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ORIGINAL PAPER

Extremal Union-Closed Set Families

Guantao Chen[1] · Hein van der Holst[1] · Alexandr Kostochka[2,3] · Nana Li[4]

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Abstract

A family of finite sets is called *union-closed* if it contains the union of any two sets in it. The *Union-Closed Sets Conjecture* of Frankl from 1979 states that each union-closed family contains an element that belongs to at least half of the members of the family. In this paper, we study structural properties of union-closed families. It is known that under the inclusion relation, every union-closed family forms a *lattice*. We call two union-closed families *isomorphic* if their corresponding lattices are isomorphic. Let $\mathcal{F}$ be a union-closed family and $\bigcup_{F\in\mathcal{F}} F$ be the *universe* of $\mathcal{F}$. Among all union-closed families isomorphic to $\mathcal{F}$, we develop an algorithm to find one with a maximum universe, and an algorithm to find one with a minimum universe. We also study properties of these two extremal union-closed families in connection with the Union-Closed Set Conjecture. More specifically, a lower bound of sizes of sets of a minimum counterexample to the dual version of the Union-Closed Sets Conjecture is obtained.

Keywords Family of sets · Union-closed sets · Normal and irreducible families

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✉ Guantao Chen
gchen@gsu.edu

Hein van der Holst
hvanderholst@gsu.edu

Alexandr Kostochka
kostochk@math.uiuc.edu

Nana Li
nana.li@umu.se

1 Department of Mathematics and Statistics, Georgia State University, Atlanta, GA 30303, USA

2 Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA

3 Sobolev Institute of Mathematics, Novosibirsk 630090, Russia

4 Department of Computing Science, Umeå University, 90187 Umeå, Sweden

1 Introduction

In this paper, we use calligraphic letters $\mathcal{A}, \mathcal{B}, \ldots$ to denote families of sets, capital letters $A, B, \ldots$ to denote set members of a set family, and lowercase letters $a, b, \ldots$ to denote elements in a set. Let $\mathcal{F}$ be a family of sets. The *universe* $X_{\mathcal{F}}$ of $\mathcal{F}$ is $\cup_{F \in \mathcal{F}} F$. A family of sets is called *finite* if $|X_{\mathcal{F}}| < \infty$, which in turn gives $|\mathcal{F}| < \infty$. We call $\mathcal{F}$ union-closed if $A \cup B \in \mathcal{F}$ whenever $A, B \in \mathcal{F}$. For any $x \in X_{\mathcal{F}}$, let $\mathcal{F}_x = \{F \in \mathcal{F} : x \in F\}$ be the subfamily of sets containing x. The following conjecture is commonly attributed to Frankl in 1979 [7].

Conjecture 1 (Union-closed sets Conjecture [7]) *For any finite union-closed family $\mathcal{F}$ with $\mathcal{F} \neq \{\emptyset\}$, there is an element $x \in X_{\mathcal{F}}$ such that $|\mathcal{F}_x| \geq |\mathcal{F}|/2$.*

The conjecture has drawn a lot of attention recently and is still wide open. Knill [8] proved that for any union-closed family $\mathcal{F}$ with $|\mathcal{F}| = m$, there is some element contained in at least $\frac{m}{\log_2 m}$ members of $\mathcal{F}$. Wójcik [11] improved this by a multiplicative constant. Reimer [9] proposed an up compression argument to obtain a sharp lower bound on the average set size of $\mathcal{F}_x$ as a function of $|\mathcal{F}|$. Roberts and Simpson [10] showed that if $\mathcal{F}$ is a counterexample with $|X_{\mathcal{F}}|$ minimum, then $|\mathcal{F}| \geq 4|X_{\mathcal{F}}| - 1$. Bruhn, Charbit, and Telle [2] showed that Frankl's conjecture is equivalent to the conjecture that in a finite non-trivial bipartite graph there are two adjacent vertices each belonging to at most half of the maximal stable sets. They also showed that some special class of bipartite graphs satisfies this bipartite graph property. Bruhn and Schaudt [3] showed that for every fixed edge-probability, almost every random bipartite graph almost surely satisfies Frankl's conjecture. Balla, Bollobás, and Eccles [1] determined the minimum average size of a union-closed family consisting of m subsets of a universe with n elements precisely, verifying a conjecture of Czédli, Maróti and Schmidt [5]. Consequently, they proved that the Union-Closed Sets Conjecture holds if $m \geq \frac{2}{3}2^n$. Falgas-Ravry [6] improved Reimer's bound with the help of some additional information about $X_{\mathcal{F}}$. We refer to Bruhn and Schaudt [4] for literature and recent results on the conjecture.

If in a family $\mathcal{F}$ of sets, there are two elements x and y such that $\mathcal{F}_x = \mathcal{F}_y$, then we can remove x or y to simplify our consideration. We call $\mathcal{F}$ *separating* if $\mathcal{F}_x \neq \mathcal{F}_y$ for any two distinct elements $x, y \in X_{\mathcal{F}}$. In this paper, we consider *separating* families of sets that contain the empty set $\emptyset$.

In his work on improving Knill's lower bound, Wójcik [11] considered the dual of a union-closed family. Given a family of sets $\mathcal{B}$, we denote by $\langle \mathcal{B} \rangle$ the union-closed family generated by $\mathcal{B}$, i.e.,

$$\langle \mathcal{B} \rangle = \{A : A = \cup_{B \in \mathcal{C}} B \text{ for some } \mathcal{C} \subseteq \mathcal{B}\}.$$

Given a union-closed family $\mathcal{F}$, the union-closed family $\mathcal{F}^* = \langle \{\mathcal{F}_x : x \in X_{\mathcal{F}}\} \rangle \cup \{\emptyset\}$ is called the *dual family* of $\mathcal{F}$, where $\mathcal{F}_x$ is viewed as a set and the sets in $\mathcal{F}_x$ are viewed as elements when considering the fual family. For each $X \in \mathcal{F}$, let $\mathcal{F}_{\not\subseteq X} = \{F \in \mathcal{F} : F \not\subseteq X\}$. Notice that the F_x sets are distinct since $F_x = F_y$ would contradict the assumption that $\mathcal{F}$ is separating. For each $x \in X_{\mathcal{F}}$, let $F_x = \bigcup_{F \in \mathcal{F} : x \notin F} F$. Since

$\emptyset \in \mathcal{F}$, F_x is well defined. Since $\mathcal{F}$ is union-closed, F_x is the maximal set of $\mathcal{F}$ not containing x. By the maximality of F_x, we have $\mathcal{F}_x = \mathcal{F}_{\nsubseteq F_x}$. Moreover, we will show in Sect. 3 that the dual $\mathcal{F}^* = \{\mathcal{F}_{\nsubseteq F} : F \in \mathcal{F}\}$. Combining with $\emptyset \in \mathcal{F}$, we have $|X_{\mathcal{F}^*}| = |\mathcal{F}| - 1$. Thus, $|\mathcal{F}^*| = |\mathcal{F}| = |X_{\mathcal{F}^*}| + 1$. Based on this observation, we call a union-closed family $\mathcal{F}$ *normal* if $|\mathcal{F}| = |X_{\mathcal{F}}| + 1$.

A set G in a union-closed family $\mathcal{F}$ is called a *generator* if G is not the union of two proper subsets in $\mathcal{F}$. We denote by $G(\mathcal{F})$ the family of all generators in $\mathcal{F}$. Following Bruhn and Schaudt [4], we call a generator G of a union-closed family $\mathcal{F}$ *abundant* if $|G| \geq |\mathcal{F}|/2$. Wójcik made the following conjecture and showed its equivalence to Conjecture 1.

Conjecture 2 *Every normal union-closed family $\mathcal{F}$ contains an abundant generator*

The inclusion relation of a family $\mathcal{F}$ of sets yields a *poset* on $\mathcal{F}$. Moreover, if the family is union-closed, the generated poset is a *lattice*—a poset in which every pair of elements A and B has a unique least upper bound $A \vee B$ and a unique greatest lower bound $A \wedge B$. More specifically, for every pair of sets A and B in any given separating union-closed family $\mathcal{F}$, we have $A \vee B = A \cup B$ and $A \wedge B = \cup_{F \in \mathcal{F}: F \subseteq A \cap B} F$. Two union-closed families $\mathcal{F}$ and $\mathcal{H}$ are *isomorphic* if their corresponding generated lattices are isomorphic. We will show that, under isomorphism, a union-closed family $\mathcal{F}$ is normal if and only if $|X_{\mathcal{F}}|$ is maximum. We call a union-closed family $\mathcal{F}$ *irreducible* if $|X_{\mathcal{F}}|$ is *minimum* among all isomorphic union-closed families. In Sect. 2, we show that by adding new elements to some sets, we can transform, up to isomorphism, a union-closed set to a normal family, and by deleting some elements from some sets, we can transform, up to isomorphism, a union-closed set to an irreducible family. In Sect. 3, we study the relationship between a union-closed family and its dual. In Sect. 4, we will show some properties of minimum counterexamples to the dual version of Conjecture 1. We start with some elementary properties of a union-closed family in the next section.

2 Enlarging and Reducing Universes

Let $\mathcal{F}$ be a separating union-closed family. Recall that for each $x \in X_{\mathcal{F}}$, F_x is the unique maximal set in $\mathcal{F}$ not containing x. Let f_F be the mapping from $X_{\mathcal{F}}$ to $\mathcal{F}$ defined by $f_F(x) = F_x$ for all $x \in X_{\mathcal{F}}$. As $\mathcal{F}$ is separating, $F_x \neq F_y$ for any two distinct elements $x, y \in X_{\mathcal{F}}$. So, f_F is injective.

For any two sets $F, G \in \mathcal{F}$, we call G a *parent* of F and F a *child* of G if $F \subsetneq G$ and $H \in \mathcal{F}$ with $F \subseteq H \subseteq G$ implies that either $H = F$ or $H = G$. Clearly, every $F \in \mathcal{F}$ with $F \neq X_{\mathcal{F}}$ has a parent. A set $F \in \mathcal{F}$ is called a *single-parent* set if it has only one parent.

Lemma 1 *Let $\mathcal{F}$ be a union-closed family. If F is a single-parent set and G is the parent of F, then $|G - F| = 1$. Moreover, if x is the unique element in $G - F$, then $F = F_x$.*

Proof Suppose to the contrary that there are two distinct elements $x, y \in G - F$. Then $\mathcal{F}_x = \mathcal{F}_y$. For otherwise there exists a set $X \in \mathcal{F}$ such that $x \in X$ and $y \notin X$, or there exists a set $X \in \mathcal{F}$ such that $y \in X$ and $x \notin X$; we may assume the former. Clearly, $X \cup F \supset F$. Since $y \in G$ and $y \notin X$, we have $G \nsubseteq X \cup F$. So, F has another parent $H \subseteq X \cup F$, different from G. This contradicts the assumption that F is a singe-parent set.

Let x be the unique element in $G - F$. Since $x \notin F$, $F \subseteq F_x$. Then $F = F_x$. For otherwise F has a parent H with $x \notin H$. So, F has two parents G and H, contradicting that F is a single-parent set. □

Let $\mathcal{F}$ be a union-closed family. Denote by $\Phi_{\mathcal{F}}$ the subfamily of all single-parent sets. By Lemma 1, $|X_{\mathcal{F}}| \geq |\Phi_{\mathcal{F}}|$. So, if $|X_{\mathcal{F}}| = |\Phi_{\mathcal{F}}|$, then $|X_{\mathcal{F}}|$ is minimum among all union-closed families isomorphic to $\mathcal{F}$. In this case, we call $\mathcal{F}$ *irreducible*. On the other hand, if $\mathcal{F}$ is normal, then $|X_{\mathcal{F}}| = |\mathcal{F}| - 1$. So, for every $F \in \mathcal{F}$ with $F \neq X_{\mathcal{F}}$, there exists an $x \in X_{\mathcal{F}}$ such that $F = F_x$. Clearly, in this case, $|X_{\mathcal{F}}|$ is maximum among all union-closed families isomorphic to $\mathcal{F}$. If $\mathcal{F}$ is not a normal family, then there exists a set $A \in \mathcal{F}$ such that $A \neq F_x$ for all $x \notin X_{\mathcal{F}}$. The following result provides an algorithm that adds a new element to $X_{\mathcal{F}}$ and maintains the same laddice structure until the family becomes normal.

Theorem 1 *Let $\mathcal{F}$ be a non-normal union-closed family and let $A \in \mathcal{F}$ such that $A \neq F_x$ for all $x \in X_{\mathcal{F}}$. Let y be an element not in $X_{\mathcal{F}}$. If $\mathcal{H}$ is a family consisting of $F \in \mathcal{F}$ if $F \subseteq A$ and $F \cup \{y\}$ otherwise, then, $\mathcal{H}$ is a union-closed family isomorphic to $\mathcal{F}$.*

Proof Let $\mathcal{H}$ be a family consisting of $F \in \mathcal{F}$ if $F \subseteq A$ and $F \cup \{y\}$ otherwise. We claim that $G \cup H \in \mathcal{H}$ for any two sets $G, H \in \mathcal{H}$ by considering two cases. Suppose first that $y \notin G \cup H$. In this case, we have both $G, H \in \mathcal{F}$, which in turn gives $G \cup H \in \mathcal{F}$ since $\mathcal{F}$ is union-closed. By the definition of $\mathcal{H}$, we have $G \cup H \in \mathcal{H}$ since $y \notin G \cup H$. Suppose next that $y \in G \cup H$. Since $G - y, H - y \in \mathcal{F}$, $(H - y) \cup (G - y) \in \mathcal{F}$. Since $y \in G \cup H$, by symmetry we may assume $y \in H$. By the definition of $\mathcal{H}$, $H - y \nsubseteq A$, so $G \cup H - y = (G - y) \cup (H - y) \nsubseteq A$. Hence, $G \cup H = (G - y) \cup (H - y) \cup \{y\}$ is a set in $\mathcal{H}$.

Moreover, it is easy to verify that $G \subset H$ if and only if $G - y \subset H - y$ since $y \notin X_{\mathcal{F}}$. Hence, the lattices generated by $\mathcal{F}$ and $\mathcal{H}$ are isomorphic. We note that $A = H_y$, which is the maximal set not containing y in $\mathcal{H}$. So, $\mathcal{H}$ is a separating union-closed set family. □

We now turn our attention to reducing the universe of a union-closed family. For a union-closed family $\mathcal{F}$ and an element $x \in X_{\mathcal{F}}$, let $\mathcal{F}^{-x} = \{F - \{x\} : F \in \mathcal{F}\}$. The following theorem provides an algorithm that deletes elements and maintains the same laddice structure until the family becomes irreducible. For example, let $\mathcal{F} = \{\emptyset, \{1\}, \{1, 2, 4\}, \{1, 3, 4\}, \{1, 2, 3, 4\}\}$. Clearly, $F_4 = \{1\}$ is not a single-parent set. $\mathcal{F}^{-4} = \{\emptyset, \{1\}, \{1, 2, \}, \{1, 3, \}, \{1, 2, 3, \}\}$ is isomophic to $\mathcal{F}$, and irreducible.

Theorem 2 *If a union-closed family $\mathcal{F}$ is not irreducible, then there exists an element $x \in X_{\mathcal{F}}$ such that $\mathcal{F}^{-x}$ is isomorphic to $\mathcal{F}$.*

Proof Since $\mathcal{F}$ is not irreducible, there exists an $x \in X_{\mathcal{F}}$ such that F_x is not a single-parent set, i.e., $F_x \notin \Phi_{\mathcal{F}}$. We claim that $\mathcal{F}^{-x}$ is isomorphic to $\mathcal{F}$. Let ψ be the mapping from $\mathcal{F}$ to $\mathcal{F}^{-x}$ defined by $\psi(F) = F - x$ for every $F \in \mathcal{F}$. We will show that ψ is a bijection and preserves the inclusion relation of $\mathcal{F}$.

We first show that ψ is a bijection. Clearly, ψ is an onto mapping, so we only need to show it is an injection. Suppose on the contrary that there exist $A, B \in \mathcal{F}$ with $A \neq B$ such that $A - x = B - x$. Then, we have either $A = B - x$ or $B = A - x$. By symmetry, we assume $A = B - x$. Since $x \notin A$, $A \subseteq F_x$ by the maximality of F_x. Since $B = A \cup \{x\}$, we have $F_x \cup B = F_x \cup (A \cup \{x\}) = F_x \cup \{x\} \in \mathcal{F}$, which in turn shows that $F_x \cup B$ is a parent of F_x. Since $F_x \notin \Phi_{\mathcal{F}}$, it has another parent G. By the definition of parent, we have $F_x \subset G$ and $F_x \cup \{x\} \not\subset G$. So, $x \notin G$, which gives a contradiction to the maximality of F_x.

We now show that ψ preserves the inclusion relation of $\mathcal{F}$. For any two distinct sets $A, B \in \mathcal{F}$, we will show that $A \subsetneq B$ if and only if $A - x \subsetneq B - x$. We first assume $A \subsetneq B$, which gives $A - x \subseteq B - x$. Since ψ is a bijection, the equality does not hold, so $A - x \subsetneq B - x$. Conversely, suppose $A - x \subsetneq B - x$. In this case, if $A \not\subset B$, then we have $x \in A$ and $x \notin B$. Since $x \notin B$, we have $B \subseteq F_x$. Since $A - x \subsetneq B \subseteq F_x$, $F_x \cup \{x\} = (F_x \cup (A - x)) \cup \{x\} = F_x \cup A$. Since $\mathcal{F}$ is union-closed, we have $F_x \cup \{x\} \in \mathcal{F}$. So, $F_x \cup \{x\}$ is a parent of F_x. Since $F_x \notin \Phi_{\mathcal{F}}$, F_x has another parent G. If $x \in G$, then $G \supset F_x \cup \{x\}$, contradicting G is a parent of F. Thus, $x \notin G$, contradicting that F_x is the maximum set in $\mathcal{F}$ not containing x. □

Corollary 1 *A union-closed family $\mathcal{F}$ can be reduced to an irreducible isomorphic union-closed family Γ by consecutively removing vertices $x \in X_{\mathcal{F}}$ with $F_x \notin \Phi_{\mathcal{F}}$.*

3 Duality and Normality

Let $\mathcal{F}$ be a union-closed family. Recall that the dual $\mathcal{F}^*$ of $\mathcal{F}$ is the union-closed family generated by $\{\mathcal{F}_x : x \in X_{\mathcal{F}}\}$, where $\mathcal{F}_x = \{F \in \mathcal{F} : x \in F\}$ for $x \in X_{\mathcal{F}}$. In this section, we give a complete description of the sets in $\mathcal{F}^*$ and use this to give a characterization of normal families in terms of its dual families, as well as in terms of generators. Recall that, under the inclusion relation, a union-closed family $\mathcal{F}$ forms a lattice, where $A \wedge B$ denotes the greatest lower bound of A and B for any $A, B \in \mathcal{F}$. Clearly, $A \wedge B = \cup_{F \in \mathcal{F}: F \subseteq A \cap B} F$. Since we assume $\emptyset \in \mathcal{F}$, $A \wedge B$ is well-defined. As noticed before, by the maximality of F_x, $\mathcal{F}_x = \mathcal{F}_{\not\subseteq F_x}$.

Lemma 2 *If $\mathcal{F}$ is a union-closed family, then for any $F, G \in \mathcal{F}$, $\mathcal{F}_{\not\subseteq F} \cup \mathcal{F}_{\not\subseteq G} = \mathcal{F}_{\not\subseteq F \wedge G}$. This implies that $\{\mathcal{F}_{\subseteq F} : F \in \mathcal{F}\}$ is union-closed since $F \wedge G \in \mathcal{F}$.*

Proof Let $H = F \wedge G$. Since $H \subseteq F$ and $H \subseteq G$, we have $\mathcal{F}_{\not\subseteq H} \supseteq \mathcal{F}_{\not\subseteq F} \cup \mathcal{F}_{\not\subseteq G}$. To prove equality, suppose for a contradiction that there exists a set $A \in \mathcal{F}_{\not\subseteq H} - \mathcal{F}_{\not\subseteq F} \cup \mathcal{F}_{\not\subseteq G}$. Since $A \in \mathcal{F}_{\not\subseteq H}$, $A \not\subseteq H$. So, $A \cup H$ contains H as a proper subset. Since $A \notin \mathcal{F}_{\not\subseteq F} \cup \mathcal{F}_{\not\subseteq G}$, we have $A \subseteq F$ and $A \subseteq G$, that is, $A \subseteq F \cap G$. So, $A \cup H \subseteq F \cap G$, contradicting that $H = F \wedge G$. □

Theorem 3 *For any union-closed family $\mathcal{F}$, we have $\mathcal{F}^* = \{\mathcal{F}_{\not\subseteq F} : F \in \mathcal{F}\}$.*

Proof Let $\mathcal{H} = \{\mathcal{F}_{\nsubseteq F} : F \in \mathcal{F}\}$. By Lemma 2, $\mathcal{H}$ is union-closed. Since $\mathcal{F}_{\nsubseteq X_{\mathcal{F}}} = \emptyset$, we have $\emptyset \in \mathcal{H}$. Since $\mathcal{F}_x = \mathcal{F}_{\nsubseteq F_x}$ for every $x \in X_{\mathcal{F}}$ and $\mathcal{F}^* = \langle\{\mathcal{F}_x : x \in X_{\mathcal{F}}\}\rangle$, by Lemma 2, $\mathcal{F}^* \subseteq \mathcal{H}$. We will show that equality holds.

Let $F \in \mathcal{F}$ and $X = \{x : F \subseteq F_x\}$. We claim $F = \wedge_{x \in X} F_x$, which in turn shows $\mathcal{F}_{\nsubseteq F} = \cup_{x \in X} \mathcal{F}_{\nsubseteq F_x} \in \mathcal{F}^*$ and completes the proof.

Let $Y = X_{\mathcal{F}} - X$. For each $y \in Y$, since $F \nsubseteq F_y$, we have $y \in F$ by the maximality of F_y. So, $Y \subseteq F$. On the other hand, for each $z \in F$, we have $F \nsubseteq F_z$, so $z \in Y$. Hence, $F = Y$. Since $F \subseteq F_x$ for each $x \in F$, we have $F \subseteq \wedge_{x \in X} F_x$. On the other hand, for each $x \in X$, since $x \notin F_x$, we have $x \notin \wedge_{z \in X} F_z$. Consequently, $Y \supseteq \wedge_{x \in X} F_x$. So, $F = Y = \wedge_{x \in X} F_x$. □

Corollary 2 *A union-closed family $\mathcal{F}$ is normal if and only if $\mathcal{F}^* = \{\mathcal{F}_x : x \in X_{\mathcal{F}}\} \cup \{\emptyset\}$.*

Proof By Theorem 3, we have $|\mathcal{F}^*| = |\{\mathcal{F}_{\nsubseteq F} : F \in \mathcal{F}\}| = |\mathcal{F}|$. By the definition of normal families, $\{\mathcal{F}_x : x \in X_{\mathcal{F}}\} \cup \{\emptyset\} \subseteq \mathcal{F}^*$. So, equality holds if and only if $|X_{\mathcal{F}}| + 1 = |\mathcal{F}|$, i.e., if and only if $\mathcal{F}$ is normal. □

Recall that a generator G of $\mathcal{F}$ is a set in $\mathcal{F}$ that cannot be expressed as the union of two proper subsets in $\mathcal{F}$, and that $G(\mathcal{F})$ denotes the subfamily of all generators of $\mathcal{F}$. For each $x \in X_{\mathcal{F}}$, let $G(\mathcal{F})_x$ denote the family of all generators containing x.

Lemma 3 *Let $\mathcal{F}$ be a union-closed family and let x, y and z be three elements of $X_{\mathcal{F}}$. Then, $\mathcal{F}_z = \mathcal{F}_x \cup \mathcal{F}_y$ if and only if $G(\mathcal{F})_z = G(\mathcal{F})_x \cup G(\mathcal{F})_y$.*

Proof We first show the "only if part". Suppose $\mathcal{F}_z = \mathcal{F}_x \cup \mathcal{F}_y$. Then for every $F \in \mathcal{F}$, $z \in F$ if and only if $x \in F$ or $y \in F$. This especially holds for every generator $F \in G(\mathcal{F})$, so $G(\mathcal{F})_z = G(\mathcal{F})_x \cup G(\mathcal{F})_y$.

To show the "if" part, we assume $G(\mathcal{F})_z = G(\mathcal{F})_x \cup G(\mathcal{F})_y$, so $G(\mathcal{F})_z \supseteq G(\mathcal{F})_x$ and $G(\mathcal{F})_z \supseteq G(\mathcal{F})_y$. We claim $F_z \subseteq F_x$. Suppose not. Then $x \in F_z$. Let $G \subseteq F_z$ be a generator with $x \in G$. Then, $G \in G(\mathcal{F})_x - G(\mathcal{F})_z$ giving a contradiction to $G(\mathcal{F})_z \subseteq G(\mathcal{F})_x$. Similarly, we have $F_z \subseteq F_y$. So, $F_z \subseteq F_x \wedge F_y$. We claim that equality holds. Suppose not. Then $F_z \subset F_x \wedge F_y$. By the maximality of F_z, we have $z \in F_x \wedge F_y$. So, there is a generator $G \subseteq F_x \cap F_y$ containing z. Since $G \subseteq F_x \wedge F_y \subseteq F_x \cap F_y$, $x, y \notin G$. So $G \notin G(\mathcal{F})_x \cup G(\mathcal{F})_y$. Since $z \in G$, we have $G \in G(\mathcal{F})_z$, giving a contradiction to $G(\mathcal{F})_z = G(\mathcal{F})_x \cup G(\mathcal{F})_y$. On the other hand, by Lemma 2, $\mathcal{F}_z = \mathcal{F}_x \cup \mathcal{F}_y$ if and only if $F_z = F_x \wedge F_y$. The proof is completed. □

Theorem 4 *A union-closed family $\mathcal{F}$ is normal if and only if $\{G(\mathcal{F})_x : x \in X_{\mathcal{F}}\} \cup \{\emptyset\}$ is union-closed.*

Proof By Corollary 2, $\mathcal{F}$ is normal if and only if $\mathcal{F}^* = \{\mathcal{F}_x : x \in X_{\mathcal{F}}\} \cup \{\emptyset\}$, which is equivalent to that $\{\mathcal{F}_x : x \in X_{\mathcal{F}}\} \cup \{\emptyset\}$ is union-closed. By Lemma 3, this is equivalent to $\{G(\mathcal{F})_x : x \in X_{\mathcal{F}}\} \cup \{\emptyset\}$ is union-closed. □

4 Generators in Minimal Counterexamples to Conjecture 2

Conjecture 2 claims that every nontrivial normal union-closed family has an abundant generator. As mentioned in the introduction, the sizes $|\mathcal{F}|$ and $|X_{\mathcal{F}}|$ of minimal counterexamples to Conjecture 1 have been actively studied. In this section, we study the size of nontrivial sets in a minimum counterexample to Conjecture 2.

Theorem 5 *If $\mathcal{F}$ is a counterexample to Conjecture 2 with $|X_{\mathcal{F}}|$ minimum, then $|F| \geq 7$ for all $F \in \mathcal{F}$ with $F \neq \emptyset$.*

Proof Let $\mathcal{F}$ be a union-closed family with no heavy generator and $|X_{\mathcal{F}}|$ is the minimum under this assumption. Denote $\mathcal{F} = \{F_1, F_2, \ldots, F_{n+1}\}$ such that F_i is the maximal set in $\mathcal{F}$ not containing i for each $i = 1, 2, \ldots, n+1$, where $F_1 = \emptyset$ and $F_{n+1} = X_{\mathcal{F}} = \{1, 2, \ldots, n\}$. Moreover, we assume that $F_2 = [1, k+1] - \{2\}$ is an arbitrary generator. We will show that $k \geq 7$, which will complete the proof.

Recall $\mathcal{F}_{\supseteq F_2} = \{F \in \mathcal{F} \;:\; F \supseteq F_2\}$. Clearly, $\mathcal{F}_{\supseteq F_2} = \{F_i \;:\; i \notin F_2 \cup \{2\}\}$. Let $\mathcal{L} = \{F - F_2 \;:\; F \in \mathcal{F}_{\supseteq F_2}\}$. We have $|\mathcal{L}| = n + 1 - k$ and F_2 becomes the empty set. Since $(F_i - F_2) \cup (F_j - F_2) = F_i \cup F_j - F_2$ for any two $F_i, F_j \in \mathcal{F}$, $\mathcal{L}$ is union-closed. Moreover, for any two distinct sets $F_i, F_j \supseteq F_2$, we have $F_i - F_2 \neq F_i - F_2$. So, $\mathcal{L}$ is normalized.

By the minimality of $\mathcal{F}$, $\mathcal{L}$ has a generator G with $|G| \geq \frac{n+1-k}{2}$. Let $G' = G \cup F_2$. If G' is a generator in $\mathcal{F}$, then we are done. So, it is not.

Claim There is $j \in F_2$ such that $G' = F_2 \cup F_j$. Moreover, one can choose such j with the property that F_j is a generator in $\mathcal{F}$.

Proof Since G' is not a generator in $\mathcal{F}$, there are incomparable sets $B_1, B_2 \in \mathcal{F}$ such that $G' = B_1 \cup B_2$. Furthermore, since G is a generator in $\mathcal{L}$, we may assume that $B_2 \notin \mathcal{F}_{\supseteq F_2}$.

Case 1: $B_1 \notin \mathcal{F}_{\supseteq F_2}$. For $j = 1, 2$, let $C_j = F_2 \cup B_j$. Since $B_1, B_2 \subseteq G'$ and $F_2 \subset G'$, $C_1, C_2 \subseteq G'$. If at least one of C_1 and C_2 is G', then we are done. Otherwise, $G = (C_1 - F_2) \cup (C_2 - F_2)$, giving a contradiction to the fact that G is a generator in $\mathcal{L}$.

Case 2: $B_1 \in \mathcal{F}_{\supseteq F_2} - \{F_2\}$. Let $C_2 = F_2 \cup B_2$. If $C_2 = G'$, then we are done. Otherwise, $Y = (B_1 - F_2) \cup (C_2 - F_2)$, giving a contradiction to the fact that G is a generator in $\mathcal{L}$.

This proves that there is $j \in F_2$ such that $G' = F_2 \cup F_j$. Among all such F_j choose one of the smallest size. If it is a generator in $\mathcal{F}$, then we are done. Otherwise, there are noncomparable $B_1, B_2 \notin \mathcal{F}_{\supseteq F_2}$ such that $F_j = B_1 \cup B_2$. For $j = 1, 2$, let $C_j = F_2 \cup B_j$. By the minimality of $|F_j|$, $C_1 \neq G' \neq C_2$. Then $G = (C_1 - F_2) \cup (C_2 \cup F_2)$, giving a contradiction to the fact that it is a generator in $\mathcal{F}_{\supseteq F_2}$. □

Since $\mathcal{F} - \mathcal{F}_{\supseteq F_2}$ contains F_j not contained in F_2, $k \geq 2$. Let F_3 be a largest generator F_j not contained in $\mathcal{F}_{\supseteq F_2}$ such that $G' = F_2 \cup F_j$. Since $1 \in F_3$, $|F_3| \geq 1 + |G| \geq \frac{n-k+3}{2}$. Since $\mathcal{F}$ is a counterexample, $-k + 3 \leq 0$, i.e. $k \geq 3$.

Claim $k \geq 5$.

If $|F_3 \cap F_2| \geq 2$, then $|F_3| \geq 2 + |G| \geq \frac{n-k+5}{2}$, which yields $k \geq 5$. So, suppose $F_3 \cap F_2 = \{1\}$. Then, for each $j \in F_2 - \{1, 3\}$, $F_3 \subset F_j$. Since $k \geq 3$, there exists at least one such set F_j. So, the smallest among such sets F_j is a generator in $\mathcal{F}$ and has size at least $|F_3| + 1 \geq \frac{n-k+5}{2}$, which again yields $k \geq 5$. □

Claim $k \geq 7$.

Proof If $|F_3 \cap F_2| \geq 3$, then $|F_3| \geq 3 + |G| \geq \frac{n-k+7}{2}$, which yields $k \geq 7$. So, $1 \leq |F_3 \cap F_2| \leq 2$.

Case 1: $|F_3 \cap F_2| = 2$. We may assume $F_3 \cap F_2 = \{1, 4\}$ and $|F_5| \leq \cdots \leq |F_{k+1}|$. Since $5 \notin F_3$, $F_3 \subset F_5$. So $|F_5| \geq 1 + |F_3| \geq 3 + |G| \geq \frac{n-k+7}{2}$. Thus if $k \leq 6$, then F_5 is not a generator. By its choice, the only possibility is that $F_5 = F_3 \cup F_4$. Since $k \geq 5$, set F_6 exists and is not the union of any two of F_3, F_4 and F_5. Then F_6 is a generator in $\mathcal{F}$ of size at least $1 + |F_3| \geq 3 + |G| \geq \frac{n-k+7}{2}$.

Case 2: $F_3 \cap F_2 = \{1\}$. Then every member of $\mathcal{F} - \mathcal{F}_{\supseteq F_2} - F_1 - F_3$ contains F_3. Let $\mathcal{M}'_2$ consist of the inclusion minimal members of $\mathcal{F} - \mathcal{F}_{\supseteq F_2} - F_1 - F_3$, and let $F_4 \in \mathcal{M}'_2$. Then F_4 is a generator in $\mathcal{F}$. By construction, $3 \in F_4$. If $F_4 = F_3 \cup \{3\}$, then $F_4 \subset F'$, giving a contradiction to the maximality of F_3. So, $|F_4| \geq 2 + |F_3| \geq 3 + |G| \geq \frac{n-k+7}{2}$, which yields $k \geq 7$. □

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