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ORIGINAL PAPER

Characterizing the Difference Between Graph Classes Defined by Forbidden Pairs Including the Claw

Guantao Chen[1] · **Michitaka Furuya**[2] · **Songling Shan**[3] · **Shoichi Tsuchiya**[4] · **Ping Yang**[1]

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Abstract

For two graphs A and B, a graph G is called $\{A, B\}$-free if G contains neither A nor B as an induced subgraph. Let P_n denote the path of order n. For nonnegative integers k, ℓ and m, let $N_{k,\ell,m}$ be the graph obtained from K_3 and three vertex-disjoint paths P_{k+1}, $P_{\ell+1}$, P_{m+1} by identifying each of the vertices of K_3 with one endvertex of one of the paths. Let $Z_k = N_{k,0,0}$ and $B_{k,\ell} = N_{k,\ell,0}$. Bedrossian characterized all pairs $\{A, B\}$ of connected graphs such that every 2-connected $\{A, B\}$-free graph is Hamiltonian. All pairs appearing in the characterization involve the claw ($K_{1,3}$) and one of $N_{1,1,1}$, P_6 and $B_{1,2}$. In this paper, we characterize connected graphs that are (i) $\{K_{1,3}, Z_2\}$-free but not $B_{1,1}$-free, (ii) $\{K_{1,3}, B_{1,1}\}$-free but not P_5-free, or (iii) $\{K_{1,3}, B_{1,2}\}$-free but not P_6-free. The third result is closely related to Bedrossian's characterization. Furthermore, we apply our characterizations to some forbidden pair problems.

Keywords Forbidden subgraph · Hamiltonian cycle · Halin graph

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✉ Michitaka Furuya
michitaka.furuya@gmail.com

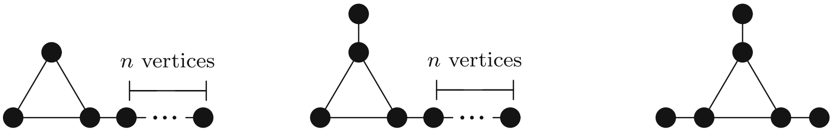
Shoichi Tsuchiya
s.tsuchiya@isc.senshu-u.ac.jp

1 Department of Mathematics and Statistics, Georgia State University, Atlanta, USA

2 College of Liberal Arts and Sciences, Kitasato University, Sagamihara, Kanagawa, Japan

3 Department of Mathematics, Illinois State University, Normal, USA

4 School of Network and Information, Senshu University, Kawasaki, Kanagawa, Japan



Z_n $B_{1,n}$ N

Fig. 1 Graphs Z_n, $B_{1,n}$ and N

1 Introduction

Our notation and terminology are standard, and mostly taken from [6]. We consider only simple and finite graphs. Let G be a graph. For $v \in V(G)$, we let $N_G(v)$ denote the *neighborhood* of v in G. For a set U, we let $G[U]$ denote the subgraph of G induced by $U \cap V(G)$.

Let $\mathcal{F}$ be a family of connected graphs. A graph G is said to be $\mathcal{F}$*-free* if G contains no member of $\mathcal{F}$ as an induced subgraph. The members of $\mathcal{F}$ are called *forbidden subgraphs*. If G is $\{F\}$-free, then G is simply said to be F*-free*. A family $\mathcal{F}$ of forbidden subgraphs is called a *forbidden pair* if $|\mathcal{F}| = 2$. For two families $\mathcal{F}_1$ and $\mathcal{F}_2$ of forbidden subgraphs, we write $\mathcal{F}_1 \leq \mathcal{F}_2$ if for every $F_2 \in \mathcal{F}_2$, there exists $F_1 \in \mathcal{F}_1$ such that F_1 is an induced subgraph of F_2. Note that if $\mathcal{F}_1 \leq \mathcal{F}_2$, then every $\mathcal{F}_1$-free graph is also $\mathcal{F}_2$-free.

Let $K_{1,3}$ denote the star with three leaves, and let K_n and P_n denote the complete graph and the path of order n, respectively. For nonnegative integers k, ℓ and m, let $N_{k,\ell,m}$ be the graph obtained from K_3 and three vertex-disjoint paths P_{k+1}, $P_{\ell+1}$, P_{m+1} by identifying each of the vertices of K_3 with one endvertex of one of the paths. The graphs $N_{k,0,0}$ and $N_{k,\ell,0}$ are denoted by Z_k and $B_{k,\ell}$, respectively. The graph $N_{1,1,1}$ is usually denoted by N (see Fig. 1).

Bedrossian [1] characterized all pairs $\{A, B\}$ of connected graphs such that every 2-connected $\{A, B\}$-free graph is Hamiltonian.

Theorem 1 (Bedrossian [1]) *Let $\mathcal{F}$ be a pair of connected graphs. Then every 2-connected $\mathcal{F}$-free graph has a Hamiltonian cycle if and only if $\mathcal{F} \leq \{K_{1,3}, N\}$, $\mathcal{F} \leq \{K_{1,3}, P_6\}$ or $\mathcal{F} \leq \{K_{1,3}, B_{1,2}\}$.*

For two different graphs $B, B' \in \{N, B_{1,2}, P_6\}$, usually it takes independent work to show that 2-connected $\{K_{1,3}, B\}$-free graphs are Hamiltonian, and 2-connected $\{K_{1,3}, B'\}$-free graphs are Hamiltonian, and the proof of one case may be harder than the other. This situation happens in many research concerning forbidden subgraphs. This naturally raises a question of investigating the difference between $\{K_{1,3}, B\}$-free graphs and $\{K_{1,3}, B'\}$-free graphs. Since a characterization of the difference together with existing results on $\{K_{1,3}, B\}$-free graphs will shed light on new properties of $\{K_{1,3}, B'\}$-free graphs.

Olariu in [11] showed that every connected Z_1-free but not K_3-free graph is a complete multipartite graph with at least three partite sets. His result is useful when we investigate the class of Z_1-free graphs (for example, the characterization was used for research of perfect Z_1-free graphs in [11]). Recently, Furuya and Tsuchiya [9]

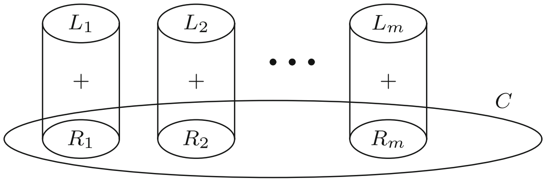


Fig. 2 Generalized comb

focused on forbidden pairs appearing in Theorem 1, and obtained a characterization similar to Olariu's result.

A graph H is called a *generalized comb* if for an integer $m \geq 3$, H consists of m disjoint cliques, say L_i $(1 \leq i \leq m)$, and a clique C containing m disjoint subcliques R_i $(1 \leq i \leq m)$ such that every vertex in L_i is adjacent to every vertex in R_i (see Fig. 2). In this context, L_i is called a *leaf-clique* and R_i is called the *root* of L_i.

Theorem 2 (Furuya and Tsuchiya [9]) *A connected graph G is $\{K_{1,3}, B_{1,2}\}$-free but not N-free if and only if G is a generalized comb.*

We will characterize new families of connected graphs that are $\{K_{1,3}, Z_2\}$-free but not $B_{1,1}$-free or $\{K_{1,3}, B_{1,m}\}$-free but not $P_{\max\{3m,m+4\}}$-free for some integer $m \geq 1$. We start with some definitions.

A generalized comb is *pointed* if all of its leaf-cliques consist of exactly one vertex. Let $\mathcal{H}_0$ be the family of pointed generalized combs. For each i $(1 \leq i \leq 8)$, let H_i be the graph depicted in Fig. 3, where the vertices of H_i $(1 \leq i \leq 5)$ indicated by the open circles are called *expandable vertices*, and denote by U_i the set of expandable vertices of H_i. Given a family $\mathcal{C} = \{C_a \mid a \in U_i\}$ of disjoint cliques indexed by a, the graph $H_i(\mathcal{C})$ is obtained from H_i by replacing each vertex $a \in U_i$ with the clique C_a and adding all edges between $u \in V(H_i) - \{a\}$ and C_a if $au \in E(H_i)$. Let

$$\mathcal{H}_i = \{H_i(\mathcal{C}) \mid \mathcal{C} = \{C_a \mid a \in U_i\} \text{ is a family of disjoint cliques indexed by } a\}.$$

Clearly, $H_i \in \mathcal{H}_i$ for each i. For convention, let $\mathcal{H}_j = \{H_j\}$ for each j $(6 \leq j \leq 8)$. The graphs in $\bigcup_{0 \leq i \leq 8} \mathcal{H}_i$ are exactly all connected graphs that are $\{K_{1,3}, Z_2\}$-free but not $B_{1,1}$-free as follows.

Theorem 3 *A connected graph G is $\{K_{1,3}, Z_2\}$-free but not $B_{1,1}$-free if and only if $G \in \bigcup_{0 \leq i \leq 8} \mathcal{H}_i$.*

For $\ell \geq 5$, let $L_0, L_1, \ldots, L_\ell$ be $\ell + 1$ disjoint cliques. Let $F_p = F_p(L_1, \ldots, L_\ell)$ be the graph obtained from a path $v_1 v_2 \cdots v_\ell$ by blowing up each v_i with L_i and joining every vertex of L_i to all vertices of L_{i+1} for $1 \leq i \leq \ell - 1$. We call F_p a *fat ℓ-path* (or simply a *fat path*). In this context, L_i $(1 \leq i \leq \ell)$ are called *fundamental cliques* of F_p. Let $F_c = F_c(L_0, \ldots, L_\ell)$ be the graph obtained from a cycle $v_0 v_1 \cdots v_\ell v_0$ by blowing up each v_i with L_i and joining every vertex of L_i to all vertices of L_{i+1} for $0 \leq i \leq \ell$ where the indices are calculated modulo $\ell + 1$. We call F_c a *fat ℓ-cycle* (or simply a *fat cycle*). In this context, L_i $(0 \leq i \leq \ell)$ are called *fundamental cliques* of F_c. Note that fat ℓ-paths have ℓ fundamental cliques but fat ℓ-cycles have $\ell + 1$ fundamental cliques. Let $\mathcal{P}(\ell)$ be the family of fat i-paths and fat i-cycles for all $i \geq \ell$.

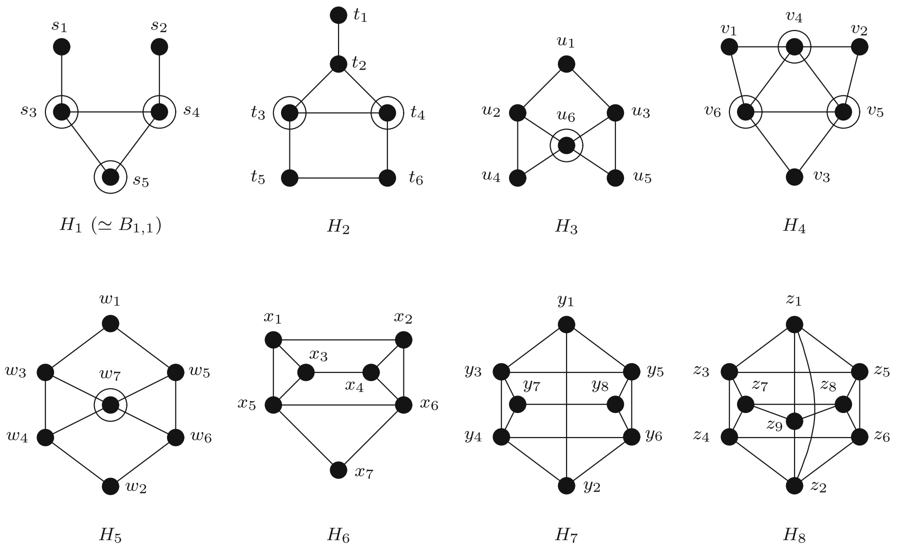


Fig. 3 Graphs H_i

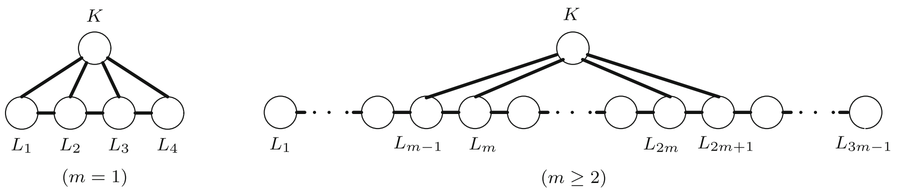


Fig. 4 Graph F'

Theorem 4 *A connected graph G is $\{K_{1,3}, B_{1,1}\}$-free but not P_5-free if and only if $G \in \mathcal{P}(5)$.*

Theorem 5 *A connected graph G is $\{K_{1,3}, B_{1,2}\}$-free but not P_6-free if and only if $G \in \mathcal{P}(6)$.*

In fact, we will prove a more general result stated below.

Theorem 6 *For an integer $m \geq 1$, a connected graph G is $\{K_{1,3}, B_{1,m}\}$-free but not $P_{\max\{3m,m+4\}}$-free if and only if $G \in \mathcal{P}(\max\{3m, m+4\})$.*

The order of the path in Theorem 6 is best possible if we use only fat paths and fat cycles in a characterization. Fix an integer $m \geq 1$. Let $F = F_p(L_1, \dots, L_{\max\{3m-1,m+3\}})$ be a fat path, and let K be a clique with $V(F) \cap K = \emptyset$. Let F' be the graph obtained from $F \cup K$ by joining each vertex of K to each vertex of $L_{\max\{m-1,1\}} \cup L_{\max\{m,2\}} \cup L_{\max\{2m,3\}} \cup L_{\max\{2m+1,4\}}$ (see Fig. 4). Then we see that F' is a connected $\{K_{1,3}, B_{1,m}\}$-free but not $P_{\max\{3m-1,m+3\}}$-free graph.

The rest of the paper is organized as follows. We give some applications of Theorems 3 and 6 in Sect. 2. In Sect. 3, we prove Theorems 3 and 6. In the last section, we prove two results that will be given in Sect. 2.

2 Applications of Theorems 3 and 6

In this section, we introduce three applications of Theorems 3 and 6. We first focus on the existence of Hamiltonian cycle. Theorem 1 is a combination of the following three results.

Theorem 7 (Duffus et al. [7]) *Every* 2*-connected* $\{K_{1,3}, N\}$*-free graph has a Hamiltonian cycle.*

Theorem 8 (Broersma and Veldman [2]) *Every* 2*-connected* $\{K_{1,3}, P_6\}$*-free graph has a Hamiltonian cycle.*

Theorem 9 (Bedrossian [1]) *Every* 2*-connected* $\{K_{1,3}, B_{1,2}\}$*-free graph has a Hamiltonian cycle.*

In [12], Ryjáček proved that if a 2-connected graph is $\{K_{1,3}, B_{1,2}\}$-free or $\{K_{1,3}, P_6\}$-free, then its closure is $\{K_{1,3}, N\}$-free. By using fundamental properties for closure, Ryjáček's result shows that Theorems 8 and 9 can be obtained from Theorem 7. Theorem 5 gives a more detailed relation among Theorems 7–9. Since all 2-connected fat i-paths and all 2-connected fat i-cycles have a Hamiltonian cycle for $i \geq 3$, Theorems 5 and 8 provide an alternative proof of Theorem 9.

We next consider the pancyclicity of graphs. Let k and m be integers with $0 \leq k \leq m$. A graph G of order $n \geq m$ is (k, m)*-pancyclic* if for every $X \subseteq V(G)$ with $|X| = k$ and every $m \leq i \leq n$, there exists a cycle of G with length i containing all vertices in X. The (k, m)-pancyclicity was defined by Faudree et al. [8] to generalize some pancyclic-type concepts, and Crane [4] recently proved the following result.

Theorem 10 (Crane [4]) *If G is a* 2*-connected* $\{K_{1,3}, P_5\}$*-free graph of order $n \geq 5$, then G is* $(1, 5)$*-pancyclic and* $(k, 3k)$*-pancyclic for all $k \geq 2$.*

Combining Theorems 4 and 10, we obtain the following result.

Theorem 11 *Let G be a* 2*-connected* $\{K_{1,3}, B_{1,1}\}$*-free graph of order $n \geq 5$. If G does not belong to $\mathcal{P}(5)$, then G is* $(1, 5)$*-pancyclic and* $(k, 3k)$*-pancyclic for all $k \geq 2$.*

Crane [5] also gave a result concerning (k, m)-pancyclicity for $\{K_{1,3}, P_6\}$-free graphs, and so we obtain a similar result as Theorem 11 for $\{K_{1,3}, B_{1,2}\}$-free graphs.

We conclude this section with the spanning Halin subgraph problem. A graph is *planar* if it can be embedded in the plane without edge-crossing, and such an embedded graph is called a *plane graph*. A *Halin graph*, named after Halin who introduced this concept in [10], is a plane graph consisting of a tree T without vertices of degree 2 and a cycle C whose vertex set is equal to the set of the leaves of T (and we often write a Halin graph H as $H = T \cup C$). It is known that 3-connectedness is a trivial necessary condition for a graph to have a spanning Halin subgraph. In [3], the following conjecture was proposed.

Conjecture 1 (Chen et al. [3]) *Let $\mathcal{H}$ be a forbidden pair. Then every* 3*-connected $\mathcal{H}$-free graph has a spanning Halin subgraph if and only if either $\mathcal{H} \leq \{K_{1,3}, Z_3\}$ or $\mathcal{H} \leq \{K_{1,3}, B_{1,2}\}$.*

The “only if” part of Conjecture 1 was already proved in [3]. Also, as a partial answer for the “if” part of the conjecture, the following theorem was proved.

Theorem 12 (Chen et al. [3]) *Every 3-connected $\{K_{1,3}, P_5\}$-free graph has a spanning Halin subgraph.*

As corollaries of Theorems 3, 4 and 12, we obtain other partial answers for the “if” part of Conjecture 1 which will be proved in Sect. 4.

Theorem 13 *Every 3-connected $\{K_{1,3}, B_{1,1}\}$-free graph has a spanning Halin subgraph.*

Theorem 14 *Every 3-connected $\{K_{1,3}, Z_2\}$-free graph has a spanning Halin subgraph.*

3 Proof of Main Results

In this section, we prove Theorems 3 and 6.

3.1 Proof of Theorem 3

Lemma 1 *Let G be a connected $\{K_{1,3}, Z_2\}$-free graph which contains an induced subgraph N. Then G is a pointed generalized comb.*

Proof Since G is Z_2-free and Z_2 is an induced subgraph of $B_{1,2}$, G is also $B_{1,2}$-free. This, together with Theorem 2, implies that G is a generalized comb. We only show that every leaf-clique consists of a single vertex. Let L_i $(1 \leq i \leq m)$ be the leaf-cliques of G, and let R_i be the root of L_i. On the contrary, we may assume that $|L_1| \geq 2$. Let $a_1, a_2 \in L_1$ with $a_1 \neq a_2$, $a_3 \in R_1$, $a_4 \in R_2$ and $a_5 \in L_2$. Then $G[\{a_1, a_2, a_3, a_4, a_5\}] \cong Z_2$, giving a contradiction. Hence G is a pointed generalized comb. □

In the following lemmas (Lemmas 2–8), we follow the labels given in Fig. 3.

Lemma 2 *Let G be a connected $\{K_{1,3}, Z_2, N\}$-free graph which contains an induced subgraph $H = H_1(\{C_{s_3}, C_{s_4}, C_{s_5}\})$, where C_{s_3}, C_{s_4} and C_{s_5} are disjoint cliques. Then for each vertex $a \in V(G) - V(H)$ with $N_G(a) \cap V(H) \neq \emptyset$, one of the following holds:*

- (i) *$C_{s_3} \cup C_{s_4} \cup C_{s_5} \subseteq N_G(a) \cap V(H)$ and $s_i \notin N_G(a) \cap V(H)$ for some $i \in \{1, 2\}$ (and so $G[V(H) \cup \{a\}] \in \mathcal{H}_1$),*
- (ii) *for some $i \in \{1, 2\}$, $N_G(a) \cap V(H) = \{s_i\} \cup C_{s_5}$ and $|C_{s_{5-i}}| = 1$ (and so $G[V(H) \cup \{a\}] \in \mathcal{H}_2$),*
- (iii) *for some $i \in \{3, 4\}$, $N_G(a) \cap V(H) = \{s_1, s_2\} \cup C_{s_i}$ and $|C_{s_{7-i}}| = |C_{s_5}| = 1$ (and so $G[V(H) \cup \{a\}] \in \mathcal{H}_3$), or*
- (iv) *$N_G(a) \cap V(H) = \{s_1, s_2\} \cup C_{s_3} \cup C_{s_4}$ and $|C_{s_5}| = 1$ (and so $G[V(H) \cup \{a\}] \in \mathcal{H}_4$).*

Proof For each $i \in \{3, 4, 5\}$, we take a vertex b_i as follows: if $N_G(a) \cap C_{s_i} \neq \emptyset$, let $b_i \in N_G(a) \cap C_{s_i}$; otherwise (i.e., $N_G(a) \cap C_{s_i} = \emptyset$), let $b_i \in C_{s_i}$.
Case 1: $N_G(a) \cap C_{s_5} \neq \emptyset$.

If $as_1, as_2 \in E(G)$, then $G[\{a, s_1, s_2, b_5\}] \cong K_{1,3}$, giving a contradiction. Thus $as_1 \notin E(G)$ or $as_2 \notin E(G)$. We may assume that $as_1 \notin E(G)$.

If $N_G(a) \cap C_{s_3} \neq \emptyset$ and $ab \notin E(G)$ for some $b \in C_{s_4}$, then $G[\{b_3, a, b, s_1\}] \cong K_{1,3}$; if $N_G(a) \cap C_{s_4} \neq \emptyset$ and $ab \notin E(G)$ for some $b \in C_{s_3}$, then $as_2 \in E(G)$ because $G[\{b_4, a, b, s_2\}] \not\cong K_{1,3}$, and hence $G[\{a, s_2, b_4, b, s_1\}] \cong Z_2$. In either case, we get a contradiction. This implies that either $C_{s_3} \cup C_{s_4} \subseteq N_G(a)$ or $N_G(a) \cap (C_{s_3} \cup C_{s_4}) = \emptyset$.
Subcase 1.1: $C_{s_3} \cup C_{s_4} \subseteq N_G(a)$.

If $ab \notin E(G)$ for some $b \in C_{s_5}$, then $G[\{b_3, a, s_1, b\}] \cong K_{1,3}$, giving a contradiction. Thus $C_{s_5} \subseteq N_G(a)$. If $as_2 \in E(G)$, let $C'_{s_i} = C_{s_i}$ $(i \in \{3, 5\})$ and $C'_{s_4} = C_{s_4} \cup \{a\}$; if $as_2 \notin E(G)$, let $C'_{s_i} = C_{s_i}$ $(i \in \{3, 4\})$ and $C'_{s_5} = C_{s_5} \cup \{a\}$. Then $G[V(H) \cup \{a\}] = H_1(\{C'_{s_3}, C'_{s_4}, C'_{s_5}\}) \in \mathcal{H}_1$, and so (i) holds.
Subcase 1.2: $N_G(a) \cap (C_{s_3} \cup C_{s_4}) = \emptyset$.

Since $G[\{a, s_1, s_2, b_5, b_3, b_4\}] \not\cong N$, we have $as_2 \in E(G)$. If $ab \notin E(G)$ for some $b \in C_{s_5}$, then $G[\{b, b_3, b_4, s_2, a\}] \cong Z_2$, giving a contradiction. Thus $C_{s_5} \subseteq N_G(a)$, and hence $N_G(a) \cap V(H) = \{s_2\} \cup C_{s_5}$. If $|C_{s_3}| \geq 2$, then $G[\{b_3, b, b_4, s_2, a\}] \cong Z_2$ where $b \in C_{s_3} - \{b_3\}$, giving a contradiction. Thus $|C_{s_3}| = 1$, and so (ii) holds.
Case 2: $N_G(a) \cap C_{s_5} = \emptyset$ (i.e., $ab_5 \notin E(G)$). □

Claim For each $i \in \{3, 4\}$, if $N_G(a) \cap C_{s_i} \neq \emptyset$, then $N_G(a) \supseteq \{s_1, s_2\} \cup C_{s_i}$.

Proof We may assume $i = 3$. Since $G[\{b_3, a, b_5, s_1\}] \not\cong K_{1,3}$, we have $as_1 \in E(G)$. By the same argument, if $N_G(a) \cap C_{s_4} \neq \emptyset$, then $as_2 \in E(G)$. Since $G[\{a, s_1, b_3, b_4, s_2\}] \not\cong Z_2$, we have $as_2 \in E(G)$ or $ab_4 \in E(G)$. In either case, we have $as_2 \in E(G)$. If $ab \notin E(G)$ for some $b \in C_{s_3}$, then $G[\{b_5, b, b_3, a, s_2\}] \cong Z_2$, giving a contradiction. Thus $C_{s_3} \subseteq N_G(a)$. □

Suppose $N_G(a) \cap (C_{s_3} \cup C_{s_4}) = \emptyset$. Since $N_G(a) \cap V(H) \neq \emptyset$, we have $as_i \in E(G)$ for some $i \in \{1, 2\}$. Hence $G[\{b_5, b_{5-i}, b_{i+2}, s_i, a\}] \cong Z_2$, giving a contradiction. Thus $N_G(a) \cap (C_{s_3} \cup C_{s_4}) \neq \emptyset$. We may assume that $N_G(a) \cap C_{s_3} \neq \emptyset$. This together with Claim 3.1 forces $\{s_1, s_2\} \cup C_{s_3} \subseteq N_G(a)$. If $|C_{s_5}| \geq 2$, then $G[\{b_5, b, b_3, a, s_2\}] \cong Z_2$ where $b \in C_{s_5} - \{b_5\}$, giving a contradiction. Thus $|C_{s_5}| = 1$.

If $N_G(a) \cap C_{s_4} \neq \emptyset$, then $C_{s_4} \subseteq N_G(a)$ by Claim 3.1, and hence (iv) holds. Thus we may assume that $N_G(a) \cap C_{s_4} = \emptyset$ (i.e., $N_G(a) \cap V(H) = \{s_1, s_2\} \cup C_{s_3}$). If $|C_{s_4}| \geq 2$, then $G[\{b_4, b, s_2, a, s_1\}] \cong Z_2$ in G where $b \in C_{s_4} - \{b_4\}$, giving a contradiction. Hence $|C_{s_4}| = 1$, and so (iii) holds. □

Lemma 3 *Let G be a connected $\{K_{1,3}, Z_2, N\}$-free graph which contains an induced subgraph $H = H_2(\{C_{t_3}, C_{t_4}\})$, where C_{t_3} and C_{t_4} are disjoint cliques. Then for each vertex $a \in V(G) - V(H)$ with $N_G(a) \cap V(H) \neq \emptyset$, one of the following holds:*

- (i) *for some $i \in \{5, 6\}$, $N_G(a) \cap V(H) = \{t_2, t_i\} \cup C_{t_3} \cup C_{t_4}$ (and so $G[V(H) \cup \{a\}] \in \mathcal{H}_2$),*
- (ii) *for some $i \in \{3, 4\}$, $N_G(a) \cap V(H) = \{t_1, t_{i+2}\} \cup C_{t_i}$ and $|C_{t_{7-i}}| = 1$ (and so $G[V(H) \cup \{a\}] \in \mathcal{H}_5$), or*

(iii) $N_G(a) \cap V(H) = \{t_1, t_2, t_5, t_6\}$ *and* $|C_{t_3}| = |C_{t_4}| = 1$ *(and so* $G[V(H) \cup \{a\}] \in \mathcal{H}_6$*).*

Proof For each $i \in \{3, 4\}$, let $b_i \in C_{t_i}$. For each $i \in \{5, 6\}$, we note that the graph $B_i := H - t_i$ belongs to $\mathcal{H}_1$.

Case 1: $N_G(a) \cap (C_{t_3} \cup C_{t_4}) = \emptyset$.

Since $N_G(a) \cap V(H) \neq \emptyset$, $N_G(a) \cap V(B_i) \neq \emptyset$ for some $i \in \{5, 6\}$. We may assume that $N_G(a) \cap V(B_5) \neq \emptyset$. Since $N_G(a) \cap (C_{t_3} \cup C_{t_4}) = \emptyset$, we have $N_G(a) \cap V(B_5) = \{t_1, t_2, t_6\}$ and $|C_{t_4}| = 1$ by Lemma 2. In particular, $N_G(a) \cap V(B_6) \neq \emptyset$. Then again by Lemma 2, $N_G(a) \cap V(B_6) = \{t_1, t_2, t_5\}$ and $|C_{t_3}| = 1$. This implies that $N_G(a) \cap V(H) = \{t_1, t_2, t_5, t_6\}$ and $|C_{t_3}| = |C_{t_4}| = 1$, and so (iii) holds.

Case 2: $N_G(a) \cap (C_{t_3} \cup C_{t_4}) \neq \emptyset$.

We may assume that $N_G(a) \cap C_{t_3} \neq \emptyset$. If $N_G(a) \cap V(B_5) = \{t_6\} \cup C_{t_3}$, then either $N_G(a) \cap V(B_6) = C_{t_3}$ or $N_G(a) \cap V(B_6) = \{t_5\} \cup C_{t_3}$, which contradicts Lemma 2. Thus, by Lemma 2, we have either $G[V(B_5) \cup \{a\}] \in \mathcal{H}_1$, or $N_G(a) \cap V(B_5) = \{t_1\} \cup C_{t_3}$ and $|C_{t_4}| = 1$.

Subcase 2.1: $G[V(B_5) \cup \{a\}] \in \mathcal{H}_1$.

We see that $\{t_2\} \cup C_{t_3} \cup C_{t_4} \subseteq N_G(a)$. Since $G[\{a, t_2, b_4, t_6, t_5\}] \ncong Z_2$, we have $at_5 \in E(G)$ or $at_6 \in E(G)$. We may assume that $at_5 \in E(G)$. If $at_1 \in E(G)$, then $G[\{a, t_1, b_4, t_5\}] \cong K_{1,3}$, giving a contradiction. Thus $at_1 \notin E(G)$. So, $at_6 \notin E(G)$ because $G[\{t_5, t_6, a, t_2, t_1\}] \ncong Z_2$. Hence we get $N_G(a) \cap V(H) = \{t_2, t_5\} \cup C_{t_3} \cup C_{t_4}$. Consequently, $G[V(H) \cup \{a\}] = H_2(\{C'_{t_3}, C'_{t_4}\})$ where $C'_{s_3} = C_{s_3} \cup \{a\}$ and $C'_{s_4} = C_{s_4}$, and so (i) holds.

Subcase 2.2: $N_G(a) \cap V(B_5) = \{t_1\} \cup C_{t_3}$ and $|C_{t_4}| = 1$.

Since $G[\{b_3, a, t_2, t_5\}] \ncong K_{1,3}$, we have $at_5 \in E(G)$. Hence $N_G(a) \cap V(H) = \{t_1, t_5\} \cup C_{t_3}$ and $|C_{t_4}| = 1$, and so (ii) holds. □

Lemma 4 *Let G be a connected* $\{K_{1,3}, Z_2, N\}$*-free graph which contains an induced subgraph* $H = H_3(\{C_{u_6}\})$*, where* C_{u_6} *is a clique. Then for each vertex* $a \in V(G) - V(H)$ *with* $N_G(a) \cap V(H) \neq \emptyset$*, one of the following holds:*

(i) $N_G(a) \cap V(H) = \{u_2, u_3, u_4, u_5\} \cup C_{u_6}$ *(and so* $G[V(H) \cup \{a\}] \in \mathcal{H}_3$*),*
(ii) $N_G(a) \cap V(H) = \{u_1, u_i, u_{7-i}\}$ *for some* $i \in \{2, 3\}$ *and* $|C_{u_6}| = 1$ *(and so* $G[V(H) \cup \{a\}] \in \mathcal{H}_6$*), or*
(iii) $N_G(a) \cap V(H) = \{u_4, u_5\}$ *(and so* $G[V(H) \cup \{a\}] \in \mathcal{H}_5$*).*

Proof For each $i \in \{2, 3\}$, we note that the graph $B_i := H - u_i$ belongs to $\mathcal{H}_1$. Since $N_G(a) \cap V(H) \neq \emptyset$, $N_G(a) \cap V(B_i) \neq \emptyset$ for some $i \in \{2, 3\}$. If $au_4, au_5 \notin E(G)$, then $N_G(a) \cap V(B_i) \subseteq \{u_1, u_{u_{5-i}}\} \cup C_{u_6}$ for each $i \in \{2, 3\}$, which contradicts Lemma 2. Thus, $au_4 \in E(G)$ or $au_5 \in E(G)$. We may assume that $au_4 \in E(G)$. Then by Lemma 2, we have either $G[V(B_3) \cup \{a\}] \in \mathcal{H}_1$, or $N_G(a) \cap V(B_3) = \{u_1, u_4\}$ and $|C_{u_6}| = 1$, or $N_G(a) \cap V(B_3) = \{u_4, u_5\}$.

Case 1: $G[V(B_3) \cup \{a\}] \in \mathcal{H}_1$.

In this case, we have $\{u_2, u_4\} \cup C_{u_6} \subseteq N_G(a)$. Then again by Lemma 2, we have either $G[V(B_2) \cup \{a\}] \in \mathcal{H}_1$ or $N_G(a) \cap V(B_2) = \{u_1, u_4\} \cup C_{u_6}$ or $N_G(a) \cap V(B_2) = \{u_1, u_3, u_4\} \cup C_{u_6}$. If $N_G(a) \cap V(B_2) = \{u_1, u_4\} \cup C_{u_6}$ (i.e., $N_G(a) \cap V(H) = \{u_1, u_2, u_4\} \cup C_{u_6}$), then $G[\{u_2, a, u_1, u_3, u_5\}] \cong Z_2$; if

$N_G(a) \cap V(B_2) = \{u_1, u_3, u_4\} \cup C_{u_6}$ (i.e., $N_G(a) \cap V(H) = \{u_1, u_2, u_3, u_4\} \cup C_{u_6}$), then $G[\{u_2, u_4, a, u_3, u_5\}] \cong Z_2$. In either case, we get a contradiction. Thus $G[V(B_2) \cup \{a\}] \in \mathcal{H}_1$. Since $au_4 \in E(G)$, we see that $N_G(a) \cap V(B_2) = \{u_3, u_4, u_5\} \cup C_{u_6}$, and hence $N_G(a) \cap V(H) = \{u_2, u_3, u_4, u_5\} \cup C_{u_6}$. Consequently, $G[V(H) \cup \{a\}] = H_3(\{C'_{u_6}\})$ where $C'_{u_6} = C_{u_6} \cup \{a\}$, and so (i) holds.

Case 2: $N_G(a) \cap V(B_3) = \{u_1, u_4\}$ and $|C_{u_6}| = 1$.

Since $au_2 \notin E(G)$ and $G[\{u_1, u_2, u_3, a\}] \ncong K_{1,3}$, we have $au_3 \in E(G)$. Hence $N_G(a) \cap V(H) = \{u_1, u_3, u_4\}$ and $|C_{u_6}| = 1$, and so (ii) holds.

Case 3: $N_G(a) \cap V(B_3) = \{u_4, u_5\}$.

Since $G[\{u_3, u_1, b, a\}] \ncong K_{1,3}$ for any $b \in C_{u_6}$, we have $au_3 \notin E(G)$. Hence $N_G(a) \cap V(H) = \{u_4, u_5\}$, and so (iii) holds. □

Lemma 5 *Let G be a connected $\{K_{1,3}, Z_2, N\}$-free graph which contains an induced subgraph $H = H_4(\{C_{v_4}, C_{v_5}, C_{v_6}\})$, where C_{v_4}, C_{v_5} and C_{v_6} are disjoint cliques. Then for each vertex $a \in V(G) - V(H)$ with $N_G(a) \cap V(H) \neq \emptyset$, we have $N_G(a) \cap V(H) = (\{v_1, v_2, v_3\} - \{v_i\}) \cup C_{v_4} \cup C_{v_5} \cup C_{v_6}$ for some $i \in \{1, 2, 3\}$. Consequently, $G[V(H) \cup \{a\}] \in \mathcal{H}_4$.*

Proof For each $i \in \{4, 5, 6\}$, let $b_i \in C_{v_i}$. For each $i \in \{5, 6\}$, we note that the graph $B_i := H - C_{v_i}$ belongs to $\mathcal{H}_1$.

Suppose $N_G(a) \cap \{v_1, v_2, v_3\} = \emptyset$. Since $N_G(a) \cap V(H) \neq \emptyset$, we may assume that $ab_4 \in E(G)$. Then $G[\{b_4, a, v_1, v_2\}] \cong K_{1,3}$, giving a contradiction. Thus, $N_G(a) \cap \{v_1, v_2, v_3\} \neq \emptyset$. We may assume that $av_1 \in E(G)$. Then, by Lemma 2, we have $G[V(B_5) \cup \{a\}] \in \mathcal{H}_1$ or $N_G(a) \cap V(B_5) = \{v_1, v_i\}$ for some $i \in \{2, 3\}$.

Suppose that $N_G(a) \cap V(B_5) = \{v_1, v_i\}$ for some $i \in \{2, 3\}$. In this case, we may assume that $N_G(a) \cap V(B_5) = \{v_1, v_2\}$. Then by Lemma 2, $N_G(a) \cap V(B_6) = \{v_1, v_2\}$. In particular, $N_G(a) \cap V(H) = \{v_1, v_2\}$. Then $G[\{b_5, v_3, b_6, v_1, a\}] \cong Z_2$, giving a contradiction. Thus $G[V(B_5) \cup \{a\}] \in \mathcal{H}_1$.

Hence we have $N_G(a) \cap V(B_5) = \{v_1\} \cup C_{v_4} \cup C_{v_6}$ or $N_G(a) \cap V(B_5) = \{v_1, v_i\} \cup C_{v_4} \cup C_{v_6}$ for some $i \in \{2, 3\}$. If $N_G(a) \cap V(B_5) = \{v_1\} \cup C_{v_4} \cup C_{v_6}$, then $ab_5 \in E(G)$ because $G[\{a, v_1, b_6, b_5, v_2\}] \ncong Z_2$, and hence $G[\{b_5, a, v_2, v_3\}] \cong K_{1,3}$, giving a contradiction. Thus $N_G(a) \cap V(B_5) = \{v_1, v_i\} \cup C_{v_4} \cup C_{v_6}$ for some $i \in \{2, 3\}$. We may assume that $N_G(a) \cap V(B_5) = \{v_1, v_2\} \cup C_{v_4} \cup C_{v_6}$. Here we focus on the subgraph of G induced by $V(B_6) \cup \{a\}$. Since B_6 belongs to $\mathcal{H}_1$, it follows from Lemma 2 that $C_{v_5} \subseteq N_G(a)$. In particular, $N_G(a) \cap V(H) = \{v_1, v_2\} \cup C_{v_4} \cup C_{v_5} \cup C_{v_6}$. Therefore $G[V(H) \cup \{a\}] = H_4(\{C'_{v_4}, C'_{v_5}, C'_{v_6}\})$ where $C'_{s_4} = C_{s_4} \cup \{a\}$ and $C'_{s_i} = C_{s_i}$ $(i \in \{5, 6\})$. □

Lemma 6 *Let G be a connected $\{K_{1,3}, Z_2, N\}$-free graph which contains an induced subgraph $H = H_5(\{C_{w_7}\})$, where C_{w_7} is a clique. Then for each vertex $a \in V(G) - V(H)$ with $N_G(a) \cap V(H) \neq \emptyset$, one of the following holds:*

(i) *$N_G(a) \cap V(H) = \{w_3, w_4, w_5, w_6\} \cup C_{w_7}$ (and so $G[V(H) \cup \{a\}] \in \mathcal{H}_5$), or*
(ii) *$N_G(a) \cap V(H) = \{w_1, w_2, w_i, w_{9-i}\}$ for some $i \in \{3, 4\}$ and $|C_{w_7}| = 1$ (and so $G[V(H) \cup \{a\}] \cong H_7$).*

Proof For each $i \in \{1, 2\}$, we note that the graph $B_i := H - w_i$ belongs to $\mathcal{H}_3$. Since $N_G(a) \cap V(H) \neq \emptyset$, $N_G(a) \cap V(B_i) \neq \emptyset$ for some $i \in \{1, 2\}$. We may assume that

$N_G(a) \cap V(B_1) \neq \emptyset$. If $N_G(a) \cap V(B_1) = \{w_3, w_5\}$, then either $N_G(a) \cap V(B_2) = \{w_3, w_5\}$ or $N_G(a) \cap V(B_2) = \{w_1, w_3, w_5\}$, which contradicts Lemma 4. This, together with Lemma 4, implies that either $G[V(B_1) \cup \{a\}] \in \mathcal{H}_3$ or $N_G(a) \cap V(B_1) = \{w_2, w_i, w_{9-i}\}$ for some $i \in \{3, 4\}$ and $|C_{w_7}| = 1$.

Case 1: $G[V(B_1) \cup \{a\}] \in \mathcal{H}_3$.

Note that we have either $N_G(a) \cap V(B_2) = \{w_3, w_4, w_5, w_6\} \cup C_{w_7}$ or $N_G(a) \cap V(B_2) = \{w_1, w_3, w_4, w_5, w_6\} \cup C_{w_7}$. This, together with Lemma 4, leads to $N_G(a) \cap V(H) = \{w_3, w_4, w_5, w_6\} \cup C_{w_7}$. Hence, $G[V(H) \cup \{a\}] = H_5(\{C'_{w_7}\}) \in \mathcal{H}_5$ where $C'_{w_7} = C_{w_7} \cup \{a\}$, and so (i) holds.

Case 2: $N_G(a) \cap V(B_1) = \{w_2, w_i, w_{9-i}\}$ for some $i \in \{3, 4\}$ and $|C_{w_7}| = 1$.

We may assume that $N_G(a) \cap V(B_1) = \{w_2, w_3, w_6\}$. Then $N_G(a) \cap V(B_2) = \{w_3, w_6\}$ or $N_G(a) \cap V(B_2) = \{w_1, w_3, w_6\}$. This, together with Lemma 4, leads to $N_G(a) \cap V(H) = \{w_1, w_2, w_3, w_6\}$, and so (ii) holds. □

Lemma 7 *Let G be a connected $\{K_{1,3}, Z_2, N\}$-free graph which contains an induced subgraph $H = H_6$. Then for each vertex $a \in V(G) - V(H)$ with $N_G(a) \cap V(H) \neq \emptyset$, $N_G(a) \cap V(H) = \{x_i, x_{i+1}, x_7\}$ for some $i \in \{1, 3\}$. Consequently, $G[V(H) \cup \{a\}] \cong H_7$.*

Proof We note that the graph $B := H - x_1$ belongs to $\mathcal{H}_3$, and the graph $B^* := H - x_5$ belongs to $\mathcal{H}_2$.

We first suppose that $ax_i, ax_{i+2} \in E(G)$ for some $i \in \{1, 2\}$. We may assume that $ax_1, ax_3 \in E(G)$. Then by Lemma 3, we have $N_G(a) \cap V(B^*) = \{x_1, x_3, x_6, x_7\}$, and hence either $N_G(a) \cap V(B) = \{x_3, x_6, x_7\}$ or $N_G(a) \cap V(B) = \{x_3, x_5, x_6, x_7\}$, which contradicts Lemma 4. Thus,

$$\text{for each } i \in \{1, 2\}, \text{ either } ax_i \notin E(G) \text{ or } ax_{i+2} \notin E(G). \tag{1}$$

If $N_G(a) \cap V(B^*) \neq \emptyset$, then $|N_G(a) \cap V(B^*)| \geq 2$ by Lemma 3. In particular, we have $N_G(a) \cap V(B) \neq \emptyset$. If $N_G(a) \cap \{x_1, x_2, x_3, x_4\} = \emptyset$, then $N_G(a) \cap V(B) \subseteq \{x_5, x_6, x_7\}$, which contradicts Lemma 4. Thus $N_G(a) \cap \{x_1, x_2, x_3, x_4\} \neq \emptyset$. By the symmetry of $x_1, \ldots, x_4$, we may assume that $ax_1 \in E(G)$. By (1), $ax_3 \notin E(G)$. Since $G[\{x_1, a, x_2, x_3\}] \not\cong K_{1,3}$, we have $ax_2 \in E(G)$. So, $ax_4 \notin E(G)$ by (1). Then, by Lemma 4, $N_G(a) \cap V(B) = \{x_2, x_7\}$. Consequently, $N_G(a) \cap V(H) = \{x_1, x_2, x_7\}$. □

Lemma 8 *Let G be a connected $\{K_{1,3}, Z_2, N\}$-free graph which contains an induced subgraph $H = H_7$. Then for each vertex $a \in V(G) - V(H)$ with $N_G(a) \cap V(H) \neq \emptyset$, $N_G(a) \cap V(H) = \{y_1, y_2, y_7, y_8\}$. Consequently, $G[V(H) \cup \{a\}] \cong H_8$.*

Proof For each $i \in \{1, 2\}$, we note that the graph $B_i := H - y_i$ is isomorphic to H_6. Since $N_G(a) \cap V(H) \neq \emptyset$, we have $N_G(a) \cap V(B_i) \neq \emptyset$ for some $i \in \{1, 2\}$. We may assume that $N_G(a) \cap V(B_1) \neq \emptyset$. Then, by Lemma 7, $N_G(a) \cap V(B_1) = \{y_2, y_3, y_5\}$ or $N_G(a) \cap V(B_1) = \{y_2, y_7, y_8\}$. In particular, $\{y_3, y_5\} \subseteq N_G(a) \cap V(B_2)$ or $\{y_7, y_8\} \subseteq N_G(a) \cap V(B_2)$. This, together with Lemma 7, leads to $N_G(a) \cap V(B_1) = \{y_2, y_7, y_8\}$ and $N_G(a) \cap V(B_2) = \{y_1, y_7, y_8\}$. So, $N_G(a) = \{y_1, y_2, y_7, y_8\}$. □

Proof of Theorem 3 By a routine but tedious argument, we can verify that every graph in $\bigcup_{0\le i\le 8}\mathcal{H}_i$ is $\{K_{1,3}, Z_2\}$-free but not $B_{1,1}$-free (and we omit its detail). Thus it suffices to show that, if a connected $\{K_{1,3}, Z_2\}$-free graph G is not $B_{1,1}$-free (i.e., G contains $B_{1,1}$ as an induced subgraph), then G belongs to $\bigcup_{0\le i\le 8}\mathcal{H}_i$.

If G is not N-free, then by Lemma 1, $G \in \mathcal{H}_0$, as desired. Thus we may assume that G is $\{K_{1,3}, Z_2, N\}$-free. Assume that G contains $B_{1,1}$ ($\in \mathcal{H}_1$) as an induced subgraph. Then G contains a graph $H \in \bigcup_{1\le i\le 8}\mathcal{H}_i$ as an induced subgraph. Choose H so that $|V(H)|$ is as large as possible. It suffices to show that $G = H$. By way of contradiction, suppose that $G \ne H$ (i.e., $V(G) - V(H) \ne \emptyset$). Since G is connected, there exists a vertex $a \in V(G) - V(H)$ which is adjacent to a vertex in $V(H)$. By the maximality of H, $G[V(H) \cup \{a\}] \notin \bigcup_{1\le i\le 8}\mathcal{H}_i$. This, together with Lemmas 2–8, gives $H = H_8$. For each $i \in \{7, 8, 9\}$, we note that the graph $B_i := H - z_i$ is isomorphic to H_7. Since $N_G(a) \cap V(H) \ne \emptyset$, we have $N_G(a) \cap V(B_i) \ne \emptyset$ for some $i \in \{7, 8\}$. We may assume that $N_G(a) \cap V(B_7) \ne \emptyset$. Then by Lemma 8, $N_G(a) \cap V(B_7) = \{z_3, z_4, z_8, z_9\}$. In particular, $az_3 \in E(G)$. On the other hand, since $N_G(a) \cap V(B_9) \ne \emptyset$, $N_G(a) \cap V(B_9) = \{z_1, z_2, z_7, z_8\}$, and so $az_3 \notin E(G)$, giving a contradiction.

This completes the proof of Theorem 3. □

3.2 Proof of Theorem 6

In order to prove Theorem 6, we give a further definition. For two integers s and t, we let $[s, t] = \{i \in \mathbb{N} \mid s \le i \le t\}$. Note that if $s > t$, then $[s, t] = \emptyset$.

Here we prove Theorem 6. We can easily verify that every graph in $\mathcal{P}(\max\{3m, m+4\})$ is $\{K_{1,3}, B_{1,m}\}$-free but not $P_{\max\{3m,m+4\}}$-free. Thus it suffices to show that if a connected $\{K_{1,3}, B_{1,m}\}$-free graph G is not $P_{\max\{3m,m+4\}}$-free (i.e., G contains $P_{\max\{3m,m+4\}}$ as an induced subgraph), then G belongs to $\mathcal{P}(\max\{3m, m + 4\})$.

Assume that G contains $P_{\max\{3m,m+4\}}$ as an induced subgraph. Then G contains a graph $H \in \mathcal{P}(\max\{3m, m + 4\})$ as an induced subgraph. Choose H so that $|V(H)|$ is as large as possible. It suffices to show that $G = H$. Otherwise, there exists a vertex $a \in V(G) - V(H)$ such that $N_G(a) \cap V(H) \ne \emptyset$. Let ℓ be the integer such that H is either a fat ℓ-path or a fat ℓ-cycle. Then we can write either $H = F_p(L_1, \ldots, L_\ell)$ or $H = F_c(L_0, \ldots, L_\ell)$ for some disjoint cliques $L_0, \ldots, L_\ell$. Let $I = \{i \mid N_G(a) \cap L_i \ne \emptyset\}$.

Claim $|I| \le 4$.

Proof Suppose that there are five fundamental cliques $L^{(1)}, \ldots, L^{(5)}$ of H with $N_G(a) \cap L^{(i)} \ne \emptyset$ $(1 \le i \le 5)$. For each i $(1 \le i \le 5)$, let $b^{(i)} \in N_G(a) \cap L^{(i)}$. Since $\max\{3m, m + 4\} \ge 5$, if H is a fat cycle, then H has at least six fundamental cliques. Thus $G[\{b^{(i)} \mid 1 \le i \le 5\}]$ has no cycle, and so is a forest of order five and maximum degree at most two. Then we can easily check that $G[\{b^{(i)} \mid 1 \le i \le 5\}]$ has an independent set B with $|B| = 3$, and hence $G[\{a\} \cup B] \cong K_{1,3}$, giving a contradiction. □

If H is a fat cycle, then $N_G(a) \cap L_i = \emptyset$ for some $0 \le i \le \ell$ by Claim 3.2. By relabeling $L_0, \ldots, L_\ell$ if necessary, we may assume that

(L1) $0 \notin I$, and
(L2) subject to (L1), $|I \cap \{1, \ell\}|$ is as small as possible.

Thus, if H is a fat cycle and there exists an integer i $(1 \le i \le \ell-2)$ with $i, i+1, i+2 \notin I$, then $I \cap \{0, 1, \ell\} = \emptyset$.

For each i $(1 \le i \le \ell)$, we take a vertex b_i as follows: if $i \in I$, let $b_i \in N_G(a) \cap L_i$; otherwise (i.e., $i \notin I$), let $b_i \in L_i$. Note that, by our choices of indices, $b_1 b_\ell \notin E(G)$ regardless of H being a fat path or a fat cycle.

Claim Assume that there exists an index j $(2 \le j \le l-2)$ such that $I \cap [2, l-1] = \{j, j+1\}$. Then either $j = 2$ and $ab_1 \in E(G)$ or $j = l-2$ and $ab_\ell \in E(G)$.

Proof Recall that $\ell \ge \max\{3m, m+4\}$. We first consider the case $\ell - m - 1 \le j \le m+1$. Then $\ell \le 2m+2$. Since $\ell \ge 3m$, we have $m \le 2$; since $\ell \ge m+4$, we have $m \ge 2$. Hence $m = 2$, and this forces $\ell = 6$ and $j = 3$. By the assumption of the claim, $ab_2, ab_5 \notin E(G)$. Since $G[\{b_2, b_3, a, b_4, b_5, b_6\}] \ncong B_{1,2}$ and $G[\{b_5, b_4, a, b_3, b_2, b_1\}] \ncong B_{1,2}$, we have $ab_1, ab_6 \in E(G)$. Then $G[\{a, b_1, b_3, b_6\}] \cong K_{1,3}$, giving a contradiction. Thus either $j \ge m+2$ or $j \le \ell - m - 2$.

We now consider the case $j \ge m+2$ (i.e., $j - m \ge 2$). Then $ab_i \notin E(G)$ for every $j - m \le i \le j-1$. Since $G[\{b_{j+2}, b_{j+1}, a, b_j, b_{j-1}, \dots, b_{j-m}\}] \ncong B_{1,m}$, this forces $b_{j+2} = b_l$ (i.e., $j = \ell - 2$) and $ab_\ell \in E(G)$, as desired. Thus we may assume that $j \le \ell - m - 2$ (i.e., $j + m + 1 \le \ell - 1$). Then $ab_i \notin E(G)$ for every $j+2 \le i \le j+m+1$. Since $G[\{b_{j-1}, b_j, a, b_{j+1}, b_{j+2}, \dots, b_{j+m+1}\}] \ncong B_{1,m}$, this forces $b_{j-1} = b_1$ (i.e., $j = 2$) and $ab_1 \in E(G)$, as desired. □

Claim For each $j \in I$, there exists an index j' $(j' \ne j)$ such that $|j - j'| = 1$ and $j' \in I$.

Proof If $2 \le j \le \ell - 1$ and $j-1, j+1 \notin I$, then $G[\{b_j, a, b_{j-1}, b_{j+1}\}] \cong K_{1,3}$, giving a contradiction. Hence if $2 \le j \le \ell - 1$, then the desired conclusion holds. Thus we may assume that $j \in \{1, \ell\}$ by (L1).

For the moment, we assume that $j = 1$ and $2 \notin I$. We further suppose that there exists an index i $(3 \le i \le \ell - 1)$ with $i \in I$. Choose i so that i is as small as possible. Then, $i+1 \in I$ since $3 \le i \le \ell - 1$. If $i+1 \le \ell - 1$ and $I \cap [3, \ell-1] = \{i, i+1\}$, then $i = l-2$ and $\ell \in I$ by Claim 3.2. This implies that if $i+1 \le \ell - 1$ (i.e., $i \le \ell - 2$), then there are three indices i_1, i_2, i_3 $(3 \le i_1 < i_2 < i_3 \le \ell)$ with $i_1, i_2, i_3 \in I$, and hence $G[\{a, b_1, b_{i_1}, b_{i_3}\}] \cong K_{1,3}$, giving a contradiction. Thus $i \ge \ell - 1$, and so $i = \ell - 1$. Note that $I \cap [1, \ell] = \{1, \ell-1, \ell\}$. This, together with the fact $\ell - m - 1 \ge 3$, implies that $G[\{b_1, a, b_\ell, b_{\ell-1}, \dots, b_{\ell-m-1}\}] \cong B_{1,m}$, giving a contradiction. Thus $I \cap [2, \ell-1] = \emptyset$. By (L1) and (L2), we see that H is a fat path. If there exists a vertex $u \in L_1$ with $au \notin E(G)$, then, since $\ell \ge m+4$, $G[\{a, b_1, u, b_2, \dots, b_{m+2}\}] \cong B_{1,m}$, giving a contradiction. Thus $L_1 \subseteq N_G(a)$. By the symmetry and the fact $\ell - 1 \notin I$, if $\ell \in I$, then $L_\ell \subseteq N_G(a)$. Hence either $N_G(a) \cap V(H) = L_1$ or $N_G(a) \cap V(H) = L_1 \cup L_\ell$. If $N_G(a) \cap V(H) = L_1$, then $G[V(H) \cup \{a\}] = F_p(\{a\}, L_1, \dots, L_\ell)$; if $N_G(a) \cap V(H) = L_1 \cup L_\ell$, then $G[V(H) \cup \{a\}] = F_c(\{a\}, L_1, \dots, L_\ell)$. In either case, $G[V(H) \cup \{a\}] \in \mathcal{P}(\max\{3m, m+4\})$, which contradicts the maximality of H. Thus if $j = 1$, then $2 \in I$. By the symmetry, if $j = \ell$, then $\ell - 1 \in I$. □

Let $i_1 = \min\{i \mid i \in I\}$ and $i_2 = \max\{i \mid i \in I\}$. By Claim 3.2, $i_1 + 1, i_2 - 1 \in I$.

Claim If $i_1 \neq 1$, then $L_{i_1+1} \subseteq N_G(a)$. If $i_2 \neq l$, then $L_{i_2-1} \subseteq N_G(a)$.

Proof If $i_1 \neq 1$ and $L_{i_1+1} \nsubseteq N_G(a)$, say $u \in L_{i_1+1} - N_G(a)$, then $G[\{b_{i_1}, b_{i_1-1}, a, u\}] \cong K_{1,3}$, giving a contradiction. Thus if $i_1 \neq 1$, then $L_{i_1+1} \subseteq N_G(a)$. By the symmetry, we have $L_{i_2-1} \subseteq N_G(a)$ if $i_2 \neq \ell$. □

Since $|I| \leq 4$, we divide the rest of the proof into three cases according to $|I| \leq 2$, $|I| = 3$, and $|I| = 4$.

Case 1: $|I| \leq 2$.

By Claim 3.2, $I = \{i_1, i_2\} = \{i_1, i_1 + 1\}$. If $|I \cap [2, \ell - 1]| = 2$, then either $1 \in I$ or $\ell \in I$ by Claim 3.2, and so $|I| \geq 3$, giving a contradiction. Thus $|I \cap [2, \ell - 1]| \leq 1$, which implies either $I = \{1, 2\}$ or $I = \{\ell - 1, \ell\}$. We may assume that $I = \{1, 2\}$. By (L2), H is a fat path. If $L_2 \nsubseteq N_G(a)$, say $u \in L_2 - N_G(a)$, then $G[\{a, b_2, u, b_3, b_4, \ldots, b_{m+3}\}] \cong B_{1,m}$, giving a contradiction. Thus $L_2 \subseteq N_G(a)$. This, together with Claim 3.2, leads to $N_G(a) \cap V(H) = L_1 \cup L_2$, and hence $G[V(H) \cup \{a\}] = F_p(L_1 \cup \{a\}, L_2, \ldots, L_\ell) \in \mathcal{P}(\max\{3m, m + 4\})$, which contradicts the maximality of H.

Case 2: $|I| = 3$.

In this case, $I = \{i_1, i_1 + 1\ (= i_2 - 1), i_2\} = \{i_1, i_1 + 1, i_1 + 2\}$. By (L1) and (L2), either H is a fat cycle and $I \cap \{0, 1, \ell\} = \emptyset$ or H is a fat path. Since either $i_1 \neq 1$ or $i_2 \neq l$, $L_{i_1+1}\ (= L_{i_2-1}) \subseteq N_G(a)$ by Claim 3.2. Suppose that either $L_{i_1} \nsubseteq N_G(a)$ or $L_{i_2} \nsubseteq N_G(a)$. We may assume that $L_{i_1} \nsubseteq N_G(a)$. Let $u \in L_{i_1} - N_G(a)$. If $i_1 \leq \ell - m - 2$ (i.e., $i_1 + m + 2 \leq l$), then $G[\{u, b_{i_1+1}, a, b_{i_1+2}, \ldots, b_{i_1+m+2}\}] \cong B_{1,m}$; if $i_1 \geq m + 2$ (i.e., $i_1 - m - 1 \geq 1$), then $G[\{a, b_{i_1}, u, b_{i_1-1}, \ldots, b_{i_1-m-1}\}] \cong B_{1,m}$. In either case, we get a contradiction. Thus $\ell - m - 1 \leq i_1 \leq m + 1$. This, together with the assumption $\ell \geq \max\{3m, m + 4\}$, leads to $m = 2$, $\ell = 6$ and $i_1 = 3$. Then $G[\{b_1, b_2, u, b_3, a, b_5\}] \cong B_{1,2}$, giving a contradiction. Thus $L_{i_1} \cup L_{i_2} \subseteq N_G(a)$ (i.e., $N_G(a) \cap V(H) = L_{i_1} \cup L_{i_1+1} \cup L_{i_2}$). Hence

$$G[V(H) \cup \{a\}] = F_c(L_0, L_1, \ldots, L_{i_1}, L_{i_1+1} \cup \{a\}, L_{i_2}, \ldots, L_\ell)$$

or

$$G[V(H) \cup \{a\}] = F_p(L_1, \ldots, L_{i_1}, L_{i_1+1} \cup \{a\}, L_{i_2}, \ldots, L_\ell)$$

according as H is a fat cycle or a fat path, which contradicts the maximality of H.

Case 3: $|I| = 4$.

In this case, $i_1 + 1 < i_2 - 1$ and $I = \{i_1, i_1 + 1, i_2 - 1, i_2\}$. Let $J_1 = [1, i_1 - 1]$, $J_2 = [i_1 + 2, i_2 - 2]$ and $J_3 = [i_2 + 1, \ell]$ (where J_i may be empty). If $|J_1| \geq m$, then $i_1 - m \geq 1$, and hence $G[\{b_{i_2}, a, b_{i_1+1}, b_{i_1}, b_{i_1-1}, \ldots, b_{i_1-m}\}] \cong B_{1,m}$; if $|J_2| \geq m$, then $i_1 + m + 1 \leq i_2 - 2$, and hence $G[\{b_{i_2}, a, b_{i_1}, b_{i_1+1}, \ldots, b_{i_1+m+1}\}] \cong B_{1,m}$; if $|J_3| \geq m$, then $i_2 + m \leq \ell$, and hence $G[\{b_{i_1}, a, b_{i_2-1}, b_{i_2}, b_{i_2+1}, \ldots, b_{i_2+m}\}] \cong B_{1,m}$. In either case, we get a contradiction. Thus $\max\{|J_1|, |J_2|, |J_3|\} \leq m - 1$. On the other hand, $|J_1| + |J_2| + |J_3| = |[1, \ell] - \{i_1, i_1 + 1, i_2 - 1, i_2\}| = \ell - 4 \geq \max\{3m - 4, m\}$. Hence we see that $m \geq 2$ and $|J_i| = |J_{i'}| = m - 1$ for some $i, i' \in \{1, 2, 3\}$

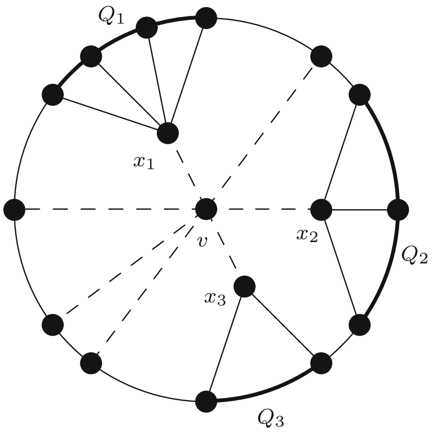


Fig. 5 A fan-cycle system

with $i \neq i'$. Without loss of generality, we may assume that $|J_1| = m - 1$ (i.e., $i_1 = m$). If $|J_2| = m - 1$, then $G[\{b_{i_2-2}, b_{i_2-1}, b_{i_2}, a, b_m, b_{m-1}, \ldots, b_1\}] \cong B_{1,m}$; if $|J_3| = m - 1$, then $G[\{b_{i_2+1}, b_{i_2}, b_{i_2-1}, a, b_m, b_{m-1}, \ldots, b_1\}] \cong B_{1,m}$. In either case, we again get a contradiction.

This completes the proof of Theorem 6.

4 The Existence of Halin Subgraph

In this section, we prove Theorems 13 and 14.

Let G be a graph. A sequence $(C : v; Q_1, \ldots, Q_m; x_1, \ldots, x_m)$ is a *fan-cycle system* of G if

1. C is a cycle of G,
2. $Q_1, \ldots, Q_m$ are vertex-disjoint paths of order at least two on C,
3. $v, x_1, \ldots, x_m$ are distinct vertices with $V(G) - V(C) = \{v, x_1, \ldots, x_m\}$,
4. $|V(C) - \bigcup_{1 \leq i \leq m} V(Q_i)| + m \geq 3$,
5. v is adjacent to every vertex in $(V(C) - \bigcup_{1 \leq i \leq m} V(Q_i)) \cup \{x_1, \ldots, x_m\}$, and
6. for i $(1 \leq i \leq m)$, x_i is adjacent to every vertex of Q_i

(see Fig. 5). In [3], the following lemma was proved in order to construct a spanning Halin subgraph.

Lemma 9 (Chen et al. [3]) *If a graph G has a fan-cycle system, then G has a spanning Halin subgraph.*

Now we show that all 3-connected graphs in $\bigcup_{0 \leq i \leq 8} \mathcal{H}_i$ have a spanning Halin subgraph.

Lemma 10 *For $G \in \bigcup_{0 \leq i \leq 8} \mathcal{H}_i$, if G is 3-connected, then G has a spanning Halin subgraph.*

Proof Since all graphs in $\bigcup_{i\in\{2,3,5,6\}}\mathcal{H}_i$ are not 3-connected, $G \in \mathcal{H}_i$ for some $i \in \{0, 1, 4, 7, 8\}$. By Lemma 9, it suffices to show that G has a fan-cycle system.

Case 1: $G \in \mathcal{H}_0$.

Let $L_1, \ldots, L_m$ be the leaf-cliques of G, and let R_i be the root of L_i. For each i $(1 \le i \le m)$, let $v_i \in R_i$. Since G is 3-connected, $|R_i - \{v_i\}| \ge 2$ for all i, and hence $G - \{v_i \mid 1 \le i \le m\}$ has a Hamiltonian cycle C containing $m - 1$ vertex-disjoint paths $Q_2, \ldots, Q_m$ with $V(Q_i) = L_i \cup (R_i - \{v_i\})$ $(2 \le i \le m)$. Then $(C : v_1; Q_2, \ldots, Q_m; v_2, \ldots, v_m)$ is a fan-cycle system of G.

Case 2: $G \in \mathcal{H}_1$.

Write $G = H_1(\{C_{s_3}, C_{s_4}, C_{s_5}\})$. For each $i \in \{3, 4\}$, let $a_i \in C_{s_i}$. Since G is 3-connected, $|C_{s_i}| \ge 3$ for $i \in \{3, 4\}$, and hence $G - \{a_3, a_4\}$ has a Hamiltonian cycle C containing a path Q with $V(Q) = (C_{s_4} - \{a_4\}) \cup \{s_2\}$. Then $(C : a_3; Q; a_4)$ is a fan-cycle system of G.

Case 3: $G \in \mathcal{H}_4$.

Write $G = H_1(\{C_{v_4}, C_{v_5}, C_{v_6}\})$. Since G is 3-connected, $|C_{v_i} \cup C_{v_j}| \ge 3$ for $i, j \in \{4, 5, 6\}$ with $i \ne j$. By symmetry, we may assume that $|C_{v_4}| \ge 2$ and $|C_{v_5}| \ge 2$. For each $i \in \{4, 5\}$, let $a_i \in C_{v_i}$. Then $G - \{a_4, a_5\}$ has a Hamiltonian cycle C containing a path Q with $V(Q) = (C_{v_5} - \{a_5\}) \cup \{v_3\}$, and hence G has a fan-cycle system $(C : a_4; Q; a_5)$.

Case 4: $G \in \mathcal{H}_7$.

Let $C = y_2y_4y_7y_8y_6$ be a cycle of G, and $Q_1 = y_4y_7$ and $Q_2 = y_8y_6$ be paths on C. Then $(C : y_1; Q_1, Q_2; y_3, y_5)$ is a fan-cycle system of G.

Case 5: $G \in \mathcal{H}_8$.

Let $C = z_2z_4z_7z_9z_8z_6$ be a cycle of G, and $Q_1 = z_4z_7$ and $Q_2 = z_8z_6$ be paths on C. Then $(C : z_1; Q_1, Q_2; z_3, z_5)$ is a fan-cycle system of G.

This completes the proof of Lemma 10. □

Lemma 11 *For $G \in \mathcal{P}(5)$, if G is 3-connected, then G has a spanning Halin subgraph.*

Proof We first suppose that G is a fat path, and write $G = F_p(L_1, \ldots, L_\ell)$. For each i $(2 \le i \le \ell - 1)$, let $a_i \in L_i$. Since G is 3-connected, $|L_i - \{a_i\}| \ge 2$ for i $(2 \le i \le \ell - 1)$, and hence $G - \{a_2, \ldots, a_{\ell-1}\}$ has a Hamiltonian cycle C such that $C[L_i]$ has exactly two components for every i $(2 \le i \le \ell - 1)$. We take the spanning tree T of G such that $N_T(a_2) = L_1 \cup (L_2 - \{a_2\}) \cup \{a_3\}$, $N_T(a_{\ell-1}) = L_\ell \cup (L_{\ell-1} - \{a_{\ell-1}\}) \cup \{a_{\ell-2}\}$ and $N_T(a_i) = (L_i - \{a_i\}) \cup \{a_{i-1}, a_{i+1}\}$ $(3 \le i \le \ell - 2)$. Then T has no vertices of degree 2 and $V(G) - \{a_2, \ldots, a_{\ell-1}\}$ is the set of leaves of T. Hence $T \cup C$ is a spanning Halin subgraph of G.

We next suppose that G is a fat cycle, and write $G = F_c(L_0, \ldots, L_\ell)$. Since G is 3-connected, G has at most two fundamental cliques of order one. Furthermore, if G has exactly two fundamental cliques of order one, then such cliques are consecutive. By symmetry, we may assume that $|L_i| \ge 2$ for every i $(1 \le i \le \ell - 1)$. For each i $(1 \le i \le \ell - 1)$, let $a_i \in L_i$. Then $G - \{a_1, \ldots, a_{\ell-1}\}$ has a Hamiltonian cycle C such that $C[L_i]$ has exactly one component for every i $(0 \le i \le \ell)$. We take a spanning tree T of G such that $N_T(a_1) = L_0 \cup (L_1 - \{a_1\}) \cup \{a_2\}$, $N_T(a_{\ell-1}) = L_\ell \cup (L_{\ell-1} - \{a_{\ell-1}\}) \cup \{a_{\ell-2}\}$ and $N_T(a_i) = (L_i - \{a_i\}) \cup \{a_{i-1}, a_{i+1}\}$ $(2 \le i \le \ell - 2)$. Then T has no vertices of degree 2 and $V(G) - \{a_1, \ldots, a_{\ell-1}\}$ is the set of leaves of T. Hence $T \cup C$ is a spanning Halin subgraph of G. □

Theorems 4, 12 and Lemma 11 lead to Theorem 13. Theorems 3, 13 and Lemma 10 lead to Theorem 14.

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