Various Topologies of Coupled-Mode Structures Exhibiting Exceptional Points of Degeneracy

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Abstract— **We investigate the modal characteristics of coupled-mode guiding structures in which the supported eigenmodes coalesce; the condition we refer to as an exceptional point of degeneracy (EPD). EPD is a point in a system parameter space at which the system eigenmodes coalesce in both their eigenvalues and *eigenvectors*, where the number of coalescing eigenmodes at the EPD defines the order of the degeneracy. First, we investigate the prospects of gain/loss balance and how it is related to realizing an EPD. Under geometrical symmetry in coupled resonators or coupled waveguides such scheme is often attributed to PT-symmetry; however, we generalize the concept of PT-symmetry to coupled waveguides exhibiting EPDs that do not necessarily have perfect geometrical symmetry. Secondly, we explore the conditions that lead to the existence of EPDs in periodically coupled waveguides that may be lossless and gainless. In general, we investigate properties associated to the emergence of EPDs in various cases: i) uniform, and ii) periodic, lossy or lossless, coupled-mode structures. Generally, the EPD condition is very sensitive to perturbations; however, it was shown recently with experimental and theoretical studies that EPDs' unconventional properties exist even in the presence of loss and fabrication errors. Extraordinary properties of such systems at EPDs, such as the giant scaling of the quality factor and the high sensitivity to perturbation, provide opportunities for various applications in traveling wave tubes, pulse compressors and generators, oscillators, switches, modulators, lasers, and extremely sensitive sensors.**

Keywords— Exceptional points, coupled waveguides, periodic structures, PT-symmetry.

I. Introduction

Exceptional points of degeneracy (EPDs) in electromagnetics are points at which two or more eigenmodes coalesce in both their eigenvalues and eigenvectors. At these points, the matrix describing the wave propagation is *defective*, i.e., it has degenerate eigenvalues associated with degenerate eigenvectors and the matrix is similar to a Jordan block or to a matrix containing at least a non-trivial Jordan block. The simplest EPD in a spatially uniform (i.e., z-invariant) waveguide can be found at the cutoff frequency where two modes, the forward and backward modes, coalesce at the propagation wavenumber k=0. The number of coalescing eigenmodes at the EPD defines the order of the degeneracy. For instance, the system exhibits 2nd order EPD if only two eigenmodes are coalescing at the EPD, and we call it the regular band edge (RBE). Similarly, we define the stationary inflection points (SIP), and the degenerate band edge (DBE) as the 3rd and 4th order EPDs, respectively [1]–[3] in systems with loss and gain. For a structures exhibiting an EPD with order m (m eigenmodes coalesce into one eigenmode), the system dispersion relation has an asymptotic behavior of $(\omega-\omega_e)\propto(k-k_e)^m$ near the EPD, where ω and k are the angular frequency and the guided wavenumber, respectively, and the EPD is designated with the subscript e. Such dispersion behavior is accompanied by severe reduction in the group velocity of waves propagating in those structures (slow wave structures) and tremendous improvement in the local density of states [4] resulting in a giant increase in the loaded quality factor of the structure [5]. Recently, structures exhibiting exceptional points were proposed in developing and enhancing wide range of applications such as sensors [6], high power devices [7], RF oscillators [8] and lasers [9].

EPD conditions can be engineered in various structures such as photonic crystals [1], [10], [11], coupled transmission lines (CTL) [12], [13], and ladder circuits [8], [14]. For natural uniform CTLs, where each line supports forward propagating modes, a second order EPDs may be obtained (beyond the cutoff frequency) based on the concept of parity-time (PT-) symmetry by using balanced and symmetrical gain and loss or even with asymmetric distributions of gain and loss [12]. However, for uniform CTLs where one line supports propagating modes and the other one supports evanescent modes, EPDs may be obtained without the need of gain or loss.

In this paper, we investigate general properties of the emergence of EPDs in various cases: uniform or periodic, lossy or lossless, coupled mode structures. The EPD condition, in general, is very sensitive to perturbations; however it was shown recently with experimental and theoretical studies that EPDs is robust to the presence of loss and fabrication errors [15], [16]. The extraordinary properties of the systems at EPDs, such as the giant scaling of the quality factor and the high sensitivity of the system eigenvalue to perturbation, provides eccentric opportunities for applications such as sensors [6], switches, modulators, Q-switching devices, pulse generation and lasers [17], [18].

II. Mathematical Description of EPDs in Coupled Mode Structures

To mathematically explain the existence of EPDs in coupled mode waveguides, guided wave dynamics are represented by two-dimensional vector mode amplitudes $\mathbf{a}(z)$ and $\mathbf{b}(z)$ representing multiple waves propagating in the $+z$ and $-z$ directions, respectively. Hence, the space evolution of the waves along the waveguide direction is described by the state vector $\boldsymbol{\psi}$ where $\boldsymbol{\psi}(z)=\begin{bmatrix}\mathbf{a}^T(z) & \mathbf{b}^T(z)\end{bmatrix}^T$ and T denotes the transpose operation. The space-evolution equation that describes the system wave dynamics is written as $\partial\boldsymbol{\psi}/\partial z=i\,\underline{\mathbf{M}}(z)\boldsymbol{\psi}$, where $\underline{\mathbf{M}}(z)$ represents the system matrix.

Uniform coupled mode systems: For uniform systems (non-periodic), the EPD is a point in the system parameter space at which the eigenvalues and eigenvectors of the matrix

$\underline{\mathbf{M}}$ coalesce, i.e., there is an eigenvalue of $\underline{\mathbf{M}}$ whose geometric multiplicity is less that its algebraic multiplicity [19]. Hence at the EPD, $\underline{\mathbf{M}}$ is non-diagonalizable yet similar to a matrix that has one or more Jordan blocks [1], [19]. This is the general condition that leads to an EPD, where we cannot represent the system evolution using regular eigenvectors. Instead, generalized eigenvectors are used to represent the system evolution, hence the solutions algebraically diverge along the *z*-direction as $\mathbf{\Psi}(z) \propto z^{q-1} e^{ikz}\, \mathbf{\Psi}_q(0)$, where $q = 2,3,\ldots,m$ with *m* being the order of the EPD, and we have adopted the $e^{-i\omega t}$ time convention.

Periodic coupled mode systems: Analogous EPD conditions may arise in periodic coupled waveguides. Within a unit cell, the state vector is translated between points *z* and *z*+*d*, where *d* is the period, as $\boldsymbol{\psi}(z+d) = \underline{\mathbf{T}}(z+d, z)\boldsymbol{\psi}(z)$. Similar to the uniform case, the EPD emerges in this periodic system when the transfer matrix $\underline{\mathbf{T}}$ is non-diagonalizable, i.e., it is similar to a non-trivial Jordan block. Moreover, the degenerate eigenmode solution in such periodic systems algebraically diverges along the *z*-direction. In this case the system may have EPD also when it is lossless and gainless, and still $\underline{\mathbf{T}}$ is similar to a Jordan block.

In proximity of an EPD at a wavenumber k_e , a degenerate eigenmode wavenumber *k* follows the law $(k - k_e)^m \propto \delta$, where *m* is the order of the EPD, and δ is a small detuning in the parameter space of the system [20] (such as gain/loss, coupling coefficient, or frequency, etc.).

III. Different Topologies of Coupled Mode Structures to Realize EPDs

Here we show examples of EPDs manifested in different topologies of coupled mode structures both in the case of uniform structures, as well as in periodic structures.

Uniform loss-gain balanced coupled mode structures: Here we focus on the emergence of EPD in coupled uniform waveguides with "balanced gain and loss" (a system with parity-time (PT-) symmetry is a particular case) as shown in Fig. 1(a) [12]. In fact, such coupled waveguides through the proper design of the introduced gain and loss may exhibit a second order or higher order EPDs at which the system matrix $\underline{\mathbf{M}}$ can be reduced to a nontrivial Jordan block. Introducing gain into naturally lossy structures provides the conditions whereby exceptional points of non-Hermitian degeneracies can be manifested, such as in PT-symmetric structures.

Uniform lossless and gainless coupled mode structures: Let us consider two coupled uniform lossless and gainless waveguides, where each waveguide (when uncoupled) may support forward propagating, backward propagating, or evanescent modes as shown in Fig. 1(b). The novel idea is that a fourth order EPD may emerge in such structure by a proper design of one the coupled waveguides to support evanescent modes so that the system matrix $\underline{\mathbf{M}}$ is similar to a nontrivial Jordan block. Fig. 2(a) shows a configuration of two CTLs supporting propagating and evanescent modes when uncoupled and therefore it may exhibit an EPD. The 3D dispersion diagram of such system is shown in Fig. 2(b) where the system parameters are given in the figure caption. This dispersion diagram shows a fourth order EPD at frequency 5 GHz, at which the four wavenumbers coalesce as $k_1 = k_2 = k_3 = k_4 = 0$.

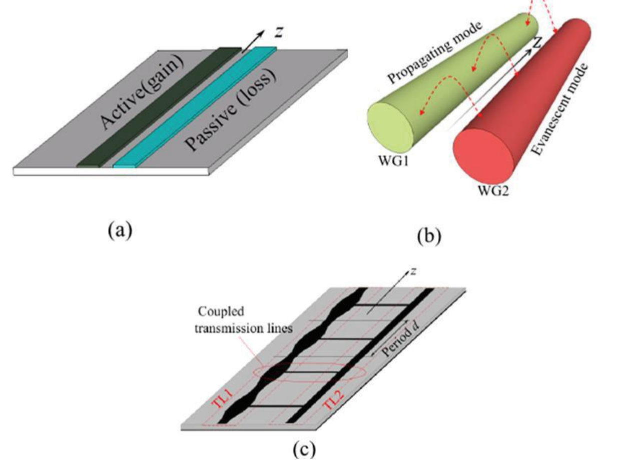

Fig. 1. (a) Uniform coupled waveguides with balanced gain and loss (not necessarily topologically-symmetric) develop an EPD of order 2 for a proper combination of parameters. (b) Uniform lossless and gainless coupled waveguides where EPD emerges due to the coupling between propagating and evanescent modes. (c) Periodic Coupled TLs, capable of supporting EPDs of orders 2 and 4.

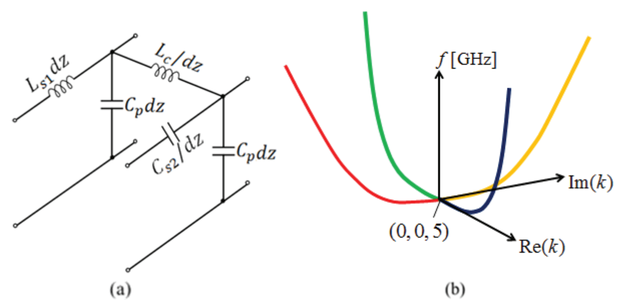

Fig. 2. (a) Uniform lossless and gainless two CTL exhibiting EPD at frequency 5 GHz with the per unit length parameters $C_p = 0.12\,\mathrm{pF/m}$, $L_{s1} = 0.2\,\mu\mathrm{H/m}$, $C_{s2} = 5.06\,\mathrm{fF \cdot m}$ and $L_c = 16.88\,\mathrm{nH \cdot m}$. (b) The 3D dispersion diagram of the system where the four modes coalesce at a point where $k = 0$.

Periodic lossless and gainless coupled mode structures: we also demonstrate that the concept of EPD applies to lossless and gainless periodic structures. For instance, consider the periodic coupled transmission lines shown in Fig. 1(c). Such periodic structure can support EPDs of different orders in the $k{-}\omega$ dispersion; by proper engineering of the different system parameters so that the transfer matrix $\underline{\mathbf{T}}$ is similar to a Jordan block [16], [19].

IV. Conclusion

We have illustrated the general characteristics of coupled mode structures to exhibit exceptional points of degeneracy (EPD) and we have provided mathematical conditions for the EPDs to exist. Different topologies of coupled mode systems, either lossless or with gain-loss balance are considered. Such EPD concepts provide unprecedented opportunities for enhancing slow-light characteristics in coupled systems for various applications including lasers, high power sources, pulse compressors and sensors.

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