Earning and Utility Limits in Fisher Markets

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Earning limits and utility limits are novel aspects in the classic Fisher market model. Sellers with earning limits have bounds on their income and lower the supply they bring to the market if income exceeds the limit. Buyers with utility limits have an upper bound on the amount of utility that they want to derive and lower the budget they bring to the market if utility exceeds the limit. Markets with these properties can have multiple equilibria with different characteristics.

We analyze earning limits and utility limits in markets with linear and spending-constraint utilities. For markets with earning limits and spending-constraint utilities, we show that equilibrium price vectors form a lattice and the spending of buyers is unique in non-degenerate markets. We provide a scaling-based algorithm to compute an equilibrium in time $O(n^3 \ell \log(\ell + nU))$, where n is the number of agents, $\ell \geq n$ a bound on the segments in the utility functions, and U the largest integer in the market representation. We show how to refine any equilibrium in polynomial time to one with minimal prices or one with maximal prices (if it exists). Moreover, our algorithm can be used to obtain in polynomial time a 2-approximation for maximizing Nash social welfare in multi-unit markets with indivisible items that come in multiple copies.

For markets with utility limits and linear utilities, we show similar results—lattice structure of price vectors, uniqueness of allocation in non-degenerate markets, and polynomial-time refinement procedures to obtain equilibria with minimal and maximal prices. We complement these positive results with hardness results for related computational questions. We prove that it is NP-hard to compute a market equilibrium that maximizes social welfare, and it is PPAD-hard to find any market equilibrium with utility functions with separate satiation points for each buyer and each good.

CCS Concepts: • **Theory of computation → Market equilibria**;

Additional Key Words and Phrases: Market equilibrium, earning limits, utility limits, equilibrium computation, spending-constraint utilities

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1 INTRODUCTION

The concept of market equilibrium is a fundamental and well-established notion in economics to analyze and predict the outcomes of strategic interaction in large markets. Initiated by Walras in 1874, the study of market equilibrium has become a cornerstone of microeconomic analysis, mostly due to general results that established existence under very mild conditions [3]. Since efficient computation is a fundamental criterion to evaluate the plausibility of equilibrium concepts, the algorithmic aspects of market equilibrium are one of the central domains in algorithmic game theory. Over the past decade, several new algorithmic approaches to compute market equilibria were discovered. Efficient algorithms based on convex programming techniques can compute equilibria in a large variety of domains [14, 27, 30]. More importantly, several approaches were proposed that avoid the use of heavy algorithmic machinery and follow combinatorial strategies [19, 22, 23, 31, 35, 41] or even work as a tâtonnement process in unknown market environments [5, 13, 16]. Designing such combinatorial algorithms is useful also beyond the study of markets, since the underlying ideas can be applied in other areas. Variants of these algorithms were shown to solve scheduling [29] and cloud computing problems [18] or can be used for fair allocation of indivisible items [17].

Fisher markets are a fundamental model to study competitive allocation of goods among rational agents. In a Fisher market, there is a set B of buyers and a set G of divisible goods. Each buyer $i \in B$ has a budget $m_i > 0$ of money and a utility function u_i that maps any bundle of goods to a non-negative utility value. Each good $j \in G$ is assumed to come in unit supply and to be sold by a separate seller. A *competitive* or *market equilibrium* is an allocation vector of goods and a vector of prices, such that (1) every buyer spends his budget to buy an optimal bundle of goods, and (2) supply equals demand. Fisher markets have been studied intensively in algorithmic game theory. For many strictly increasing and concave utility functions, market equilibria can be described by convex programs [24, 37]. There are a variety of algorithms for computing market equilibria [19, 20, 30, 42]. For linear markets, there are even algorithms that run in strongly polynomial time [35, 41]. Moreover, simple tâtonnement [13, 16] or proportional response dynamics [8, 43] converge to equilibrium (quickly).

A common assumption in all this work is that utility functions are non-satiated, that is, the utility of every buyer strictly increases with the amount of good allocated to it, and the utility of every seller strictly increases with the money earned by it. Consequently, when buyers and sellers are price-taking agents, it is in their best interest to spend their entire budget and bring all supply to the market, respectively. In this article, we study new variants of linear Fisher markets with satiated utility functions recently proposed in [15].

First, we consider markets in which each seller has an earning limit, which gives him an incentive to be *thrifty* in equilibrium, i.e., to possibly reduce the supply of his good in the market to meet his earning limit under equilibrium prices. This is a natural property in many domains, e.g., when sellers have revenue targets. Many properties of such markets are not well understood. Interestingly, thrifty equilibria in Fisher markets with earning limits also relate closely to *fair allocations of indivisible* items. There has been a surge of interest in allocating indivisible items to maximize Nash social welfare. Very recent work [1, 17, 26] has provided the first constant-factor approximation algorithms for this important problem. The algorithms first compute and then cleverly round a thrifty equilibrium of a Fisher market with earning limits. The tools and techniques for computing market equilibria are a key component in this approach.

In this article, we consider algorithmic and structural properties of markets with earning limits and spending-constraint utilities. Spending-constraint utilities are a natural generalization of linear utilities with many additional applications [20, 39]. We show structural properties of thrifty

equilibria and provide new and improved polynomial-time algorithms for computation. Moreover, we show how these algorithms can be used to approximate Nash social welfare in markets where each item j is provided in s_j copies (where s_j is a given integer). As a result, we obtain the first polynomial-time approximation algorithms for multi-unit markets.

The second generalization of the Fisher market model that we consider has buyers with linear utilities and *utility limits*. That is, there is a *happiness cap* $c_i > 0$ for each buyer i, and her utility function is $u_i(\mathbf{x}_i) = \min\{c_i, \sum_{j \in G} u_{ij} x_{ij}\}$, where $\mathbf{x}_i = (x_{ij})_{j \in G}$ is any bundle of goods assigned to buyer i. Such utility functions with happiness caps are also known as budget-additive utility functions. They are a simple class of submodular and concave functions and a natural generalization of the standard and well-understood case of linear utilities. These utility functions arise naturally in cases where agents have an intrinsic upper bound on their utility. For example, if the goods are food and the utility of a food item for a particular buyer is its calorie content, then calories above a certain threshold do not increase the utility of the buyer. In addition, there are a variety of further applications in adword auctions and revenue maximization problems [2, 4, 9, 11]. Recently, market models where agents have linear utilities with utility limits attracted a significant amount of research interest, e.g., for the allocation of indivisible goods in offline [2, 4, 11] and online [9, 32] scenarios, for truthful mechanism design [10], and for the study of Walrasian equilibrium with quasi-linear utilities [21, 25, 36]. As simple variants of submodular functions, they capture many of the inherent difficulties of more general domains. Given this amount of interest, it is perhaps surprising that they are not well understood within the classic Fisher and exchange markets.

If buyers have utility limits, then it is natural to assume that they are modest and thrifty, i.e., do not ask for bundles whose utility exceeds their utility cap and still spend money in the most economical way. We show that the thrifty and modest equilibria form a lattice and design two procedures, using which we can turn any thrifty and modest equilibrium into one with smallest prices (minimum revenue) and one with largest prices (maximum revenue), respectively. We also give a number of hardness results for related computational questions. In particular, we prove that it is NP-hard to compute a (not necessarily thrifty and modest) market equilibrium that maximizes social welfare, and it is PPAD-hard to find any market equilibrium with utility functions with separate satiation points for each buyer and each good.

1.1 Contribution and Outline

After formal discussion of the market model, we discuss some preliminaries in Section 1.3, including a formal condition for the existence of a thrifty equilibrium. In Section 2, we study the structure of thrifty equilibria in markets with earning limits and thrifty and modest equilibria in markets with utility limits. In particular, we show that the set of equilibrium price vectors always forms a lattice. Moreover, in non-degenerate markets (for a formal definition see Section 1.3) the spending of buyers is unique accross equilibria.

In Section 3, we focus on markets with earning limits and spending-constraint utilities and outline a novel algorithm to compute a thrifty equilibrium in time $O(n^3 \ell \log(\ell + nU))$, where n is the total number of agents, ℓ is the maximum number of segments in the description of the utility functions that is incident to any buyer or any good, and U is the largest integer in the representation of utilities, budgets, and earning limits. For linear markets, the running time simplifies to $O(n^4 \log nU))$. Our algorithm uses a scaling technique with decreasing prices and maintains assignments in which buyers overspend their money. A technical challenge is to maintain rounded versions of the spending restrictions in the utility functions. The algorithm runs until the maximum overspending of all buyers becomes tiny and then rounds the outcome to

an exact equilibrium. Given an arbitrary thrifty equilibrium, we show how to find in polynomial time a thrifty equilibrium with smallest prices, or one with largest prices (if it exists).

Next, we show in Section 3.3 how to round a thrifty equilibrium in linear markets with earning limits to an allocation of indivisible goods in a multi-unit market to approximate the Nash social welfare. In these markets, for each item j there are a number s_j of available copies. The direct application of existing algorithms [1, 17] would require pseudo-polynomial time. Instead, we show how to adjust the rounding procedure in [17] to run in strongly polynomial time. The resulting algorithm yields a 2-approximation and runs in time $O(n^4 \log(nU))$, which is polynomial in the input size.

In Section 4, we turn to markets with utility limits. First in Section 4.1, we exploit the lattice structure of thrifty and modest equilibria and design two procedures, using which we can turn any thrifty and modest equilibrium into one with smallest prices (minimum revenue) and one with largest prices (maximum revenue), respectively.

Next, we study two extensions in Section 4.2. When we drop the assumption of thrifty and modest buyers, then we face multiple market equilibria. A natural goal is to compute an allocation that maximizes utilitarian social welfare. We prove that this problem is NP-hard, even when social welfare is measured by a k-norm of the vector of buyer utilities, for any constant $k > 0$. Moreover, we consider a variant of linear utilities with a utility limit for *each buyer and each good*. They constitute a special class of separable piecewise-linear concave (SPLC) utilities, where each piecewise-linear component consists of two segments with the second one being constant. We show that even in this very special case computing any market equilibrium becomes PPAD-hard.

1.2 Related Work

For Fisher markets we focus on some directly related work about computation of market equilibria. For markets with linear utilities a number of polynomial-time algorithms have been derived [19, 30, 42], including ones that run in strongly polynomial time [35, 41]. For spending-constraint utilities in exchange markets [20] a polynomial-time algorithm was recently obtained [5]. For Fisher markets with spending-constraint utilities, the algorithm by Vegh [41] runs in strongly polynomial time.

Linear markets with either earning or utility limits were studied only recently [6, 15]. The equilibria solve standard convex programs. The Shmyrev program [37] for earning limits also applies to spending-constraint utilities. For utility limits, the framework of [40] provides an (arbitrary) equilibrium in time $O(n^5 \log(nU))$. For earning limits, our algorithm runs in time $O(n^3 \ell \log(\ell + nU))$ for spending-constraint and $O(n^4 \log(nU))$ for linear utilities. It computes an approximate solution that can be rounded to an exact equilibrium. An approximate solution could also be obtained with classic algorithms for separable convex optimization [28, 33]. These algorithms have slower running times—in particular, the algorithm of [33] obtains the required precision only in time $O(n^3 \ell^2 \log(\ell) \log(\ell + nU))$.

An interesting open problem are strongly polynomial-time algorithms for arbitrary earning limits. A non-trivial challenge in adjusting [35] is the precision of intermediate prices. For the framework of [41] the challenge lies in generalizing the Error-method to markets with earning limits.

Approximating optimal allocations of indivisible items that maximize Nash social welfare has been studied recently for markets with additive [15, 17], separable concave [1], and budget-additive valuations [26]. Here equilibria of markets with earning limits can be rounded to yield a 2-approximation. We extend this approach to markets with multi-unit items, where each item j comes in s_j copies (and the input includes s_j in standard logarithmic coding). In contrast to the

direct, pseudo-polynomial extensions of previous work, we show how to obtain a 2-approximation in polynomial time.

Some of the results in this article have appeared previously in extended abstracts in the proceedings of ESA 2016 [6] and SAGT 2017 [7].

1.3 Preliminaries

In a Fisher market, there is a set B of buyers and a set G of goods. Every buyer $i \in B$ has a *budget* $m_i > 0$ of money and a utility function u_i. We consider two types of utility functions in this article.

Linear Utilities. The utility function u_i of buyer i from a bundle $\mathbf{x}_i = (x_{ij})_{j \in G}$ of goods is defined as $u_i(\mathbf{x}_i) = \sum_{j \in G} u_{ij} x_{ij}$, where u_{ij} is his utility from one unit of good j and x_{ij} is the amount of good j assigned to this buyer.

Spending-Constraint Utilities. The utility function u_i of buyer i is a *spending-constraint function* given by non-empty sets of *segments* $K_{ij} = \{(i,j,k) \mid 1 \le k \le \ell_{ij}\}$ for each good $j \in G$. Each segment $(i,j,k) \in K_{ij}$ comes with a *utility value* u_{ijk} and a *spending limit* $c_{ijk} > 0$. We assume that the utility function is piecewise linear and concave, i.e., $u_{ijk} > u_{ij,k+1} > 0$ for all $\ell_{ij} - 2 \ge k \ge 1$. W.l.o.g. we assume that the last segment has $u_{ij\ell_{ij}} = 0$ and $c_{ijk} = \infty$. Note that a linear utility function can be viewed as a special case of spending-constraint function with only one segment and an unlimited spending limit.

Buyer i can spend at most an amount of c_{ijk} of money on segment (i,j,k). We use $\mathbf{f} = (f_{ijk})_{(i,j,k) \in K_{ij}}$ to denote the spending of money on segments. $\mathbf{f}$ is termed *money flow*. A segment is *closed* if $f_{ijk} = c_{ijk}$, otherwise *open*. For notational convenience, we let $f_{ij} = \sum_{(i,j,k) \in K_{ij}} f_{ijk}$.

Given a vector $\mathbf{p} = (p_j)_{j \in G}$ of strictly positive *prices* for goods, a money flow results in an allocation $x_{ij} = \sum_k f_{ijk}/p_j$ of good j. The *bang-per-buck ratio* of segment (i,j,k) is $\alpha_{ijk} = u_{ijk}/p_j$. To maximize his utility, buyer i spends his budget m_i on segments in non-increasing order of bang-per-buck ratio, while respecting the spending limits. A bundle $\mathbf{x}_i = (x_{ij})_{j \in G}$ that results from this approach is termed a *demand bundle* and denoted by $\mathbf{x}_i^*$. The corresponding money flow on the segments is termed *demand flow* $\mathbf{f}_i^*$.

Demand bundles and flows might not be unique, but they differ only on the allocated segments with smallest bang-per-buck ratio. This smallest ratio is termed *maximum bang-per-back (MBB) ratio* and denoted by α_i. Note that α_i is unique given $\mathbf{p}$. All segments with $\alpha_{ijk} \ge \alpha_i$ are termed *MBB segments*. The segments with $\alpha_{ijk} = \alpha_i$ are termed *active segments*. We assume w.l.o.g. $m_i \le \sum_{j,k:u_{ijk}>0} c_{ijk}$, since no buyer would spend more.

1.3.1 *Markets with Earning Limits.*

We consider a natural condition on seller supplies. Each good is owned by a different seller, and the seller has a maximum endowment of 1. Seller j comes with an *earning limit* d_j. He only brings a supply $e_j \le 1$ that suffices to reach this earning limit under the given prices. Intuitively, while each seller has utility $\min\{d_j, e_j p_j\}$, we also assume that he has a tiny utility for unsold parts of his good. Hence, he only brings a supply to earn d_j. More formally, the *active price* of good j is given by $p_j^a = \min(d_j, p_j)$. His good is *capped* if $p_j^a = d_j$ and *uncapped* otherwise. A *thrifty supply* is $e_j = p_j^a/p_j$, which guarantees $e_j p_j \le d_j$, i.e., the earning limit holds when market clears.

We consider *thrifty equilibria*.

Definition 1.1. A pair $(\mathbf{x}, \mathbf{p})$ is a *thrifty equilibrium* if (1) $\mathbf{x}_i$ is a demand bundle under prices $\mathbf{p}$ for every $i \in B$, and (2) $\sum_i x_{ij} p_j = \sum_{i,k} f_{ijk} = p_j^a$, for every $j \in G$.

Note that when a set of strictly positive prices is fixed, the money flow $\mathbf{f}$ can be used to uniquely determine the allocation $\mathbf{x}$ by $x_{ij} = f_{ij}/p_j$, and vice versa. It turns out that every thrifty

equilibrium has strictly positive prices for all goods. By an abuse of notation, we also call $(\mathbf{f}, \mathbf{p})$ a thrifty equilibrium if its corresponding allocation $\mathbf{x}$ and $\mathbf{p}$ satisfy the above conditions.

1.3.2 Markets with Utility Limits. We consider another variation with limits on the buyer's side. We consider linear utility functions for buyers and assume that each buyer has a *utility limit*. That is, there is a *happiness cap* $c_i > 0$ for each buyer i, and the utility function is $u_i(\mathbf{x}_i) = \min\{c_i, \sum_{j \in G} u_{ij} x_{ij}\}$, where $\mathbf{x}_i = (x_{ij})_{j \in G}$ is any bundle of goods assigned to buyer i. Such utility functions with happiness caps are also know as budget-additive utility functions.

For a given price vector $\mathbf{p}$ and buyer i, the MBB ratio is simplified as $\alpha_i = \max_j u_{ij}/p_j$, where we make the assumption that $0/0 = 0$. Such utilities strictly generalize linear utilities: When all c_i's are large enough, they are equivalent to linear utilities. If buyer i is uncapped in a market equilibrium $(\mathbf{x}, \mathbf{p})$, then it behaves as in the linear case, spends all its budget, and buys only MBB goods ($x_{ij} > 0$ only if $u_{ij}/p_j = \alpha_i$). Otherwise, if buyer i is capped in $(\mathbf{x}, \mathbf{p})$, then it might buy non-MBB goods and not spend all of its budget. This implies that unlike the case of linear utilities, market equilibrium prices, and utilities are not unique when buyers have utility limits.

It is easy to see that we can obtain one market equilibrium by simply ignoring the happiness caps and treating the market as a linear one. However, this equilibrium is often undesirable, since it is not always Pareto-optimal. The main challenges here arise from capped buyers, who may possibly have multiple choices for the demand bundle. Next let us introduce two convenient restrictions on the allocation to capped buyers.

- An allocation $\mathbf{x}_i$ for buyer i is called *modest* if $\sum_j u_{ij} x_{ij} \le c_i$. By definition, for uncapped buyers every demand bundle is modest. For capped buyers, a modest bundle of goods $\mathbf{x}_i$ is such that utility breaks even between the linear part and c_i, i.e., $c_i = \sum_j u_{ij} x_{ij}$.
- A demand bundle $\mathbf{x}_i$ is called *thrifty* or *MBB* if it consists of only MBB goods: $x_{ij} > 0$ only if $u_{ij}/p_j = \alpha_i$. As noted above, for uncapped buyers every demand bundle is thrifty.

Definition 1.2. A pair $(\mathbf{x}, \mathbf{p})$ is a *thrifty and modest equilibrium* if (1) $\mathbf{x}_i$ is a thrifty and modest demand bundle under prices $\mathbf{p}$ for every buyer $i \in B$, and (2) $\sum_i x_{ij} p_j = p_j$ for every $j \in G$.

Note that here the equilibrium definition explicitly gives an allocation $\mathbf{x}$ instead of the money flow $\mathbf{f}$. This is because in markets with utility limits, an equilibrium may have some good j with price zero. In this case, all money flows f_{ij} are zero, but there might be positive allocation $x_{ij} > 0$ toward some buyer i. In this case, deriving the allocation from money flow via $x_{ij} = f_{ij}/p_j$ would not be well defined.

Thrifty and modest equilibria are desirable, because they capture the behavioral assumption that each buyer spends the least amount of money to obtain a utility maximizing bundle of goods. We observe below that these equilibria also have allocations $\mathbf{x}$ that are Pareto-optimal.

2 STRUCTURE OF EQUILIBRIA

2.1 Earning Limits

We first look at the structure of thrifty equilibria in spending-constraint Fisher markets with earning limits. Recall that, by definition, in any thrifty equilibrium uncapped goods are available in full supply, capped goods in thrifty supply.

Proposition 2.1. *Across all thrifty equilibria: (1) The seller incomes are unique, and (2) there is a unique set of uncapped goods, and their prices are unique.*

These uniqueness properties are a direct consequence of the fact that thrifty equilibria are the solutions of the following convex program [15]:

$$\begin{aligned}
\text{Max.} \quad & \sum_{i\in B}\sum_{j\in G}\sum_{(i,j,k)\in K_{ij}} f_{ijk}\log u_{ijk} - \sum_{j\in G}(q_j \log q_j - q_j) \\
\text{s.t.} \quad & \sum_{j\in G}\sum_{(i,j,k)\in K_{ij}} f_{ijk} = m_i \quad \forall i \in B \\
& \sum_{i\in B}\sum_{(i,j,k)\in K_{ij}} f_{ijk} = q_j \quad \forall j \in G \\
& f_{ijk} \le c_{ijk} \quad \forall (i,j,k) \in K_{ij} \\
& q_j \le d_j \quad \forall j \in G \\
& f_{ijk} \ge 0 \quad \forall i \in B,\ j \in G, (i,j,k) \in K_{ij}.
\end{aligned} \tag{1}$$

The incomes of sellers and, consequently, the sets of capped and uncapped goods are unique in all thrifty equilibria. The money flow, allocation, and prices of capped goods might not be unique.

Buyers always spend all their budget, but this can be impossible when every seller must not earn more than its limit.[1] Then a thrifty equilibrium does not exist. This, however, turns out to be the only obstruction to nonexistence.

Let $\hat{B} \subseteq B$ be a set of buyers, and $N(\hat{B}) = \{j \in G \mid u_{ij1} > 0, i \in \hat{B}\}$ be the set of goods such that there is at least one buyer in $\hat{B}$ with positive utility on its first segment for the good. The following *money clearing* condition states that buyers can spend their money without violating the earning limits.

Definition 2.1 (Money Clearing). A market is *money clearing* if for every subset of buyers $\hat{B} \subseteq B$ there is a flow f such that

$$\begin{aligned}
f_{ij} &\le \sum_{k=1}^{k^+} c_{ijk}, && \forall i \in \hat{B}, \forall j \in N(\hat{B}), k^+ = \max\{k \mid u_{ijk} > 0\} \\
\sum_{i\in\hat{B}} f_{ij} &\le d_j, && \forall j \in N(\hat{B}) \quad \text{and} \quad \sum_{j\in N(\hat{B})} f_{ij} \ge m_i, \quad \forall i \in \hat{B}.
\end{aligned} \tag{MC}$$

Money clearing is clearly necessary for the existence of a thrifty equilibrium. It is also sufficient, since, e.g., our algorithm in Section 3 will successfully compute an equilibrium iff money clearing holds. Alternatively, it can be verified that this is the unique necessary and sufficient feasibility condition for convex program (1). It is easy to check condition (MC) by a max-flow computation. We therefore assume that our market instance satisfies it.

Lemma 2.1. *A thrifty equilibrium exists iff the market is money clearing.*

Let us define some more useful concepts for the analysis. For any pair $(\mathbf{f}, \mathbf{p})$ the *surplus of buyer* i is given by $s(i) = \sum_{j\in G} f_{ij} - m_i$, and the *surplus of good* j is $s(j) = p_j^a - \sum_{i\in B} f_{ij}$. The *active-segment graph* $G(\mathbf{p})$ is a bipartite graph $(B \cup G, E)$ that contains edge (i, j) iff there is some active segment (i, j, k). Note that there can be at most one active segment (i, j, k) for an (i, j). A market is called *non-degenerate* if the active segment graph for any non-zero $\mathbf{p}$ is a forest. Non-degeneracy can always be obtained by a perturbation of the utilities without changing the set of equilibria, as shown in [22]. This is because the spending of a buyer on non-active MBB segments does not change with this perturbation, and hence the market on the active segment graph is effectively a linear market.

[1] Consider the example of a linear market with one buyer and one good. The utility is $u_{11} > 0$, the buyer has a budget $m_1 = 2$, and the good has an earning limit $d_1 = 1$.

2.1.1 Lattice Structure—Some Intuition. We start by providing some intuition for the structural results in the case where all utility functions are linear, i.e., with a single segment in every K_{ij}. Consider a thrifty equilibrium $(\mathbf{f}, \mathbf{p})$. Call an edge (i, j) $\mathbf{p}$-MBB if $u_{ij}/p_j = \alpha_i$. The active-segment graph here simplifies to an *MBB graph* $G(\mathbf{p})$.

We will first argue that when a connected component C of $G(\mathbf{p})$ contains only capped goods then it is possible to change the prices of goods in C while maintaining an equilibrium. Then, using this, we argue that it is possible to merge components to arrive at an equilibrium where each component of the MBB graph contains at least one uncapped good.

Let C be any connected component of the MBB graph. The buyers in C spend all budget on the goods in C, and no other buyer spends money on the goods in C. Thus,

$$\sum_{i \in C \cap B} m_i = \sum_{j \in C \cap G} p_j^a = \sum_{j \in C \cap G_u} p_j + \sum_{j \in C \cap G_c} d_j,$$

where G_c and G_u are the sets of capped and uncapped goods, respectively.

First, assume all goods in C are capped. Let r be a positive real and consider the pair $(\mathbf{f}, \mathbf{p}')$, where $p'_j = r \cdot p_j$ if $j \in C \cap G_c$ and $p'_j = p_j$ otherwise. Note that the allocations for any good $j \in C \cap G_c$ are scaled by $1/r$. The pair $(\mathbf{f}, \mathbf{p}')$ is an equilibrium provided that all edges with positive allocation are also $\mathbf{p}'$-MBB and $p'_j \geq d_j$ for all $j \in C \cap G_c$. This certainly holds for $r > 1$ and $r - 1$ sufficiently small. If $p_j > d_j$ for all $j \in C$, then this also holds for $r < 1$ and $1 - r$ sufficiently small. Thus, there is some freedom in choosing the prices in components containing only capped goods even for a fixed MBB graph. For non-degenerate instances, the money flow is unique (but not the allocation).

Now assume that there is at least one uncapped good in C, and let j_u be such an uncapped good. The price of any other good j in the component is linearly related to the price j_u, i.e., $p_j = \gamma_j p_{j_u}$, where γ_j is a rational number whose numerator and denominator is a product of utilities. Thus,

$$\sum_{i \in C \cap B} m_i = \sum_{j \in C \cap G} p_j^a = \sum_{j \in C \cap G_u} \gamma_j p_{j_u} + \sum_{j \in C \cap G_c} d_j,$$

and the reference price is uniquely determined. All prices in the component are uniquely determined. For a non-degenerate instance the money flow and allocation are also uniquely determined.

Suppose in a component C containing only capped goods we increase the prices by a common factor $r > 1$. We raise r continuously until a new MBB edge arises. If we can raise r indefinitely, then no buyer in the component is interested in any good outside the component. Otherwise, a new MBB edge arises, and then C is united with some other component. At this moment, the money flow over the new MBB edge is zero. If the newly formed component contains an uncapped good, then prices in the component are fixed and money flow is exactly as in the moment of joining the components. Otherwise, we raise all prices in the newly formed component, and so on. If the market is non-degenerate, then money flow is unique, and money will never flow on the new MBB edge.

If the component contains only capped goods j with $p_j > d_j$, then we can decrease prices continuously by a common factor $r < 1$ until a new MBB edge arises. If no MBB edge ever arises, then no buyer outside the component is interested in any good in the component, which allows to argue as above.

We have so far described how the prices in a component of the MBB graph of an equilibrium are determined if at least one good is uncapped and how the prices can be scaled by a common factor if all goods are capped. We have also discussed how components are merged and that the new MBB edge arising in a merge will never carry nonzero flow. Components can also be split if they contain an edge with zero flow.

Consider an equilibrium $(\mathbf{f}, \mathbf{p})$ and assume $f_{ij} = 0$ for some edge (i, j) of the MBB graph w.r.t. $\mathbf{p}$. Let C be the component containing (i, j), and let C_1 and C_2 be the components of $C \setminus \{i, j\}$. Let the instance be non-degenerate. Hence, the MBB graph is a forest. If we want to retain all MBB edges within C_1 and C_2 and only drop (i, j), then we have to either increase all prices in the subcomponent containing j or decrease all prices in the subcomponent containing i. Both options are infeasible if both components contain a good with price strictly below its earning limit. The first option is feasible if the component containing j contains only goods with prices at least their earning limits. The latter option is feasible if the component containing i contains only goods with prices strictly larger than their earning limits. The split does not affect the money flow.

If the above described changes allow to change any equilibrium into any other equilibrium, then money flow should be unique across all equilibria. Moreover, the set of edges carrying flow should be the same in all equilibria. The MBB graph for an equilibrium contains these edges, and maybe some more edges that do not carry flow. Next, we prove that this intuition captures the truth, even for the general case of spending-constraint utility functions.

2.1.2 Lattice Structure—Formal Proofs. We characterize the set of price vectors of thrifty equilibria, which we denote by

$$\mathcal{P} = \{\mathbf{p} \mid \exists \mathbf{f} \text{ s.t. } (\mathbf{f}, \mathbf{p}) \text{ is a thrifty equilibrium}\}.$$

For money clearing markets, we establish two results: (1) The set of equilibrium price vectors forms a lattice and (2) the money flow is unique in non-degenerate markets. The proof relies on the following structural properties. Given $\mathbf{p}$ and $\mathbf{p}'$, we partition the set of goods into sets $S_r = \{j \mid p'_j = r \cdot p_j\}$, for $r > 0$. For a price vector $\mathbf{p}$, let segment (i, j, k) be $\mathbf{p}$-MBB if $u_{ijk}/p_j \geq \alpha_i$ and $\mathbf{p}$-active if $u_{ijk}/p_j = \alpha_i$. For a set T of goods and an equilibrium $(\mathbf{f}, \mathbf{p})$, let

$$K(T, \mathbf{p}) = \{(i, j, k) \mid \text{segment is } \mathbf{p}\text{-MBB for some } j \in T\},$$
$$K_a(T, \mathbf{p}) = \{(i, j, k) \mid f_{ijk} > 0 \text{ for some } j \in T \text{ and some equilibrium } (\mathbf{f}, \mathbf{p})\},$$

where the sets denote the set of $\mathbf{p}$-MBB segments for goods in T and the ones on which some good in T is allocated. Note that $K_a(T, \mathbf{p}) \subseteq K(T, \mathbf{p})$.

Lemma 2.2. *For any two thrifty equilibria $E = (\mathbf{f}, \mathbf{p})$ and $E' = (\mathbf{f}', \mathbf{p}')$:*

(1) $K_a(S_r, \mathbf{p}) = K_a(S_r, \mathbf{p}')$ for every $r > 0$, i.e., for each of the two price vectors the union of all possible equilibrium money flows will use the same set of segments.
(2) $K_a(S_r, \mathbf{p}) = K_a(S_r, \mathbf{p}') \subseteq K(S_r, \mathbf{p}') \subseteq K(S_r, \mathbf{p})$ for $r > 1$. Similarly, $K_a(S_r, \mathbf{p}') = K_a(S_r, \mathbf{p}) \subseteq K(S_r, \mathbf{p}) \subseteq K(S_r, \mathbf{p}')$ for $r < 1$.
(3) If $f_{ijk} > 0$ for $(i, j, k) \in K_a(S_r, \mathbf{p})$ with $r > 1$, then (i, j, k) is $\mathbf{p}'$-MBB. If $f'_{ijk} > 0$ for $(i, j, k) \in K_a(S_r, \mathbf{p}')$ with $r < 1$, then (i, j, k) is $\mathbf{p}$-MBB.

Proof. For the analysis we also consider

$$B(T, \mathbf{p}) = \{i \mid \exists (i, j, k) \in K(T, \mathbf{p})\},$$
$$B_a(T, \mathbf{p}) = \{i \mid \exists (i, j, k) \in K_a(T, \mathbf{p})\},$$

as the sets of buyers corresponding to $K(T, \mathbf{p})$ and $K_a(T, \mathbf{p})$, where $B_a(T, \mathbf{p}) \subseteq B(T, \mathbf{p})$.

We first focus on S_{r_1} with $r_1 = \max_j p'_j/p_j$, i.e., the set of goods with largest factor of price increase from $\mathbf{p}$ to $\mathbf{p}'$. For any $i \in B(S_{r_1}, \mathbf{p}')$, there is some $(i, j, k) \in K(S_{r_1}, \mathbf{p}')$ such that $u_{ijk}/p'_j \geq u_{ij'k'}/p'_{j'}$ for all $\mathbf{p}'$-active (i, j', k') with $j' \notin S_{r_1}$. Since $u_{ijk}/p_j = r_1 u_{ijk}/p'_j$ and $r_1 u_{ij'k'}/p'_{j'} > u_{ij'k'}/p_{j'}$ we conclude $K(S_{r_1}, \mathbf{p}') \subseteq K(S_{r_1}, \mathbf{p})$.

Next we analyze the total money spent on segments with $j \in S_{r_1}$ by buyers in $B(S_{r_1}, \mathbf{p}')$, with respect to equilibria E and E'. Since the prices p'_j of goods $j \in S_{r_1}$ decrease by the largest factor, the spending on these goods in E can only increase. In fact, we have that

$$\sum_{(i,j,k)\in K(S_{r_1},\mathbf{p})} f_{ijk} \geq \sum_{(i,j,k)\in K(S_{r_1},\mathbf{p})} f'_{ijk} \ , \quad \text{for every buyer } i \in B. \tag{2}$$

This implies

$$\begin{aligned} \sum_{j\in S_{r_1}} p_j^a &= \sum_{i\in B(S_{r_1},\mathbf{p})} \sum_{(i,j,k)\in K(S_{r_1},\mathbf{p})} f_{ijk} \\ &\geq \sum_{i\in B(S_{r_1},\mathbf{p}')} \sum_{(i,j,k)\in K(S_{r_1},\mathbf{p}')} f'_{ijk} = \sum_{j\in S_{r_1}} p'^a_j. \end{aligned} \tag{3}$$

However, since $p'_j > p_j$ for every $j \in S_{r_1}$, this can only be fulfilled when the inequalities in Equations (2) and (3) are equalities. In particular, all goods in S_{r_1} must exactly reach their earning limit in both E and E' (as already observed in Proposition 2.1 part 2). Moreover, in E, no $i \in B(S_{r_1}, \mathbf{p}) \setminus B(S_{r_1}, \mathbf{p}')$ can ever receive allocation from goods in S_{r_1}. Hence, $B_a(S_{r_1}, \mathbf{p}) = B_a(S_{r_1}, \mathbf{p}')$.

In both E and E' each buyer $i \in B_a(S_{r_1}, \mathbf{p})$ spends the same amount of money on S_{r_1}, which we denote by $m_i(S_{r_1})$. Every buyer spends on segments in non-increasing order of u_{ijk}/p_j. This implies that a segment is $\mathbf{p}$-MBB iff it is $\mathbf{p}'$-MBB. The possible allocations are the solution of a transportation problem, where each good $j \in S_{r_1}$ receives d_j flow, and each buyer $i \in B_a(S_{r_1}, \mathbf{p})$ emits $m_i(S_{r_1})$ flow, routed over the same set of MBB edges in non-increasing order of bang-per-buck ratio. Every such allocation is a possible spending in both equilibria. This implies $K_a(S_{r_1}, \mathbf{p}) = K_a(S_{r_1}, \mathbf{p}')$. Note that $K_a(S_{r_1}, \mathbf{p}') \subset K(S_{r_1}, \mathbf{p}')$ when there are two $\mathbf{p}'$-active segments $(i, j, k), (i, j', k') \in K(S_{r_1}, \mathbf{p}')$ with $f_{ijk} = 0$ and $f_{ij'k'} > 0$.

In this sense, the spending and the way goods are allocated in $\mathbf{p}$ remains a feasible assignment on $\mathbf{p}'$-MBB segments. As such, we can drop the goods from S_1 from consideration. Then, we can apply the analysis in the same way for $r_2 = \max_{j\notin S_{r_1}} p'_j/p_j$ and S_{r_2}. Iterative application shows the properties for all S_r with $r > 1$; that is, $K_a(S_r, \mathbf{p}) = K_a(S_r, \mathbf{p}')$ and $K_a(S_r, \mathbf{p}) \subseteq K(S_r, \mathbf{p}') \subseteq K(S_r, \mathbf{p})$. Reversing the role of $E = (\mathbf{f}, \mathbf{p})$ and $E' = (\mathbf{f}', \mathbf{p}')$, we obtain the same claims for sets S_r with $r < 1$. That is, $K_a(S_r, \mathbf{p}') = K_a(S_r, \mathbf{p})$, $K_a(S_r, \mathbf{p}) \subseteq K(S_r, \mathbf{p}) \subseteq K(S_r, \mathbf{p}')$. Finally, since all segments $K_a(S_r, \mathbf{p}) = K_a(S_r, \mathbf{p}')$ and all buyer sets $B_a(S_r, \mathbf{p}) = B_a(S_r, \mathbf{p}')$, for every $r \neq 1$, this must also hold for $r = 1$. This proves parts 1 and 2. Part 3 is a consequence of part 2—since $K_a(S_r, \mathbf{p}) = K_a(S_r, \mathbf{p}')$, every $\mathbf{p}$-MBB segment with $f_{ijk} > 0$ is $\mathbf{p}'$-MBB and vice versa. This proves part 3 and concludes the proof. □

For $r = 1$ and the goods S_1, part 1 of the lemma implies $K_a(S_1, \mathbf{p}) = K_a(S_1, \mathbf{p}')$. Inspecting part 2 of the lemma, the reader might be tempted to believe that $K(S_1, \mathbf{p}) = K(S_1, \mathbf{p}')$ as well. This, however, is not necessarily the case.

Example 2.1. Consider a linear market with two buyers and two goods. Buyer 1 has $u_{11} = 15$, $u_{12} = 1$; buyer 2 has $u_{21} = 0$, $u_{22} = 1$. The budgets are $m_1 = m_2 = 1$, and the earning limits $d_1 = 1$, $d_2 = \infty$. Prices $\mathbf{p} = (15, 1)$ and $\mathbf{p}' = (14, 1)$ correspond to thrifty equilibria. The flow in both equilibria is given by $\mathbf{f} = \mathbf{f}'$ with $f_{11} = f_{22} = 1$ and $f_{12} = f_{21} = 0$. Note that $S_1 = \{2\}$. Now $K(S_1, \mathbf{p}) = \{(1, 2), (2, 2)\}$ and $K(S_1, \mathbf{p}') = \{(2, 2)\}$. Hence $K(S_1, \mathbf{p}) \neq K(S_1, \mathbf{p}')$, although we have $K_a(S_1, \mathbf{p}) = K_a(S_1, \mathbf{p}') = \{(2, 2)\}$.

For the main result of this section, we consider the coordinate-wise comparison of price vectors, i.e., $\mathbf{p} \geq \mathbf{p}'$ iff $p_j \geq p'_j, \forall j \in G$. Moreover, for price vectors $\mathbf{p}$ and $\mathbf{p}'$ we consider the supremum $\overline{\mathbf{p}}$ and the infimum $\underline{\mathbf{p}}$, i.e., $\overline{p}_j = \max(p_j, p'_j)$ and $\underline{p}_j = \min(p_j, p'_j)$.

Theorem 2.1. *The pair $(\mathcal{P}, \geq)$ is a lattice.*

Proof. Consider any two thrifty equilibria $E = (\mathbf{f}, \mathbf{p})$ and $E' = (\mathbf{f}', \mathbf{p}')$. We show that supremum $\overline{\mathbf{p}}$ and infimum $\underline{\mathbf{p}}$ are price vectors of thrifty equilibria. This property implies the lattice structure.

We first consider the pair $(\mathbf{f}, \overline{\mathbf{p}})$. Due to Proposition 2.1 part 2, this state is feasible with respect to earning limits and has thrifty supplies. It remains to show that the allocation is MBB. Compared to $\mathbf{p}$, $\overline{\mathbf{p}}$ has higher prices for the goods in S_r with $r > 1$. Hence the allocations to the goods in S_r with $r \leq 1$ are still MBB. Consider any good $j \in S_r$ with $r > 1$. If $f_{ijk} > 0$, then (i, j, k) is $\mathbf{p}'$-MBB by part 3 of Lemma 2.2. Thus, $u_{ijk}/\overline{p}_j = u_{ijk}/p'_j = \alpha'_j \geq u_{ij'k'}/p'_{j'}$ for all $\mathbf{p}'$-active segments (i, j', k'). Since $p'_{j'} = \overline{p}_{j'}$ for $\ell \in S_r$ with $r > 1$ and $p'_{j'} \leq p_j = \overline{p}_j$ for $j' \in S_r$ with $r \leq 1$, we observe (i, j, k) is $\overline{\mathbf{p}}$-MBB. We conclude that $(\mathbf{f}, \overline{\mathbf{p}})$ is a thrifty equilibrium.

Let us now consider the pair $(\underline{\mathbf{f}}, \underline{\mathbf{p}})$ with spending $\underline{\mathbf{f}}$ defined as

$$\underline{f}_{ijk} = \begin{cases} f_{ijk} & \text{if } j \in S_r \text{ with } r > 1 \\ f'_{ijk} & \text{if } j \in S_r \text{ with } r \leq 1 \end{cases}.$$

Again, the new state $(\underline{\mathbf{f}}, \underline{\mathbf{p}})$ is feasible with respect to earning limits and has thrifty supplies. It remains to show that the allocation is MBB.

Consider the goods in S_r with $r > 1$ and a buyer $i \in B_a(S_r, \mathbf{p})$. For prices $\mathbf{p}'$, we know by part 3 of Lemma 2.2 that for buyer i every segment (i, j, k) with $f_{ijk} > 0$ is $\mathbf{p}'$-MBB. Now, to reach $\underline{\mathbf{p}}$, we keep prices of S_r with $r \leq 1$ as in $\mathbf{p}'$ and decrease the prices of S_r with $r > 1$ to $\mathbf{p}$. As such, i does not obtain new MBB segments for goods in S_r with $r \leq 1$. For the remaining goods in S_r with $r \geq 1$, however, the allocation for i is MBB, since prices and spending for these goods are as in equilibrium E.

Similarly, consider the goods in S_r with $r < 1$ and a buyer $i \in B_a(S_r, \mathbf{p}')$. For prices $\mathbf{p}$, we know by part 3 of Lemma 2.2 that for buyer i every segment (i, j, k) with $f'_{ijk} > 0$ is $\mathbf{p}$-MBB. Now, to reach $\underline{\mathbf{p}}$, we keep prices of S_r with $r \geq 1$ as in $\mathbf{p}$ and decrease the prices of S_r with $r < 1$ to $\mathbf{p}'$. As such, i does not obtain new MBB segments for goods in S_r with $r \geq 1$. For the remaining goods in S_r with $r < 1$, however, the allocation for i is MBB, since rices and spending for these goods are as in equilibrium E'.

Finally, consider the goods in S_1 and a buyer $i \in B_a(S_r, \mathbf{p}') = B_a(S_r, \mathbf{p})$. Hence, since for $j \in S_1$ we have $p_j = p'_j$, a segment (i, j, k) is $\mathbf{p}$-MBB iff it is $\mathbf{p}'$-MBB. Repeating the above arguments for $r > 1$ and $r < 1$, we observe that for buyer i no new MBB segments evolve in S_r with $r \neq 1$. Hence, the spending $\underline{f}_{ij}$ is MBB for i.

We conclude that $(\underline{\mathbf{f}}, \underline{\mathbf{p}})$ is a thrifty equilibrium. □

Corollary 2.1. *There exists a thrifty equilibrium with coordinate-wise lowest prices. Among all thrifty equilibria, it yields the largest supply in the market and the maximum utility for every buyer.*

Theorem 2.2. *In a non-degenerate market, all thrifty equilibria have the same money flow.*

We first observe the following fact about transportation problems on forests.

Lemma 2.3. *The solution for a transportation problem on a forest is unique.*

Proof. Let $e = (x, y)$ be any edge of the forest. Removal of e splits the tree containing e into two sets X and Y with $x \in X$ and $y \in Y$. The flow across e in the direction from x to y is $\sum_{u \in X} b(u) = -\sum_{v \in Y} b(y)$. Note that $\sum_{w \in X \cup Y} b(w) = 0$.

Alternatively, we may consider any edge (x, y) incident to a leaf x in the forest. Then the flow across the edge (x, y) is equal to $b(x)$. We add $b(x)$ to $b(y)$, remove x, and iterate. □

Proof of Theorem 2.2. Consider the equilibrium $E = (\mathbf{f}, \mathbf{p})$ with smallest prices. Suppose there is another equilibrium $E' = (\mathbf{f}', \mathbf{p}')$ with prices $\mathbf{p}' \geq \mathbf{p}$. By Lemma 2.3, there are unique money flows in $\mathbf{f}$ and $\mathbf{f}'$ in E and E', respectively. Every good $j \in S_1$ with $p_j = p'_j$ has inflow p^a_j in both equilibria. Every good with $p'_j > p_j$ has inflow d_j in both equilibria due to Proposition 2.1 part 2. Every MBB segment (i, j, k) with $f_{ijk} > 0$ remains MBB under $\mathbf{p}'$ due to Lemma 2.2 part 3. Thus, $\mathbf{f}$ remains a feasible flow for E'. Since by Lemma 2.3 money flows are unique, we have $\mathbf{f} = \mathbf{f}'$. □

The convex program implies that there is a unique income for each seller. This is consistent with our observation that a good can have different prices in two equilibria only when income equals its earning limit.

While existence of an equilibrium with smallest prices is guaranteed, we might or might not have an equilibrium with coordinate-wise largest prices (e.g., when all goods are capped in equilibrium, prices can be raised indefinitely).

Consider a linear market with two buyers and two goods. Let $u_{11} = u_{22} = 1$, $u_{21} = 1/2$ and $u_{12} = 0$. Let $m_1 = m_2 = 1$ and $d_2 = 1$. Then $x_{11} = x_{22} = 1$, $x_{21} = 0$, $p_1 = p_2 = 1$ is a thrifty equilibrium with largest buyer utility. All thrifty equilibria have the same allocation, all have price $p_1 = 1$, and $p_2 \in [1, 2]$. Hence, in this market $(p_1, p_2) = (1, 2)$ yield largest prices and smallest buyer utility in any thrifty equilibrium.

Now consider a linear market with a single buyer and a single good. Let $u_{11} = 1$, $m_1 = 1$ and $d_1 = 1$. Then $x_{11} = 1$, $p_1 = 1$ is a thrifty equilibrium with largest buyer utility. There are infinitely many other equilibria $x_{11} = 1$, $p_1 \geq 1$, and there is no equilibrium with largest prices.

2.2 Utility Limits

Consider the following Eisenberg-Gale-type program (4), which allows us to find a modest and Pareto-optimal allocation,

$$\begin{aligned} \text{Max.} \quad & \sum_{i \in B} m_i \log \sum_{j \in G} u_{ij} x_{ij} & & & \\ \text{s.t.} \quad & \sum_{j \in G} u_{ij} x_{ij} & \leq & c_i & i \in B \\ & \sum_{i \in B} x_{ij} & \leq & 1 & j \in G \\ & x_{ij} & \geq & 0 & i \in B,\ j \in G. \end{aligned} \tag{4}$$

By standard arguments, we consider the dual for Equation (4) using dual variables γ_i and p_j for the first two constraints, respectively, and the KKT conditions read:

(1) $p_j/u_{ij} \geq m_i/u_i - \gamma_i$
(2) $x_{ij} > 0 \Rightarrow p_j/u_{ij} = m_i/u_i - \gamma_i$
(3) $p_j \geq 0$ and $p_j > 0 \Rightarrow \sum_{i \in B} x_{ij} = 1$
(4) $\gamma_i \geq 0$ and $\gamma_i > 0 \Rightarrow u_i = c_i$.

Observe that the Lagrange multiplier γ_i indicates if the cap c_i represents a tight constraint in the optimum solution. The dual variables p_j can be interpreted as prices. Note that conditions 1 and 2 imply that $x_{ij} > 0$ if and only if $j \in \arg\min_{j'} p_{j'}/u_{ij'} = \arg\max_{j'} u_{ij'}/p_{j'}$, i.e., all agents purchase goods with maximum bang-per-buck. Hence, similarly as for linear markets [38], the KKT conditions imply that an optimal solution to the EG program (4) and corresponding dual prices constitute a market equilibrium, in which every agent buys goods that have maximum bang-per-buck. The KKT conditions postulate this also for agents whose utility reaches the cap. Thus, the

optimal solution to this program is a thrifty and modest equilibrium. Furthermore, we obtain the following favorable analytical properties.

Proposition 2.2. *The optimal solutions to Equation (4) are exactly the thrifty and modest equilibria. The utility vector is unique across all such equilibria and each such equilibrium is Pareto-optimal. In particular, there is a unique set of capped buyers. Non-capped buyers spend all their money. Capped buyers do not overspend.*

Proof. We observe first that there is an interior feasible solution to Equation (4). Simply set $x_{ij} = \epsilon > 0$ for all i and j, where ϵ is small enough such all constraints in Equation (4) are satisfied with inequality. The existence of an interior feasible solution guarantees that the KKT conditions are necessary and sufficient for an optimal solution to Equation (4).

Let $\mathbf{x}$ and $\mathbf{x}'$ be two optimal solutions to Equation (4) and assume that $u_h(\mathbf{x}) \neq u_h(\mathbf{x}')$ for some buyer h. Consider the allocation $\mathbf{x}'' = (\mathbf{x} + \mathbf{x}')/2$. It is clearly feasible. Also,

$$\sum_{i \in B} m_i \log u_i(\mathbf{x}'') > \left(\sum_{i \in B} m_i \log u_i(\mathbf{x}) + \sum_{i \in B} m_i \log u_i(\mathbf{x}') \right) /2,$$

a contradiction to the optimality of the allocation. The inequality follows from the concavity of the log-function. We have now shown that the utilities of the buyers are unique among all optimal solutions of Equation (4). Thus, every optimal solution to Equation (4) is thrifty, modest, and Pareto-optimal.

Conversely, let $(\mathbf{x}, \mathbf{p})$ be a thrifty and modest equilibrium. We show that $\mathbf{x}$ is an optimal solution to Equation (4). $\mathbf{x}$ is feasible, since it is modest and does not overallocate any good. Since $\mathbf{x}_i$ is a thrifty demand bundle for buyer i, we have $u_{ij}/p_j = \alpha_i = \max_\ell u_{i\ell}/p_\ell$ whenever $x_{ij} > 0$. Thus

$$m_i \geq \sum_j p_j x_{ij} = \sum_j \frac{u_{ij}}{\alpha_i} x_{ij} = \frac{u_i(\mathbf{x})}{\alpha_i},$$

and hence $m_i/u_i(\mathbf{x}) \geq 1/\alpha_i$. Let $\gamma_i = m_i/u_i - 1/\alpha_i$. Then $\gamma_i \geq 0$. We show that the KKT conditions hold. For any j, we have $p_j/u_{ij} \geq 1/\alpha_i = m_i/u_i - \gamma_i$. If $x_{ij} > 0$, then $p_j/u_{ij} = 1/\alpha_i = m_i/u_i - \gamma_i$. Prices are non-negative by definition and $p_j > 0$ implies that good j is completely allocated by Walras's law. Finally, assume $\gamma_i > 0$. Then $m_i/u_i(\mathbf{x}) > 1/\alpha_i$ and hence

$$m_i > \frac{u_i(\mathbf{x})}{\alpha_i} = \frac{\sum_j u_{ij} x_{ij}}{\alpha_i} = \sum_j p_j x_{ij},$$

where the first equality follows from the fact that the allocation is modest. Let $\delta = m_i / \sum_j p_j x_{ij}$. Then buyer i could afford the bundle $\delta \mathbf{x}_i$. Since $\mathbf{x}_i$ is a demand bundle for buyer i, we must have $c_i \leq \sum_j u_{ij} x_{ij}$. Since the allocation is modest, we have equality. □

While utilities are unique, allocation and prices of thrifty and modest equilibria might not be unique. Consider a market with two identical buyers and two goods, where $u_{11} = u_{12} = u_{21} = u_{22} = 1$, $c_1 = c_2 = 1$, and $m_1 = m_2 = 5$. The unique equilibrium utility of both buyers is $u_1 = u_2 = 1$, which can be obtained for any $p_1 = p_2 = p$, where $p \in [0, 5]$ and allocation $\mathbf{x}$ satisfying $x_{11} + x_{12} = 1$; $x_{21} + x_{22} = 1$; $x_{11} + x_{21} = 1$; $x_{12} + x_{22} = 1$.

2.2.1 Lattice Structure. Our proofs roughly proceed along the lines of Section 2.1, however, with several notable differences. We characterize the set of price vectors of thrifty and modest equilibria, for which we again use the notation

$$\mathcal{P} = \{\mathbf{p} \mid \exists \mathbf{x} \text{ s.t. } (\mathbf{x}, \mathbf{p}) \text{ is a thrifty and modest equilibrium } \}.$$

For markets with utility limits, we establish three results, which in part mirror the results for earning limits above: (1) For utility limits, the market decomposes into the market for the goods G_0 that have price zero in every equilibrium and the market for the remaining goods. (2) The set of equilibrium price vectors forms a lattice. (3) The allocation of the goods in $G \setminus G_0$ is unique in all nondegenerate markets.

Theorem 2.3. *Let G_0 be the goods that have price zero in every equilibrium and let B_0 be the set of buyers that have positive utility for some good in G_0. Then*

$$\mathcal{P}_{(B,G)} = \mathcal{P}_{(B\setminus B_0, G\setminus G_0)} \times \mathcal{P}_{(B_0,G_0)}.$$

Proof. Let $(\mathbf{x}, \mathbf{p})$ be an equilibrium for the market with buyers $B \setminus B_0$ and goods $G \setminus G_0$ and $(\mathbf{x}', \mathbf{0})$ be an equilibrium for the market with buyers B_0 and goods G_0. Then $((\mathbf{x}, \mathbf{x}'), (\mathbf{p}, \mathbf{0}))$ is an equilibrium for the market with buyers B and goods G. We only have to show that all allocations are along MBB edges. This is true for the goods in G_0 as the price of these goods is zero. This is also true for the goods in $G \setminus G_0$ as the buyers in $B \setminus B_0$ have utility zero for the goods in G_0.

Conversely, let $(\mathbf{x}, (\mathbf{p}, \mathbf{0}))$ be an equilibrium for the market with buyers B and goods G. We need to show that no good in $G \setminus G_0$ is allocated to a buyer in B_0. Assume otherwise, say, $x_{ij} > 0$ for $i \in B_0$ and $j \in G \setminus G_0$. Let $(\mathbf{x}', \mathbf{p}')$ be an equilibrium for a price vector in which the good of price j is positive. In the equilibrium $((\mathbf{x} + \mathbf{x}')/2, (\mathbf{p} + \mathbf{p}')/2)$, j is partially allocated to i, j has a positive price, and the goods in G_0 have price zero. This contradicts the fact that i has nonzero utility for some good in G_0. □

For the other results, we again show a structural lemma. Given $\mathbf{p}$ and $\mathbf{p}'$, we here rely on a partition of the set of goods into three sets: $S_= = \{j \mid p_j = p'_j\}$, $S_< = \{j \mid p_j < p'_j\}$ and $S_> = \{j \mid p_j > p'_j\}$. For a price vector $\mathbf{p}$, call (i, j) $\mathbf{p}$-MBB if $u_{ij}/p_j = \alpha_i$. For a set T of goods and an equilibrium $(\mathbf{x}, \mathbf{p})$, we again a notation similar as in the proof of Lemma 2.2. More concretely, we use

$$\begin{aligned} B(T, \mathbf{p}) &= \{i \mid (i, j) \text{ is } \mathbf{p}\text{-MBB for some } j \in T\}, \\ B_a(T, \mathbf{x}) &= \{i \mid x_{ij} > 0 \text{ for some } j \in T\}, \\ B_a(T, \mathbf{p}) &= \{i \mid x_{ij} > 0 \text{ for some } j \in T \text{ and some equilibrium } (\mathbf{x}, \mathbf{p})\}, \end{aligned}$$

to denote the set of buyers who are connected to T through an $\mathbf{p}$-MBB edge, who are allocated some good in T, and who are some good in T in some equilibrium $(\mathbf{x}, \mathbf{p})$. Clearly $B_a(T, \mathbf{x}) \subseteq B_a(T, \mathbf{p}) \subseteq B(T, \mathbf{p})$.

Lemma 2.4. *Given any two thrifty and modest equilibria $E = (\mathbf{x}, \mathbf{p})$ and $E' = (\mathbf{x}', \mathbf{p}')$, we have*

1. *$B_a(S_=, \mathbf{p}) = B_a(S_=, \mathbf{p}')$, $B_a(S_<, \mathbf{p}) = B_a(S_<, \mathbf{p}')$, and $B_a(S_>, \mathbf{p}) = B_a(S_> \mathbf{p}')$, i.e., the goods are allocated to the same set of buyers in both equilibria.*
2. *$B_a(S_=, \mathbf{p})$, $B_a(S_<, \mathbf{p})$, and $B_a(S_>, \mathbf{p})$ as well as $B_a(S_=, \mathbf{p}')$, $B_a(S_<, \mathbf{p}')$, and $B_a(S_>, \mathbf{p}')$ are mutually disjoint.*
3. *$B_a(S_<, \mathbf{x}) = B_a(S_<, \mathbf{x}') = B_a(S_<, \mathbf{p}') = B_a(S_<, \mathbf{p}) = B(S_<, \mathbf{p}') \subseteq B(S_<, \mathbf{p})$ and $B_a(S_>, \mathbf{x}) = B_a(S_>, \mathbf{x}') = B_a(S_>, \mathbf{p}') = B_a(S_>, \mathbf{p}) = B(S_>, \mathbf{p}) \subseteq B(S_>, \mathbf{p}')$*
4. *All buyers in $B_a(S_<, \mathbf{p})$ and $B_a(S_>, \mathbf{p})$ are capped buyers in both equilibria.*
5. *If $x_{ij} > 0$ for $i \in B_a(S_<, \mathbf{p})$, then (i, j) is $\mathbf{p}'$-MBB. If $x'_{ij} > 0$ for $i \in B_a(S_>, \mathbf{p}')$, then (i, j) is $\mathbf{p}$-MBB.*

Proof. We first focus on $S_<$, the set of goods whose prices strictly increase from $\mathbf{p}$ to $\mathbf{p}'$. For any $i \in \Gamma(S_<, \mathbf{p}')$, there is some $j \in S_<$ such that $u_{ij}/p'_j \geq u_{i\ell}/p'_\ell$ for all $\ell \notin S_<$. Since $u_{ij}/p_j > u_{ij}/p'_j$ and $u_{i\ell}/p'_\ell \geq u_{ij}/p_\ell$ we conclude

(a) No edge (i, j) is $\mathbf{p}$-MBB for $i \in B(S_<, \mathbf{p}')$ and $j \notin S_<$, and
(b) $B(S_<, \mathbf{p}') \subseteq B_a(S_<, \mathbf{x})$. Indeed, for every buyer $i \in B(S_<, \mathbf{p}')$ all incident $\mathbf{p}$-MBB edges connect it to a good in $S_<$ by part (a) and every buyer has at least one incident edge with positive allocation.

Next we analyze the total money spent on goods in set $S_<$ by buyers in $B(S_<, \mathbf{p}')$, with respect to equilibria E and E'. Due to fact (a), buyers in $B(S_<, \mathbf{p}')$ buy only goods in the set $S_<$ in E. Thus,

$$\sum_{i \in B(S_<, \mathbf{p}')} m_i^a = \sum_{i \in B(S_<, \mathbf{p}')} \sum_{j \in S_<} x_{ij} p_j,$$

where m_i^a is the money spent by buyer i in E. Let $m_i^{a\prime}$ be the money spent by buyer i in E'. Consider any buyer $i \in B(S_<, \mathbf{p}')$. If i is uncapped in E, then clearly $m_i^{a\prime} \le m_i = m_i^a$. If i is capped in E, then he spends $\sum_{j \in S_<} x_{ij} p_j$ and obtains utility c_i. In E', the cost of this bundle is $\sum_{j \in S_<} x_{ij} p_j'$ and hence i spends at most this amount. Hence, the total increase of spending of buyers in $B(S_<, \mathbf{p}')$ from $\mathbf{p}$ to $\mathbf{p}'$ will be no more than

$$\begin{aligned}\sum_{i \in B(S_<, \mathbf{p}')} (m_i^{a\prime} - m_i^a) &\le \sum_{\substack{i \in B(S_<, \mathbf{p}') \\ i \text{ is capped in } E}} (m_i^{a\prime} - m_i^a) \\ &\le \sum_{\substack{i \in B(S_<, \mathbf{p}') \\ i \text{ is capped in } E}} \sum_{j \in S_<} (x_{ij} p_j' - x_{ij} p_j).\end{aligned}$$

Further, due to the definition of $B(S_<, \mathbf{p}')$ and the fact that $(\mathbf{x}', \mathbf{p}')$ is a market equilibrium, we have

$$\sum_{j \in S_<} p_j' \le \sum_{i \in B(S_<, \mathbf{p}')} m_i^{a\prime}.$$

Combining these inequalities, we obtain

$$\sum_{j \in S_<} p_j' \le \sum_{i \in B(S_<, \mathbf{p}')} m_i^{a\prime} \tag{5}$$

$$= \sum_{i \in B(S_<, \mathbf{p}')} m_i^a + \sum_{\substack{i \in B(S_<, \mathbf{p}') \\ i \text{ is uncapped in } E}} (m_i^{a\prime} - m_i^a) + \sum_{\substack{i \in B(S_<, \mathbf{p}') \\ i \text{ is capped in } E}} (m_i^{a\prime} - m_i^a)$$

$$\le \sum_{i \in B(S_<, \mathbf{p}')} \sum_{j \in S_<} x_{ij} p_j + \sum_{\substack{i \in B(S_<, \mathbf{p}') \\ i \text{ is capped in } E}} \sum_{j \in S_<} (x_{ij} p_j' - x_{ij} p_j) \tag{6}$$

$$= \sum_{j \in S_<} \left(\sum_{\substack{i \in B(S_<, \mathbf{p}') \\ i \text{ is capped in } E}} x_{ij} \right) p_j' + \sum_{j \in S_<} \left(\sum_{\substack{i \in B(S_<, \mathbf{p}') \\ i \text{ is uncapped in } E}} x_{ij} \right) \left(p_j' + (p_j - p_j') \right)$$

$$= \sum_{j \in S_<} \sum_{i \in B(S_<, \mathbf{p}')} x_{ij} p_j' - \left(\sum_{\substack{i \in B(S_<, \mathbf{p}') \\ i \text{ is uncapped in } E}} x_{ij} \right) (p_j' - p_j)$$

$$\le \sum_{j \in S_<} p_j'. \tag{7}$$

We must have equality throughout. Equality (5) implies that the buyers in $B(S_<, \mathbf{p}')$ spend all their money on goods in $S_<$ in equilibrium E'. Thus any good outside $S_<$ allocated to a buyer in

$B(S_<, \mathbf{p}')$ would have to have price zero in $\mathbf{p}'$. Since it is impossible that a buyer has MBB edges to a good of price zero and a good of positive price, it follows that goods outside $S_<$ are completely allocated to buyers outside $B(S_<, \mathbf{p}')$. Hence $B_a(S_<, \mathbf{p}') = B(S_<, \mathbf{p}')$ and $B(S_<, \mathbf{p}')$ is disjoint from $B_a(S_=, \mathbf{p}') \cup B_a(S_>, \mathbf{p}')$. Equality (6) implies that buyers in $B(S_<, \mathbf{p}')$ that are uncapped in E are also uncapped in E' and those that are capped in E spend $\sum_{j \in S_<} x_{ij} p'_j$ in E' and hence can afford a bundle of utility c_i; thus, they are also capped in E'. Equality (7) implies

$$\sum_{i \in B(S_<, \mathbf{p}')} x_{ij} = 1 \quad \text{and} \quad \sum_{\substack{i \in B(S_<, \mathbf{p}') \\ i \text{ is uncapped in } E}} x_{ij} = 0 \tag{8}$$

for all $j \in S_<$, i.e., in equilibrium E the goods in $S_<$ are completely allocated to capped buyers in $B(S_<, \mathbf{p}')$. Thus, $B_a(S_<, \mathbf{p}) \subseteq B(S_<, \mathbf{p}')$, and all buyers in $B_a(S_<, \mathbf{p})$ are capped in E, and, by the above, also in E'. Combining the inclusions among $B_a(S_<, \mathbf{p})$, $B(S_<, \mathbf{p})$, $B_a(S_<, \mathbf{p}')$, and $B(S_<, \mathbf{p}')$, we obtain $B_a(S_<, \mathbf{p}) = B_a(S_<, \mathbf{p}') = B(S_<, \mathbf{p}') \subseteq B(S_<, \mathbf{p})$. The latter inclusion may be proper as Example 2.3 shows.

Since the buyers in $B_a(S_<, \mathbf{p})$ are capped in E and they achieve their cap by the allocation of goods in $S_<$, these buyers are not assigned any goods from $S_= \cup S_>$. Thus $B_a(S_<, \mathbf{p})$ is disjoint from $B_a(S_=, \mathbf{p}) \cup B_a(S_>, \mathbf{p})$.

We next exploit that $m_i^{a\prime} = \sum_{j \in S_<} x_{ij} p'_j$ for $i \in B(S_<, \mathbf{p}')$ and that these buyers are capped in E'. Let α'_i be the MBB-ratio for i in E'. Then $\alpha'_i = c_i / m_i^{a\prime} \geq u_{ij}/p'_j$ for all j and hence (note that $m_i^{a\prime} > 0$ and therefore $0 < \alpha'_i < \infty$),

$$c_i = \alpha'_i m_i^{a\prime} = \alpha'_i \sum_{j \in S_<} x_{ij} p'_j \leq \alpha'_i \sum_{j \in S_<} x_{ij} u_{ij} / \alpha'_i = c_i.$$

The inequality is thus an equality, and we conclude that $x_{ij} > 0$ implies that (i, j) is $\mathbf{p}'$-MBB for $i \in \Gamma(S_<, \mathbf{p}')$.

Reversing the role of $(\mathbf{x}, \mathbf{p})$ and $(\mathbf{x}', \mathbf{p}')$, we obtain the same claims for set $S_>$. That is, $B_a(S_>, \mathbf{p}) = B_a(S_>, \mathbf{p}')$, $B_a(S_>, \mathbf{p})$ has no overlap with $B_a(S_=, \mathbf{p}) \cup B_a(S_<, \mathbf{p})$, $B_1(S_>, \mathbf{p}')$ has no overlap with $B_a(S_=, \mathbf{p}') \cup B_a(S_>, \mathbf{p}')$, all buyers in $B_a(S_>, \mathbf{p})$ are capped buyers in both equilibria, and $x'_{ij} > 0$ implies that (i, j) is $\mathbf{p}$-MBB for $i \in B(S_>, \mathbf{p})$.

By the above, the sets $B_a(S_<, \mathbf{p})$, $B_a(S_=, \mathbf{p}), B_a(S_>, \mathbf{p})$ as well as the sets $B_1(S_<, \mathbf{p}')$, $B_a(S_=, \mathbf{p}'), B_a(S_>, \mathbf{p}')$ are disjoint and hence are partitions of the set of buyers. Since the first and last sets in both partitions are the same, the middle set is also the same.

Clearly, $B_a(S_<, \mathbf{x}) \subseteq B_a(S_<, \mathbf{p}), B_a(S_>, \mathbf{x}) \subseteq B_1(S_>, \mathbf{p})$, and $B_a(S_=, \mathbf{x}) \subseteq B_a, (S_=, \mathbf{p})$. Since the sets on the right are disjoint and the union of the sets on the left is equal to the set of buyers, we must have equality. The same argument holds true for the primed vectors.

This concludes the proof. □

The lattice structure applies again with respect to the coordinate-wise comparison, i.e., $\mathbf{p} \geq \mathbf{p}'$ iff $p_j \geq p'_j$ for all $j \in G$.

Theorem 2.4. *The pair $(\mathcal{P}, \geq)$ is a lattice.*

Proof. Consider any two thrifty and modest equilibria $(\mathbf{x}, \mathbf{p})$ and $(\mathbf{x}', \mathbf{p}')$. We again consider the supremum $\overline{\mathbf{p}}$ and the infimum $\underline{\mathbf{p}}$ for the price vectors, i.e., $\overline{p}_j = \max(p_j, p'_j)$ and $\underline{p}_j = \min(p_j, p'_j)$. We show that $\overline{\mathbf{p}}$ and $\underline{\mathbf{p}}$ are price vectors of thrifty and modest equilibria.

We first show that $(\mathbf{x}, \overline{\mathbf{p}})$ is a thrifty and modest equilibrium. We only need to show that all allocations are MBB. Compared to $\mathbf{p}$, $\overline{\mathbf{p}}$ has higher prices for the goods in $S_<$. Hence, the allocations to the goods in $S_= \cup S_>$ are still MBB. Consider any good $j \in S_<$. If $x_{ij} > 0$, then (i, j) is $\mathbf{p}'$-MBB

by part 4 of Lemma 2.4. Thus, $u_{ij}/\overline{p}_j = u_{ij}/p'_j = \alpha'_j \geq u_{i\ell}/p'_\ell$ for all ℓ. Since $p'_\ell = \overline{p}_\ell$ for $\ell \in S_<$ and $p'_\ell \leq p_\ell = \overline{p}_\ell$ for $\ell \in S_= \cup S_>$, we conclude that (i, j) is $\overline{\mathbf{p}}$-MBB.

We next show that $(\underline{\mathbf{x}}, \underline{\mathbf{p}})$ is a thrifty and modest equilibrium, where $\underline{\mathbf{x}}$ is defined as

$$\underline{x}_{ij} = \begin{cases} x_{ij} & \text{if } j \in S_< \\ x'_{ij} & \text{if } j \in S_= \cup S_>. \end{cases}$$

Goods are allocated as in one of the equilibria and hence no good is overallocated and goods with positive price are completely allocated. We only need to show that the allocation is MBB. Consider any edge (i, j) with $\underline{x}_{ij} > 0$. We need to show (*) $u_{ij}/\underline{p}_j \geq u_{i\ell}/\underline{p}_\ell$ for all ℓ.

Assume first that $j \in S_<$ and hence $i \in B_a(S_<, \mathbf{p}) = B_a(S_<, \mathbf{p}')$, where the equality is by part 1 of Lemma 2.4. This means we have $\underline{x}_{ij} = x_{ij} > 0$ and there exists some $j' \in S_<$ such that $x'_{ij'} > 0$. Since $x_{ij} > 0$ we have $u_{ij}/p_j \geq u_{i\ell}/p_\ell$ for all ℓ. This establishes (*) for $\ell \in S_< \cup S_=$. For $\ell \in S_>$, we have $u_{ij}/\underline{p}_j = u_{ij}/p_j \geq \max_{k \in S_<}\{u_{ik}/p_k\} \geq \max_{k \in S_<}\{u_{ik}/p'_k\} = u_{ij'}/p'_{j'} \geq u_{i\ell}/p'_\ell = u_{i\ell}/\underline{p}_\ell$.

Assume next that $j \in S_>$ and hence $i \in B_a(S_>, \mathbf{p}') = B_a(S_>, \mathbf{p}')$. We use the same argument as above. We have $\underline{p}_j = p'_j, \underline{x}_{ij} = x'_{ij} > 0$, and there exists some $j' \in S_>$ such that $x_{ij'} > 0$. Since $x'_{ij} > 0$, we have $u_{ij}/p'_j \geq u_{i\ell}/p'_\ell$ for all ℓ. This establishes (*) for $\ell \in S_> \cup S_=$. For $\ell \in S_<$, we have $u_{ij}/\underline{p}_j = u_{ij}/p'_j = \max_{k \in S_>}\{u_{ik}/p'_k\} \geq \max_{k \in S_>} u_{ik}/p_k = u_{ij'}/p_{j'} \geq u_{i\ell}/p_\ell = u_{i\ell}/\underline{p}_\ell$.

Finally, for $j \in S_=$ and hence $i \in B_a(S_=, \mathbf{p}') = B_a(S_=, \mathbf{p})$. Then $\underline{p}_j = p_j = p'_j, \underline{x}_{ij} = x'_{ij} > 0$, and there exists some $j' \in S_=$ such that $x_{ij'} > 0$. Since $x'_{ij} > 0$ we have $u_{ij}/p'_j \geq u_{i\ell}/p'_\ell$ for all ℓ. This establishes (*) for $\ell \in S_> \cup S_=$. For $\ell \in S_<$, we have $u_{ij}/\underline{p}_j = u_{ij}/p'_j = \max_{k \in S_=}\{u_{ik}/p'_k\} = \max_{k \in S_=} u_{ik}/p_k = u_{ij'}/p_{j'} \geq u_{i\ell}/p_\ell = u_{i\ell}/\underline{p}_\ell$.

We conclude that $(\underline{\mathbf{x}}, \underline{\mathbf{p}})$ is a thrifty and modest equilibrium. □

Corollary 2.2. *There exists a thrifty and modest equilibrium with coordinate-wise highest (respectively, lowest) prices. It yields the maximum (respectively, minimum) revenue for the seller among all thrifty and modest equilibria.*

Example 2.2. Consider the following market with two buyers and two goods. Let $u_{11} = u_{12} = u_{22} = 1$ and $u_{21} = 0$. Let $m_1 = m_2 = 1$ and $c_1 = 1$. Then $x_{11} = x_{22} = 1, x_{12} = 0, p_1 = p_2 = 1$ is a thrifty and modest equilibrium with maximum revenue. A thrifty and modest equilibrium with minimum revenue has the same allocation and $p_1 = 0$ and $p_2 = 1$.

Example 2.3. Consider the following market with two buyers and two goods. Let $u_{11} = u_{21} = 1, u_{22} = 2$ and $u_{12} = 0$. Let $m_1 = m_2 = 1$ and $c_1 = c_2 = 1$. Then $x_{11} = 1, x_{22} = 1/2, x_{21} = 0, \mathbf{p}' = (1, 0)$ is a thrifty and modest equilibrium with maximum revenue. A thrifty and modest equilibrium with minimum revenue has the same allocation and $\mathbf{p} = (0, 0)$. Then $S_< = \{1\}, \Gamma(S_<, \mathbf{p}') = \{1\}$, and $\Gamma(S_<, \mathbf{p}) = \{1, 2\}$.

To show our uniqueness result, we again need Lemma 2.3 on unique solutions for a transportation problem in a forest.

Theorem 2.5. *Let G_0 be the goods that have prize zero in all equilibria and assume that the instance is non-degenerate. The allocation of the goods in $G \setminus G_0$ is unique.*

Proof. Let B_0 be the buyers that have positive utility for some good in G_0. By Theorem 2.3, the goods in G_0 are allocated to the buyers in B_0 and the goods in $G \setminus G_0$ are allocated to the buyers in $B \setminus B_0$. We may therefore assume that G_0 is empty. We use induction on the number of goods.

Due to Theorem 2.4, there is an equilibrium vector of maximum prices. We here denote it by $\mathbf{p}$. For each equilibrium price vector $\mathbf{q} \neq \mathbf{p}$, we consider the set $S = \{j \mid q_j < p_j\}$. We denote the set of

these sets by

$$\mathcal{S} = \{S \mid S \neq \emptyset \text{ and there is an equilibrium price vector } \mathbf{q} \text{ with } q_j < p_j \text{ iff } j \in S\}.$$

Assume first that $\mathcal{S}$ is empty. Then either the market is empty, i.e., no goods and no buyers, or all prices and all budgets are fixed. Thus the money flow is unique by Lemma 2.3 and hence the allocation is fixed.

Assume next that $\mathcal{S}$ is nonempty. Let S be a minimal element (under set inclusion) of $\mathcal{S}$. Then $S \subseteq S'$ or $S \cap S' = \emptyset$ for all $S' \in \mathcal{S}$. Assume otherwise, so there is S' in $\mathcal{S}$ such that $\emptyset \neq S \cap S'$ and $S \cap S'$ is a proper subset of S. Let $\mathbf{q}$ and $\mathbf{q}'$ be price vectors defining S and S'. Observe that $\max(\mathbf{q}, \mathbf{q}')$ is an equilibrium price vector; it defines $S \cap S'$. This represents a contradiction to the minimality of S.

We apply Lemma 2.4 to $\mathbf{p}$ and $\mathbf{p}' = \mathbf{q}$. Then $S_< = \emptyset$ and $S_>$ is the set of goods j with $q_j < p_j$. Let $\mathbf{x}$ and $\mathbf{x}'$ be allocations compatible to $\mathbf{p}$ and $\mathbf{p}'$, respectively. □

Claim 2.1. *There is a real $\beta < 1$ such that $p'_j/p_j = \beta$ for all $j \in S_>$.*

Proof. Assume otherwise and consider the price vector $\mathbf{r}$ defined as

$$r_j = \begin{cases} p_j & \text{if } j \notin S_> \\ \gamma p'_j & \text{if } j \in S_> \end{cases},$$

where $\gamma > 1$ is such that $\mathbf{r} \leq \mathbf{p}$. We show that $(\mathbf{x}', \mathbf{r})$ is a thrifty and modest equilibrium. We only need to show that all allocations are MBB. Consider any pair (i, j) with $x'_{ij} > 0$. If $j \notin S_>$, then the allocation is MBB, since $r_j = p'_j$ and $\mathbf{r} \geq \mathbf{p}'$. If $j \in S_>$, then (i, j) is $\mathbf{p}$-MBB by Part 5 of Lemma 2.4. Thus $u_{ij}/p_j \geq u_{i\ell}/p_\ell$ for all ℓ and hence $u_{ij}/r_j \geq u_{ij}/p_j \geq u_{i\ell}/p_\ell = u_{i\ell}/r_\ell$ for $\ell \notin S_>$. For $\ell \in S_>$, we have $u_{ij}/(\gamma p'_j) = (1/\gamma)u_{ij}/p'_j \geq (1/\gamma)u_{i\ell}/p'_\ell$.

Consider now the minimal γ for which $\gamma p'_j = p_j$ for some j. Then the price vector $\mathbf{r}$ implies that a proper non-empty subset of S belongs to $\mathcal{S}$, a contradiction. □

Since $B_a(S_>, \mathbf{x}') = B(S_>, p)$ for every equilibrium $(\mathbf{x}', \mathbf{r})$ by Part 3 of Lemma 2.4, the sum of the prices of the goods in $S_>$ must be equal to the budgets of the buyers in $B(S_>, \mathbf{p})$ for every γ. The prices of the goods in $S_>$ and the budgets of the buyers in $B(S_>, \mathbf{p})$ scale with γ. For every fixed γ, the money flow is unique by Lemma 2.3. Moreover, it scales with γ. Thus the allocation of the goods in $S_>$ to the buyers in $B(S_>, \mathbf{p})$ is unique for all price vectors $\mathbf{r}$.

We have now shown that the allocation of the goods in $S = S_>$ is unique for all price vectors where only the prices of the goods in S are decreased. Consider now a price vector $\mathbf{q}'$ where the prices of the goods in S and some other goods are decreased. As above, let $\mathbf{q}$ be a price vector where only the prices of the goods in S are decreased. We may assume $q'_j \leq q_j$ for all $j \in S_>$, because we already showed that with respect to $\mathbf{q}$, the prices of the goods in $S_>$ can be scaled up. Let $\mathbf{x}$ and $\mathbf{x}'$ be allocations compatible with $\mathbf{q}$ and $\mathbf{q}'$. In the proof that $\mathcal{P}$ is an upper lattice, we showed that $(\mathbf{x}', \max(\mathbf{q}, \mathbf{q}')) = (\mathbf{x}', \mathbf{q})$ is an equilibrium. Since, with respect to $\mathbf{q}$, the allocation of the goods in $S_>$ is unique, $x'_{ij} = x_{ij}$ for all $j \in S_>$. We have now shown that the allocation of the goods in $S_>$ is unique across all equilibria.

We remove the goods in $S_>$ and the buyers in $B(S_>, \mathbf{p})$ from the market. Note that the allocation of the goods in $S_>$ to the buyers in $B(S_>, \mathbf{p})$ satiates the buyers and they need no further allocation. By induction hypothesis, the allocation in the market $(B \setminus B(S_>, \mathbf{p}), G \setminus S_>)$ is unique. Thus, the overall allocation is unique.

3 ALGORITHMS FOR MARKETS WITH EARNING LIMITS

3.1 Scaling Algorithm to Compute a Thrifty Equilibrium

In this section, we propose and discuss a polynomial-time scaling algorithm to compute a thrifty equilibrium. We begin with defining some useful tools and concepts. The *active-segment network* $N(\mathbf{p}) = (\{s,t\} \cup B \cup G, E)$ contains a node for each buyer and each good, along with two additional nodes s and t. It contains every edge (s,i) for $i \in B$ with capacity $m_i - c_i^c$, where $c_i^c = \sum_{(i,j,k) \text{ closed}} c_{ijk}$. Also, it contains every (j,t) for $j \in G$ with capacity $p_j^a - c_j^c$, where $c_j^c = \sum_{(i,j,k) \text{ closed}} c_{ijk}$. It contains edge (i,j) with infinite capacity iff there is some active segment (i,j,k). Finally, the *active-residual network* $G_r(\mathbf{f},\mathbf{p})$ contains a node for each buyer and each good. It contains forward edge (i,j) iff there is some active segment (i,j,k) with $f_{ijk} < c_{ijk}$ and contains backward edge (j,i) iff there is some active segment (i,j,k) with $f_{ijk} > 0$. Moreover, $G_r(\mathbf{f},\mathbf{p},i)$ is the subgraph of $G_r(\mathbf{f},\mathbf{p})$ induced by the set of all buyers $i' \in G_r(\mathbf{f},\mathbf{p})$ such that there is an augmenting path from i' to i.

Our algorithm uses Δ-*discrete capacities* $\hat{c}_{ijk} = \lceil c_{ijk}/\Delta \rceil \cdot \Delta$ for all $i \in B, j \in G$ and $(i,j,k) \in K_{ij}$, where we iteratively decrease Δ. Initially, the algorithm overestimates the budget of buyer i, where it assumes the buyer has $r\Delta$ money and every segment has Δ-discrete capacities. Then $\mathbf{f}_i$ is a (Δ, r)-*discrete demand* for buyer i iff it is a demand flow for buyer i under these conditions.

We also adjust the definitions of MBB ratio, active segments, active-segment graph, network, and residual network to the case of Δ-discrete capacities. We denote these adjusted versions by $\hat{\alpha}$, $\hat{\mathcal{G}}(\mathbf{p})$, $\hat{N}(\mathbf{p})$, $\hat{\mathcal{G}}_r(\mathbf{f},\mathbf{p})$, and $\hat{\mathcal{G}}_r(\mathbf{f},\mathbf{p},j)$, respectively.

Finally, we make a number of assumptions to simplify the stated bound on the running time. We assume w.l.o.g. that $|B| = |G|$ (by adding dummy buyers and/or goods) and define $n = |B| + |G|$. Moreover, we let $K_i = \bigcup_{j \in G} K_{ij}$ and $K_j = \bigcup_{i \in B} K_{ij}$ and assume w.l.o.g. that $\ell = |K_i| = |K_j| \geq n$ for every buyer i and every good j (by adding dummy segments with 0 utility).

Algorithm 1 computes a thrifty equilibrium in polynomial time. We call a run of the outer while-loop a Δ-*phase*. The precision parameter Δ is halved in each phase until it is decreased to exponentially small size. Then a final rounding procedure PostProcessing rounds the solution to an exact equilibrium. In each Δ-phase, the surplus of all buyers is decreased to at most Δ by decreasing prices and rerouting flow.

For the analysis, we use the following notion of Δ-feasible solution.

Definition 3.1. Given a value $\Delta > 0$, a pair $(\mathbf{f},\mathbf{p})$ of flow and prices with $\mathbf{p} \geq \mathbf{0}$ and $\mathbf{f} \geq \mathbf{0}$ is a Δ-*feasible solution* if

- $\ell\Delta \leq s(i) \leq (\ell+1)\Delta, \forall i \in B$.
- $\forall j \in G$: If $p_j < p_j^0$, then $0 \leq s(j) \leq \Delta$. If $p_j = p_j^0$, then $-\infty < s(j) \leq \Delta$.
- $\mathbf{f}$ is Δ-integral, and $f_{ijk} > 0$ only if (i,j,k) is a closed or open MBB segment w.r.t. Δ-discretized capacities.

For the running time, note that prices are non-increasing. Once a capped good becomes uncapped, it remains uncapped. We refer to an execution of the repeat loop in Algorithm 1 as an iteration. After the initialization, there may be goods j for which d_j is smaller than the initial value of Δ and that receive flow from some buyer. As long as their surplus is negative, these goods keep their initial price. The following observations are useful to prove a bound on the running time. We also observe that the precision of prices and flow values is always bounded.

Lemma 3.1. *Once the surplus of a good is non-negative, it stays non-negative. If the surplus of a good is negative, then its price is the initial price.*

ALGORITHM 1: Scaling Algorithm for Markets with Earning Limits

Input: Fisher market $\mathcal{M}$ with spending constraint utilities and earning limits
Budget m_i, earning limits d_j, and parameters u_{ijk}, c_{ijk}
Output: Thrifty equilibrium $(\mathbf{f}, \mathbf{p})$

$\Delta \leftarrow U^{n+1} \sum_{i \in B} m_i$; $p_j^0 \leftarrow n(\ell+1)\Delta, \forall j \in G$; $\mathbf{p} \leftarrow \mathbf{p}^0$
$f_i \leftarrow (\Delta, \ell+1)$-discrete demand for buyer i
while $\Delta > 1/(2\ell(2nU)^{4n})$ **do**
  $\Delta \leftarrow \Delta/2$;
  for *each closed segment* (i,j,k) **do** $f_{ijk} \leftarrow \lceil c_{ijk}/\Delta \rceil \cdot \Delta$
  for *each* $i \in B$ *with* $s(i) > (\ell+1)\Delta$ **do**
    Pick any active segment (i,j,k) with $f_{ijk} > 0$ and set $f_{ijk} \leftarrow f_{ijk} - \Delta$
  while *there is a good* j' *with* $s(j') > \Delta$ **do** // Δ-phase
    repeat // iteration
      $(\hat{B}, \hat{G}) \leftarrow$ Set of (buyers, goods) in $\hat{\mathcal{G}}_r(\mathbf{f}, \mathbf{p}, j')$
      $x \leftarrow 1$; Define $p_j \leftarrow x p_j, \forall j \in \hat{G}$ // active prices & surpluses change, too
      Decrease x continuously down from 1 until one of the following events occurs
        Event 1: $s(j') = \Delta$
        Event 2: $s(j) \le 0$ for a $j \in \hat{G}$
        $P \leftarrow$ path from j to j' in $\hat{\mathcal{G}}_r(\mathbf{f}, \mathbf{p}, j')$ // Δ-augmentation
        Update $\mathbf{f}$: $f_{ijk} = \begin{cases} f_{ijk} + \Delta & \text{if } (i,j) \text{ is a forward arc in } P \\ f_{ijk} - \Delta & \text{if } (i,j) \text{ is a backward arc in } P \\ f_{ijk} & \text{otherwise} \end{cases}$
        Event 3: A capped good becomes uncapped
        Event 4: New active segment (i,j,k) with $i \notin \hat{B}, j \in \hat{G}, f_{ijk} < \hat{c}_{ijk}$
    until *Event 1 or 2 occurs*
$(\mathbf{f}, \mathbf{p}) \leftarrow$ PostProcessing$(\mathbf{f}, \mathbf{p})$

Proof. The surplus of a good can only decrease if its price decreases or if additional money flow is pushed into it—in particular, observe that the adjustment of the flow to Δ-discrete capacities only increases the surplus of each good. If additional money flow is pushed into a good, then its surplus before the push is at least Δ. Hence, it is non-negative after the push. Price decreases stop once there is a good with a non-positive surplus, so a non-negative surplus cannot become negative. □

Lemma 3.2. *The first run of the outer while-loop in Algorithm 1 takes* $O(n^3\ell)$ *time, every subsequent one takes* $O(n^2\ell)$ *time. At the end of each* Δ*-phase, the pair* $(\mathbf{f}, \mathbf{p})$ *is a* Δ*-feasible solution.*

Proof. After initialization, all buyers have surplus $\ell\Delta \le s(i) \le (\ell+1)\Delta$, and all goods have surplus $s(j) \le n(\ell+1)\Delta$. In the beginning of the outer while-loop, we reduce Δ to half and adjust the flow to Δ-discrete capacities. Due to reduction of Δ, all buyers have surplus $2\ell\Delta \le s(i) \le 2(\ell+1)\Delta$, and all goods have surplus $s(j) \le 2n(\ell+1)\Delta$. Due to adjustment of the flow to Δ-discrete capacities, $s(i)$ decreases by at most $\ell\Delta$, and $s(j)$ increases by at most $\ell\Delta$, for every $i \in B, j \in G$. This results in $\ell\Delta \le s(i) \le 2(\ell+1)\Delta$ and $s(j) \le 2n(\ell+1)\Delta + \ell\Delta$. In the following loop, we reduce the surplus of all buyers to $\ell\Delta \le s(i) \le (\ell+1)\Delta$, which takes at most $n(\ell+1)$ iterations. This implies that every buyer surplus satisfies the conditions of a Δ-feasible solution. Every buyer surplus stays unchanged in the Δ-phase. In the subsequent Δ-phase, we reduce the surplus of every good to at most Δ. All prices are non-increasing; hence, without flow adjustment, all surpluses of goods are non-increasing. In a flow adjustment along path P, we keep every surplus of intermediate

ALGORITHM 2: PostProcessing($\mathbf{f}, \mathbf{p}$)

Input: ε-feasible solution $(\mathbf{f}, \mathbf{p})$ for $\varepsilon = 1/(2\ell(2nU)^{4n})$
Output: Market equilibrium $(\mathbf{f}', \mathbf{p}')$

$\hat{\mathcal{G}}(\mathbf{p}) = (B \cup G, E) \leftarrow$ active-segment graph at $\mathbf{p}$ w.r.t. Δ-discrete capacities $\hat{c}_{ijk}$
while $\exists$ *a component* C *in* $\hat{\mathcal{G}}(\mathbf{p})$ *s.t. all goods are capped* **do**
  $x \leftarrow 1$; Define prices as $p_j \leftarrow x p_j, \forall j \in C \cap G$
  Decrease x continuously down from 1 until one of the following events occurs
  Event 1: A capped good becomes uncapped
  Event 2: A new segment (i, j, k) becomes active `// components merge`
Recompute active-segment graph $\hat{\mathcal{G}}(\mathbf{p})$ and let C be the set of its components
Let K^c be the set of closed segments in $(\mathbf{f}, \mathbf{p})$ w.r.t. Δ-discrete capacities
$\hat{c}^c_j \leftarrow \sum_{(i,j,k)\in K^c} \hat{c}_{ijk}$ for every $j \in G$
$\hat{c}^c_i \leftarrow \sum_{(i,j,k)\in K^c} \hat{c}_{ijk}$ for every $i \in B$
for *each component* $C \in \mathcal{C}$ **do**
  Set prices $\mathbf{p}$ as solution of the following system of equations
  (1) $u_{ijk}p_{j'} = u_{ij'k'}p_j$ (for active segments from a buyer i to goods j and j')
  (2) $\sum_{j\in C\cap G}(p^a_j - \hat{c}^c_j) - \sum_{i\in C\cap B}(m_i - \hat{c}^c_i) = \sum_{u\in C} s(u)$ (sum of surpluses)
  // $-n(\ell+1)\Delta \le s(u) \le (\ell+1)\Delta$
Let $\mathbf{Ap} = \mathbf{b}$ be the matrix form of the above system
Let $\mathbf{Ap}' = \mathbf{b}'$ be the system where $\mathbf{b}'$ is obtained from $\mathbf{b}$ after substituting $s(u) = 0$ and using c^c_u based on original c_{ijk}, for all $u \in B \cup G$
$\mathbf{f}' \leftarrow$ maximum s-t-flow in network $N(\mathbf{p}')$
return $(\mathbf{f}', \mathbf{p}')$

goods the same. We reduce the surplus of good j' and increase the surplus of good j by Δ. Since good j has non-positive surplus, this never increases the surplus beyond Δ. Since good j' has surplus more than Δ, this never makes the surplus of good j' negative. Hence, in the first Δ-phase there can be at most n iterations that terminate with Event 1, and at most $2n^2(\ell+1) + n\ell$ that terminate with Event 2. Furthermore, since prices are decreasing, Event 3 happens at most n times overall. Moreover, since the residual network $\mathcal{G}_r$ expands at most n times by including a new buyer, Event 4 happens at most n times in each iteration. Overall, the first Δ-phase takes time at most $n(\ell+1) + n(n + 2n^2(\ell+1) + n\ell) + n = O(n^3\ell)$.

Note that at the end of the Δ-phase, we have a Δ-feasible solution. The conditions for the surplus of all buyers hold, since they were unchanged during the Δ-phase. By Lemma 3.1, we have negative surplus only for goods whose price has not been touched in the process. By termination of the Δ-phase, it follows that every good surplus satisfies the conditions of Δ-feasible solution.

Hence, in every subsequent run of the outer while-loop, we start with $s(j) \le \Delta$ for all goods. After adjustment of Δ and the flow to Δ-discrete capacities, we have $s(j) \le (\ell+2)\Delta$ for every $j \in G$ and $\ell\Delta \le s(i) \le 2(\ell+1)\Delta$ for every $i \in B$. The next for-loop then guarantees $s(i) \le (\ell+1)\Delta$ for all buyers. By repeating the arguments above, the following Δ-phase takes time $O(n^2\ell)$. □

In addition to the running time, we also show that the precision of intermediate prices and flow values is bounded.

Lemma 3.3. *If all budgets, earning limits and utility values are integers bounded by U, then all flow values and prices at the end of each iteration are rational numbers whose denominators are at most poly$(1/\Delta, n, U^n)$.*

Proof. Note that the flow values are always Δ-integral, and hence they are rational numbers with the desired size. Also, the starting prices are rational numbers of the desired size. At the end of each iteration, one of the four events occurs. In all cases, we show that the prices remain polynomially bounded if they are so at the beginning of the iteration. This will complete the proof.

In case of Event 3, a capped good j becomes uncapped, so $p_j = d_j$ and the ratio of any other price in the active component and p_j can be written as the ratio of products of at most n utility values. Hence, they are polynomially bounded. The other prices are not touched, so they remain same.

In case of Event 4, a new active segment arises, and therefore we can again write any price in the active component in terms of a price variable that has not been touched. All prices are polynomially bounded.

Event 1 can happen only if $p_k^a = p_k$. In that case, p_k is Δ-integral and all other prices in the active component can be expressed in terms of p_k using the MBB relation. Hence, all prices are of the desired size.

In case of Event 2, if $s(j) < 0$, then this implies that p_j has not been decreased since the beginning, so all prices are again fine. For the other case, $s(j) = 0$ and that implies p_j is Δ-integral. Therefore, all prices are polynomially bounded. □

Finally, for correctness of the algorithm, it maintains the following condition resulting from (MC) for active prices.

Lemma 3.4. *Let $\hat{B} \subseteq B$ be a set of buyers and let $N(\hat{B})$ be the goods having positive utility for some buyer in $\hat{B}$. At all times $\sum_{j \in N(\hat{B})} p_j^a - \sum_{i \in \hat{B}} m_i \geq 0$.*

Proof. Consider the connected components of the bipartite graph $(B \cup G, E)$, where $E = \{(i,j) \in B \times G \mid f_{ij} > 0\}$. We show the claim for each connected component C separately. If there is a good j with negative surplus, then $p_j = p_j^0$. This implies that $p_h \geq d_h$ and $p_h^a = d_h$ for all goods $h \in C \cap G$. Hence the claim follows from (MC). If all goods have non-negative surplus, then

$$\begin{aligned}
\sum_{j \in N(\hat{B})} p_j^a - \sum_{i \in \hat{B}} m_i &= \sum_{j \in N(\hat{B})} \left(p_j^a - \sum_{i \in \hat{B}} f_{ij} \right) + \sum_{i \in \hat{B}} \left(\sum_{j \in N(\hat{B})} f_{ij} - m_i \right) \\
&\geq \sum_{j \in N(\hat{B})} \left(p_j^a - \sum_{i \in B} f_{ij} \right) + \sum_{i \in \hat{B}} \left(\sum_{j \in N(\hat{B})} f_{ij} - m_i \right) \\
&= \sum_{u \in \hat{B} \cup N(\hat{B})} s(u) \geq 0.
\end{aligned}$$

□

Lemma 3.5. *Let $(\mathbf{f}, \mathbf{p})$ be the flow and price vector computed by the outer while-loop in Algorithm 1. The pair is Δ-feasible for $\Delta = 1/(2\ell(2nU)^{4n})$ and $-n(\ell+1)\Delta \leq s(j) \leq \Delta$ for all $j \in G$.*

Proof. The first claim follows from Lemma 3.2. Thus, $s(j) \leq \Delta$ for every good j. By Lemma 3.4

$$0 \leq \sum_{j \in G} p_j^a - \sum_{i \in B} m_i = \sum_{u \in B \cup G} s(u) \leq n(\ell+1)\Delta - \sum_{u;\, s(u) < 0} \|s(u)\|,$$

and hence $s(j) \geq -n(\ell+1)\Delta$ for every good j. □

Note that by Lemma 3.5 we call PostProcessing with a pair $(\mathbf{f}, \mathbf{p})$ that is Δ-feasible for $\Delta = 1/(2\ell(2nU)^{4n})$. Also $-n(\ell+1)\Delta \leq s(u) \leq (\ell+1)\Delta$ for every $u \in B \cup G$.

The while-loop in PostProcessing ensures that all components of the active-segment graph $\hat{\mathcal{G}}(\mathbf{p})$ contain an uncapped good. For each component C of $\hat{\mathcal{G}}(\mathbf{p})$, the algorithm sets up a system of linear equations in price variables of the form $A\mathbf{p} = \mathbf{b}$, and we show that after perturbing $\mathbf{b}$ slightly, we

get an equilibrium. Since we apply the same procedure on each component separately, we assume without loss of generality that there is exactly one component C of $\hat{\mathcal{G}}(\mathbf{p})$.

All goods in C are connected with each other through a set of active MBB edges. Whenever there are two active segments (i, j, k) and (i, j', k') for a buyer i and two goods j, j', we have the following relation between p_j and $p_{j'}$:

$$u_{ijk} p_{j'} = u_{ij'k'} p_j. \tag{9}$$

It is easy to check that there are $|C \cap G| - 1$ of these MBB relations, which are linearly independent, and there is essentially one free price variable. Additionally, we have a condition for C on the sum of surpluses:

$$\sum_{j \in C \cap G} (p_j^a - \hat{c}_j^c) - \sum_{i \in C \cap B} (m_i - \hat{c}_i^c) = \sum_{u \in C} s(u). \tag{10}$$

Since there is at least one uncapped good, the set of active prices p_j^a can be divided into a set for capped goods and a set for uncapped goods; $p_j^a = p_j$ for each uncapped j, and $p_j^a = d_j$ for each capped j. We can rewrite (10) as:

$$\sum_{j \in C \cap G \text{ uncapped}} p_j = \sum_{u \in C} s(u) - \sum_{j \in C \cap G \text{ capped}} d_j + \sum_{i \in C \cap B} m_i + \sum_{j \in C \cap G} \hat{c}_j^c - \sum_{i \in C \cap B} \hat{c}_i^c. \tag{11}$$

We can write the system of Equations (9) and (11) in matrix form as $\mathbf{Ap} = \mathbf{b}$. All entries of $\mathbf{A}$ are integers due to our assumption on the input parameters, and $\mathbf{b}$ has exactly one non-zero entry resulting from Equation (11). Now consider another system $\mathbf{Ap'} = \mathbf{b'}$ for a price vector $\mathbf{p'}$, where $\mathbf{b'}$ is obtained after setting $s(u) = 0$ and using c_u^c that sums the original capacities of closed segments. Next we show that $\mathbf{p'}$ gives an equilibrium. For this, we show that there is a feasible flow in the active-segment network $N(\mathbf{p'})$ with min-cuts $(s, B \cup G \cup t)$ and $(s \cup B \cup G, t)$. The proof is based on an adaption of a similar result in [23].

Note that all entries of $\mathbf{A}$ are integers in $[-U, U]$. For $\mathbf{b'}$ all entries are integers in $[-2n\ell U, 2n\ell U]$. By Cramer's rule, the solution of $\mathbf{Ap'} = \mathbf{b'}$ is a vector of rational numbers with common denominator $D \leq (nU)^n$. That is, all p'_j are of form q_j/D, where both q_j and D are integers. Let $\varepsilon = n(\ell + 1)\Delta + n\ell\Delta$. Since $||\mathbf{b}| - |\mathbf{b'}|| < \varepsilon$, we have $|p_j - p'_j| \leq \varepsilon D, \forall j$. Let $\varepsilon' = \varepsilon D^2$, then $|Dp_j - q_j| = |D(p_j - p'_j)| \leq \varepsilon D^2 = \varepsilon'$.

Lemma 3.6. *Every MBB segment with respect to* $\mathbf{p}$ *is also an MBB segment with respect to* $\mathbf{p'}$. *Furthermore, the set of capped and uncapped goods with respect to* $\mathbf{p}$ *and* $\mathbf{p'}$ *are the same.*

Proof. Suppose for two segments (i, j, k) and (i, j', k') we have $u_{ijk} p_{j'} \leq u_{ij'k'} p_j$, then

$$u_{ijk} q_{j'} \leq u_{ijk} D p_{j'} \leq D u_{ij'k'} p_j \leq D u_{ij'k'} (p'_j + \varepsilon D) \leq u_{ij'k'} q_j + \varepsilon' u_{ij'k'} < u_{ij'k'} q_j + 1.$$

Since both $u_{ijk} q_{j'}$ and $u_{ij'k'} q_j$ are integers, we have $u_{ijk} q_{j'} \leq u_{ij'k'} q_j$. This implies that all bang-per-buck relations for segments in the market are preserved. In particular, a segment is MBB w.r.t. $\mathbf{p}$ iff it is MBB w.r.t. $\mathbf{p'}$. The capped goods w.r.t. $\mathbf{p}$ remain capped w.r.t. $\mathbf{p'}$. Suppose $p_j \geq d_j$, then $q_j \geq D p_j - \varepsilon' > D d_j - 1$. Since q_j and d_j are integers, we have that $q_j \geq D d_j$ and $p'_j \geq d_j$. Similarly, if $p_j \leq d_j$, then $q_j \leq D p_j + \varepsilon' < D d_j + 1$. Again, $q_j \leq D d_j$ and $p'_j \leq d_j$. □

Note that after the rounding, all (active) prices $\mathbf{p'}$ are rational numbers with common denominator D. We assign to all closed segments the full amount c_{ijk}. For the active segments, consider the network $N(\mathbf{p'})$, and let c be the capacity of cut $(s, B \cup G \cup t)$ in $N(\mathbf{p'})$. Suppose there is a min-cut in $N(\mathbf{p'})$ with value less than c. Then that value is at most $c - 1/D$. This same cut in $\hat{N}(\mathbf{p})$ will have

ALGORITHM 3: MinPrices for Earning Limits

Input: Market parameters and any thrifty equilibrium $(\mathbf{f}, \mathbf{p})$
Output: Thrifty equilibrium with smallest prices

$E(\mathbf{f}) \leftarrow \{(i,j,k) \mid f_{ijk} > 0\}$
$G_c \leftarrow$ Set of capped goods at $(\mathbf{f}, \mathbf{p})$
Solve an LP in q_j and λ_i:

$$\begin{array}{ll} \min \sum_i \lambda_i & \\ q_j \le u_{ijk}\lambda_i, & \text{for segment } (i,j,k) \in E(\mathbf{f}) \\ q_j = p_j, & \forall j \in G \setminus G_c \\ q_j \ge d_j, & \forall j \in G_c \\ \lambda_i, q_j \ge 0 & \forall i \in B, j \in G \end{array}$$

return $(\mathbf{f}, \mathbf{q})$

value at most $c - 1/D + \varepsilon D|G| + \varepsilon|B|$. Also the capacity of the cut $(s, B \cup G \cup t)$ in $\hat{N}(\mathbf{p})$ is at least $c - \varepsilon|B|$. Therefore the total surplus in $\hat{N}(\mathbf{p})$ is at least

$$c - \varepsilon|B| - (c - 1/D + \varepsilon D|G| + \varepsilon|B|) \ge 1/D - n\varepsilon D > \varepsilon,$$

which is a contradiction. Hence $(s, B \cup G \cup t)$ is a min-cut in $N(\mathbf{p}')$. Hence, after removing the money allocated to closed segments from buyer budgets and prices of goods, the remaining money on the active segments allows an allocation that clears the market. This shows that PostProcessing works correctly. Hence, we know the algorithm is correct, requires only bounded precision, and runs in polynomial time.

The following theorem is the main result of this section.

Theorem 3.1. *Algorithm 1 computes a thrifty equilibrium for money-clearing markets with earning limits in $O(n^3 \ell \log(\ell + nU))$ time.*

Proof. At the beginning, $\Delta \le U^{n+1}$ and Δ is reduced to $\Delta/2$ until $\Delta < 1/(2\ell(2nU)^{4n})$. Therefore, since $\ell \ge n$, the total number of Δ-phases is $O(n \log(\ell + nU))$. While the first phase takes time $O(n^3\ell)$, each subsequent phase takes time $O(n^2\ell)$. Further, PostProcessing takes $O(n^4 \log(nU))$ time [23]. The total running time of Algorithm 1 is $O(n^3 \ell \log(\ell + nU))$. □

3.2 Equilibria with Extremal Prices

In this section, we provide algorithms to refine arbitrary thrifty equilibria to ones with smallest or largest prices in polynomial time. Given an arbitrary thrifty equilibrium, Algorithm 3 computes a thrifty equilibrium with smallest prices. Algorithm 4 computes a thrifty equilibrium with largest prices if it exists. Otherwise, it yields a set S of goods for which prices can be raised indefinitely.

Theorem 3.2. *Algorithm 3 computes a thrifty equilibrium with smallest prices.*

Proof. By Lemma 2.2 part 3, we know that if $f_{ijk} > 0$, then (i,j,k) is an MBB segment in every thrifty equilibrium. Let $E(\mathbf{f}) = \{(i,j,k) \mid f_{ijk} > 0\}$, and let G_c and G_u be the (unique) sets of capped and uncapped goods in thrifty equilibria, respectively. Note that a vector $\mathbf{q}$ of pointwise smallest prices implies a pointwise largest MBB ratio α_i for all buyers $i \in B$. Using $\lambda_i = 1/\alpha_i > 0$, Algorithm 3 optimizes the LP to find the minimal λ_i with prices $\mathbf{q}$ that preserve the MBB segments. The prices $\mathbf{q}$ then determine all active segments, and they determine the flow on all segments that are non-active and MBB (and, thus, closed). For the active ones, the feasible flows are exactly the solutions to a straightforward transportation problem. In particular, the original flow $\mathbf{f}$ stays an equilibrium flow, since all edges that carry flow in $\mathbf{f}$ stay MBB, the outflow of every buyer i is m_i,

ALGORITHM 4: MaxPrices for Earning Limits

Input: Market parameters and any thrifty equilibrium $(\mathbf{f}, \mathbf{p})$
Output: Thrifty equilibrium with largest prices

Initialize active price $p_j^a \leftarrow \min\{d_j, p_j\}$ for every good $j \in G$
$S \leftarrow \{j \mid p_j > 0$ and j is not connected to any uncapped good in $G(\boldsymbol{p})\}$
while $S \neq \emptyset$ **do**
  $x \leftarrow 1$
  Set prices $p_j \leftarrow x p_j$, $\forall j \in S$
  Increase x continuously from 1 until a new active segment appears
  Recompute S
return $(\mathbf{f}, \mathbf{p})$

the inflow of every good j is p_j^a. Moreover, $\mathbf{f}$ saturates non-active MBB segments under prices $\mathbf{q}$, which is directly implied by the proof of Lemma 2.2, part 2. □

Theorem 3.3. *Algorithm 4 computes a thrifty equilibrium with largest prices if it exists.*

Proof. It is easy to check that throughout the algorithm, $(\mathbf{f}, \mathbf{p})$ always remains a thrifty equilibrium. Assume by contradiction that at the end of the algorithm, $(\mathbf{f}, \mathbf{p})$ is not an equilibrium with largest prices. Let $E' = (\mathbf{f}', \mathbf{p}')$ be an equilibrium with largest prices, and define $S_1 = \{j \mid p'_j > p_j\}$. By Proposition 2.1 part 2, all goods in S_1 are capped goods. Moreover, by Lemma 2.2 part 3 every segment with $f_{ijk} > 0$ for jŁ $\in S_1$ is also an MBB segment in E'. Because prices of goods in S_1 strictly decrease from $\mathbf{p}'$ to $\mathbf{p}$, every buyer i with active edges in S_1 in the active segment graph with prices $\mathbf{p}'$ will have active edges only to S_1 with prices $\mathbf{p}$. Therefore, set S is nonempty for the While loop, and the algorithm should not terminate. □

3.3 Nash Social Welfare in Additive Multi-Unit Markets

Using our algorithm to compute a thrifty equilibrium in linear markets with earning limits, we can approximate the optimal Nash social welfare for additive valuations, indivisible items, and multiple copies for each item. Here there are n agents and m items. For item j, there are $s_j \in \mathbb{N}$ copies. The valuation of agent i for an assignment x of goods is $v_i(x) = \sum_j v_{ij} x_{ij}$, where x_{ij} denotes the number of copies of item j that agent i receives. The goal is to find an assignment such that the Nash social welfare $(\prod_i v_i(x))^{1/n}$ is maximized.

Suppose for each item there is only a single copy. In this case, the algorithm of [17] provides a 2-approximation [15]. It finds an equilibrium for a linear market, where each agent i is a buyer with a budget $m_i = 1$, and each item j is a good with earning limit $d_j = 1$. The allocation of the equilibrium gets rounded to an integral assignment.

The direct adjustment of this approach to handle $s_j \geq 1$ copies is to represent each copy of item j by a separate auxiliary item with unit supply (all valued exactly the same way as item j). Then run the algorithm from [17] using a linear Fisher market with an earning limit of 1 for each copy. The number of copies corresponds directly to the total earning limit of (all auxiliary items of) the good.

A similar approach is used in [1] to provide a 2-approximation for separable concave utilities. This, however, yields a running time polynomial in $\max_j s_j$, which is only pseudo-polynomial for multi-unit markets (due to standard logarithmic coding of s_j's). We here outline a way to make the algorithm efficient.

Proposition 3.1. *There is a polynomial-time 2-approximation algorithm for maximizing Nash social welfare in multi-unit markets with additive valuations.*

Proof. First, we replace each item with $s_j \le 2n$ by s_j auxiliary items with supply 1 as in the direct adjustment. Each of these gets an auxiliary good with earning limit 1 in the market. For each item with $s_j > 2n$, we introduce an uncapped good in the market. For every auxiliary good, we assume that every buyer i has utility $u_{ij} = v_{ij}$. For every uncapped good, we assume every buyer i has utility $u_{ij} = v_{ij}s_j$. Then we use our algorithm above to compute a thrifty equilibrium for this market in time $O((nm)^4 \log((nm) \cdot \max_{i,j} v_{ij}s_j))$. Let $(\mathbf{x}, \mathbf{p})$ be this equilibrium.

The subsequent rounding of the equilibrium allocation follows ideas laid out in [17]. Consider the spending graph, i.e., the subgraph of the MBB graph where buyers spend their money. Because of non-degeneracy of the MBB graph [17, 35], the spending graph is a forest. To handle the uncapped goods, we first present an inefficient approach and then observe how to implement it implicitly in polynomial time.

Given an uncapped good j, let us expand the spending graph in the following way: Introduce s_j many copies, each with price $p'_j = p_j/s_j$. The valuation of buyer i for each copy is v_{ij}. Since good j is uncapped, we know $p_j \le \sum_i m_i = n$. Moreover, since $s_j > 2n$, this implies $p'_j < 1/2$. Let $f_{ij} = x_{ij}p_j$ be the money that agent i spends on good j. The parent agent i_0 in the spending graph becomes direct parent of $\lceil f_{i_0,j}/p'_j \rceil$ many copies. If $f_{i_0,j}/p'_j$ is not an integer, then the parent pays p'_j to $\lfloor f_{i_0,j}/p'_j \rfloor$ many copies, and the rest to one additional copy. The first child i_1 of good j is assigned to contribute the missing money for this additional copy (until it is fully paid for) and becomes its child. Then, if i_1 still has remaining money, then it contributes this money to purchase further copies, for which it becomes the parent. Also, it remains parent of any other goods $j' \ne j$ for which it is a parent in the spending graph. Naturally, if i_0 exactly pays an integer number of copies, then i_1 becomes the root of a new tree component and purchases additional copies of good j in the same way.

More formally, i_1 becomes parent of

$$\max\left(0, \left\lceil \left(f_{i_1,j} - \left(p'_j \cdot \lceil f_{i_0,j}/p'_j \rceil - f_{i_0,j}\right)\right)/p'_j \right\rceil\right)$$

further copies of good j. We continue this expansion process, in which child agent i_k of good j becomes parent of

$$\max\left(0, \left\lceil \left(f_{i_k,j} - \left(p'_j \cdot \left\lceil \sum_{\ell=0}^{k-1} f_{i_\ell,j}/p'_j \right\rceil - \sum_{\ell=0}^{k-1} f_{i_\ell,j}\right)\right)/p'_j \right\rceil\right). \tag{12}$$

many copies of good j. Since prices and utilities are both scaled by s_j, it is easy to verify that this represents an equilibrium assignment for the market where we introduce s_j auxiliary goods for good j, each with earning limit 1.

Now, since $p'_j < 1/2$, the rounding procedure in [17] applied to this expanded spending graph will assign all copies of item j to the parent agent of the corresponding good and remove them from the graph. Thus, the rounding procedure simply removes good j from the spending graph and assigns the number of copies given by Equation (12) to parent buyer i_0 and children i_ℓ, for $\ell = 1, 2, \ldots$. Obviously, this can be done directly for each uncapped good j in $O(n)$ time without explicit expansion of the spending graph. Consequently, our adjusted algorithm achieves a running time of $O(n^4 \log nU)$—our algorithm to compute an equilibrium takes $O(n^4 \log nU)$ time and the rounding procedure takes $O(n^4)$ time. □

4 ALGORITHMS FOR MARKETS WITH UTILITY LIMITS

4.1 Equilibria with Extremal Prices

As mentioned above, in markets with utility limits, the framework in [40] provides an (arbitrary) equilibrium in time $O(n^5 \log(nU))$. In this section, we show how to transform in

ALGORITHM 5: MinPrices for Utility Limits

Input: Market parameters and thrifty and modest equilibrium $(\mathbf{x}, \mathbf{p})$
Output: Thrifty and modest equilibrium with smallest prices

Initialize active budget $m_i^a \leftarrow \min\{m_i, \min_j c_i p_j / u_{ij}\}$ for every buyer $i \in B$
$S \leftarrow \{j \mid p_j > 0$ and j does not have incident equality edges to any uncapped buyer$\}$
$B' \leftarrow$ Set of buyers who have incident equality edges to S
while $S \neq \emptyset$ **do**
    $\gamma \leftarrow 1$
    Define prices and active budgets as follows:
    $p_j \leftarrow \gamma \cdot p_j,\ \forall j \in S;\ m_i^a \leftarrow \gamma \cdot m_i^a,\ \forall i \in B'$
    Decrease γ continuously down from 1 until one of the following events occurs:
    Event 1: γ becomes zero
    Event 2: A new equality edge appears
    Recompute S and B'
return $(\mathbf{x}, \mathbf{p})$ `// x remains same as in the input`

polynomial time any thrifty and modest equilibrium into one with minimum and maximum revenue using the postprocessing procedures Algorithm 5 and Algorithm 6, respectively.

Theorem 4.1. *Algorithm 5 computes a thrifty and modest equilibrium with smallest prices.*

Proof. It is easy to check that throughout the algorithm, $(\mathbf{x}, \mathbf{p})$ always remains a thrifty and modest equilibrium. Assume by contradiction that at the end of the algorithm, $(\mathbf{x}, \mathbf{p})$ is not an equilibrium with smallest prices. Let $(\mathbf{x}', \mathbf{p}')$ be an equilibrium with smallest prices, and define $S_1 = \{j \mid p_j > p'_j\}$. By Lemma 2.4 property (3), all buyers in $\Gamma(S_1, \mathbf{p})$ are capped buyers. Because prices of goods in set S_1 decrease from $\mathbf{p}$ to $\mathbf{p}'$, every buyer i incident to S_1 in the equality graph with prices $\mathbf{p}$ will only have equality edges to S_1 with prices $\mathbf{p}'$. Therefore, we have $i \in \Gamma(S_1, \mathbf{p}') = \Gamma(S_1, \mathbf{p})$ (the equality is again by Lemma 2.4). This implies that $\Gamma(S_1, \mathbf{p})$ is also the set of buyers who have incident equality edges to S_1 with prices $\mathbf{p}$. Hence, set S is nonempty for the while-loop, and the algorithm should not terminate. □

Theorem 4.2. *Algorithm 6 computes a thrifty and modest equilibrium with largest prices.*

Proof. Algorithm 6 takes a thrifty and modest equilibrium $(\mathbf{x}, \mathbf{p})$ and outputs an equilibrium with largest prices. We first process the zero priced goods. Let G_0 be the set of goods whose prices are zero at $\mathbf{p}$, B_0 be the set of buyers who derive positive utility from a good in G_0, and E_0 be the set of tuples (i, j) in $(B_0 \times G_0)$ such that buyer i derives positive utility from good j. We next construct a bipartite graph $((B_0, G_0), E_0)$ formed by the set of nodes B_0 and G_0 and edges in E_0.

We do the following for each connected component C of the bipartite graph. Let B_C and G_C denote the set of buyers and goods in C, respectively. Note that all buyers in B_C are capped. We solve the linear program LP1 to determine whether it is possible to achieve utility limits for all buyers in B_C without consuming all of G_C. If the answer is yes, i.e., $opt1$ is positive, then we claim that goods of G_C are priced zero in all equilibria. For a contradiction, suppose there is an equilibrium $(\tilde{\mathbf{x}}, \tilde{\mathbf{p}})$ where prices of goods in G_C are positive. At $\tilde{\mathbf{p}}$, let m_i^a be the active budget of buyer i in B_C. Note that active budget of a buyer is the least amount of budget needed to achieve its utility limit, and the total prices are equal to total active budget. Let $\mathbf{x}'$ be an allocation where each buyer in B_C achieves its utility limit and goods of G_C are not fully consumed. If we use $\mathbf{x}'$ at prices $\tilde{\mathbf{p}}$, then the total money spent by each buyer i is at least m_i^a, which implies that the total

$$
\begin{array}{ll}
\max \sum_{j \in G_C} p_j & \\
\lambda_i = p_j / u_{ij}, & \forall (i,j) \in E_C \\
\lambda_i \le p_j / u_{ij}, & \forall i \in B_C, \forall j \in G \\
\sum_{j \in G_C} f_{ij} = c_i \lambda_i, & \forall i \in B_C \\
\sum_{i \in B_C} f_{ij} = p_j, & \forall j \in G_C \\
f_{ij} = 0, & \forall (i,j) \notin E_C \\
p_j \ge 0, f_{ij} \ge 0, & \forall i \in B_C, \forall j \in G_C
\end{array}
$$

Fig. 1. Linear program LP2 for Algorithm 6.

ALGORITHM 6: MaxPrices for Utility Limits

Input: Market parameters and thrifty and modest equilibrium $(\mathbf{x}, \mathbf{p})$
Output: Thrifty and modest equilibrium with largest prices

/* First, process zero priced goods.*/
$G_0 \leftarrow \{j \in G \mid p_j = 0\}$
$B_0 \leftarrow \{i \in B \mid u_{ij} > 0, j \in G_0\}$
$E_0 \leftarrow \{(i,j) \in (B_0 \times G_0) \mid u_{ij} > 0\}$
foreach *connected component* C *of bipartite graph* $((B_0, G_0), E_0)$ **do**
  $B_C \leftarrow$ Set of buyers in C
  $G_C \leftarrow$ Set of goods in C
  Let $opt1$ be the optimal value of the following LP in variables y_{ij}s:

$$
\begin{array}{ll}
\max |G_C| - \sum_{i \in B_C, j \in G_C} y_{ij} & \\
\sum_{j \in G_C} u_{ij} y_{ij} = c_i & \forall i \in B_C \\
\sum_{i \in B_C} y_{ij} \le 1 & \forall j \in G_C \\
y_{ij} \ge 0 & \forall i \in B, j \in G
\end{array}
$$

  if $opt1 = 0$ **then**
    $E_C \leftarrow \{(i,j) \in (B_C \times G_C) \mid x_{ij} > 0\}$
    $(p_j)_{j \in G_C} \leftarrow$ Optimal solution of LP2 in Figure 1
$E \leftarrow \{(i,j) \in (B \setminus B_0 \times G \setminus G_0) \mid x_{ij} > 0\}$
$(C_1, \ldots, C_k) \leftarrow$ connected components of $(B \setminus B_0, G \setminus G_0, E)$ where all buyers are capped
$C \leftarrow C_1 \cup \cdots \cup C_k$
$B_C \leftarrow$ Set of buyers in C
$G_C \leftarrow$ Set of goods in C
$E_C \leftarrow \{(i,j) \in (B_C \times G_C) \mid x_{ij} > 0\}$
$(p_j)_{j \in G_C} \leftarrow$ Optimal solution of LP2 in Figure 1
return $(\mathbf{x}, \mathbf{p})$ `// x remains same as in the input`

incoming money to goods in G_C at $\mathbf{x}'$ is at least the total prices of G_C. However, the goods of G_C are not fully consumed, which is a contradiction.

If $opt1$ is zero, then we find the maximum possible prices of goods in G_C using another linear program LP2, given in Figure 1, while maintaining $\mathbf{x}$ to be on MBB edges. For that, we first find the set of edges $(i,j) \in (B_C \times G_C)$ with positive allocation. The variables in LP2 are λ_i's, p_j's, and f_{ij}'s, where λ_i denote the minimum budget buyer i needs for a unit utility at prices $\mathbf{p}$ assuming that the edges in E_C remain MBB at $\mathbf{p}$. This is captured in the first two constraints. The f_{ij} denote the money spent by buyer i on good j. The money balance is captured in the third and fourth

constraints. The third and fifth constraints together capture that each buyer i in B_C achieves its utility limit. We claim that the output of LP2 gives the largest possible prices for goods in G_C.

Let $\mathbf{p}'$ be the prices obtained from the solution of LP2. Suppose there is an equilibrium $(\tilde{\mathbf{x}}, \tilde{\mathbf{p}})$ where goods in G_C give more revenue than the revenue obtained from them at $\mathbf{p}'$. Note that the total active budget of buyers in B_C with respect to prices $\tilde{\mathbf{p}}$ is same as total prices of goods in G_C at $\tilde{\mathbf{p}}$. Further, observe that $\mathbf{p}'$ are the maximum prices given the constraint that $\mathbf{x}$ remains MBB for goods in G_C. Hence, there is a tuple (i, j) such that $x_{ij} > 0$ and (i, j) is not an MBB edge with respect to prices $\tilde{\mathbf{p}}$. Each buyer in B_C achieves its utility limit at both $\mathbf{x}$ and $\tilde{\mathbf{x}}$. These together imply that the total budget spent by buyers in B_C at $\mathbf{x}$ with respect to prices $\tilde{\mathbf{p}}$ is strictly more than the total prices at $\tilde{\mathbf{p}}$, which is a contradiction. Thus, $\mathbf{p}'$ is the vector of largest prices that goods in G_C can achieve in any equilibrium.

Finally, we process positive priced goods. For that, let E denote the set of tuples (i, j) in the remaining set of buyers and goods such that there is a positive flow from buyer i to good j at $\mathbf{x}$. Now consider the bipartite graph between buyers $B \setminus B_0$ and goods $G \setminus G_0$ with edges in E. Recall that Lemma 2.4 shows that if there is an uncapped buyer in a connected component C of bipartite graph $((B \setminus B_0, G \setminus G_0), E)$, then the prices of goods in C are unique in all equilibria. Using this, we only need to process the components where all buyers are capped. Let $C_1, \ldots, C_k$ be the connected components where all buyers are capped. Let C be the union of all these components, and B_C and G_C be the set of buyers and goods in C, respectively. Further, let E_C be the restriction of E on B_C and G_C. We now solve LP2 for the goods in C to find the maximum prices such that the allocation $\mathbf{x}$ remains on the MBB goods. Using a similar argument as in the last paragraph above, we can conclude that this gives the largest possible prices for goods in G_C. □

4.2 Extensions

All thrifty and modest equilibria have a Pareto-optimal allocation. Note, however, that the standard definition of market equilibrium requires that buyers obtain a demand bundle with maximum utility and market clears. Whenever a capped buyer achieves the utility cap, he or she buys a demand bundle, even if he or she does so in a non-thrifty way by spending his or her money on non-MBB goods. If we allow non-thrifty spending, then we obtain market equilibria outside the set of thrifty and modest equilibria, for which utilities are not uniquely determined. In fact, we show that market equilibria with maximum social welfare might not be thrifty and modest equilibria, and computing such optimal equilibria becomes NP-hard. As a corollary, we note that the proof can also be used to show NP-hardness for optimizing any constant norm of utility values.

Theorem 4.3. *It is* NP*-hard to compute a market equilibrium that maximizes social welfare.*

Proof. We reduce from 3-Dimensional Matching. Consider an instance I composed of three disjoint sets A, B, C of elements and a set $T \subseteq A \times B \times C$ of triples. Let $n = |A| = |B| = |C|$ be the number of elements in each set and $m = |T|$ the number of triples. W.l.o.g. assume $m \geq n$. Now we construct a Fisher market based on I as follows. For each element $i \in A \cup B \cup C$ we introduce an *element agent* i with budget 1. For each triple $j \in T$ we introduce a good j and an *auxiliary agent* i_j with budget 1. All these agents have linear utility functions. In addition, there is a single *decision agent* i_d with a budget-additive utility function and a budget of $4m^2(m - n)$.

For the utility values, for each agent $i \in A \cup B \cup C$ we assume $u_{ij} = 1$ if triple j contains i and 0 otherwise. For auxiliary agent i_j the utility is $u_{i_j j} = 1/m^3$ and 0 for all other goods. Finally, the decision agent i_d has utility $u_{i_d j} = 1/m^3$ for every good j and a cap of $c_{i_d} = (m - n)/(m(m^2 + 1))$. Our claim is that a market equilibrium with social welfare of $W = 3n \cdot (1/4) + n \cdot (1/4m^3) + (m - n)/m^3$ exists if and only if the instance I has a solution.

First, suppose I has a solution $S \subseteq T$. Then we set the prices to be $p_j = 4$ for every $j \in S$ and $p_j = m^2 + 1$ for every $j \notin S$. As for the allocation, each agent i spends its entire budget of 1 on the good $j \in S$ that includes him. Each auxiliary agent spends its budget on the corresponding good. Finally, the decision agent i_d spends a budget of m^2 on each of the $m - n$ goods $j \notin S$. Observe that all goods are allocated, and (since w.l.o.g. we can assume $m > 2$) every agent with linear utility function spends its entire budget on an MBB good. The decision agent has optimal utility $(m - n) \cdot (1/m^3) \cdot m^2/(m^2 + 1) = c_{i_d}$. As such, we obtain a market equilibrium. Straightforward inspection reveals that the social welfare in this state is indeed W.

However, assume that a market equilibrium achieves a social welfare of at least W. Note that for each good j, the auxiliary agent can at most obtain a utility of $1/m^3$ by getting all of good j. Similarly, the decision agent can obtain at most m goods and get a utility of c_{i_d} for all of them. Thus, by giving all goods to auxiliary and decision agents, together they can contribute at most $1/m^2$ to the social welfare.

We first observe that in every market equilibrium the decision agent obtains a utility of c_{i_d}. Consider any good j and let us broadly overestimate the price in equilibrium by assuming that the auxiliary agent and all three element agents spend a total budget of 4 on j. This is clearly an upper bound on the money that is spent by the element and auxiliary agents on good j. To derive an upper bound, assume this happens on every good j. Even in this case, the decision agent can contribute a budget of $4m^2$ to any set of $m - n$ goods. Since in a market equilibrium, the goods must be shared in proportion to the money spent, the decision agent would thereby be able to obtain a share of $4m^2/(4m^2 + 4) = m^2/(m^2 + 1)$ from each good it contributes to. In total this yields a utility of $(m - n) \cdot (m^2/(m^2 + 1)) \cdot (1/m^3) = c_{i_d}$. Hence, in every market equilibrium the decision agent obtains at least a total share of $(m - n) \cdot m^2/(m^2 + 1)$ of all the goods. Thus, the total remaining supply of goods that can be allocated to the remaining agents is at most $n + (m - n)/(m^2 + 1)$.

Let us now discuss how to distribute this remaining supply optimally among the agents. For every good j, any equilibrium allocation must be proportional to the incoming money. We remove the fraction obtained by the decision agent, denote the remaining supply by s_j, and note $s_j \geq 0$ and $\sum_j s_j \leq n + (m - n)/(m^2 + 1)$. The auxiliary agent always spends its budget of 1 on j. Let y_j be the money spent by element agents on good j, so $3 \geq y_j \geq 0$ and $\sum_j y_j = 3n$. The welfare obtained from good j by auxiliary and element agents in any equilibrium is

$$s_j \left(\frac{y_j}{y_j + 1} + \frac{1}{y_j + 1} \cdot \frac{1}{4m^3} \right).$$

Hence, the social welfare obtained by element and auxiliary agents in any market equilibrium is upper bounded by the optimum solution to the following optimization problem:

$$\begin{aligned} \text{Max.} \quad & \sum_{j \in [m]} s_j \frac{y_j + 1/(4m^3)}{y_j + 1} \\ \text{s.t.} \quad & \sum_{j \in [m]} s_j = n + \tfrac{m-n}{m^2+1} \\ & \sum_{j \in [m]} y_j = 3n \\ & y_j \leq 3 \quad \forall j \in [m] \\ & s_j \leq 1 \quad \forall j \in [m]. \end{aligned} \tag{13}$$

The objective function is linear in the s_j and concave in the y_j, the constraints are concave, and the equality constraints are affine and their gradients are linearly independent. The feasible solution $y_j = 3/m$ and $s_j = (n + (m - n)/(m^2 + 1))/m$ satisfies the inequality constraints with strict

inequality. Hence, the KKT-conditions characterize the unique optimal solution. We use dual variables α and β for the equality constraints and λ_j and μ_j for the inequality constraints. The optimal solution must satisfy

$$-\frac{y_j + 1/(4m^3)}{y_j + 1} + \alpha + \mu_j = 0$$
$$-s_j \frac{1 - 1/(4m^3)}{(y_j + 1)^2} + \beta + \lambda_j = 0$$
$$\lambda_j(y_j - 3) = 0 \text{ and } \lambda_j \geq 0 \text{ for all } j$$
$$\mu_j(s_j - 1) = 0 \text{ and } \mu_j \geq 0 \text{ for all } j.$$

Thus, $\alpha \leq (y_j + 1/(4m^3))/(y_j + 1)$ for all j. Note that $\alpha < (y_j + 1/(4m^3))(y_j + 1)$ implies $\mu_j > 0$ and $s_j = 1$. Also $\beta_j \leq s_j(1 - 1/(4m^3))/(y_j + 1)^2$ for all j. Similarly, $\beta_j < s_j(1 - 1/(4m^3))/(y_j + 1)^2$ implies $\lambda_j > 0$ and $y_j = 3$. We number the y's such that $y_1 \geq y_2 \geq \cdots \geq y_m$. Let ℓ be such that $y_1 \geq \cdots \geq y_\ell > y_{\ell+1} = \cdots = y_m$. Then $s_j = 1$ for $1 \leq j \leq \ell$ and hence $\ell \leq n$. Let $k \leq \ell$ be such that $y_1 = \cdots = y_k > y_{k+1}$. Then $y_j = 3$ for $1 \leq j \leq k$. If $k < \ell$, then $y_{k+2} > 0$, since $\sum_j y_j = 3n$, and we increase the objective by increasing y_{k+1}. Thus $k = \ell$. If $\ell < n$, then $s_{\ell+2} > 0$ and we increase the objective by increasing $s_{\ell+1}$ and $y_{\ell+1}$. Thus $\ell = n$, and the unique optimum is $y_1 = \cdots = y_n = 3$, $y_{n+1} = \cdots = y_m = 0$, $s_1 = \cdots = s_n = 1$.

This proves that in the optimum there are n goods to which the decision player does not contribute ($s_j = 1$) and for which there are exactly three element players that can contribute all their budget to this good ($y_j = 3$). Thus, the upper bound on the social welfare is attained only when the decision player contributes to exactly $m - n$ goods such that the remaining n goods correspond to a partition of the $3n$ agents into n disjoint triples. By straightforward inspection, we see that the upper bound on the social welfare amounts to exactly W. A market equilibrium of social welfare W can exist only if there is a solution to the underlying instance I. This concludes the proof. □

Corollary 4.1. *It is NP-hard to compute a market equilibrium* $(\mathbf{x}, \mathbf{p})$ *that maximizes* $\sum_i (u_i(\mathbf{x}))^\rho$ *for every constant* $\rho > 0$.

Proof. For $\rho > 1$, we can use exactly the same reduction. The optimum coincides with the optimum for social welfare, since we still want to maximize the share of goods assigned to the element agents. For constant $0 < \rho < 1$ and sufficiently large m, the common factor $1/(4m^3)$ is strong enough to keep the incentive of maximizing the share of the element agents. □

There are several ways of introducing satiation points into the utility function. Instead of a global cap, let us assume there is a cap c_{ij} for the utility buyer i can obtain from good j. A *good-based budget-additive utility* of buyer i is then $u_i(\mathbf{x}_i) = \sum_j \min(c_{ij}, u_{ij}x_{ij})$. This variant turns out to be an elementary special case of separable piecewise-linear concave (SPLC) utilities, in which every piece consists of a linear segment followed by a constant segment. We show that even finding a single market equilibrium here becomes PPAD-hard. The proof adjusts a construction put forward in [12].

Theorem 4.4. *It is* PPAD*-hard to compute a market equilibrium in Fisher markets with good-based budget-additive utilities.*

Proof. We adapt the construction of Chen and Teng [12] to prove the theorem. They show PPAD-completeness of computing an approximate equilibrium in Fisher markets under SPLC utilities where each PLC function has at most two segments. Here the second segment can have positive rate of utility, i.e., non-zero slope, hence PPAD-hardness for Fisher markets under

good-based budget-additive utilities where the second segment has zero slope, i.e., no utility, requires adjustment in their construction.

Chen and Teng [12] reduce the PPAD-hard problem of computing an approximate Nash equilibrium in a two-player game to the problem of computing an approximate equilibrium in Fisher markets under SPLC utilities. Their main idea is to construct a family of *price-regulating* markets $\mathcal{M}_n$ for each $n \geq 1$, which has n buyers and $2n$ goods. In $\mathcal{M}_n$, each buyer has budget of three units and each good has supply of one unit, and every approximate equilibrium price vector p satisfies the following *price-regulation* property:

$$\frac{1}{2} \leq \frac{p_{2k-1}}{p_{2k}} \leq 2 \quad \text{and} \quad p_{2k-1} + p_{2k} \approx 3 \quad \text{for every } 1 \leq k \leq n. \tag{14}$$

Next for a given two-player game, additional buyers are inserted in the price-regulating market and game parameters are embedded into their budget and utility functions. These new buyers are given very small budget so that the price-regulation property is still satisfied.

First, we modify the family of price-regulating markets $\mathcal{M}_n$ for each $n \geq 1$ so that each PLC function is either linear or linear with a threshold. In the construction in [12], each buyer k derives non-zero utility only from goods $2k-1$ and $2k$. Its utility function for good $2k-1$ is linear with slope 2 (utility per unit amount), and for good $2k$ it is linear with slope 4 until unit amount and then linear with slope 1. Since the slope of the second segment is 1, it is not a good-based budget-additive utility function. Simply decreasing the slope of the second segment from 1 to 0 does not work. We get only one inequality:

$$\frac{1}{2} \leq \frac{p_{2k-1}}{p_{2k}}.$$

To construct a correct reduction, we use two buyers, say, $(k, 1)$ and $(k, 2)$, instead of one buyer, k. We set the supply of each good to two units instead of one. Both buyers $(k, 1)$ and $(k, 2)$ have budget of three units each, and both derive non-zero utility only from goods $2k-1$ and $2k$. We set the utility function of buyer $(k, 1)$ as follows: For good $2k-1$, it is linear with slope 2, and for good $2k$, it is linear with slope 4 to the unit amount and then linear with zero slope. Similarly, the utility function of buyer $(k, 2)$ is set as follows: For good $2k$, it is linear with slope 2, and for good $2k-1$, it is linear with slope 4 to the unit amount and then linear with zero slope. We claim that this enforces the price-regulation property (14) on every equilibrium price vector $\mathbf{p}$.

Suppose $p_{2k-1}/p_{2k} > 2$ and then buyer $(k, 2)$ demands only good $2k$. This results in more demand of good $2k$ and less demand of good $2k-1$ and hence does not give an equilibrium. Similarly, we get contradiction for the case $p_{2k-1}/p_{2k} < 1/2$. When $\frac{1}{2} \leq \frac{p_{2k-1}}{p_{2k}} \leq 2$, then buyer $(k, 1)$ demands one unit of good $2k-1$ and one unit of good $2k$ and the same for buyer $2k$. This yields an equilibrium.

Next, for the additional buyers who embed the game parameters, we simply change the slope of the second segment from positive to zero for each utility function. We claim that this works, because these buyers do not buy any good on the second segment in the original construction in [12]. Hence, it has no effect on equilibrium when the slope of the second segment is decreased. This concludes the proof. □

5 CONCLUSION

In this article, we analyze Fisher markets with linear utilities and either earning limits or utility limits. We concentrate on the structure, computation, and complexity of thrifty equilibria for earning limits and thrifty and modest equilibria for utility limits. In both market models, these equilibria can be described as optimal solutions to a convex program. They have a number of desirable properties, e.g., unique seller incomes (with earning limits) or buyer utilities (with utility

limits) as well as lattice structure of price vectors. Moreover, in both market models we provide algorithms to compute a thrifty or thrifty and modest equilibrium with smallest or largest prices.

For markets with earning limits, we prove our results even in the much more general domain with spending-constraint utilities. For these markets, we also present a new and improved scaling algorithm to compute a thrifty equilibrium. Moreover, we apply our results to approximating the Nash social welfare with indivisible items in multi-unit markets. For markets with utility limits, we show that closely related variants of the problem suffer from computational hardness results.

There are a number of intriguing open problems for market models with satiated utilities. Recent work [15] shows the existence of equilibria with utility limits even for more general CES functions via an Eisenberg-Gale convex program similar to Equation (4). It would be interesting to see whether we can compute these equilibria faster than by applying standard algorithms to solve the convex program. In addition, for markets with CES functions and earning limits or for markets with spending-constraint utilities and utility limits there is nothing known about existence, structure, or computation.

Equilibria in linear markets with both earning and utility limits are known to have intriguing non-convex structure [26]. Moreover, they can be rounded to a constant-factor approximation of Nash social welfare in markets with indivisible goods and budget-additive utilities. It would be interesting to see whether Fisher markets with earning or utility limits can yield good approximation algorithms for this problem even beyond (rather direct generalizations of) linear utilities, for which this has been a successful approach over the past several years [1, 15, 17, 26].

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