Cambridge Journal of Mathematics
Volume 7, Number 3, 283–318, 2019

Global well-posedness and scattering for nonlinear Schrödinger equations with algebraic nonlinearity when $d = 2, 3$ and u_0 is radial

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In this paper we discuss global well-posedness and scattering for some initial value problems that are $\dot{H}^1$ subcritical. We prove global well-posedness and scattering for radial data in H^s, $s > s_c$, where the initial value problem is $\dot{H}^{s_c}$-critical. We make use of the long time Strichartz estimates of [13] to do this.

1. Introduction

In this paper we examine the three dimensional initial value problem

$$(i\partial_t + \Delta)u = F(u) = |u|^2 u, \qquad u(0,x) = u_0 \in H_x^s(\mathbf{R}^3), \tag{1.1}$$

as well as the two dimensional initial value problems

$$(i\partial_t + \Delta)u = |u|^{2k} u, \qquad u(0,x) = u_0 \in H_x^s(\mathbf{R}^2), \tag{1.2}$$

where k may be any positive integer. In each case u_0 is a radial function.

Solutions to (1.1) and (1.2) give rise to a family of solutions via the scaling,

$$u(t,x) \mapsto u_\lambda(t,x) = \lambda^{\frac{1}{k}} u(\lambda^2 t, \lambda x). \tag{1.3}$$

Under this scaling, for any $s \in \mathbf{R}$,

$$\|u_\lambda(0,x)\|_{\dot{H}_x^s(\mathbf{R}^d)} = \lambda^{-\frac{d}{2}+s+\frac{1}{k}} \|u(0,x)\|_{\dot{H}_x^s(\mathbf{R}^d)}. \tag{1.4}$$

Thus, (1.1) is called $\dot{H}^{1/2}$-critical since under (1.4),

$$\|u_\lambda(0,x)\|_{\dot{H}_x^{1/2}(\mathbf{R}^3)} = \|u(0,x)\|_{\dot{H}_x^{1/2}(\mathbf{R}^3)}. \tag{1.5}$$

Likewise, (1.2) is called $\dot{H}^{s_c}$-critical, where $s_c = 1 - \frac{1}{k}$.

This scaling is crucial to local well-posedness. Recall the usual definition of well-posedness.

Definition 1.1 (Well-posedness)**.** *The initial value problem* (1.1) *is well-posed on an open interval* $I \subset \mathbf{R}$, $0 \in I$, *for* $u_0 \in H_x^s(\mathbf{R}^3)$, *if*

1. (1.1) *has a unique solution* u *lying in* $C_t^0(I; H_x^s(\mathbf{R}^3))$,
2. *The solution satisfies the Duhamel formula*

$$u(t) = e^{it\Delta} u_0 - i \int_0^t e^{i(t-\tau)\Delta}(|u|^2 u)(\tau) d\tau, \tag{1.6}$$

3. *For any compact* $J \subset I$, *the map* $u_0 \mapsto L_{t,x}^5(J \times \mathbf{R}^3)$ *is continuous.*

The definition of global well-posedness for (1.2) *corresponds to* $(1) - (3)$ *above, although* $L_{t,x}^5$ *should be replaced by* $L_{t,x}^{4k}$ *and* $\mathbf{R}^3$ *should be replaced by* $\mathbf{R}^2$.

Also recall the definition of scattering.

Definition 1.2 (Scattering)**.** *A global solution to* (1.1) *and* (1.2) *with initial data* u_0 *is said to scatter forward in time to some* $u_+ \in H_x^s(\mathbf{R}^d)$ *if*

$$\lim_{t \to +\infty} \| u(t) - e^{it\Delta} u_+ \|_{H^s(\mathbf{R}^d)} = 0. \tag{1.7}$$

Analogously, u *is said to scatter backward in time to some* $u_- \in H_x^s(\mathbf{R}^3)$ *if*

$$\lim_{t \to -\infty} \| u(t) - e^{it\Delta} u_- \|_{H^s(\mathbf{R}^d)} = 0. \tag{1.8}$$

(1.1) *is said to be scattering for initial data lying in a certain set* X *if for each* $u_0 \in X$ *there exists* u_+ *and* u_- *such that* (1.7) *and* (1.8) *hold, and furthermore, the maps* $u_0 \mapsto u_+$ *and* $u_0 \mapsto u_-$ *are continuous as functions of* u_0.

Remark. Scattering for (1.1) corresponds to $\|u\|_{L_{t,x}^5(\mathbf{R} \times \mathbf{R}^3)} < \infty$ and scattering for (1.2) corresponds to $\|u\|_{L_{t,x}^{4k}(\mathbf{R} \times \mathbf{R}^2)} < \infty$.

Theorem 1.1. (1.1) *is locally well-posed for any* $u_0 \in H^s(\mathbf{R}^3)$, $s > \frac{1}{2}$ *on some interval* $[-T, T]$, $T(\|u_0\|_{H^s}, s) > 0$. *If* $u_0 \in \dot{H}^{1/2}(\mathbf{R}^3)$ *then* (1.1) *is locally well-posed on some interval* $[-T, T]$, $T(u_0) > 0$, *where* $T(u_0)$ *depends on the profile of the initial data and not just its size. Moreover, for* $\|u_0\|_{\dot{H}^{1/2}(\mathbf{R}^3)}$ *small,* (1.1) *is globally well-posed and scattering.*

The corresponding results also hold for (1.2) *and the critical space* $\dot{H}^{1-\frac{1}{k}}(\mathbf{R}^2)$.

Proof. See [5], [6]. □

Remark. [7] and [8] proved that Theorem 1.1 is sharp.

Remark. [21] proved that (1.1) is globally well-posed and scattering if and only if $\|u(t)\|_{\dot{H}_x^{1/2}(\mathbf{R}^3)}$ is uniformly bounded on its interval of existence. See [48] for the same result when $k = 1$ and $d = 2$.

(1.2) with $k = 1$ is now completely solved. [24] proved that (1.2) is globally well-posed and scattering for any $u_0 \in L^2(\mathbf{R}^2)$, u_0 radial. [16] extended this to nonradial data.

In this paper we show that (1.1) and (1.2) are globally well-posed and scattering for $u_0 \in H_x^s(\mathbf{R}^d)$, u_0 radial, $s > \frac{1}{2}$, and $s > 1 - \frac{1}{k}$ respectively.

We begin with the cubic problem in three dimensions.

Theorem 1.2. *The initial value problem* (1.1) *is globally well-posed and scattering for any* $s > \frac{1}{2}$ *for* u_0 *radial.*

Next, we will prove an explicit upper bound on the scattering size, or $L_{t,x}^4$ norm, for a solution to the two dimensional, cubic problem ($k = 1$) for u_0 radial lying in a subspace of $L_x^2(\mathbf{R}^2)$. Scattering for the two dimensional, radial, cubic problem has already been proved for [24]. See [16] for a proof in the nonradial case. However, no explicit norm was computed in [24] or [16], which we will do here.

Theorem 1.3. *When* $k = 1$ *and* u_0 *is radial,* (1.2) *has a global solution with*

(1.9)
$$\|u\|_{L_{t,x}^4(\mathbf{R}\times\mathbf{R}^2)}^4 \lesssim (\|u_0\|_{\dot{H}^s(\mathbf{R}^2)} + \||x|^s u_0\|_{L^2(\mathbf{R}^2)})^{\frac{8(1-s)}{s}+2}(1 + \|u_0\|_{L^2})^{\frac{4(1-s)}{s}+1}.$$

A solution to the focusing problem

$$iu_t + \Delta u = -|u|^2 u, \qquad u(0,x) = u_0, \tag{1.10}$$

has the scattering size bound

(1.11)
$$\|u\|_{L_{t,x}^4(\mathbf{R}\times\mathbf{R}^2)}^4 \lesssim (\|u_0\|_{\dot{H}^s(\mathbf{R}^2)} + \||x|^s u_0\|_{L^2(\mathbf{R}^2)})^{\frac{8(1-s)}{s}+2}(1 + \|u_0\|_{L^2})^{\frac{4(1-s)}{s}+1} \times (1 - \frac{\|u_0\|_{L^2}^2}{\|Q\|_{L^2}^2})^{-\frac{1}{s}}.$$

Q is the ground state of the focusing problem, that is, the positive solution to

$$\Delta Q + Q^3 = Q. \tag{1.12}$$

Remark. [29] proved a result in this form for $s = 1$.

Finally we prove two dimensional scattering results for (1.2) when $k > 1$.

Theorem 1.4. *The initial value problem (1.2) is globally well-posed and scattering for any $s > 1 - \frac{1}{k}$, u_0 radial.*

1.1. Method of proof

The I-method is used to prove Theorems 1.2, 1.3, and 1.4. A solution to (1.1) conserves the quantities mass,

$$M(u(t)) = \int |u(t,x)|^2 dx = M(u(0)), \tag{1.13}$$

and energy,

$$E(u(t)) = \frac{1}{2} \int |\nabla u(t,x)|^2 dx + \frac{1}{4} \int |u(t,x)|^4 dx. \tag{1.14}$$

Likewise, a solution to (1.2) conserves mass (1.13) and the energy

$$E(u(t)) = \frac{1}{2} \int |\nabla u(t,x)|^2 dx + \frac{1}{2k+2} \int |u(t,x)|^{2k+2} dx. \tag{1.15}$$

(1.14) and (1.15) combined with the local well-posedness theorem (Theorem 1.1) proves that (1.1) and (1.2) are globally well-posed for data in H^1. See [18], [28] for a proof of scattering in the radial case; [10], [12], and [34] for a proof of scattering in the nonradial case for $u_0 \in H^1$.

The reason for the gap between the local well-posedness result of Theorem 1.1 and the regularity needed to prove a global result in [18] ($s = 1$) is due to an absence of a conserved quantity that controls $\|u(t)\|_{\dot{H}^s}$ for $0 < s < 1$. It is true that the momentum, a $\dot{H}^{1/2}$-critical quantity, is conserved, but this quantity does not control the $\dot{H}^{1/2}$ norm.

The first progress in extending the global well-posedness results for data in H^1 to H^s, $s < 1$ came from the Fourier truncation method. [3] proved that the cubic nonlinear initial value problem is globally well-posed in two dimensions for data in H^s, $s > \frac{3}{5}$ when $d = 2$. In three dimensions [4] proved global well-posedness for $s > \frac{11}{13}$ and global well-posedness and scattering for $s > \frac{5}{7}$ when u_0 is radial. In fact, [3], [4] proved something more, namely that for s in the appropriate interval

$$u(t) - e^{it\Delta} u_0 \in H^1(\mathbf{R}^d). \tag{1.16}$$

It was precisely (1.16) that lead to the development of the I-method since (1.16) is false for many dispersive partial differential equations. See [23] for example. Instead, [11] defined an operator $I : H^s(\mathbf{R}^d) \to H^1(\mathbf{R}^d)$. Tracking the change of $E(Iu(t))$, [11] proved global well-posedness for the cubic nonlinear Schrödinger equation when $d = 2$ for $s > \frac{4}{7}$, and when $d = 3$ for $s > \frac{5}{6}$. [12] extended the $d = 3$ result to $s > \frac{5}{6}$. [14] extended this to $s > \frac{5}{7}$, and then [40] extended this result to $s > \frac{2}{3}$.

Both [14] and [40] utilized the linear-nonlinear decomposition. See also [36] for this method in the context of the wave equation. Here we will use the long time Strichartz estimates of [13]. We show that for radial data, the long time Strichartz estimates decay rapidly, and thus can beat any polynomial power of N arising from the I-operator.

The outline of the paper is as follows. In §2 we will recall some linear estimates needed in the proof. In §3 we will describe the I-method and outline the proof of Theorems 1.2 and 1.4. In §4 we will make an induction on frequency argument and prove long time Strichartz estimates for $d = 3$. In §5 we will prove the energy increment in $d = 3$, yielding Theorem 1.2. In §6 we prove Theorem 1.3, obtaining scattering size for the cubic problem in dimension $d = 2$. Then in §7 we will make an induction on frequency argument and prove long-time Strichartz estimates for $d = 2$ when $k > 1$. Finally in §8 we will prove the energy increment in $d = 2$, yielding Theorem 1.4.

At this point it is necessary to mention some notation used in the paper. This notation was used in [11]. The expression $A \lesssim_B D$ indicates $A \leq C(B)D$, where $C(B)$ is some constant. When we say $A \lesssim_{\|u_0\|_{H^s}} B$ or $A \lesssim_{\|u_0\|_{H^s},k} B$ we mean that $A \leq C(\|u_0\|_{H^s}, s, k)B$. $A \sim B$ denotes $A \lesssim B$ and $B \lesssim A$.

We will also use the notation $A \lesssim B^{a+}$. This means that for any $\epsilon > 0$, there exists $C(\epsilon)$ such that $A \leq C(\epsilon)B^{a+\epsilon}$. We will also use expressions like $\|u\|_{L^{p+}} \lesssim A$, which means that $\|u\|_{L^{p+\epsilon}} \leq C(\epsilon)A$. $\|u\|_{L^{p+}} \gtrsim A$ has the obvious definition.

Throughout the paper it is unnecessary to distinguish between u and $\bar{u}$. Therefore, we will often write expressions like $|u|^2 u$ as u^3 for convenience.

2. Linear estimates

In this section we mention a number of estimates for the linear Schrödinger equation. None of the results in this section are new.

2.1. Sobolev spaces

Definition 2.1 (Littlewood-Paley decomposition)**.** *Take* $\psi \in C_0^\infty(\mathbf{R}^d)$*,* $\psi(x) = 1$ *for* $|x| \leq 1$*,* $\psi = 0$ *for* $|x| > 2$*, where* $\psi(x)$ *is radial and decreasing. Then for any* j *let*

$$\phi_j(x) = \psi(2^{-j}x) - \psi(2^{-j+1}x). \tag{2.1}$$

Let P_j *be the Fourier multiplier given by*

$$\widehat{P_j f}(\xi) = \phi_j(\xi)\hat{f}(\xi). \tag{2.2}$$

This gives the Littlewood-Paley decomposition

$$f = \sum_{j=-\infty}^{\infty} P_j f, \tag{2.3}$$

at least in the L^2 *sense.*

The Littlewood-Paley decomposition is quite useful since

Theorem 2.1 (Littlewood-Paley theorem)**.** *For any* $1 < p < \infty$*,*

$$\|f\|_{L^p(\mathbf{R}^d)} \sim_{p,d} \|(\sum_{j=-\infty}^{\infty} |P_j f|^2)^{1/2}\|_{L^p(\mathbf{R}^d)}. \tag{2.4}$$

Definition 2.2 (Sobolev spaces)**.** *For* $s \in \mathbf{R}$ *the Sobolev space* $\dot{H}^s(\mathbf{R}^d)$ *is the space of functions whose Fourier transform has finite weighted* L^2 *norm,*

$$\|f\|_{\dot{H}^s(\mathbf{R}^d)} = \||\xi|^s \hat{f}(\xi)\|_{L^2(\mathbf{R}^d)}, \tag{2.5}$$

where

$$\hat{f}(\xi) = (2\pi)^{-d/2} \int e^{-ix\cdot\xi} f(x) dx. \tag{2.6}$$

We define the inhomogeneous space

$$\|f\|_{H^s(\mathbf{R}^d)} = \|(1 + |\xi|^2)^{s/2} \hat{f}(\xi)\|_{L^2(\mathbf{R}^d)}. \tag{2.7}$$

Notice that

(2.8)

$$\|P_j f\|_{L^2(\mathbf{R}^d)} \lesssim 2^{-js} \|f\|_{\dot{H}^s(\mathbf{R}^d)}, \qquad \|P_j f\|_{L^2(\mathbf{R}^d)} \lesssim \inf(2^{-js}, 1) \|f\|_{H^s(\mathbf{R}^d)}.$$

Remark. (2.8) is called Bernstein's inequality.

It follows from Hölder's inequality that for $2 \leq p \leq \infty$,

$$\|P_j f\|_{L^p(\mathbf{R}^d)} \lesssim_d 2^{jd(\frac{1}{2} - \frac{1}{p})} \|P_j f\|_{L^2(\mathbf{R}^d)}. \tag{2.9}$$

Then for $1 < p < \infty$, $s = d(\frac{1}{2} - \frac{1}{p})$,

$$\|f\|_{L^p(\mathbf{R}^d)} \lesssim_{s,d} \|f\|_{\dot{H}^s(\mathbf{R}^d)}. \tag{2.10}$$

We also have the radial Sobolev embedding

$$\||x| P_j f\|_{L^\infty(\mathbf{R}^3)} \lesssim \|P_j f\|_{\dot{H}^{1/2}(\mathbf{R}^3)}. \tag{2.11}$$

See [37], [38], [44], [45], and many other sources for more details on Sobolev spaces.

2.2. Strichartz estimates

Theorem 2.2. *Let $e^{it\Delta}$ be the solution operator to the linear evolution equation $(i\partial_t + \Delta)u = 0$. That is, $u = e^{it\Delta} u_0$ solves*

$$(i\partial_t + \Delta)u = 0, \qquad u(0, x) = u_0. \tag{2.12}$$

When $d = 3$ define

$$(p, q) \in \mathcal{A}_3 \quad \text{if and only if} \quad 2 \leq p \leq \infty, \quad \text{and} \quad \frac{2}{p} = 3(\frac{1}{2} - \frac{1}{p}). \tag{2.13}$$

When $d = 2$ define

$$(p, q) \in \mathcal{A}_2 \quad \text{if and only if} \quad 2 < p \leq \infty, \quad \text{and} \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{2}. \tag{2.14}$$

If (p, q) lies in $\mathcal{A}_d$ then we say that (p, q) is an admissible pair.

Let p' denote the Lebesgue dual, that is $\frac{1}{p} + \frac{1}{p'} = 1$. Then if (p,q) and $(\tilde{p},\tilde{q})$ are admissible pairs, when $d = 2$,

$$\begin{aligned} \|e^{it\Delta}u_0\|_{L_t^p L_x^q(\mathbf{R}\times\mathbf{R}^2)} &\lesssim_p \|u_0\|_{L^2(\mathbf{R}^2)}, \\ \|\int_0^t e^{i(t-\tau)\Delta}F(\tau)d\tau\|_{L_t^p L_x^q(I\times\mathbf{R}^2)} &\lesssim_{p,\tilde{p}} \|F\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}(I\times\mathbf{R}^2)}, \end{aligned} \tag{2.15}$$

and when $d = 3$,

$$\begin{aligned} \|e^{it\Delta}u_0\|_{L_t^p L_x^q(\mathbf{R}\times\mathbf{R}^3)} &\lesssim \|u_0\|_{L^2(\mathbf{R}^3)}, \\ \|\int_0^t e^{i(t-\tau)\Delta}F(\tau)d\tau\|_{L_t^p L_x^q(I\times\mathbf{R}^3)} &\lesssim \|F\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}(I\times\mathbf{R}^3)}, \end{aligned} \tag{2.16}$$

Proof. [39] proved this theorem in the case $p = q$, $\tilde{p} = \tilde{q}$. See [9], [19], and [47] for a proof of the general result, $p > 2$. [22] proved the endpoint result $p = 2$ when $d = 3$. [41] gives a nice description of the overall theory. □

Because of this fact it is convenient, especially in three dimensions, to work with the Strichartz space and the dual Strichartz space.

Definition 2.3 (Strichartz space)**.** *Let S^0 be the Strichartz space*

$$S^0(I\times\mathbf{R}^3) = L_t^\infty L_x^2(I\times\mathbf{R}^3) \cap L_t^2 L_x^6(I\times\mathbf{R}^3). \tag{2.17}$$

Let N^0 be the dual

$$N^0(I\times\mathbf{R}^3) = L_t^1 L_x^2(I\times\mathbf{R}^3) + L_t^2 L_x^{6/5}(I\times\mathbf{R}^3). \tag{2.18}$$

Then Theorem 2.2 implies

$$\begin{aligned} \|e^{it\Delta}u_0\|_{S^0(\mathbf{R}\times\mathbf{R}^3)} &\lesssim \|u_0\|_{L^2(\mathbf{R}^3)}, \\ \|\int_0^t e^{i(t-\tau)\Delta}F(\tau)d\tau\|_{S^0(I\times\mathbf{R}^3)} &\lesssim \|F\|_{N^0(I\times\mathbf{R}^3)}. \end{aligned} \tag{2.19}$$

We will also utilize the local smoothing estimate of [35]. Suppose ψ is the same ψ as in Definition 2.1. Then

$$\|\psi(\frac{x}{R})e^{it\Delta}(P_j u_0)\|_{L^2_{t,x}(I\times\mathbf{R}^d)} \lesssim 2^{-j/2}R^{1/2}\|P_j u_0\|_{L^2(\mathbf{R}^d)}. \tag{2.20}$$

The dual of (2.20) is

$$\|\int_I e^{-it\Delta}\psi(\frac{x}{R})(P_j F(\tau))\|_{L^2_x(\mathbf{R}^d)} \lesssim 2^{-j/2}R^{1/2}\|\psi(\frac{x}{R})P_j F\|_{L^2_{t,x}(I\times\mathbf{R}^d)}. \tag{2.21}$$

Interpolating (2.15) and (2.21), for any $q < 2$, if F is supported on $|x| \leq R$,

$$\| |\nabla|^{1-\frac{1}{q}} \int_0^\infty e^{-it\Delta} F(t,x) dt \|_{L^2(\mathbf{R}^3)} \lesssim R^{1-\frac{1}{q}} \|F\|_{L_t^q L_x^2(\mathbf{R}\times\mathbf{R}^3)}. \tag{2.22}$$

Now let $\chi(x) = \psi(\frac{x}{2}) - \psi(x)$. For any $0 < R < \infty$ and $x \in \mathbf{R}^3$,

$$1 = \psi(Rx) + \sum_{j=0}^{\infty} \chi(2^{-j} Rx). \tag{2.23}$$

Then by (2.22),

$$\begin{aligned} \| |\nabla|^{1-\frac{1}{q}} \int_I e^{-it\Delta} F(t) dt \|_{L_x^2(\mathbf{R}^3)} \\ \lesssim R^{\frac{1}{q}-1} \|\psi(Rx)F\|_{L_t^q L_x^2} + R^{\frac{1}{q}-1} \sum_{j\geq 0} 2^{j(1-\frac{1}{q})} \|\chi(2^{-j}Rx)F\|_{L_t^q L_x^2}. \end{aligned} \tag{2.24}$$

To simplify notation, let

$$\|F\|_{X_R(I\times\mathbf{R}^d)} = R^{\frac{1}{q}-1} \|\psi(Rx)F\|_{L_t^q L_x^2} + R^{\frac{1}{q}-1} \sum_{j\geq 0} 2^{j(1-\frac{1}{q})} \|\chi(2^{-j}Rx)F\|_{L_t^q L_x^2}. \tag{2.25}$$

2.3. U_Δ^2 spaces

The U_Δ^2 spaces are a class of function spaces first introduced in [43] to study wave maps. [26] and [27] applied these spaces to nonlinear Schrödinger problems. See [20] for a general description of these spaces. These spaces are quite useful to critical problems since the $X^{s,b}$ spaces of [1] and [2] (see also [17]) are not scale invariant except at $b = \frac{1}{2}$, which has the same difficulty as the failure of the embedding $\dot{H}^{1/2}(\mathbf{R}) \subset L^\infty(\mathbf{R})$.

Definition 2.4 (U_Δ^p spaces)**.** *Let $1 \leq p < \infty$. Let U_Δ^p be an atomic space whose atoms are piecewise solutions to the linear equation,*

$$u_\lambda = \sum_k 1_{[t_k, t_{k+1})} e^{it\Delta} u_k, \qquad \sum_k \|u_k\|_{L^2}^p = 1. \tag{2.26}$$

Then for any $1 \leq p < \infty$,

$$\|u\|_{U_\Delta^p} = \inf\{\sum_\lambda |c_\lambda| : u = \sum_\lambda c_\lambda u_\lambda, u_\lambda \text{ are } U_\Delta^p \text{ atoms}\}. \tag{2.27}$$

For any $1 \leq p < \infty$, $U^p_\Delta \subset L^\infty L^2$. Additionally, U^p_Δ functions are continuous except at countably many points and right continuous everywhere.

Theorem 2.3. *If u solves*

$$iu_t + \Delta u = F_1 + F_2, \qquad u(0,x) = u_0, \tag{2.28}$$

on the interval $0 \in I \subset \mathbf{R}$, then for $q < 2$,

$$\| |\nabla|^{1-\frac{1}{q}} u \|_{U^2_\Delta(I \times \mathbf{R}^d)} \lesssim_q \| |\nabla|^{1-\frac{1}{q}} u_0 \|_{L^2(\mathbf{R}^d)} + \|F_1\|_{X_R(I \times \mathbf{R}^d)} + \| |\nabla|^{1-\frac{1}{q}} F_2 \|_{L_t^{2+} L_x^{\frac{2d}{d-2}-}(I \times \mathbf{R}^d)}. \tag{2.29}$$

Proof. This is proved using Strichartz estimates, (2.21), and the Christ-Kiselev lemma (see [9]). □

Remark. The notation

$$\|A\|_{L_t^{p+} L_x^{q-}} \lesssim B, \tag{2.30}$$

means that for admissible pair $(\tilde{p}, \tilde{q})$ close to (p,q) with $\tilde{p} > p$ and $\tilde{q} < q$,

$$\|A\|_{L_t^{\tilde{p}} L_x^{\tilde{q}}} \lesssim B, \tag{2.31}$$

for an admissible pair $(\tilde{p}, \tilde{q})$, where the implicit constant can go to infinity as $(\tilde{p}, \tilde{q}) \to (p,q)$.

3. Description of the I-method and outline of the proof

Since there are no known conserved quantities that control $\|u\|_{\dot{H}^s}$ for $0 < s < 1$, we utilize the by now well known modified energy of [11], $E(Iu(t))$, where I is the I-operator and E is the usual energy.

Definition 3.1 (I-operator)**.** *Let $I : H^s(\mathbf{R}^d) \to H^1(\mathbf{R}^d)$ be the Fourier multiplier*

$$\widehat{If}(\xi) = m_N(\xi) \hat{f}(\xi), \tag{3.1}$$

where

$$m_N(\xi) = \begin{cases} 1 & \text{if } |\xi| \leq N, \\ \frac{N^{1-s}}{|\xi|^{1-s}} & \text{if } |\xi| \geq 2N. \end{cases} \tag{3.2}$$

To simplify notation N is suppressed for the rest of the paper.

Remark. It is convenient to write

$$P_{\leq N}u = \sum_{j:2^j \leq N} P_j u. \tag{3.3}$$

Let $P_{>N}u = u - P_{\leq N}u$. This notation may be abbreviated $u_{\leq N} = P_{\leq N}u$.

There is an obvious tradeoff here. Taking $N = \infty$, $E(Iu(t))$ is the energy of u, which is a conserved quantity. However, for a general $u \in \dot{H}^s$, $s < 1$, $E(Iu(0)) = \infty$. In general, as N increases, $\frac{d}{dt}E(Iu(t))$ decreases and $E(Iu(t))$ increases. Therefore, the question of global well-posedness revolves around which side will win this tug of war. More precisely, by the Sobolev embedding theorem, when $d = 3$,

$$E(Iu(t)) \lesssim \|Iu\|_{\dot{H}^1(\mathbf{R}^3)}^2 + \|Iu\|_{\dot{H}^1(\mathbf{R}^3)}^2 \|u\|_{\dot{H}^{1/2}(\mathbf{R}^3)}^2, \tag{3.4}$$

and when $d = 2$,

$$E(Iu(t)) \lesssim \|Iu\|_{\dot{H}^1(\mathbf{R}^2)}^2 + \|Iu\|_{\dot{H}^1(\mathbf{R}^2)}^2 \|u\|_{\dot{H}^{1-\frac{1}{k}}(\mathbf{R}^2)}^{2k}. \tag{3.5}$$

Therefore,

$$E(Iu(0)) \lesssim C(\|u(0)\|_{H^s}) N^{2(1-s)}. \tag{3.6}$$

Meanwhile,

$$\|u(t)\|_{H^s(\mathbf{R}^d)}^2 \lesssim E(Iu(t)) + M(Iu(t)). \tag{3.7}$$

Since $M(Iu(t)) \leq M(u(t)) = M(u(0))$, a uniform bound on $E(Iu(t))$ for all t yields a uniform bound on $\|u(t)\|_{H^s(\mathbf{R}^3)}$.

It is convenient to use the rescaling in (1.3) so that $E(Iu(t)) \leq 1$. Indeed, by (1.4), when $d = 2$ there exists $\lambda^{s-1+\frac{1}{k}} \sim C(\|u(0)\|_{H^s(\mathbf{R}^2)})N^{s-1}$ and when $d = 3$ there exists $\lambda^{s-\frac{1}{2}} \sim C(\|u(0)\|_{H^s(\mathbf{R}^3)})N^{s-1}$ such that

$$E(Iu_\lambda(0)) \leq \frac{1}{2}. \tag{3.8}$$

Remark. $C(\|u(0)\|_{H^s(\mathbf{R}^d)})$ is a constant that may change from line to line.

Then by (1.4),

$$\|u_{\lambda}(0)\|_{L_x^2(\mathbf{R}^3)} \lesssim C(\|u(0)\|_{H^s(\mathbf{R}^3)}) N^{\frac{1-s}{2s-1}} \|u(0)\|_{L_x^2(\mathbf{R}^3)}, \tag{3.9}$$

and

$$\|u_{\lambda}(0)\|_{L_x^2(\mathbf{R}^2)} \lesssim C(\|u(0)\|_{H^s(\mathbf{R}^2)}) N^{\frac{k-1}{k} \cdot \frac{1-s}{s-1+\frac{1}{k}}} \|u(0)\|_{L_x^2(\mathbf{R}^2)}. \tag{3.10}$$

λ is suppressed until the end of the paper, so for now u refers to u_{λ} until otherwise indicated.

Next recall the interaction Morawetz estimate.

Theorem 3.1 (Interaction Morawetz estimate)**.** *Suppose u is a solution to (1.1) or (1.2) on some interval J. Then*

$$\| |\nabla|^{\frac{3-d}{2}} |u|^2 \|_{L_{t,x}^2(J \times \mathbf{R}^d)}^2 \lesssim \|u\|_{L_t^{\infty} L_x^2(J \times \mathbf{R}^d)}^2 \|u\|_{L_t^{\infty} \dot{H}_x^{1/2}(J \times \mathbf{R}^d)}^2. \tag{3.11}$$

Proof. This was proved in three dimensions by [12]. [10] and [34] independently proved (3.11) in dimensions one and two. [42] proved the interaction Morawetz estimate in dimensions $d \geq 4$, a result that will not be needed here. □

(3.11) is extremely useful due to a local well-posedness result of [12].

Lemma 3.2. *If $E(Iu(a_l)) \leq 1$, $J_l = [a_l, b_l]$, and $\|u\|_{L_{t,x}^4(J_l \times \mathbf{R}^3)} \leq \epsilon$ for some $\epsilon > 0$ sufficiently small, then*

$$\|\nabla I u\|_{S^0(J_l \times \mathbf{R}^3)} \lesssim 1. \tag{3.12}$$

Proof. See [12] or [14]. □

A similar result is available in dimension $d = 2$.

Lemma 3.3. *If $k > 1$, $E(Iu(a_l)) \leq 1$, $J_l = [a_l, b_l]$, and $\| |\nabla|^{1/2} |u|^2 \|_{L_{t,x}^2(J_l \times \mathbf{R}^2)} \leq \epsilon$ for some $\epsilon(k) > 0$ sufficiently small, then for $(p, q) \in \mathcal{A}_2$,*

$$\|\nabla I u\|_{L_t^p L_x^q(J_l \times \mathbf{R}^2)} \lesssim_{p,k} 1. \tag{3.13}$$

Proof. By the Sobolev embedding theorem

$$\|u\|_{L_t^4 L_x^8(J_l \times \mathbf{R}^2)}^2 \lesssim \| |\nabla|^{1/2} |u|^2 \|_{L_{t,x}^2(J_l \times \mathbf{R}^2)} \leq \epsilon. \tag{3.14}$$

Interpolating (3.14) with $\|P_j u\|_{L_x^\infty(J_l\times\mathbf{R}^2)} \lesssim 2^j \|P_j u\|_{L_x^2(\mathbf{R}^2)}$, combined with the Littlewood-Paley theorem proves

$$\|Iu\|_{L_t^{3k}L_x^{6k}(J_l\times\mathbf{R}^2)} \lesssim \epsilon^{\frac{4}{3k}} \|\nabla Iu\|^{1-\frac{4}{3k}}_{L_t^\infty L_x^2(J_l\times\mathbf{R}^2)}. \tag{3.15}$$

Also by Bernstein's inequality and (3.2),

$$\|(1-I)u\|_{L_t^{3k}L_x^{6k}(J_l\times\mathbf{R}^2)} \lesssim N^{-\frac{1}{k}} \|\nabla Iu\|_{L_t^{3k}L_x^{\frac{6k}{3k-2}}(J_l\times\mathbf{R}^2)}. \tag{3.16}$$

Then by Strichartz estimates (Theorem 2.2),

$$\begin{aligned} &\|\nabla Iu\|_{L_t^{3k}L_x^{\frac{6k}{3k-2}}\cap L_t^\infty L_x^2(J_l\times\mathbf{R}^2)} \\ &\quad\lesssim \|\nabla Iu(a_l)\|_{L_x^2(\mathbf{R}^2)} + (\epsilon^{\frac{4}{3k}}\|\nabla Iu\|^{\frac{3k-4}{3k}}_{L_t^\infty L_x^2(J_l\times\mathbf{R}^2)} \\ &\qquad + N^{-\frac{1}{k}}\|\nabla Iu\|_{L_t^{3k}L_x^{\frac{6k}{3k-2}}(J_l\times\mathbf{R}^2)})^{2k}\|\nabla Iu\|_{L_t^\infty L_x^2(J_l\times\mathbf{R}^2)}. \end{aligned} \tag{3.17}$$

Since N is large and $\epsilon>0$ is small the proof is complete. □

(3.17) also implies

$$\|\nabla Iu\|_{U^2_\Delta(J_l\times\mathbf{R}^2)} \lesssim_k 1, \tag{3.18}$$

and similarly Lemma 3.3 implies

$$\|\nabla Iu\|_{U^2_\Delta(J_l\times\mathbf{R}^3)} \lesssim 1. \tag{3.19}$$

Theorems 1.2, 1.3, and 1.4 are then proved by a bootstrapping estimate. Let

$$J = \{t : E(Iu(\tau)) \leq 1 \quad \text{for all} \quad 0\leq\tau\leq t\}. \tag{3.20}$$

J is clearly nonempty since $0\in J$. Moreover, standard local well-posedness theory implies that J is a closed interval. Therefore, to prove $J=[0,\infty)$ it suffices to show that J is open. By (3.2), interpolation, and Bernstein's inequality,

$$\|P_{\leq N}u(t)\|_{\dot H^{1/2}(\mathbf{R}^d)} \lesssim \|Iu(t)\|^{1/2}_{\dot H^1(\mathbf{R}^d)}\|P_{\leq N}u(t)\|^{1/2}_{L^2(\mathbf{R}^d)}, \tag{3.21}$$

and

$$\|P_{>N}u(t)\|_{\dot H^{1/2}(\mathbf{R}^d)} \lesssim N^{-1/2}\|Iu\|_{\dot H^1(\mathbf{R}^d)}. \tag{3.22}$$

Therefore if J is an interval such that $E(Iu(t)) \leq 1$ on J, then (3.9), (3.11), (3.21), (3.22), and the conservation of mass imply that

$$\|u\|_{L^4_{t,x}(J\times\mathbf{R}^3)}^4 \lesssim C(\|u(0)\|_{H^s(\mathbf{R}^3)}) N^{\frac{3(1-s)}{2s-1}}, \tag{3.23}$$

and

$$\|u\|_{L_t^4 L_x^8(J\times\mathbf{R}^2)}^4 \lesssim C(\|u(0)\|_{H^s(\mathbf{R}^2)}, k) N^{\frac{k-1}{k}\cdot\frac{3(1-s)}{s-1+\frac{1}{k}}}. \tag{3.24}$$

To close the bootstrap, we prove that for $N(d, k, \|u(0)\|_{H^s})$ sufficiently large,

$$\int_J \frac{d}{dt} E(Iu(t))dt \leq \frac{1}{10}. \tag{3.25}$$

(3.25) and Theorem 1.1 imply that for any $T > 0$ there exists $\delta(T) > 0$ such that if $[0,T] \subset J$, $[0, T+\delta) \subset J$. Therefore J is open and thus $J = [0, \infty)$. Finally, we can recover the $\|u(t)\|_{H^s}$ bound by rescaling back and then computing the $\|u(t)\|_{H^s}$ norm from the bounds on $M(u(t))$ and $E(Iu(t))$ after rescaling.

(3.25) is proved using long time Strichartz estimates. Estimates of this form were introduced in [13] within the context of the mass-critical nonlinear Schrödinger initial value problem. The long time Strichartz estimates have been utilized in subsequent papers ([15], [16], [25], [30], [31], [32], [46]).

4. Induction on frequency and long time Strichartz estimates in three dimensions

Theorem 4.1. *Let $0 \in J$ be an interval such that $E(Iu(t)) \leq 1$ on J. Then for $N(s, \|u_0\|_{H^s})$ sufficiently large,*

$$\|P_{>\frac{N}{8}} \nabla Iu\|_{L_t^2 L_x^6(J\times\mathbf{R}^3)} \lesssim 1. \tag{4.1}$$

Proof. As in [13], this theorem is proved using an induction on frequency argument. First observe that

$$P_{>M}(|u_{\leq\frac{M}{8}}|^2 u_{\leq\frac{M}{8}}) = 0. \tag{4.2}$$

Remark. This fact is why this method does not immediately carry over to a non-algebraic nonlinearity, in other words, when p is not equal to $2k$ for some positive integer k.

Decompose

$$P_{>M}F(u) = P_{>M}O((u_{>\frac{M}{8}})(u_{\leq \frac{M}{8}})^2) + P_{>M}O((u_{>\frac{M}{8}})^2 u). \tag{4.3}$$

By the product rule and the fact that ∇I is a Fourier multiplier whose symbol is increasing as $|\xi| \nearrow \infty$, if $M \leq N$,

$$\begin{aligned} &\nabla I P_{>M} O((u_{>\frac{M}{8}})(u_{\leq \frac{M}{8}})^2) \\ &\quad = O((\nabla I P_{>\frac{M}{8}} u)(P_{\leq \frac{M}{8}} u)^2) + O((I P_{>\frac{M}{8}} u)(\nabla u_{\leq \frac{M}{8}})(u_{\leq \frac{M}{8}})). \end{aligned} \tag{4.4}$$

Then by (2.29),

$$\begin{aligned} &\|\nabla I P_{>M} u(t)\|_{U^2_\Delta(J\times\mathbf{R}^3)} \\ &\quad\lesssim \|\nabla I P_{>M} u(0)\|_{L^2_x(\mathbf{R}^3)} + \|\nabla I P_{>M} O((u_{>\frac{M}{8}})^2 u)\|_{L_t^{2-} L_x^{6/5+}(J\times\mathbf{R}^3)} \\ &\quad\quad + \|P_{>M} O((u_{>\frac{M}{8}})(\nabla u_{\leq \frac{M}{8}})(u_{\leq \frac{M}{8}}))\|_{L_t^{2-} L_x^{6/5+}} \\ &\quad\quad + \frac{1}{M^{1-\frac{1}{q}}} \|P_{>M} O((\nabla I u_{>\frac{M}{8}})(u_{\leq \frac{M}{8}})^2)\|_{X_R}, \end{aligned} \tag{4.5}$$

for some R to be specified later.

First observe that since $E(Iu(t)) \leq 1$ for all $t \in J$,

$$\|\nabla I P_{>M} u(0)\|_{L^2_x(\mathbf{R}^3)} \lesssim 1. \tag{4.6}$$

Again using the properties of ∇I, choosing $\delta(\epsilon) > 0$ so that $(2-\epsilon, \frac{6}{5} + \delta(\epsilon))$ is the dual of an admissible pair, and subsequent ϵ and $\delta(\epsilon)$ to correspond with Hölder's inequality,

(4.7)
$$\begin{aligned} &\|\nabla I ((P_{>\frac{M}{8}} u)^2 u)\|_{L_t^{2-\epsilon} L_x^{6/5+\delta(\epsilon)}(J\times\mathbf{R}^3)} \\ &\quad\lesssim \|\nabla I u\|_{L_t^{\infty-\epsilon} L_x^{2+\delta(\epsilon)}(J\times\mathbf{R}^3)} \|P_{>\frac{M}{8}} u\|^2_{L_t^4 L_x^6(J\times\mathbf{R}^3)} \end{aligned}$$

(4.8)
$$+ \|\nabla I P_{>\frac{M}{8}} u\|_{L_t^2 L_x^6(J\times\mathbf{R}^3)} \|P_{>\frac{M}{8}} u\|_{L_t^\infty L_x^2(J\times\mathbf{R}^3)} \|P_{\leq N} u\|_{L_t^{\infty-\epsilon} L_x^{6+\delta(\epsilon)}(J\times\mathbf{R}^3)}$$

(4.9)
$$+ \|\nabla I P_{>\frac{M}{8}} u\|_{L_t^2 L_x^6(J\times\mathbf{R}^3)} \|P_{>\frac{M}{8}} u\|_{L_t^\infty L_x^3(J\times\mathbf{R}^3)} \|P_{>N} u\|_{L_t^{\infty-\epsilon} L_x^{3+\delta(\epsilon)}(J\times\mathbf{R}^3)}.$$

Remark. The notation $\infty - \epsilon$ refers to a very large number, specifically $\frac{2(2-\epsilon)}{\epsilon}$.

Now by the Sobolev embedding theorem and interpolation,

(4.10)
$$\|P_{>\frac{M}{8}}u\|_{L_t^4L_x^6(J\times\mathbf{R}^3)} \lesssim \||\nabla|^{1/2}P_{>\frac{M}{8}}u\|_{L_t^2L_x^6(J\times\mathbf{R}^3)}^{1/2}\||\nabla|^{1/2}P_{>\frac{M}{8}}u\|_{L_t^\infty L_x^2(J\times\mathbf{R}^3)}^{1/2}.$$

Therefore, by Bernstein's inequality, $E(Iu(t))\leq 1$, and (3.2),

$$(4.10) \lesssim M^{-1/2}\|\nabla IP_{>\frac{M}{8}}u\|_{L_t^2L_x^6(J\times\mathbf{R}^3)}^{1/2}. \tag{4.11}$$

Therefore,

$$\begin{aligned}(4.7)+(4.8)+(4.9) &\lesssim \|\nabla IP_{>\frac{M}{8}}u\|_{L_t^2L_x^6(J\times\mathbf{R}^3)}(M^{-1}\|\nabla Iu\|_{L_t^{\infty-\epsilon}L_x^{2+\delta(\epsilon)}} \\ &\quad + M^{-1}\|P_{\leq N}u\|_{L_t^{\infty-\epsilon}L_x^{6+\delta(\epsilon)}} + M^{-1/2}\|P_{>N}u\|_{L_t^{\infty-\epsilon}L_x^{3+\delta(\epsilon)}}).\end{aligned} \tag{4.12}$$

Now by (3.19), (3.23), $E(Iu(t))\leq 1$ on J, and Lemma 3.3,

(4.13)
$$\|Iu\|_{L_t^{\infty-\epsilon}L_x^{6+\delta(\epsilon)}(J\times\mathbf{R}^3)} + \|\nabla Iu\|_{L_t^{\infty-\epsilon}L_x^{2+\delta(\epsilon)}(J\times\mathbf{R}^3)} \lesssim_{s,\|u_0\|_{H^s}(\mathbf{R}^3)} N^{\frac{3(1-s)}{2s-1}\cdot\frac{\epsilon}{2(2-\epsilon)}}$$

and

$$\|P_{>N}u\|_{L_t^{\infty-\epsilon}L_x^{3+\delta(\epsilon)}(J\times\mathbf{R}^3)} \lesssim N^{-\frac{1}{2}+\frac{3(1-s)}{2s-1}\cdot\frac{\epsilon}{2(2-\epsilon)}}. \tag{4.14}$$

Therefore, if $M\leq N$,

$$(4.12) \lesssim_{s,\|u_0\|_{H^s(\mathbf{R}^3)}} M^{-1}N^{\frac{3(1-s)}{2s-1}\cdot\frac{\epsilon}{2(2-\epsilon)}}\|\nabla IP_{>\frac{M}{8}}u\|_{U_\Delta^2(J\times\mathbf{R}^3)}. \tag{4.15}$$

Similarly,

$$\begin{aligned} &\|P_{>M}O((u_{>\frac{M}{8}})(\nabla u_{\leq\frac{M}{8}})(u_{\leq\frac{M}{8}}))\|_{L_t^{2-\epsilon}L_x^{6/5+\delta(\epsilon)}} \\ &\lesssim \|\nabla u_{\leq\frac{M}{8}}\|_{L_t^{\infty-\epsilon}L_x^{2+\delta(\epsilon)}}\|u_{>\frac{M}{8}}\|_{L_t^2L_x^6}\|u_{\leq\frac{M}{8}}\|_{L_t^\infty L_x^6} \\ &\lesssim_{s,\|u_0\|_{H^s(\mathbf{R}^3)}} M^{-1}N^{\frac{3(1-s)}{2s-1}\cdot\frac{\epsilon}{2(2-\epsilon)}}\|\nabla IP_{>\frac{M}{8}}u\|_{U_\Delta^2(J\times\mathbf{R}^3)}.\end{aligned} \tag{4.16}$$

It only remains to analyze

$$\frac{1}{M^{1-\frac{1}{q}}}\|(\nabla IP_{>\frac{M}{8}}u)(u_{\leq\frac{M}{8}})^2\|_{X_R}. \tag{4.17}$$

Remark. For notational convenience, choose $q=2-\epsilon$.

It is here that the radial symmetry of u is utilized. Recall that for any $\frac{1}{2} < s < \frac{3}{2}$, the radial Sobolev embedding implies

$$\| |x|^{3/2-s} u \|_{L^\infty_x(\mathbf{R}^3)} \lesssim \|u\|_{\dot{H}^s(\mathbf{R}^3)}. \tag{4.18}$$

Interpolating this with

$$\|Iu\|^4_{L^4_t L^\infty_x(J\times\mathbf{R}^3)} \lesssim_{\|u_0\|_{H^s(\mathbf{R}^3)}} N^{\frac{3(1-s)}{2s-1}}, \tag{4.19}$$

which is a consequence of (3.19) and Strichartz estimates, along with (3.9), implies that

$$\| |x|^{1/2} Iu \|_{L^{\infty-\epsilon}_t L^\infty_x(J\times\mathbf{R}^3)} \lesssim_{s,\|u_0\|_{H^s(\mathbf{R}^3)}} N^{\frac{3(1-s)}{2s-1}\cdot\frac{\epsilon}{2(2-\epsilon)}} N^{\frac{\epsilon}{2-3\epsilon}\cdot\frac{1-s}{2s-1}}. \tag{4.20}$$

Now choose $R = N$. By (2.20), (4.20), and (4.18),

$$\begin{aligned}
&R^{\frac{1}{q}-1} M^{\frac{1}{q}-1} \|\psi(Rx)(\nabla I P_{>\frac{M}{8}} u)(u_{\leq \frac{M}{8}})^2\|_{L^q_t L^2_x(J\times\mathbf{R}^3)} \\
&\lesssim R^{\frac{1}{q}-1} M^{\frac{1}{q}-1} \|\psi(Rx)(\nabla I P_{>\frac{M}{8}} u)\|_{L^2_{t,x}(J\times\mathbf{R}^3)} \|u_{\leq\frac{M}{8}}\|^{\frac{2\epsilon}{(2-\epsilon)}}_{L^4_t L^\infty_x} \|u_{\leq\frac{M}{8}}\|^{\frac{4-4\epsilon}{2-\epsilon}}_{L^\infty_{t,x}} \\
&\lesssim_{s,\|u_0\|_{H^s}} R^{\frac{1}{q}-1} R^{-\frac{1}{2}} M^{\frac{1}{q}-1} M^{-\frac{1}{2}} N^{\frac{3(1-s)}{2s-1}\cdot\frac{2\epsilon}{2-\epsilon}} M^{\frac{2-2\epsilon}{2-\epsilon}} \|\nabla I P_{>\frac{M}{8}} u\|_{U^2_\Delta(J\times\mathbf{R}^3)} \\
&= N^{\frac{-4+3\epsilon}{2(2-\epsilon)}} M^{\frac{-4+3\epsilon}{2(2-\epsilon)}} N^{\frac{3(1-s)}{2s-1}\cdot\frac{2\epsilon}{2-\epsilon}} M^{\frac{2-2\epsilon}{2-\epsilon}} \|\nabla I P_{>\frac{M}{8}} u\|_{U^2_\Delta(J\times\mathbf{R}^3)}.
\end{aligned} \tag{4.21}$$

Also, by (2.20) and (4.20),

$$\begin{aligned}
&M^{\frac{1}{q}-1} \sum_{j\geq 0} R^{\frac{1}{q}-1} 2^{j(1-\frac{1}{q})} \|\chi(2^{-j}Rx)(\nabla I P_{>\frac{M}{8}} u)(u_{\leq\frac{M}{8}})^2\|_{L^q_t L^2_x} \\
&\lesssim M^{\frac{1}{q}-1} \sum_{j\geq 0} R 2^{-j} R^{\frac{1}{q}-1} 2^{j(1-\frac{1}{q})} \|\chi(2^{-j}Rx)(\nabla I P_{>\frac{M}{8}} u)\|_{L^2_{t,x}} \\
&\quad \times \| |x|^{1/2} Iu \|_{L^{\infty-\epsilon}_t L^\infty_x} \| |x|^{1/2} Iu \|_{L^\infty_{t,x}} \\
&\lesssim_{s,\|u_0\|_{H^s(\mathbf{R}^3)}} R^{\frac{1}{q}-\frac{1}{2}} M^{\frac{1}{q}-\frac{3}{2}} \sum_{j\geq 0} 2^{j(\frac{1}{2}-\frac{1}{q})} N^{\frac{3(1-s)}{2s-1}\cdot\frac{\epsilon}{2(2-\epsilon)}} N^{\frac{\epsilon}{2-3\epsilon}\cdot\frac{1-s}{2s-1}} \\
&\quad \times \|\nabla I P_{>\frac{M}{8}} u\|_{U^2_\Delta(J\times\mathbf{R}^3)} \\
&\lesssim N^{\frac{\epsilon}{2(2-\epsilon)}} M^{\frac{\epsilon}{2(2-\epsilon)}-1} N^{\frac{3(1-s)}{2s-1}\cdot\frac{\epsilon}{2(2-\epsilon)}} N^{\frac{\epsilon}{2-3\epsilon}\cdot\frac{1-s}{2s-1}} \|\nabla I P_{>\frac{M}{8}} u\|_{U^2_\Delta(J\times\mathbf{R}^3)}.
\end{aligned} \tag{4.22}$$

Combining (4.5), (4.15), (4.16), (4.21), and (4.22),

$$\|\nabla I P_{>M} u\|_{U^2_\Delta(J\times\mathbf{R}^3)} \lesssim_{s,\|u_0\|_{H^s(\mathbf{R}^3)},\epsilon} 1 + \frac{N^{C_1(s)\epsilon}}{M^{1-C_2(s)\epsilon}} \|\nabla I P_{>\frac{M}{8}} u\|_{U^2_\Delta(J\times\mathbf{R}^3)}. \tag{4.23}$$

Remark. It is not too important to compute exactly what $C_1(s)$ and $C_2(s)$ are, except to know that they are constant. This means that for any s, after taking $\epsilon(s) > 0$ sufficiently small, $C_1(s)\epsilon < \frac{1}{4}$ and $C_2(s)\epsilon < \frac{1}{4}$.

Now we argue by induction on frequency. If $M \geq C(s, \|u_0\|_{H^s})N^{2/3}$, then (4.23) implies

$$\begin{aligned} \|\nabla I P_{>M} u\|_{U^2_\Delta(J\times\mathbf{R}^3)} \\ \lesssim_{s,\|u_0\|_{H^s(\mathbf{R}^3)}} 1 + N^{-\frac{1}{4}} C(s, \|u_0\|_{H^s})^{-3/4} \|\nabla I P_{>\frac{M}{8}} u\|_{U^2_\Delta(J\times\mathbf{R}^3)}. \end{aligned} \tag{4.24}$$

Also, by (3.19),

$$\|\nabla I P_{>C(s,\|u_0\|_{H^s})N^{2/3}} u\|_{U^2_\Delta(J\times\mathbf{R}^3)} \lesssim N^{\frac{3(1-s)}{4s-2}}. \tag{4.25}$$

Therefore, by induction, for $C(s, \|u_0\|_{H^s})$ sufficiently large,

$$\|\nabla I P_{>\frac{N}{8}} u\|_{U^2_\Delta(J\times\mathbf{R}^3)} \lesssim_{\|u_0\|_{H^s},s} 1 + N^{\frac{3(1-s)}{4s-2}} N^{-c\ln(N)}. \tag{4.26}$$

Therefore, choosing N sufficiently large, say $\ln(N) = C_0 \frac{1-s}{s-\frac{1}{2}} + \ln(C(s, \|u_0\|_{H^s}))$, for some constants C_0 and $C(s, \|u_0\|_{H^s})$,

$$\|\nabla I P_{>\frac{N}{8}} u\|_{U^2_\Delta(J\times\mathbf{R}^3)} \lesssim_{\|u_0\|_{H^s(\mathbf{R}^3)}} 1. \tag{4.27}$$

□

5. Energy increment in three dimensions

Now we show a bound on the modified energy increment when $d = 3$.

Lemma 5.1. *For N sufficiently large so that $\ln(N) \geq C_0 \frac{1-s}{s-1/2} + \ln(C(s, \|u_0\|_{H^s}))$,*

$$\int_J |\frac{d}{dt} E(Iu(t))| dt \lesssim \frac{1}{N^{1-}}. \tag{5.1}$$

Proof. Because I is a Fourier multiplier which is constant in time and Δ commutes with I, (1.1) implies

$$iIu_t + \Delta Iu = |Iu|^2(Iu) + I(|u|^2 u) - |Iu|^2(Iu). \tag{5.2}$$

Therefore,

$$\frac{d}{dt} E(Iu(t)) = \langle Iu_t, |Iu|^2(Iu) - I(|u|^2 u)\rangle. \tag{5.3}$$

Then by (5.2) and integrating by parts,

$$\frac{d}{dt}E(Iu(t)) = -\langle i\nabla Iu, \nabla(|Iu|^2(Iu) - I(|u|^2u))\rangle \\ -\langle iI(|u|^2u), (|Iu|^2(Iu) - I(|u|^2u))\rangle. \tag{5.4}$$

First estimate $\langle i\nabla Iu, \nabla(|Iu|^2(Iu) - I(|u|^2u))\rangle$. As was mentioned before, it is unnecessary to distinguish between polynomial terms involving u and $\bar{u}$. Observe that

$$(IP_{\leq \frac{N}{8}}u)^3 - I((P_{\leq \frac{N}{8}}u)^3) = 0. \tag{5.5}$$

Next,

$$(IP_{> \frac{N}{8}}u)(IP_{\leq \frac{N}{8}}u)^2 - I((P_{> \frac{N}{8}}u)(P_{\leq \frac{N}{8}}u)^2) \\ = (IP_{> \frac{N}{2}}u)(P_{\leq \frac{N}{8}}u)^2 - I((P_{> \frac{N}{2}}u)(P_{\leq \frac{N}{8}}u)^2). \tag{5.6}$$

By the fundamental theorem of calculus,

$$|m(\xi_2 + \xi_3 + \xi_4) - m(\xi_2)| \lesssim \frac{|\xi_3 + \xi_4|}{|\xi_2|}. \tag{5.7}$$

Moreover, (5.6) implies

$$I((P_{> \frac{N}{8}}u)(P_{\leq \frac{N}{8}}u)^2) - (IP_{> \frac{N}{8}}u)(P_{\leq \frac{N}{8}}u)^2 \tag{5.8}$$

has a Fourier transform supported on $|\xi| \geq \frac{N}{8}$. Then by (5.7), $E(Iu(t)) \leq 1$, and Theorem 4.1,

$$\int_J \langle i\nabla Iu, \nabla((IP_{> \frac{N}{8}}u)(P_{\leq \frac{N}{8}}u)^2 - I((P_{> \frac{N}{8}}u)(P_{\leq \frac{N}{8}}u)^2))\rangle dt \tag{5.9}$$

$$\lesssim \frac{1}{N}\|\nabla IP_{> \frac{N}{8}}u\|^2_{L_t^2 L_x^6(J\times\mathbf{R}^3)}\|\nabla Iu\|_{L_t^\infty L_x^2(J\times\mathbf{R}^3)}\|Iu\|_{L_t^\infty L_x^6(J\times\mathbf{R}^3)} \lesssim \frac{1}{N}. \tag{5.10}$$

Also since $E(Iu(t)) \leq 1$,

$$\int_J \langle i\nabla Iu, \nabla((IP_{> \frac{N}{8}}u)^2(P_{\leq \frac{N}{8}}u) - I((P_{> \frac{N}{8}}u)^2(P_{\leq \frac{N}{8}}u)))\rangle dt \tag{5.11}$$

$$\lesssim \|\nabla Iu\|_{L_t^\infty L_x^2}\|\nabla IP_{> \frac{N}{8}}u\|_{L_t^2 L_x^6}\|IP_{> \frac{N}{8}}u\|_{L_t^2 L_x^6}\|P_{\leq \frac{N}{8}}u\|_{L_t^\infty L_x^6} \lesssim \frac{1}{N}. \tag{5.12}$$

Finally, by (4.10) and (4.11),

$$\int_J \langle i\nabla Iu, \nabla((IP_{>\frac{N}{8}}u)^3 - I((P_{>\frac{N}{8}}u)^3))\rangle dt \tag{5.13}$$

$$\lesssim \|\nabla Iu\|_{L_t^\infty L_x^2} \|\nabla IP_{>\frac{N}{8}}u\|_{L_t^2 L_x^6} \|P_{>\frac{N}{8}}u\|_{L_t^4 L_x^6}^2 \lesssim \frac{1}{N}. \tag{5.14}$$

This takes care of the first term in (5.4). Now consider the term

$$\int_J \langle I(|u|^2 u), I(|u|^2 u) - |Iu|^2(Iu)\rangle dt. \tag{5.15}$$

(5.5) and (5.6) imply that this six-linear term must have at least two $P_{>\frac{N}{8}}u$ terms. By the Sobolev embedding theorem, Bernstein's inequality, and (3.2),

$$\|I((P_{>\frac{N}{8}}u)^3)\|_{L_{t,x}^2} \lesssim \|\nabla IP_{>\frac{N}{8}}u\|_{L_t^2 L_x^6} \|P_{>\frac{N}{8}}u\|_{L_t^\infty L_x^3}^2 \lesssim \frac{1}{N}. \tag{5.16}$$

Therefore,

$$\int_J \langle I((P_{>\frac{N}{8}}u)^3), I((P_{>\frac{N}{8}}u)^3) + (IP_{>\frac{N}{8}}u)^3\rangle dt \lesssim \frac{1}{N^2}. \tag{5.17}$$

Next,

$$\begin{aligned} &\int_J \langle I((P_{>\frac{N}{8}}u)^3), (P_{>\frac{N}{8}}u)^2 (P_{\leq\frac{N}{8}}u)\rangle dt \\ &\lesssim \|I(P_{>\frac{N}{8}}u)^3\|_{L_{t,x}^2} \|P_{>\frac{N}{8}}u\|_{L_t^2 L_x^6} \|P_{\leq\frac{N}{8}}u\|_{L_{t,x}^\infty} \|P_{>\frac{N}{8}}u\|_{L_t^\infty L_x^3} \lesssim \frac{1}{N^2}. \end{aligned} \tag{5.18}$$

Finally,

$$\int_J \int (P_{>\frac{N}{8}}u)^2 (P_{\leq\frac{N}{8}}u)^2 u^2 dx dt \tag{5.19}$$

$$\lesssim \|P_{>\frac{N}{8}}u\|_{L_t^2 L_x^6}^2 \|P_{\leq\frac{N}{8}}u\|_{L_t^\infty L_x^6}^4 + \|P_{>\frac{N}{8}}u\|_{L_t^4 L_x^6}^4 \|P_{\leq\frac{N}{8}}u\|_{L_t^\infty L_x^6} \lesssim \frac{1}{N^2}. \tag{5.20}$$

This proves Lemma 5.1. □

Rescaling back, we have proved

$$\|u(t)\|_{L_x^2(\mathbf{R}^3)} = \|u(0)\|_{L_x^2(\mathbf{R}^3)}, \tag{5.21}$$

and

$$\|u(t)\|_{\dot{H}^s(\mathbf{R}^3)} \lesssim \|u(0)\|_{L^2(\mathbf{R}^3)} + N^{\frac{1-s}{2s-1}} \|u(0)\|_{\dot{H}^s(\mathbf{R}^3)}. \tag{5.22}$$

Therefore, by (4.26),

$$\|u(t)\|_{H^s(\mathbf{R}^3)} \lesssim C(s, \|u_0\|_{H^s(\mathbf{R}^3)}) \|u_0\|_{H^s(\mathbf{R}^3)}, \tag{5.23}$$

where C behaves like $e^{C_1 \frac{1-s}{2s-1}}$ for some constant C_1 as $s \searrow \frac{1}{2}$. This completes the proof of Theorem 1.2 since (5.23) gives a bound on $\|u\|_{L^4_{t,x}}$ by Theorem 3.1. Interpolating this with the uniform bound on $\|u(t)\|_{H^s}$ implies a bound on $L_t^p L_x^q$, where (p,q) is a $\frac{1}{2}$-admissible pair, that is $\frac{2}{p} = 3(\frac{1}{2} - \frac{1}{q} - \frac{1}{6})$. Since $s > \frac{1}{2}$, $p < \infty$. Partitioning $\mathbf{R}$ into finitely many pieces with $\|u\|_{L_t^p L_x^q(J_l \times \mathbf{R}^3)} < \epsilon$ and making a perturbation argument, $\|u\|_{L^5_{t,x}(\mathbf{R} \times \mathbf{R}^3)} < \infty$, which implies scattering. □

6. A computed mass-critical bound

In two dimensions the cubic problem

$$iu_t + \Delta u = \mu |u|^2 u, \quad u(0,x) = u_0, \quad \mu = \pm 1, \tag{6.1}$$

is mass-critical (see (1.4)). $\mu = +1$ is the defocusing case and $\mu = -1$ is the focusing case.

[24] proved that (6.1) was globally well-posed and scattering in the defocusing case ($\mu = 1$) and in the focusing case ($\mu = -1$) with mass less than the mass of the ground state. This result was extended to the nonradial case by [16]. However, [24] and [16] did not compute an explicit bound, which we will do here for initial data lying in $\dot{H}^s \cap |x|^s L^2 \subset L^2$.

Theorem 6.1. *If u_0 is a radially symmetric function with $u_0 \in \dot{H}^s(\mathbf{R}^2)$, $s > 0$, then the defocusing initial value problem*

$$iu_t + \Delta u = |u|^2 u, \quad u(0,x) = u_0, \tag{6.2}$$

has a solution on $[0,1]$ with

$$\|u\|^4_{L^4_{t,x}([0,1] \times \mathbf{R}^2)} \lesssim_s \|u_0\|_{\dot{H}^s(\mathbf{R}^2)}^{\frac{8(1-s)}{s}+2} (1 + \|u_0\|_{L^2})^{\frac{4(1-s)}{s}+2}. \tag{6.3}$$

The focusing initial value problem

$$iu_t + \Delta u = -|u|^2 u, \quad u(0,x) = u_0, \tag{6.4}$$

has a solution on $[0,1]$ *with*

$$\|u\|^4_{L^4_{t,x}([0,1]\times\mathbf{R}^2)} \lesssim_s \|u_0\|_{\dot{H}^s(\mathbf{R}^2)}^{\frac{8(1-s)}{s}+2}(1+\|u_0\|_{L^2})^{\frac{4(1-s)}{s}+2}(1-\frac{\|u_0\|^2_{L^2}}{\|Q\|^2_{L^2}})^{-\frac{1}{s}}, \tag{6.5}$$

where Q *is the soliton for* (6.4)*, that is* Q *solves the elliptic problem*

$$\Delta Q + Q^3 = Q. \tag{6.6}$$

This gives a scattering result.

Corollary 6.2. *The initial value problem* (6.2) *is globally well-posed and scattering for initial data lying in* $\dot{H}^s(\mathbf{R}^2)\cap |x|^s L^2(\mathbf{R}^2)$*.*

Proof of Corollary. By time reversal symmetry it suffices to prove

$$\|u\|_{L^4_{t,x}([0,\infty)\times\mathbf{R}^2)} < \infty. \tag{6.7}$$

Rescale so that $\|u_0\|_{\dot{H}^s} = \||x|^s u_0\|_{L^2}$. Shift $t=0$ to $t=1$ and then make the pseudoconformal transformation, for $t>0$,

$$v(t,x) = \frac{1}{t}e^{i\frac{|x|^2}{4t}}u(\frac{-1}{t},\frac{x}{t}). \tag{6.8}$$

Then v also solves (6.2) with initial data

$$\|v(-1,x)\|_{\dot{H}^s(\mathbf{R}^2)} \lesssim \|u_0\|_{\dot{H}^s(\mathbf{R}^2)} + \||x|^s u_0\|_{L^2(\mathbf{R}^2)}. \tag{6.9}$$

Then by Theorem 6.1,

$$\|v\|_{L^4_{t,x}([-1,0]\times\mathbf{R}^2)} < \infty. \tag{6.10}$$

It is easy to verify by direct computation that

$$\|v\|_{L^4_{t,x}([-1,0]\times\mathbf{R}^2)} = \|u\|_{L^4_{t,x}([1,\infty)\times\mathbf{R}^2)}. \tag{6.11}$$

Then shifting $t=1$ back to $t=0$ gives (6.7). ☐

Proof of Theorem 6.1. Without loss of generality suppose that $\|u_0\|_{\dot{H}^s}\gtrsim 1$. Otherwise, (6.3) could be proved by a small data argument. Next choose

$$\lambda \sim_s \|u_0\|_{\dot{H}^s}^{-\frac{4(1-s)}{s}-1}(1+\|u_0\|_{L^2})^{-\frac{2(1-s)}{s}-1}. \tag{6.12}$$

Then after rescaling by (1.3),

$$E(Iu(0)) \lesssim N^{2(1-s)} \|u_0\|_{\dot{H}^s}^{-8(1-s)} (1 + \|u_0\|_{L^2})^{-4(1-s)}. \tag{6.13}$$

Then if we choose

$$N \sim_s \|u_0\|_{\dot{H}^s}^4 (1 + \|u_0\|_{L^2})^2, \tag{6.14}$$

$$E(Iu(0)) \leq \frac{1}{2}. \tag{6.15}$$

Then to prove Theorem 6.1 it suffices to prove

$$E(Iu(t)) \leq 1, \quad \text{for all} \quad t \in [0, \lambda^{-2}]. \tag{6.16}$$

As in the cubic problem in three dimensions, this result will be proved using a long-time Strichartz estimate. First observe that if J is an interval, $J \subset [0, \lambda^{-2}]$, and $E(Iu(t)) \leq 1$ for all $t \in J$, then

$$\|Iu\|_{L^4_{t,x}(J \times \mathbf{R}^2)}^4 \lesssim \lambda^{-2}. \tag{6.17}$$

Bernstein's inequality and $E(Iu(t)) \leq 1$ on J implies that $\|(1 - I)u\|_{L^\infty_t L^2_x(J \times \mathbf{R}^2)} \lesssim N^{-1}$, so then by standard perturbative arguments, if $\|Iu\|_{L^4_{t,x}(I_j \times \mathbf{R}^2)} \leq \epsilon$, then for N sufficiently large,

$$\|u\|_{L^4_{t,x}(I_j \times \mathbf{R}^2)} \leq 2\epsilon. \tag{6.18}$$

Therefore,

$$\|u\|_{L^4_{t,x}(J \times \mathbf{R}^2)}^4 \lesssim \lambda^{-2}, \tag{6.19}$$

and thus since $E(Iu(t)) \leq 1$ on J,

$$\|\nabla Iu\|_{U^2_\Delta(J \times \mathbf{R}^2)} \lesssim \lambda^{-1}. \tag{6.20}$$

Theorem 6.3 (Long time Strichartz estimate)**.** *If $E(Iu(t)) \leq 1$ on J, then*

$$\|\nabla IP_{> \frac{N}{8}} u\|_{U^2_\Delta(J \times \mathbf{R}^2)} \lesssim 1. \tag{6.21}$$

Proof. Again by (2.29), for any $M \leq N$,

$$\|\nabla IP_{>M} u\|_{U^2_\Delta(J \times \mathbf{R}^2)} \tag{6.22}$$

$$\begin{aligned}&\lesssim \|\nabla I P_{>\frac{M}{8}} u(0)\|_{L^2} + \|\nabla I((P_{>\frac{M}{8}} u)^2 u)\|_{L_t^{2-} L_x^{1+}(J \times \mathbf{R}^2)} \\ &\quad + \|(\nabla I P_{\leq \frac{M}{8}} u)(P_{>\frac{M}{8}} u)(P_{\leq \frac{M}{8}} u)\|_{L_t^{2-} L_x^{1+}(J \times \mathbf{R}^2)} \\ &\quad + \frac{1}{M^{1-\frac{1}{q}}} \|(\nabla I P_{>\frac{M}{8}} u)(P_{\leq \frac{M}{8}} u)^2\|_{X_R}.\end{aligned}$$

Remark. Here we will use the $+$ and $-$ notation instead of $L_t^{2-\epsilon} L_x^{1+\delta(\epsilon)}$, and will not explicitly compute the ϵ dependence of the exponents. The interested reader could use the analysis in section four as a template, since the computations are quite similar. The important fact is that the powers will be bounded by a constant times $\epsilon > 0$.

First, since $E(Iu(t)) \leq 1$ for all $t \in J$, $\|\nabla I P_{>\frac{M}{8}} u(0)\|_{L^2} \lesssim 1$. Next, by (6.19) and Bernstein's inequality,

$$\|\nabla I((P_{>\frac{M}{8}} u)^2 u)\|_{L_t^{2-} L_x^{1+}(J \times \mathbf{R}^2)} \lesssim \frac{1}{M \lambda^{0+}} \|\nabla I P_{>\frac{M}{8}} u\|_{U_\Delta^2(J \times \mathbf{R}^2)}, \tag{6.23}$$

and

$$\|(\nabla I P_{\leq \frac{M}{8}} u)(P_{>\frac{M}{8}} u)(P_{\leq \frac{M}{8}} u)\|_{L_t^{2-} L_x^{1+}(J \times \mathbf{R}^2)} \lesssim \frac{1}{M \lambda^{0+}} \|\nabla I P_{>\frac{M}{8}} u\|_{U_\Delta^2(J \times \mathbf{R}^2)}. \tag{6.24}$$

Finally, by the fundamental theorem of calculus and $E(Iu(t)) \leq 1$,

$$|x| |P_{\leq \frac{M}{8}} u|^2 \leq \int_{|x|}^\infty r \partial_r (|P_{\leq \frac{M}{8}} u|^2) dr \lesssim \|\nabla I u\|_{L^2} \|I u\|_{L^2} \lesssim \|u_0\|_{L^2}. \tag{6.25}$$

Then for any $j \geq 0$ and R, by (6.20),

$$\begin{aligned}&\frac{R^{1-\frac{1}{q}} 2^{j(1-\frac{1}{q})}}{M^{1-\frac{1}{q}}} \|\chi(\frac{x}{2^j R})(\nabla I P_{>\frac{M}{8}} u)(P_{\leq \frac{M}{8}} u)^2\|_{L_t^q L_x^2(J \times \mathbf{R}^2)} \\ &\qquad \lesssim \frac{\lambda^{\frac{1}{2}-\frac{1}{q}}}{M^{\frac{3}{2}-\frac{1}{q}}} \|u_0\|_{L^2} \|\nabla I P_{>\frac{M}{8}} u\|_{U_\Delta^2(J \times \mathbf{R}^2)}.\end{aligned} \tag{6.26}$$

Also by (6.25) and $J \subset [0, \lambda^{-2}]$,

$$\begin{aligned}&R^{1-\frac{1}{q}} 2^{j(1-\frac{1}{q})} \|\chi(\frac{x}{2^j R})(\nabla I P_{>\frac{M}{8}} u)(P_{\leq \frac{M}{8}} u)^2\|_{L_t^q L_x^2(J \times \mathbf{R}^2)} \\ &\qquad \lesssim R^{\frac{1}{2}-\frac{1}{q}} 2^{j(\frac{1}{2}-\frac{1}{q})} \frac{\lambda^{1-\frac{2}{q}}}{M^{\frac{1}{2}}} \|u_0\|_{L^2} \|\nabla I P_{>\frac{M}{8}} u\|_{U_\Delta^2(J \times \mathbf{R}^2)}.\end{aligned} \tag{6.27}$$

Also by (6.19) and (6.20),

$$R^{1-\frac{1}{q}}\|\psi(\frac{x}{R})(\nabla IP_{>\frac{M}{8}}u)(P_{\leq\frac{M}{8}}u)^2\|_{L_t^qL_x^2(J\times\mathbf{R}^2)} \lesssim \frac{\lambda^{1-\frac{2}{q}}R^{1-\frac{1}{q}}}{M^{\frac{1}{2}}}\|\nabla IP_{>\frac{M}{8}}u\|_{U_\Delta^2(J\times\mathbf{R}^2)}. \tag{6.28}$$

Then taking $R=1$ and using (6.27) and (6.28) to sum over j, combined with the fact that $\lambda\sim N^{\frac{-(1-s)}{s}}$, and taking q arbitrarily close to 2,

$$\|\nabla IP_{>M}u\|_{U_\Delta^2(J\times\mathbf{R}^2)}\lesssim 1+\frac{N^{0+}}{M^{1-}}\|u_0\|_{L^2}\|\nabla IP_{>\frac{M}{8}}u\|_{U_\Delta^2(J\times\mathbf{R}^2)}. \tag{6.29}$$

Then making an induction on frequency argument, starting with $M=N^{3/4}$, implies that

$$\|\nabla IP_{>\frac{N}{8}}u\|_{U_\Delta^2(J\times\mathbf{R}^2)}\lesssim 1+N^{-c\ln(N)}\|u_0\|_{\dot{H}^s}^{\frac{4(1-s)}{s}+1}(1+\|u_0\|_{L^2})^{\frac{2(1-s)}{s}+1}. \tag{6.30}$$

Then if N is given by (6.14), the proof is complete. □

This gives a bound on the growth of $E(Iu(t))$.

Theorem 6.4. *$E(Iu(t))\leq 1$ for all $t\in[0,\lambda^{-2}]$.*

Proof. Since $E(Iu(0))\leq\frac{1}{2}$, it remains to bound the time integral of $\frac{d}{dt}E(Iu(t))$. Much of the analysis in section five may be copied directly to this situation as well. However, there are some differences due to the difference in dimension, and thus there are different exponents due to different Sobolev embeddings. For example, instead of (5.10), estimate

$$\begin{aligned}&\|\nabla Iu\|_{L_t^\infty L_x^2(J\times\mathbf{R}^2)}\|\nabla IP_{>\frac{N}{8}}u\|_{L_t^{2+}L_x^{\infty-}(J\times\mathbf{R}^2)}\|P_{>\frac{N}{8}}u\|_{L_t^{2+}L_x^{\infty-}(J\times\mathbf{R}^2)}\\&\quad\times\|P_{\leq\frac{N}{8}}u\|_{L_t^{\infty-}L_x^{2+}(J\times\mathbf{R}^2)}\\&\lesssim\frac{1}{N}\frac{1}{\lambda^{0+}}\|u_0\|_{L^2}.\end{aligned} \tag{6.31}$$

This finishes the proof of Theorem 6.1 in the defocusing case. □

For the focusing problem use the Gagliardo-Nirenberg inequality (see [33]),

$$\|u\|_{L_x^4(\mathbf{R}^2)}^4\leq\frac{1}{2}\frac{\|u\|_{L_x^2(\mathbf{R}^2)}^2}{\|Q\|_{L_x^2(\mathbf{R}^2)}^2}\|\nabla u\|_{L_x^2(\mathbf{R}^2)}^2. \tag{6.32}$$

Therefore,

$$\|\nabla Iu\|_{L_x^2(\mathbf{R}^2)}^2 (1 - \frac{\|u_0\|_{L^2}^2}{\|Q\|_{L^2}^2}) \lesssim E(Iu), \tag{6.33}$$

where in this case

$$E(Iu(t)) = \frac{1}{2}\int |\nabla Iu(t,x)|^2 dx - \frac{1}{4}\int |Iu(t,x)|^4 dx. \tag{6.34}$$

Then replace (6.12) and (6.14) with

$$\lambda \sim_s \|u_0\|_{\dot{H}^s}^{-\frac{4(1-s)}{s}-1} (1 + \|u_0\|_{L^2})^{-\frac{2(1-s)}{s}-1} (1 - \frac{\|u_0\|_{L^2}^2}{\|Q\|_{L^2}^2})^{\frac{1}{s}} \tag{6.35}$$

and

$$N \sim_s \|u_0\|_{\dot{H}^s}^4 (1 + \|u_0\|_{L^2})^2 (1 - \frac{\|u_0\|_{L^2}^2}{\|Q\|_{L^2}^2})^{-1}, \tag{6.36}$$

respectively. Then proceed as in the defocusing case. □

7. Induction on frequency in two dimensions

Now turn to the two dimensional problem (1.2) with $k > 1$, $k \in \mathbf{Z}$. Here the critical space is $\dot{H}^{s_c}$, $s_c = \frac{k-1}{k}$. Once again take the I operator as defined in (3.2). Then,

$$E(Iu(0)) \lesssim_{k, \|u_0\|_{H^s(\mathbf{R}^2)}} N^{2(1-s)}. \tag{7.1}$$

Rescale with $\lambda \sim_{\|u_0\|_{H^s}, k} N^{\frac{1-s}{s-s_c}}$ so that $E(Iu(0)) = \frac{1}{2}$. After rescaling $M(Iu(0)) \lesssim N^{\frac{1-s}{2(s-s_c)}}$. Suppose J is an interval with $E(Iu(t)) \leq 1$ for all $t \in J$. Recalling (3.23),

$$\|u\|_{L_t^4 L_x^8(J \times \mathbf{R}^2)}^4 \lesssim \||\nabla|^{1/2}|u|^2\|_{L_{t,x}^2}^2 \lesssim_{\|u(0)\|_{H^s(\mathbf{R}^2)}, k} N^{s_c \cdot \frac{3(1-s)}{s-s_c}}. \tag{7.2}$$

Then, by Lemma 3.3,

$$\|\nabla Iu\|_{U_\Delta^2(J \times \mathbf{R}^2)} \lesssim_{\|u(0)\|_{H^s(\mathbf{R}^2)}, k} N^{s_c \frac{3(1-s)}{2(s-s_c)}}. \tag{7.3}$$

Once again make an induction on frequency argument to prove long time Strichartz estimates.

Theorem 7.1. *Let $0 \in J$ be an interval such that $E(Iu(t)) \leq 1$ on J. Then for any $s > s_c$, there exists $N(s, k, \|u_0\|_{H^s}) < \infty$ such that*

$$\|\nabla I P_{> \frac{N}{8k}} u\|_{U_\Delta^2(J \times \mathbf{R}^2)} \lesssim_p 1. \tag{7.4}$$

Proof. Again by (2.29), if $M \leq N$,

$$\begin{aligned} &\|\nabla I P_{>M} u(t)\|_{U_\Delta^2(J \times \mathbf{R}^2)} \\ &\lesssim \|\nabla I P_{>M} u(0)\|_{L_x^2(\mathbf{R}^2)} + \|\nabla I P_{>M}((P_{>\frac{M}{8k}} u)^2 u^{2k-1})\|_{L_t^{2-} L_x^{1+}(J \times \mathbf{R}^2)} \\ &\quad + \|(I P_{>\frac{M}{8k}} u)(\nabla P_{\leq \frac{M}{8k}} u)(P_{\leq \frac{M}{8k}} u)^{2k-1}\|_{L_t^{2-} L_x^{1+}(J \times \mathbf{R}^2)} \\ &\quad + \frac{1}{M^{1-\frac{1}{q}}} \|(\nabla I P_{>\frac{M}{8k}} u)(P_{\leq \frac{M}{8k}} u)^{2k}\|_{X_R}. \end{aligned} \tag{7.5}$$

Once again, since the nonlinearity is algebraic,

$$P_{>M}(|u_{\leq \frac{M}{8k}}|^{2k} u_{\leq \frac{M}{8k}}) = 0. \tag{7.6}$$

Once again it is also perfectly fine to not distinguish between u and $\bar{u}$. Now again since the Fourier multiplier of ∇I is increasing as $|\xi| \nearrow \infty$,

$$\|\nabla I((P_{>\frac{M}{8k}} u)^2 u^{2k-1})\|_{L_t^{2-} L_x^{1+}(J \times \mathbf{R}^2)} \tag{7.7}$$

$$\lesssim \|\nabla I P_{>\frac{M}{8k}} u\|_{L_t^2 L_x^{\infty-}} \|P_{>\frac{M}{8k}} u\|_{L_t^{\infty-} L_x^{2+}} \|Iu\|_{L_t^{\infty-} L_x^{\infty}}^{k-2} \|Iu\|_{L_t^{\infty} L_x^{2k+2}}^{k+1} \tag{7.8}$$

$$+ \|\nabla I P_{>\frac{M}{8k}} u\|_{L_t^{2+} L_x^{\infty-}} \|P_{>\frac{M}{8k}} u\|_{L_t^{\infty} L_x^{2k}} \|P_{>N} u\|_{L_t^{\infty-} L_x^{2k+}}^{2k-1} \tag{7.9}$$

$$+ \|P_{>\frac{M}{8k}} u\|_{L_{t,x}^4}^2 \|\nabla I u\|_{L_t^{\infty-} L_x^{2+}} \|Iu\|_{L_t^{\infty-} L_x^{\infty}}^{2k-1} \tag{7.10}$$

$$+ \|P_{>\frac{M}{8k}} u\|_{L_t^{4k} L_x^{4k}}^2 \|\nabla I u\|_{L_t^{\infty-} L_x^{2+}} \|P_{>N} u\|_{L_t^{4k} L_x^{4k}}^{2k-2}. \tag{7.11}$$

Remark. Once again, we will use the $+$ and $-$ notation, rather than explicitly computing the ϵ dependence in the exponents.

Now by Lemma 3.3 and (7.3),

$$\|Iu\|_{L_t^{\infty-} L_x^{\infty}} + \|\nabla I u\|_{L_t^{\infty-} L_x^{2+}} \lesssim N^+. \tag{7.12}$$

Also by interpolation and Bernstein's inequality,

$$\|P_{>\frac{M}{8k}} u\|_{L_{t,x}^4}^2 \lesssim \frac{1}{M} \|\nabla I u\|_{L_t^{\infty} L_x^2} \|\nabla I u\|_{U_\Delta^2}, \tag{7.13}$$

and

$$\|P_{>\frac{M}{8k}}u\|_{L^{4k}_{t,x}}^{2k} \lesssim \frac{1}{M}\|\nabla Iu\|_{L_t^\infty L_x^2}^{2k-1}\|\nabla Iu\|_{U^2_\Delta}. \tag{7.14}$$

Making an argument almost identical to the estimates when $d = 3$,

$$(7.7) \lesssim_{k,\|u_0\|_{H^s(\mathbf{R}^2)}} \frac{N^+}{M^{1-}}\|\nabla IP_{>\frac{M}{8k}}u\|_{U^2_\Delta(J\times\mathbf{R}^2)}. \tag{7.15}$$

Similarly, since $M \leq N$ and $E(Iu(t)) \leq 1$,

$$\begin{aligned} &\|(IP_{>\frac{M}{8k}}u)(\nabla P_{\leq\frac{M}{8k}}u)(P_{\leq\frac{M}{8k}}u)^{2k-1}\|_{L_t^{2-}L_x^{1+}(J\times\mathbf{R}^2)} \\ &\lesssim \frac{1}{M}\|\nabla IP_{>\frac{M}{8k}}u\|_{L_t^{2+}L_x^{\infty-}}\|\nabla Iu\|_{L_t^{\infty-}L_x^{2+}}\|Iu\|_{L_t^{\infty-}L_x^\infty}^{k-2}\|Iu\|_{L_t^\infty L_x^{2k+2}}^{k+1} \\ &\lesssim \frac{N^+}{M}\|\nabla IP_{>\frac{M}{8k}}u\|_{U^2_\Delta(J\times\mathbf{R}^2)}. \end{aligned} \tag{7.16}$$

Once again,

$$\frac{1}{M^{1-\frac{1}{q}}}\|(\nabla IP_{>\frac{M}{8k}}u)(P_{\leq\frac{M}{8k}}u)^{2k}\|_{X_R(J\times\mathbf{R}^2)} \tag{7.17}$$

is estimated by the local smoothing estimate

$$\|\nabla IP_{>\frac{M}{8k}}u\|_{L^2_{t,x}(J\times\{x:|x|\leq R\})} \lesssim \frac{R^{1/2}}{M^{1/2}}\|\nabla IP_{>\frac{M}{8k}}u\|_{U^2_\Delta(J\times\mathbf{R}^2)}. \tag{7.18}$$

Then by the fundamental theorem of calculus,

$$|Iu(x)|^{2k} \lesssim \frac{1}{|x|}\int_{|x|}^\infty r\partial_r(|Iu(r)|^{2k})dr \lesssim \frac{1}{|x|}\|\nabla Iu\|_{L^2}\|Iu\|_{L_x^{4k-2}}^{2k-1} \lesssim \frac{1}{|x|}. \tag{7.19}$$

The last inequality follows from the fact that $E(Iu(t)) \leq 1$ along with the interpolation (for $k > 1$)

$$\|Iu\|_{L_x^{4k-2}} \lesssim \|\nabla Iu\|_{L^2}^\theta\|Iu\|_{L^{2k+2}}^{1-\theta}. \tag{7.20}$$

It is not particularly important what θ is. Then by (7.19) and (7.12), for any $j \geq 0$,

$$\frac{2^{j(1-\frac{1}{q})}R^{1-\frac{1}{q}}}{M^{1-\frac{1}{q}}}\|\chi(\frac{2^{-j}x}{R})(P_{<\frac{M}{8k}}u)^{2k}(\nabla IP_{>\frac{M}{8k}}u)\|_{L_t^qL_x^2} \tag{7.21}$$

$$\lesssim \frac{N^{+}}{M^{\frac{3}{2}-\frac{1}{q}}}\|\nabla IP_{>\frac{M}{8k}}u\|_{U_{\Delta}^{2}(J\times\mathbf{R}^{2})}.$$

Then for j very large, for $k \geq 2$,
(7.22)

$$\sum_{j\geq J}\frac{2^{j(1-\frac{1}{q})}R^{1-\frac{1}{q}}}{M^{1-\frac{1}{q}}}\|\chi(\frac{2^{-j}x}{R})(P_{<\frac{M}{8k}}u)^{2k}(\nabla IP_{>\frac{M}{8k}}u)\|_{L_t^qL_x^2(J\times\mathbf{R}^2)}$$

$$\lesssim \sum_{j\geq J}\frac{2^{j(\frac{1}{q}-\frac{1}{2})}R^{\frac{1}{q}-\frac{1}{2}}}{M^{\frac{3}{2}-\frac{1}{q}}}\|Iu\|_{L_t^{\infty}L^2}\|Iu\|_{L_t^{\infty}\dot{H}^1}\|\nabla IP_{>\frac{M}{8k}}u\|_{U_{\Delta}^2(J\times\mathbf{R}^2)}\|Iu\|_{L_t^{\infty-}L_x^{\infty}}^{2k-2}$$

$$\lesssim \sum_{j\geq J}\frac{2^{j(\frac{1}{q}-\frac{1}{2})}R^{\frac{1}{q}-\frac{1}{2}}}{M^{\frac{3}{2}-\frac{1}{q}}}N^{\frac{1-s}{2(s-s_c)}}N^{+}\|\nabla IP_{>\frac{M}{8k}}u\|_{U_{\Delta}^2(J\times\mathbf{R}^2)}.$$

Then taking $J(N,s,s_c,k,R)$ sufficiently large and q arbitrarily close to 2,

$$\lesssim \frac{N^{+}}{M^{1-}}\|\nabla IP_{>\frac{M}{8k}}u\|_{U_{\Delta}^2(J\times\mathbf{R}^2)}. \tag{7.23}$$

Finally taking $R(N)$ sufficiently small, by the Sobolev embedding theorem, and (7.12),

$$\begin{aligned}&\frac{R^{1-\frac{1}{q}}}{M^{1-\frac{1}{q}}}\|\psi(\frac{x}{R})(\nabla IP_{>\frac{M}{8k}}u)(Iu)^{2k}\|_{L_t^qL_x^2(J\times\mathbf{R}^2)}\\ &\quad\lesssim \frac{R^{1-\frac{1}{q}}}{M^{\frac{3}{2}-\frac{1}{q}}}N^{+}\|\nabla IP_{>\frac{M}{8k}}u\|_{U_{\Delta}^2(J\times\mathbf{R}^2)}.\end{aligned} \tag{7.24}$$

This time we starting the induction at $C(s,\|u_0\|_{H^s},k)N^{3/4}$ for $C(s,\|u_0\|_{H^s},k)$ sufficiently large,

$$\|\nabla IP_{>\frac{N}{8k}}u\|_{U_{\Delta}^2(J\times\mathbf{R}^2)} \lesssim_{\|u_0\|_{H^s},s,k} 1 + N^{\frac{3(1-s)}{4(s-s_c)}s_c}N^{-\frac{c}{6}\ln(N)}, \tag{7.25}$$

for some constant $c > 0$.

Remark. We could replace $\frac{c}{6}$ with $\frac{c}{q}$ for any $q > 4$. Therefore, choosing N sufficiently large,

$$\|\nabla IP_{>\frac{N}{8k}}u\|_{U_{\Delta}^2(J\times\mathbf{R}^3)} \lesssim_{\|u_0\|_{H^s},k} 1. \tag{7.26}$$

□

8. Energy increment

To complete the proof of Theorem 1.4 it remains to prove the usual bound on the growth of $E(Iu(t))$.

Lemma 8.1. *If J is an interval with $E(Iu(t)) \leq 1$ on J,*

$$\int_J |\frac{d}{dt} E(Iu(t))| dt \lesssim_k \frac{1}{N^{1-}}. \tag{8.1}$$

Proof. We compute

$$\begin{aligned} \frac{d}{dt} E(Iu(t)) &= \langle Iu_t, |Iu|^{2k}(Iu) - I(|u|^{2k}u)\rangle \\ &= -\langle i\nabla Iu, \nabla(|Iu|^{2k}(Iu) - I(|u|^{2k}u))\rangle \\ &-\langle iI(|u|^{2k}u), (|Iu|^{2k}(Iu) - I(|u|^{2k}u))\rangle. \end{aligned} \tag{8.2}$$

Once again,

$$(IP_{\leq \frac{N}{8k}} u)^{2k+1} - I((P_{\leq \frac{N}{8k}} u)^{2k+1}) = 0. \tag{8.3}$$

Also,

$$\begin{aligned} &(IP_{> \frac{N}{8k}} u)(IP_{\leq \frac{N}{8k}} u)^{2k} - I((P_{> \frac{N}{8k}} u)(P_{\leq \frac{N}{8k}} u)^{2k}) \\ &= (IP_{> \frac{N}{2}} u)(P_{\leq \frac{N}{8k}} u)^{2k} - I((P_{> \frac{N}{2}} u)(P_{\leq \frac{N}{8k}} u)^{2k}). \end{aligned} \tag{8.4}$$

As before in (5.7),

$$-\int_J \langle i\nabla Iu, \nabla((IP_{> \frac{N}{8k}} u)(P_{\leq \frac{N}{8k}} u)^{2k} - I((P_{> \frac{N}{8k}} u)(P_{\leq \frac{N}{8k}} u)^{2k}))\rangle dt \tag{8.5}$$

$$\begin{aligned} &\lesssim \frac{1}{N} \|\nabla IP_{> \frac{N}{8k}} u\|^2_{L_t^{2+} L_x^{\infty-}(J \times \mathbf{R}^2)} \|\nabla Iu\|_{L_t^{\infty-} L_x^{2+}(J \times \mathbf{R}^2)} \|Iu\|^{2k-1}_{L_t^\infty L_x^{(4k-2)}(J \times \mathbf{R}^2)} \\ &\lesssim \frac{1}{N^{1-}}. \end{aligned} \tag{8.6}$$

This follows from (7.20), (7.26) to estimate $\|\nabla IP_{> \frac{N}{8k}} u\|_{L_t^{2+} L_x^{\infty-}(J \times \mathbf{R}^2)}$ and (7.12) to estimate $\|\nabla Iu\|_{L_t^{\infty-} L_x^{2+}(J \times \mathbf{R}^2)}$.

Next, since $E(Iu(t)) \leq 1$,

$$\int_J \langle i\nabla Iu, \nabla((IP_{> \frac{N}{8k}} u)^2 (P_{\leq \frac{N}{8k}} u)^{2k-1} - I((P_{> \frac{N}{8k}} u)^2 (P_{\leq \frac{N}{8k}} u)^{2k-1}))\rangle dt \tag{8.7}$$

(8.8)

$$\lesssim \|\nabla Iu\|_{L_t^{\infty-}L_x^{2+}} \|\nabla IP_{>\frac{N}{8k}}u\|_{L_t^{2+}L_x^{\infty-}} \|IP_{>\frac{N}{8k}}u\|_{L_t^{2+}L_x^{\infty-}} \|P_{\leq\frac{N}{8k}}u\|_{L_t^{\infty}L_x^{4k-2}}^{2k-1} \lesssim \frac{1}{N^{1-}}.$$

Finally, we skip ahead to

$$\int_J \langle i\nabla Iu, \nabla((IP_{>\frac{N}{8k}}u)^{2k+1} - I((P_{>\frac{N}{8k}}u)^{2k+1}))\rangle dt \tag{8.9}$$

$$\lesssim \|\nabla Iu\|_{L_t^{\infty-}L_x^{2+}} \|\nabla IP_{>\frac{N}{8k}}u\|_{L_t^{2+}L_x^{\infty-}} \|P_{>\frac{N}{8k}}u\|_{L_{t,x}^{4k}}^{2k} \lesssim \frac{1}{N^{1-}}. \tag{8.10}$$

Remark. The other terms can be handled in a similar manner.

Then by (7.26), $E(Iu(t)) \leq 1$, we are done with the first term in (8.2). Now we consider the term

$$\int_J \langle I(u^{2k+1}), I(u^{2k+1}) - (Iu)^{2k+1}\rangle dt. \tag{8.11}$$

Once again this term must have at least two $P_{>\frac{N}{8k}}u$ terms. By the Sobolev embedding theorem, $E(Iu(t)) \leq 1$, Bernstein's inequality, (7.12), and (7.26),

(8.12)

$$\|I((P_{>\frac{N}{8k}}u)^{2k+1})\|_{L_{t,x}^2} \lesssim \|\nabla IP_{>\frac{N}{8k}}u\|_{L_t^{2+}L_x^{\infty-}} \|P_{>\frac{N}{8k}}u\|_{L_t^{\infty-}L_x^{2k+}}^{2k} \lesssim \frac{1}{N^{1-}}.$$

Therefore,

$$\int_J \langle I((P_{>\frac{N}{8k}}u)^{2k+1}), I((P_{>\frac{N}{8k}}u)^{2k+1}) - (IP_{>\frac{N}{8k}}u)^{2k+1}\rangle dt \lesssim \frac{1}{N^{2-}}. \tag{8.13}$$

Next,

$$\begin{aligned} &\int_J \langle I((P_{>\frac{N}{8k}}u)^{2k+1}), (P_{>\frac{N}{8k}}u)^{2k}(P_{\leq\frac{N}{8k}}u)\rangle dt \\ &\lesssim \|I(P_{>\frac{N}{8k}}u)^{2k+1}\|_{L_{t,x}^2} \|P_{>\frac{N}{8k}}u\|_{L_{t,x}^{4k}}^{2k} \|P_{\leq\frac{N}{8k}}u\|_{L_{t,x}^{\infty}} \lesssim \frac{1}{N^{2-}}. \end{aligned} \tag{8.14}$$

Finally,

$$\begin{aligned} &\int_J\int (P_{>\frac{N}{8k}}u)^2(P_{\leq\frac{N}{8k}}u)^2 u^{4k-2} dx dt \\ &\lesssim \int_J\int (P_{>\frac{N}{8k}}u)^{4k+2} dx dt + \int_J\int (P_{>\frac{N}{8k}}u)^2 (P_{\leq\frac{N}{8k}}u)^{4k} dx dt. \end{aligned} \tag{8.15}$$

Interpolating the L_x^{2k+2} and $\dot{H}^1$ norms, since $E(Iu(t)) \leq 1$,

$$\|Iu\|_{L_t^\infty L_x^{4k}} \lesssim 1. \tag{8.16}$$

This proves Lemma 8.1. □

Rescaling back, we have proved

$$\|u(t)\|_{L_x^2(\mathbf{R}^2)} = \|u(0)\|_{L_x^2(\mathbf{R}^2)}, \tag{8.17}$$

and

$$\|u(t)\|_{\dot{H}^s(\mathbf{R}^2)} \lesssim \|u(0)\|_{L^2(\mathbf{R}^2)} + N^{s_c \cdot \frac{1-s}{s-s_c}} \|u(0)\|_{\dot{H}^s(\mathbf{R}^2)}. \tag{8.18}$$

Therefore,

$$\|u(t)\|_{H^s(\mathbf{R}^2)} \lesssim C(\|u_0\|_{H^s(\mathbf{R}^2)}, k) \|u_0\|_{H^s(\mathbf{R}^2)}, \tag{8.19}$$

where C behaves like $e^{C_1 \frac{s_c(1-s)}{s-s_c}}$ for some constant C_1.

Acknowledgements

During the time of writing this paper, the author was supported by an NSF postdoc, as well as NSF grants DMS-1500424 and DMS-1764358. The author was also a member of the Institute for Advanced Study as a von Neumann fellow during part of the writing of this paper. Finally, the author would like to acknowledge the helpful suggestions of the anonymous referee.

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Received May 3, 2017