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ORIGINAL PAPER

The Chromatic Number of Graphs with No Induced Subdivision of K_4

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Abstract

In 2012, Lévêque, Maffray and Trotignon conjectured that if a graph does not contain an induced subdivision of K_4, then it is 4-colorable. Recently, Le showed that every such graph is 24-colorable. In this paper, we improve the upper bound to 8.

Keywords Chromatic number · Subdivision · ISK4

Mathematics Subject Classification 05C15 · 05C12

1 Introduction

All graphs in this paper are finite and simple. Let G be a graph. A *subdivision* of G is a graph obtained from G by replacing the edges of G with independent paths of length at least one between their end vertices. For a graph H, we say that G *contains* H if H is isomorphic to an induced subgraph of G, and otherwise, G is H-*free*. For a family $\mathcal{F}$ of graphs, we say that G is $\mathcal{F}$-*free* if G is F-free for every graph $F \in \mathcal{F}$. An *ISK4* of G is an induced subgraph of G that is ismorphic to a subdivision of K_4, where K_4 denotes the complete graph on four vertices. A graph is *ISK4-free* if it does not contain any induced subdivision of K_4.

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Gyárfás [2] defined that a class of graphs $\mathcal{G}$ is *χ-bounded* if there exists a χ-bounding function f such that $\chi(G) \leq f(\omega(G))$ holds for every graph $G \in \mathcal{G}$, where $\chi(G)$ is the chromatic number of G and $\omega(G)$ is the maximum size of a clique in G. Scott [7] conjectured that for any graph H, the class of those graphs that do not contain any subdivision of H is *χ-bounded*, and proved this conjecture in the case that H is a forest. However, this conjecture was disproved by Pawlik et al. [6]. In 2012, Lévêque, Maffray and Trotignon [5] showed that the chromatic number of ISK4-free graphs is bounded by a constant c, which implies that Scott's conjecture is true when $H = K_4$. The proof is based on a decomposition theorem for ISK4-free graphs in [5] and a result of Kühn and Osthus [3], from which it follows that c is at least $2^{2^{2^{25}}}$. Since no example of an ISK4-free graph whose chromatic number is 5 or more is known, Lévêque, Maffray and Trotignon [5] proposed the following conjecture.

Conjecture 1.1 (Lévêque, Maffray and Trotignon [5]) *If G is an ISK4-free graph, then $\chi(G) \leq 4$.*

A *hole* of a graph is an induced cycle of length at least four. A *wheel* is a graph that consists of a hole H plus a vertex which has at least three neighbors on H. Lévêque, Maffray and Trotignon [5] proved that every {ISK4, wheel}-free graph is 3-colorable. Trotignon and Vušković [8] showed that every ISK4-free graph with girth at least 5 is 3-colorable. In the same paper, they further conjectured that every {ISK4, triangle}-free graph is also 3-colorable. Le [4] showed that the chromatic number of every {ISK4, triangle}-free graph is at most 4. Recently, Chudnovsky et al. [1] confirmed the conjecture of Trotignon and Vušković [8] and they obtained the following result.

Theorem 1.2 (Chudnovsky et al. [1]) *If G is an* {*ISK*4, triangle}*-free graph, then $\chi(G) \leq 3$.*

By using a layering approach, Le [4] gave a new upper bound for the chromatic number of ISK4-free graphs. He proved the following theorem.

Theorem 1.3 (Le [4]) *If G is an ISK4-free graph, then $\chi(G) \leq 24$.*

In [4], Le mentioned that the bound 24 could be slightly improved by excluding more structures in each layer. We actually improve the upper bound to 8 by applying a similar idea in [4] and by Theorem 1.2.

Theorem 1.4 *If G is an ISK4-free graph, then $\chi(G) \leq 8$.*

In order to prove Theorem 1.4, we apply the layering approach introduced in [4]. We mainly consider the properties of the triangles within layers and then find an independent set whose deletion results in a triangle-free subgraph in each layer. It follows from Theorem 1.2 and a result in [4] that each layer can be 4-colored, and hence the whole graph is 8-colorable.

2 Preliminaries

We follow the notation used in [4, 5]. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For any $S \subseteq V(G)$ and $C \subseteq V(G)\backslash S$, we denote by $N_C(S)$ the set of neighbors of S in C. Let $N(S) = N_{V(G)\backslash S}(S)$. We say that S *dominates* C if $N_C(S) = C$. We denote by $G - S$ the graph obtained from G by deleting the vertices in S together with their incident edges and denote by $G[S]$ the subgraph of G induced by S.

The *line graph* $L(G)$ of G is the graph with vertex set $E(G)$, where two vertices are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in G. A *complete bipartite* (resp., *complete tripartite*) graph $K_{p,q}$ (resp., $K_{p,q,r}$) is a graph whose vertex set can be partitioned into two (resp., three) independent sets of size p and q (resp., p, q and r) such that every pair of vertices from two different independent sets are adjacent. A complete bipartite or complete tripartite graph is *thick* if it contains a $K_{3,3}$.

A *square* $S = \{v_1, v_2, v_3, v_4\}$ is an induced cycle C of length four such that v_1, v_2, v_3, v_4 occur on C in this order. For a path $P = x_1x_2...x_n$, we say that x_1 and x_n are the *end vertices* of P and $x_2, ..., x_{n-1}$ are the *interior vertices* of P. A *link* of S is an induced path P of G with end vertices p and p' such that either $p = p'$ and $N_S(p) = S$, or $N_S(p) = \{v_1, v_2\}$ and $N_S(p') = \{v_3, v_4\}$, or $N_S(p) = \{v_1, v_4\}$ and $N_S(p') = \{v_2, v_3\}$, and there are no edges between S and the interior vertices of P. A *rich square* is a graph K that contains a square S such that there are at least two components in $K - S$, each of which is a link of S. It is easy to see that $K_{2,2,2}$ is the smallest rich square. A *prism* is a graph consisting of three vertex-disjoint induced paths $P_1 = x_1...y_1$, $P_2 = x_2...y_2$, $P_3 = x_3...y_3$ of length at least 1, such that $x_1x_2x_3$ and $y_1y_2y_3$ are triangles and there are no edges between these paths except those of the two triangles.

For any nonnegative integer k, a *k-cutset* in a graph G is a subset $S \subsetneq V(G)$ of size k such that $G - S$ is disconnected. A *clique cutset* S is a cutset which is also a clique. A *proper 2-cutset* of a graph G is a pair of nonadjacent vertices $\{a, b\}$ if $V(G)\backslash\{a, b\}$ can be partitioned into two sets X and Y satisfying that there is no edge between X and Y and neither $G[X \cup \{a, b\}]$ nor $G[Y \cup \{a, b\}]$ is a path from a to b.

Lévêque, Maffray and Trotignon [5] proved the following two decomposition theorems for ISK4-free graphs, which will be used in our later proof.

Lemma 2.1 (Lévêque, Maffray and Trotignon [5]) *Let G be an ISK4-free graph. If G contains a $K_{3,3}$, then either G is a thick complete bipartite or complete tripartite graph, or G has a clique cutset of size at most* 3.

Lemma 2.2 (Lévêque, Maffray and Trotignon [5]) *Let G be an ISK4-free graph. If G contains a rich square or a prism, then either G is the line graph of a graph with maximum degree* 3 *or a rich square, or G has a clique cutset of size at most* 3, *or G has a proper* 2*-cutset.*

Le [4] verified that two classes of graphs mentioned in Lemma 2.2 are 4-colorable.

Lemma 2.3 (Le [4]) *If G is the line graph of a graph with maximum degree 3 or a rich square, then $\chi(G) \leq 4$.*

Note that every thick complete bipartite or complete tripartite graph is 3-colorable, and the line graph of a graph with maximum degree 3 or a rich square is 4-colorable (by Lemma 2.3). Hence by applying Lemmas 2.1 and 2.2, we may assume the existence of a $K_{3,3}$, a prism or a $K_{2,2,2}$ in an ISK4-free graph G always implies that G has a clique cutset of size at most 3 or a proper 2-cutset. It is easy to see that if G contains such a cutset K, then $V(G)\backslash K$ can be partitioned into two vertex-disjoint sets such that there are no edges between them. It follows from the proof of Theorem 1.4 in [5] that we can immediately show $\chi(G) \leq 8$ by applying the induction method (see Sect. 3 for more details). Therefore, we can mainly consider the class of {ISK4, $K_{3,3}$, prism, $K_{2,2,2}$}-free graphs when proving Theorem 1.4.

For two distinct vertices s and t of G, the *distance* $d(s, t)$ between s and t is the number of edges in a shortest path from s to t in G. Let $u \in V(G)$, we define $V_0(u) = \{u\}$ and $V_i(u) = \{v | d(v, u) = i\}$ for each $i \geq 1$. Obviously, there are no edges between $V_i(u)$ and $V_j(u)$ for all i, j with $|i - j| \geq 2$.

In [4], Le introduced two special induced subgraphs called the *upstairs path* and the *confluence*. We will use these two structures to connect two or three vertices in the same layer through only the upper layers. The following three results were observed in [4].

Lemma 2.4 (Le [4]) *Let G be a graph and let $u \in V(G)$. If x and y are two distinct vertices in $V_i(u)$, then there exists an induced path P in G from x to y such that $V(P) \subseteq \{u\} \cup V_1(u) \cup \cdots \cup V_i(u)$ and $|V(P) \cap V_j(u)| \leq 2$ for each $1 \leq j \leq i$.*

The path P satisfying the conditions in Lemma 2.4 is called the *upstairs path* of $\{x, y\}$. For three distinct vertices x, y, z of a graph G, a graph H is called a *confluence* of $\{x, y, z\}$ if it is one of the following two types:

(1) Type 1: H consists of three internally vertex-disjoint induced paths P_x, P_y, P_z from a vertex u to x, y, z, respectively, such that there are no edges between these paths.
(2) Type 2: H consists of a triangle $x'y'z'$ and three vertex-disjoint induced paths P_x, P_y, P_z connecting x and x', y and y', z and z', respectively, such that there are no edges between these paths except those of the triangle $x'y'z'$.

We call u the *center* of H if it is of Type 1 and $x'y'z'$ the *center triangle* of H if it is of Type 2. Note that it is possible the length of the path P_x is 0 when $x = u$ (Type 1) or $x = x'$ (Type 2).

Lemma 2.5 (Le [4]) *Let G be a graph and let $u \in V(G)$. If x, y, z are three distinct vertices in $V_i(u)$, then there exists a set $S \subseteq \{u\} \cup V_1(u) \cup \cdots \cup V_{i-1}(u)$ such that $G[S \cup \{x, y, z\}]$ is a confluence of $\{x, y, z\}$.*

Lemma 2.6 (Le [4]) *Let G be a graph and let $u \in V(G)$. Then*

$$\chi(G) \leq \max_{i\ odd} \chi(G[V_i(u)]) + \max_{j\ even} \chi(G[V_j(u)]).$$

The final lemma in this section is about the class of {ISK4, triangle, $K_{3,3}$}-free graphs. It demonstrates that if there is a set S that dominates $V(C)$ where C is a hole, then there must exist some vertices in S which have only one or two neighbors in $V(C)$.

Lemma 2.7 (Le [4]) *Let G be an $\{ISK4, triangle, K_{3,3}\}$-free graph and let C be a hole in G. If there is a subset $S \subseteq V(G) \setminus V(C)$ such that S dominates $V(C)$, then one of the following holds*:

(i) *there exist four distinct vertices u_1, u_2, u_3, u_4 in S and four distinct vertices v_1, v_2, v_3, v_4 in $V(C)$ such that for each $1 \leq i \leq 4$, $N_{V(C)}(u_i) = \{v_i\}$;*

(ii) *there exist three distinct vertices u_1, u_2, u_3 in S and three distinct vertices v_1, v_2, v_3 in $V(C)$ such that for each $1 \leq i \leq 3$, $N_{V(C)}(u_i) = \{v_i\}$ and v_1, v_2, v_3 are pairwise non-adjacent;*

(iii) *there exist three distinct vertices u_1, u_2, u_3 in S and four distinct vertices v_1, v_2, v_3, v_3' in $V(C)$ such that $N_{V(C)}(u_1) = \{v_1\}$, $N_{V(C)}(u_2) = \{v_2\}$, $N_{V(C)}(u_3) = \{v_3, v_3'\}$ and v_1, v_3, v_2, v_3' occur on C in this order.*

3 Proof of Theorem 1.4

In this section, we prove Theorem 1.4 by applying the layering approach in [4] and by considering the structures of the triangles in each layer.

Proof of Theorem 1.4 We prove the theorem by induction on $|V(G)|$. Since $\chi(G) \leq |V(G)|$, we may assume that $|V(G)| \geq 9$ and the result holds for smaller values of $|V(G)|$.

In the following, we will show we can suppose that G contains none of $\{K_{3,3}$, prism, $K_{2,2,2}\}$. Note that the idea used here is the same as the proof of Theorem 1.4 in [5] (as well as the proof of Theorem 4 in [4]). However, for the sake of completeness, we give the proof here.

Suppose that G contains a $K_{3,3}$, a prism or a $K_{2,2,2}$ (a rich square). Since every thick complete bipartite or complete tripartite graph is 3-colorable and the line graph of a graph with maximum degree 3 or a rich square is 4-colorable (by Lemma 2.3), then by Lemmas 2.1 and 2.2, we may assume that either G has a clique cutset of size at most 3 or G has a proper 2-cutset. In both cases, there exists a cutset K in G such that $V(G) \setminus K$ can be partitioned into two sets X and Y and there are no edges between them.

If K is a clique cutset of size at most 3, then by the induction hypothesis, the two subgraphs of G induced by $X \cup K$ and $Y \cup K$ are 8-colorable. We can combine these two 8-colorings so that they coincide on K and obtain an 8-coloring of G. So we may assume that G has no clique cutset of size at most 3 and hence K is a proper 2-

cutset, say $K = \{a, b\}$. This implies that G is 2-connected and there exists an induced path P_Y with end vertices a and b and with interior vertices in Y. Let G'_X be the subgraph of G induced by $X \cup V(P_Y)$. Then every interior vertex of P_Y has degree 2 in G'_X. Let G''_X be the graph obtained from G'_X by deleting all the interior vertices of P_Y and adding a new edge between a and b. Similarly, we can define the graphs G'_Y and G''_Y . Since G'_X is an induced subgraph of G, we know that G'_X is an ISK4-free graph. Then it is easy to see that G''_X is also an ISK4-free graph. The same holds for G''_Y. By the induction hypothesis, both G''_X and G''_Y admit an 8-coloring. Since a and b receive different colors in both colorings, we can combine them so that they coincide on $\{a, b\}$ and obtain an 8-coloring of G again. Hence by the arguments as above, we may assume that G is an {ISK4, $K_{3,3}$, prism, $K_{2,2,2}$}-free graph from now on.

Let u be a vertex of G. Suppose that $G[V_i(u)]$ is triangle-free for each $i \geq 1$. By Theorem 1.2, we have $\chi(G[V_i(u)]) \leq 3$. By applying Lemma 2.6, we conclude that $\chi(G) \leq 6 < 8$ and the assertion of the theorem holds. So there must exist a triangle $x_1x_2x_3$ in $G[V_i(u)]$ for some i. Note that $i \geq 2$; otherwise, $\{u, x_1, x_2, x_3\}$ induces a K_4, a contradiction. For each x_k $(1 \leq k \leq 3)$, we call a vertex $y \in V_{i-1}(u)$ a *private neighbor* of x_k if $N(y) \cap \{x_1, x_2, x_3\} = \{x_k\}$.

Claim 1. We may assume that $|N_{V_{i-1}(u)}(x_1)| = |N_{V_{i-1}(u)}(x_3)| = 1$ with $N_{V_{i-1}(u)}(x_1) \neq N_{V_{i-1}(u)}(x_3)$, $|N_{V_{i-1}(u)}(x_2)| \geq 2$, $N_{V_{i-1}(u)}(x_1) \subseteq N_{V_{i-1}(u)}(x_2)$ and $N_{V_{i-1}(u)}(x_3) \subseteq N_{V_{i-1}(u)}(x_2)$.

We first suppose that each x_j has a private neighbor y_j in $V_{i-1}(u)$ for $1 \leq j \leq 3$. By Lemma 2.5, there exists a confluence H of $\{y_1, y_2, y_3\}$. If H is of Type 1, then $V(H) \cup \{x_1, x_2, x_3\}$ induces an ISK4, a contradiction. If H is of Type 2, then $V(H) \cup \{x_1, x_2, x_3\}$ induces a prism, again a contradiction.

So we assume without loss of generality that x_1 and x_2 share a common neighbor $v \in V_{i-1}(u)$. Then $vx_3 \notin E(G)$; otherwise, $\{v, x_1, x_2, x_3\}$ induces a K_4, a contradiction.

Let w be a neighbor of x_3 in $V_{i-1}(u)$. If w is a private neighbor of x_3, then by Lemma 2.4, there exists an upstairs path P of $\{v, w\}$ and $V(P) \cup \{x_1, x_2, x_3\}$ induces an ISK4, a contradiction. Note that w is not adjacent to every vertex in $\{x_1, x_2, x_3\}$; otherwise, $\{w, x_1, x_2, x_3\}$ induces a K_4, again a contradiction. By symmetry between x_1 and x_2, we may assume that $wx_2 \in E(G)$ and $wx_1 \notin E(G)$.

We now show that $N_{V_{i-1}(u)}(x_3) = \{w\}$. Suppose on the contrary that there exists another neighbor z of x_3 in $V_{i-1}(u)$. Then by the same argument as above for w, we see that z is also not a private neighbor of x_3. Note that z is not adjacent to every vertex in $\{x_1, x_2, x_3\}$; otherwise, $\{z, x_1, x_2, x_3\}$ induces a K_4, a contradiction. Then z has exactly one neighbor in $\{x_1, x_2\}$.

If $zx_1 \notin E(G)$ and $zx_2 \in E(G)$, then by Lemma 2.4, there exists an upstairs path P of $\{w, z\}$ and $V(P) \cup \{x_2, x_3\}$ induces an ISK4, a contradiction. So we may assume that $zx_1 \in E(G)$ and $zx_2 \notin E(G)$.

By applying Lemma 2.5, we find a confluence H of $\{v, w, z\}$. Suppose first that H is of Type 1. Let v^* be the center of H. By symmetry, we may assume that $v^* \neq w$ and $v^* \neq z$. Then $V(H) \cup \{x_2, x_3\}$ induces an ISK4, a contradiction. Hence we may further assume that H is of Type 2. Let $v'w'z'$ be the center triangle of H. If

$\{v, w, z\} = \{v', w', z'\}$, then $V(H) \cup \{x_1, x_2, x_3\}$ induces a $K_{2,2,2}$, a contradiction. So we have $|\{v, w, z\} \cap \{v', w', z'\}| \leq 2$. By symmetry, we may assume that $w \notin \{v', w', z'\}$. Therefore $V(H) \cup \{x_2, x_3\}$ induces a prism, again a contradiction. This implies that $N_{V_{i-1}(u)}(x_3) = \{w\}$. Similarly, we can prove that $N_{V_{i-1}(u)}(x_1) = \{v\}$. So Claim 1 holds.

Following the notation of Claim 1, for each triangle $T = x_1x_2x_3$ in $G[V_i(u)]$, we call x_2 the *core* of T and x_1 and x_3 the *non-cores* of T. Note that every triangle in $G[V_i(u)]$ has a unique core.

Claim 2. For two distinct triangles in $G[V_i(u)]$ which share a common edge, the core of these two triangles must be one end vertex of the common edge.

Let xyz and $x'yz$ be two triangles in $G[N_i(u)]$ which share the common edge yz. To prove Claim 2, it suffices to show that the core of xyz and $x'yz$ is y or z. Suppose on the contrary that the cores of xyz and $x'yz$ are x and x', respectively. By Claim 1, let y^* and z^* be the unique neighbors of y and z in $V_{i-1}(u)$, respectively. Then we have $xy^*, xz^*, x'y^*, x'z^* \in E(G)$. Note that $xx' \notin E(G)$; otherwise, $\{x, x', y, z\}$ induces a K_4, a contradiction. If $y^*z^* \in E(G)$, then $\{x, x', y, z, y^*, z^*\}$ induces a $K_{2,2,2}$, a contradiction. If $y^*z^* \notin E(G)$, then $\{x, x', y, y^*, z^*\}$ induces an ISK4, again a contradiction. This proves Claim 2.

Claim 3. For two triangles in $G[V_i(u)]$ with exactly one common vertex, the core of these two triangles must be this common vertex.

Let xyz and $xy'z'$ be two triangles in $G[V_i(u)]$ which share the common vertex x. We need to show that the core of xyz and $xy'z'$ is x. Suppose on the contrary that the cores of xyz and $xy'z'$ are y and y', respectively. Then by Claim 1, let x^*, z^*, z'^* be the unique neighbors of x, z, z' in $V_{i-1}(u)$, respectively. Hence we have $yx^*, yz^*, y'x^*, y'z'^* \in E(G)$. Note that it is possible $z^* = z'^*$. It is easy to see that $yy' \notin E(G)$; otherwise, $\{x, x^*, y, y'\}$ induces a K_4, a contradiction. By Claim 2, we know that $yz', y'z \notin E(G)$. If $z^* = z'^*$, then $\{x, y, y', z, z^*\}$ induces an ISK4, a contradiction. So we may assume that $z^* \neq z'^*$. Therefore, we have $yz'^* \in E(G)$ or $y'z^* \in E(G)$; as otherwise, by Lemma 2.4, there exists an upstairs path P of $\{z^*, z'^*\}$ and $V(P) \cup \{x, y, y', z\}$ induces an ISK4, a contradiction. But now, we notice that $\{x, y, y', z', z'^*\}$ (if $yz'^* \in E(G)$) or $\{x, y, y', z, z^*\}$ (if $y'z^* \in E(G)$) induces an ISK4, again a contradiction. So Claim 3 holds.

Claim 4. For two triangles in $G[V_i(u)]$ with no common vertex, there are no edges between their non-core vertices of different triangles.

Let $x_1x_2x_3$ and $x_4x_5x_6$ be two triangles in $G[V_i(u)]$ with no common vertex. Let x_2 and x_5 be the cores of $x_1x_2x_3$ and $x_4x_5x_6$, respectively. We now show that $x_1x_4, x_1x_6, x_3x_4, x_3x_6 \notin E(G)$. Suppose on the contrary that one of these four edges exists, say $x_3x_4 \in E(G)$. By Claim 3, we have $x_1x_4, x_2x_4, x_3x_5, x_3x_6 \notin E(G)$. Applying Claim 1 again, let y_1, y_2, y_3 and y_4 be the unique neighbors of x_1, x_3, x_4 and x_6 in $V_{i-1}(u)$, respectively. Then we have $x_2y_1, x_2y_2, x_5y_3, x_5y_4 \in E(G)$. For convenience, by Lemma 2.4, let $P_{j,k}$ be the upstairs path of $\{y_j, y_k\}$ for $1 \leq j, k \leq 4$ and $j \neq k$. We consider two cases.

Case 1. $|\{y_1, y_2\} \cap \{y_3, y_4\}| \leq 1$.

First, suppose that $y_2 = y_3$. We claim that $x_2x_6 \notin E(G)$; otherwise, $\{x_2, x_3, x_4, x_6, y_2\}$ induces an ISK4, giving a contradiction. We also have $x_2y_4 \in$

$E(G)$ or $x_1x_6 \in E(G)$; for otherwise, $V(P_{1,4}) \cup \{x_1, x_2, x_3, x_4, x_6\}$ induces an ISK4, a contradiction. Suppose that $x_2y_4 \in E(G)$. Then by symmetry, we may assume that $x_5y_1 \in E(G)$. But now, we see that $V(P_{1,4}) \cup \{x_2, x_5\}$ (if $x_2x_5 \in E(G)$) or $V(P_{1,4}) \cup \{x_2, x_3, x_4, x_5\}$ (if $x_2x_5 \notin E(G)$) induces an ISK4, giving a contradiction. So we may assume that $x_2y_4 \notin E(G)$ and hence $x_1x_6 \in E(G)$. In this case, we conclude that $V(P_{2,4}) \cup \{x_1, x_2, x_3, x_6\}$ induces an ISK4, again a contradiction.

Next, assume that $y_2 \neq y_3$. By symmetry, we only need to consider the following three cases: (a) $\{y_1, y_2\} \cap \{y_3, y_4\} = \emptyset$; or (b) $y_1 = y_4$; or (c) $y_2 = y_4$. In all three cases, we have $x_2y_3 \in E(G)$; for otherwise, $V(P_{1,3}) \cup \{x_1, x_2, x_3, x_4\}$ induces an ISK4, a contradiction. But then, $V(P_{2,3}) \cup \{x_2, x_3, x_4\}$ induces an ISK4, again a contradiction.

Case 2. $|\{y_1, y_2\} \cap \{y_3, y_4\}| = 2$.

In this case, we have $\{y_1, y_2\} = \{y_3, y_4\}$. If $y_1 = y_3$ and $y_2 = y_4$, then $V(P_{1,2}) \cup \{x_2, x_3, x_4\}$ induces an ISK4, a contradiction. So we deduce that $y_1 = y_4$ and $y_2 = y_3$. Note that $x_2x_5 \notin E(G)$; otherwise, $V(P_{1,2}) \cup \{x_2, x_5\}$ induces an ISK4, a contradiction. We also have $x_1x_5 \in E(G)$; as otherwise, $\{x_1, x_2, x_3, x_4, x_5, y_1\}$ induces an ISK4, a contradiction. Now, we see that $\{x_1, x_3, x_4, x_5, y_2\}$ induces an ISK4, again a contradiction. This proves Claim 4.

Let $\mathscr{C}$ be the set of the cores of all triangles in $G[V_i(u)]$. For each $v \in \mathscr{C}$, let G_v be the subgraph of $G[V_i(u)]$ induced by all the vertices of the triangles containing v.

Claim 5. $G_v - \{v\}$ is a forest.

Suppose on the contrary that $G_v - \{v\}$ is not a forest. Let C be an induced cycle in $G_v - \{v\}$ and let S be the set of all the neighbors of $V(C)$ in $V_{i-1}(u)$. Then by Claim 1, we know that every vertex of $V(C)$ has exactly one neighbor in S and v is adjacent to all the vertices of $V(C) \cup S$.

We consider the graph $G[V(C) \cup S]$. If there exist three vertices in $V(C) \cup S$ such that these three vertices induces a triangle in $G[V(C) \cup S]$, then these three vertices together with the vertex v induce a K_4 in G, a contradiction. So we see that $G[V(C) \cup S]$ is triangle-free, which implies that C is a hole in $G[V(C)] \cup S$. Since $G[V(C) \cup S]$ is an induced subgraph of G, we deduce that $G[V(C) \cup S]$ is an {ISK4, triangle, $K_{3,3}$}-free graph. We now apply a weaker version of Lemma 2.7 for $G[V(C) \cup S]$, S and C. Note that the idea used here is inspired by the proof of Lemma 8 in [4].

If (i) or (ii) of Lemma 2.7 holds, then there exist three distinct vertices s_1, s_2, s_3 in S and three distinct vertices c_1, c_2, c_3 in $V(C)$ such that for each $1 \leq j \leq 3$, $N_{V(C)}(s_j) = \{c_j\}$. By applying Lemma 2.5, there exists a confluence F of $\{s_1, s_2, s_3\}$ in G. If F is of Type 1, then $V(F) \cup V(C)$ induces an ISK4, a contradiction. If F is of Type 2, then $V(F) \cup \{v\}$ induces an ISK4, a contradiction.

If (iii) of Lemma 2.7 holds, then there exist two distinct vertices s_1, s_2 in S and three distinct vertices c_1, c_2, c_2' in $V(C)$ such that $N_{V(C)}(s_1) = \{c_1\}$ and $N_{V(C)}(s_2) = \{c_2, c_2'\}$. By Lemma 2.4, there exists an upstairs path P of $\{s_1, s_2\}$ in G and $V(P) \cup V(C)$ induces an ISK4, again a contradiction. So Claim 5 holds.

Now let $\mathscr{C} = \{v_1, v_2, \ldots, v_m\}$. Then by Claim 5, for each $1 \leq j \leq m$, we see that $G_{v_j} - \{v_j\}$ is a forest and hence admits a bipartition (V_1^j, V_2^j) such that both V_1^j and V_2^j are independent sets in $G_{v_j} - \{v_j\}$. By Claim 4, there are no edges between

$G_{v_j} - \{v_j\}$ and $G_{v_k} - \{v_k\}$ for $1 \leq j, k \leq m$ and $j \neq k$. This implies that $V^* = \bigcup_{j=1}^{m} V_1^j$ is an independent set in $G[V_i(u)]$. Moreover, $G[V_i(u)] - V^*$ is an {ISK4, triangle}-free graph. Then by Theorem 1.2, $G[V_i(u)] - V^*$ is 3-colorable. Hence we can show a 4-coloring of $G[V_i(u)]$ as follows: assign colors 1, 2, 3 to $V_i(u) \backslash V^*$ and color 4 to V^*. By Lemma 2.6, we conclude that $\chi(G) \leq 8$. This completes the proof of Theorem 1.4. □

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