B. Dodson (2018) “Global Well-Posedness for the Defocusing, Cubic, Nonlinear Wave Equation in Three Dimensions for Radial Initial Data in $\dot{H}^s \times \dot{H}^{s-1}$, $s > \frac{1}{2}$,”
International Mathematics Research Notices, Vol. 00, No. 0, pp. 1–21
Advance Access publication on Month xx, 2018
doi:10.1093/imrn/rnx323

Global Well-Posedness for the Defocusing, Cubic, Nonlinear Wave Equation in Three Dimensions for Radial Initial Data in $\dot{H}^s \times \dot{H}^{s-1}$, $s > \frac{1}{2}$

Benjamin Dodson*

Johns Hopkins University, 404 Krieger Hall, 3400 N. Charles Street, Baltimore, MD, 21218

**Correspondence to be sent to: bdodson4@jhu.edu*

In this paper we study the defocusing, cubic nonlinear wave equation in three dimensions with radial initial data. The critical space is $\dot{H}^{1/2} \times \dot{H}^{-1/2}$. We show that if the initial data is radial and lies in $\left(\dot{H}^s \times \dot{H}^{s-1}\right) \cap \left(\dot{H}^{1/2} \times \dot{H}^{-1/2}\right)$ for some $s > \frac{1}{2}$, then the cubic initial value problem is globally well-posed. The proof utilizes the I-method, long time Strichartz estimates, and local energy decay. This method is quite similar to the method used in [11].

1 Introduction

In this paper we study the defocusing, cubic wave equation

$$u_{tt} - \Delta u = F(u) = -u^3, u(0,x) = u_0, u_t(0,x) = u_1, u : \mathbf{R} \times \mathbf{R}^3 \to \mathbf{R}. \tag{1}$$

A solution to (1) actually produces a family of solutions due to scaling. Indeed, if u solves (1) with initial data $(u(0), u_t(0))$ then for any $\lambda > 0$,

$$u(t,x) \mapsto \lambda u(\lambda t, \lambda x), \tag{2}$$

Received May 12, 2017; Revised October 3, 2017; Accepted November 8, 2017

is a solution to (1) with initial data $\left(\lambda u(0,\lambda x), \lambda^2 u_t(0,\lambda x)\right)$. Equation (2) preserves the $\dot{H}^{1/2}\left(\mathbf{R}^3\right)$ norm of u and the $\dot{H}^{-1/2}\left(\mathbf{R}^3\right)$ norm of u_t, and thus (1) is called $\dot{H}^{1/2}$-critical.

Study of dispersive partial differential equations with initial data lying in the critical Sobolev space is currently an important topic of research. References [23] and [24] proved a sharp counterexample to well-posedness of (1) for data lying in a Sobolev space less regular than the critical Sobolev space. See [5] for similar results for a number of dispersive equations, including (1).

On the other hand, positive results have been obtained for a number of initial value problems with initial data lying in the critical Sobolev space. Reference [24] proved a local well-posedness result for (1) with initial data in $\dot{H}^{1/2} \times \dot{H}^{-1/2}$.

For the energy-critical, defocusing wave equation in three dimensions (quintic), global existence of smooth, radially symmetric solutions was proved in [30]. Reference [15] extended this result to the general case. Reference [27] extended this result to dimensions $3 \leq d \leq 7$. Global well-posedness for initial data lying in the energy space was proved by [28] and [16].

Remark. This question has also been completely worked out for the defocusing energy-critical (quintic) Schrödinger equation [3, 9], and the defocusing, mass-critical Schrödinger equation [10, 21]. In each case scattering has also been proved.

Remark. The above discussion was not intended to be a complete discussion of defocusing energy-critical and mass-critical problems. For one thing, discussion of dimensions other than $d = 3$ was omitted entirely. Discussion of the focusing problem, see for example [18], was also completely omitted.

What unites the energy-critical wave equation, the energy-critical Schrödinger equation, and the mass-critical Schrödinger equation is the presence of a conserved quantity that controls the critical Sobolev norm. For example, if u solves the wave equation

$$u_{tt} - \Delta u = -|u|^p u, \tag{3}$$

then the energy

$$E(u(t)) = \frac{1}{2}\int_{\mathbf{R}^d} |\nabla u(t,x)|^2 \,dx + \frac{1}{2}\int |\partial_t u(t,x)|^2 \,dx + \frac{1}{p+2}\int |u(t,x)|^{p+2} dx \tag{4}$$

is conserved. Therefore for (1) the energy is given by

$$E(u(t)) = \frac{1}{2}\int |\nabla u(t,x)|^2\,dx + \frac{1}{2}\int |u_t(t,x)|^2\,dx + \frac{1}{4}\int |u(t,x)|^4\,dx = E(u(0)). \tag{5}$$

However, there is no known conserved quantity that controls $\|u(t)\|_{\dot{H}^{1/2}(\mathbf{R}^3)}$ or $\|u_t(t)\|_{\dot{H}^{-1/2}(\mathbf{R}^3)}$. In fact this is the only obstacle to proving global well-posedness and scattering for (1) with radial data.

Theorem 1.1. Suppose u solves (1) on an interval I, I is the maximal interval of existence of the solution, and

$$\|u\|_{L_t^\infty \dot{H}^{1/2}(I\times\mathbf{R}^3)} + \|u_t\|_{L_t^\infty \dot{H}^{-1/2}(I\times\mathbf{R}^3)} < \infty. \tag{6}$$

Then u is global, that is $I = \mathbf{R}$, and u scatters to a free solution both forward and backward in time.

Proof. See [12]. ■

The definitions of well-posedness and scattering that are used here are the standard definitions.

Definition 1.2. (Well-posedness). The initial value problem (1) is well-posed on an open interval $I \subset \mathbf{R}$, $0 \in I$, for $(u_0, u_1) \in \left(\dot{H}^s \cap \dot{H}^{1/2}\right) \times \left(\dot{H}^{s-1} \cap \dot{H}^{-1/2}\right) = X$ if

(i) (1) has a unique solution u lying in $C_t^0(I; X)$,

(ii) The solution satisfies the Duhamel formula

$$(u(t), u_t(t)) = S(t)(u_0, u_1) - \int_0^t S(t-\tau)(0, u^3)\,d\tau, \tag{7}$$

where $S(t)(f, g)$ is the solution operator to the linear wave equation $u_{tt} - \Delta u = 0$, $u(0, x) = f(x)$, $u_t(0, x) = g(x)$.

(iii) For any compact $J \subset I$, the map $(u_0, u_1) \mapsto L_{t,x}^4\left(J \times \mathbf{R}^3\right)$ is continuous.

Equation (1) is said to be globally well-posed if $I = \mathbf{R}$.

Definition 1.3. (Scattering). A global solution to (1) with initial data $(u_0, u_1) \in X$ is said to scatter forward in time to some $(u_0, u_1)^+ \in X$ if

$$\lim_{t\to+\infty} \left\|(u(t), u_t(t)) - S(t)(u_0, u_1)^+\right\|_X = 0. \tag{8}$$

Analogously, u is said to scatter backward in time to some $(u_0, u_1)^- \in X$ if

$$\lim_{t\to-\infty} \|(u(t), u_t(t)) - S(t)(u_0, u_1)^-\|_X = 0. \tag{9}$$

Equation (1) is said to be scattering for initial data lying in a certain set if for each (u_0, u_1) lying in that set there exists $(u_0, u_1)^+$ and $(u_0, u_1)^-$ such that (8) and (9) hold, and furthermore, the maps $(u_0, u_1) \mapsto (u_0, u_1)^+$ and $(u_0, u_1) \mapsto (u_0, u_1)^-$ are continuous as functions of (u_0, u_1). □

For a number of focusing, dispersive partial differential equations, there exist solutions with bounded critical Sobolev norm that fail to be global or fail to scatter. This phenomenon is called type-two blowup. Excluding type-two blowup, such as in the proof of Theorem 1.1 of [12], usually utilizes concentration compactness arguments.
These arguments are very similar to arguments used to prove global well-posedness and scattering for energy-critical wave and Schrödinger equations, and mass-critical Schrodinger equations. In fact, given a conserved quantity that controls the critical Sobolev norm, all that is left is to exclude type-two blowup. Thus, when [19] proved global well-posedness and scattering for the cubic nonlinear Schrödinger equation with bounded $\dot{H}^{1/2}(\mathbf{R}^3)$, this introduced a number of techniques that were very instrumental in the proofs of energy-critical and mass-critical scattering results.
To the author's knowledge there are no known methods for proving global well-posedness and scattering for dispersive equations without either assuming the existence of a quantity that conserves the critical Sobolev norm or in fact having such a quantity.
In this paper we utilize the I-method to prove that for any $s > \frac{1}{2}$ the $\dot{H}^s \times \dot{H}^{s-1}$ norm of $(u(t), u_t(t))$ is bounded on any finite compact subset of $\mathbf{R}$. This is enough to prove global well-posedness.

Theorem 1.4. (Main theorem). Equation (1) is globally well-posed for any radial initial data $(u(0), u_t(0)) = (u_0, u_1) \in \dot{H}^s(\mathbf{R}^3) \times \dot{H}^{s-1}(\mathbf{R}^3) \cap \dot{H}^{1/2} \times \dot{H}^{-1/2}$, $s > \frac{1}{2}$.

Remark. By finite propagation speed, this also implies that (1) is globally well-posed for initial data lying in $\dot{H}^s \times \dot{H}^{s-1}$ for any $s > \frac{1}{2}$.

The I-method has its roots in the Fourier truncation method. The Fourier truncation method was introduced by [2] for the cubic nonlinear Schrödinger equation and by [20] for (1), proving (1) is globally well-posed for $u(0) \in \dot{H}^s(\mathbf{R}^3) \cap L^4(\mathbf{R}^3)$, $u_t(0) \in \dot{H}^s(\mathbf{R}^3)$, $s > \frac{3}{4}$. See [1] for $s \geq \frac{3}{4}$ and [13].

The I-method is an improvement over the Fourier truncation method. For example [7] was able to improve the results of [2] for the nonlinear Schrödinger equation. On the wave equation side, [25] and [26] extended the results of [20] to $s > \frac{13}{18}$ and to $s > \frac{7}{10}$ if u has radial symmetry (although in both cases with inhomogeneous initial data to avoid technical complications). Perhaps more importantly, [17] proved a well-posedness result that was technically unattainable via the Fourier truncation method. See [11] for a more detailed discussion of the history of the I-method.

To prove our result we make use of the long-time Strichartz estimates. The long time Strichartz estimates were introduced in [11] and were actually inspired in large part by the linear–nonlinear decomposition of [26]. Basically, the idea is that if u solves (1) on an interval $[0, T]$, on which we have some a priori bound on the $\|u(t)\|_{L_t^\infty \dot{H}^s([0,T]\times\mathbf{R}^3)}$ norm for some $s > \frac{1}{2}$, then we can show that at high frequencies, the solution u is dominated by the free evolution from initial data $(u(0), u_t(0))$.

We then take the usual modified energy

$$E(Iu(t)) = \frac{1}{2}\int |\nabla Iu(t,x)|^2 \,dx + \frac{1}{2}\int |Iu_t(t,x)|^2 \,dx + \frac{1}{4}\int |Iu(t,x)|^4 \,dx, \tag{10}$$

where I is a smoothing Fourier multiplier

$$I : \dot{H}^s\left(\mathbf{R}^3\right) \to \dot{H}^1\left(\mathbf{R}^3\right), \qquad I : \dot{H}^{s-1}\left(\mathbf{R}^3\right) \to L^2\left(\mathbf{R}^3\right). \tag{11}$$

Direct computation shows that $\frac{d}{dt}E(Iu(t))$ is a quadrilinear integral operator on u that has at least two terms at high frequencies. Using the long-time Strichartz estimates, we can then show that the integral of $\frac{d}{dt}E(Iu(t))$ over the interval $[0, T]$ is small, which in turn implies that $E(Iu(t))$ is pretty close to $E(Iu(0))$. Meanwhile, an a priori upper bound on $E(Iu(t))$ gives us good control over $\|u(t)\|_{\dot{H}^s}$, allowing us to make a bootstrap argument that proves Theorem 1.4.

This argument is extremely similar to the scattering argument in [11]. There are two main reasons we do not prove scattering here. The first is the lack of an interaction

Morawetz estimate for the wave equation, unlike the interaction Morawetz estimate for the nonlinear Schrödinger equation in [8]. The second is that the L^2 norm of u is not conserved for the nonlinear wave equation (1), as it is for the nonlinear Schrödinger equation. Observe that [11] assumed that the initial data lay in $L^2(\mathbf{R}^d)$.

2 Linear Estimates for the Wave Equation

In this section we prove some Strichartz-type estimates on solutions to linear wave equations that will be needed in the proof of Theorem 1.4. We begin with a discussion of the Littlewood–Paley partition of unity.

Definition 2.1. (Littlewood–Paley partition of unity). Suppose $\psi \in C_0^\infty(\mathbf{R}^3)$ is a radial, decreasing function supported on $|x| \leq 2$, $\psi = 1$ on $|x| \leq 1$. Then for any N we define the Littlewood–Paley projection

$$(P_N f)(x) = \mathcal{F}^{-1}\left(\left(\psi\left(\frac{\xi}{N}\right) - \psi\left(\frac{2\xi}{N}\right)\right)\hat{f}(\xi)\right)(x), \tag{12}$$

where

$$\mathcal{F}^{-1}(\hat{f}(\xi))(x) = (2\pi)^{-3/2}\int e^{ix\cdot\xi}\hat{f}(\xi)\,\mathrm{d}\xi. \tag{13}$$

Also define the operators

$$(P_{\leq N} f)(x) = \mathcal{F}^{-1}\left(\psi\left(\frac{\xi}{N}\right)\hat{f}(\xi)\right)(x), \tag{14}$$

and $P_{>N} = 1 - P_{\leq N}$. □

Remark. Since ψ is a $C_0^\infty(\mathbf{R}^3)$ function, $P_N f$ is the convolution of f with a Schwartz function that is $\lesssim_l N^3(1 + N|x|)^{-l}$ for any $l \in \mathbf{Z}$.

Next recall the Strichartz estimates of [29].

Theorem 2.2. (Strichartz estimate). If u solves $u_{tt} - \Delta u = F$ on an interval I, with $t_0 \in I$, then

$$\|u(t)\|_{L^4_{t,x}(I\times\mathbf{R}^3)} \lesssim \|u(t_0)\|_{\dot{H}^{1/2}(\mathbf{R}^3)} + \|u_t(t_0)\|_{\dot{H}^{-1/2}(\mathbf{R}^3)} + \|F\|_{L^{4/3}_{t,x}(I\times\mathbf{R}^3)}. \tag{15}$$

□

Reference [14] extended Strichartz estimates to all admissible pairs when $d = 3$. Combining Strichartz estimates with local energy decay yields the following estimate.

Theorem 2.3. (Linear estimates). If u solves the wave equation

$$u_{tt} - \Delta u = F_1 + F_2 + F_3, \qquad u(0) = u_0, u_t(0) = u_1, \tag{16}$$

then

$$\begin{aligned} &\||\nabla|^{1/2}u\|_{L^4_{t,x}(I\times\mathbf{R}^3)} + \left(\sup_R \frac{1}{R^{1/2}}\|\nabla u\|_{L^2_{t,x}(I\times\{|x|\leq R\})}\right) \\ &+ \left\||\nabla|^{-1/2}u_t\right\|_{L^4_{t,x}(I\times\mathbf{R}^3)} + \left(\sup_R \frac{1}{R^{1/2}}\|u_t\|_{L^2_{t,x}(I\times\{|x|\leq R\})}\right) \\ &\lesssim \|u_0\|_{\dot{H}^1(\mathbf{R}^3)} + \|u_1\|_{L^2(\mathbf{R}^3)} + \|F_3\|_{L^1_tL^2_x(I\times\mathbf{R}^3)} \\ &+ \left\||\nabla|^{1/2}F_1\right\|_{L^{4/3}_{t,x}(I\times\mathbf{R}^3)} + \sum_{j=-\infty}^{\infty} 2^{j/2}\|F_2\|_{L^2_{t,x}(I\times\{2^j\leq|x|\leq 2^{j+1}\})}. \end{aligned} \tag{17}$$

□

Proof. Again let $S(t)(u_0, u_1)$ be the solution operator to (16) with $F_1 = F_2 = F_3 = 0$,

$$\cos(t\sqrt{-\Delta})u_0 + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}u_1. \tag{18}$$

By Strichartz estimates and the sharp Huygens principle,

$$\begin{aligned} &\left\||\nabla|^{1/2}S(t)(u_0,u_1)\right\|_{L^4_{t,x}(\mathbf{R}\times\mathbf{R}^3)} + \left(\sup_R \frac{1}{R^{1/2}}\|\nabla S(t)(u_0,u_1)\|_{L^2_{t,x}(\mathbf{R}\times\{|x|\leq R\})}\right) \\ &+ \left\||\nabla|^{-1/2}\partial_t S(t)(u_0,u_1)\right\|_{L^4_{t,x}(\mathbf{R}\times\mathbf{R}^3)} + \left(\sup_R \frac{1}{R^{1/2}}\|\partial_t S(t)(u_0,u_1)\|_{L^2_{t,x}(\mathbf{R}\times\{|x|\leq R\})}\right) \\ &\lesssim \|u_0\|_{\dot{H}^1(\mathbf{R}^3)} + \|u_1\|_{L^2(\mathbf{R}^3)}. \end{aligned} \tag{19}$$

To prove this, it suffices to prove

$$\int_{\mathbf{R}} \int_{|x| \leq 1} |\nabla u(t,x)|^2 + (\partial_t u(t,x))^2 \, dx \, dt \lesssim \|u_0\|_{\dot{H}^1(\mathbf{R}^3)}^2 + \|u_1\|_{L^2(\mathbf{R}^3)}^2, \tag{20}$$

and then rescale. The proof of (20) is quite standard. See for example Theorem 3 of [4]. Then by duality, (19), and the Strichartz estimates of [29],

$$\left\| \nabla \int \frac{\sin(-\tau\sqrt{-\Delta})}{\sqrt{-\Delta}} F(\tau) \, d\tau \right\|_{L_x^2(\mathbf{R}^3)} + \left\| \partial_t \int \frac{\sin((t-\tau)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(\tau) \, d\tau \right\|_{L_x^2(\mathbf{R}^3)}$$

$$\lesssim \left\| |\nabla|^{1/2} F_1 \right\|_{L_{t,x}^{4/3}(I \times \mathbf{R}^3)} + \sum_{j=-\infty}^{\infty} 2^{j/2} \|F_2\|_{L_{t,x}^2(I \times \{2^j \leq |x| \leq 2^{j+1}\})} + \|F_3\|_{L_t^1 L_x^2(I \times \mathbf{R}^3)}. \tag{21}$$

Therefore, by the Christ–Kiselev lemma of [6], when $u_0 = u_1 = 0$,

$$\left\| |\nabla|^{1/2} u \right\|_{L_{t,x}^4(I \times \mathbf{R}^3)} + \left\| |\nabla|^{-1/2} u_t \right\|_{L_{t,x}^4(I \times \mathbf{R}^3)} \lesssim \|u_0\|_{\dot{H}^1(\mathbf{R}^3)} + \|u_1\|_{L^2(\mathbf{R}^3)}$$

$$+ \left\| |\nabla|^{1/2} F_1 \right\|_{L_{t,x}^{4/3}(I \times \mathbf{R}^3)} + \sum_{j=-\infty}^{\infty} 2^{j/2} \|F_2\|_{L_{t,x}^2(I \times \{2^j \leq |x| \leq 2^{j+1}\})} + \|F_3\|_{L_t^1 L_x^2(I \times \mathbf{R}^3)}, \tag{22}$$

and

$$\left(\sup_R \frac{1}{R^{1/2}} \left\| \int_0^t \frac{\sin((t-\tau)\sqrt{-\Delta})}{\sqrt{-\Delta}} (F_1 + F_3)(\tau) \, d\tau \right\|_{L_{t,x}^2(\mathbf{R} \times \{|x| \leq R\})} \right)$$

$$+ \left(\sup_R \frac{1}{R^{1/2}} \left\| \partial_t \int_0^t \frac{\sin((t-\tau)\sqrt{-\Delta})}{\sqrt{-\Delta}} (F_1 + F_3)(\tau) \, d\tau \right\|_{L_{t,x}^2(\mathbf{R} \times \{|x| \leq R\})} \right)$$

$$\lesssim \left\| |\nabla|^{1/2} F_1 \right\|_{L_{t,x}^{4/3}(I \times \mathbf{R}^3)} + \|F_3\|_{L_t^1 L_x^2(I \times \mathbf{R}^3)}. \tag{23}$$

Therefore it only remains to show

$$\left(\sup_{R>0}\frac{1}{R^{1/2}}\left\|\nabla\int_0^t\frac{\sin((t-\tau)\sqrt{-\Delta})}{\sqrt{-\Delta}}F_2(\tau)\,\mathrm{d}\tau\right\|_{L^2_{t,x}(I\times\{x:|x|\le R\})}\right)$$
$$+\left(\sup_{R>0}\frac{1}{R^{1/2}}\left\|\partial_t\int_0^t\frac{\sin((t-\tau)\sqrt{-\Delta})}{\sqrt{-\Delta}}F_2(\tau)\,\mathrm{d}\tau\right\|_{L^2_{t,x}(I\times\{x:|x|\le R\})}\right)$$
$$\lesssim\sum_{j=-\infty}^{\infty}2^{j/2}\|F_2\|_{L^2_{t,x}(I\times\{2^j\le|x|\le 2^{j+1}\})}. \tag{24}$$

Finally, if $u_0 = u_1 = F_1 = F_3 = 0$ and F_2 is supported on $\{x : |x| \le R\}$, then the sharp Huygens principle implies that the supports of

$$\int_{\tau\in[0,t]\cap[kR,(k+1)R]} S(t-\tau)(0,F_2)\,\mathrm{d}\tau \tag{25}$$

are finitely overlapping. Since Hölder's inequality implies

$$\|F_2\|_{L^1_tL^2_x([kR,(k+1)R]\times\mathbf{R}^3)}\lesssim R^{1/2}\|F_2\|_{L^2_tL^2_x([kR,(k+1)R]\times\mathbf{R}^3)}, \tag{26}$$

$$\left(\sup_{R>0}\frac{1}{R^{1/2}}\|\nabla u\|_{L^2_{t,x}(I\times\{x:|x|\le R\})}\right)+\left(\sup_{R>0}\frac{1}{R^{1/2}}\|u_t\|_{L^2_{t,x}(I\times\{x:|x|\le R\})}\right)$$
$$\lesssim\sum_{j=-\infty}^{\infty}2^{j/2}\|F_2\|_{L^2_{t,x}(I\times\{2^j\le|x|\le 2^{j+1}\})} \tag{27}$$

follows from (23). This completes the proof of Theorem 2.3. ∎

Remark. The same argument also implies that if P_N is a Littlewood–Paley multiplier,

$$\left(\sup_{R>0}N\|P_Nu\|_{L^2_{t,x}(\mathbf{R}\times\{x:|x|\le R\})}\right)\lesssim\|u_0\|_{\dot H^1(\mathbf{R}^3)}+\|u_1\|_{L^2(\mathbf{R}^3)}$$
$$+\|\nabla F_1\|_{L^2_tL^1_x(I\times\mathbf{R}^3)}+\sum_{j=-\infty}^{\infty}2^{j/2}\|F_2\|_{L^2_{t,x}(I\times\{2^j\le|x|\le 2^{j+1}\})}, \tag{28}$$

with constant independent of N.

We will also utilize the endpoint Strichartz estimate of [22].

Theorem 2.4. (Endpoint Strichartz estimates). For u_0, u_1 radial,

$$\|S(t)(u_0, u_1)\|_{L_t^2 L_x^\infty(\mathbf{R}\times\mathbf{R}^3)} \lesssim \|u_0\|_{\dot{H}^1(\mathbf{R}^3)} + \|u_1\|_{L_x^2(\mathbf{R}^3)}. \tag{29}$$

Also, by duality, if F is radial,

$$\left\| \int_{\mathbf{R}} S(-t)(0, F)(t)\, \mathrm{d}t \right\|_{L_x^2(\mathbf{R}^3)} \lesssim \|F\|_{L_t^2 L_x^1(\mathbf{R}\times\mathbf{R}^3)}. \tag{30}$$

Proof. See [22]. ■

3 Proof of the Main Theorem

We follow the work of [7] and later [25] and [26], and define the I-operator $I : H^s \to H^1$, where I is given by the Fourier multiplier

$$m(\xi) = \begin{cases} 1 & \text{if } |\xi| \leq N \\ \frac{N^{1-s}}{|\xi|^{1-s}} & \text{if } |\xi| > 2N. \end{cases} \tag{31}$$

By the Sobolev embedding theorem,

$$\begin{aligned} E(Iu(0)) &\lesssim \|\nabla Iu(0,x)\|_{L^2}^2 + \|Iu_t(0,x)\|_{L^2}^2 + \|Iu(0,x)\|_{L^6(\mathbf{R}^3)}^2 \|u(0,x)\|_{L^3(\mathbf{R}^3)}^2 \\ &\lesssim_{\|u_0\|_{\dot{H}^{1/2}}} \|\nabla Iu(t,x)\|_{L^2}^2 + \|Iu_t(t,x)\|_{L^2}^2. \end{aligned} \tag{32}$$

Therefore,

$$E(Iu(0)) \leq C\left(\|u_0\|_{\dot{H}^{1/2}} + \|u_1\|_{\dot{H}^{-1/2}}, \|u_0\|_{\dot{H}^s} + \|u_1\|_{\dot{H}^{s-1}}\right) N^{2(1-s)}. \tag{33}$$

To prove global well-posedness it suffices to prove that for any compact interval $[0, T_0] \subset \mathbf{R}$, there exists an $N(T_0)$ sufficiently large so that

$$E(Iu(t)) \leq 2CN^{2(1-s)}. \tag{34}$$

We prove this with a standard bootstrap argument. Suppose that for some interval $[0, T] \subset [0, T_0]$,

$$\sup_{t\in[0,T]} E(Iu(t)) \leq 2CN^{2(1-s)}. \tag{35}$$

Then we show that for $N(T_0)$ sufficiently large,

$$E(Iu(t)) \leq \frac{3}{2} C N^{2(1-s)}, \tag{36}$$

which implies $E(Iu(t)) \leq 2CN^{2(1-s)}$ on $[0, T_0]$.

Definition 3.1. Let I be the Fourier multiplier with a fixed N. For $1 \leq M \leq N$, let

$$\begin{aligned} \mathcal{S}(T, M) = & \left\| P_{>M} |\nabla|^{1/2} Iu \right\|_{L^4_{t,x}([0,T]\times \mathbf{R}^3)} + \left\| P_{>M} |\nabla|^{-1/2} Iu_t \right\|_{L^4_{t,x}([0,T]\times \mathbf{R}^3)} \\ & + \sup_{N^{-1} \leq R \leq 4T_0} \frac{1}{R^{1/2}} \| P_{>M} \nabla Iu \|_{L^2_{t,x}([0,T]\times\{|x|\leq R\})} \\ & + \sup_{N^{-1} \leq R \leq 4T_0} \frac{1}{R^{1/2}} \| P_{>M} Iu_t \|_{L^2_{t,x}([0,T]\times\{|x|\leq R\})} \\ & + \sup_{N^{-1} \leq R \leq 4T_0} \frac{M}{R^{1/2}} \| P_{>M} Iu \|_{L^2_{t,x}([0,T]\times\{|x|\leq R\})}. \end{aligned} \tag{37}$$

□

Theorem 3.2. (Long-time Strichartz estimate). Suppose $E(Iu(t)) \leq 2CN^{2(1-s)}$ on $[0, T]$. Then there exists a small constant $c(s, \|u_0\|_{\dot{H}^{1/2}}, \|u_1\|_{\dot{H}^{-1/2}}) > 0$ such that if

$$\ln(N) \gtrsim \frac{1-s}{c\left(\frac{1}{2} - s\right)} + \sqrt{\frac{\ln(T_0)}{c\left(\frac{1}{2} - s\right)}}, \tag{38}$$

then

$$\mathcal{S}\left(T, \frac{N}{8}\right) \lesssim \sqrt{C} N^{1-s}. \tag{39}$$

□

Proof of Theorem 3.2. For a large, fixed constant C_1, let

$$\mathcal{T} = \left\{ T' \in [0, T] : \mathcal{S}\left(T', \frac{N}{8}\right) \leq C_1 N^{1-s} \right\}. \tag{40}$$

It is clear from Hölder's inequality in time and the uniform bound on $E(Iu(t))$ that $\mathcal{T}$ is nonempty. Also, by the Lebesgue dominated convergence theorem, $\mathcal{T}$ is a closed set. Therefore, to prove Theorem 3.2 it suffices to prove that $\mathcal{T}$ is open in $[0, T]$.

The radial Sobolev embedding theorem implies a bilinear estimate on $[0, T']$ with $T' \in \mathcal{T}$.

Lemma 3.3. (Bilinear estimate). For $M \leq N$, if $E(Iu(t)) \leq 2CN^{2(1-s)}$ on $[0, T']$,

$$\left\| \left(P_{> \frac{M}{8}} \nabla Iu\right) (P_{<N} u) \right\|_{L^2_{t,x}([0,T'] \times \{x : |x| \leq 4T_0\}} \lesssim (\ln(T_0) + \ln(N))^{1/2} \mathcal{S}\left(T', \frac{M}{8}\right) \sqrt{C} N^{1-s}, \tag{41}$$

$$\left\| \left(P_{> \frac{M}{8}} u\right) (P_{<N} u) \right\|_{L^2_{t,x}([0,T'] \times \{x : |x| \leq 4T_0\})} \lesssim \frac{1}{M} (\ln(T_0) + \ln(N))^{1/2} \mathcal{S}\left(T', \frac{M}{8}\right) \sqrt{C} N^{1-s}, \tag{42}$$

and

$$\left\| \left(P_{> \frac{M}{8}} Iu_t\right) (P_{<N} u) \right\|_{L^2_{t,x}([0,T'] \times \{x : |x| \leq 4T_0\}} \lesssim (\ln(T_0) + \ln(N))^{1/2} \mathcal{S}\left(T', \frac{M}{8}\right) \sqrt{C} N^{1-s}. \tag{43}$$

■

Remark. Because $E(Iu(t)) \leq 2CN^{2(1-s)}$, $\|P_{<N} u\|_{L^\infty_t \dot{H}^1([0,T'] \times \mathbf{R}^3)} \lesssim \sqrt{C} N^{1-s}$.

Proof of Lemma 3.3. By definition of $\mathcal{S}\left(T', \frac{M}{8}\right)$,

$$\left\| P_{> \frac{M}{8}} \nabla Iu \right\|_{L^2_{t,x}\left([0,T'] \times \left\{|x| \leq \frac{1}{N}\right\}\right)} \lesssim N^{-1/2} \mathcal{S}\left(T', \frac{M}{8}\right), \tag{44}$$

so by the Sobolev embedding theorem $\|P_{<N} u\|_{L^\infty} \lesssim N^{1/2} \|\nabla P_{<N} u\|_{L^2}$,

$$\begin{aligned} &\left\| \left(P_{> \frac{M}{8}} \nabla Iu\right) (P_{<N} u) \right\|_{L^2_{t,x}([0,T'] \times \left\{|x| \leq \frac{1}{N}\right\}} \\ &\lesssim \mathcal{S}\left(T', \frac{M}{8}\right) \|\nabla Iu\|_{L^\infty_t L^2_x([0,T'] \times \mathbf{R}^3)} \lesssim \sqrt{C} N^{1-s} \mathcal{S}\left(T', \frac{M}{8}\right). \end{aligned} \tag{45}$$

Now partition $\left\{\frac{1}{N} \leq |x| \leq 4T_0\right\}$ into $\lesssim \ln(N) + \ln(T_0)$ annuli $\{x : 2^j \leq |x| \leq 2^{j+1}\}$, where $\frac{1}{N} \leq 2^j \leq 4T_0$. On each annulus, by definition of $\mathcal{S}\left(T', \frac{M}{8}\right)$,

$$\left\| P_{> \frac{M}{8}} \nabla Iu \right\|_{L^2_{t,x}([0,T'] \times \{2^j \leq |x| \leq 2^{j+1}\})} \lesssim \mathcal{S}\left(T', \frac{M}{8}\right) 2^{j/2}, \tag{46}$$

while by the radial Sobolev embedding theorem,

$$2^{j/2} \|Iu\|_{L^\infty_{t,x}([0,T'] \times \{2^j \leq |x| \leq 2^{j+1}\})} \lesssim \|\nabla Iu\|_{L^\infty_t L^2_x([0,T'] \times \mathbf{R}^3)} \lesssim \sqrt{C} N^{1-s}. \tag{47}$$

The arguments to prove (42) and (43) are identical, (42) makes use of Bernstein's inequality. ■

Now by Duhamel's formula, (7),

$$(Iu(t), Iu_t(t)) = S(t)(Iu_0, Iu_1) - \int_0^t S(t-\tau)\left(0, Iu^3\right) d\tau. \tag{48}$$

Decompose

$$\begin{aligned} P_{>M}u^3 &= 3P_{>M}\left(\left(P_{>\frac{M}{8}}u\right)\left(P_{<\frac{M}{8}}u\right)^2\right) + 3P_{>M}\left(\left(P_{>\frac{M}{8}}u\right)^2\left(P_{<\frac{M}{8}}u\right)\right) + P_{>M}\left(\left(P_{>\frac{M}{8}}u\right)^3\right) \\ &= 3P_{>M}\left(\left(P_{>\frac{M}{8}}u\right)\left(P_{<\frac{M}{8}}u\right)(P_{<N}u)\right) \\ &\quad + 3P_{>M}\left(\left(P_{>\frac{M}{8}}u\right)\left(P_{<\frac{M}{8}}u\right)(P_{>N}u)\right) + P_{>M}\left(\left(P_{>\frac{M}{8}}u\right)^3\right). \end{aligned} \tag{49}$$

Take $\psi \in C_0^\infty\left(\mathbf{R}^3\right)$, $\psi(x) = 1$ for $|x| \leq 1$, and $\psi(x)$ is supported on $|x| \leq 2$. Now by (42) and the fact that the Littlewood–Paley kernel is rapidly decreasing, $T_0 >> 1$ and $N >> 1$, $E(Iu(t)) \leq 2CN^{2(1-s)}$, and the Sobolev embedding theorems, both radial and standard,

$$\begin{aligned} &\sum_{2^j \leq 4T_0} 2^{j/2}\left\| IP_{>M}\psi\left(\frac{x}{2T_0}\right)\left(\left(P_{>\frac{M}{8}}u\right)(P_{<N}u)\left(P_{<\frac{M}{8}}u\right)\right)\right\|_{L^2_{t,x}([0,T']\times\{x:2^j\leq|x|\leq 2^{j+1}\})} \\ &\lesssim \mathcal{S}\left(T', \frac{M}{8}\right)\frac{\sqrt{C}N^{1-s}}{M}(\ln(T_0)+\ln(N))^{1/2}\left(\sum_{2^j\leq 4T_0} 2^j\left\|P_{<\frac{M}{8}}u\right\|^2_{L^\infty_{t,x}([0,T']\times\{x:2^j\leq|x|\leq 2^{j+1}\})}\right)^{1/2}, \end{aligned} \tag{50}$$

$$\begin{aligned} &\lesssim \mathcal{S}\left(T', \frac{M}{8}\right)\frac{\sqrt{C}N^{1-s}}{M}(\ln(T_0)+\ln(N))\left\||x|^{1/2}u\right\|_{L^\infty_{t,x}([0,T']\times\mathbf{R}^3)} \\ &\lesssim \mathcal{S}\left(T', \frac{M}{8}\right)\frac{\sqrt{C}N^{1-s}}{M}(\ln(T_0)+\ln(N))\|u\|_{L^\infty_t\dot{H}^1([0,T']\times\mathbf{R}^3)} \lesssim \mathcal{S}\left(T', \frac{M}{8}\right)\frac{CN^{2(1-s)}}{M}(\ln(T_0)+\ln(N)). \end{aligned} \tag{51}$$

Also, by the radial Sobolev embedding theorem, Holder's inequality in time, and Bernstein's inequality,

$$\begin{aligned} &\left\| IP_{>M}\left(\left(1-\psi\left(\frac{x}{2T_0}\right)\right)\right)\left(P_{>\frac{M}{8}}u\right)\left(P_{<\frac{M}{8}}u\right)(P_{<N}u)\right\|_{L^1_tL^2_x([0,T']\times\{x:2^j\leq|x|\leq 2^{j+1}\})} \\ &\lesssim \frac{1}{M}\left\||x|^{1/2}P_{<\frac{M}{8}}u\right\|_{L^\infty_tL^\infty_x([0,T']\times\mathbf{R}^3)}\left\||x|^{1/2}P_{<N}u\right\|_{L^\infty_tL^\infty_x([0,T']\times\mathbf{R}^3)}\|Iu\|_{L^\infty_t\dot{H}^1([0,T']\times\mathbf{R}^3)} \lesssim \frac{C^{3/2}N^{3(1-s)}}{M}. \end{aligned} \tag{52}$$

Similarly, since

$$\left\| \left(P_{>\frac{M}{8}} u\right) (P_{>N} u) \right\|_{L^2_{t,x}([0,T'] \times \mathbf{R}^3)} \lesssim \frac{1}{N^{1/2} M^{1/2}} \mathcal{S}\left(T', \frac{M}{8}\right) \mathcal{S}(T', N), \tag{53}$$

$$\sum_{2^j \leq 4T_0} 2^{j/2} \left\| I P_{>M} \psi\left(\frac{x}{2T_0}\right) \left(\left(P_{>\frac{M}{8}} u\right) (P_{>N} u) \left(P_{<\frac{M}{8}} u\right) \right) \right\|_{L^2_{t,x}([0,T'] \times \{x: 2^j \leq |x| \leq 2^{j+1}\})}$$

$$\lesssim \mathcal{S}\left(T', \frac{M}{8}\right) \mathcal{S}(T', N) \frac{1}{N^{1/2} M^{1/2}} \left(\sum_{2^j \leq 4T_0} 2^j \left\| P_{<\frac{M}{8}} u \right\|^2_{L^\infty_{t,x}([0,T'] \times \{x: 2^j \leq |x| \leq 2^{j+1}\})} \right)^{1/2}, \tag{54}$$

$$\lesssim \mathcal{S}\left(T', \frac{M}{8}\right) \mathcal{S}(T', N) \frac{\sqrt{C} N^{(1-s)}}{M} (\ln(T_0) + \ln(N)). \tag{55}$$

Also, by the radial Sobolev embedding theorem, Holder's inequality in time, and Bernstein's inequality,

$$\left\| I P_{>M} \left(\left(1 - \psi\left(\frac{x}{2T_0}\right)\right) \left(P_{>\frac{M}{8}} u\right) \left(P_{<\frac{M}{8}} u\right) (P_{>N} u) \right) \right\|_{L^1_t L^2_x([0,T'] \times \{x: 2^j \leq |x| \leq 2^{j+1}\})}$$

$$\lesssim \frac{1}{M^{1/2} N^{1/2}} \left\| \left(P_{>\frac{M}{8}} u\right) (P_{>N} u) \right\|_{L^2_{t,x}([0,T'] \times \mathbf{R}^3)} \|Iu\|_{L^\infty_t \dot{H}^1([0,T'] \times \mathbf{R}^3)} \lesssim \frac{C^{3/2} N^{3(1-s)}}{M}. \tag{56}$$

Finally split

$$\left(P_{>\frac{M}{8}} u\right)^3 = \left(P_{>\frac{M}{8}} u\right) \left(P_{\frac{M}{8} < \cdot < N} u\right)^2 + 2 \left(P_{>\frac{M}{8}} u\right) (P_{>N} u) \left(P_{\frac{M}{8} < \cdot < N} u\right) + \left(P_{>\frac{M}{8}} u\right) (P_{>N} u)^2. \tag{57}$$

Because $P_{\frac{M}{8} < \cdot < N} u = P_{<N} P_{>\frac{M}{8}} u$, the above computations may also be applied to the first two terms in (57), apply (50)–(52) to the first term and (53)–(56) to the second term.

To estimate the last term, since $|\xi|^{1/2} m(\xi)$ is increasing as $|\xi| \to \infty$, $|\nabla|^{1/2} I$ obeys a Leibniz-type rule (use the paraproduct estimates of [31]). Therefore, by Bernstein's inequality and the definition of $\mathcal{S}$,

$$\left\| |\nabla|^{1/2} I \left(\left(P_{>\frac{M}{8}} u(t)\right) (P_{>N} u(t))^2 \right) \right\|_{L^{4/3}_{t,x}([0,T'] \times \mathbf{R}^3)}$$

$$\lesssim \left\| |\nabla|^{1/2} I P_{>\frac{M}{8}} u \right\|_{L^4_{t,x}([0,T'] \times \mathbf{R}^3)} \|P_{>N} u\|^2_{L^4_{t,x}([0,T'] \times \mathbf{R}^3)}$$

$$+ \left\| P_{>\frac{M}{8}} u \right\|_{L^4_{t,x}([0,T'] \times \mathbf{R}^3)} \left\| |\nabla|^{1/2} I P_{>N} u \right\|_{L^4_{t,x}([0,T'] \times \mathbf{R}^3)} \|P_{>N} u\|_{L^4_{t,x}([0,T'] \times \mathbf{R}^3)}$$

$$\lesssim \frac{1}{N^{1/2}} \frac{1}{M^{1/2}} \mathcal{S}\left(T', \frac{M}{8}\right) \mathcal{S}(T', N)^2. \tag{58}$$

Therefore, by Theorem 2.3, (48)–(57), and $E(Iu(t)) \leq 2CN^{2(1-s)}$ on $[0, T]$,

$$\mathcal{S}(T', M) \lesssim C^{1/2}N^{1-s} + \frac{C^{3/2}N^{3(1-s)}}{M} + \mathcal{S}(T', M)\frac{CN^{2(1-s)}}{M} + \frac{1}{M^{1/2}N^{1/2}}\mathcal{S}(T', M)\mathcal{S}(T', N)^2, \quad (59)$$

so, by the bootstrap assumption, $\mathcal{S}\left(T', \frac{N}{8}\right) \leq C_1 N^{1-s}$, if $M \geq N^{\frac{3}{2}-s}$,

$$\mathcal{S}(T', M) \lesssim C^{1/2}N^{1-s} + C^{3/2}N^{\frac{3}{2}-2s} + \mathcal{S}(T', M)CN^{\frac{1}{2}-s} + C_1^2 N^{\frac{3}{4}-\frac{3}{2}s}\mathcal{S}(T', M). \quad (60)$$

Therefore, for some $c > 0$ sufficiently small, for T_0 large and N satisfying

$$\ln(N) \geq \frac{1-s}{c\left(s-\frac{1}{2}\right)} + \sqrt{\frac{\ln(T_0)}{c\left(s-\frac{1}{2}\right)}}, \quad (61)$$

$$\mathcal{S}\left(T', \frac{N}{8}\right) \lesssim \mathcal{S}\left(T', N^{\frac{3}{2}-s}\right)\ln(T_0)N^{c\ln(N)\left(\frac{1}{2}-s\right)} + CN^{1-s}. \quad (62)$$

Then Theorem 3.2 follows from the following lemma.

Lemma 3.4. If u solves (1) and $E(Iu(t)) \leq 2CN^{2(1-s)}$ on $[0, T']$, then

$$\mathcal{S}\left(T', N^{\frac{3}{2}-s}\right) \lesssim CN^{2(1-s)}T_0^{1/2}. \quad (63)$$

Indeed, returning to the proof of Theorem 3.2, plugging (63) into (62),

$$\mathcal{S}\left(T', \frac{N}{8}\right) << C_1 N^{1-s}, \quad (64)$$

and therefore $\mathcal{T}'$ is both open and closed in $[0, T]$. Since $\mathcal{T}'$ is nonempty, $\mathcal{T}' = [0, T]$.

Proof of Lemma 3.4. Since $E(Iu(t)) \leq 2CN^{2(1-s)}$ for $t \in [0, T] \subset [0, T_0]$,

$$\|Iu\|_{L_{t,x}^4([0,T]\times\mathbf{R}^3)}^4 \lesssim T_0 2CN^{2(1-s)}. \quad (65)$$

Partition $[0, T]$ into $\lesssim \frac{2}{\eta}CT_0N^{2(1-s)}$ subintervals I_j such that $|I_j| \leq \frac{\eta}{CN^{2(1-s)}}$, for some small constant η. Then on each interval $\|u_{\leq N}\|_{L_{t,x}^4(I_j\times\mathbf{R}^3)} \lesssim \eta^{1/4}$.

Then by Theorem 2.2, $E(Iu(t)) \leq 2CN^{2(1-s)}$, Bernstein's inequality, and the fact that $|\xi|^{1/2}m(\xi)$ is increasing in $|\xi|$,

$$\left\| |\nabla|^{1/2} Iu \right\|_{L^4_{t,x}(I_j \times \mathbf{R}^3)} \lesssim \|Iu\|_{L^\infty_t \dot{H}^1(I_j \times \mathbf{R}^3)} + \|Iu_t\|_{L^\infty_t L^2_x(I_j \times \mathbf{R}^3)} + \left\| |\nabla|^{1/2} Iu \right\|_{L^4_{t,x}(I_j \times \mathbf{R}^3)} \|u\|^2_{L^4_{t,x}(I_j \times \mathbf{R}^3)}$$
$$\lesssim N^{1-s} + \eta^{1/2} \left\| |\nabla|^{1/2} Iu \right\|_{L^4_{t,x}(I_j \times \mathbf{R}^3)} + \frac{1}{N} \left\| |\nabla|^{1/2} Iu \right\|^3_{L^4_{t,x}(I_j \times \mathbf{R}^3)}. \tag{66}$$

Then since N is large, $\left\| |\nabla|^{1/2} Iu \right\|_{L^4_{t,x}(I_j \times \mathbf{R}^3)} \lesssim C^{1/2} N^{1-s}$, and by Bernstein's inequality,

$$\|u\|_{L^4_{t,x}(I_j \times \mathbf{R}^3)} \lesssim \|Iu\|_{L^4_{t,x}(I_j \times \mathbf{R}^3)} + \|(1-I)u\|_{L^4_{t,x}(I_j \times \mathbf{R}^3)} \lesssim \eta + C^{1/2} N^{\frac{1}{2}-s} \lesssim \eta. \tag{67}$$

Therefore, by Theorem 2.3 and $E(Iu(t)) \leq 2CN^{1-s}$ on I_j,

$$\left\| |\nabla|^{1/2} Iu \right\|_{L^4_{t,x}(I_j \times \mathbf{R}^2)} + \left(\sup_R R^{-1/2} \|\nabla Iu\|_{L^2_{t,x}(I_j \times \{|x| \leq R\})} \right) \lesssim C^{1/2} N^{1-s}. \tag{68}$$

This proves Lemma 3.4. ■

Theorem 3.2 provides a bound on the growth of $E(Iu(t))$.

Lemma 3.5. For any $t \in [0, T_0]$, $E(Iu(t)) \leq \frac{3}{2} CN^{1-s}$. □

Proof. Again make a bootstrap argument. Let

$$\mathcal{T} = \left\{ T \in [0, T_0] : E(Iu(t)) \leq \frac{3}{2} CN^{1-s} \text{ for all } t \in [0, T] \right\}. \tag{69}$$

Because $E(Iu(0)) = CN^{1-s}$, $\mathcal{T}$ is clearly nonempty. Also, since $E(Iu(t))$ is a continuous function of time $\mathcal{T}$ is closed. Therefore, it only remains to show that $\mathcal{T}$ is open in $[0, T_0]$. Then compute

$$\frac{\mathrm{d}}{\mathrm{d}t} E(Iu(t)) = \int (Iu_t)(t,x)(I(u^3)(t,x) - (Iu)^3(t,x))\, \mathrm{d}x. \tag{70}$$

Splitting $u = u_h + u_l$, $u_l = P_{<\frac{N}{8}} u$, the Fourier support of u_l implies that

$$I\left(u_l^3\right) - (Iu_l)^3 = 0. \tag{71}$$

Also,

$$I\left(u_l^2 P_{<\frac{N}{2}} u\right) - (Iu_l)^2 IP_{<\frac{N}{2}} u = 0, \tag{72}$$

which implies that

$$\begin{aligned}(70) = 3\int IP_h u_t(t,x)\left(I\left(u_l^2 u_h\right)(t,x) - (Iu_h(t,x))(Iu_l(t,x))^2\right) \mathrm{d}x \\ + O\left(\int Iu_t(t,x)\left(I\left(u_h^2 u\right)(t,x) - (Iu_h)^2(t,x) Iu(t,x)\right) \mathrm{d}x\right).\end{aligned} \tag{73}$$

Then by Theorem 3.2 and Lemma 3.3, for N sufficiently large,

$$\begin{aligned}\int_0^T \int_{|x|\leq 4T_0} IP_h u_t(t,x)\left(I\left(u_l^2 u_h\right)(t,x) - (Iu_h(t,x))(Iu_l(t,x))^2\right) \mathrm{d}x\,\mathrm{d}t \\ \lesssim (\ln(N) + \ln(T_0)) C_1^2 N^{2(1-s)} \frac{N^{2(1-s)}}{N} << N^{2(1-s)}.\end{aligned} \tag{74}$$

Meanwhile, by the radial Sobolev embedding theorem, Bernstein's inequality, the fact that $\frac{1}{N} << T_0$, and that the Littlewood–Paley kernel of I is rapidly decreasing outside the ball $|x| \lesssim \frac{1}{N}$,

$$\begin{aligned}&\int_0^T \int_{|x|>4T_0} IP_h u_t(t,x)\left(I\left(u_l^2 u_h\right)(t,x) - (Iu_h(t,x))(Iu_l(t,x))^2\right) \mathrm{d}x\ \mathrm{d}t \\ &\lesssim \left\| |x|^{1/2} u_l \right\|_{L^\infty_{t,x}([0,T]\times\mathbf{R}^3)}^2 \|\nabla Iu\|_{L_t^\infty L_x^2([0,T]\times\mathbf{R}^3)} \|Iu_t\|_{L_t^\infty L_x^2([0,T]\times\mathbf{R}^3)} \\ &\lesssim C^4 N^{4(1-s)} \frac{1}{N} << N^{2(1-s)}.\end{aligned} \tag{75}$$

Next, integrating by parts, again by Theorem 3.2 and the fact that $|\nabla|^{1/2} I$ satisfies the Leibniz-type rule, and Bernstein's inequality

$$\begin{aligned}&\int_0^T \int (I\partial_t u_h)(t,x)\left(I(u_h(t,x)^3) - \left(Iu_h(t,x)\right)^3\right) \mathrm{d}x\ \mathrm{d}t \\ &= \int_0^T \int |\nabla|^{-1/2}(I\partial_t u_h)(t,x) |\nabla|^{1/2}\left(I\left(u_h^3\right)(t,x) - (Iu_h(t,x))^3\right) \mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{76}$$

$$\begin{aligned}&\lesssim \left\| |\nabla|^{-1/2} IP_h u_t \right\|_{L^4_{t,x}([0,T]\times\mathbf{R}^3)} \||\nabla|^{1/2} Iu_h\|_{L^4_{t,x}(\mathbf{R}\times\mathbf{R}^3)} \|u_h\|_{L^4_{t,x}(\mathbf{R}\times\mathbf{R}^3)}^2 \\ &\lesssim \frac{C_1^4}{N} N^{4(1-s)} << N^{2(1-s)}.\end{aligned} \tag{77}$$

Meanwhile, by Lemma 3.3 and the fact that $\|IP_{<N}u_t\|_{\dot{H}^1} \lesssim N\|Iu_t\|_{L^2} \lesssim CNN^{1-s}$,

$$\int_0 \int_{|x|\leq 4T_0} (I\partial_t u_l)(t,x) \left(I(u_h(t,x))^3\right) - \left(Iu_h(t,x))^3\right) \mathrm{d}x\, \mathrm{d}t$$

$$\lesssim \|u_h\|^2_{L^4_{t,x}([0,T]\times\mathbf{R}^3)} \|u_h(P_{<N}Iu_t)\|_{L^2_{t,x}([0,T]\times\{x:|x|\leq 4T_0\})} \tag{78}$$

$$\lesssim \frac{C_1^2 N^{2(1-s)}}{N} C_1^2 N^{2(1-s)} << N^{2(1-s)}. \tag{79}$$

Finally, by the radial Sobolev embedding theorem and Bernstein's inequality,

$$\int_0^T \int_{|x|>4T_0} (I\partial_t u_l)(t,x) \left(I\left(u_h(t,x)^3\right) - (Iu_h(t,x))^3\right) \mathrm{d}x\, \mathrm{d}t$$

$$\lesssim \|u_h\|^2_{L^4_{t,x}([0,T]\times\mathbf{R}^3)} \||x|^{1/2} I\partial_t u_l\|_{L^\infty_{t,x}([0,T]\times\mathbf{R}^3)} \|u_h\|_{L^\infty_t L^2_x([0,T]\times\mathbf{R}^3)} \tag{80}$$

$$\lesssim \frac{C_1^2 N^{2(1-s)}}{N} C^2 N^{2(1-s)} << N^{2(1-s)}. \tag{81}$$

The term

$$\int_0^T \int (Iu_t)(t,x) \left(I\left(u_h^2 u_l\right)(t,x) - (Iu_h)^2(t,x) Iu_l(t,x)\right) \mathrm{d}x\, \mathrm{d}t \tag{82}$$

can be treated as an interpolation of terms with the cubic nonlinearity in the form $u_l^2 u_h$ with terms in the cubic nonlinearity of the form u_h^3.

Therefore, $\int_0^T \left|\frac{\mathrm{d}}{\mathrm{d}t} E(Iu(t))\right| \mathrm{d}t << N^{2(1-s)}$, which implies that $E(Iu(t)) \leq \frac{3}{2} CN^{2(1-s)}$, so $[0, T]$ is open in T_0. Therefore, $E(Iu(t)) \leq CN^{2(1-s)}$ on $[0, T_0]$. ∎

Proof of Theorem 1.4. By Bernstein's inequality,

$$\|u_{>N}\|_{L^\infty_t \dot{H}^s([0,T_0]\times\mathbf{R}^3)} + \|\partial_t u_{>N}\|_{L^\infty_t \dot{H}^{s-1}([0,T_0]\times\mathbf{R}^3)}$$

$$\lesssim \frac{1}{N^{1-s}} \|\nabla Iu\|_{L^\infty_t L^2_x([0,T_0]\times\mathbf{R}^3)} + \|Iu_t\|_{L^\infty_t L^2_x([0,T_0]\times\mathbf{R}^3)} \lesssim 1. \tag{83}$$

Also,

$$\|u_{<N}\|_{L^\infty_t \dot{H}^1([0,T_0]\times\mathbf{R}^3)} + \|u_{<N}\|_{L^\infty_t L^2_x([0,T_0]\times\mathbf{R}^3)} \lesssim N^{1-s}. \tag{84}$$

Interpolating this bound with the trivial bound

$$\|Iu(t) - Iu(0)\|_{L^2(\mathbf{R}^3)} \le \int_0^t \|\partial_t Iu(\tau)\|_{L^2(\mathbf{R}^3)} \, \mathrm{d}\tau \lesssim T_0 N^{1-s}, \tag{85}$$

proves that for $T_0 > 1$,

$$\|Iu(t)\|_{L_t^\infty \dot{H}^s([0,T_0]\times\mathbf{R}^3)} \lesssim T_0^{1-s} N^{1-s} \tag{86}$$

and

$$\|Iu(t)\|_{L_t^\infty \dot{H}^{1/2}([0,T_0]\times\mathbf{R}^3)} \lesssim T_0^{1/2} N^{1-s}. \tag{87}$$

Also for $\frac{3}{p} = \frac{7}{2} - s$, by the Sobolev embedding theorem and definition of I,

$$\|Iu_t(t) - Iu_t(0)\|_{\dot{H}^{-1}(\mathbf{R}^3)} \le \int_0^t \|\Delta Iu(\tau)\|_{\dot{H}^{-1}(\mathbf{R}^3)} \, \mathrm{d}\tau + \int_0^t \left\|Iu^3(\tau)\right\|_{L_x^{6/5}(\mathbf{R}^3)} \mathrm{d}\tau \tag{88}$$

$$\lesssim \int_0^t N^{1-s} \, \mathrm{d}\tau + \int_0^t \|u(\tau)\|_{L_x^3(\mathbf{R}^3)}^2 \|P_{<N} u(\tau)\|_{L_x^6(\mathbf{R}^3)}$$
$$+ N^{1-s} \int_0^t \|P_{>N} u(\tau)\|_{L_x^{3p}(\mathbf{R}^3)}^3 \, \mathrm{d}\tau \lesssim T_0 N^{1-s} + T_0^{3/2} N^{3(1-s)}. \tag{89}$$

Therefore, by interpolation if $t \in [0, T_0]$, $T_0 > 1$,

$$\|u_t(t)\|_{\dot{H}^{s-1}(\mathbf{R}^3)} \lesssim T_0^{\frac{3}{2}(1-s)} N^{3(1-s)}, \tag{90}$$

and

$$\|u_t(t)\|_{\dot{H}^{-1/2}(\mathbf{R}^3)} \lesssim T_0^{3/4} N^{3/2}. \tag{91}$$

Thus the $\dot{H}^s \cap \dot{H}^{1/2} \times \dot{H}^{s-1} \cap \dot{H}^{-1/2}$ norm is uniformly bounded on any compact subset of $\mathbf{R}$. Global well-posedness then follows from the local result of [24]. ■

Funding

This work was partially supported by National Science Foundation [grant number DMS-1500424 to B.D.] during the time of writing this paper.

Acknowledgments

The author is deeply indebted to Tristan Roy for his many helpful comments on preliminary version of this paper. The author also wishes to thank the anonymous referees for their corrections.

References

[1] Bahouri, H. and J. Chemin. "On global well-posedness for defocusing cubic wave equation." *Int. Math. Res. Not.* (2006): Art. ID 54873.

[2] Bourgain, J. "Refinements of Strichartz' inequality and applications to 2D-NLS with critical nonlinearity." *Int. Math. Res. Not.* 5 (1998): 253–83.

[3] Bourgain, J. "Global wellposedness of defocusing critical nonlinear Schrödinger equation in the radial case." *J. Amer. Math. Soc.* 12, no. 1 (1999): 145–71.

[4] Burq, N. "Global Strichartz estimates for nontrapping geometries: about an article by H. F. Smith and C. D. Sogge." *Comm. Part. Diff. Equations* 28, no. 9–10 (2003): 1675–83.

[5] Christ, M., J. Colliander, and T. Tao. "Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations." *Amer. J. Math.* 125, no. 6 (2003): 1235–93.

[6] Christ, M. and A. Kiselev. "Maximal functions associated to filtrations." *J. Funct. Anal.* 179, no. 2 (2001): 409–25.

[7] Colliander, J., M. Keel, G. Staffilani, H. Takaoka, and T. Tao. "Almost conservation laws and global rough solutions to a nonlinear Schrödinger equation." *Math. Res. Lett.* 9, no. 5–6 (2002): 659–82.

[8] Colliander, J., M. Keel, G. Staffilani, H. Takaoka, and T. Tao. "Global existence and scattering for rough solutions of a nonlinear Schrödinger equation on $\mathbf{R}^3$." *Comm. Pure Appl. Math.* 57, no. 8 (2004): 987–1014.

[9] Colliander, J., M. Keel, G. Staffilani, H. Takaoka, and T. Tao. "Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation on $\mathbf{R}^3$." *Ann. Math. (2)* 167, no. 3 (2008): 767–865.

[10] Dodson, B. "Global well-posedness and scattering for the defocusing L^2-critical nonlinear Schrödinger equation when $d \geq 3$." *J. Amer. Math. Soc.* 25, no. 2 (2008): 429–63.

[11] Dodson, B. "Global well-posedness and scattering for nonlinear Schrödinger equations with algebraic nonlinearity when $d = 2, 3$, u_0 radial." arxiv:1405.0218.

[12] Dodson, B. and A. Lawrie. "Scattering for the radial 3D cubic wave equation." *Anal. PDE* 8, no. 2 (2015): 467–97.

[13] Gallagher, I. and F. Planchon. "On global solutions to a defocusing semi - linear wave equation." *Rev. Mat. Iberoam.* 19, no. 1 (2003): 161–77.

[14] Ginibre, J. and G. Velo. "Smoothing properties and retarded estimates for some dispersive evolution equations." *Comm. Math. Phys.* 144, no. 1 (1992): 163–88.

[15] Grillakis, M. "Regularity for the wave equation with a critical nonlinearity." *Comm. Pure Appl. Math.* 45, no. 6 (1992): 749–74.

[16] Kapitanski, L. "Global and unique weak solutions of nonlinear wave equations." *Math. Res. Lett.* 1, no. 2 (1994): 211–23.

[17] Keel, M. and T. Tao. "Local and global well posedness of wave maps on R^{1+1} for rough data." *Int. Mat. Res. Not.* 21 (1998): 1117–56.

[18] Kenig, C. and M. Merle. "Global well-posedness, scattering and blow-up for the energy-critical focusing non-linear wave equation." *Acta Math.* 201, no. 2 (2008): 147–212.

[19] Kenig, C. and M. Merle. "Scattering for $H^{1/2}$ bounded solutions to the cubic, defocusing NLS in 3 dimensions." *Trans. Amer. Math. Soc.* 362, no. 4 (2010): 1937–62.

[20] Kenig, C., G. Ponce, and L. Vega. "Global well-posedness for semi-linear wave equations." *Comm. Part. Diff. Equations* 25, no. 9–10 (2000): 1741–52.

[21] Killip, R., M. Visan, and X. Zhang. "The mass-critical nonlinear Schrödinger equation with radial data in dimensions three and higher." *Anal. PDE* 1, no. 2 (2008): 229–66.

[22] Klainerman, S. and M. Machedon. "Space – time estimates for null forms and the local existence theorem." *Comm. Pure Appl. Math.* 46, no. 9 (1993): 1221–68.

[23] Lindblad, H. "A sharp counterexample to the local existence of low-regularity solutions to nonlinear wave equations." *Duke Math. J.* 72, no. 2 (1993): 503–39.

[24] Lindblad, H. and C. Sogge. "On existence and scattering with minimal regularity for semilinear wave equations." *J. Funct. Anal.* 130, no. 2 (1995): 357–426.

[25] Roy, T. "Global well-posedness for the radial defocusing cubic wave equation on $\mathbf{R}^3$ and for rough data." *Electron. J. Dif. Equations* 2007, no. 166, 22 pp.

[26] Roy, T. "Adapted linear – nonlinear decomposition and global well-posedness for solutions to the defocusing cubic wave equation on $\mathbf{R}^3$." *Discrete Contin. Dyn. Syst.* 24, no. 4 (2009): 1307–23.

[27] Shatah, J. and M. Struwe. "Regularity results for nonlinear wave equations." *Ann. Math. (2)* 138, no. 3 (1993): 503–18.

[28] Shatah, J. and M. Struwe. "Well - posedness in the energy space for semilinear wave equations with critical growth." *Int. Math. Res. Not.* 1994, no. 7: 303ff.

[29] Strichartz, R. S. "Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations." *Duke Math. J*. 44, no. 3 (1977): 705–14.

[30] Struwe, M. "Globally regular solutions to the u^5 Klein-Gordon equation." *Ann. Sc. Norm. Sup. Pisa Cl. Sci. (4)* 15, no. 3 (1988): 495–513.

[31] Taylor, M. "Tools for PDE." *Mathematics Surveys and Monographs.* 81, Providence, RI: American Mathematical Society, 2000.