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Hamiltonicity of edge-chromatic critical graphs

Yan Cao [a], Guantao Chen [a,1], Suyun Jiang [b,1,*], Huiqing Liu [c,1], Fuliang Lu [d,1]

[a] Department of Mathematics and Statistics, Georgia State University, Atlanta, GA 30303, United States
[b] Institute for Interdisciplinary Research, Jianghan University, Wuhan, Hubei 430056, China
[c] Hubei Key Laboratory of Applied Mathematics, Faculty of Mathematics and Statistic, Hubei University, Wuhan, Hubei 430062, China
[d] School of Mathematics and Statistics, Minnan Normal University, Zhangzhou 363000, China

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ABSTRACT

Given a graph G, denote by $\Delta(G)$ and $\chi'(G)$ the maximum degree and the chromatic index of G, respectively. A simple graph G is called *edge-Δ-critical* if $\Delta(G) = \Delta$, $\chi'(G) = \Delta + 1$ and $\chi'(H) \le \Delta$ for every proper subgraph H of G. We prove that every edge-Δ-critical graph of order n with maximum degree at least $\frac{2n}{3} + 12$ is Hamiltonian.

1. Introduction

All graphs in this paper are finite and simple. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Denote by $\Delta(G)$, $\delta(G)$ and $\alpha(G)$ the maximum degree, the minimum degree and the independence number of G, respectively. An *edge-k-coloring* of G is a mapping $\varphi : E(G) \to \{1, 2, \ldots, k\}$ such that $\varphi(e) \ne \varphi(f)$ for any two adjacent edges e and f. The codomain $\{1, 2, \ldots, k\}$ is called the *color set* of φ. Denote by $\mathcal{C}^k(G)$ the set of all edge-k-colorings of G. The *chromatic index* $\chi'(G)$ is the least integer $k \ge 0$ such that $\mathcal{C}^k(G) \ne \emptyset$. We call graph G *class one* if $\chi'(G) = \Delta(G)$ and *class two* otherwise. Vizing [13] proved $\chi'(G) = \Delta(G) + 1$ if G is class two. An edge e of G is called *critical* if $\chi'(G - e) < \chi'(G)$, where $G - e$ is the subgraph obtained from G by removing the edge e. A graph G is called *(edge-)Δ-critical* if $\Delta(G) = \Delta$, $\chi'(G) = \Delta + 1$ and $\chi'(H) \le \Delta$ for any proper subgraph H of G. Clearly, if G is Δ-critical, then G is connected and $\chi'(G - e) = \Delta(G)$ for any $e \in E(G)$.

In 1965, Vizing [14] proposed the following conjecture about structure properties of Δ-critical graphs.

Conjecture 1 (*Vizing [14]*)**.** *Every Δ-critical graph with chromatic index at least 3 contains a 2-factor.*

In 1968, Vizing [15] proposed a weaker conjecture on the independence number of Δ-critical graphs as follows.

Conjecture 2 (*Vizing [15]*)**.** *For every Δ-critical graph G of order n, $\alpha(G) \le \frac{n}{2}$.*

* Corresponding author.
E-mail addresses: ycao17@gsu.edu (Y. Cao), gchen@gsu.edu (G. Chen), jiang.suyun@163.com (S. Jiang), hql_2008@163.com (H. Liu), flianglu@163.com (F. Lu).

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Conjecture 2 was verified by Luo and Zhao [9] for Δ-critical graphs of order n with maximum degree at least $\frac{n}{2}$, and by Grünewald and Steffen [6] for Δ-critical graphs with many edges, including all overfull graphs.

Chen and Shan [5] verified Conjecture 1 for Δ-critical graphs of order n with maximum degree at least $\frac{n}{2}$. Obviously, if a graph is Hamiltonian, then it contains a 2-factor. Luo and Zhao [10] proved that a Δ-critical graph G of order n with $\Delta(G) \geq \frac{6n}{7}$ is Hamiltonian. Furthermore, Luo, Miao and Zhao [8] showed that a Δ-critical graph G of order n with $\Delta(G) \geq \frac{4n}{5}$ is Hamiltonian. Recently, Chen, Chen and Zhao [3] improved the lower bound to $\Delta(G) \geq \frac{3n}{4}$. In this paper, we give the following result about the Hamiltonicity of Δ-critical graphs.

Theorem 1. *If G is a Δ-critical graph of order n with $\Delta(G) \geq \frac{2n}{3} + 12$, then G is Hamiltonian.*

It would be nice to know the minimum number α $(0 < \alpha < 1)$ such that every Δ-critical graph G of order n with $\Delta(G) \geq \alpha n$ is Hamiltonian. Our main techniques applied to prove Theorem 1 are the following: (1) extending Woodall's Lemma ($q = 2\Delta(G) - d(x) - d(y) + 2$, see Lemma 3) to an arbitrary q with $q \leq \Delta(G) - 10$ (see Lemma 4); (2) extending Woodall's Lemma (consider the neighbor of a vertex x, see Lemma 3) to Lemma 5 (consider the neighbor of two adjacent vertices).

Let G be a graph and x be a vertex of G. Denote by $N_G(x)$ and $d_G(x)$ the neighborhood and degree of x in G, respectively. We always drop the subscript G and simply write $N(x)$ and $d(x)$ if there is no ambiguity. For any nonnegative integer k, we call a vertex x a *k-vertex* if $d(x) = k$, a *$(< k)$-vertex* if $d(x) < k$, and a *$(> k)$-vertex* if $d(x) > k$. Similarly, we call a neighbor y of x a *k-neighbor*, a *$(< k)$-neighbor* and a *$(> k)$-neighbor* if $d(y) = k$, $< k$ and $> k$, respectively. Denote by $V_{\geq k}(G)$ the subset of $V(G)$ of vertices with degree at least k. Let k be a positive integer and e_0 an edge of G such that $\mathcal{C}^k(G - e_0) \neq \emptyset$, and let $\varphi \in \mathcal{C}^k(G - e_0)$ and $v \in V(G)$. Let $\varphi(v) = \{\varphi(e) : e \text{ is incident with } v\}$ and $\bar{\varphi}(v) = \{1, \ldots, k\} \setminus \varphi(v)$. We call $\varphi(v)$ the set of colors *seen* by v and $\bar{\varphi}(v)$ the set of colors *missing* at v. A set $X \subseteq V(G)$ is called *elementary* with respect to φ if $\bar{\varphi}(u) \cap \bar{\varphi}(v) = \emptyset$ for every two distinct vertices $u, v \in X$. For any color α, let E_α denote the set of edges assigned color α. Clearly, E_α is a matching of G. For any two colors α and β, the components of the spanning subgraph of G with edge set $E_\alpha \cup E_\beta$, named *(α, β)-chains*, are even cycles and paths with alternating colors α and β. For a vertex v of G, we denote by $P_v(\alpha, \beta, \varphi)$ the unique (α, β)-chain that contains the vertex v. Let $\varphi/P_v(\alpha, \beta, \varphi)$ denote the edge-k-coloring obtained from φ by switching colors α and β on the edges on $P_v(\alpha, \beta, \varphi)$. If v is not incident with any edge of color α or β, then $P_v(\alpha, \beta, \varphi) = \{v\}$ (a path of length 0), and $\varphi/P_v(\alpha, \beta, \varphi) = \varphi$.

We will give a few technical lemmas in Section 2 and prove Theorem 1 in Section 3. Due to the length of the proofs of Lemmas 4 and 5, we will prove Lemmas 4 and 5 in Section 4.

2. Lemmas

Let q be a positive number, G be a Δ-critical graph and $x \in V(G)$. For each $y \in N(x)$, let $\sigma_q(x, y) = |\{z \in N(y) \setminus \{x\} : d(z) \geq q\}|$, the number of neighbors of y (except x) with degree at least q. Vizing studied the case $q = \Delta$ and obtained the following result.

Lemma 1 (*Vizing's Adjacency Lemma [14]*)**.** *Let G be a Δ-critical graph. Then $\sigma_\Delta(x, y) \geq \Delta - d(x) + 1$ for every $xy \in E(G)$.*

Woodall [16] studied $\sigma_q(x, y)$ for the case $q = 2\Delta - d(x) - d(y) + 2$ and obtained the following two results. For convenience, we let $\sigma(x, y) = \sigma_q(x, y)$ when $q = 2\Delta - d(x) - d(y) + 2$.

Lemma 2 (*Woodall [16]*)**.** *Let xy be an edge in a Δ-critical graph G. Then there are at least $\Delta - \sigma(x, y) \geq \Delta - d(y) + 1$ vertices $z \in N(x) \setminus \{y\}$ such that $\sigma(x, z) \geq 2\Delta - d(x) - \sigma(x, y)$.*

Furthermore, Woodall defined the following two parameters.

$$p_{\min}(x) := \min_{y \in N(x)} \sigma(x, y) - \Delta + d(x) - 1 \quad \text{and}$$

$$p(x) := \min\left\{ p_{\min}(x), \left\lfloor \frac{d(x)}{2} \right\rfloor - 1 \right\}.$$

Clearly, $p(x) \leq d(x)/2 - 1$. As a corollary, the following lemma shows that there are at least $d(x)/2$ neighbors y of x such that $\sigma(x, y) \geq \Delta/2$.

Lemma 3 (*Woodall [16]*)**.** *Every vertex x in a Δ-critical graph has at least $d(x) - p(x) - 1$ neighbors y for which $\sigma(x, y) \geq \Delta - p(x) - 1$.*

Our proof of Theorem 1 uses the following two lemmas, which will be proved in Section 4.

Lemma 4. *Let G be a Δ-critical graph. For a vertex $x \in V(G)$ and a positive number q, if $d(x) < \frac{\Delta}{2}$ and $q \leq \Delta - 10$, then for any $y \in N(x)$, there exists another neighbor z of x such that $\sigma_q(x, y) + \sigma_q(x, z) > 2\Delta - d(x) - \frac{2(d(x)-1)}{\Delta - q} - \lceil \frac{4(d(x)-1)}{\Delta - q} + \frac{8(d(x)-1)}{(\Delta - q)^2} \rceil$.*

Lemma 5. *Let G be a Δ-critical graph and q be a positive number such that $q \le \Delta - 10$ and minimum degree $\delta(G) > \frac{\Delta}{2} - 2$. For an edge $x_1x_2 \in E(G)$, if $d(x_1)+d(x_2) \le \frac{3}{2}\Delta - 2$, then there exist two distinct vertices $z, y \in V(G)\setminus\{x_1, x_2\}$ with $z \in N(x_1)$ and $y \in N(x_2)$ such that $\sigma_q(x_1, z)+\sigma_q(x_2, y) > 3\Delta - d(x_1) - d(x_2) - \frac{2(d(x_1)+d(x_2)-\Delta-2)}{\Delta-q} - \lceil \frac{4(d(x_1)+d(x_2)-\Delta+2)}{\Delta-q} + \frac{8(d(x_1)+d(x_2)-\Delta-2)}{(\Delta-q)^2} \rceil - 2$.*

Our approach is inspired by the recent development of the Tashkinov tree technique for multigraphs. Let G be a multigraph without loops, e_1 be an edge of G with endvertices y_0 and y_1 and $\varphi \in \mathcal{C}^k(G - e_1)$. A *Tashkinov tree* T with respect to G, e_1 and φ is an alternating sequence $T = (y_0, e_1, y_1, \ldots, e_p, y_p)$ with $p \ge 1$ consisting of edges $e_1, e_2, \ldots, e_p$ and vertices $y_0, y_1, \ldots, y_p$ such that the following two conditions hold.

(1) The vertices $y_0, y_1, \ldots, y_p$ are distinct and $e_i = y_ry_i$ for each $1 \le i \le p$, where $r < i$;
(2) For every edge e_i with $2 \le i \le p$, there is a vertex y_h with $0 \le h < i$ such that $\varphi(e_i) \in \bar{\varphi}(y_h)$.

Clearly, a Tashkinov tree is indeed a tree of G. Tashkinov [12] proved that if G is k-critical with $k \ge \Delta(G)+1$, then $V(T)$ is elementary. In the above definition, if we change condition (1) to say that the edges $e_1, e_2, \ldots, e_p$ are distinct and $e_i = y_0y_i$ for every i, then T is called a *multi-fan*, as defined in [11]. Stiebitz et al. [11] showed that the vertex set of a multi-fan is elementary. In the definition of Tashkinov tree, if $e_i = y_{i-1}y_i$ for every i, i.e., T is a path with endvertices y_0 and y_p, then T is called a *Kierstead path*, which was introduced by Kierstead [7]. Kierstead proved that for every Kierstead path P the set $V(P)$ is elementary if G is k-critical with $k \ge \Delta(G) + 1$. For simple graphs, following Kierstead's proof, Zhang [17] noticed the following Lemma.

Lemma 6 (*Kierstead [7], Zhang [17]*)**.** *Let G be a class two graph with maximum degree Δ. If $e_1 \in E(G)$ is a critical edge and $K = (y_0, e_1, y_1, \ldots, y_{p-1}, e_p, y_p)$ is a Kierstead path with respect to e_1 and a coloring $\varphi \in \mathcal{C}^{\Delta}(G - e_1)$ such that $d(y_j) < \Delta$ for $j = 2, \ldots, p$, then $V(K)$ is elementary with respect to φ.*

Kostochka and Stiebitz considered elementary property of Kierstead paths with four vertices and showed the following Lemma.

Lemma 7 (*Kostochka and Stiebitz [11]*)**.** *Let G be a class two graph with maximum degree Δ. Let e_1 be a critical edge of G and $\varphi \in \mathcal{C}^{\Delta}(G - e_1)$. If $K = (y_0, e_1, y_1, e_2, y_2, e_3, y_3)$ is a Kierstead path with respect to e_1 and φ, then the following statements hold:*

(1) $\bar{\varphi}(y_0) \cap \bar{\varphi}(y_1) = \emptyset$;
(2) if $d(y_2) < \Delta$, then $V(K)$ is elementary with respect to φ;
(3) if $d(y_1) < \Delta$, then $V(K)$ is elementary with respect to φ;
(4) $|\bar{\varphi}(y_3) \cap (\bar{\varphi}(y_0) \cup \bar{\varphi}(y_1))| \le 1$.

In the definition of Tashkinov tree $T = (y_0, e_1, y_1, e_2, y_2, \ldots, y_p)$, we call T a *broom* if $e_2 = y_1y_2$ and for each $i \ge 3$, $e_i = y_2y_i$, i.e., y_2 is one of the endvertices of e_i for each $i \ge 3$. Moreover, we call a broom T a *simple broom* if $\varphi(e_i) \in \bar{\varphi}(y_0) \cup \bar{\varphi}(y_1)$ for each $i \ge 3$, i.e., $(y_0, e_1, y_1, e_2, y_2, e_i, y_i)$ is a Kierstead path.

Lemma 8 (*Chen, Chen and Zhao [3]*)**.** *Let G be a Δ-critical graph, $e_1 = y_0y_1 \in E(G)$ and $\varphi \in \mathcal{C}^{\Delta}(G - e_1)$. Let $B = \{y_0, e_1, y_1, e_2, y_2, \ldots, e_p, y_p\}$ be a simple broom with respect to e_1 and φ. If $|\bar{\varphi}(y_0) \cup \bar{\varphi}(y_1)| \ge 4$ and $\min\{d(y_1), d(y_2)\} < \Delta$, then $V(B)$ is elementary with respect to φ.*

The *circumference* of a graph is the length of longest cycles of the graph.

Lemma 9 (*Brandt and Veldman [2]*)**.** *Let $G \ne K_{1,n-1}$ be a graph of order n. If $d(x) + d(y) \ge n$ for every edge xy of G, then the circumference of G is $n - \max\{|S| - |N(S)| + 1, 0\}$, where S is an independent set of G with $S \cup N(S) \ne V(G)$.*

Using Lemma 9, Chen, Chen and Zhao obtained the following result.

Lemma 10 (*Chen, Chen and Zhao [3]*). *Let G be a Δ-critical graph of order n. If $d(x) + d(y) \ge n$ for every edge xy of G, then G is Hamiltonian.*

The *Bondy–Chvátal closure* $C(G)$ of a graph G with order n, defined by Bondy and Chvátal [1], is the maximal graph obtained from G by consecutively adding the edges xy if the degree sum of x and y is at least n. They proved that $C(G)$ is well-defined and $C(G)$ is Hamiltonian if and only if G is Hamiltonian.

Lemma 11 (*Chen, Ellingham, Saito and Shan [4]*)**.** *Let G be a bipartite graph with partite sets X and Y. If for every $S \subseteq X$, $|N(S)| \ge \frac{3|S|}{2}$, then G has a subgraph H covering X such that for every $x \in X$, $d_H(x) = 2$ and for every $y \in Y$, $d_H(y) \le 2$.*

3. Proof of Theorem 1

Suppose on the contrary that there exists a non-Hamiltonian Δ-critical graph G of order n with maximum degree $\Delta \geq \frac{2}{3}n + 12$. Solving $n > \Delta \geq \frac{2}{3}n + 12$, we get $\Delta > 36$. Recall that $C(G)$ is the Bondy–Chvátal closure of G.

Before proceeding with the proof, we give a brief outline of our proof strategy. We first show that there is a positive number $r_1 := r_1(\Delta)$ such that $|V_{\geq r_1}(G)| \geq \frac{n}{2}$. We then show that there is another positive number $r_2 := r_2(\Delta)$, which is smaller than r_1, such that for any $u \in V_{>r_2}(G)$ and any $v \in V_{\geq r_1}(G)$, we have that $d_{C(G)}(u) + d_{C(G)}(v) \geq n$, which in turn shows that $d_{C(G)}(u) \geq |V_{\geq r_1}(G)| \geq \frac{n}{2}$. So $V_{>r_2}(G)$ is a clique in $C(G)$. We finally show that $V_{\leq r_2}(G)$ is an independent set of G and is covered by paths intersecting $V_{>r_2}(G)$ and all endvertices of these paths are in $V_{>r_2}(G)$, which shows that $C(G)$ is Hamiltonian, so does G.

We first prove a general result.

Claim 3.1. *Let q be a positive number with $q \leq \Delta - 10$. Then*

$$|V_{\geq q}(G)| > \begin{cases} \frac{3\Delta}{4} - \frac{3\Delta-18}{2(\Delta-q)} - \frac{2\Delta-12}{(\Delta-q)^2} + \frac{1}{2} & \text{if } \delta(G) \leq \frac{\Delta}{2} - 2; \\ \frac{3\Delta}{4} - \frac{3\Delta-110}{2(\Delta-q)} - \frac{2\Delta-84}{(\Delta-q)^2} + 8 & \text{if } \delta(G) > \frac{\Delta}{2} - 2. \end{cases}$$

Consequently, we have

$$|V_{\geq q}(G)| > \frac{3\Delta}{4} - \frac{3\Delta}{2(\Delta - q)} - \frac{2\Delta}{(\Delta - q)^2}.$$

Proof. Assume first that $\delta(G) \leq \frac{\Delta}{2} - 2$, and let x be a vertex such that $d(x) = \delta(G)$. By Lemma 4, there exists a vertex $y \in N(x)$ such that

$$\sigma_q(x, y) > \Delta - \frac{d(x)}{2} - \frac{3(d(x) - 1)}{\Delta - q} - \frac{4(d(x) - 1)}{(\Delta - q)^2} - \frac{1}{2}. \tag{1}$$

Clearly, $|V_{\geq q}(G)| \geq \sigma_q(x, y)$. Using $d(x) \leq \frac{\Delta}{2} - 2$ in (1), we obtain the lower bound.

Assume now that $\delta(G) > \frac{\Delta}{2} - 2$. Note that $n \leq \frac{3}{2}\Delta - 18$ since $\Delta \geq \frac{2n}{3} + 12$. Let xy be an edge of G such that $d(x) + d(y) \leq n - 1 < \frac{3}{2}\Delta - 2$, such an edge exists as otherwise G would be Hamiltonian by Lemma 10. Applying Lemma 5, we may assume that there exists a neighbor z of x such that

$$\sigma_q(x, z) > \frac{3\Delta - d(x) - d(y) - 3}{2} - \frac{3(d(x) + d(y) - \Delta) + 2}{\Delta - q} - \frac{4(d(x) + d(y) - \Delta - 2)}{(\Delta - q)^2}.$$

Since $d(x) + d(y) \leq n - 1 \leq \frac{3}{2}\Delta - 19$, we get the desired lower bound. □

Applying Claim 3.1 to $q = \Delta - 17 < \Delta - 10$, we obtain the following inequality

$$|V_{\geq \Delta-17}(G)| > \frac{3\Delta}{4} - \frac{3\Delta}{34} - \frac{2\Delta}{17^2} = \frac{757}{1156}\Delta. \tag{2}$$

Let

$$r_1 := r_1(\Delta) = \begin{cases} \Delta - 17 & \text{if } \Delta \leq 94; \\ (1 - \frac{179}{1156})\Delta & \text{if } \Delta \geq 95. \end{cases}$$

Claim 3.2. $|V_{\geq r_1}(G)| \geq \frac{n}{2}$.

Proof. If $\Delta \leq 94$, then by (2) we have $|V_{\geq r_1}(G)| = |V_{\geq \Delta-17}(G)| > \frac{757}{1156}\Delta > \frac{3}{4}\Delta - 9 \geq \frac{3}{4}(\frac{2}{3}n + 12) - 9 = \frac{n}{2}$.

Suppose $\Delta \geq 95$. Let $q = (1 - \frac{179}{1156})\Delta < \Delta - 10$. If $\delta(G) > \frac{\Delta}{2} - 2$, then by Claim 3.1 we have

$$|V_{\geq(1-\frac{179}{1156})\Delta}(G)| > \frac{3\Delta}{4} - \frac{3 \cdot 1156\Delta}{2 \cdot 179\Delta} - \frac{2 \cdot 1156^2\Delta}{179^2\Delta^2} + 8 > \frac{n}{2} + 9 + 8 - 10 - 2 > \frac{n}{2},$$

where we use the following inequalities $\frac{3}{4}\Delta \geq \frac{n}{2} + 9$, $\frac{3\cdot1156}{2\cdot179} = \frac{3468}{358} < 10$, $\Delta > 9^2$, and $\frac{2\cdot1156^2}{179^2\Delta} < 2(\frac{1156}{1611})^2 < 2$. If $\delta(G) \leq \frac{\Delta}{2} - 2$, then by Claim 3.1,

$$\begin{aligned} |V_{\geq(1-\frac{179}{1156})\Delta}(G)| &> \frac{3\Delta}{4} - \frac{1156(3\Delta - 18)}{2 \cdot 179\Delta} - \frac{1156^2(2\Delta - 12)}{179^2\Delta^2} + \frac{1}{2} \\ &\geq \frac{n}{2} + 9 + \frac{1}{2} - \frac{1734}{179} + \left(\frac{1156 \cdot 9}{179\Delta} - \frac{1156^2 \cdot 2}{179^2\Delta}\right) + \frac{1156^2 \cdot 12}{179^2\Delta^2} \\ &= \frac{n}{2} - A - B(\Delta), \end{aligned}$$

where $A = -9 - \frac{1}{2} + \frac{1734}{179} = \frac{67}{358} < 0.2$, and

$$B(\Delta) = \frac{1156}{179^2}\left(\frac{-179\cdot 9 + 1156\cdot 2}{\Delta} - \frac{1156\cdot 12}{\Delta^2}\right) = \frac{1156}{179^2}\left(\frac{701}{\Delta} - \frac{13872}{\Delta^2}\right).$$

As a function of Δ, $B(\Delta)$ has a unique maximum at $\frac{2\cdot 13872}{701}$, which is less than 94, and so, for $\Delta \geq 95$,

$$B(\Delta) < B(94) = \frac{1156}{179^2}\cdot\frac{52022}{94^2} < \frac{1200}{170^2}\cdot\frac{170^2\cdot 2}{90^2} < 0.3.$$

Thus $|V_{\geq(1-\frac{179}{1156})\Delta}(G)| > \frac{n}{2} - 0.2 - 0.3 = \frac{n-1}{2}$, and so, being an integer, $|V_{\geq(1-\frac{179}{1156})\Delta}(G)| \geq \frac{n}{2}$. □

Let $r_2 := r_2(\Delta) = \frac{\Delta}{2} - 2$.

Claim 3.3. *For any $u \in V_{>r_2}(G)$ and any $v \in V_{\geq r_1}(G)$, $d_{C(G)}(u) + d_{C(G)}(v) \geq n$.*

Proof. If $\Delta \leq 94$, we have $r_1 = \Delta - 17$, and so $d_{C(G)}(u) + d_{C(G)}(v) \geq d(u) + d(v) > \frac{3\Delta}{2} - 19 \geq n - 1$.

Suppose $\Delta \geq 95$. For any vertex $w \in V_{\geq\Delta-17}(G)$, we have $d(u) + d(w) > \frac{3\Delta}{2} - 19 \geq n - 1$. So $uw \in E(C(G))$. Thus by (2) we have $d_{C(G)}(u) \geq |V_{\geq\Delta-17}(G)| > \frac{757}{1156}\Delta$. Then

$$d_{C(G)}(u) + d_{C(G)}(v) \geq d_{C(G)}(u) + d(v) > \frac{757}{1156}\Delta + (1 - \frac{179}{1156})\Delta = \frac{3}{2}\Delta > n.$$

This completes the proof of Claim 3.3. □

By Claims 3.2 and 3.3, we have $d_{C(G)}(u) \geq |V_{\geq r_1}(G)| \geq \frac{n}{2}$. So $V_{>r_2}(G)$ is a clique in $C(G)$.

Claim 3.4. *$|N(X)| \geq 2|X|$ for any $X \subseteq V_{<\frac{\Delta}{2}+1}(G)$.*

Proof. Let X be a subset of $V_{<\frac{\Delta}{2}+1}(G)$. Since G is Δ-critical, each edge uv of G has $d(u) + d(v) \geq \Delta + 2$. Thus X is an independent set of G, so $X \cap N(X) = \emptyset$. Let H be the bipartite graph induced by the edges with one endvertex in X and the other in $N(X)$. For each $x \in X$, let $N_1(x) = \{y \in N(x) : \sigma(x, y) \geq \Delta - p(x) - 1\}$ and $N_2(x) = N(x) \setminus N_1(x)$, where $p(x)$ is defined before Lemma 3.

Let $x \in X$. By Lemma 3, x has at least $d(x) - p(x) - 1$ neighbors y for which $\sigma(x, y) \geq \Delta - p(x) - 1$. Thus $|N_1(x)| \geq d(x) - p(x) - 1$. Since $2\Delta - d(x) - d(y) + 2 > \frac{\Delta}{2} + 1$, we have $\sigma(x, y) \leq \sigma_{\frac{\Delta}{2}+1}(x, y) \leq d(y) - d_H(y)$. Thus for each $y \in N_1(x)$ we have

$$d_H(y) \leq d(y) - \sigma(x, y) \leq d(y) - (\Delta - p(x) - 1) \leq p(x) + 1,$$

and for each $y \in N(x)$ we have

$$d_H(y) \leq d(y) - \sigma(x, y) \leq d(y) - (\Delta - d(x) + p(x) + 1) \leq d(x) - p(x) - 1.$$

For each edge $xy \in E(H)$ with $x \in X$ and $y \in N(X)$, we define $M(x, y) = \frac{1}{d_H(y)}$. Then we have

$$\sum_{xy\in E(H)} M(x, y) = \sum_{y\in N(X)}\sum_{x\in N(y)}\frac{1}{d_H(y)} = \sum_{y\in N(X)} 1 = |N(X)|.$$

On the other hand,

$$\sum_{xy\in E(H)} M(x, y) = \sum_{x\in X}\sum_{y\in N(x)}\frac{1}{d_H(y)} = \sum_{x\in X}\left(\sum_{y\in N_1(x)}\frac{1}{d_H(y)} + \sum_{y\in N_2(x)}\frac{1}{d_H(y)}\right)$$
$$\geq \sum_{x\in X}\left(\frac{d(x) - p(x) - 1}{p(x) + 1} + \frac{p(x) + 1}{d(x) - p(x) - 1}\right) \geq \sum_{x\in X} 2 = 2|X|.$$

Therefore $|N(X)| \geq 2|X|$. □

By Claim 3.4 and Lemma 11, G has a subgraph H covering $V_{\leq r_2}(G)$ such that for every $x \in V_{\leq r_2}(G)$, $d_H(x) = 2$ and for every $y \in V_{>r_2}(G)$, $d_H(y) \leq 2$. That is, there exist some vertex-disjoint paths $P_1, \ldots, P_s$ covering $V_{\leq r_2}(G)$ such that the endvertices of P_i belong to $V_{>r_2}(G)$ for all $1 \leq i \leq s$. Therefore, we can insert each vertex of $V_{\leq r_2}(G)$ into the subgraph induced by $V_{>r_2}(G)$. Since $V_{>r_2}(G)$ is a clique of $C(G)$, $C(G)$ is Hamiltonian. So G is Hamiltonian, a contradiction.

4. Proofs of Lemmas 4 and 5

4.1. Proof of Lemma 4

Lemma 4. *Let G be a Δ-critical graph. For a vertex $x \in V(G)$ and a positive number q, if $d(x) < \frac{\Delta}{2}$ and $q \le \Delta - 10$, then for any $y \in N(x)$, there exists another neighbor z of x such that $\sigma_q(x,y) + \sigma_q(x,z) > 2\Delta - d(x) - \frac{2(d(x)-1)}{\Delta - q} - \lceil \frac{4(d(x)-1)}{\Delta-q} + \frac{8(d(x)-1)}{(\Delta-q)^2} \rceil$.*

Proof. Let y be a neighbor of x. A vertex $z \in N(x) \setminus \{y\}$ is called *feasible* if there exists a coloring $\varphi \in \mathcal{C}^{\Delta}(G - xy)$ such that $\varphi(xz) \in \bar{\varphi}(y)$, and such a coloring φ is called *z-feasible*. Denote by $\mathcal{C}_z$ the set of all z-feasible colorings. For each feasible vertex z and each z-feasible coloring $\varphi \in \mathcal{C}_z$, let

$$
\begin{aligned}
Z(\varphi) &= \{v \in N(z) \setminus \{x\} : \varphi(vz) \in \bar{\varphi}(x) \cup \bar{\varphi}(y)\},\\
C_z(\varphi) &= \{\varphi(vz) : v \in Z(\varphi) \text{ and } d(v) < q\},\\
Y(\varphi) &= \{v \in N(y) \setminus \{x\} : \varphi(vy) \in \bar{\varphi}(x) \cup \bar{\varphi}(z)\}, \text{ and}\\
C_y(\varphi) &= \{\varphi(vy) : v \in Y(\varphi) \text{ and } d(v) < q\}.
\end{aligned}
$$

Note that $Z(\varphi)$ and $Y(\varphi)$ are vertex sets while $C_z(\varphi)$ and $C_y(\varphi)$ are color sets. Clearly, $C_z(\varphi) \subseteq \bar{\varphi}(x) \cup \bar{\varphi}(y)$ and $C_y(\varphi) \subseteq \bar{\varphi}(x) \cup \bar{\varphi}(z)$. For each color $\alpha \in \varphi(z)$, let $z_\alpha \in N(z)$ such that $\varphi(zz_\alpha) = \alpha$. For each color $\beta \in \varphi(y)$, let $y_\beta \in N(y)$ such that $\varphi(yy_\beta) = \beta$. Let

$$T_0(\varphi) = \{\alpha \in \varphi(x) \cap \varphi(y) \cap \varphi(z) : d(y_\alpha) < q \text{ and } d(z_\alpha) < q\}.$$

Since $(\varphi(x) \cap \varphi(y)) \cap (\bar{\varphi}(x) \cup \bar{\varphi}(y)) = \emptyset$ and $(\varphi(x) \cap \varphi(z)) \cap (\bar{\varphi}(x) \cup \bar{\varphi}(z)) = \emptyset$, we obtain that $T_0(\varphi) \cap (C_z(\varphi) \cup C_y(\varphi)) = \emptyset$.

Since G is Δ-critical and φ is z-feasible, $\{x, y, z\}$ is elementary with respect to φ. So $\bar{\varphi}(x)$, $\bar{\varphi}(y)$, $\bar{\varphi}(z)$ and $\varphi(x) \cap \varphi(y) \cap \varphi(z)$ are mutually exclusive. It is not difficult to see that

$$|Z(\varphi)| = |\bar{\varphi}(x)| + |\bar{\varphi}(y)| - 1 \text{ and } |Y(\varphi)| = |\bar{\varphi}(x)| + |\bar{\varphi}(z)|. \tag{3}$$

Also,

$$\bar{\varphi}(x) \cup \bar{\varphi}(y) \cup \bar{\varphi}(z) \cup (\varphi(x) \cap \varphi(y) \cap \varphi(z)) = \{1, 2, \ldots, \Delta\}. \tag{4}$$

Recall that $\sigma_q(x, y)$ and $\sigma_q(x, z)$ are the numbers of vertices with degree at least q in $N(y) \setminus \{x\}$ and $N(z) \setminus \{x\}$, respectively. So, by Eqs. (3) and (4), we have

$$
\begin{aligned}
&\sigma_q(x,y) + \sigma_q(x,z)\\
\ge\ & |Y(\varphi)| - |C_y(\varphi)| + |Z(\varphi)| - |C_z(\varphi)| + |\varphi(x) \cap \varphi(y) \cap \varphi(z)| - |T_0(\varphi)|\\
=\ & |\bar{\varphi}(x)| + |\bar{\varphi}(z)| + |\bar{\varphi}(x)| + |\bar{\varphi}(y)| - 1 + |\varphi(x) \cap \varphi(y) \cap \varphi(z)|\\
& - |C_y(\varphi)| - |C_z(\varphi)| - |T_0(\varphi)|\\
=\ & \Delta + |\bar{\varphi}(x)| - 1 - |C_y(\varphi)| - |C_z(\varphi)| - |T_0(\varphi)|\\
=\ & 2\Delta - d(x) - |C_y(\varphi)| - |C_z(\varphi)| - |T_0(\varphi)|.
\end{aligned}
$$

So, Lemma 4 follows from the two statements below.

I. For any $\varphi \in \mathcal{C}_z$, $|C_z(\varphi)| < \frac{d(x)-1}{\Delta-q}$ and $|C_y(\varphi)| < \frac{d(x)-1}{\Delta-q}$;
II. there exists a $\varphi \in \mathcal{C}_z$ such that $|T_0(\varphi)| \le \lceil \frac{4(d(x)-1)}{\Delta-q} + \frac{8(d(x)-1)}{(\Delta-q)^2} \rceil$.

For every z-feasible coloring $\varphi \in \mathcal{C}^{\Delta}(G - xy)$, let $\varphi^d \in \mathcal{C}^{\Delta}(G - xz)$ be obtained from φ by uncoloring edge xz and assigning $\varphi(xz)$ to edge xy, and keeping all colors on other edges unchanged. Clearly, φ^d is y-feasible and $Z(\varphi^d) = Z(\varphi)$, $Y(\varphi^d) = Y(\varphi)$, $C_z(\varphi^d) = C_z(\varphi)$ and $C_y(\varphi^d) = C_y(\varphi)$. We call φ^d the *dual coloring* of φ. Considering dual colorings, we see that some properties that hold for vertex z also hold for vertex y.

Let $z \in N(x) \setminus \{y\}$ be a feasible vertex and $\varphi \in \mathcal{C}_z$ be a corresponding coloring. By the definition of $Z(\varphi)$, $\{y, x, z\} \cup Z(\varphi)$ is the vertex set of a simple broom. Since $1 \le d(x) < \frac{\Delta}{2}$, we have $|\bar{\varphi}(x) \cup \bar{\varphi}(y)| \ge \Delta - d(x) + 2 \ge 4$. Thus by Lemma 8 the set $\{y, x, z\} \cup Z(\varphi)$ is elementary with respect to φ. Counting the number of missing colors of vertices in the set $\{y, x, z\} \cup Z(\varphi)$, since y may be a vertex of $Z(\varphi)$, we obtain $(\Delta - q)|C_z(\varphi)| + |\bar{\varphi}(x)| < \sum_{v \in \{y,x,z\} \cup Z(\varphi)} |\bar{\varphi}(v)| \le \Delta$, which implies that $|C_z(\varphi)| < \frac{d(x)-1}{\Delta-q}$. By considering its dual coloring φ^d, we have $|C_y(\varphi)| = |C_y(\varphi^d)| < \frac{d(x)-1}{\Delta-q}$. Hence, **I** holds.

The proof of **II** is much more complicated and will be placed in a separate section. A coloring $\varphi \in \mathcal{C}_z$ is called *optimal* if over all z-feasible colorings the followings hold:
1. $|C_z(\varphi)| + |C_y(\varphi)|$ is maximum;
2. subject to 1, $|C_z(\varphi) \cap C_y(\varphi)|$ is minimum.

Note that a z-feasible coloring φ is optimal if and only if its dual coloring φ^d is an optimal y-feasible coloring.

4.1.1. Proof of Statement II

Suppose to the contrary that $|T_0(\varphi)| > \lceil \frac{4(d(x)-1)}{\Delta-q} + \frac{8(d(x)-1)}{(\Delta-q)^2} \rceil$ for every $\varphi \in \mathcal{C}_z$. For each z-feasible coloring φ, let $R(\varphi) = C_z(\varphi) \cup C_y(\varphi)$, $T_0(\varphi) = \{k_1, \ldots, k_{|T_0(\varphi)|}\}$ and

$$V(T_0(\varphi)) = \{z_{k_i} \in N(z) : k_i \in T_0(\varphi)\} \cup \{y_{k_i} \in N(y) : k_i \in T_0(\varphi)\}.$$

Let φ be an optimal z-feasible coloring and assume, without loss of generality, $\varphi(xz) = 1$. For convenience, we let $Z = Z(\varphi)$, $Y = Y(\varphi)$, $C_z = C_z(\varphi)$, $C_y = C_y(\varphi)$, $T_0 = T_0(\varphi)$ and $R = R(\varphi)$. Note that $1 \notin R \cup T_0$ and $R \cap T_0 = \emptyset$.

Claim A. *For each $i \in \bar{\varphi}(x) \setminus R$ and $k \in T_0$, $P_x(i, k, \varphi)$ contains both y and z.*

Proof. We first show that $z \in V(P_x(i, k, \varphi))$. Otherwise, $P_z(i, k, \varphi)$ is disjoint from $P_x(i, k, \varphi)$. Let $\varphi' = \varphi/P_z(i, k, \varphi)$. Since $1 \notin \{i, k\}$, φ' is also z-feasible. Note that $\varphi'(zz_k) = \varphi(zz_i) = i$. Since $d(z_k) < q$ and $i \in \bar{\varphi}(x) \setminus R$, we have $i \in C_z(\varphi') \setminus C_z$. Since neither i nor k is in R, we have $C_z(\varphi') \supseteq C_z \cup \{i\}$ and $C_y(\varphi') \supseteq C_y$, giving a contradiction to the maximality of $|C_y| + |C_z|$. By considering the dual coloring φ^d, we can verify that $y \in V(P_x(i, k, \varphi))$. □

Claim B. *Suppose that there exist three vertices $u_1, u_2, u_3 \in V(T_0) \setminus \{y, z\}$ and two distinct colors α, β with $\alpha \in \bar{\varphi}(u_1) \cap \bar{\varphi}(u_2) \cap \bar{\varphi}(u_3)$ and $\beta \in \bar{\varphi}(x) \setminus R$. Then there exists a vertex $u \in \{u_1, u_2, u_3\}$ with $x \notin V(P_u(\alpha, \beta, \varphi))$ such that the coloring $\varphi' = \varphi/P_u(\alpha, \beta, \varphi)$ is z-feasible and optimal and has the following properties: $\varphi'(x) = \varphi(x)$, $C_y(\varphi') = C_y$, $C_z(\varphi') = C_z$, $R(\varphi') = R$, $T_0(\varphi') \supseteq T_0$ and $\beta \in \bar{\varphi}'(u)$.*

Proof. For each $i \in \{1, 2, 3\}$, let $P_i = P_{u_i}(\alpha, \beta, \varphi)$ and $\varphi_i = \varphi/P_i$. Clearly none of u_1, u_2, u_3, x can be an internal vertex (of degree 2) in an (α, β)-chain, and so there are at least two values of i such that $x \notin V(P_i)$; let $i = 1$ be one of them. For each i such that $x \notin V(P_i)$, we have the following observations.

- $\varphi_i(x) = \varphi(x)$ since $x \notin V(P_i)$.
- $\beta \in \bar{\varphi}_i(u_i)$ since $\alpha \in \bar{\varphi}(u_i)$.
- All the conclusions of Claim B hold (with $\varphi' = \varphi_i$ and $T_0(\varphi_i) = T_0$) if $V(P_i) \cap \{x, y, z\} = \emptyset$.
- Since $\beta \notin T_0$, we have $T_0(\varphi_i) \supseteq T_0$ if $\alpha \notin T_0$. But if $\alpha \in T_0$ then it follows from Claim A that $V(P_i) \cap \{x, y, z\} = \emptyset$, and so $T_0(\varphi_i) = T_0$.
- Since $\beta \neq 1$, we have $\varphi_i(xz) = 1 \in \bar{\varphi}_i(y)$ if $\alpha \neq 1$, and so φ_i is z-feasible. But $y \in P_x(1, \beta, \varphi)$, as otherwise we could get an edge-Δ-coloring of G from $\varphi/P_x(1, \beta, \varphi)$ by coloring xy with 1; thus if $\alpha = 1$ then $V(P_i) \cap \{x, y, z\} = \emptyset$ and all the conclusions hold.

Thus it suffices to assume $\alpha \neq 1$ and to prove that there exists a number $i \in \{1, 2, 3\}$ such that $x \notin V(P_i)$, $C_y(\varphi_i) = C_y$ and $C_z(\varphi_i) = C_z$, as these imply that φ_i is optimal and $R(\varphi_i) = R$. We consider the following four cases.

Case 1: $\alpha \in \varphi(x) \setminus R$ and $\alpha \neq 1$.

Since $\alpha, \beta \notin R = C_y \cup C_z$, it follows that $C_y(\varphi_1) \supseteq C_y$ and $C_z(\varphi_1) \supseteq C_z$. Since φ is optimal, $C_y(\varphi_1) = C_y$ and $C_z(\varphi_1) = C_z$, which is all we need to prove.

Case 2: $\alpha \in \varphi(x) \cap R$.

Assume first $\alpha \in C_y$. By the definition of C_y, $\alpha \in \bar{\varphi}(z)$. If $z \notin V(P_x(\alpha, \beta, \varphi))$ then we could get an edge-Δ-coloring of G from $\varphi^d/P_x(\alpha, \beta, \varphi)$ by coloring xz with α. Thus $P_x(\alpha, \beta, \varphi) = P_z(\alpha, \beta, \varphi)$, and so $x, z \notin V(P_i)$ for all $i \in \{1, 2, 3\}$. Note that there exists a path P_j ($j \in \{1, 2, 3\}$) such that $y \notin V(P_j)$. So $V(P_j) \cap \{x, y, z\} = \emptyset$, which is all we need.

If $\alpha \in C_z$, then by interchanging y and z in the above argument, and replacing φ^d by φ in the second line, we get the same conclusion.

Case 3: $\alpha \in \bar{\varphi}(x) \setminus (C_y \cap C_z)$.

Since $\{x, y, z\}$ is an elementary set with respect to φ, $\alpha, \beta \in \bar{\varphi}(x) \cap \varphi(y) \cap \varphi(z)$. Since $\alpha \notin C_y \cap C_z$ and $\beta \notin R$, either $\alpha, \beta \notin C_y$ or $\alpha, \beta \notin C_z$ (or both). Without loss of generality, we assume $\alpha, \beta \notin C_y$ (the other case is similar). Then the neighbors y_α and y_β of y both have degree at least q. Clearly $x \notin V(P_i)$ for all $i \in \{1, 2, 3\}$, and so we can choose P_j ($j \in \{1, 2, 3\}$) so that $z \notin V(P_j)$. Then $C_y(\varphi_j) = C_y$ and $C_z(\varphi_j) = C_z$, regardless of whether or not $y \in V(P_j)$.

Case 4: $\alpha \in C_z \cap C_y$.

In this case, $\alpha \in \bar{\varphi}(x)$. Clearly $x \notin V(P_i)$ for all $i \in \{1, 2, 3\}$. We claim that $P_z(\alpha, \beta, \varphi) = P_y(\alpha, \beta, \varphi)$. For otherwise, let $\varphi_0 = \varphi/P_z(\alpha, \beta, \varphi)$. Since $\alpha, \beta \in \bar{\varphi}(x)$, $P_x(\alpha, \beta, \varphi) = \{x\}$, which is disjoint from $P_z(\alpha, \beta, \varphi)$, and so $\varphi_0(x) = \varphi(x)$. Since $\alpha \in C_z$ and $\beta \in \bar{\varphi}(x) \setminus R$, we have $d(z_\alpha) < q$ and $d(z_\beta) \geq q$, so $C_z(\varphi_0) = (C_z \cup \{\beta\}) \setminus \{\alpha\}$ and $C_y(\varphi_0) = C_y$, which implies that $|C_z(\varphi_0)| + |C_y(\varphi_0)| = |C_z| + |C_y|$ and $|C_z(\varphi_0) \cap C_y(\varphi_0)| = |C_z \cap C_y| - 1$, which contradicts the minimality of $|C_z \cap C_y|$. Thus $P_z(\alpha, \beta, \varphi) = P_y(\alpha, \beta, \varphi)$. So we can choose P_j ($j \in \{1, 2, 3\}$) so that $V(P_j) \cap \{x, y, z\} = \emptyset$, which is all we need. □

Let $t = \lceil \frac{4(d(x)-1)}{\Delta-q} + \frac{8(d(x)-1)}{(\Delta-q)^2} \rceil$. Recall that we have assumed $|T_0(\varphi)| > t$. If $yz \in E(G)$, then let $T_0'(\varphi) = T_0(\varphi) \setminus \{\varphi(yz)\}$, otherwise, let $T_0'(\varphi) = T_0(\varphi)$. Clearly, $|T_0'(\varphi)| \geq t$. For each t-element subset T of $T_0'(\varphi)$, say $T = \{k_1, \dots, k_t\}$, let

$$\begin{aligned} V(T,\varphi) &= \{z_{k_1}, z_{k_2}, \dots, z_{k_t}\} \cup \{y_{k_1}, y_{k_2}, \dots, y_{k_t}\}, \\ W(T,\varphi) &= \{u \in V(T,\varphi) : \bar\varphi(u) \cap (\bar\varphi(x) \setminus R(\varphi)) = \emptyset\}, \\ M(T,\varphi) &= \{u \in V(T,\varphi) : \bar\varphi(u) \cap (\bar\varphi(x) \setminus R(\varphi)) \neq \emptyset\}, \\ E(T,\varphi) &= \{zz_{k_1}, zz_{k_2}, \dots, zz_{k_t}, yy_{k_1}, yy_{k_2}, \dots, yy_{k_t}\}, \\ E_W(T,\varphi) &= \{e \in E(T,\varphi) :\ e \text{ is incident with a vertex in } W(T,\varphi)\}, \text{ and} \\ E_M(T,\varphi) &= \{e \in E(T,\varphi) :\ e \text{ is incident with a vertex in } M(T,\varphi)\}. \end{aligned}$$

Clearly, $V(T,\varphi) = W(T,\varphi) \uplus M(T,\varphi)$ and $E(T,\varphi) = E_W(T,\varphi) \uplus E_M(T,\varphi)$, where $\uplus$ denotes disjoint union. For convenience, we let $W = W(T,\varphi)$, $M = M(T,\varphi)$, $E_W = E_W(T,\varphi)$ and $E_M = E_M(T,\varphi)$ if T and φ are clear. Note that $\{z_{k_1}, \dots, z_{k_t}\} \cap \{y_{k_1}, \dots, y_{k_t}\}$ may not be empty, but $|T| \leq |V(T,\varphi)| \leq |E(T,\varphi)| = 2|T|$, $\frac{|E_W|}{2} \leq |W| \leq |E_W|$ and $\frac{|E_M|}{2} \leq |M| \leq |E_M|$.

We assume that $|E_M(T,\varphi)|$ is maximum over all optimal z-feasible colorings φ and all t-element subsets T of $T_0'(\varphi)$. Let $Y_M = \{y_k : k \in T \text{ and } z_k, y_k \in M\}$. For any $Y' \subseteq Y_M$, let $Z(Y') = \{z_k : y_k \in Y'\}$, and let $C_M(Y')$ be the union of all single-element sets of the form $\bar\varphi(v) \cap (\bar\varphi(x) \setminus R)$ with $v \in Y' \cup Z(Y')$, ignoring any sets of this form with more than one element. Moreover, let C_M be the union of all single-element sets of the form $\bar\varphi(v) \cap (\bar\varphi(x) \setminus R)$ with $v \in M$. Clearly, $|C_M(Y_M)| \leq |C_M| \leq |M|$.

Claim C. *The following three statements hold.*

(a) If $k \in T_0$, $i, j \in \bar\varphi(x) \setminus R$, $i \in \bar\varphi(z_k)$ and $j \in \bar\varphi(y_k)$, then $i \neq j$.

(b) If $y_k \in Y_M$ then there exist distinct colors i, j as in (a).

(c) If $y_k \in Y_M$ and $|\bar\varphi(y_k) \cap (\bar\varphi(x) \setminus R)| \geq 2$, then there exist distinct colors $i, j, l \in \bar\varphi(x) \setminus R$ such that $i \in \bar\varphi(z_k)$ and $j, l \in \bar\varphi(y_k)$.

Proof. If $i \in \bar\varphi(y_k) \cap \bar\varphi(z_k)$, then neither y_k nor z_k can be an internal vertex (with degree 2) of $P_x(i, k, \varphi)$, whereas Claim A implies that at least one of y_k and z_k must be an internal vertex of this path. Thus $i \neq j$, which proves (a). (b) follows because, by the definitions of Y_M and M, there exist colors $i, j \in \bar\varphi(x) \setminus R$ such that $i \in \bar\varphi(z_k)$ and $j \in \bar\varphi(y_k)$; and (c) holds for the same reason. □

Claim D. *The following two statements hold.*

(a) The hypotheses of Claim B cannot hold with $u_1, u_2, u_3 \in W$ and $\beta \notin C_M$.

(b) If the hypotheses of Claim B hold, and u and $\varphi' = \varphi/P_u(\alpha, \beta, \varphi)$ are given by Claim B, then

(i) if $\beta \notin C_M$ then $u \notin W$;

(ii) if $\alpha \in \varphi(x) \cup R$, $u \in Y_M$ and $\beta \notin C_M(\{u\})$, then there is a color $k \in T \subseteq T_0'(\varphi')$ for which the following holds: there are three distinct colors $i, j, l \in \bar\varphi'(x) \setminus R(\varphi')$ such that $i \in \bar\varphi'(z_k)$ and $j, l \in \bar\varphi'(y_k)$.

Proof. Clearly (a) follows from (b)(i).

By Claim B, $\varphi'(x) = \varphi(x)$, $R(\varphi') = R$, $T_0 \subseteq T_0(\varphi')$, and $\beta \in \bar\varphi'(u)$. As we remarked in the proof of Claim B, $\beta \notin T_0$, and if $\alpha \in T_0$ then $V(P_u(\alpha, \beta, \varphi)) \cap \{x, y, z\} = \emptyset$. Thus $T_0'(\varphi') \supseteq T_0'(\varphi) \supseteq T$ and for each $k \in T$ we have $\varphi'(yy_k) = \varphi'(zz_k) = \varphi(yy_k) = \varphi(zz_k) = k$.

To prove (b)(i), suppose that $\beta \notin C_M$ and $u \in W$. Since $\beta \in \bar\varphi(x) \setminus R = \bar\varphi'(x) \setminus R(\varphi')$ and $u \in W$, it follows that $u \in M(T,\varphi') \setminus M$. To avoid the contradiction $|E_M(T,\varphi')| > |E_M|$, it must be that $P_u(\alpha, \beta, \varphi)$ ends with an edge of color α at a vertex $v \in M$ such that β is the unique color in $\bar\varphi(v) \cap (\bar\varphi(x) \setminus R)$, so that $v \notin M(T,\varphi')$. But then $\beta \in C_M$, which is a contradiction.

To prove (b)(ii), suppose that $u = y_k \in Y_M$, where $k \in T \subseteq T_0'(\varphi')$. Since $\beta \notin C_M(\{u\}) = C_M(\{y_k\})$, it is not possible that $\bar\varphi(z_k) \cap (\bar\varphi(x) \setminus R) = \{\beta\}$ or $\bar\varphi(y_k) \cap (\bar\varphi(x) \setminus R) = \{\beta\}$. By Claim C, there exist distinct colors $i, j \in \bar\varphi(x) \setminus R = \bar\varphi'(x) \setminus R(\varphi')$ such that $i \in \bar\varphi(z_k)$, $j \in \bar\varphi(y_k)$, and $\beta \notin \{i, j\}$. Since $\alpha \in \varphi(x) \cup R$, it follows that $\alpha \notin \{i, j\}$. Thus $i \in \bar\varphi'(z_k)$ and $j \in \bar\varphi'(y_k)$. Since also $\beta \in \bar\varphi'(u) = \bar\varphi'(y_k)$, the result follows with $l = \beta$. □

Recall that the statement **I** states that for any $\varphi \in \mathcal{C}_z$ we have $|C_z(\varphi)| < \frac{d(x)-1}{\Delta-q}$ and $|C_y(\varphi)| < \frac{d(x)-1}{\Delta-q}$. Since $|T| = \lceil \frac{4(d(x)-1)}{\Delta-q} + \frac{8(d(x)-1)}{(\Delta-q)^2} \rceil$, we have

$$\frac{(\Delta-q)|T|}{2} \geq 2d(x) - 2 + \frac{4(d(x)-1)}{\Delta-q} > 2|\varphi(x) \cup R|, \tag{5}$$

where the second inequality follows from **I**. Since $d(x) < \frac{\Delta}{2}$ and $q \leq \Delta - 10$, the second inequality of (5) implies $2|\varphi(x) \cup R| < \frac{6\Delta-12}{5}$. By **I** again, we have $|\bar\varphi(x) \setminus R| > \Delta - d(x) + 1 - 2(\frac{d(x)-1}{\Delta-q}) > \frac{2\Delta+6}{5}$, and $|T| + \frac{2|\varphi(x)\cup R|}{\Delta-q-1} < \frac{\Delta-2}{5} + \frac{\Delta-2}{25} + \frac{6\Delta-12}{45} + 1 = \frac{84\Delta+57}{225}$. Since $\frac{2\Delta+6}{5} > \frac{84\Delta+57}{225}$ and $\frac{2\Delta+6}{5} > \frac{2(6\Delta-12)}{45} + 2$ as $\Delta > 2$, we have

$$|\bar\varphi(x) \setminus R| > |T| + \frac{2|\varphi(x) \cup R|}{\Delta-q-1} \tag{6}$$

and

$$|\bar{\varphi}(x) \setminus R| > \frac{4|\varphi(x) \cup R|}{\Delta - q - 1} + 2. \tag{7}$$

Claim E. *There exist an optimal z-feasible coloring φ' and a color $k \in T \subseteq T_0'(\varphi')$ for which the following holds: there are three distinct colors $i, j, l \in \bar{\varphi}'(x) \setminus R(\varphi')$ such that $i \in \bar{\varphi}'(z_k)$ and $j, l \in \bar{\varphi}'(y_k)$.*

Proof. We consider the following two cases, according to the value of $|E_M|$.

Case 1: $|E_M| \leq |T| + \frac{2|\varphi(x)\cup R|}{\Delta-q-1}$.

Since $|C_M| \leq |E_M|$, it follows from (6) that there exists a color $\beta \in \bar{\varphi}(x) \setminus (R \cup C_M)$.

We first claim that $|E_M| > |T|$. If not, then $|E_M| \leq |T|$, which in turn gives $|E_W| \geq |T|$ since $|E_M| + |E_W| = 2|T|$. Thus $|W| \geq \frac{|T|}{2}$. By the definition of T_0, we have $d(v) < q$ for every vertex $v \in V(T, \varphi)$, which includes all $v \in W$. So $\sum_{v\in W} |\bar{\varphi}(v)| > (\Delta - q)\frac{|T|}{2} > 2|\varphi(x) \cup R|$, by (5). By the definition of W, $\bar{\varphi}(v) \subseteq \varphi(x) \cup R$ for every $v \in W$. By the Pigeonhole Principle, there exist three vertices $u_1, u_2, u_3 \in W$ and a color $\alpha \in \varphi(x) \cup R$ such that $\alpha \in \bar{\varphi}(u_1) \cap \bar{\varphi}(u_2) \cap \bar{\varphi}(u_3)$. But this contradicts Claim D(a).

So we may assume that $|E_M| = |T| + p$, so that $|E_W| = |T| - p$, where $p > 0$. Since $|E_M| = |T| + p$, it follows that $|Y_M| \geq p$. We may assume that $|\bar{\varphi}(y_k) \cap (\bar{\varphi}(x) \setminus R)| = 1$ for all $y_k \in Y_M$, as otherwise the result holds by Claim C(c). Thus $|\bar{\varphi}(v) \cap (\varphi(x) \cup R)| = |\bar{\varphi}(v)| - 1 > \Delta - q - 1$ for all $v \in Y_M$, while $|\bar{\varphi}(v) \cap (\varphi(x) \cup R)| = |\bar{\varphi}(v)| > \Delta - q$ for all $v \in W$. Since $|W| \geq |E_W|/2$, $|E_W| = |T| - p$, and $\Delta - q > 2$, we have $\sum_{v\in W\cup Y_M} |\bar{\varphi}(v) \cap (\varphi(x) \cup R)| > (\Delta - q)\frac{|T|-p}{2} + (\Delta - q - 1)p > (\Delta - q)\frac{|T|}{2} > 2|\varphi(x) \cup R|$, by (5). Hence there exist three vertices $u_1, u_2, u_3 \in W \cup Y_M$ and a color $\alpha \in \varphi(x) \cup R$ such that $\alpha \in \bar{\varphi}(u_1) \cap \bar{\varphi}(u_2) \cap \bar{\varphi}(u_3)$. Since $\beta \notin C_M$, the result follows from Claim D(b).

Case 2: $|E_M| > |T| + \frac{2|\varphi(x)\cup R|}{\Delta-q-1}$.

Since $|E_M| > |T| + \frac{2|\varphi(x)\cup R|}{\Delta-q-1}$, it follows that $|Y_M| > \frac{2|\varphi(x)\cup R|}{\Delta-q-1}$. We may assume that $|\bar{\varphi}(y_k) \cap (\bar{\varphi}(x) \setminus R)| = 1$ for all $y_k \in Y_M$, as otherwise the result holds by Claim C(c). Let Y' be a subset of Y_M with $|Y'| = \left\lceil \frac{2|\varphi(x)\cup R|}{\Delta-q-1} \right\rceil$. Then we have $\sum_{v\in Y'} |\bar{\varphi}(v) \cap (\varphi(x) \cup R)| > (\Delta - q - 1)|Y'| \geq 2|\varphi(x) \cup R|$. Thus there exist three vertices $u_1, u_2, u_3 \in Y'$ and a color $\alpha \in \varphi(x) \cup R$ such that $\alpha \in \bar{\varphi}(u_1) \cap \bar{\varphi}(u_2) \cap \bar{\varphi}(u_3)$. Clearly, $|C_M(Y')| \leq 2|Y'|$. By (7), we have $|\bar{\varphi}(x) \setminus R| > 2|Y'| \geq |C_M(Y')|$, thus there exists a color $\beta \in \bar{\varphi}(x) \setminus (R \cup C_M(Y'))$, and the result follows from Claim D(b). □

Let k, i, j, l, φ' be as stated in Claim E. By the proofs of Claim B, D and E, we know that $\varphi'(x) = \varphi(x)$, $\varphi'(xz) = 1 \in \bar{\varphi}'(y)$, $C_y(\varphi') = C_y$ and $C_z(\varphi') = C_z$. Clearly, $l \neq 1$. So $P_x(l, 1, \varphi') = P_y(l, 1, \varphi')$, and they are disjoint from $P_{y_k}(l, 1, \varphi')$. If $1 \notin \bar{\varphi}'(y_k)$, we consider the coloring $\varphi'/P_{y_k}(l, 1, \varphi')$, and rename it as φ'. So we may assume $1 \in \bar{\varphi}'(y_k)$.

By Claim A, the paths $P_x(i, k, \varphi')$ and $P_x(j, k, \varphi')$ both contain y, z. Since $\varphi'(yy_k) = \varphi'(zz_k) = k$, these two paths also contain y_k, z_k. Since $i \in \bar{\varphi}'(z_k)$, x and z_k are the two endvertices of $P_x(i, k, \varphi')$. So, $i \in \varphi'(y) \cap \varphi'(z) \cap \varphi'(y_k)$. Similarly, $j \in \varphi'(y) \cap \varphi'(z) \cap \varphi'(z_k)$. We now consider the following sequence of colorings of $G - xy$.

Let φ_1 be obtained from φ' by assigning $\varphi_1(yy_k) = 1$. Since 1 was missing at both y and y_k, φ_1 is an edge-Δ-coloring of $G - xy$. Now k is missing at y and y_k, and i is still missing at x and z_k. Note that $P_x(i, k, \varphi_1) = P_y(i, k, \varphi_1)$, as otherwise we could get an edge-Δ-coloring of G from $\varphi_1/P_x(i, k, \varphi_1)$ by coloring xy with k. Furthermore, $z_k, y_k \notin V(P_x(i, k, \varphi_1))$ since either i or k is missing at these two vertices, which in turn shows that $z \notin V(P_x(i, k, \varphi_1))$ since $\varphi_1(zz_k) = k$.

Let $\varphi_2 = \varphi_1/P_x(i, k, \varphi_1)$. We have $k \in \bar{\varphi}_2(x)$, $i \in \bar{\varphi}_2(y) \cap \bar{\varphi}_2(z_k)$ and $j \in \bar{\varphi}_2(x) \cap \bar{\varphi}_2(y_k)$. Since G is not edge-Δ-colorable, $P_x(i, j, \varphi_2) = P_y(i, j, \varphi_2)$ which contains neither y_k nor z_k.

Let $\varphi_3 = \varphi_2/P_x(i, j, \varphi_2)$. Then $k \in \bar{\varphi}_3(x)$ and $j \in \bar{\varphi}_3(y) \cap \bar{\varphi}_3(y_k)$.

Let φ_4 be obtained from φ_3 by recoloring yy_k with j. Then $\varphi_4(xz) = 1 \in \bar{\varphi}_4(y)$ and $\varphi_4(zz_k) = k \in \bar{\varphi}_4(x)$; the first of these implies that φ_4 is z-feasible, and the second implies that $k \in C_z(\varphi_4)$, since $d(z_k) < q$. Since $1, i, j, k \notin R = C_y \cup C_z$, the colors in R are unchanged during this sequence of recolorings, and so $C_y(\varphi_4) \supseteq C_y$ and $C_z(\varphi_4) \supseteq C_z \cup \{k\}$. Therefore, $|C_y(\varphi_4)| + |C_z(\varphi_4)| \geq |C_y| + |C_z| + 1$, giving a contradiction. So **II** holds. □

4.2. Proof of Lemma 5

The proof of Lemma 5 has a similar structure to that of Lemma 4, but the differences are sufficiently great that we include it in full.

Lemma 5. *Let G be a Δ-critical graph and q be a positive number such that $q \leq \Delta - 10$ and minimum degree $\delta(G) > \frac{\Delta}{2} - 2$. For an edge $x_1x_2 \in E(G)$, if $d(x_1) + d(x_2) \leq \frac{3}{2}\Delta - 2$, then there exist two distinct vertices $z, y \in V(G) \setminus \{x_1, x_2\}$ with $z \in N(x_1)$ and $y \in N(x_2)$ such that $\sigma_q(x_1, z) + \sigma_q(x_2, y) > 3\Delta - d(x_1) - d(x_2) - \frac{2(d(x_1)+d(x_2)-\Delta-2)}{\Delta-q} - \lceil \frac{4(d(x_1)+d(x_2)-\Delta+2)}{\Delta-q} + \frac{8(d(x_1)+d(x_2)-\Delta-2)}{(\Delta-q)^2} \rceil - 2$.*

Proof. Let edge $x_1x_2 \in E(G)$ be defined as in Lemma 5. A pair of distinct vertices $z, y \in V(G) \setminus \{x_1, x_2\}$ with $z \in N(x_1)$ and $y \in N(x_2)$ is called *feasible* if there exists a coloring $\varphi \in \mathcal{C}^\Delta(G - x_1x_2)$ such that $\varphi(x_1z) \in \bar{\varphi}(x_2)$ and $\varphi(x_2y) \in \bar{\varphi}(x_1)$, and such a coloring φ is called *zy-feasible*. Denote by $\mathcal{C}_{zy}$ the set of all *zy*-feasible colorings. For each pair of feasible vertices z, y and each *zy*-feasible coloring $\varphi \in \mathcal{C}_{zy}$, let

$$\begin{aligned}
Z(\varphi) &= \{v \in N(z) \setminus \{x_1\} : \varphi(vz) \in \bar{\varphi}(x_1) \cup \bar{\varphi}(x_2) \cup \bar{\varphi}(y)\},\\
C_z(\varphi) &= \{\varphi(vz) : v \in Z(\varphi) \text{ and } d(v) < q\},\\
Y(\varphi) &= \{v \in N(y) \setminus \{x_2\} : \varphi(vy) \in \bar{\varphi}(x_1) \cup \bar{\varphi}(x_2) \cup \bar{\varphi}(z)\}, \text{ and}\\
C_y(\varphi) &= \{\varphi(vy) : v \in Y(\varphi) \text{ and } d(v) < q\}.
\end{aligned}$$

Note that $Z(\varphi)$ and $Y(\varphi)$ are vertex sets while $C_z(\varphi)$ and $C_y(\varphi)$ are color sets. Clearly, $C_z(\varphi) \subseteq \bar{\varphi}(x_1) \cup \bar{\varphi}(x_2) \cup \bar{\varphi}(y)$ and $C_y(\varphi) \subseteq \bar{\varphi}(x_1) \cup \bar{\varphi}(x_2) \cup \bar{\varphi}(z)$. For each color $\alpha \in \varphi(z)$, let $z_\alpha \in N(z)$ such that $\varphi(zz_\alpha) = \alpha$. For each color $\beta \in \varphi(y)$, let $y_\beta \in N(y)$ such that $\varphi(yy_\beta) = \beta$. Let

$$T_0(\varphi) = \{\alpha \in \varphi(x_1) \cap \varphi(x_2) \cap \varphi(y) \cap \varphi(z) : d(y_\alpha) < q \text{ and } d(z_\alpha) < q\}.$$

Since $d(x_1)+d(x_2) \le \frac{3}{2}\Delta - 2$ and $\delta(G) > \frac{\Delta}{2} - 2$, we have $d(x_1) < \Delta$ and $d(x_2) < \Delta$. We assume that $\varphi(x_1z) = 1 \in \bar{\varphi}(x_2)$ and $\varphi(x_2y) = 2 \in \bar{\varphi}(x_1)$.

First, we claim that $\bar{\varphi}(x_1)$, $\bar{\varphi}(x_2)$, $\bar{\varphi}(y)$, $\bar{\varphi}(z)$ and $\varphi(x_1) \cap \varphi(x_2) \cap \varphi(y) \cap \varphi(z)$ are mutually exclusive. Let φ_0 be a coloring obtained from φ by uncoloring x_1z and coloring x_1x_2 with 1. In the new coloring φ_0, we have $\bar{\varphi}_0(z) = \bar{\varphi}(z) \cup \{1\}$, $\bar{\varphi}_0(x_1) = \bar{\varphi}(x_1)$, $\bar{\varphi}_0(x_2) = \bar{\varphi}(x_2) \setminus \{1\}$ and $\bar{\varphi}_0(y) = \bar{\varphi}(y)$. Since $\varphi_0(x_1x_2) = 1 \in \bar{\varphi}_0(z)$ and $\varphi_0(x_2y) = 2 \in \bar{\varphi}_0(x_1)$, $\{z, x_1, x_2, y\}$ forms a Kierstead path with respect to φ_0. By Lemma 7, $\{z, x_1, x_2, y\}$ is elementary with respect to φ_0 as $d(x_1) < \Delta$. It follows that $\bar{\varphi}_0(x_1)$, $\bar{\varphi}_0(x_2)$, $\bar{\varphi}_0(y)$ and $\bar{\varphi}_0(z)$ are mutually exclusive. Clearly, $\bar{\varphi}(x_1)$, $\bar{\varphi}(x_2)$, $\bar{\varphi}(y)$ and $\bar{\varphi}(z)$ also are mutually exclusive, and then the claim holds.

It is not difficult to see that

$$|Z(\varphi)| = |\bar{\varphi}(x_1)| + |\bar{\varphi}(x_2)| + |\bar{\varphi}(y)| - 1, \ |Y(\varphi)| = |\bar{\varphi}(x_1)| + |\bar{\varphi}(x_2)| + |\bar{\varphi}(z)| - 1, \tag{8}$$

and $T_0(\varphi) \cap (C_z(\varphi) \cup C_y(\varphi)) = \emptyset$. Also,

$$\bar{\varphi}(x_1) \cup \bar{\varphi}(x_2) \cup \bar{\varphi}(y) \cup \bar{\varphi}(z) \cup (\varphi(x_1) \cap \varphi(x_2) \cap \varphi(y) \cap \varphi(z)) = \{1, 2, \dots, \Delta\}. \tag{9}$$

Recall that $\sigma_q(x_2, y)$ and $\sigma_q(x_1, z)$ are the numbers of vertices with degree at least q in $N(y) \setminus \{x_2\}$ and $N(z) \setminus \{x_1\}$, respectively. So, by Eqs. (8) and (9), we have

$$\begin{aligned}
&\sigma_q(x_2, y) + \sigma_q(x_1, z)\\
\ge\ & |Y(\varphi)| - |C_y(\varphi)| + |Z(\varphi)| - |C_z(\varphi)| + |\varphi(x_1) \cap \varphi(x_2) \cap \varphi(y) \cap \varphi(z)| - |T_0(\varphi)|\\
=\ & |\bar{\varphi}(x_1)| + |\bar{\varphi}(x_2)| + |\bar{\varphi}(z)| - 1 + |\bar{\varphi}(x_1)| + |\bar{\varphi}(x_2)| + |\bar{\varphi}(y)| - 1\\
&+ |\varphi(x_1) \cap \varphi(x_2) \cap \varphi(y) \cap \varphi(z)| - |C_y(\varphi)| - |C_z(\varphi)| - |T_0(\varphi)|\\
=\ & \Delta + |\bar{\varphi}(x_1)| + |\bar{\varphi}(x_2)| - |C_y(\varphi)| - |C_z(\varphi)| - |T_0(\varphi)| - 2\\
=\ & 3\Delta - d(x_1) - d(x_2) - |C_y(\varphi)| - |C_z(\varphi)| - |T_0(\varphi)|.
\end{aligned}$$

For any edge $e \notin E(G)$, we let $\{\varphi(e)\} = \emptyset$. So, Lemma 5 follows from the two statements below as $T_0(\varphi) \cap (C_z(\varphi) \cup C_y(\varphi)) = \emptyset$.

I. For any $\varphi \in \mathcal{C}_{zy}$, $|C_z(\varphi) \setminus \{\varphi(zx_2)\}| < \frac{d(x_1)+d(x_2)-\Delta-2}{\Delta-q}$ and $|C_y(\varphi) \setminus \{\varphi(yx_1)\}| < \frac{d(x_1)+d(x_2)-\Delta-2}{\Delta-q}$;
II. there exists a $\varphi \in \mathcal{C}_{zy}$ such that $|T_0(\varphi) \setminus \{\varphi(zx_2), \varphi(yx_1)\}| \le \lceil \frac{4(d(x_1)+d(x_2)-\Delta+2)}{\Delta-q} + \frac{8(d(x_1)+d(x_2)-\Delta-2)}{(\Delta-q)^2} \rceil$.

To prove statements **I** and **II**, we first give the following claim.

Claim A1. *Under a coloring $\varphi \in \mathcal{C}_{zy}$, the following two statements hold.*
(a) $\{z, x_1, x_2, y\}$ is elementary with respect to φ.
(b) Assume $\varphi(x_1z) = 1 \in \bar{\varphi}(x_2)$ and $\varphi(x_2y) = 2 \in \bar{\varphi}(x_1)$. For every two distinct vertices $w_1, w_2 \in \{z, x_1, x_2, y\}$ and two distinct colors α, β with $\alpha \in \bar{\varphi}(w_1)$ and $\beta \in \bar{\varphi}(w_2)$,
(i) if $\{\alpha, \beta\} \cap \{1, 2\} = \emptyset$, then $P_{w_1}(\alpha, \beta, \varphi) = P_{w_2}(\alpha, \beta, \varphi)$;
(ii) if $\{\alpha, \beta\} \cap \{1, 2\} \ne \emptyset$, $w_1 = x_1$ and $w_2 = x_2$, then $P_{x_1}(\alpha, \beta, \varphi) = P_{x_2}(\alpha, \beta, \varphi)$.

Proof. Note that $\bar{\varphi}(x_1)$, $\bar{\varphi}(x_2)$, $\bar{\varphi}(y)$ and $\bar{\varphi}(z)$ are mutually exclusive. By the definition of elementary, (a) holds.

To prove (b)(i), suppose that $P_{w_1}(\alpha, \beta, \varphi) \ne P_{w_2}(\alpha, \beta, \varphi)$. Let $\varphi_1 = \varphi/P_{w_1}(\alpha, \beta, \varphi)$. Then $\beta \in \bar{\varphi}_1(w_1) \cap \bar{\varphi}_1(w_2)$. Let φ_2 be obtained from φ_1 by uncoloring zx_1 and coloring x_1x_2 with 1. Then $\varphi_2(x_1x_2) = 1 \in \bar{\varphi}_2(z)$, $\varphi_2(x_2y) = 2 \in \bar{\varphi}_2(x_1)$ and $\beta \in \bar{\varphi}_2(w_1) \cap \bar{\varphi}_2(w_2)$. Thus $\{z, x_1, x_2, y\}$ forms a Kierstead path, and by Lemma 7, $\{z, x_1, x_2, y\}$ is elementary with respect to φ_2 as $d(x_1) < \Delta$. This contradicts the fact that $\beta \in \bar{\varphi}_2(w_1) \cap \bar{\varphi}_2(w_2)$.

To prove (b)(ii), if $P_{x_1}(\alpha, \beta, \varphi) \ne P_{x_2}(\alpha, \beta, \varphi)$, then we could get an edge-Δ-coloring of G from $\varphi_1 = \varphi/P_{x_1}(\alpha, \beta, \varphi)$ by coloring x_1x_2 with color β, a contradiction. □

4.2.1. Proof of Statement I

First we prove the following claim.

Claim B1. *For every $\varphi \in \mathcal{C}_{zy}$, the sets $\{x_2, x_1, z\} \cup Z(\varphi)$ and $\{x_1, x_2, y\} \cup Y(\varphi)$ are both elementary with respect to φ.*

Proof. Let φ be an arbitrary coloring in $\mathcal{C}_{zy}$. Since y and z are symmetric, we only need to show that $\{x_2, x_1, z\} \cup Z(\varphi)$ is elementary with respect to φ, that is, we only need to prove that

$$\text{for any } v \in Z(\varphi) \setminus \{x_2\}, \{x_2, x_1, z, v\} \text{ is elementary with respect to } \varphi, \tag{10}$$

and

$$\bar{\varphi}(v) \cap \bar{\varphi}(v') = \emptyset \quad \text{for every two distinct vertices } v, v' \in Z(\varphi) \setminus \{x_2\}. \tag{11}$$

We first show that (10) holds. If $\varphi(vz) \in \bar{\varphi}(x_1) \cup \bar{\varphi}(x_2)$, then $\{x_2, x_1, z, v\}$ forms a Kierstead path as $\varphi(zx_1) \in \bar{\varphi}(x_2)$. Since $d(x_1) < \Delta$, by Lemma 7, $\{x_2, x_1, z, v\}$ is elementary with respect to φ, so (10) holds. Then we suppose that $k = \varphi(vz) \in \varphi(x_1) \cap \varphi(x_2) \cap \bar{\varphi}(y)$. Clearly, $d(y) < \Delta$. By Claim A1(a), $\{x_2, x_1, z\}$ is elementary with respect to φ. So if (10) does not hold, then there exists a color η in $\bar{\varphi}(v) \cap (\bar{\varphi}(x_1) \cup \bar{\varphi}(x_2) \cup \bar{\varphi}(z))$. We can choose η to satisfy one of the following three cases.

Case 1: $\eta \in \bar{\varphi}(v) \cap \bar{\varphi}(x_1) \setminus \{2\}$.

Note that $k \in \bar{\varphi}(y)$ and $\{\eta, k\} \cap \{1, 2\} = \emptyset$. By Claim A1(b)(i), $P_{x_1}(\eta, k, \varphi) = P_y(\eta, k, \varphi)$. Since v is an endvertex of an (η, k)-chain, $P_{x_1}(\eta, k, \varphi)$ is disjoint from $P_v(\eta, k, \varphi)$. Let $\varphi_1 = \varphi/P_{x_1}(\eta, k, \varphi)$. Then $k \in \bar{\varphi}_1(x)$ and $\eta \in \bar{\varphi}_1(y) \cap \bar{\varphi}_1(v)$. Let φ_2 be obtained from φ_1 by uncoloring vz, zx_1 and coloring zx_1, x_1x_2 with k, 1, respectively. Then $\eta \in \bar{\varphi}_2(y) \cap \bar{\varphi}_2(v)$, $\varphi_2(zx_1) = k \in \bar{\varphi}_2(v)$, $\varphi_2(x_1x_2) = 1 \in \bar{\varphi}_2(z)$ and $\varphi_2(x_2y) = 2 \in \bar{\varphi}_2(x_1)$. It follows that $\{v, z, x_1, x_2, y\}$ forms a Kierstead path with respect to φ_2. Since x_1, x_2, y are $(< \Delta)$-vertices, by Lemma 6, the set $\{v, z, x_1, x_2, y\}$ is elementary with respect to φ_2, contradicting the fact that $\eta \in \bar{\varphi}_2(y) \cap \bar{\varphi}_2(v)$.

Case 2: $\eta \in \{1, 2\}$ and $\bar{\varphi}(v) \cap \bar{\varphi}(x_1) \setminus \{2\} = \emptyset$.

By Claim A1(b)(ii), $P_{x_1}(1, 2, \varphi) = P_{x_2}(1, 2, \varphi)$. Since $\varphi(zx_1) = 1$ and $\varphi(x_2y) = 2$, $V(P_v(1, 2, \varphi)) \cap \{z, x_1, x_2, y\} = \emptyset$. If $\eta = 2$, let $\varphi' = \varphi/P_v(1, 2, \varphi)$, then $1 \in \bar{\varphi}'(v)$. So we may assume that $\eta = 1$. Since $d(x_1) < \Delta$ and colors $1, k \in \varphi(x_1)$, there exists a color $\delta \in \bar{\varphi}(x_1)$ such that $\delta \notin \{1, 2, k\}$. By Claim A1(b)(ii), $P_{x_1}(1, \delta, \varphi) = P_{x_2}(1, \delta, \varphi)$. So $P_{x_1}(1, \delta, \varphi)$ is disjoint from $P_v(1, \delta, \varphi)$. Let $\varphi_1 = \varphi/P_v(1, \delta, \varphi)$. Then $\delta \in \bar{\varphi}_1(v) \cap \bar{\varphi}_1(x_1) \setminus \{2\}$. By the similar argument of Case 1 (replace η by δ in Case 1), we are done.

Case 3: $\eta \in \bar{\varphi}(v) \cap (\bar{\varphi}(x_2) \cup \bar{\varphi}(z)) \setminus \{1\}$ and $\bar{\varphi}(v) \cap \bar{\varphi}(x_1) = \emptyset$.

Recall that $d(x_1) < \Delta$ and colors $1, k \in \varphi(x_1)$, and $\eta \in \varphi(x_1)$ since $\bar{\varphi}(v) \cap \bar{\varphi}(x_1) = \emptyset$. Thus there exists a color $\delta \in \bar{\varphi}(x_1)$ such that $\delta \notin \{1, 2, k, \eta\}$. Since $\eta \in \bar{\varphi}(x_2) \cup \bar{\varphi}(z) \setminus \{1\}$, by Claim A1(b)(i), $P_z(\delta, \eta, \varphi) = P_{x_1}(\delta, \eta, \varphi)$ or $P_{x_2}(\delta, \eta, \varphi) = P_{x_1}(\delta, \eta, \varphi)$. So $P_{x_1}(\delta, \eta, \varphi)$ is disjoint from $P_v(\delta, \eta, \varphi)$. Let $\varphi_1 = \varphi/P_v(\delta, \eta, \varphi)$. Then $\delta \in \bar{\varphi}_1(x_1) \cap \bar{\varphi}_1(v) \setminus \{2\}$. By the similar argument of Case 1, we are done.

We now show that (11) holds. If not, let $\alpha \in \bar{\varphi}(v) \cap \bar{\varphi}(v')$. By (10) and the fact that $1 \in \bar{\varphi}(x_2)$ and $2 \in \bar{\varphi}(x_1)$, we have $\alpha \notin \{1, 2, \varphi(vz), \varphi(v'z)\}$. Since $d(x_1)+d(x_2) \le \frac{3}{2}\Delta - 2$ and $\Delta \ge q+10 > 10$, we have $|\bar{\varphi}(x_1)|+|\bar{\varphi}(x_2)| \ge 2\Delta - (\frac{3}{2}\Delta - 2)+2 = \frac{1}{2}\Delta + 4 > 9$, which implies that for some $i \in \{1, 2\}$ there exists a color $\beta \in \bar{\varphi}(x_i)$ such that $\beta \notin \{1, 2, \varphi(vz), \varphi(v'z)\}$. Clearly, each of v, v', x_i must be an endvertex in an (α, β)-chain, so there exists a vertex in $\{v, v'\}$, assume v, not in the path $P_{x_i}(\alpha, \beta, \varphi)$. Let $\varphi_1 = \varphi/P_{x_i}(\alpha, \beta, \varphi)$. Then $\alpha \in \bar{\varphi}_1(v) \cap \bar{\varphi}_1(x_i)$. On the other hand, we will show that $\bar{\varphi}_1(v) \cap \bar{\varphi}_1(x_i) = \emptyset$. Since $\alpha, \beta \notin \{1, 2, \varphi(vz), \varphi(v'z)\}$, we have $v \in Z(\varphi_1) \setminus \{x_2\}$ and φ_1 is zy-feasible. Therefore, $\{v, z, x_1, x_2\}$ is elementary with respect to φ_1 by (10), a contradiction.

This completes the proof of Claim B1. □

Note that x_2 may be a vertex of $Z(\varphi)$ and x_1 may be a vertex of $Y(\varphi)$. By Claim B1 and the definitions of $C_y(\varphi)$ and $C_z(\varphi)$, we have

$$(\Delta - q)|C_z(\varphi) \setminus \{\varphi(zx_2)\}| + |\bar{\varphi}(x_1)| + |\bar{\varphi}(x_2)| < \sum_{v \in \{x_1, x_2, z\} \cup Z(\varphi)} |\bar{\varphi}(v)| \le \Delta$$

and

$$(\Delta - q)|C_y(\varphi) \setminus \{\varphi(yx_1)\}| + |\bar{\varphi}(x_1)| + |\bar{\varphi}(x_2)| < \sum_{v \in \{x_1, x_2, y\} \cup Y(\varphi)} |\bar{\varphi}(v)| \le \Delta.$$

It follows that $|C_z(\varphi) \setminus \{\varphi(zx_2)\}| < \frac{d(x_1)+d(x_2)-\Delta-2}{\Delta-q}$ and $|C_y(\varphi) \setminus \{\varphi(yx_1)\}| < \frac{d(x_1)+d(x_2)-\Delta-2}{\Delta-q}$. Hence, **I** holds.

4.2.2. Proof of Statement II

Suppose to the contrary that for every $\varphi \in \mathcal{C}_{zy}$ we have $|T_0(\varphi) \setminus \{\varphi(zx_2), \varphi(yx_1)\}| > \lceil \frac{4(d(x_1)+d(x_2)-\Delta+2)}{\Delta-q} + \frac{8(d(x_1)+d(x_2)-\Delta-2)}{(\Delta-q)^2} \rceil$. For each coloring $\varphi \in \mathcal{C}_{zy}$, let $R(\varphi) = C_z(\varphi) \cup C_y(\varphi)$, $R'(\varphi) = R(\varphi) \cup \{1, 2\}$, $T_0(\varphi) = \{k_1, \ldots, k_{|T_0(\varphi)|}\}$ and

$$V(T_0(\varphi)) = \{z_{k_i} \in N(z) : k_i \in T_0(\varphi)\} \cup \{y_{k_i} \in N(y) : k_i \in T_0(\varphi)\}.$$

Note that $\{x_1, x_2\} \cap V(T_0(\varphi))$ may not be empty. If x_1 (x_2) is in $V(T_0(\varphi))$, then $yx_1 \in E(G)$ ($zx_2 \in E(G)$).

A coloring $\varphi \in \mathcal{C}_{zy}$ is called *optimal* if over all zy-feasible colorings the followings hold:

1. $|C_z(\varphi)| + |C_y(\varphi)|$ is maximum;
2. subject to 1, $|C_z(\varphi) \cap C_y(\varphi)|$ is minimum.

Let φ be an optimal zy-feasible coloring and assume that $\varphi(x_1z) = 1 \in \bar{\varphi}(x_2)$ and $\varphi(x_2y) = 2 \in \bar{\varphi}(x_1)$. For convenience, we let $Z = Z(\varphi)$, $Y = Y(\varphi)$, $C_z = C_z(\varphi)$, $C_y = C_y(\varphi)$, $T_0 = T_0(\varphi)$, $R = R(\varphi)$ and $R' = R'(\varphi)$. Note that $1 \notin C_z \cup T_0$, $2 \notin C_y \cup T_0$ and $R \cap T_0 = \emptyset$.

Claim C1. *Let k be an arbitrary color in T_0. For each $i \in \bar{\varphi}(x_2) \setminus (R \cup \{1\})$, $P_{x_2}(i, k, \varphi)$ contains both y and z; and for each $i \in \bar{\varphi}(x_1) \setminus (R \cup \{2\})$, $P_{x_1}(i, k, \varphi)$ contains both y and z.*

Proof. Since x_1 and x_2 are symmetric, we only show the first part of the statement. Let $\{u, v\} = \{y, z\}$. Suppose that $u \notin V(P_{x_2}(i, k, \varphi))$. Then $P_u(i, k, \varphi)$ is disjoint from $P_{x_2}(i, k, \varphi)$. Let $\varphi' = \varphi/P_u(i, k, \varphi)$. Since $\{i, k\} \cap \{1, 2\} = \emptyset$, φ' is also zy-feasible. Note that $\varphi'(uu_k) = \varphi(uu_i) = i$. Since $d(u_k) < q$ and $i \in \bar{\varphi}(x_2) \setminus (R \cup \{1\})$, we have $i \in C_u(\varphi') \setminus C_u$. Since $k \in T_0$, $R \cap T_0 = \emptyset$ and $i \notin R$, the colors in R are unchanged, it follows that $C_u(\varphi') \supseteq C_u \cup \{i\}$ and $C_v(\varphi') \supseteq C_v$, giving a contradiction to the maximality of $|C_y| + |C_z|$. Thus $P_{x_2}(i, k, \varphi)$ contains both y and z. □

Let φ^0 be obtained from φ by interchanging the colors 1 and 2 on all $(1, 2)$-chains except the one connecting x_1 to x_2. Since $\{z, x_1, x_2, y\} \subseteq V(P_{x_2}(1, 2, \varphi))$, we have $\varphi^0(v) = \varphi(v)$ for every vertex $v \in \{z, x_1, x_2, y\}$. It is easy to see that φ^0 is an optimal zy-feasible coloring if and only if φ is an optimal zy-feasible coloring. Recall that $R' = C_y \cup C_z \cup \{1, 2\}$.

Claim D1. *Suppose that there exist three vertices $u_1, u_2, u_3 \in V(T_0) \setminus \{z, x_1, x_2, y\}$ and two distinct colors α, β with $\alpha \in \bar{\varphi}(u_1) \cap \bar{\varphi}(u_2) \cap \bar{\varphi}(u_3)$ and $\beta \in (\bar{\varphi}(x_1) \cup \bar{\varphi}(x_2)) \setminus R'$. For a vertex $u \in \{u_1, u_2, u_3\}$, let $\varphi'_u = \varphi^0/P_u(i, \beta, \varphi^0)$ if $\beta \in \bar{\varphi}(x_i) \setminus R'$ and $\alpha = j$, where $\{i, j\} = \{1, 2\}$, and let $\varphi'_u = \varphi/P_u(\alpha, \beta, \varphi)$ otherwise. Then there exists a vertex $u \in \{u_1, u_2, u_3\}$ with $\{x_1, x_2\} \nsubseteq V(P_u(\alpha, \beta, \varphi))$ such that the coloring $\varphi' = \varphi'_u$ is zy-feasible and optimal and has the following properties: $\varphi'(x_1) = \varphi(x_1)$, $\varphi'(x_2) = \varphi(x_2)$, $C_y(\varphi') = C_y$, $C_z(\varphi') = C_z$, $R(\varphi') = R$, $T_0(\varphi') \supseteq T_0$ and $\beta \in \bar{\varphi}'(u)$.*

Proof. Since $\bar{\varphi}(x_1) \cap \bar{\varphi}(x_2) = \emptyset$ and x_1 and x_2 are symmetric, we assume $\beta \in \bar{\varphi}(x_1) \setminus R'$ and therefore $\beta \in \varphi(x_2)$. For $i \in \{1, 2, 3\}$, let $P_i = P_{u_i}(\alpha, \beta, \varphi)$ and $\varphi_i = \varphi/P_i$. Clearly none of u_1, u_2, u_3, x_1 can be an internal vertex (of degree 2) in an (α, β)-chain, and so there are at least two values of i such that $x_1 \notin V(P_i)$; let $i = 1$ be one of them. For each i such that $x_1 \notin V(P_i)$, we have the following observations.

- $\varphi_i(x_1) = \varphi(x_1)$ since $x_1 \notin V(P_i)$.
- $\varphi_i(x_2) = \varphi(x_2)$. This is obvious if $\alpha \in \varphi(x_2)$, since $\beta \in \varphi(x_2)$. And if $\alpha \in \bar{\varphi}(x_2)$ then $P_{x_1}(\alpha, \beta, \varphi) = P_{x_2}(\alpha, \beta, \varphi)$ by Claim A1(b), and so $x_2 \notin V(P_i)$.
- $\beta \in \bar{\varphi}_i(u_i)$ since $\alpha \in \bar{\varphi}(u_i)$.
- All the conclusions of Claim D1 hold (with $\varphi' = \varphi'_{u_i}$ and $T_0(\varphi'_{u_i}) = T_0$) if $V(P_i) \cap \{x_1, y, z\} = \emptyset$.
- Since $\beta \notin T_0$, we have $T_0(\varphi'_{u_i}) \supseteq T_0$ if $\alpha \notin T_0$. But if $\alpha \in T_0$ then it follows from Claim C1 that $V(P_i) \cap \{x_1, y, z\} = \emptyset$, and so $T_0(\varphi'_{u_i}) = T_0$.
- The result holds if $\alpha = 1$. For, by Claim A1(b) and the fact that $\varphi(x_1z) = 1$, we have $P_z(1, \beta, \varphi) = P_{x_1}(1, \beta, \varphi) = P_{x_2}(1, \beta, \varphi)$. So if we choose j so that $y \notin P_j$, then $V(P_j) \cap \{x_1, x_2, y, z\} = \emptyset$, which is more than we need.
- The result holds if $\alpha = 2$. In this case $\varphi' = \varphi^0/P_u(1, \beta, \varphi^0)$ by definition, where $u \in \{u_1, u_2, u_3\}$. Note that $1 \in \bar{\varphi}^0(u_j)$ for all $j \in \{1, 2, 3\}$, and $\varphi^0(v) = \varphi(v)$ for every vertex $v \in \{z, x_1, x_2, y\}$, and so the result follows by applying the case $\alpha = 1$ to φ^0.

Thus it suffices to assume $\alpha \notin \{1, 2\}$ and to prove that there exists a number $i \in \{1, 2, 3\}$ such that $x_1 \notin V(P_i)$, $C_y(\varphi_i) = C_y$ and $C_z(\varphi_i) = C_z$, as these imply that φ_i is optimal and $R(\varphi_i) = R$. Note that φ_i is zy-feasible, since $\{\alpha, \beta\} \cap \{1, 2\} = \emptyset$. We consider the following four cases.

Case 1: $\alpha \in (\varphi(x_1) \cap \varphi(x_2)) \setminus R'$.

Since $\alpha, \beta \notin R' = C_y \cup C_z \cup \{1, 2\}$, it follows that $C_y(\varphi_1) \supseteq C_y$ and $C_z(\varphi_1) \supseteq C_z$. Since φ is optimal, $C_y(\varphi_1) = C_y$ and $C_z(\varphi_1) = C_z$, which is all we need to prove.

Case 2: $\alpha \in \varphi(x_1) \cap \varphi(x_2) \cap R'$.

Clearly, $\alpha \notin \{1, 2\}$. Assume first $\alpha \in C_y$. By the definition of C_y, $\alpha \in \bar{\varphi}(z)$. It follows that $P_{x_1}(\alpha, \beta, \varphi) = P_z(\alpha, \beta, \varphi)$ by Claim A1(b). So $x_1, z \notin V(P_i)$ for all $i \in \{1, 2, 3\}$. Note that there exists a path P_j ($j \in \{1, 2, 3\}$) such that $y \notin V(P_j)$. So $V(P_j) \cap \{x_1, y, z\} = \emptyset$, which is all we need.

If $\alpha \in C_z$, then by interchanging y and z in the above argument, we get the same conclusion.

Case 3: $\alpha \in (\bar{\varphi}(x_1) \cup \bar{\varphi}(x_2)) \setminus (C_y \cap C_z)$ and $\alpha \notin \{1, 2\}$.

By Claim A1(a), $\{z, x_1, x_2, y\}$ is elementary with respect to φ. Then $\alpha, \beta \in (\bar{\varphi}(x_1) \cup \bar{\varphi}(x_2)) \cap \varphi(y) \cap \varphi(z)$. Since $\alpha \notin C_y \cap C_z$ and $\beta \notin R$, either $\alpha, \beta \notin C_y$ or $\alpha, \beta \notin C_z$ (or both). Without loss of generality, we assume $\alpha, \beta \notin C_y$ (the other case is similar). Then the neighbors y_α and y_β of y both have degree at least q. Note that by Claim A1(b) we have $P_{x_1}(\alpha,\beta,\varphi) = P_{x_2}(\alpha,\beta,\varphi)$ if $\alpha \in \bar\varphi(x_2)$, and $P_{x_1}(\alpha,\beta,\varphi) = \{x_1\}$ if $\alpha \in \bar\varphi(x_1)$. It follows that $x_1 \notin V(P_i)$ for all $i \in \{1,2,3\}$, and so we can choose P_j ($j \in \{1,2,3\}$) so that $z \notin V(P_j)$. Then $C_y(\varphi_j) = C_y$ and $C_z(\varphi_j) = C_z$, regardless of whether or not $y \in V(P_j)$.

Case 4: $\alpha \in C_z \cap C_y$ and $\alpha \notin \{1,2\}$.

In this case, $\alpha \in \bar\varphi(x_1) \cup \bar\varphi(x_2)$. By the same argument as in Case 3, $x_1 \notin V(P_i)$ for all $i \in \{1,2,3\}$. Since $\alpha \in C_z \cap C_y$ and $\beta \in \bar\varphi(x_1) \setminus R \subseteq \varphi(y) \cap \varphi(z)$, the vertices y_α and z_α have degree less than q and the vertices y_β and z_β have degree at least q. Let $\varphi_0 = \varphi/P_z(\alpha,\beta,\varphi)$. Then either $\varphi_0(x_1) = \varphi(x_1)$ and $\varphi_0(x_2) = \varphi(x_2)$ (if $\alpha \in \bar\varphi(x_1)$ or $P_{x_1}(\alpha,\beta,\varphi)$ does not pass through z), or else $\varphi_0(x_1) = (\varphi(x_1) \cup \{\beta\}) \setminus \{\alpha\}$ and $\varphi_0(x_2) = (\varphi(x_2) \cup \{\alpha\}) \setminus \{\beta\}$ (if $\alpha \in \bar\varphi(x_2)$ and $P_{x_1}(\alpha,\beta,\varphi)$ passes through z). So $\alpha,\beta \in \bar\varphi_0(x_1) \cup \bar\varphi_0(x_2) = \bar\varphi(x_1) \cup \bar\varphi(x_2)$, $Z(\varphi_0) = Z(\varphi)$, and $Y(\varphi_0) = Y(\varphi)$. If $P_z(\alpha,\beta,\varphi)$ does not pass through y, then $C_z(\varphi_0) = (C_z \cup \{\beta\}) \setminus \{\alpha\}$ and $C_y(\varphi_0) = C_y$, which implies that $|C_z(\varphi_0)| + |C_y(\varphi_0)| = |C_z| + |C_y|$ and $|C_z(\varphi_0) \cap C_y(\varphi_0)| = |C_z \cap C_y| - 1$; this contradicts the minimality of $|C_z \cap C_y|$, and this contradiction shows that $P_z(\alpha,\beta,\varphi) = P_y(\alpha,\beta,\varphi)$. So we can choose j so that P_j does not pass through y or z, and then $V(P_j) \cap \{x_1, y, z\} = \emptyset$, which is all we need. □

Let $t = \lceil \frac{4(d(x_1)+d(x_2)-\Delta+2)}{\Delta-q} + \frac{8(d(x_1)+d(x_2)-\Delta-2)}{(\Delta-q)^2} \rceil$. Recall that $|T_0(\varphi) \setminus \{\varphi(zx_2), \varphi(yx_1)\}| > t$. Let $T_0'(\varphi) = T_0(\varphi) \setminus \{\varphi(yz), \varphi(yx_1), \varphi(zx_2)\}$. Note that yz may not be an edge of G. It follows that $|T_0'(\varphi)| \ge |T_0(\varphi) \setminus \{\varphi(zx_2), \varphi(yx_1)\}| - 1$. Thus $|T_0'(\varphi)| \ge t$. For each t-element subset $T = \{k_1, \ldots, k_t\}$ of $T_0'(\varphi)$, let

$$\begin{aligned}
V(T,\varphi) &= \{z_{k_1}, z_{k_2}, \ldots, z_{k_t}\} \cup \{y_{k_1}, y_{k_2}, \ldots, y_{k_t}\},\\
W(T,\varphi) &= \{u \in V(T,\varphi) : \bar\varphi(u) \cap ((\bar\varphi(x_1) \cup \bar\varphi(x_2)) \setminus R'(\varphi)) = \emptyset\},\\
M(T,\varphi) &= \{u \in V(T,\varphi) : \bar\varphi(u) \cap ((\bar\varphi(x_1) \cup \bar\varphi(x_2)) \setminus R'(\varphi)) \neq \emptyset\},\\
E(T,\varphi) &= \{zz_{k_1}, zz_{k_2}, \ldots, zz_{k_t}, yy_{k_1}, yy_{k_2}, \ldots, yy_{k_t}\},\\
E_W(T,\varphi) &= \{e \in E(T,\varphi) : \ e \text{ is incident with a vertex in } W(T,\varphi)\}, \text{ and}\\
E_M(T,\varphi) &= \{e \in E(T,\varphi) : \ e \text{ is incident with a vertex in } M(T,\varphi)\}.
\end{aligned}$$

Clearly, $V(T,\varphi) = W(T,\varphi) \uplus M(T,\varphi)$ and $E(T,\varphi) = E_W(T,\varphi) \uplus E_M(T,\varphi)$, where $\uplus$ denotes disjoint union. For convenience, we let $W = W(T,\varphi), M = M(T,\varphi), E_W = E_W(T,\varphi)$ and $E_M = E_M(T,\varphi)$ if T and φ are clear. Note that $\{z_{k_1}, \ldots, z_{k_t}\} \cap \{y_{k_1}, \ldots, y_{k_t}\}$ may not be empty, but $|T| \le |V(T,\varphi)| \le |E(T,\varphi)| = 2|T|$, $\frac{|E_W|}{2} \le |W| \le |E_W|$ and $\frac{|E_M|}{2} \le |M| \le |E_M|$.

We assume that $|E_M(T,\varphi)|$ is maximum over all optimal zy-feasible colorings φ and all t-element subsets T of $T_0'(\varphi)$. Let $Y_M = \{y_k : k \in T \text{ and } z_k, y_k \in M\}$. For any $Y' \subseteq Y_M$, let $Z(Y') = \{z_k : y_k \in Y'\}$, and let $C_M(Y')$ be the union of all single-element sets of the form $\bar\varphi(v) \cap ((\bar\varphi(x_1) \cup \bar\varphi(x_2)) \setminus R')$ with $v \in Y' \cup Z(Y')$, ignoring any sets of this form with more than one element. Moreover, let C_M be the union of all single-element sets of the form $\bar\varphi(v) \cap ((\bar\varphi(x_1) \cup \bar\varphi(x_2)) \setminus R')$ with $v \in M$. Clearly, $|C_M(Y_M)| \le |C_M| \le |M|$.

Claim E1. *The following three statements hold.*

(a) If $k \in T_0'(\varphi)$, $i, j \in (\bar\varphi(x_1) \cup \bar\varphi(x_2)) \setminus R'$, $i \in \bar\varphi(z_k)$ and $j \in \bar\varphi(y_k)$, then $i \neq j$.

(b) If $y_k \in Y_M$ then there exist distinct colors i, j as in (a).

(c) If $y_k \in Y_M$ and $|\bar\varphi(y_k) \cap ((\bar\varphi(x_1) \cup \bar\varphi(x_2)) \setminus R')| \ge 2$, then there exist distinct colors $i, j, l \in (\bar\varphi(x_1) \cup \bar\varphi(x_2)) \setminus R'$ such that $i \in \bar\varphi(z_k)$ and $j, l \in \bar\varphi(y_k)$.

Proof. If $i \in \bar\varphi(y_k) \cap \bar\varphi(z_k)$, then neither y_k nor z_k can be an internal vertex (with degree 2) of $P_{x_1}(i,k,\varphi)$ or $P_{x_2}(i,k,\varphi)$, whereas Claim C1 implies that at least one of y_k and z_k must be an internal vertex of one of these paths. Thus $i \neq j$, which proves (a). (b) follows because, by the definitions of Y_M and M, there exist colors $i, j \in (\bar\varphi(x_1) \cup \bar\varphi(x_2)) \setminus R'$ such that $i \in \bar\varphi(z_k)$ and $j \in \bar\varphi(y_k)$; and (c) holds for the same reason. □

Recall that φ^0 is obtained from φ by interchanging the colors 1 and 2 on all $(1,2)$-chains except the one connecting x_1 to x_2. Since $1, 2 \notin T_0(\varphi)$, it follows that $T_0(\varphi^0) = T_0(\varphi)$ and $T \subseteq T_0'(\varphi^0) = T_0'(\varphi)$. Hence $V(T,\varphi^0) = V(T,\varphi)$. It is clear that $R(\varphi^0)$ is the same as $R(\varphi)$ except possibly for changing 1 to 2 or vice versa, and so $R'(\varphi^0) = R'(\varphi) = R(\varphi) \cup \{1,2\}$. Thus, for each vertex $v \in V(T,\varphi)$, $\bar\varphi^0(v) \cap ((\bar\varphi^0(x_1) \cup \bar\varphi^0(x_2)) \setminus R'(\varphi^0)) = \bar\varphi(v) \cap ((\bar\varphi(x_1) \cup \bar\varphi(x_2)) \setminus R')$.

Claim F1. *The following two statements hold.*

(a) The hypotheses of Claim D1 cannot hold with $u_1, u_2, u_3 \in W$ and $\beta \notin C_M$.

(b) Assume that the hypotheses of Claim D1 hold, and u and φ' are given by Claim D1, where $\varphi' = \varphi/P_u(\alpha,\beta,\varphi)$, $\varphi^0/P_u(1,\beta,\varphi^0)$ or $\varphi^0/P_u(2,\beta,\varphi^0)$ as appropriate. Then

(i) if $\beta \notin C_M$ then $u \notin W$;

(ii) if $\alpha \notin (\bar\varphi(x_1) \cup \bar\varphi(x_2)) \setminus R'$, $u \in Y_M$ and $\beta \notin C_M(\{u\})$, then there is a color $k \in T \subseteq T_0'(\varphi')$ for which the following holds: there are three distinct colors $i, j, l \in (\bar\varphi'(x_1) \cup \bar\varphi'(x_2)) \setminus R'(\varphi')$ such that $i \in \bar\varphi'(z_k)$ and $j, l \in \bar\varphi'(y_k)$.

Proof. Clearly (a) follows from (b)(i). In the rest of this proof we will assume by symmetry that $\beta \in \bar{\varphi}(x_1) \setminus R'$.

By Claim D1, $\varphi'(x_1) = \varphi(x_1)$, $\varphi'(x_2) = \varphi(x_2)$, $R(\varphi') = R$, $T_0 \subseteq T_0(\varphi')$, and $\beta \in \bar{\varphi}'(u)$. As we remarked in the proof of Claim D1, $\beta \notin T_0$, and if $\alpha \in T_0$ then $V(P_u(\alpha, \beta, \varphi)) \cap \{x_1, y, z\} = \emptyset$. Thus $T_0'(\varphi') \supseteq T_0'(\varphi) \supseteq T$ and for each $k \in T$ we have $\varphi'(yy_k) = \varphi'(zz_k) = \varphi(yy_k) = \varphi(zz_k) = k$.

To prove (b)(i), suppose that $\beta \notin C_M$ and $u \in W$. Since $\beta \in (\bar{\varphi}(x_1) \cup \bar{\varphi}(x_2)) \setminus R' = (\bar{\varphi}'(x_1) \cup \bar{\varphi}'(x_2)) \setminus R'(\varphi')$ and $u \in W$, it follows that $u \in M(T, \varphi') \setminus M$. To avoid the contradiction $|E_M(T, \varphi')| > |E_M|$, it must be that $P_u(\alpha, \beta, \varphi)$ or $P_u(1, \beta, \varphi^0)$ ends with an edge of color α at a vertex $v \in M$ such that β is the unique color in $\bar{\varphi}(v) \cap ((\bar{\varphi}(x_1) \cup \bar{\varphi}(x_2)) \setminus R')$, so that $v \notin M(T, \varphi')$. But then $\beta \in C_M$, which is a contradiction.

To prove (b)(ii), suppose that $u = y_k \in Y_M$, where $k \in T \subseteq T_0'(\varphi')$. Since $\beta \notin C_M(\{u\}) = C_M(\{y_k\})$, it is not possible that $\bar{\varphi}(z_k) \cap ((\bar{\varphi}(x_1) \cup \bar{\varphi}(x_2)) \setminus R') = \{\beta\}$ or $\bar{\varphi}(y_k) \cap ((\bar{\varphi}(x_1) \cup \bar{\varphi}(x_2)) \setminus R') = \{\beta\}$. By Claim E1, there exist distinct colors $i, j \in (\bar{\varphi}(x_1) \cup \bar{\varphi}(x_2)) \setminus R' = (\bar{\varphi}'(x_1) \cup \bar{\varphi}'(x_2)) \setminus R'(\varphi')$ such that $i \in \bar{\varphi}(z_k), j \in \bar{\varphi}(y_k)$, and $\beta \notin \{i, j\}$. Since $\alpha \notin (\bar{\varphi}(x_1) \cup \bar{\varphi}(x_2)) \setminus R'$, it follows that $\alpha \notin \{i, j\}$. Thus $i \in \bar{\varphi}'(z_k)$ and $j \in \bar{\varphi}'(y_k)$. Since also $\beta \in \bar{\varphi}'(u) = \bar{\varphi}'(y_k)$, the result follows with $l = \beta$. □

For convenience, let $X = (\bar{\varphi}(x_1) \cup \bar{\varphi}(x_2)) \setminus R'$; then $|X| \ge 2\Delta - d(x_1) - d(x_2) - |R|$. Recall that the statement **I** states that for any $\varphi \in \mathcal{C}_{zy}$ we have $|C_z(\varphi) \setminus \{\varphi(zx_2)\}| < \frac{d(x_1)+d(x_2)-\Delta-2}{\Delta-q}$ and $|C_y(\varphi) \setminus \{\varphi(yx_1)\}| < \frac{d(x_1)+d(x_2)-\Delta-2}{\Delta-q}$, so that

$$|R| < \frac{2(d(x_1)+d(x_2)-\Delta-2)}{\Delta-q} + 2. \tag{12}$$

Since $|T| = \lceil \frac{4(d(x_1)+d(x_2)-\Delta+2)}{\Delta-q} + \frac{8(d(x_1)+d(x_2)-\Delta-2)}{(\Delta-q)^2} \rceil$, we have

$$\frac{(\Delta-q)|T|}{2} > 2(d(x_1)+d(x_2)-\Delta+2) + 2(|R|-2) \ge 2(\Delta-|X|), \tag{13}$$

where the first inequality uses (12). Since $d(x_1)+d(x_2) \le \frac{3}{2}\Delta - 2$ and $q \le \Delta - 10$, we have $|T| \le \frac{4(\frac{1}{2}\Delta)}{10} + \frac{8(\frac{1}{2}\Delta-4)}{100} + 1 = \frac{\Delta}{5} + \frac{\Delta-8}{25} + 1 = \frac{6\Delta+17}{25}$. By (12) again, we have

$$\begin{aligned} |X| &> 2\Delta - d(x_1) - d(x_2) - \frac{2(d(x_1)+d(x_2)-\Delta-2)}{\Delta-q} - 2 \\ &\ge \frac{1}{2}\Delta + 2 - \frac{2(\frac{1}{2}\Delta-4)}{10} - 2 = \frac{2\Delta+4}{5}, \end{aligned}$$

$\frac{2(\Delta-|X|)}{\Delta-q-1} < \frac{2}{9}(\frac{3\Delta-4}{5}) = \frac{6\Delta-8}{45}$, and $|T| + \frac{2(\Delta-|X|)}{\Delta-q-1} \le \frac{6\Delta+17}{25} + \frac{6\Delta-8}{45} = \frac{84\Delta+113}{225}$. Since $\frac{84\Delta+113}{225} < \frac{2\Delta+4}{5}$ and $\frac{2(6\Delta-8)}{45} + 2 < \frac{2\Delta+4}{5}$ as $\Delta \ge 7$, we have

$$|X| > |T| + \frac{2(\Delta-|X|)}{\Delta-q-1}, \tag{14}$$

and

$$|X| > 2\left(\frac{2(\Delta-|X|)}{\Delta-q-1}\right) + 2. \tag{15}$$

Claim G1. *There exist an optimal zy-feasible coloring φ' and a color $k \in T \subseteq T_0'(\varphi')$ for which the following holds: there are three distinct colors $i, j, l \in (\bar{\varphi}'(x_1) \cup \bar{\varphi}'(x_2)) \setminus R'(\varphi')$ such that $i \in \bar{\varphi}'(z_k)$ and $j, l \in \bar{\varphi}'(y_k)$.*

Proof. We consider the following two cases, according to the value of $|E_M|$.

Case 1: $|E_M| \le |T| + \frac{2(\Delta-|X|)}{\Delta-q-1}$. Since $|C_M| \le |E_M|$, it follows from (14) that there exists a color $\beta \in X \setminus C_M$.

First we claim that $|E_M| > |T|$. If not, then $|E_M| \le |T|$, which implies that $|E_W| \ge |T|$ since $|E_M| + |E_W| = 2|T|$. Thus $|W| \ge \frac{|T|}{2}$. By the definition of T_0, we have $d(v) < q$ for every vertex $v \in V(T, \varphi)$, which includes all $v \in W$. So $\sum_{v \in W} |\bar{\varphi}(v)| > (\Delta - q)\frac{|T|}{2} > 2(\Delta - |X|)$, by (13). By the definition of W, $\bar{\varphi}(v) \cap X = \emptyset$ for every $v \in W$. So, by the Pigeonhole Principle, there exist three vertices $u_1, u_2, u_3 \in W$ and a color $\alpha \notin X$ such that $\alpha \in \bar{\varphi}(u_1) \cap \bar{\varphi}(u_2) \cap \bar{\varphi}(u_3)$. But this contradicts Claim F1(a).

So we may assume that $|E_M| = |T| + p$, so that $|E_W| = |T| - p$, where $p > 0$. Since $|E_M| = |T| + p$, it follows that $|Y_M| \ge p$. We may assume that $|\bar{\varphi}(y_k) \cap X| = 1$ for all $y_k \in Y_M$, as otherwise the result holds by Claim E1(c). Thus $|\bar{\varphi}(v) \setminus X| = |\bar{\varphi}(v)| - 1 > \Delta - q - 1$ for all $v \in Y_M$, while $|\bar{\varphi}(v) \setminus X| = |\bar{\varphi}(v)| > \Delta - q$ for all $v \in W$. Since $|W| \ge |E_W|/2$, $|E_W| = |T| - p$, and $\Delta - q > 2$, we have $\sum_{v \in W \cup Y_M} |\bar{\varphi}(v) \setminus X| > (\Delta - q)\frac{|T|-p}{2} + (\Delta - q - 1)p > (\Delta - q)\frac{|T|}{2} > 2(\Delta - |X|)$, by (13). Hence there exist three vertices $u_1, u_2, u_3 \in W \cup Y_M$ and a color $\alpha \notin X$ such that $\alpha \in \bar{\varphi}(u_1) \cap \bar{\varphi}(u_2) \cap \bar{\varphi}(u_3)$. Since $\beta \notin C_M$, the result follows from Claim F1(b).

Case 2: $|E_M| > |T| + \frac{2(\Delta-|X|)}{\Delta-q-1}$.

Since $|E_M| > |T| + \frac{2(\Delta-|X|)}{\Delta-q-1}$, it follows that $|Y_M| > \frac{2(\Delta-|X|)}{\Delta-q-1}$. We may assume that $|\bar{\varphi}(y_k) \cap X| = 1$ for all $y_k \in Y_M$, as otherwise the result holds by Claim E1(c). Let Y' be a subset of Y_M with $|Y'| = \left\lceil \frac{2(\Delta-|X|)}{\Delta-q-1} \right\rceil$. Then we have $\sum_{v \in Y'} |\bar{\varphi}(v) \setminus X| > (\Delta - q - 1)|Y'| \ge 2(\Delta - |X|)$. Thus there exist three vertices $u_1, u_2, u_3 \in Y'$ and a color $\alpha \notin X$

such that $\alpha \in \bar{\varphi}(u_1) \cap \bar{\varphi}(u_2) \cap \bar{\varphi}(u_3)$. Clearly, $|C_M(Y')| \le 2|Y'|$. By (15), we have $|X| > 2|Y'| \ge |C_M(Y')|$, thus there exists a color $\beta \in X \setminus C_M(Y')$, and the result follows from Claim F1(b). □

Let k, i, j, l, φ' be as stated in Claim G1 and assume $d(x_2) \le d(x_1)$. Since $d(x_1) + d(x_2) \le \frac{3}{2}\Delta - 2$, $d(x_2) \le d(x_1)$ and $q \le \Delta - 10$, we have

$$\begin{aligned} |\bar{\varphi}'(x_2) \setminus (R(\varphi') \cup \{1\})| &\ge \Delta - d(x_2) - |C_z(\varphi') \setminus \{\varphi'(zx_2)\}| - |C_y(\varphi')| \\ &> \Delta - d(x_2) - \frac{2(d(x_1) + d(x_2) - \Delta - 2)}{\Delta - q} - 1 \\ &\ge \frac{\Delta}{4} + 1 - \frac{\Delta - 8}{10} - 1 = \frac{3\Delta + 16}{20} \ge 2, \end{aligned}$$

where the second inequality follows from **I** and the last inequality holds because $\Delta \ge 8$. Thus

$$|\bar{\varphi}'(x_2) \setminus (R(\varphi') \cup \{1\})| \ge 3. \tag{16}$$

First we claim that

> there exists an optimal zy-feasible coloring φ^*, such that $i, j, l \in \bar{\varphi}^*(x_2) \setminus (R(\varphi^*) \cup \{1\})$, $k \in T \subseteq T_0'(\varphi^*)$, $i \in \bar{\varphi}^*(z_k)$ and $j, l \in \bar{\varphi}^*(y_k)$.

For otherwise, by Claim G1, at least one of i, j and l must be in $\bar{\varphi}'(x_1) \setminus (R(\varphi') \cup \{2\})$ rather than in $\bar{\varphi}'(x_2) \setminus (R(\varphi') \cup \{1\})$. Suppose first that $i \in \bar{\varphi}'(x_1) \setminus (R(\varphi') \cup \{2\})$. By (16), there exists a color $\delta \in \bar{\varphi}'(x_2) \setminus (R(\varphi') \cup \{1\})$ such that $\delta \notin \{j, l\}$. Note that $P_{x_1}(i, \delta, \varphi') = P_{x_2}(i, \delta, \varphi')$ by Claim A1(b). Then $z_k \notin V(P_{x_1}(i, \delta, \varphi'))$ as $i \in \bar{\varphi}'(z_k)$. Let $\varphi_1' = \varphi'/P_{x_1}(i, \delta, \varphi')$. Then $i \in \bar{\varphi}_1{}'(x_2) \setminus (R(\varphi_1') \cup \{1\})$ and $i \in \bar{\varphi}_1{}'(z_k)$. Since $\{i, \delta\} \cap (R'(\varphi') \cup \{j, l\}) = \emptyset$, we have $j, l \in \bar{\varphi}_1{}'(y_k)$, $C_z(\varphi_1') = C_z(\varphi')$, $C_y(\varphi_1') = C_y(\varphi')$ and φ_1' is zy-feasible, which implies that φ_1' is an optimal zy-feasible coloring. Note that $\bar{\varphi}_1'(x_2) = (\bar{\varphi}'(x_2) \setminus \{\delta\}) \cup \{i\}$, and so $|\bar{\varphi}_1'(x_2) \setminus (R(\varphi_1') \cup \{1\})| \ge 3$ by (16). Using the same method above twice again, we can find an optimal zy-feasible coloring φ^* such that $i, j, l \in \bar{\varphi}^*(x_2) \setminus (R(\varphi^*) \cup \{1\})$, $k \in T \subseteq T_0'(\varphi^*)$, $i \in \bar{\varphi}^*(z_k)$, and $j, l \in \bar{\varphi}^*(y_k)$.

If $2 \in \bar{\varphi}^*(y_k)$, we let $\varphi_1 = \varphi^*$. Otherwise, we let φ_1 be obtained from φ^* by interchanging the colors 2 and l on all $(2, l)$-chains except $P_{x_2}(2, l, \varphi^*)$. Note that $P_{x_1}(2, l, \varphi^*) = P_{x_2}(2, l, \varphi^*)$ by Claim A1(b) and $l \in \bar{\varphi}^*(y_k)$. Thus $2 \in \bar{\varphi}_1(y_k)$. If $2 \in C_z(\varphi^*)$ and $P_{x_2}(2, l, \varphi^*)$ does not pass through z, then $C_z(\varphi_1) \supseteq (C_z(\varphi^*) \cup \{l\}) \setminus \{2\}$; otherwise, $C_z(\varphi_1) \supseteq C_z(\varphi^*)$. Also, $C_y(\varphi_1) \supseteq C_y(\varphi^*)$. It follows from the optimality of φ^* that equality holds and φ_1 is an optimal zy-feasible coloring.

Let φ_2 be the coloring obtained from φ_1 by uncoloring x_2y and coloring x_1x_2 with color 2. Then $2 \in \bar{\varphi}_2(y)$.

Let φ_3 be obtained from φ_2 by assigning $\varphi_3(yy_k) = 2$. Now k is missing at y and y_k and i is still missing at z_k and x_2. Note that $P_{x_2}(i, k, \varphi_3) = P_y(i, k, \varphi_3)$, as otherwise we could get an edge-Δ-coloring of G from $\varphi_3/P_y(i, k, \varphi_3)$ by coloring x_2y with i. Furthermore, $z_k, y_k \notin P_{x_2}(i, k, \varphi_3)$ since either i or k is missing at these two vertices, which additionally shows that $z \notin P_{x_2}(i, k, \varphi_3)$ since $\varphi_3(zz_k) = k$.

Let $\varphi_4 = \varphi_3/P_{x_2}(i, k, \varphi_3)$. We have $k, j \in \bar{\varphi_4}(x_2) \cap \bar{\varphi_4}(y_k)$ and $i \in \bar{\varphi_4}(y) \cap \bar{\varphi_4}(z_k)$. Since G is not edge-Δ-colorable, $P_{x_2}(i, j, \varphi_4) = P_y(i, j, \varphi_4)$ which contains neither z_k nor y_k.

Let $\varphi_5 = \varphi_4/P_{x_2}(i, j, \varphi_4)$. Then $k \in \bar{\varphi_5}(x_2)$ and $j \in \bar{\varphi_5}(y) \cap \bar{\varphi_5}(y_k)$.

Let φ_6 be obtained from φ_5 by recoloring yy_k with j. Then $2 \in \bar{\varphi}_6(y)$.

Let φ_7 be the coloring obtained from φ_6 by uncoloring x_1x_2 and coloring x_2y with color 2. Then $\varphi_7(x_1z) = 1 \in \bar{\varphi_7}(x_2)$, $\varphi_7(x_2y) = 2 \in \bar{\varphi}_7(x_1)$, and $\varphi_7(zz_k) = k \in \bar{\varphi_7}(x_2)$. Thus φ_7 is zy-feasible, and $k \in C_z(\varphi_7)$ since $d(z_k) < q$. Since $i, j, k \notin R(\varphi_1)$ and $\varphi_7(x_2y) = \varphi_1(x_2y) = 2$, the colors in $R(\varphi_1)$ are unchanged as a result of this sequence of recolorings, and so $C_z(\varphi_7) \supseteq C_z(\varphi_1) \cup \{k\}$ and $C_y(\varphi_7) \supseteq C_y(\varphi_1)$. It follows that $|C_y(\varphi_7)| + |C_z(\varphi_7)| \ge |C_y(\varphi_1)| + |C_z(\varphi_1)| + 1$, and this contradicts the optimality of φ_1. So **II** holds.

This completes the proof of Lemma 5, and hence of Theorem 1. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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