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Spanning bipartite graphs with high degree sum in graphs

Guantao Chen [a], Shuya Chiba [b], Ronald J. Gould [c], Xiaofeng Gu [d], Akira Saito [e,*], Masao Tsugaki, Tomoki Yamashita [f]

[a] Department of Mathematics and Statistics, Georgia State University, Atlanta, GA 30303, United States of America
[b] Applied Mathematics, Faculty of Advanced Science and Technology, Kumamoto University, 2–39–1 Kurokami, Kumamoto 860-8555, Japan
[c] Department of Mathematics and Computer Science, Emory University, Atlanta, GA 30322, United States of America
[d] Department of Mathematics, University of West Georgia, Carrollton, GA 30118, United States of America
[e] Department of Information Science, Nihon University, Sakurajosui 3-25-40, Setagaya-Ku, Tokyo 156-8550, Japan
[f] Department of Science, Kindai University, 3–4–1 Kowakae, Higashi-Osaka, Osaka 577-8502, Japan

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ABSTRACT

The classical Ore's Theorem states that every graph G of order $n \geq 3$ with $\sigma_2(G) \geq n$ is hamiltonian, where $\sigma_2(G) = \min\{d_G(x) + d_G(y): x, y \in V(G), x \neq y, xy \notin E(G)\}$. Recently, Ferrara, Jacobson and Powell (Discrete Math. 312 (2012), 459–461) extended the Moon–Moser Theorem and characterized the non-hamiltonian balanced bipartite graphs H of order $2n \geq 4$ with partite sets X and Y satisfying $\sigma_{1,1}(H) \geq n$, where $\sigma_{1,1}(H) = \min\{d_H(x)+d_H(y): x \in X, y \in Y, xy \notin E(H)\}$. Though the latter result apparently deals with a narrower class of graphs, we prove in this paper that it implies Ore's Theorem for graphs of even order.

1. Introduction

In this paper, we only consider finite simple graphs. For standard graph-theoretic notation and terminology not explained in this paper, we refer the reader to [1]. For $v \in V(G)$, let $N_G(v)$ and $d_G(v)$ denote the neighborhood and the degree of v in G, respectively. If H is a subgraph of G, we write $H \subseteq G$. For a graph G and $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of G induced by X.

Let X and Y be disjoint sets of vertices in G. Then let $E_G(X, Y)$ denote the set of edges $e = xy$ with $x \in X$ and $y \in Y$, and let $e_G(X, Y) = |E_G(X, Y)|$. Furthermore, $G[X, Y]$ is the graph defined by $V(G[X, Y]) = X \cup Y$ and $E(G[X, Y]) = E_G(X, Y)$. Note that $G[X, Y]$ is a bipartite graph with partite sets X and Y.

When no confusion results, we often identify a singleton set with its element. For example, if $x \in V(G)$, we write $e_G(x, Y)$ instead of $e_G(\{x\}, Y)$. If, in addition, $Y = \{y\}$, we write $e_G(x, y)$ instead of $e_G(\{x\}, \{y\})$. Note that the value of $e_G(x, y)$ is either 0 or 1 since we only consider simple graphs.

Degree sum is a topic which has been studied actively in the theory of hamiltonicity. It deals with the minimum sum of degrees of vertices in certain independent sets and relates with hamiltonian properties of graphs. One of the most well-known results in this topic is Ore's Theorem. For a non-complete graph G, we define $\sigma_2(G)$ by

$$\sigma_2(G) = \min\{d_G(x) + d_G(y): x, y \in V(G), x \neq y, xy \notin E(G)\}.$$

If G is a complete graph, we define $\sigma_2(G) = +\infty$.

* Corresponding author.
E-mail addresses: gchen@gsu.edu (G. Chen), schiba@kumamoto-u.ac.jp (S. Chiba), rg@mathcs.emory.edu (R.J. Gould), xgu@westga.edu (X. Gu), asaito@chs.nihon-u.ac.jp (A. Saito), tsugaki@hotmail.com (M. Tsugaki), yamashita@math.kindai.ac.jp (T. Yamashita).

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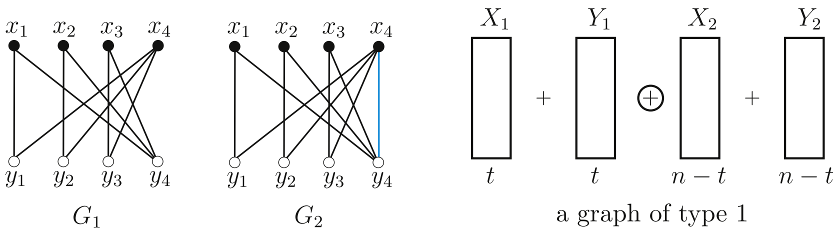

Fig. 1. Graphs of type 1 and type 2.

Theorem A (*Ore's Theorem [4]*). *For $n \geq 3$, every graph G of order n with $\sigma_2(G) \geq n$ is hamiltonian.*

Moon and Moser [3] investigated a degree sum condition for hamiltonicity in bipartite graphs. A bipartite graph is said to be *balanced* if its partite sets have the same order. Trivially, a bipartite graph contains a hamiltonian cycle only if it is balanced. Also, according to the spirit of Ore's Theorem, it may not be appropriate to incorporate the degree sum of vertices chosen from the same partite set. Actually, Moon and Moser only considered the degree sum of pairs of vertices taken from different partite sets. Let G be a bipartite graph with partite sets X and Y. If G is not a complete bipartite graph, we define $\sigma_{1,1}(G)$ by

$$\sigma_{1,1}(G) = \min\{d_G(x) + d_G(y) : x \in X, y \in Y, xy \notin E(G)\}.$$

If G is a complete bipartite graph, we define $\sigma_{1,1}(G) = +\infty$.

Theorem B (*Moon and Moser [3]*). *For $n \geq 2$, every balanced bipartite graph G of order $2n$ with $\sigma_{1,1}(G) \geq n+1$ is hamiltonian.*

Observing Theorems A and B, we may want to relax the condition $\sigma_{1,1}(G) \geq n+1$ in Theorem B to $\sigma_{1,1}(G) \geq n$. However, we cannot do it without allowing exceptions. Let n and t be integers with $n \geq 2$ and $1 \leq t \leq n-1$. Then following [2], we define $H_{t,n-t}$ to be the graph formed from $K_{t,t} \cup K_{n-t,n-t}$ by selecting one partite set of each complete bipartite graph and adding all possible edges between these sets. Then every graph G with $K_{t,t} \cup K_{n-t,n-t} \subseteq G \subseteq H_{t,n-t}$ is a bipartite graph of order $2n$ and satisfies $\sigma_{1,1}(G) = n$, but it is not hamiltonian. Also, let G_1 and G_2 be the graphs depicted in Fig. 1. Then G_i ($i = 1, 2$) is a bipartite graph of order 8 and satisfies $\sigma_{1,1}(G_i) = 4$, but it is not hamiltonian.

The above graphs arise as counterexamples if we relax the degree sum condition $\sigma_{1,1}(G) \geq n+1$ to $\sigma_{1,1}(G) \geq n$. However, Ferrara, Jacobson and Powell [2] proved that these are the only exceptions.

Theorem C (*[2]*). *Let n be an integer with $n \geq 2$ and let G be a balanced bipartite graph of order $2n$ with $\sigma_{1,1}(G) \geq n$. Then one of the following holds.*

(1) G is hamiltonian.
(2) $K_{t,t} \cup K_{n-t,n-t} \subseteq G \subseteq H_{t,n-t}$ for some integer t with $1 \leq t \leq n-1$.
(3) G is isomorphic to G_1 or G_2.

In this paper, we study the relationship between Ore's Theorem and Theorem C. Theorem C only deals with bipartite graphs, while Ore's Theorem handles both bipartite and non-bipartite graphs. Apparently, Ore's Theorem concerns a broader class of graphs. However, we prove that Theorem C implies Ore's Theorem. If a graph G of order $2n$ satisfies $K_{t,t} \cup K_{n-t,n-t} \subseteq G \subseteq H_{t,n-t}$ for some t with $1 \leq t \leq n-1$, we call G a graph *of type* 1. Also, we say that a graph G is *of type* 2 if G is isomorphic to either G_1 or G_2. See Fig. 1, where the symbol '+ ' means that every vertex on the left is joined to every vertex on the right by an edge, while '$\oplus$' means that there may exist an edge joining a vertex on the left and a vertex on the right.

Theorem 1. *Let n be an integer with $n \geq 2$ and let G be a graph of order $2n$. If $\sigma_2(G) \geq 2n$, then G contains a spanning balanced bipartite graph H such that*

(1) $\sigma_{1,1}(H) \geq n$, and
(2) H is of neither type 1 nor type 2.

For a graph of even order satisfying Ore's condition, Theorem 1 gives more detailed information than the existence of a hamiltonian cycle.

In the next section, we give a proof to Theorem 1. In Section 3, we give concluding remarks.

2. Proof of Theorem 1

In the subsequent arguments, we frequently use the following observations. The proof is an easy calculation and we omit it.

Lemma 2. *Let G be a graph and let X and Y be disjoint nonempty subsets of $V(G)$. Let $x \in X$ and $y \in Y$. Then*

(1) $e_G(X \setminus \{x\}, Y \cup \{x\}) = e_G(X, Y) - e_G(x, Y) + e_G(x, X \setminus \{x\})$ *and*
(2) $e_G\big((X \setminus \{x\}) \cup \{y\}, (Y \setminus \{y\}) \cup \{x\}\big) = e_G(X, Y) - e_G(x, Y) - e_G(y, X) + e_G(x, X \setminus \{x\}) + e_G(y, Y \setminus \{y\}) + 2e_G(x, y)$.

A partition $\{X, Y\}$ of the vertex set $V(G)$ of a graph G of even order is said to be *balanced* if $|X| = |Y| = \frac{1}{2}|V(G)|$. A balanced partition $\{X, Y\}$ is said to be a *maximal partition* of G if $e_G(X', Y') \le e_G(X, Y)$ holds for every balanced partition $\{X', Y'\}$ of $V(G)$. The next lemma acts as a basis of our proof.

Lemma 3. *Let G be a graph of even order. Then $\sigma_{1,1}(G[X, Y]) \ge \frac{1}{2}\sigma_2(G)$ holds for every maximal partition $\{X, Y\}$ of G.*

Proof. Let $\{X, Y\}$ be a maximal partition of G, and let $H = G[X, Y]$. Since there is nothing to prove if H is a complete bipartite graph, we assume that H is not a complete bipartite graph. Let $x \in X$ and $y \in Y$ with $xy \notin E(H)$ and $d_H(x) + d_H(y) = \sigma_{1,1}(H)$. Note $d_H(x) = e_G(x, Y)$ and $d_H(y) = e_G(y, X)$.

Let $X' = (X \setminus \{x\}) \cup \{y\}$ and $Y' = (Y \setminus \{y\}) \cup \{x\}$. By Lemma 2 (2),

$$e_G(X', Y') = e_G(X, Y) + e_G(x, X \setminus \{x\}) + e_G(y, Y \setminus \{y\}) - e_G(x, Y) - e_G(y, X).$$

Since $d_G(x) = e_G(x, X \setminus \{x\}) + e_G(x, Y) = e_G(x, X \setminus \{x\}) + d_H(x)$ and $d_G(y) = e_G(y, Y \setminus \{y\}) + e_G(y, X) = e_G(y, Y \setminus \{y\}) + d_H(y)$, it follows that $e_G(X', Y') - e_G(X, Y) = d_G(x) + d_G(y) - 2(d_H(x) + d_H(y))$. Moreover, since $\{X, Y\}$ is a maximal partition of G, we have $e_G(X', Y') \le e_G(X, Y)$. Therefore, $d_G(x) + d_G(y) - 2(d_H(x) + d_H(y)) \le 0$, which yields

$$2\sigma_{1,1}(H) = 2(d_H(x) + d_H(y)) \ge d_G(x) + d_G(y) \ge \sigma_2(G). \quad \square$$

By Lemma 3, if G is a graph of order $2n$ with $\sigma_2(G) \ge 2n$, then $\sigma_{1,1}(G[X, Y]) \ge n$ holds for every maximal partition $\{X, Y\}$ of G.

In the proof of Theorem 1, we will find a required graph as $G[X, Y]$ for some balanced partition $\{X, Y\}$. The next lemma says that when we deal with maximal partitions in the proof, a graph of type 2 does not arise.

Lemma 4. *Let G be a graph of order 8 with $\sigma_2(G) \ge 8$. Then $G[X, Y]$ is not a graph of type 2 for any maximal partition $\{X, Y\}$ of G.*

Proof. Assume $G[X, Y]$ is a graph of type 2 for some maximal partition $\{X, Y\}$ of G. Let $H = G[X, Y]$. Label the vertices of H as in G_1 in Fig. 1, where possibly the edge x_4y_4 exists as in G_2. We may assume $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_1, y_2, y_3, y_4\}$.

Claim. *For each pair of distinct indices i and j with $\{i, j\} \subseteq \{1, 2, 3\}$, $e_G(x_i, X \setminus \{x_i\}) + e_G(y_j, Y \setminus \{y_j\}) = 4$.*

Proof. Note $x_iy_j \notin E(G)$ and $e_G(x_i, Y) = e_G(y_j, X) = 2$. Let $X' = (X \setminus \{x_i\}) \cup \{y_j\}$ and $Y' = (Y \setminus \{y_j\}) \cup \{x_i\}$. By Lemma 2 (2),

$$\begin{aligned} e_G(X', Y') &= e_G(X, Y) - e_G(x_i, Y) - e_G(y_j, X) + e_G(x_i, X \setminus \{x_i\}) + e_G(y_j, Y \setminus \{y_j\}) \\ &= e_G(X, Y) + e_G(x_i, X \setminus \{x_i\}) + e_G(y_j, Y \setminus \{y_j\}) - 4. \end{aligned}$$

Since $\{X, Y\}$ is a maximal partition, we have $e_G(X', Y') \le e_G(X, Y)$, which implies $e_G(x_i, X \setminus \{x_i\}) + e_G(y_j, Y \setminus \{y_j\}) \le 4$.

Since $\sigma_2(G) \ge 8$ and $x_iy_j \notin E(G)$, we have $d_G(x_i) + d_G(y_j) \ge 8$. On the other hand,

$$\begin{aligned} d_G(x_i) + d_G(y_j) &= e_G(x_i, X \setminus \{x_i\}) + e_G(x_i, Y) + e_G(y_j, Y \setminus \{y_j\}) + e_G(y_j, X) \\ &= e_G(x_i, X \setminus \{x_i\}) + e_G(y_j, Y \setminus \{y_j\}) + 4. \end{aligned}$$

These imply $e_G(x_i, X \setminus \{x_i\}) + e_G(y_j, Y \setminus \{y_1\}) \ge 4$. Therefore, we have $e_G(x_i, X \setminus \{x_i\}) + e_G(y_j, Y \setminus \{y_j\}) = 4$. $\square$

By applying Claim with $(i, j) = (1, 2)$ and $(i, j) = (3, 1)$, we have $e_G(x_1, X \setminus \{x_1\}) + e_G(y_2, Y \setminus \{y_2\}) = 4$ and $e_G(x_3, X \setminus \{x_3\}) + e_G(y_1, Y \setminus \{y_1\}) = 4$. By adding them, we have

$$e_G(x_1, X \setminus \{x_1\}) + e_G(y_1, Y \setminus \{y_1\}) + e_G(x_3, X \setminus \{x_3\}) + e_G(y_2, Y \setminus \{y_2\}) = 8.$$

On the other hand, $e_G(x_3, X \setminus \{x_3\}) + e_G(y_2, Y \setminus \{y_2\}) = 4$ by Claim with $(i, j) = (3, 2)$. Therefore, we have $e_G(x_1, X \setminus \{x_1\}) + e_G(y_1, Y \setminus \{y_1\}) = 4$.

Now let $X'' = (X \setminus \{x_1\}) \cup \{y_1\}$ and $Y'' = (Y \setminus \{y_1\}) \cup \{x_1\}$, and apply Lemma 2 (2) to (X'', Y''). Since $x_1y_1 \in E(G)$, we have

$$\begin{aligned} e_G(X'', Y'') &= e_G(X, Y) - e_G(x_1, Y) - e_G(y_1, X) + e_G(x_1, X \setminus \{x_1\}) + e_G(y_1, Y \setminus \{y_1\}) + 2 \\ &= e_G(X, Y) - 2 - 2 + 4 + 2 = e_G(X, Y) + 2. \end{aligned}$$

This contradicts the maximality of $\{X, Y\}$, and hence the lemma follows. $\square$

We now prove Theorem 1. By Lemmas 3 and 4, if we take a maximal partition $\{X, Y\}$ in a graph G of order $2n$ with $\sigma_2(G) \ge 2n$, then $\sigma_{1,1}(G[X, Y]) \ge n$ and $G[X, Y]$ is not a graph of type 2. Based on this observation, in the proof of

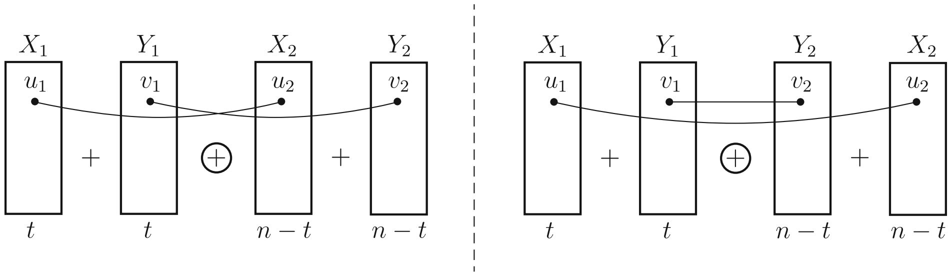

Fig. 2. Proof of Claim 2.

Theorem 1, we first try to find a maximal partition $\{X, Y\}$ such that $G[X, Y]$ is not a graph of type 1. If we find one, $G[X, Y]$ is a required spanning subgraph of G. However, in some cases, we fail to find such a maximal partition. If it happens, we will search for a required partition $\{X, Y\}$ in the broader set of balanced partitions. In this case, Lemmas 3 and 4 do not help us, and we will give a specific proof to confirm that $G[X, Y]$ has the required property.

Proof of Theorem 1. Let G be a graph of order $2n$ with $\sigma_2(G) \geq 2n$, and assume G does not satisfy the conclusion. Then for every balanced partition $\{X, Y\}$ of G, either $\sigma_{1,1}(G[X, Y]) < n$ or $G[X, Y]$ is a graph of either type 1 or type 2.

Take a maximal partition $\{X, Y\}$ of G. Then by Lemmas 3 and 4, $G[X, Y]$ is a graph of type 1, which means $K_{t,t} \cup K_{n-t,n-t} \subseteq G[X, Y] \subseteq H_{t,n-t}$ for some t with $1 \leq t \leq n-t$. Let X_1 and Y_1 be the partite sets of $K_{t,t}$ and X_2 and Y_2 be the partite sets of $K_{n-t,n-t}$. By symmetry, we may assume $E_G(Y_1, X_2) \subseteq E(H_{t,n-t})$ (see Fig. 1). Then we have

(C1) $xy \in E(G)$ for every $x \in X_1$ and $y \in Y_1$,
(C2) $xy \in E(G)$ for every $x \in X_2$ and $y \in Y_2$, and
(C3) $E_G(X_1, Y_2) = \emptyset$.

We may assume $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$.

Now we take arbitrary vertices $x_1 \in X_1$ and $y_2 \in Y_2$ and fix them. Also we define X', Y', X_1^- and Y_2^- by

$$\begin{aligned} X' &= \big(X \setminus \{x_1\}\big) \cup \{y_2\}, \\ Y' &= \big(Y \setminus \{y_2\}\big) \cup \{x_1\}, \\ X_1^- &= X_1 \setminus \{x_1\} \text{ and} \\ Y_2^- &= Y_2 \setminus \{y_2\}. \end{aligned}$$

We will prove a series of claims.

Claim 1. *$e_G(x_1, X \setminus \{x_1\}) + e_G(y_2, Y \setminus \{y_2\}) = n$, and $\{X', Y'\}$ is a maximal partition.*

Proof. Note $x_1y_2 \notin E(G)$ by (C3). Hence $d_G(x_1)+d_G(y_2) \geq 2n$ by the hypothesis. Also note $e_G(x_1, Y) = t$ and $e_G(y_2, X) = n-t$ by (C1), (C2), and (C3). Therefore,

$$\begin{aligned} e_G(x_1, X \setminus \{x_1\}) + e_G(y_2, Y \setminus \{y_2\}) &= d_G(x_1) - e_G(x_1, Y) + d_G(y_2) - e_G(y_2, X) \\ &\geq 2n - t - (n-t) = n. \end{aligned}$$

On the other hand, by Lemma 2 (2),

$$\begin{aligned} e_G(X', Y') &= e_G(X, Y) - e_G(x_1, Y) - e_G(y_2, X) + e_G(x_1, X \setminus \{x_1\}) + e_G(y_2, Y \setminus \{y_2\}) \\ &= e_G(X, Y) - t - (n-t) + e_G(x_1, X \setminus \{x_1\}) + e_G(y_2, Y \setminus \{y_2\}) \\ &= e_G(X, Y) - n + e_G(x_1, X \setminus \{x_1\}) + e_G(y_2, Y \setminus \{y_2\}). \end{aligned}$$

Since $\{X, Y\}$ is a maximal partition, $e_G(X', Y') \leq e_G(X, Y)$ and hence we have $e_G(x_1, X \setminus \{x_1\}) + e_G(y_2, Y \setminus \{y_2\}) \leq n$. Thus, we have $e_G(x_1, X \setminus \{x_1\}) + e_G(y_2, Y \setminus \{y_2\}) = n$ and $e_G(X', Y') = e_G(X, Y)$. In particular, $\{X', Y'\}$ is a maximal partition. □

Claim 2. *$e_G(X_1, X_2) = 0$ or $e_G(Y_1, Y_2) = 0$.*

Proof. Assume $e_G(X_1, X_2) > 0$ and $e_G(Y_1, Y_2) > 0$. Then there exist vertices $u_1 \in X_1$, $u_2 \in X_2$, $v_1 \in Y_1$ and $v_2 \in Y_2$ with $u_1u_2, v_1v_2 \in E(G)$. Let $\hat{X} = X_1 \cup Y_2$, $\hat{Y} = Y_1 \cup X_2$ and $\hat{H} = G[\hat{X}, \hat{Y}]$. Note that $\{\hat{X}, \hat{Y}\}$ is a balanced partition of $V(G)$ though we do not know whether it is a maximal partition of G (see Fig. 2).

Take $\hat{x} \in \hat{X}$ and $\hat{y} \in \hat{Y}$ with $\hat{x}\hat{y} \notin E(\hat{H})$. By (C1) and (C2), $\hat{H}[X_1, Y_1] = H[X_1, Y_1]$ and $\hat{H}[Y_2, X_2] = H[X_2, Y_2]$ are balanced complete bipartite graphs of order $2t$ and $2(n-t)$, respectively. Therefore, $\{\hat{x}, \hat{y}\} \not\subseteq X_1 \cup Y_1$ and $\{\hat{x}, \hat{y}\} \not\subseteq X_2 \cup Y_2$, which

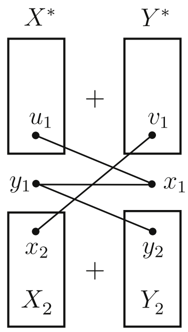

Fig. 3. H^*.

imply $\{\hat{x}, \hat{y}\} \cap (X_1 \cup Y_1) \neq \emptyset$ and $\{\hat{x}, \hat{y}\} \cap (X_2 \cup Y_2) \neq \emptyset$. Hence we have $d_{\hat{H}}(\hat{x}) + d_{\hat{H}}(\hat{y}) \geq t + (n - t) = n$, which implies $\sigma_{1,1}(\hat{H}) \geq n$. Moreover, $\hat{H}[X_i, Y_i]$ contains a hamiltonian path P_i joining u_i and v_i for $i \in \{1, 2\}$. Then $u_1P_1v_1v_2P_2u_2u_1$ is a hamiltonian cycle in $\hat{H}$. Thus, $\hat{H}$ is of neither type 1 nor type 2. This is a contradiction. □

By Claim 2 and the symmetry, we may assume

$$e_G(X_1, X_2) = 0. \qquad (*)$$

Claim 3. *We have $e_G(x_1, X_1^-) \geq 1$ and $e_G(y_2, Y_1) \geq 2$. In particular, $|X_1| \geq 2$.*

Proof. By Claim 1, $e_G(x_1, X \setminus \{x_1\}) + e_G(y_2, Y \setminus \{y_2\}) = n$. Since $|Y \setminus \{y_2\}| = n - 1$, we have $e_G(y_2, Y \setminus \{y_2\}) \leq n - 1$. Therefore, since $e_G(X_1, X_2) = 0$ by (∗), $e_G(x_1, X_1^-) = e_G(x_1, X \setminus \{x_1\}) = n - e_G(y_2, Y \setminus \{y_2\}) \geq 1$.

Since $e_G(x_1, X \setminus \{x_1\}) = e_G(x_1, X_1 \setminus \{x_1\}) \leq |X_1| - 1 = t - 1$ and $e_G(y_2, Y_2 \setminus \{y_2\}) \leq |Y_2| - 1 = n - t - 1$, we have

$$\begin{aligned} n &= e_G(x_1, X \setminus \{x_1\}) + e_G(y_2, Y \setminus \{y_2\}) \\ &= e_G(x_1, X \setminus \{x_1\}) + e_G(y_2, Y_1) + e_G(y_2, Y_2 \setminus \{y_2\}) \\ &\leq t - 1 + (n - t - 1) + e_G(y_2, Y_1) = n - 2 + e_G(y_2, Y_1), \end{aligned}$$

which yields $e_G(y_2, Y_1) \geq 2$. □

For a pair of distinct vertices y_1, v_1 in Y_1, we call (y_1, v_1) a *violating pair* if it satisfies the following two conditions (P1) and (P2), where $X^* = (X \setminus \{x_1\}) \cup \{y_1\}$, $Y^* = (Y \setminus \{y_1\}) \cup \{x_1\}$ and $H^* = G[X^*, Y^*]$.

(P1) $Y_2 \subseteq N_G(y_1)$ and $X_2 \subseteq N_G(v_1)$.
(P2) $d_{H^*}(x_1) + d_{H^*}(x^*) \geq n$ for any $x^* \in X^* \setminus \{y_1\}$ with $x_1x^* \notin E(G)$.

Claim 4. *There does not exist a violating pair.*

Proof. Assume that Y_1 contains a violating pair (y_1, v_1). Let $X^* = (X \setminus \{x_1\}) \cup \{y_1\}$, $Y^* = (Y \setminus \{y_1\}) \cup \{x_1\}$ and $H^* = G[X^*, Y^*]$. Note that $\{X^*, Y^*\}$ is a balanced partition of $V(G)$ though we do not know whether it is a maximal partition of G.

By (C2) and (P1), $Y_2 \cup \{v_1\} \subseteq N_G(x_2^*)$ for each $x_2^* \in X_2$ and $X_2 \cup \{y_1\} \subseteq N_G(y_2^*)$ for each $y_2^* \in Y_2$. In particular, we have

- $d_{H^*}(x_2^*) \geq n - t + 1$ for each $x_2^* \in X_2$, and
- $d_{H^*}(y_2^*) \geq n - t + 1$ for each $y_2^* \in Y_2$.

Moreover, by (C1) $x_1 \in N_G(y_1)$ and hence, by (P1), $Y_2 \cup \{x_1\} \subseteq N_G(y_1)$. In particular.

- $d_{H^*}(y_1) \geq n - t + 1$.

We claim $\sigma_{1,1}(H^*) \geq n$. Take $x^* \in X^*$ and $y^* \in Y^*$ with $x^*y^* \notin E(H^*)$. Note $X^* = (X_1 \setminus \{x_1\}) \cup X_2 \cup \{y_1\}$ and $Y^* = (Y_1 \setminus \{y_1\}) \cup Y_2 \cup \{x_1\}$ (see Fig. 3).

If $y^* \in Y_1 \setminus \{y_1\}$, then since $X_1 \setminus \{x_1\} \subseteq N_G(y^*)$ by (C1), $d_{H^*}(y^*) \geq t - 1$ and $x^* \notin X_1 \setminus \{x_1\}$. Hence $x^* \in X_2 \cup \{y_1\}$, which implies $d_{H^*}(x^*) \geq n - t + 1$. Thus, we have $d_{H^*}(x^*) + d_{H^*}(y^*) \geq n$.

If $y^* \in Y_2$, then $d_{H^*}(y^*) \geq n - t + 1$. Moreover, since $X_2 \subseteq N_G(y^*)$ by (C2) and $Y_2 \subseteq N_G(y_1)$ by (P1), we have $x^* \notin X_2 \cup \{y_1\}$, which implies $x^* \in X_1 \setminus \{x_1\}$. Then $Y_1 \setminus \{y_1\} \subseteq N_G(x^*)$ and hence $d_{H^*}(x^*) \geq t - 1$. Thus, we have $d_{H^*}(x^*) + d_{H^*}(y^*) \geq n$.

Finally, if $y^* = x_1$, then since $y_1 \in N_G(x_1)$, we have $x^* \neq y_1$. Then $d_{H^*}(x^*) + d_{H^*}(y^*) = d_{H^*}(x^*) + d_{H^*}(x_1) \geq n$ by (P2). Therefore, we have $\sigma_{1,1}(H^*) \geq n$.

Take $x_2 \in X_2$. By (P1), $\{y_1y_2, v_1x_2\} \subseteq E(G)$ and hence $\{y_1y_2, v_1x_2\} \subseteq E(H^*)$. Also, $x_1y_1 \in E(H^*)$ by (C1). By Claim 3, there exists a vertex $u_1 \in X_1 \setminus \{x_1\}$ with $u_1x_1 \in E(G)$, which implies $u_1x_1 \in E(H^*)$.

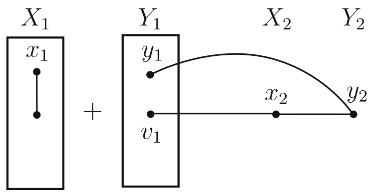

Fig. 4. Proof of Claim 5.

Since $H^*[X_1\setminus\{x_1\}, Y_1\setminus\{y_1\}] = G[X_1\setminus\{x_1\}, Y_1\setminus\{y_1\}]$ is a balanced complete bipartite graph, it contains a hamiltonian path P_1 joining u_1 and v_1. Also since $H^*[X_2, Y_2] = G[X_2, Y_2]$ is a balanced complete bipartite graph, it contains a hamiltonian path P_2 joining x_2 and y_2. Then $u_1P_1v_1x_2P_2y_2y_1x_1u_1$ is a hamiltonian cycle of H^*. This implies that H^* is of neither type 1 nor type 2, which is a contradiction. □

Claim 5. $|Y_2| \geq 2$.

Proof. Assume the contrary. Then $Y_2 = \{y_2\}$ and $t = |X_1| = n - 1$. Let $X_2 = \{x_2\}$.

Assume $N_G(x_2) \cap Y_1 = \emptyset$. Let $\hat{X} = X_1 \cup \{y_2\} = \big(X \setminus \{x_2\}\big) \cup \{y_2\}$ and $\hat{Y} = Y_1 \cup \{x_2\} = \big(Y \setminus \{y_2\}\big) \cup \{x_2\}$. Then $\{\hat{X}, \hat{Y}\}$ is a balanced partition of $V(G)$. We have $e_G(x_2, Y) = e_G(x_2, y_2) = 1$ by the assumption, $e_G(y_2, X) = e_G(y_2, x_2) = 1$ by (C2) and (C3), $e_G(x_2, X\setminus\{x_2\}) = e_G(x_2, X_1) = 0$ by (∗) and $e_G(y_2, Y\setminus\{y_2\}) = e_G(y_2, Y_1) \geq 2$ by Claim 3. Then by applying Lemma 2 (2) to $\hat{X}$ and $\hat{Y}$, we have

$$\begin{aligned} e_G(\hat{X}, \hat{Y}) &= e_G(X, Y) - e_G(x_2, Y) - e_G(y_2, X) + e_G(x_2, X \setminus \{x_2\}) + e_G(y_2, Y \setminus \{y_2\}) + 2e_G(x_2, y_2) \\ &\geq e_G(X, Y) - 1 - 1 + 0 + 2 + 2 = e_G(X, Y) + 2. \end{aligned}$$

This contradicts the maximality of $\{X, Y\}$. Hence we have $N_G(x_2) \cap Y_1 \neq \emptyset$. Take $v_1 \in N_G(x_2) \cap Y_1$. By Claim 3, we can take $y_1 \in N_G(y_2) \cap (Y_1 \setminus \{v_1\})$ (see Fig. 4).

We claim that (y_1, v_1) is a violating pair. Since $X_2 = \{x_2\}$ and $Y_2 = \{y_2\}$, y_1 and v_1 satisfy (P1). Let $X^* = \big(X\setminus\{x_1\}\big)\cup\{y_1\}$, $Y^* = \big(Y \setminus \{y_1\}\big) \cup \{x_1\}$ and $H^* = G[X^*, Y^*]$. By Claim 3, $N_G(x_1) \cap (X_1 \setminus \{x_1\}) \neq \emptyset$. We also have $x_1y_1 \in E(G)$ by (C1) and hence $d_{H^*}(x_1) \geq 2$.

Let $x^* \in X^* \setminus \{y_1\} = X \setminus \{x_1\} = \big(X_1 \setminus \{x_1\}\big) \cup \{x_2\}$ with $x_1x^* \notin E(G)$. If $x^* \in X_1 \setminus \{x_1\}$, then $d_{H^*}(x^*) \geq |Y_1 \setminus \{y_1\}| = n - 2$ by (C1) and $d_{H^*}(x_1) + d_{H^*}(x^*) \geq 2 + (n - 2) = n$.

Suppose $x^* = x_2$. Then by (∗), $x_1x_2 \notin E(G)$ and hence $d_G(x_1)+d_G(x_2) \geq 2n$ since $\sigma_2(G) \geq 2n$. On the other hand, by (C1) and (C3), $e_G(x_1, Y \setminus \{y_1\}) = e_G(x_1, Y_1 \setminus \{y_1\}) = |Y_1| - 1 = n - 2$, and by (∗), $e_G(x_2, (X_1 \setminus \{x_1\}) \cup \{y_1\}) = e_G(x_2, y_1) \leq 1$. Therefore,

$$\begin{aligned} d_{H^*}(x_1) + d_{H^*}(x_2) &= d_G(x_1) - e_G(x_1, Y \setminus \{y_1\}) + d_G(x_2) - e_G(x_2, (X_1 \setminus \{x_1\}) \cup \{y_1\}) \\ &\geq 2n - (n - 2) - 1 = n + 1. \end{aligned}$$

Thus, (P2) is satisfied and hence $\{y_1, v_1\}$ is a violating pair. This contradicts Claim 4, and the claim follows. □

Recall $X' = \big(X \setminus \{x_1\}\big) \cup \{y_2\}$ and $Y' = \big(Y \setminus \{y_2\}\big) \cup \{x_1\}$. Since $\{X', Y'\}$ is a maximal partition of G by Claim 1, $G[X', Y']$ is a graph of type 1 by the assumption. Thus, $K_{s,s} \cup K_{n-s,n-s} \subseteq G[X', Y'] \subseteq H_{s,n-s}$ for some s with $1 \leq s \leq n - 1$. Let X'_1 and Y'_1 be the partite sets of $K_{s,s}$ and X'_2 and Y'_2 be the partite sets of $K_{n-s,n-s}$, where $E_G(Y'_1, X'_2) \subseteq E(H_{s,n-s})$ and $E_G(X'_1, Y'_2) = \emptyset$. Thus, we can apply (C1), (C2) and (C3) to (X'_1, Y'_1, X'_2, Y'_2) instead of (X_1, Y_1, X_2, Y_2). We may assume $X' = X'_1 \cup X'_2$ and $Y' = Y'_1 \cup Y'_2$.

Claim 6. *We have $X_1^- \subseteq X'_2$, $X'_1 \subseteq X_2 \cup \{y_2\}$, $Y_2^- \subseteq Y'_1$, and $Y'_2 \subseteq Y_1 \cup \{x_1\}$.*

Proof. Note $X = (X' \setminus \{y_2\}) \cup \{x_1\}$ and $Y = (Y' \setminus \{x_1\}) \cup \{y_2\}$.

We first prove the following subclaim.

Subclaim. $X'_1 \cap X_1^- = \emptyset$.

Proof. Assume $X'_1 \cap X_1^- \neq \emptyset$, and take $x'_1 \in X'_1 \cap X_1^-$. We investigate the inclusion relationships between $\{X_1^-, Y_1, X_2, Y_2^-\}$ and $\{X'_1, Y'_1, X'_2, Y'_2\}$.

Take $v_1 \in Y_1$. Then $v_1 \neq y_2$ and hence $v_1 \in Y' = Y'_1 \cup Y'_2$. Since $x'_1 \in X_1$, we have $x'_1v_1 \in E(G)$ by (C1) for (X_1, Y_1, X_2, Y_2). On the other hand, $x'_1 \in X'_1$ and hence $N_G(x'_1) \cap Y'_2 = \emptyset$ by (C3) for (X'_1, Y'_1, X'_2, Y'_2). Therefore, we have $v_1 \notin Y'_2$ and hence $v_1 \in Y'_1$. This proves $Y_1 \subseteq Y'_1$.

Take $v_2 \in Y_2^-$. Then $v_2 \in Y' = Y'_1 \cup Y'_2$. Since $x'_1 \in X_1$, $v_2 \notin N_G(x'_1)$ by (C3) for (X_1, Y_1, X_2, Y_2). On the other hand, since $x'_1 \in X'_1$, we have $Y'_1 \subseteq N_G(x'_1)$ by (C1) for (X'_1, Y'_1, X'_2, Y'_2). These imply $v_2 \notin Y'_1$ and hence $v_2 \in Y'_2$. This proves $Y_2^- \subseteq Y'_2$.

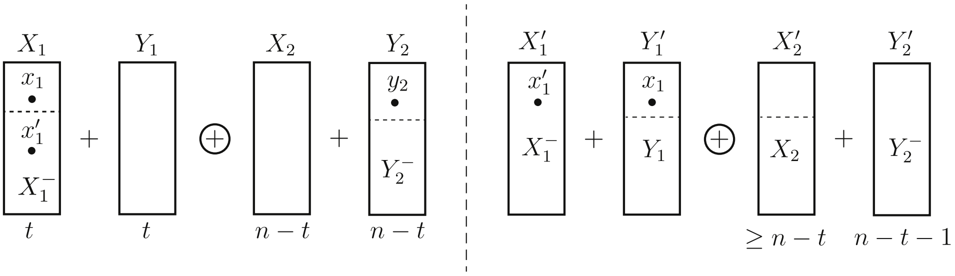

Fig. 5. Proof of Subclaim.

By Claim 5, $|Y_2| \geq 2$ and hence $Y_2^- \neq \emptyset$. Take $y'_2 \in Y_2^-$.

Take $u_1 \in X_1^-$. Then $u_1 \in X' = X'_1 \cup X'_2$. Since $u_1 \in X_1$ and $y'_2 \in Y_2$, $u_1 \notin N_G(y'_2)$ by (C3) for (X_1, Y_1, X_2, Y_2). On the other hand, since $Y_2^- \subseteq Y'_2$, we have $X'_2 \subseteq N_G(y'_2)$ by (C2) for (X'_1, Y'_1, X'_2, Y'_2). Therefore, we have $u_1 \notin X'_2$ and hence $u_1 \in X'_1$. This proves $X_1^- \subseteq X'_1$.

Take $u_2 \in X_2$. Then since $u_2 \neq x_1$, $u_2 \in X' = X'_1 \cup X'_2$. Since $y'_2 \in Y_2$, $u_2 \in N_G(y'_2)$ by (C2) for (X_1, Y_1, X_2, Y_2). On the other hand, since $Y_2^- \subseteq Y'_2$, we have $X'_1 \cap N_G(y'_2) = \emptyset$ by (C3) for (X'_1, Y'_1, X'_2, Y'_2). Therefore, we have $u_2 \notin X'_1$ and hence $u_2 \in X'_2$. This proves $X_2 \subseteq X'_2$.

By the definition of Y', $x_1 \in Y' = Y'_1 \cup Y'_2$. Also, by Claim 3, $N_G(x_1) \cap X_1^- \neq \emptyset$, and since $X_1^- \subseteq X'_1$, we have $N_G(x_1) \cap X'_1 \neq \emptyset$. On the other hand, since $E_G(X'_1, Y'_2) = \emptyset$ by (C3) for (X'_1, Y'_1, X'_2, Y'_2), we have $x_1 \notin Y'_2$. This implies $x_1 \in Y'_1$ (see Fig. 5).

At this stage, we have $Y_1 \cup \{x_1\} \subseteq Y'_1$ and $Y_2^- \subseteq Y'_2$. Since $Y'_1 \cup Y'_2 = Y' = Y_1 \cup \{x_1\} \cup Y_2^-$, these yield $Y'_1 = Y_1 \cup \{x_1\}$ and $Y'_2 = Y_2^-$. In particular, $|Y'_2| = |Y_2^-| = n - t - 1$. On the other hand, since $X_2 \subseteq X'_2$, we have $|X'_2| \geq |X_2| = n - t$, and hence $G[X'_2, Y'_2]$ is not a balanced bipartite graph. This contradicts the fact that X'_2 and Y'_2 are the partite sets of the balanced bipartite graph $K_{n-s,n-s}$. Hence the subclaim follows. □

Since $X_1^- \subseteq X \setminus \{x_1\} \subseteq X' = X'_1 \cup X'_2$ and $X'_1 \cap X_1^- = \emptyset$ by Subclaim, we have $X_1^- \subseteq X'_2$. Furthermore, since $X'_1 \subseteq X' = X_1^- \cup X_2 \cup \{y_2\}$ and $X_1^- \cap X'_1 = \emptyset$, we have $X'_1 \subseteq X_2 \cup \{y_2\}$.

Assume $Y'_2 \cap Y_2^- \neq \emptyset$ and take $y'_2 \in Y'_2 \cap Y_2^-$. Note $X_1^- \neq \emptyset$ by Claim 3. Take $u_1 \in X_1^-$. Then since $X_1^- \subseteq X'_2$, $u_1 \in X'_2$. However, since $y'_2 \in Y_2^-$ and $u_1 \in X_1^-$, $u_1y'_2 \notin E(G)$ by (C3) for (X_1, Y_1, X_2, Y_2), while since $y'_2 \in Y'_2$ and $u_1 \in X'_2$, $u_1y'_2 \in E(G)$ by (C2) for (X'_1, Y'_1, X'_2, Y'_2). This is a contradiction, and hence we have $Y'_2 \cap Y_2^- = \emptyset$.

Since $Y_2^- \subseteq Y \setminus \{y_2\} \subseteq Y' = Y'_1 \cup Y'_2$ and $Y_2^- \cap Y'_2 = \emptyset$, we have $Y_2^- \subseteq Y'_1$. Furthermore, since $Y'_2 \subseteq Y' = Y_1 \cup Y_2^- \cup \{x_1\}$ and $Y_2^- \cap Y'_2 = \emptyset$, we have $Y'_2 \subseteq Y_1 \cup \{x_1\}$. □

Claim 7. *$x_1 \in Y'_2$ and $y_2 \in X'_1$.*

Proof. Assume $x_1 \notin Y'_2$. Since $x_1 \in Y' = Y'_1 \cup Y'_2$, we have $x_1 \in Y'_1$. Then we have $X'_1 \subseteq N_G(x_1)$ by (C1) for (X'_1, Y'_1, X'_2, Y'_2). Since $X'_1 \subseteq X_2 \cup \{y_2\}$ by Claim 6, this yields $N_G(x_1) \cap (X_2 \cup \{y_2\}) \neq \emptyset$. On the other hand, since $e_G(X_1, X_2) = 0$ by (∗) and $E_G(X_1, Y_2) = \emptyset$ by (C3) for (X_1, Y_1, X_2, Y_2), $N_G(x_1) \cap (X_2 \cup \{y_2\}) = \emptyset$. This is a contradiction, and we have $x_1 \in Y'_2$.

Note $y_2 \in X' = X'_1 \cup X'_2$. Since $E_G(X_1, Y_2) = \emptyset$ by (C3) for (X_1, Y_1, X_2, Y_2), we have $y_2 \notin N_G(x_1)$. On the other hand, since $x_1 \in Y'_2$, $X'_2 \subseteq N_G(x_1)$ by (C2) for (X'_1, Y'_1, X'_2, Y'_2), This implies $y_2 \notin X'_2$ and hence $y_2 \in X'_1$. □

Claim 8. *$X'_1 = X_2 \cup \{y_2\}$ and $X'_2 = X_1^-$.*

Proof. Since $x_1 \in Y'_2$ by Claim 7, we have $X'_2 \subseteq N_G(x_1)$ by (C2) for (X'_1, Y'_1, X'_2, Y'_2). On the other hand, since $e_G(X_1, X_2) = 0$ by (∗) and $x_1 \in X_1$, $N_G(x_1) \cap X_2 = \emptyset$. These imply $X'_2 \cap X_2 = \emptyset$.

Note $X'_1 \cup X'_2 = X' = X_1^- \cup X_2 \cup \{y_2\}$. Then since $X_1^- \subseteq X'_2$ by Claim 6, $y_2 \in X'_1$ by Claim 7 and $X'_2 \cap X_2 = \emptyset$, we have $X'_2 = X_1^-$ and $X'_1 = X_2 \cup \{y_2\}$. □

Claim 9. *We have $|N_G(y_2) \cap Y_1| = 2$. Moreover, $N_G(x) \cap Y_1 = N_G(y_2) \cap Y_1$ for each $x \in X_2$.*

Proof of Claim 9. Note $Y'_1 \cup Y'_2 = Y' = Y_1 \cup Y_2^- \cup \{x_1\}$. Since $Y_2^- \subseteq Y'_1$ by Claim 6 and $x_1 \in Y'_2$ by Claim 7, we have $Y'_1 = Y_2^- \cup (Y_1 \cap Y'_1)$ and $Y'_2 = \big(Y_1 \setminus (Y_1 \cap Y'_1)\big) \cup \{x_1\}$.

Take $x \in X_2$. Note $X_2 \cup \{y_2\} = X'_1$ by Claim 8, and hence $\{x, y_2\} \subseteq X'_1$. By (C1) and (C3) for (X'_1, Y'_1, X'_2, Y'_2), $Y'_1 \subseteq N_G(x) \cap N_G(y_2)$ and $Y'_2 \cap \big(N_G(x) \cup N_G(y_2)\big) = \emptyset$. Since $Y_1 \subseteq Y'_1 \cup Y'_2$, it follows that $N_G(x) \cap Y_1 = N_G(y_2) \cap Y_1 = Y_1 \cap Y'_1$.

Moreover, since $X_2' = X_1^-$ by Claim 8, we have

$$\begin{aligned} n &= |Y_1'| + |X_2'| = |Y_2^-| + |Y_1 \cap Y_1'| + |X_1^-| \\ &= (|Y_2| - 1) + |Y_1 \cap Y_1'| + (|X_1| - 1) = |X_1| + |Y_2| + |Y_1 \cap Y_1'| - 2 \\ &= n + |Y_1 \cap Y_1'| - 2, \end{aligned}$$

which yields $|Y_1 \cap Y_1'| = 2$. Since $N_G(x) \cap Y_1 = N_G(y_2) \cap Y_1 = Y_1 \cap Y_1'$, we obtain the desired conclusion. □

Claim 10. *$X_1^- \subseteq N_G(x_1)$ and $Y_2^- \subseteq N_G(y_2)$*

Proof. Take $u_1 \in X_1^-$ and $v_2 \in Y_2^-$. Then by Claim 6, $u_1 \in X_2'$ and by Claim 7, $x_1 \in Y_2'$. Hence we have $x_1u_1 \in E(G)$ by (C2) for (X_1', Y_1', X_2', Y_2'). Moreover, by Claim 6, $v_2 \in Y_1'$ and by Claim 7, $y_2 \in X_1'$. Hence $y_2v_2 \in E(G)$ by (C1) for (X_1', Y_1', X_2', Y_2'). □

Since x_1 and y_2 are arbitrarily chosen from X_1 and Y_2, respectively, under the assumption of $(*)$, Claim 10 tells us that both $G[X_1]$ and $G[Y_2]$ are complete graphs.

By Claim 9, $|N_G(y_2) \cap Y_1| = 2$. Hence we let $N_G(y_2) \cap Y_1 = \{y_1, v_1\}$. Then $N_G(x) \cap Y_1 = \{y_1, v_2\}$ for every $x \in X_2$. Moreover, since y_2 is arbitrarily chosen from Y_2 under the assumption of $(*)$, we can apply Claim 9 to a vertex in X_2 and arbitrary vertex in Y_2 and obtain $N_G(v) \cap Y_1 = \{y_1, v_1\}$ for every vertex v in Y_2. Therefore, $X_2 \cup Y_2 \subseteq N_G(y_1) \cap N_G(v_1)$.

We now prove that (y_1, v_1) is a violating pair. Since $X_2 \cup Y_2 \subseteq N_G(y_1) \cap N_G(v_1)$, it satisfies (P1). Let $X^* = (X \setminus \{x_1\}) \cup \{y_1\}$, $Y^* = (Y \setminus \{y_1\}) \cup \{x_1\}$ and $H^* = G[X^*, Y^*]$.

Take $x^* \in X^* \setminus \{y_1\} = X \setminus \{x_1\} = \big(X_1 \setminus \{x_1\}\big) \cup X_2$ with $x^*x_1 \notin E(G)$. Since $G[X_1]$ is a complete graph, $X_1 \setminus \{x_1\} \subseteq N_{H^*}(x_1)$. This implies $x^* \notin X_1 \setminus \{x_1\}$ and hence $x^* \in X_2$. Moreover, since $x_1y_1 \in E(H^*)$, $d_{H^*}(x_1) \geq |X_1 \setminus \{x_1\}| + 1 = t$. On the other hand, by (C2), $Y_2 \subseteq N_{H^*}(x^*)$, which implies $d_{H^*}(x^*) \geq n - t$. Thus, we have $d_{H^*}(x^*) + d_{H^*}(x_1) \geq (n-t) + t = n$ and hence (P2) holds. Therefore, (y_1, v_1) is a violating pair. However, this contradicts Claim 4, and the theorem follows. □

3. Concluding remarks

In this paper, we have investigated two degree sum conditions for the existence of a hamiltonian cycle. One of them is the classical Ore's Theorem. The other is an extension of the Moon–Moser Theorem, which was proved in [2]. Though the latter only concerns bipartite graphs, we have proved that it implies Ore's Theorem.

We do not know the relationship between the Moon–Moser Theorem itself and Ore's Theorem. Hence we raise the following question.

Question. For a positive integer n, does every graph G of order $2n$ with $\sigma_2(G) \geq 2n$ contain a spanning bipartite subgraph H with $\sigma_{1,1}(H) \geq n + 1$?

We believe that the answer to this question is negative. However, we have not found such an example.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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