0),$$

which yields the claim.

We now set

$$t_1 = t_0 + \delta T^{1-\alpha}$$

and observe that since $t_0 \le \frac{T}{2}$, we may guarantee that $t_1 < T$.

We will estimate the integral in (4.4) by estimating separately the contribution of $[0, t_0]$ and $[t_0, t_1]$.

We first treat $[0, t_0]$. For $t > t_1$, we may use the dispersive estimate, Hölder's inequality, and (4.5) to estimate

$$\left\|\int_0^{t_0} e^{i(t-s)\Delta}|u|^p u\, ds\right\|_{L^\infty_x} \lesssim \int_0^{t_0} |t-s|^{-1} \|u(s)\|^{\frac{(p+2)(p-1)}{p}}_{L^{p+2}} \|u(s)\|^{\frac{2}{p}}_{L^2}\, ds$$

$$\lesssim \left[\int_0^{t_0} \int |u(s,x)|^{p+2}\, dx\, ds\right]^{\frac{p-1}{p}} \|\,|t-s|^{-1}\|_{L^p_s}$$

$$\lesssim_{E_0} T^{\frac{p-1}{p}\alpha}|t-t_0|^{-\frac{p-1}{p}} \lesssim \delta^{-\frac{p-1}{p}} T^{-(1-2\alpha)\frac{p-1}{p}}$$

yielding

$$\left\|\int_0^{t_0} e^{i(t-s)\Delta}|u|^p u\, ds\right\|_{L^\infty_{t,x}([t_1,\infty)\times\mathbb{R}^2)} \lesssim \delta^{-\frac{p-1}{p}} T^{-(1-2\alpha)\frac{p-1}{p}}.$$

On the other hand, we may write

$$i \int_0^{t_0} e^{i(t-s)\Delta}|u|^p u\, ds = e^{i(t-t_0)\Delta}u(t_0) - e^{it\Delta}u_0,$$

so that by Strichartz we have

$$\left\|\int_0^{t_0} e^{i(t-s)\Delta}|u|^p u(s)\, ds\right\|_{L^4_{t,x}(\mathbb{R}\times\mathbb{R}^2)} \lesssim_{E_0} 1.$$

Thus, by interpolation and the fact that $\frac{1}{3} < \alpha < \frac{1}{2}$, we get

$$(4.7) \qquad \left\| \int_0^{t_0} e^{i(t-s)\Delta} |u|^p u(s)\,ds \right\|_{L^{2p}_{t,x}([t_1,\infty)\times\mathbb{R}^2)}$$

$$\lesssim_{E_0} [\delta^{-\frac{p-1}{p}} T^{-(1-2\alpha)\frac{p-1}{p}}]^{1-\frac{2}{p}} \lesssim_{E_0} C(\varepsilon) T^{-\beta}$$

for some $\beta > 0$. This is an acceptable contribution to (4.4).

We next consider the contribution of $[t_0, t_1]$. Let us first show how the estimate works, employing the $a\pm$ notation; we will show how to choose exponents more precisely in Remark 4.1 below. By Sobolev embedding, Strichartz, the fractional chain rule, and (4.6),

$$\left\| \int_{t_0}^{t_1} e^{i(t-s)\Delta} |u|^p u(s)\,ds \right\|_{L^{2p}_{t,x}} \lesssim \left\| |\nabla|^{s_c} \int_{t_0}^{t_1} e^{i(t-s)\Delta} |u|^p u(s)\,ds \right\|_{L^{2p}_t L^{\frac{2p}{p-1}}_x}$$

$$\lesssim \| |\nabla|^{s_c}(|u|^p u) \|_{L^{2-}_t L^{1+}_x([t_0,t_1]\times\mathbb{R}^2)}$$

$$\lesssim \|u\|^{\frac{p+2}{2}+}_{L^{p+2}_{t,x}([t_0,t_1]\times\mathbb{R}^2)} \|u\|^{\frac{p-2}{2}-}_{L^\infty_t L^\infty_x} \| |\nabla|^{s_c} u \|_{L^\infty_t L^{2+}_x}$$

$$\lesssim_{E_0} \varepsilon^{\frac{1}{2}},$$

which is again an acceptable contribution to (4.4).

This completes the proof of (4.4) and hence of Theorem 1.1. $\qquad\square$

Remark 4.1. It is also possible (although, we contend, less transparent) to choose the exponents in the final estimate above more precisely: Let $\theta \in (0,1)$ be a small parameter satisfying $p(1-\theta) > 2$. In the estimates above, we choose the dual admissible pair

$$L^{2-}_t L^{1+}_x = L^{2-\theta}_t L^{\frac{4-2\theta}{4-3\theta}}_x$$

and estimate

$$\| |\nabla|^{s_c}(|u|^p u) \|_{L^{2-\theta}_t L^{\frac{4-2\theta}{4-3\theta}}_x([t_0,t_1]\times\mathbb{R}^2)} \lesssim \|u\|^{\frac{p+2}{2-\theta}}_{L^{p+2}_{t,x}([t_0,t_1]\times\mathbb{R}^2)} \|u\|^{\frac{p(1-\theta)-2}{2-\theta}}_{L^\infty_t L^{r_2}_x} \| |\nabla|^{s_c} u \|_{L^\infty_t L^{r_1}_x},$$

where, given a choice of r_1 we must have (by scaling)

$$r_2 = \frac{2 r_1 [p(1-\theta)-2]}{r_1(2-3\theta)-(4-2\theta)}.$$

In particular, to guarantee finiteness of r_2 we should take

$$(4.8) \qquad\qquad r_1 > \frac{4-2\theta}{2-3\theta} > 2,$$

which is compatible with $r_1 \leq \frac{2p}{p-2}$ (needed for the embedding $H^1 \subset \dot{H}^{s_c,r_1}$) provided θ also obeys $\theta \leq \frac{2}{p+1}$. It then remains to verify that we may guarantee $r_2 \geq 2$. After some rearranging, this reduces to the constraint

$$r_1 \leq \begin{cases} \frac{4-2\theta}{2-3\theta-[p(1-\theta)-2]} & \text{if } p(1-\theta) - 2 < 2 - 3\theta, \\ \infty & \text{otherwise.} \end{cases}$$

As this is compatible with (4.8), we conclude that there exist suitable choices of exponents, as claimed.

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