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DIRAC'S CONDITION FOR SPANNING HALIN SUBGRAPHS*

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Abstract. Let G be an n-vertex graph with $n \geq 3$. A classic result of Dirac from 1952 asserts that G is hamiltonian if $\delta(G) \geq n/2$. Dirac's theorem is one of the most influential results in the study of hamiltonicity and by now there are many related known results (see, e.g., [J. A. Bondy, *Handbook of Combinatorics, Vol.* 1, MIT Press, Cambridge, MA, 1995, pp. 3–110]. A *Halin graph* is a planar graph consisting of two edge-disjoint subgraphs: a spanning tree of at least four vertices and with no vertex of degree 2, and a cycle induced by the set of the leaves of the spanning tree. Halin graphs possess rich hamiltonicity properties such as being hamiltonian, hamiltonian connected, and almost pancyclic. As a continuous "generalization" of Dirac's theorem, in this paper, we show that there exists a positive integer n_0 such that any graph G with $n \geq n_0$ vertices and $\delta(G) \geq (n+1)/2$ contains a spanning and pancyclic Halin subgraph H. In addition, for every nonhamiltonian cycle C in H, there is a cycle C' longer than C such that C' contains all vertices from C and at most two more vertices not from C.

Key words. Halin graph, ladder graph, Dirac's condition, regularity lemma, blow-up lemma

AMS subject classifications. 05C35, 05C38

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1. Introduction. A classic theorem of Dirac [12] from 1952 asserts that every graph on $n\,(n \geq 3)$ vertices with minimum degree at least $n/2$ is hamiltonian. Following Dirac's result, numerous results on hamiltonicity properties on graphs with restricted degree conditions have been obtained (see, for instance, [15, 16]). Traditionally, under similar conditions, results for a graph being hamiltonian, hamiltonian-connected, and pancyclic are obtained separately. We may ask, under certain conditions, if it is possible to uniformly show a graph possessing several hamiltonicity properties. The work on finding the square of a hamiltonian cycle in a graph can be seen as an attempt in this direction. However, it requires minimum degree of $2n/3$ for an n-vertex graph G to contain the square of a hamiltonian cycle; for examples, see [7, 13, 14, 20, 25]. For bipartite graphs, finding the existence of a spanning ladder is a way of simultaneously showing the graph having many hamiltonicity properties (see [10, 11]). In this paper, we introduce another approach of uniformly showing the possession of several hamiltonicity properties in a graph: we show the existence of a spanning *Halin graph* in a graph under a given minimum degree condition.

A tree with no vertex of degree 2 is called a *homeomorphically irreducible tree* (HIT). A *Halin graph* $H = T \cup C$ is a simple planar graph consisting of an HIT T with at least four vertices and a cycle C induced by the set of leaves of T. The HIT T is called the *underlying tree* of H. A wheel graph is an example of a Halin graph, where the underlying tree is a star. Halin constructed Halin graphs in [17] for the study of minimally 3-connected graphs. Lovász and Plummer called such graphs Halin graphs in their study of planar bicritical graphs [21], which are planar graphs having a 1-factor after deleting any two vertices. Intensive research has been done on Halin

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graphs. Bondy [4] in 1975 showed that a Halin graph is hamiltonian. In the same year, Lovász and Plummer [21] showed that not only is a Halin graph itself hamiltonian, but each of its subgraph obtained by deleting a vertex is hamiltonian. In 1987, Barefoot [2] proved that Halin graphs are hamiltonian-connected, i.e., there is a hamiltonian path connecting any two vertices of the graph. Furthermore, it was proved that each edge of a Halin graph is contained in a hamiltonian cycle and is avoided by another [24]. Bondy and Lovász [6] and Skowrońska [23] independently, in 1985, showed that a Halin graph is almost pancyclic and is pancyclic if the underlying tree has no vertex of degree 3, where an n-vertex graph is *almost pancyclic* if it contains cycles of length from 3 to n with the possible exception of a single even length and is *pancyclic* if it contains cycles of length from 3 to n. Some problems that are NP-complete for general graphs have been shown to be polynomial time solvable for Halin graphs. For example, Cornuéjols, Naddef, and Pulleyblank [9] showed that in a Halin graph, a hamiltonian cycle can be found in polynomial time. Nevertheless, it is NP-complete to determine whether a graph contains a (spanning) Halin subgraph [18].

Despite all these nice properties of Halin graphs mentioned above, the problem of determining whether a graph contains a spanning Halin subgraph has not been well-studied except a conjecture proposed by Lovász and Plummer [21] in 1975. The conjecture states that *every* 4-*connected plane triangulation contains a spanning Halin subgraph* (disproved recently [8]). In this paper, we investigate the minimum degree condition that implies the existence of a spanning Halin subgraph in a graph, thereby giving another approach for uniformly showing the possession of several hamiltonicity properties in a graph under a given minimum degree condition. We obtain the following result.

THEOREM 1.1. *There exists $n_0 > 0$ such that for any graph G with $n \geq n_0$ vertices, if $\delta(G) \geq (n+1)/2$, then G contains a spanning and pancyclic Halin subgraph H. In addition, for every non-hamiltonian cycle C in H, there is a cycle C' longer than C such that C' contains all vertices from C and at most two more vertices not from C.*

Note that if $n \geq 4$, an n-vertex graph with minimum degree at least $(n+1)/2$ is 3-connected. Hence, the minimum degree condition in Theorem 1.1 implies the 3-connectedness, which is a necessary condition for a graph to contain a spanning Halin subgraph, since every Halin graph is 3-connected. A Halin graph contains a triangle, and bipartite graphs are triangle-free. Hence, $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ contains no spanning Halin subgraph. For n even, the graph obtained from two copies of $K_{\frac{n}{2}+1}$ by gluing them together on an edge is 2-connected, so it has no spanning Halin subgraph. Both these graphs have minimum degree at most $n/2$. We see that the minimum degree condition in Theorem 1.1 is best possible. For the pancyclicity property, it is worth mentioning that a result by Bondy [3] says that every n-vertex graph with minimum degree at least $\frac{n+1}{2}$ is pancyclic.

Theorem 1.1 is proved for large graphs. It might be very hard to obtain the same result for all graphs, as when constructing a Halin graph in general, we may need to find its underlying tree first. The minimum degree condition suffices for the existence of a such tree T in G (in fact, it was shown that an n-vertex graph with minimum degree at least $4\sqrt{2n}$ contains a spanning tree with no vertex of degree 2 [1]). However, the hardness lies in finding a cycle C spanning the set of the leaves of T so that $T \cup C$ is planar. In other words, when T is fixed, we have to find a cycle C in G passing through a set of given vertices in some particular order. The other way of finding a spanning Halin graph H is to find a spanning subgraph which contains H, for example, spanning structures close to ladder structures (e.g., graphs

H_1 to H_5 as defined in next section). Particularly, the square of a hamiltonian cycle contains H_1 or H_2 as a spanning subgraph, so it contains a spanning Halin subgraph. But the disadvantage of using "uniform" structures as H_i is that it makes it hard for constructing them "manually." Nevertheless, we still suspect that $(n+1)/2$ is the right condition for all graphs to contain a spanning Halin subgraph.

2. Notation and definitions. We consider simple and finite graphs only. Let G be a graph. Denote by $V(G)$ and $E(G)$ the vertex set and edge set of G, respectively, and by $e(G)$ the cardinality of $E(G)$. We denote by $\delta(G)$ the minimum degree of G and by $\Delta(G)$ the maximum degree. Let $v \in V(G)$ be a vertex and $S \subseteq V(G)$ a subset. Then $G[S]$ is the subgraph of G induced by S. Similarly, $G[F]$ is the subgraph induced by F if $F \subseteq E(G)$. The notation $\Gamma_G(v,S)$ denotes the set of neighbors of v in S, and $deg_G(v,S) = |\Gamma_G(v,S)|$. We let $\Gamma_{\overline{G}}(v,S) = S - \Gamma_G(v,S)$ and $deg_{\overline{G}}(v,S) = |\Gamma_{\overline{G}}(v,S)|$. Note that if $v \in S$, then $v \in \Gamma_{\overline{G}}(v,S)$. Given another set $U \subseteq V(G)$, define $\Gamma_G(U,S) = \bigcap_{u\in U} \Gamma_G(u,S)$, $deg_G(U,S) = |\Gamma_G(U,S)|$, and $N_G(U,S) = \bigcup_{u\in U} \Gamma_G(u,S)$. When $U = \{u_1, u_2, \ldots, u_k\}$, we may write $\Gamma_G(U,S)$, $deg_G(U,S)$, and $N_G(U,S)$ as $\Gamma_G(u_1,u_2,\ldots,u_k,S)$, $deg_G(u_1,u_2,\ldots,u_k,S)$, and $N_G(u_1,u_2,\ldots,u_k,S)$, respectively, in specifying the vertices in U. When $S = V(G)$, we only write $\Gamma_G(U)$, $deg_G(U)$, and $N_G(U)$. Let $U_1, U_2 \subseteq V(G)$ be two disjoint subsets. Then $\delta_G(U_1,U_2) = \min\{deg_G(u_1,U_2) \mid u_1 \in U_1\}$ and $\Delta_G(U_1,U_2) = \max\{deg_G(u_1,U_2) \mid u_1 \in U_1\}$. Notice that the notation $\delta_G(U_1,U_2)$ and $\Delta_G(U_1,U_2)$ is not symmetric with respect to U_1 and U_2. We denote by $E_G(U_1,U_2)$ the set of edges with one end in U_1 and the other in U_2, and the cardinality of $E_G(U_1,U_2)$ is denoted by $e_G(U_1,U_2)$. We may omit the index G if there is no risk of confusion. Let $u, v \in V(G)$ be two vertices. We write $u \sim v$ if u and v are adjacent. A path connecting u and v is called a (u,v)-path. If G is a bipartite graph with partite sets A and B, we denote G by $G(A,B)$ in emphasizing the two partite sets.

In constructing Halin graphs, we use ladder graphs and a class of "ladder-like" graphs as substructures. We give the description of these graphs below.

DEFINITION 2.1. *An n-ladder $L_n = L_n(A,B)$ is a balanced bipartite graph with*

$$A = \{a_1, a_2, \ldots, a_n\} \quad \textit{and} \quad B = \{b_1, b_2, \ldots, b_n\}$$

such that $a_i \sim b_j$ iff $|i-j| \le 1$. We call a_ib_i the ith rung of L_n. If n is even, then we call each of the shortest (a_1,b_n)-path $a_1b_2a_3b_4\cdots a_{n-1}b_n$ and (b_1,a_n)-path $b_1a_2b_3a_4\cdots b_{n-1}a_n$ a side of L_n, and if n is odd, then we call each of the shortest (a_1,a_n)-path $a_1b_2a_3b_4\cdots a_{n-1}b_{n-1}a_n$ and (b_1,b_n)-path $b_1a_2b_3a_4\cdots b_{n-2}a_{n-1}b_n$ a side of L_n.

Let L be a ladder with xy as one of its rungs. For an edge gh, we say xy and gh are *adjacent* if $x \sim g, y \sim h$ or $x \sim h, y \sim g$. Suppose L has its first rung as ab and its last rung as cd, we denote L by $ab - L - cd$ in specifying the two rungs. If a and c (and so b and d) are contained in a same side of L, we denote L by $\overrightarrow{ab} - L - \overrightarrow{cd}$. Let A and B be two disjoint vertex sets. We say the rung xy of L is *contained* in $A \times B$ if either $x \in A, y \in B$ or $x \in B, y \in A$. Let L' be another ladder vertex-disjoint with L. If the last rung of L is adjacent to the first rung of L', we write LL' for the new ladder obtained by concatenating L and L'. In particular, if $L' = gh$ is an edge, we write LL' as Lgh.

We now define five types of "ladder-like" graphs, call them H_1, H_2, H_3, H_4, and H_5, respectively. Let L_n be a ladder with a_1b_1 and a_nb_n as the first and last rungs, respectively, and x, y, z, w, u be five new vertices. Then each of H_i is obtained from

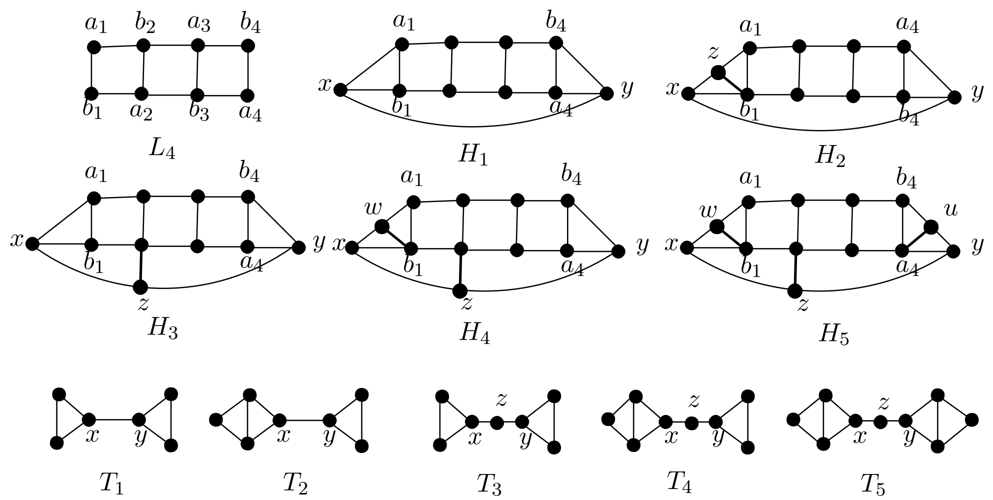

Figure 1. *L_4, H_i constructed from L_4, and T_i associated with H_i for each $i = 1, 2, \cdots, 5$*

FIG. 1. *L_4, H_i constructed from L_4, and T_i associated with H_i for each $i = 1, 2, \dots, 5$.*

In Extremal Case 1, we will show that G contains a spanning Halin subgraph isomorphic to a graph in $\mathcal{Q}$ (defined in (2.1)). In all other cases, we will construct a spanning subgraph of G isomorphic to either a wheel or some H_i for some $i \in \{1,2,3,4,5\}$. Note that a ladder graph in $\mathcal{Q}$ and each H_i is pancyclic. Furthermore, by easy checking, a wheel, each graph in $\mathcal{Q}$, and each H_i satisfies the following property: for every non-hamiltonian cycle C in it, there is a cycle C' longer than C such that C' contains all vertices from C and at most two more vertices not from C. Hence, to prove Theorem 1.1, we only need to show the existence of the mentioned graphs above. The following three theorems deal with the Non-extremal Case, and the two extremal cases, respectively, and thus give a proof of Theorem 1.1.

Theorem 3.1. *Suppose that $0 < \beta \ll 1/(20 \cdot 17^3)$ and n is a sufficiently large integer. Let G be a graph on n vertices with $\delta(G) \ge (n+1)/2$. If G is in Extremal Case 1, then G contains a spanning Halin subgraph isomorphic to a graph in $\mathcal{Q}$ (defined in (2.1)) as a subgraph.*

Theorem 3.2. *Suppose that $0 < \beta \ll 1/(20 \cdot 17^3)$ and n is a sufficiently large integer. Let G be a graph on n vertices with $\delta(G) \ge (n+1)/2$. If G is in Extremal Case 2, then G contains a spanning Halin subgraph isomorphic to either a wheel or some H_i, $i \in \{1,2,3,4,5\}$.*

Theorem 3.3. *Let n be a sufficiently large integer and G an n-vertex graph with $\delta(G) \ge (n+1)/2$. If G is in the Non-extremal Case, then G has a spanning Halin subgraph isomorphic to H_1 or H_2.*

We need the following lemma in each of the proofs of Theorems 3.1 - 3.2 in dealing with "garbage" vertices.

Lemma 3.1. *Let F be a graph such that $V(F)$ is partitioned into $S \cup R$. Suppose that (i) there are $|R|$ vertex-disjoint 3-stars (a 3-star is a copy of $K_{1,3}$) with the vertices in R as their centers and with leaves contained in S, (ii) for any two vertices $u, v \in N(R,S)$, $deg(u,v,S) \ge 6|R|$, and (iii) for any three vertices $u, v, w \in N(N(R,S),S)$, $deg(u,v,w,S) \ge 7|R|$. Then there is a ladder spanning R and some other $7|R|-2$ vertices from S. Additionally, the ladder has the vertices on its first and last rungs in S.*

Proof. Let $R = \{w_1, w_2, \cdots, w_r\}$. Consider first that $r = 1$. Choose $x_{11}, x_{12}, x_{13} \in \Gamma(w_1, S)$. By (ii), there are distinct vertices $y^1_{12} \in \Gamma(x_{11}, x_{12}, S)$ and $y^1_{23} \in \Gamma(x_{12}, x_{13}, S)$. Then the graph L on $\{w_1, x_{11}, x_{12}, x_{13}, y^1_{12}, y^1_{23}\}$ with edges in

$$\{w_1x_{11}, w_1x_{12}, w_1x_{13}, y^1_{12}x_{11}, y^1_{12}x_{12}, y^1_{23}x_{12}, y^1_{23}x_{13}\}$$

is a ladder covering R with $|V(L)| = 6$. Suppose now $r \ge 2$. By condition (i), for each i with

L_n by adding some specified vertices and edges as follows. Additionally, for each i with $1 \le i \le 5$, we define a graph T_i associated with H_i. A depiction of a ladder L_4, H_1, H_2, H_3, H_4, H_5 constructed from L_4, and the graph T_i associated with H_i are given in Figure 1.

H_1: Add two new vertices x, y and the edges xa_1, xb_1, ya_n, yb_n, and xy.

Let $T_1 = H_1[\{x, y, a_1, b_1, a_n, b_n\}]$.

H_2: Add three new vertices x, y, z and the edges $za_1, zb_1, xz, xb_1, ya_n, yb_n$, and xy.

Let $T_2 = H_2[\{x, y, z, a_1, b_1, a_n, b_n\}]$.

H_3: Add three new vertices x, y, z and the edges $xa_1, xb_1, ya_n, yb_n, xz, yz$, and either za_i or zb_i for some $1 \le i \le n$. Note that H_2 is a special case of H_3 with $i = 1$ or n.

Let $T_3 = H_3[\{x, y, z, a_1, b_1, a_n, b_n\}]$.

H_4: Add four new vertices x, y, z, w and the edges $wa_1, wb_1, xw, xb_1, ya_n, yb_n, xz, yz$, and either za_i or zb_i for some $1 \le i \le n$ such that a_i or b_i is a vertex on the side of L that has b_1 as one end.

Let $T_4 = H_4[\{x, y, z, w, a_1, b_1, a_n, b_n\}]$.

H_5: Add five new vertices x, y, z, w, u.

If n is odd, we add the edges $wa_1, wb_1, xw, xb_1, ua_n, ub_n, yu, yb_n, xz, yz$, and either za_i or zb_i for some $1 \le i \le n$ such that a_i or b_i is a vertex on the shortest (b_1, b_n)-path in L.

And if n is even, we add the edges $wa_1, wb_1, xw, xb_1, ua_n, ub_n, yu, ya_n, xz, yz$, and either za_i or zb_i for some $1 \le i \le n$ such that a_i or b_i is a vertex on the shortest (b_1, a_n)-path in L.

The graph obtained from H_5 by deleting the vertex z and adding the edge xy is identical with H_4 with $i = n$.

Let $T_5 = H_5[\{x, y, z, w, u, a_1, b_1, a_n, b_n\}]$.

Let $i = 1, 2, \dots, 5$. Notice that each of H_i is a Halin graph, and the graph obtained from H_5 by deleting the vertex z and adding the edge xy is also a Halin graph. Except H_1, each H_i has a unique underlying tree. Notice also that xy is an edge on the cycle along the leaves of any underlying tree of H_1 or H_2. For each H_i and T_i, call x the *left* *end* and y the *right end*, and call a vertex of degree at least 3 in the underlying tree of H_i a *Halin constructible vertex*. By analyzing the structure of H_i, we see that each internal vertex on a/the shortest (x, y)-path in $H_i - xy$ (for $i = 1, 2$) or $H_i - z$ (for $i = 3, 4, 5$) is a Halin constructible vertex. Note that any vertex in $V(H_1) - \{x, y\}$ can be a Halin constructible vertex. We call a_1b_1 the *head link* of T_i and a_nb_n the *tail link* of T_i, and for each of T_3, T_4, T_5, we call the vertex z not contained in any triangles the *connecting vertex*. The notation of H_i and T_i are fixed hereafter.

Let $T \in \{T_1, \ldots, T_5\}$ be a subgraph of a graph G. Suppose that T has head link ab, tail link cd, and possibly the connecting vertex z. Suppose $G - V(T)$ contains a spanning ladder L with first rung c_1d_1 and last rung c_nd_n such that c_1d_1 is adjacent to ab and c_nd_n is adjacent to cd. Additionally, if the connecting vertex z of T exists, then z has a neighbor z', which is an internal vertex on a shortest path between the two ends of T in the graph $abLcd \cup T - z$. Then $abLcd \cup T$ or $abLcd \cup T \cup \{zz'\}$ is a spanning Halin subgraph of G. This technique is frequently used later on in constructing a Halin graph. The following proposition gives another way of constructing a Halin graph based on H_1 and H_2.

For $i = 1, 2$, let $G_i \in \{H_1, H_2\}$ be disjoint and with left end x_i and right end y_i. Let $u_i \in V(G_i)$ be a Halin constructible vertex. We call the graph $Q := G_1 \cup G_2 - \{x_1y_1, x_2y_2\} \cup \{x_1x_2, y_1y_2, u_1u_2\}$ the *connection of* G_1 *and* G_2 and write $Q = G_1 \oplus G_2$. Let

$$\mathcal{Q} = \{H_1 \oplus H_1, H_1 \oplus H_2, H_2 \oplus H_2\}. \tag{2.1}$$

Proposition 1. *Every graph $Q \in \mathcal{Q}$ is a Halin graph and is pancyclic.*

Proof. We show the statement for $Q = H_1 \oplus H_2$. The other cases can be shown similarly.

We embed H_1 in the plane so that x_1 is located on the left side of u_1, and y_1 is located on the right side of u_1. We do the same for H_2 but put H_2 below the location of H_1. Now, we can remove the edges x_1y_1 and x_2y_2 and add the edges x_1x_2, u_1u_2, y_1y_2 to obtain still a plane graph, which gives a plane embedding of Q. We next show that Q can be decomposed into a homeomorphically irreducible spanning tree and a cycle induced by the leaves of the tree. Let T_i be an underlying plane tree of H_i. Then $T := T_1 \cup T_1 \cup \{u_1u_2\}$ is an HIT spanning $V(H_1) \cup V(H_2)$. Since H_i is a Halin graph and T_i is an underlying tree of H_i, the edge induced subgraph $C_i = H_i[E(H_i) - E(T_i)]$ is a cycle. Furthermore, $x_iy_i \in E(C_i)$ by the construction of H_i. Thus $C_i - x_iy_i$ is an (x_i, y_i)-path spanning the set of leaves of T_i, and so $C = (C_1 - x_1y_1) \cup (C_2 - x_2y_2) \cup \{x_1x_2, y_1y_2\}$ is a cycle spanning the set of leaves of T. Hence $Q = T \cup C$ is a Halin graph.

To see the pancyclicity of Q, suppose that H_1 has $2n_1 + 2$ vertices and H_2 has $2n_2 + 3$ vertices. It is easy to check that in H_i, there are (x_i, y_i)-paths of order from $n_i + 2$ to $|V(H_i)|$, and there are cycles of order from 3 to $|V(H_i)|$. Combining the (x_1, y_1)-paths from H_1 and (x_2, y_2)-paths from H_2 using the edges x_1x_2 and y_1y_2 gives cycles of length from $n_1 + n_2 + 4$ to $n_1 + n_2 + 5 = |V(Q)|$ in Q. Together with cycles in H_i, we know that Q contains all cycles of length from 3 to $|V(Q)|$. ∎

3. Proof of Theorem 1.1. In this section, we prove Theorem 1.1. Following the standard setup of proofs applying the regularity lemma, we divide the proof into a nonextremal case and extremal cases. For this purpose, we define the two extremal cases in the following.

Let G be an n-vertex graph and V its vertex set. Let $0 < \beta \leq 1$ be a constant. Let $W \subseteq V(G)$. We say W is an *approximate vertex-cut* of G with parameter β if there is a partition V_1 and V_2 of $V - W$ such that $e_G(V_1, V_2) \le \beta n^2$ and $\delta[G[V_i]] \ge \delta(G) - |W| - \beta n$ for each $i = 1, 2$. The two extremal cases are defined as below.

Extremal Case 1. G has an approximate vertex-cut of size at most $5\beta n$ with parameter β.

Extremal Case 2. There exists a partition $V_1 \cup V_2$ of V such that $|V_1| \ge (1/2 - 7\beta)n$ and $\Delta(G[V_1]) \le \beta n$.

Nonextremal case. We say that an n-vertex graph with minimum degree at least $(n+1)/2$ is in the *nonextremal case* if it is in neither Extremal Case 1 nor Extremal Case 2.

In Extremal Case 1, we will show that G contains a spanning Halin subgraph isomorphic to a graph in $\mathcal{Q}$ (defined in (2.1)). In all other cases, we will construct a spanning subgraph of G isomorphic to either a wheel or H_i for some $i \in \{1, 2, 3, 4, 5\}$. Note that a wheel graph, each graph in $\mathcal{Q}$, and each H_i are pancyclic. Furthermore, by easy checking, a wheel graph, each graph in $\mathcal{Q}$, and each H_i satisfy the following property: for every nonhamiltonian cycle C in it, there is a cycle C' longer than C such that C' contains all vertices from C and at most two more vertices not from C. Hence, to prove Theorem 1.1, we only need to show the existence of the mentioned graphs above. The following three theorems deal with the nonextremal case and the two extremal cases, respectively, and thus give a proof of Theorem 1.1.

THEOREM 3.1. *Suppose that* $0 < \beta \ll 1/(20 \cdot 17^3)$ *and* n *is a sufficiently large integer. Let* G *be a graph on* n *vertices with* $\delta(G) \ge (n+1)/2$*. If* G *is in Extremal Case* 1*, then* G *contains a spanning Halin subgraph isomorphic to a graph in* $\mathcal{Q}$ *(defined in* (2.1)*) as a subgraph.*

THEOREM 3.2. *Suppose that* $0 < \beta \ll 1/(20 \cdot 17^3)$ *and* n *is a sufficiently large integer. Let* G *be a graph on* n *vertices with* $\delta(G) \ge (n+1)/2$*. If* G *is in Extremal Case* 2*, then* G *contains a spanning Halin subgraph isomorphic to either a wheel or some* H_i*,* $i \in \{1, 2, 3, 4, 5\}$*.*

THEOREM 3.3. *Let* n *be a sufficiently large integer and* G *an* n*-vertex graph with* $\delta(G) \ge (n+1)/2$*. If* G *is in the nonextremal case, then* G *has a spanning Halin subgraph isomorphic to* H_1 *or* H_2*.*

We need the following lemma in each of the proofs of Theorems 3.1 and 3.2 in dealing with “garbage” vertices.

LEMMA 3.1. *Let* F *be a graph such that* $V(F)$ *is partitioned into* $S \cup R$*. Suppose that* (i) *there are* $|R|$ *vertex-disjoint* 3*-stars (a* 3*-star is a copy of* $K_{1,3}$*) with the vertices in* R *as their centers and with leaves contained in* S*,* (ii) *for any two vertices* $u, v \in N(R, S)$*,* $deg(u, v, S) \ge 6|R|$*, and* (iii) *for any three vertices* $u, v, w \in N(N(R, S), S)$*,* $deg(u, v, w, S) \ge 7|R|$*. Then there is a ladder spanning* R *and some other* $7|R| - 2$ *vertices from* S*. Additionally, the ladder has the vertices on its first and last rungs in* S*.*

Proof. Let $R = \{w_1, w_2, \ldots, w_r\}$. Consider first that $r = 1$. Choose $x_{11}, x_{12}, x_{13} \in \Gamma(w_1, S)$. By (ii), there are distinct vertices $y^1_{12} \in \Gamma(x_{11}, x_{12}, S)$ and $y^1_{23} \in \Gamma(x_{12}, x_{13}, S)$. Then the graph L on $\{w_1, x_{11}, x_{12}, x_{13}, y^1_{12}, y^1_{23}\}$ with edges in

$$\{w_1x_{11}, w_1x_{12}, w_1x_{13}, y^1_{12}x_{11}, y^1_{12}x_{12}, y^1_{23}x_{12}, y^1_{23}x_{13}\}$$

is a ladder covering R with $|V(L)| = 6$. Suppose now $r \ge 2$. By condition (i), for each i with $1 \le i \le r$, there exist distinct vertices $x_{i1}, x_{i2}, x_{i3} \in \Gamma(w_i, S)$. By (ii), we choose

$$w_i \sim x_{i1}, x_{i2}, x_{i3};\ y^i_{12} \sim x_{i1}, x_{i2};\ y^i_{23} \sim x_{i2}, x_{i3};\ z_i \sim x_{i3}, x_{i+1,1};\ u_i \sim x_{i3}, x_{i+1,1}, z_i.$$

It is easy to check that L is a ladder covering R with $|V(L)| = 8r - 2$. The ladder has its first and last rungs in S is seen by its construction. Figure 2 gives a depiction of L for $|R| = 2$. ■

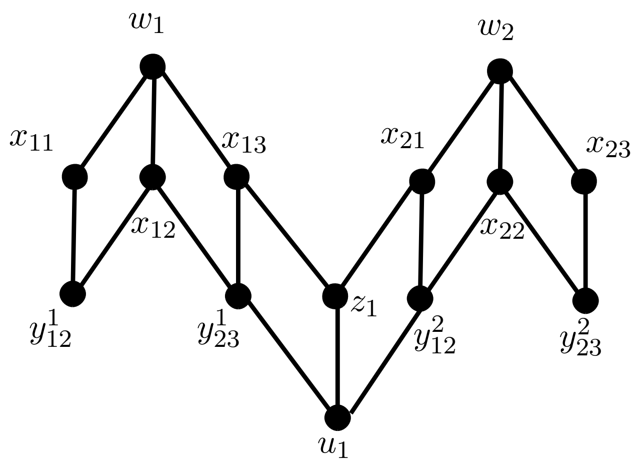

Figure 2. *Ladder L of order 14*

FIG. 2. *Ladder L of order* 14.

We will also need the bipartite version of Lemma 3.1. Since the proof is similar, we omit it.

Lemma 3.2. *Let $F(A, B)$ be a bipartite graph such that $V(F)$ is partitioned into $S \cup R$ with $R \subseteq A$. Suppose that (i) there are $|R|$ vertex-disjoint 3-stars with the vertices in R as their centers and with leaves contained in S, (ii) for any two vertices $u, v \in N(R, S)$, $deg(u, v, S) \geq 3|R|$, and (iii) for any three vertices $u, v, w \in N(N(R, S), S)$, $deg(u, v, w, S) \geq 4|R|$. Then there is a ladder spanning R and some other $7|R| - 2$ vertices from S with $3|R| - 1$ of them taken from A. Additionally, the ladder has its first and last rungs in S.*

The following simple observation is heavily used in the proofs explicitly or implicitly.

Lemma 3.3. *Let $U = \{u_1, u_2 \cdots, u_k\}, S \subseteq V(G)$ be subsets. Then $deg(u_1, u_2, \cdots, u_k, S) \geq |S| - (deg_{\overline{G}}(u_1, S) + \cdots + deg_{\overline{G}}(u_k, S)) \geq |S| - k(|S| - \delta(U, S)) = k\delta(U, S) - (k-1)|S|$.*

Extremal Case 1 is easier than the other cases, so we start with it.

3.1. Proof of Theorem 3.1. We assume that G has an approximate vertex-cut W with parameter β such that $|W| \leq 5\beta n$. Let V_1 and V_2 be the partition of $V - W$ such that $\delta(G[V_i]) \geq (1/2 - 6\beta)n$. As $\delta(G) \geq (n+1)/2$, $(1/2 - 6\beta)n \leq |V_i| \leq (1/2 + 6\beta)n$. We partition W into two subsets as follows:

$$W_1 = \{w \in W \mid deg(w, V_1) \geq (n+1)/4 - 2.5\beta n\} \quad \text{and} \quad W_2 = W - W_1.$$

As $\delta(G) \geq (n+1)/2$, we have $deg(w, V_2) \geq (n+1)/4 - 2.5\beta n$ for any $w \in W_2$. Since G is 3-connected and $(1/2 - 6\beta)n \geq 3$, there are three independent edges p_1p_2, q_1q_2, and r_1r_2 between $G[V_1 \cup W_1]$ and $G[V_2 \cup W_2]$ with $p_1, q_1, r_1 \in V_1 \cup W_1$ and $p_2, q_2, r_2 \in V_2 \cup W_2$.

distinct vertices $y^1_{12}, y^1_{23}, \ldots, y^r_{12}, y^r_{23}$ different from the existing vertices already chosen such that $y^i_{12} \in \Gamma(x_{i1}, x_{i2}, S)$ and $y^i_{23} \in \Gamma(x_{i2}, x_{i3}, S)$ for each i, and at the same time, we chose distinct vertices $z_1, z_2, \ldots, z_{r-1}$ from the unchosen vertices in B such that $z_i \in \Gamma(x_{i3}, x_{(i+1)1}, S)$ for each $1 \leq i \leq r-1$. Finally, by (iii), choose distinct vertices $u_1, u_2, \ldots, u_{r-1}$ from the unchosen vertices in S such that $u_i \in \Gamma(y^i_{23}, y^{i+1}_{12}, z_i, S)$. Let L be the graph with

$$V(L) = R \cup \{x_{i1}, x_{i2}, x_{i3}, y^i_{12}, y^i_{23}, z_i, u_i, x_{r1}, x_{r2}, x_{r3}, y^r_{12}, y^r_{23} \mid 1 \leq i \leq |R| - 1\} \text{ and}$$

$E(L)$ consisting of the edges $w_r x_{r1}, w_r x_{r2}, w_r x_{r3}, y^r_{12}x_{r1}, y^r_{12}x_{r2}, y^r_{23}x_{r2}, y^r_{23}x_{r3}$ and the edges indicated below for each $1 \leq i \leq r-1$:

$$w_i \sim x_{i1}, x_{i2}, x_{i3};\ y^i_{12} \sim x_{i1}, x_{i2};\ y^i_{23} \sim x_{i2}, x_{i3};\ z_i \sim x_{i3}, x_{i+1,1};\ u_i \sim x_{i3}, x_{i+1,1}, z_i.$$

It is easy to check that L is a ladder covering R with $|V(L)| = 8r - 2$. The ladder has its first and last rungs in S, shown by its construction. Figure 2 gives a depiction of L for $|R| = 2$. □

We will also need the bipartite version of Lemma 3.1. Since the proof is similar, we omit it.

LEMMA 3.2. *Let $F(A, B)$ be a bipartite graph such that $V(F)$ is partitioned into $S \cup R$ with $R \subseteq A$. Suppose that* (i) *there are $|R|$ vertex-disjoint 3-stars with the vertices in R as their centers and with leaves contained in S,* (ii) *for any two vertices $u, v \in N(R, S)$, $deg(u, v, S) \geq 3|R|$, and* (iii) *for any three vertices $u, v, w \in N(N(R, S), S)$, $deg(u, v, w, S) \geq 4|R|$. Then there is a ladder spanning R and some other $7|R| - 2$ vertices from S with $3|R| - 1$ of them taken from A. Additionally, the ladder has its first and last rungs in S.*

The following simple observation is heavily used in the proofs explicitly or implicitly.

LEMMA 3.3. *Let $U = \{u_1, u_2, \ldots, u_k\}, S \subseteq V(G)$ be subsets. Then $deg(u_1, u_2, \ldots, u_k, S) \geq |S| - (deg_{\overline{G}}(u_1, S) + \cdots + deg_{\overline{G}}(u_k, S)) \geq |S| - k(|S| - \delta(U, S)) = k\delta(U, S) - (k-1)|S|$.*

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As $\delta(G) \ge (n+1)/2$, we have $deg(w, V_2) \ge (n+1)/4 - 2.5\beta n$ for any $w \in W_2$. Since G is 3-connected and $(1/2 - 6\beta)n > 3$, there are three independent edges p_1p_2, q_1q_2, and r_1r_2 between $G[V_1 \cup W_1]$ and $G[V_2 \cup W_2]$ with $p_1, q_1, r_1 \in V_1 \cup W_1$ and $p_2, q_2, r_2 \in V_2 \cup W_2$.

For $i = 1, 2$, by the partition of W_i, we see that $\delta(W_i, V_i) \ge 3|W_i| + 3$. Thus, $\delta(W_i, V_i - \{p_i, q_i\}) \ge 3|W_i|$. There are $|W_i - \{p_i, q_i\}|$ vertex-disjoint 3-stars with their centers in $W_i - \{p_i, q_i\}$. By Lemma 3.3, for any $u, v, w \in V_i$, we have

$$deg(u, v, V_i - \{p_i, q_i\}) \ge 2\delta(G[V_i]) - |V_i| - 2 \ge (1/2 - 18\beta)n - 2 \ge 6|W_i|,$$
$$deg(u, v, w, V_i - \{p_i, q_i\}) \ge 3\delta(G[V_i]) - 2|V_i| - 2 \ge (1/2 - 30\beta)n - 2 \ge 7|W_i|.$$

By Lemma 3.1, we can find a ladder L_i which spans $W_i - \{p_i, q_i\}$ and another $7|W_i - \{p_i, q_i\}| - 2$ vertices from $V_i - \{p_i, q_i\}$, if $W_i - \{p_i, q_i\} \ne \emptyset$. Denote a_ib_i and c_id_i the first and last rungs of L_i (if L_i exists), respectively. Let

$$G_i = G[V_i - V(L_i)] \quad \text{and} \quad n_i = |V(G_i)|.$$

Note that $n_i \ge (n+1)/2 - 6\beta n - 7|W_i| \ge (n+1)/2 - 41\beta n$. For $i = 1, 2$,

$$deg_{G_i}(x) \ge \delta(G[V_i]) - 7|W_i| \ge (n+1)/2 - 41\beta n \text{ if } x \in V(G_i) \text{ and } x \notin \{p_i, q_i\} \cap W,$$
$$deg_{G_i}(p_i) \ge (n+1)/4 - 2.5\beta n - 7|W_i| \ge (1/4 - 41\beta)n \quad \text{if } p_i \in W,$$
$$deg_{G_i}(q_i) \ge (n+1)/4 - 2.5\beta n - 7|W_i| \ge (1/4 - 41\beta)n \quad \text{if } q_i \in W.$$

Let $i = 1, 2$. We now show that G_i contains a spanning subgraph isomorphic to either H_1 or H_2 as defined in the previous section. Since $n_i \le (1/2 + 6\beta)n$ and $deg_{G_i}(x) \ge (n+1)/2 - 41\beta n$ for any $x \in V(G_i) - W$, any subgraph of G_i induced by at least $(1/4 - 41\beta)n$ vertices not in W has minimum degree at least $(n+1)/2 - 41\beta n - (n_i - (1/4 - 41\beta)n) \ge (1/4 - 88\beta)n$ and thus has a matching of size at least 3. Hence, when n_i is even, we can choose independent edges $e_i = x_iy_i$ and $f_i = z_iw_i$ with

$$x_i, y_i \in \Gamma_{G_i}(p_i) - \{q_i\} \quad \text{and} \quad z_i, w_i \in \Gamma_{G_i}(q_i) - \{p_i\}.$$

And if n_i is odd, we can choose independent edges g_iy_i (we may assume $g_i \ne r_i$), $f_i = z_iw_i$, and a vertex x_i with

$$g_i, x_i, y_i \in \Gamma_{G_i}(p_i) - \{q_i\}, x_i \in \Gamma_{G_i}(g_i, y_i) - \{p_i, q_i\} \quad \text{and} \quad z_i, w_i \in \Gamma_{G_i}(q_i) - \{x_i, p_i\},$$

where the existence of the vertex x_i is possible since the subgraph of G_i induced by $\Gamma_{G_i}(p_i) - \{q_i\}$ has minimum degree at least $(1/2 - 41\beta)n - ((1/2 + 6\beta)n - |\Gamma_{G_i}(p_i) - \{q_i\}|) \ge |\Gamma_{G_i}(p_i) - \{q_i\}| - 47\beta n$, and hence contains a triangle. In this case, again, denote $e_i = x_iy_i$. Let

$$\begin{cases} G_i' = G_i - \{p_i, q_i\} & \text{if } n_i \text{ is even;} \\ G_i' = G_i - \{p_i, q_i, g_i\} & \text{if } n_i \text{ is odd.} \end{cases}$$

By the definition above, $|V(G_i')|$ is even.

The following claim is a modification of (1) of Lemma 2.2 in [11].

CLAIM 3.1.1. *For $i = 1, 2$, let $a'_i b'_i, c'_i d'_i \in E(G'_i)$ be two independent edges. Then G'_i contains two vertex-disjoint ladders Q_{i1} and Q_{i2} spanning $V(G'_i)$ such that Q_{i1} has $e_i = x_i y_i$ as its first rung and $a'_i b'_i$ as its last rung, and Q_{i2} has $c'_i d'_i$ as its first rung and $f_i = z_i w_i$ as its last rung, where e_i and f_i are defined prior to this claim.*

Proof. We only show the claim for $i = 1$ as the case for $i = 2$ is similar. Notice that by the definition of G'_1, $|V(G'_1)|$ is even. Since $|V(G'_1)| \le (1/2 + 6\beta)n$ and $\delta(G'_1) \ge (n + 1)/2 - 41\beta n - 3 \ge |V(G'_1)|/2 + 4$, G'_1 has a perfect matching M containing $e_1 = x_1 y_1, f_1 = z_1 w_1, a'_1 b'_1, c'_1 d'_1$. We identify a'_1 and c'_1 into a vertex called s' and identify b'_1 and d'_1 into a vertex called t'. Denote G''_1 as the resulting graph. Note that $s't' \in E(G''_1)$ by the way of identifications. Partition $V(G''_1)$ arbitrarily into U_1 and U_2 with $|U_1| = |U_2|$ such that $x_1, z_1, s' \in U_1$, $y_1, w_1, t' \in U_2$, and let $M' := (M - \{a'_1 b'_1, c'_1 d'_1\}) \cup \{s't'\}$. Since U_1 and U_2 are defined after giving the matching M, we know that $M' \subseteq E_{G''_1}(U_1, U_2)$.

Define an auxiliary graph H' with vertex set M' and edge set defined as follows. If $xy, uv \in M' - \{s't'\}$ with $x, u \in U_1$, then $xy \sim_{H'} uv$ if and only if $x \sim_{G'_1} v$ and $y \sim_{G'_1} u$ (we do not include the case that $x \sim_{G'_1} u$ and $y \sim_{G'_1} v$ as we defined a bipartition here). Additionally, for any $pq \in M' - \{s't'\}$ with $p \in U_1$,

$$pq \sim_{H'} s't' \quad \text{if and only if} \quad p \sim_{G'_1} b'_1, d'_1 \quad \text{and} \quad q \sim_{G'_1} a'_1, c'_1.$$

Notice that a ladder with rungs in M' is corresponding to a path in H' and vice versa. We next show that $\delta(H') \ge |V(H')|/2 + 1$. This will imply that H' has a hamiltonian path starting with e_1, ending with f_1, and having $s't'$ as an internal vertex. The path with $s't'$ replaced by $a'_1 b'_1$ and $c'_1 d'_1$ is corresponding to the required ladders in G'_1.

Let $u \in U_1$ and $v \in U_2$. Note that

$$\begin{aligned}
deg(u, U_2) &\ge \delta(G'_1) - |U_1| \ge (1/2 - 44\beta)n - 3,\\
deg(v, U_1) &\ge \delta(G'_1) - |U_2| \ge (1/2 - 44\beta)n - 3,\\
deg(a'_1, c'_1, U_2) &\ge 2deg(u, U_2) - |U_1| \ge (1/2 - 92\beta)n,\\
deg(b'_1, d'_1, U_1) &\ge 2deg(v, U_1) - |U_2| \ge (1/2 - 92\beta)n.
\end{aligned}$$

Let $uv \in M'$ with $u \in U_1$. If $u \ne s'$, then

$$\begin{aligned}
deg(uv, V(H')) &= |\{u'v' \in M' \mid u' \in U_1, u \sim v' \text{ and } v \sim u'\}|\\
&\ge |U_1| - \Gamma_{\overline{G''_1}}(u, U_2) - \Gamma_{\overline{G''_1}}(v, U_1)\\
&\ge 2((1/2 - 44\beta)n - 3) - |U_2|\\
&\ge (1/4 - 92\beta)n.
\end{aligned}$$

If $u = s'$, then

$$\begin{aligned}
deg(s't', V(H')) &= |\{u'v' \in M' \mid u' \in U_1, v' \sim a'_1, c'_1 \text{ and } u' \sim b'_1, d'_1\}|\\
&\ge |U_1| - |U_2 - \Gamma(a'_1, c'_1, U_2)| - |U_1 - \Gamma(b'_1, d'_1, U_1)|\\
&\ge 2((1/2 - 92\beta)n) - |U_2|\\
&\ge (1/4 - 187\beta)n.
\end{aligned}$$

Since $\beta < 1/2200$ and n is very large,

$$\delta(H') \ge (1/4 - 187\beta)n \ge |V(H')|/2 + 1,$$

as desired. ∎

be denoted as $x_2y_2 - Q_{21}L_2Q_{22} - z_2w_2$. To make r_2 a Halin constructible vertex, we let $J_2 = Q_{21}L_2Q_{22}\cup\{g_2x_2, g_2y_2, p_2g_2, p_2y_2, q_2z_2, q_2w_2\}$ if r_2 is on the shortest (y_2, w_2)-path in $Q_{21}L_2Q_{22}$, and let $J_2 = Q_{21}L_2Q_{22} \cup \{g_2x_2, g_2y_2, p_2g_2, p_2x_2, q_2z_2, q_2w_2\}$ if r_2 is on the shortest (x_2, z_2)-path (recall that $g_2, x_2, y_2 \in \Gamma_{G_2}(p_2)$). Let $H = J_1\cup J_2\cup\{p_1p_2, r_1r_2, q_1q_2\}$. Then H is a spanning Halin subgraph of G by Proposition 1 as $J_1 + p_1q_1 \cong H_1$ and $J_2 + p_2q_2 \cong H_2$. Figure 3 gives a construction of H for the above case when r_2 is on the shortest (y_2, w_2)-path in $Q_{21}L_2Q_{22}$.

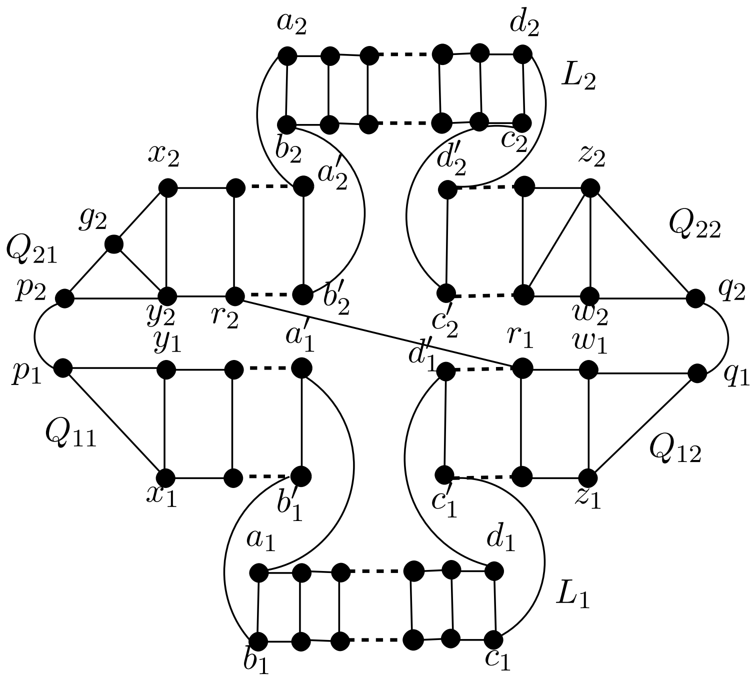

Figure 3. *A Halin graph H*

FIG. 3. *A Halin graph H.*

3.2. Proof of Theorem 3.2. Recall Extremal Case 2: There exists a partition $V_1 \cup V_2$ of V such that $|V_1| \ge (1/2 - 7\beta)n$ and $\Delta(G[V_1]) \le \beta n$. Since $\delta(G) \ge (n+1)/2$, the assumptions imply that

$$(1/2 - 7\beta)n \le |V_1| \le (1/2+\beta)n \quad \text{and} \quad (1/2-\beta)n \le |V_2| \le (1/2+7\beta)n.$$

Let β and α be real numbers satisfying $\beta \le \alpha/20$ and $\alpha \le (1/17)^3$. Set $\alpha_1 = \alpha^{1/3}$ and $\alpha_2 = \alpha^{2/3}$. We first repartition $V(G)$ as follows.

$$V_2' = \{v \in V_2 \mid deg(v, V_1) \ge (1-\alpha_1)|V_1|\}, V_{01} = \{v \in V_2 - V_2' \mid deg(v, V_2') \ge (1-\alpha_1)|V_2'|\},$$
$$V_1' = V_1 \cup V_{01}, \quad \text{and} \quad V_0 = V_2 - V_2' - V_{01}.$$

Claim 3.2.1. $|V_{01} \cup V_0| = |V_2 - V_2'| \le \alpha_2|V_2|$.

Proof. Notice that $e(V_1, V_2) \ge (1/2-\beta)n|V_1| \ge \frac{1/2-\beta}{1/2+7\beta}|V_1||V_2| \ge (1-\alpha)|V_1||V_2|$ as $\beta \le \alpha/20$. Hence,

$$(1-\alpha)|V_1||V_2| \le e(V_1, V_2) \le e(V_1, V_2') + e(V_1, V_2 - V_2') \le |V_1||V_2'| + (1-\alpha_1)|V_1||V_2 - V_2'|.$$

This gives that $|V_{01} \cup V_0| = |V_2 - V_2'| \le \alpha_2|V_2|$. ■

As a result of moving vertices from V_2 to V_1 and by Claim 3.2.1, we have the following.

$$(1/2-7\beta)n \le |V_1'| \le (1/2+\beta)n + |V_{01}| \le (1/2+\beta)n + \alpha_2(1/2+7\beta)n \le (1/2+\alpha_2)n,$$
$$(1/2-\alpha_2)n \le |V_2'| \le (1/2+7\beta)n,$$
$$\delta(V_1', V_2') \ge \min\{(1/2-\beta)n - |V_2 - V_2'|, (1-\alpha_1)|V_2'|\} \ge (1/2 - 2\alpha_1/3)n,$$
$$(1) \quad \delta(V_2', V_1') \ge (1-\alpha_1)|V_1'| \ge (1-\alpha_1)(1/2-7\beta)n \ge (1/2-2\alpha_1/3)n,$$
$$\delta(V_0, V_1') \ge (n+1)/2 - (1-\alpha_1)|V_2'| - |V_0| \ge \alpha_1 n/3 \ge 3|V_0| + 10$$

We may assume n_1 is even and n_2 is odd and construct a spanning Halin subgraph of G (the construction for the other three cases follow a similar argument). Recall that p_1p_2, q_1q_2, r_1r_2 are the three prescribed independent edges between $G[V_1 \cup W_1]$ and $G[V_2 \cup W_2]$, where $p_1, q_1, r_1 \in V_1 \cup W_1$ and $p_2, q_2, g_2, r_2 \in V_2 \cup W_2$. As shown in Figure 3, for $i = 1, 2$, by inserting L_i in between Q_{i1} and Q_{i2} to get a larger ladder, and joining p_i, g_i, and q_i to the end rungs of the new ladder, we will obtain a graph J_i, where $J_i + p_iq_i \cong H_i$ and the vertices p_i and q_i serve as the left end and the right end of $J_i + p_iq_i$. The three edges p_1p_2, q_1q_2, and r_1r_2 will be used as "connecting edges" to get $(J_1 + p_1q_1) \oplus (J_2 + p_2q_2)$, which is the connection of $J_1 + p_1q_1$ and $J_2 + p_2q_2$ as defined in (2.1).

For a uniform discussion, we may assume that both of the ladders L_1 and L_2 exist. Let $i = 1, 2$. Recall that L_i has a_ib_i as its first rung and c_id_i as its last rung. Choose $a_i' \in \Gamma_G(a_i, V(G_i'))$, $b_i' \in \Gamma_G(b_i, V(G_i'))$ such that $a_i'b_i' \in E(G)$ and $c_i' \in \Gamma_G(c_i, V(G_i'))$, $d_i' \in \Gamma_G(d_i, V(G_i'))$ such that $c_i'd_i' \in E(G)$ (a_i', b_i', c_i', d_i' are chosen mutually distinct and distinct from $x_i, y_i, z_i, w_i, g_i, r_i$). This is possible as $\delta(V_i, V(G_i')) \ge (n+1)/2 - 41\beta n \ge 2$. Let Q_{i1} and Q_{i2} be the ladders of G_i' given by Claim 3.1.1. Set $J_1 = Q_{11}L_1Q_{12}\cup\{p_1x_1, p_1y_1, q_1z_1, q_1w_1\}$. Assume $Q_{21}L_2Q_{22}$ is a ladder that can be denoted as $x_2y_2 - Q_{21}L_2Q_{22} - z_2w_2$. To make r_2 a Halin constructible vertex, we let $J_2 = Q_{21}L_2Q_{22} \cup \{g_2x_2, g_2y_2, p_2g_2, p_2y_2, q_2z_2, q_2w_2\}$ if r_2 is on the shortest (y_2, w_2)-path in $Q_{21}L_2Q_{22}$ and let $J_2 = Q_{21}L_2Q_{22}\cup\{g_2x_2, g_2y_2, p_2g_2, p_2x_2, q_2z_2, q_2w_2\}$ if r_2 is on the shortest (x_2, z_2)-path (recall that $g_2, x_2, y_2 \in \Gamma_{G_2'}(p_2)$). Let $H = J_1 \cup J_2 \cup \{p_1p_2, r_1r_2, q_1q_2\}$. Then H is a spanning Halin subgraph of G by Proposition 1 as $J_1 + p_1q_1 \cong H_1$ and $J_2 + p_2q_2 \cong H_2$. Figure 3 gives a construction of H for the above case when r_2 is on the shortest (y_2, w_2)-path in $Q_{21}L_2Q_{22}$.

3.2. Proof of Theorem 3.2. Recall Extremal Case 2: There exists a partition $V_1 \cup V_2$ of V such that $|V_1| \geq (1/2 - 7\beta)n$ and $\Delta(G[V_1]) \leq \beta n$. Since $\delta(G) \geq (n+1)/2$, the assumptions imply that

$$(1/2 - 7\beta)n \leq |V_1| \leq (1/2 + \beta)n \quad \text{and} \quad (1/2 - \beta)n \leq |V_2| \leq (1/2 + 7\beta)n.$$

Let β and α be real numbers satisfying $\beta \leq \alpha/20$ and $\alpha \leq (1/17)^3$. Set $\alpha_1 = \alpha^{1/3}$ and $\alpha_2 = \alpha^{2/3}$. We first repartition $V(G)$ as follows:

$$V_2' = \{v \in V_2 \,|\, deg(v, V_1) \geq (1-\alpha_1)|V_1|\}, V_{01}=\{v \in V_2 - V_2' \,|\, deg(v, V_2') \geq (1-\alpha_1)|V_2'|\},$$
$$V_1' = V_1 \cup V_{01}, \quad \text{and} \quad V_0 = V_2 - V_2' - V_{01}.$$

Claim 3.2.1. $|V_{01} \cup V_0| = |V_2 - V_2'| \leq \alpha_2 |V_2|$.

Proof. Notice that $e(V_1, V_2) \geq (1/2 - \beta)n|V_1| \geq \frac{1/2-\beta}{1/2+7\beta}|V_1||V_2| \geq (1-\alpha)|V_1||V_2|$ as $\beta \leq \alpha/20$. Hence,

$$(1-\alpha)|V_1||V_2| \leq e(V_1, V_2) \leq e(V_1, V_2') + e(V_1, V_2 - V_2') \leq |V_1||V_2'| + (1-\alpha_1)|V_1||V_2 - V_2'|.$$

This gives that $|V_{01} \cup V_0| = |V_2 - V_2'| \leq \alpha_2 |V_2|$. ∎

As a result of moving vertices from V_2 to V_1 and by Claim 3.2.1, we have the following:

$$(1/2 - 7\beta)n \leq |V_1'| \leq (1/2 + \beta)n + |V_{01}| \leq (1/2 + \beta)n + \alpha_2(1/2 + 7\beta)n \leq (1/2 + \alpha_2)n,$$
$$(1/2 - \alpha_2)n \leq |V_2'| \leq (1/2 + 7\beta)n,$$

$$\begin{aligned} &\delta(V_1', V_2') \geq \min\{(1/2 - \beta)n - |V_2 - V_2'|, (1-\alpha_1)|V_2'|\} \geq (1/2 - 2\alpha_1/3)n, \\ &\delta(V_2', V_1') \geq (1-\alpha_1)|V_1| \geq (1-\alpha_1)(1/2 - 7\beta)n \geq (1/2 - 2\alpha_1/3)n, \\ &\delta(V_0, V_1') \geq (n+1)/2 - (1-\alpha_1)|V_2'| - |V_0| \geq \alpha_1 n/3 \geq 3|V_0| + 10, \\ &\delta(V_0, V_2') \geq (n+1)/2 - (1-\alpha_1)|V_1| - |V_0 \cup V_{01}| \geq \alpha_1 n/3 \geq 3|V_0| + 10. \end{aligned} \tag{3.1}$$

Claim 3.2.2. *We may assume that* $\Delta(G) < n - 1$.

Proof. Suppose to the contrary and let $w \in V(G)$ such that $deg(w) = n - 1$. Then by $\delta(G) \geq (n+1)/2$ we have $\delta(G - w) \geq (n-1)/2$, and thus $G - w$ has a hamiltonian cycle. This implies that G has a spanning wheel subgraph, in particular, a spanning Halin subgraph of G. ∎

Claim 3.2.3. *Let t, m be positive integers with m sufficiently large and H be a graph with $\delta(H) \geq \max\{t+1, 3\}$, and let $V_1^0 = \{v \in V(H) \,|\, deg_H(v) \geq \alpha_1 m\}$. If $|V(H) - V_1^0| - 4(t+1) \geq m - 6\alpha_2 m$, then $H - V_1^0$ contains at least $t+2$ vertex-disjoint 3-stars.*

Proof. Assume $H - V_1^0$ contains a subgraph M of at most $s < t + 2$ disjoint 3-stars. Then $|V(M)| \leq 4(t+1)$. Note that if $t = 1$, then $\delta(H) \geq 3$. By counting the number of edges between $V(M)$ and $V(H) - V_1^0 - V(M)$ in two ways, we get that

$$\begin{aligned} \max\{1, t-1\}|V(H) - V_1^0| - 4(t+1) &\leq e_{H - V_1^0}(V(M), V(H) - V_1^0 - V(M)) \\ &\leq 4s\Delta(H - V_1^0) \leq 4s\alpha_1 m. \end{aligned} \tag{3.2}$$

Since $|V(H) - V_1^0| - 4(t+1) \geq m - 6\alpha_2 m$ and $\alpha \leq (1/8)^3$ by the assumption, inequality (3.2) gives that $s \geq 4\max\{1, t-1\} \geq t + 2$, showing a contradiction. ∎

CLAIM 3.2.4. *There exists a subgraph $T \subseteq G$ with $|V(T)| \equiv n \,(mod\, 2)$ such that T and $G - V(T)$ satisfy the following conditions.*

(i) *T is isomorphic to some graph in $\{T_1, T_2, \ldots, T_5\}$.*

(ii) *Let $2m = n - |V(T)|$. Then $G - V(T)$ contains a balanced spanning bipartite graph G' with partite sets U_1 and U_2 such that $|U_1| = |U_2| = m$.*

(iii) *There exists a subset W of $U_1 \cup U_2$ with at most $\alpha_2 n$ vertices such that $deg_{G'}(x, V(G') - W) \geq (1 - \alpha_1 - 2\alpha_2)m$ for all $x \notin W$.*

(iv) *Assume that T has head link x_1x_2 and tail link y_1y_2. There exist $x_1'x_2', y_1'y_2' \in E(G')$ such that $x_i', y_i' \in U_i - W$, $x_{3-i}' \sim x_i$, and $y_{3-i}' \sim y_i$, for $i = 1, 2$, and if T has a connecting vertex, then the connecting vertex is contained in $V_1' \cup V_2' - W$.*

(v) *There are $|W|$ vertex-disjoint 3-stars in $G' - \{x_1', x_2', y_1', y_2'\}$ with the vertices in W as their centers.*

Proof. By (3.1) and Lemma 3.3, for $i = 1, 2$, we notice that for any $u, v, w \in V_i'$,

$$deg(u, v, w, V_{3-i}') \geq 3\delta(V_i', V_{3-i}') - 2|V_{3-i}'| \geq (1/2 - 3\alpha_1)n > n/4. \tag{3.3}$$

We now separate the proof into two cases according to the parity of n.

Case 1. *n is even.* Suppose first that $\max\{|V_1'|, |V_2'|\} \leq n/2$. We arbitrarily partition V_0 into V_{10} and V_{20} such that $|V_1' \cup V_{10}| = |V_2' \cup V_{20}| = n/2$. Since $\delta(G) \geq (n+1)/2$,

$$\delta(G[V_1' \cup V_{10}]), \delta(G[V_2' \cup V_{20}]) \geq 1.$$

Let $x_1u_1 \in E(G[V_1' \cup V_{10}])$ and $y_2u_2 \in E(G[V_2' \cup V_{20}])$ such that $x_1 \in V_1'$ and $u_2 \in \Gamma(u_1, V_2')$. If there exists $u_1 \in V_1'$, then it is clear by (3.1) that $deg(x_1, u_1, V_2') \geq 2|V_0| + 10$. Thus, we assume that V_1' is an independent set in G. Then we have

$$deg(x_1, V_2') \geq (n+1)/2 - |V_0| > |V_2' \cup V_{20}| - |V_0|.$$

Since

$$deg(u_1, V_2') \geq 3|V_0| + 10$$

by (3.1), we again have that $deg(x_1, u_1, V_2') \geq 2|V_0| + 10$. Hence, there exists $x_2 \in \Gamma(x_1, u_1, V_2') - \{y_2, u_2\}$. Similarly, there exists $y_1 \in \Gamma(y_2, u_2, V_1') - \{x_1, u_1\}$. Then $G[\{x_1, u_1, x_2, y_1, u_2, y_2\}]$ contains a subgraph T isomorphic to T_1. Let

$$m = (n-6)/2, U_1 = (V_1' \cup V_{10}) - V(T), \text{ and } U_2 = (V_2' \cup V_{20}) - V(T). \tag{3.4}$$

We then have $|U_1| = |U_2| = m$.

Let $G' = (V(G) - V(T), E_G(U_1, U_2))$ be the bipartite graph with partite sets U_1 and U_2. Notice that $|W| \leq |V_0| \leq \alpha_2|V_2| < \alpha_2 n$. By (3.1), we have $deg_{G'}(x, V(G') - W) \geq (1 - \alpha_1 - 2\alpha_2)m$ for all $x \notin W$. This shows (iii). By the construction of T above, we have $x_1, y_1 \in V_1' - W$. For $i = 1, 2$, by (3.1) and the definition of U_i and W, we have

$$\delta(V_0, U_i - W) = \delta(V_0, V_i' - V(T)) \geq 3|V_0| + 6. \tag{3.5}$$

So $|\Gamma_{G'}(y_2, U_1 - W)|, |\Gamma_{G'}(x_2, U_1 - W)| \geq 3|V_0| + 6$. Applying statement (iii) and Lemma 3.3, we have $e_{G'}(\Gamma_{G'}(x_2, U_1 - W), \Gamma_{G'}(x_1, U_2 - W)), e_{G'}(\Gamma_{G'}(y_2, U_1 - W), \Gamma_{G'}(y_1, U_2 - W)) \geq (3|V_0| + 6)(1 - 2\alpha_1 - 4\alpha_2)m > 2m$. Hence, we can find independent edges $x_1'x_2'$ and $y_1'y_2'$ such that $x_i', y_i' \in U_i - W$, $x_{3-i}' \sim x_i$, and $y_{3-i}' \sim y_i$. This gives statement (iv). Finally, by (3.5), we have $\delta(V_0, U_i - W - \{x_1', x_2', y_1', y_2'\}) \geq 3|V_0| + 2$. Hence, there are vertex-disjoint 3-stars with their centers in W.

Otherwise we have $\max\{|V_1'|, |V_2'|\} > n/2$. By (3.1), we have the same lower bound for $\delta(V_1', V_2')$, $\delta(V_2', V_1')$, and $\delta(V_0, V_1')$, $\delta(V_0, V_2')$. Furthermore, all the arguments in the following will depend only on the degree conditions, so we assume, without loss of generality (w.l.o.g.), that $|V_1'| \geq n/2+1$. Then $\delta(G[V_1']) \geq 2$ and thus $G[V_1']$ contains two vertex-disjoint paths of order 3 and 2, respectively. Let $m = (n-8)/2$. Let V_1^0 be the set of vertices $u \in V_1'$ such that $deg(u, V_1') \geq \alpha_1 m$. We consider three cases here.

Case (a)*:* $|V_1'| - 5 \leq m$. Let $x_1u_1w_1, y_1v_1 \subseteq G[V_1']$ be two vertex-disjoint paths, and let $x_2 \in \Gamma(x_1, u_1, w_1, V_2'), y_2 \in \Gamma(y_1, v_1, V_2')$, and $z \in \Gamma(w_1, v_1, V_2')$ be three distinct vertices. Then $G[\{x_1, u_1, w_1, x_2, z, y_1, v_1, y_2\}]$ contains a subgraph T isomorphic to T_4. Notice that $|V_2' - V(T)| \leq m$. We arbitrarily partition V_0 into V_{10} and V_{20} such that $|(V_1' - V(T)) \cup V_{10}| = |(V_2' - V(T)) \cup V_{20}| = m$. Let

$$U_1 = (V_1' \cup V_{10}) - V(T), U_2 = (V_2' \cup V_{20}) - V(T) \text{ and } W = V_0. \tag{3.6}$$

Hence we assume $|V_1'| - 5 = m + t_1$ for some $t_1 \geq 1$. This implies that $|V_1'| = n/2 + t_1 + 1$ and thus $\delta(G[V_1']) \geq t_1 + 2$.

Case (b)*:* $|V_1^0| \geq t_1$. We form a set W with t_1 vertices from V_1^0 and all the vertices of V_0. Then $|V_1' - W| = m + 5 + t_1 - t_1 = m + 5 = n/2 + 1$, and hence $\delta(G[V_1' - W]) \geq 2$. Similarly as in Case (a), we can find a subgraph T of G contained in $G[(V_1' \cup V_2') - W]$ isomorphic to T_4. Let

$$U_1 = V_1' - V(T) - W, \quad \text{and} \quad U_2 = (V_2' \cup W) - V(T). \tag{3.7}$$

Case (c)*:* $|V_1^0| < t_1$. Suppose that $|V_1' - V_1^0| = m + 5 + t_1' = n/2 + t_1' + 1$ for some $t_1' \geq 1$. This implies that $\delta(G[V_1' - V_1^0]) \geq t_1' + 2$.

Note that $|V_1' - V_1^0| = m + 5 + t_1' = n/2 + t_1' + 1$ and $|V_1'| \leq (1/2 + \beta)n + \alpha_2|V_2'|$. Thus, $t_1' \leq |V_1'| - m - 5 \leq 2\alpha_2 m$, and $|V_1' - V_1^0| - 4(t_1' + 1)) \geq m - 3t_1' \geq m - 6\alpha_2 m$. By Claim 3.2.3, $G[V_1' - V_1^0]$ contains $t_1' + 2$ vertex-disjoint 3-stars. Let $x_1u_1w_1$ and y_1v_1 be two paths taken from two 3-stars in M. Then we can find a subgraph T of G isomorphic to T_4 in the same way as in Case (a). We take exactly t_1' 3-stars from the remaining ones in M and denote the centers of these stars by W'. Let

$$U_1 = V_1' - V_1^0 - V(T) - W', W = W' \cup V_1^0 \cup V_0, \text{ and } U_2 = (V_2' \cup W) - V(T). \tag{3.8}$$

For the partition of U_1 and U_2 defined in each of (3.6), (3.7), and (3.8), we let $G' = (V(G) - V(T), E_G(U_1, U_2))$ be the bipartite graph with partite sets U_1 and U_2. Notice that

$$\begin{aligned}
&|W| \leq |V_0| \leq \alpha_2 n \quad \text{if Case (a) occurs,}\\
&|W| \leq |V_0| + |V_1'| - m - 5 \leq (1/2+\beta)n + |V_0 \cup V_{01}| - n/2 - 1 \leq \alpha_2 n \quad \text{if Case (b) occurs,}\\
&|W| = |W' \cup V_1^0 \cup V_0| = |V_1' - U_1 - V(T)| + |V_0 \cup V_{01}|\\
&\qquad \leq (1/2 + \beta)n - (1/2 - 4)n + |V_0 \cup V_{01}| \leq \alpha_2 n \quad \text{if Case (c) occurs.}
\end{aligned}$$

(Recall that $|V_1'| \leq (1/2 + \beta)n + |V_{01}|$ and $|V_0 \cup V_{01}| \leq \alpha_2|V_2|$ from (3.1).) Since $\delta(V_2', V_1') \geq (1 - 2\alpha_1/3)n$ from (3.1) and $|V_1' - U_1| \leq 2\alpha_2 m$, we have $\delta(U_2 - W, U_1 - W) \geq (1 - \alpha_1 - 2\alpha_2)m$. On the other hand, from (3.1), $\delta(V_1', V_2') \geq (1/2 - 2\alpha_1/3)n$. This gives that $\delta(U_1 - W, U_2 - W) \geq (1 - \alpha_1 - 2\alpha_2)m$. Hence, we have $deg_{G'}(x, V(G') - W) \geq (1 - \alpha_1 - 2\alpha_2)m$ for all $x \notin W$. According to the construction of T, we have $x_1, y_1 \in V_1' - W$. Applying statement (iii), by Lemma 3.3, for any $u \in \Gamma_{G'}(x_2, U_1 - W)$, $deg(u, x_1, U_2 - W) \geq 2(1 - \alpha_1 - 2\alpha_2)m - m = (1 - 2\alpha_1 - 4\alpha_2)m$. Thus, $e_{G'}(\Gamma_{G'}(x_1, U_2 -$

$W), \Gamma_{G'}(x_2, U_1 - W)), e_{G'}(\Gamma_{G'}(y_1, U_2 - W), \Gamma_{G'}(y_2, U_1 - W)) \geq (3|V_0| + 6)(1 - 2\alpha_1 - 4\alpha_2)m > 2m$. Hence, we can find independent edges $x_1'x_2'$ and $y_1'y_2'$ such that $x_i', y_i' \in U_i - W$, $x_{3-i}' \sim x_i$, and $y_{3-i}' \sim y_i$. By the construction of T, T is isomorphic to T_4, and the connecting vertex $z \in V_2' \subseteq V_1' \cup V_2' - W$. This gives statement (iv). Finally, as

$$\begin{aligned}\delta(V_0, U_1 - W) &\geq \delta(V_0, V_1') - |V_1' - (U_1 - W)| \geq \alpha_1 n/3 - (1/2 + \alpha_2)n + n/2 - 4 - \alpha_2 n \\ &\geq (1/3\alpha_1 - 2\alpha_2)n - 4 \geq 3|W| + 5,\end{aligned}$$

we have $\delta(V_0, U_1 - W - \{x_1', x_2', y_1', y_2'\}) \geq 3|W| + 1$. By the definition of V_1^0, we have $\delta(V_1^0, V_1' - W - \{x_1', x_2', y_1', y_2'\}) \geq \alpha_1 m - \alpha_2 n - 4 \geq 3|W|$. For the vertices in W' in Case (c), we already know that there are vertex-disjoint 3-stars in G' with centers in W'. Hence, regardless of the construction of W, we can always find vertex-disjoint 3-stars with their centers in W.

Case 2. *n is odd.* Suppose first that $\max\{|V_1'|, |V_2'|\} \leq (n+1)/2$ and let $m = (n-7)/2$. We arbitrarily partition V_0 into V_{10} and V_{20} such that, w.l.o.g., say $|V_1' \cup V_{10}| = (n+1)/2$ and $|V_2' \cup V_{20}| = (n-1)/2$. (Again, here we use the symmetry of the lower bounds on $\delta(V_1', V_2')$, $\delta(V_2', V_1')$, and $\delta(V_0, V_1')$, $\delta(V_0, V_2')$ from (3.1).) We show that $G[V_1' \cup V_{10}]$ either contains two independent edges or is isomorphic to $K_{1,(n-1)/2}$. As $\delta(G) \geq (n+1)/2$, we have $\delta(G[V_1' \cup V_{10}]) \geq 1$. Since n is sufficiently large, $(n+1)/2 > 3$. Then it is easy to see that if $G[V_1' \cup V_{10}] \ncong K_{1,(n-1)/2}$, then $G[V_1' \cup V_{10}]$ contains two independent edges. Furthermore, we can choose two independent edges x_1u_1 and y_1v_1 such that $u_1, v_1 \in V_1'$. This is obvious if $|V_{10}| \leq 1$. So we assume $|V_{10}| \geq 2$. As $\delta(V_0, V_1') \geq 3|V_0| + 10$, by choosing $x_1, y_1 \in V_{10}$, we can choose distinct vertices $u_1 \in \Gamma(x_1, V_1')$ and $v_1 \in \Gamma(y_1, V_1')$. Let $x_2 \in \Gamma(x_1, u_1, V_2'), y_2 \in \Gamma(y_1, v_1, V_2')$, and $z \in \Gamma(u_1, v_1, V_2')$. Then $G[\{x_1, u_1, x_2, y_1, v_1, y_2, z\}]$ contains a subgraph T isomorphic to T_3. We assume now that $G[V_1' \cup V_{10}]$ is isomorphic to $K_{1,(n-1)/2}$. Let u_1 be the center of the star $K_{1,(n-1)/2}$. Then each leaf of the star has at least $(n-1)/2$ neighbors in $V_2' \cup V_{20}$. Since $|V_2' \cup V_{20}| = (n-1)/2$, we have $\Gamma(v, V_2' \cup V_{20}) = V_2' \cup V_{20}$ if $v \in V_1' \cup V_{10} - \{u_1\}$. By the definition of V_0, $\Delta(V_0, V_1') < (1-\alpha_1)|V_1| + |V_{01}|$ and $\Delta(V_0, V_2') < (1-\alpha_1)|V_2'|$, and so $u_1 \in V_1'$, $V_{10} = \emptyset$, and $V_{20} = \emptyset$. We claim that V_2' is not an independent set. Otherwise, by $\delta(G) \geq (n+1)/2$, for each $v \in V_2'$, $\Gamma(v, V_1') = V_1'$. This in turn shows that u_1 has degree $n-1$, showing a contradiction to Claim 3.2.2. So let $y_2v_2 \in E(G[V_2'])$ be an edge. Let $w_1 \in \Gamma(v_2, V_1') - \{u_1\}$ and $w_1u_1x_1$ be a path containing w_1. Choose $y_1 \in \Gamma(y_2, v_2, V_1') - \{w_1, u_1, x_1\}$ and $x_2 \in \Gamma(x_1, u_1, w_1, V_2') - \{y_2, v_2\}$. Then $G[\{x_1, u_1, x_2, w_1, v_2, y_2, y_1\}]$ contains a subgraph T isomorphic to T_2. Let $U_1 = (V_1' \cup V_{10}) - V(T)$ and $U_2 = (V_2' \cup V_{20}) - V(T)$ and $W = V_0 - V(T)$. We have $|U_1| = |U_2| = m$ and $|W| \leq |V_0| \leq \alpha_2 n$.

Otherwise we have $\max\{|V_1'|, |V_2'|\} \geq (n+1)/2 + 1$. By the symmetry of lower bounds on degrees related to V_1' and V_2' from (3.1), we assume, w.l.o.g., that $|V_1'| \geq (n+1)/2 + 1$. Then $\delta(G[V_1']) \geq 2$ and thus $G[V_1']$ contains two independent edges. Let $m = (n-7)/2$ and V_1^0 be the set of vertices $u \in V_1'$ such that $deg(u, V_1') \geq \alpha_1 m$. Since $|V_1'| \geq (n+1)/2 + 1 > m + 4$, we assume $|V_1'| = m + 4 + t_1$ for some $t_1 \geq 1$. We consider three cases here.

Case (a)*: Assume first that* $|V_1^0| \geq t_1$. We form a set W with t_1 vertices from V_1^0 and all the vertices of V_0. Then $|V_1' - W| = m + 4 + t_1 - (|V_1'| - 4 - m) = m + 4 = (n+1)/2$, and we have $\delta(G[V_1' - W]) \geq 1$. As any vertex $u \in V_1' - W$ is a vertex such that $deg(u, V_1') < \alpha_1 m$, we know $G[V_1' - W]$ contains two independent edges. Let $x_1u_1, y_1v_1 \subseteq E(G[V_1' - W])$ be two independent edges, and let $x_2 \in \Gamma(x_1, u_1, V_2'), y_2 \in \Gamma(y_1, v_1, V_2')$, and $z \in \Gamma(u_1, v_1, V_2')$ be three distinct vertices. Then

$G[\{x_1, u_1, x_2, z, y_1, v_1, y_2\}]$ contains a subgraph T isomorphic to T_3. Let $U_1 = V_1' - V(T) - W$, $U_2 = (V_2' \cup W) - V(T)$. Then $|U_1| = |U_2| = m$ and $|W| \le |V_0| + |V_1' - U_1| \le |V_2 - V_2'| + \beta n + 4 \le \alpha_2 n$.

Thus we assume that $|V_1^0| < t_1$. Suppose that $|V_1' - V_1^0| = m + 4 + t_1' = (n+1)/2 + t_1'$ for some $t_1' \ge 1$. This implies that $\delta(G[V_1' - V_1^0]) \ge t_1' + 1$.

Case (b)*:* $t_1' \ge 2$. Note that $|V_1' - V_1^0| = m + 4 + t_1' = (n+1)/2 + t_1'$ and $|V_1'| \le (1/2 + \beta)n + \alpha_2 |V_2'|$. Thus, $t_1' \le |V_1'| - m - 4 \le 2\alpha_2 m$, and $|V_1' - V_1^0| - 4(t_1' + 1) \ge m - 3t_1' \ge m - 6\alpha_2 m$. By Claim 3.2.3, $G[V_1' - V_1^0]$ contains a graph M of $t_1' + 2$ vertex-disjoint 3-stars. Let $x_1 u_1$ and $y_1 v_1$ be two paths taken from two 3-stars in M. Then we can find a subgraph T of G isomorphic to T_3 the same way as in Case (a). We take exactly t_1' 3-stars from the remaining ones in M and denote the centers of these stars by W'. Let $U_1 = V_1' - V_1^0 - V(T) - W'$, $W = W' \cup V_1^0 \cup V_0$, and $U_2 = (V_2' \cup W) - V(T)$. Then $|U_1| = |U_2| = m$.

Case (c)*:* $t_1' = 1$. In this case, we let $m = (n-9)/2$. If $G[V_1' - V_1^0]$ contains a vertex adjacent to all other vertices in $V_1' - V_1^0$, then the vertex would be contained in V_1^0 by the definition of V_1^0. Hence, we assume that $G[V_1' - V_1^0]$ has no vertex adjacent to all other vertices in $V_1' - V_1^0$. Then by the assumptions that $\delta(G) \ge (n+1)/2$ and $|V_1' - V_1^0| = (n+1)/2 + 1$, we can find two vertex-disjoint paths of order 3 in $G[V_1' - V_1^0]$. Let $x_1 u_1 w_1$ and $y_1 v_1 z_1$ be two paths in $G[V_1' - V_1^0]$. There exist distinct vertices $x_2 \in \Gamma(x_1, u_1, w_1, V_2')$, $y_2 \in \Gamma(y_1, v_1, z_1, V_2')$, and $z \in \Gamma(w_1, z_1, V_2')$. Then $G[\{x_1, u_1, w_1, x_2, y_1, v_1, z_1, y_2, z\}]$ contains a subgraph T isomorphic to T_5. Let $U_1 = V_1' - V_1^0 - V(T)$, $W = V_1^0 \cup V_0$, and $U_2 = (V_2' \cup W) - V(T)$. Then $|U_1| = |U_2| = m$.

For the partition of U_1 and U_2 in all the cases discussed in Case 2, we let $G' = (V(G) - V(T), E_G(U_1, U_2))$ be the bipartite graph with partite sets U_1 and U_2. Similarly as in Case 1, we can show that all the statements (i)–(v) hold. $\square$

Let $W_1 = U_1 \cap W$ and $W_2 = U_2 \cap W$. By (v) of Claim 3.2.4, we know that there are $|W_1|$ vertex-disjoint 3-stars with centers in W_1 and all other vertices in $U_2 - W_2 - \{x_1', y_1', x_2', y_2'\}$, and $|W_2|$ vertex-disjoint 3-stars with centers in W_2 and all other vertices in $U_1 - W_1 - \{x_1', y_1', x_2', y_2'\}$, and all these $|W_1| + |W_2|$ stars are vertex-disjoint. Let S be the union of the 3-stars with centers in W_2. By (iii) of Claim 3.2.4,

$$\Gamma(u, v, U_1 - W_1 - V(S) - \{x_1', x_2', y_1', y_2'\}) \ge 3|W_1| \quad \text{for any } u, v \in U_2 - W_2,$$
$$\Gamma(u, v, w, U_2 - V(S) - \{x_1', x_2', y_1', y_2'\}) \ge 4|W_1| \quad \text{for any } u, v, w \in U_1 - W_1 - V(S).$$

By Lemma 3.2, we can find a ladder L_1 disjoint from the 3-stars in S with centers in W_2 such that L_1 is spanning W_1, $4|W_1| - 1$ vertices from $U_2 - W_2 - \{x_1', x_2', y_1', y_2'\}$, and another $3|W_1| - 1$ vertices from $U_1 - W_1 - \{x_1', x_2', y_1', y_2'\}$, if $W_1 \ne \emptyset$.

Again, by (iii) of Claim 3.2.4,

$$\Gamma(u, v, U_2 - W_2 - V(L_1) - \{x_1', x_2', y_1', y_2'\}) \ge 3|W_2| \quad \text{for any } u, v \in U_1 - W_1,$$
$$\Gamma(u, v, w, U_1 - W_1 - V(L_1) - \{x_1', x_2', y_1', y_2'\}) \ge 4|W_2| \quad \text{for any } u, v, w \in U_2 - W_2.$$

By Lemma 3.2, we can find a ladder L_2 disjoint from L_1 such that L_2 is spanning W_2, $4|W_2| - 1$ vertices from $U_1 - V(L_1) - \{x_1', x_2', y_1', y_2'\}$, and another $3|W_2| - 1$ vertices from $U_2 - W_2 - V(L_1) - \{x_1', x_2', y_1', y_2'\}$, if $W_2 \ne \emptyset$.

Denote $a_{1i} a_{2i}$ and $b_{1i} b_{2i}$ the first and last rungs of L_i (if L_i exists), respectively, where $a_{1i}, b_{1i} \in U_1$. As $|U_1| = |U_2|$, and we took $4|W_1| + 4|W_2| - 2$ vertices, respectively, from U_1 and U_2 when constructing L_1 and L_2, we have $|U_1 - V(L_1 \cup L_2)| = |U_2 - V(L_1 \cup L_2)|$. Let

$$U_i' = U_i - V(L_1 \cup L_2), \qquad m' = |U_1'| = |U_2'|, \quad \text{and} \quad G'' = G''(U_1' \cup U_2', E_G(U_1', U_2')).$$

Since $|W| \le \alpha_2 n$, $m \ge (n-9)/2$, and n is sufficiently large, we have $1/n + 7|W| \le 15\alpha_2 m$. As $\delta(G'-W) \ge (1-\alpha_1-2\alpha_2)m$ and $\alpha \le (1/17)^3$, we obtain the following:

$$\delta(G'') \ge 7m'/8 + 1.$$

Let $a'_{2i} \in \Gamma(a_{1i}, U'_2)$, $a'_{1i} \in \Gamma(a_{2i}, U'_1)$ such that $a'_{1i}a'_{2i} \in E(G)$ and $b'_{2i} \in \Gamma(b_{1i}, U'_2)$, $b'_{1i} \in \Gamma(b_{2i}, U'_1)$ such that $b'_{1i}b'_{2i} \in E(G)$. We have the claim below.

Claim 3.2.5. *The balanced bipartite graph G'' contains three vertex-disjoint ladders Q_1, Q_2, and Q_3 spanning $V(G'')$ such that the first rung of Q_1 is $x'_1x'_2$ and the last rung of Q_1 is $a'_{11}a'_{21}$, the first rung of Q_2 is $b'_{11}b'_{21}$ and the last rung of Q_2 is $a'_{12}a'_{22}$, and the first rung of Q_3 is $b'_{12}b'_{22}$ and the last rung of Q_3 is $y'_1y'_2$.*

Proof. Since $\delta(G'') \ge 7m'/8+1 > m'/2+6$, G'' has a perfect matching M containing the following edges: $x'_1x'_2, a'_{11}a'_{21}, b'_{11}b'_{21}, a'_{12}a'_{22}, b'_{12}b'_{22}, y'_1y'_2$. We identify a'_{11} and b'_{11}, a'_{21} and b'_{21}, a'_{12} and b'_{12}, and a'_{22} and b'_{22} as vertices called c'_{11}, c'_{21}, c'_{12}, and c'_{22}, respectively. Denote $G^* = G^*(U^*_1, U^*_2)$ as the resulting graph and let $c'_{11}c'_{21}, c'_{12}c'_{22} \in E(G^*)$. Denote $M' := M - \{a'_{11}a'_{21}, b'_{11}b'_{21}, a'_{12}a'_{22}, b'_{12}b'_{22}\} \cup \{c'_{11}c'_{21}, c'_{12}c'_{22}\}$. Define an auxiliary graph H' on M' as follows. If $xy, uv \in M' - \{c'_{11}c'_{21}, c'_{12}c'_{22}\}$ with $x, u \in U'_1$, then $xy \sim_{H'} uv$ if and only if $x \sim_{G'} v$ and $y \sim_{G'} u$. For any $pq \in M' - \{c'_{11}c'_{21}, c'_{12}c'_{22}\}$ with $p \in U'_2$, $pq \sim_{H'} c'_{11}c'_{21}$ (resp., $pq \sim_{H'} c'_{12}c'_{22}$) if and only if $p \sim_{G'} a'_{11}, b'_{11}$ and $q \sim_{G'} a'_{21}, b'_{21}$ (resp., $p \sim_{G'} a'_{12}, b'_{12}$ and $q \sim_{G'} a'_{22}, b'_{22}$). Notice that there is a natural one-to-one correspondence between ladders with rungs in M' and paths in H'. Since $\delta_{G^*}(U^*_1, U^*_2), \delta_{G^*}(U^*_2, U^*_1) \ge 3m'/4 + 1$, we get $\delta(H') \ge m'/2 + 1$. Hence H' has a hamiltonian path starting with $x'_1x'_2$, ending with $y'_1y'_2$, and having $c'_{11}c'_{21}$ and $c'_{12}c'_{22}$ as two internal vertices. The path with the vertex $c'_{11}c'_{21}$ replaced by $a'_{11}a'_{21}$ and $b'_{11}b'_{21}$ and with the vertex $c'_{12}c'_{22}$ replaced by $a'_{12}a'_{22}$ and $b'_{12}b'_{22}$ is corresponding to the required ladders in G''. ◻

If $T \in \{T_1, T_2\}$, then

$$H = x_1x_2Q_1L_1Q_2L_2Q_3y_1y_2 \cup T$$

is a spanning Halin subgraph of G. Suppose now that $T \in \{T_3, T_4, T_5\}$ and z is the connecting vertex. Then $z \in V'_1 \cup V'_2 - W$ by Claim 3.2.4. Suppose, w.l.o.g., that $z \in V'_2 - W$. Then by (iii) of Claim 3.2.4 and $\delta(V'_2, V'_1) \ge (1/2 - 2\alpha_1/3)n$ from (3.1), we have that $deg_G(z, U'_1) \ge deg_G(z, V'_1 - V(L_1 \cup L_2) - V(T)) \ge (1-\alpha_1-10\alpha_2)m > m/2+1$. So z has a neighbor on each side of the ladder $Q_1L_1Q_2L_2Q_3$, which has m vertices on each side, and each side has at most $m/2+1$ vertices from each partition of U'_1 and U'_2. Let H' be obtained from $x_1x_2Q_1L_1Q_2L_2Q_3y_1y_2 \cup T$ by suppressing the degree 2 vertex z. Then H' is a Halin graph such that there exists one side of $Q_1L_1Q_2L_2Q_3$ with each vertex on it as a degree 3 vertex on a underlying tree of H'. Let z' be a neighbor of z such that z' has degree 3 in the underlying tree of H'. Then

$$H = x_1x_2Q_1L_1Q_2L_2Q_3y_1y_2 \cup T \cup \{zz'\}$$

is a spanning Halin subgraph of G.

3.3. Proof of Theorem 3.3. In this section, we prove Theorem 3.3. In the first subsection, we introduce the regularity lemma, the blow-up lemma, and some related results. Then we show that G contains a subgraph T isomorphic to T_1 if n is even and to T_2 if n is odd. By showing that $G - V(T)$ contains a spanning ladder L with its first rung adjacent to the head link of T and its last rung adjacent to the tail link of T, we get a spanning Halin subgraph H of G formed by $L \cup T$.

3.3.1. The regularity lemma and the blow-up lemma. For any two disjoint nonempty vertex sets A and B of a graph G, the *density* of A and B is the ratio $d(A,B) := \frac{e(A,B)}{|A|\cdot|B|}$. Let ε and δ be two positive real numbers. The pair (A,B) is called ε-regular if for every $X \subseteq A$ and $Y \subseteq B$ with $|X| > \varepsilon|A|$ and $|Y| > \varepsilon|B|$, $|d(X,Y) - d(A,B)| < \varepsilon$ holds. In addition, if $\delta(A,B) > \delta|B|$ and $\delta(B,A) > \delta|A|$, we say (A,B) is an (ε,δ)-super regular pair.

LEMMA 3.4 (regularity lemma-degree form [26]). *For every $\varepsilon > 0$ there is an $M = M(\varepsilon)$ such that if G is any graph with n vertices and $d \in [0,1]$ is any real number, then there is a partition of the vertex set $V(G)$ into $l+1$ clusters $V_0, V_1, \ldots, V_l$, and there is a spanning subgraph $G' \subseteq G$ with the following properties:*

- $l \le M$;
- $|V_0| \le \varepsilon n$, *all clusters* $|V_i| = |V_j| \le \lceil \varepsilon n \rceil$ *for all* $1 \le i \ne j \le l$;
- $deg_{G'}(v) > deg_G(v) - (d+\varepsilon)n$ *for all* $v \in V(G)$;
- $e(G'[V_i]) = 0$ *for all* $i \ge 1$;
- *in G', all pairs (V_i, V_j) $(1 \le i \ne j \le l)$ are ε-regular, each with a density either 0 or greater than d.*

LEMMA 3.5 (blow-up lemma [19]). *For every $\delta, \Delta, c > 0$, there exists an $\varepsilon = \varepsilon(\delta,\Delta,c)$ and $\gamma = \gamma(\delta,\Delta,c) > 0$ such that the following holds. Let (X,Y) be an (ε,δ)-superregular pair with $|X| = |Y| = N$. If a bipartite graph H with $\Delta(H) \le \Delta$ can be embedded in $K_{N,N}$ by a function ϕ, then H can be embedded in (X,Y). Moreover, in each $\phi^{-1}(X)$ and $\phi^{-1}(Y)$ (the inverse image of X and Y, respectively), fix at most γN special vertices z, each of which is equipped with a subset S_z of X or Y of size at least cN. The embedding of H into (X,Y) exists even if we restrict the image of z to be S_z for all special vertices z.*

Besides the above two lemmas, we also need the two lemmas below regarding regular pairs.

LEMMA 3.6. *If (A,B) is an ε-regular pair with density d, then for any $A' \subseteq A$ with $|A'| \ge \varepsilon|A|$, there are at most $\varepsilon|B|$ vertices $b \in B$ such that $deg(b, A') < (d-\varepsilon)|A'|$.*

LEMMA 3.7 (slicing lemma). *Let (A,B) be an ε-regular pair with density d, and for some $\nu > \varepsilon$, let $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \ge \nu|A|$, $|B'| \ge \nu|B|$. Then (A',B') is an ε'-regular pair of density d', where $\varepsilon' = \max\{\varepsilon/\nu, 2\varepsilon\}$ and $d' > d - \varepsilon$.*

3.3.2. Finding subgraph T.

CLAIM 3.3.1. *Let n be a sufficient large integer and G an n-vertex graph with $\delta(G) \ge (n+1)/2$. Suppose that G is not in Extremal Case 2. Then if n is even, G contains a subgraph T isomorphic to T_1, and if n is odd, G contains a subgraph T isomorphic to T_2.*

Proof. Suppose first that n is even. Let $xy \in E(G)$ be an edge. We show that $G[N(x) - \{y\}]$ contains an edge x_1x_2 and $G[N(y) - \{x\}]$ contains an edge y_1y_2 such that the two edges are independent. Since G is not in Extremal Case 2, it has no independent set of size at least $(1/2 - 7\beta)n$. Since n is sufficiently large, $|N(x) - \{y\}| \ge \frac{n+1}{2} - 1 > (1/2 - 7\beta)n$. Thus, $G[N(x) - \{y\}]$ contains an edge x_1x_2. Similarly, $G[N(y) - \{x, x_1, x_2\}]$ contains an edge y_1y_2. Therefore, $G[\{x, y, x_1, x_2, y_1, y_2\}]$ contains a subgraph T isomorphic to T_1. We then assume that n is odd. We show in the first step that G contains a subgraph isomorphic to K_4^- (K_4 with one edge removed). Let $yz \in E(G)$. As $\delta(G) \ge (n+1)/2$, there exists $y_1 \in \Gamma(y,z)$. If there exists $y_2 \in \Gamma(y,z) - \{y_1\}$, we are done. Otherwise, $(\Gamma(y) - \{y_1, z\}) \cap (\Gamma(z) - \{y_1, y\}) = \emptyset$. As $\delta(G) \ge (n+1)/2$ and $|\Gamma(y) \cup \Gamma(z) - \{y_1, y, z\}| > (n+1)/2$, y_1 is adjacent to a vertex $y_2 \in \Gamma(y) \cup \Gamma(z) - \{y_1, y, z\}$. Assume $y_2 \in \Gamma(z) - \{y_1, y\}$. Then $G[\{y, y_1, z, y_2\}]$ contains a copy of K_4^-. Choose $x \in \Gamma(y) - \{z, y_1, y_2\}$ and choose an edge $x_1x_2 \in G[\Gamma(x) - \{y, y_1, y_2, z\}]$. Then $G[\{y, y_1, z, y_2, x, x_1, x_2\}]$ contains a subgraph T isomorphic to T_2. ∎

Let T be a subgraph of G as given by Claim 3.3.1. Suppose the head link of T is x_1x_2 and the tail link of T is y_1y_2. Let $G' = G - V(T)$. We show in the next section that G' contains a spanning ladder with its first rung being adjacent to x_1x_2 and its last rung being adjacent to y_1y_2. Let $n' = |V(G')|$. Then we have $\delta(G') \geq (n+1)/2 - 7 \geq n'/2 - 4 \geq (1/2 - \beta)n'$, where β is the parameter defined in the two extremal cases.

3.3.3. Finding a spanning ladder of G' with prescribed end rungs. In this subsection, we roughly follow the following steps in finding a spanning ladder L in G' with its first rung being adjacent to x_1x_2 and its last rung being adjacent to y_1y_2.

Step 1. Apply the regularity lemma to G' to partition the vertices of G' into clusters and a "garbage" set V_0. Construct the reduced graph G_r with vertices as clusters.

Step 2. In G_r, find a hamiltonian path $X_1Y_1 \cdots X_kY_k$ such that each of $deg(x_1, X_1)$, $deg(x_2, Y_1)$, $deg(y_1, X_k)$, and $deg(y_2, Y_k)$ is large. (The ladder L will be constructed such that its vertices on the first and last rungs are contained, respectively, from vertices in $X_1 \cup Y_1$ and $X_k \cup Y_k$ so that we can concatenate x_1x_2 and y_1y_2 to its first and last rungs, respectively.)

Step 3. After regularizing each pair (X_i, Y_i) into a $(2\varepsilon, d - 3\varepsilon)$-superregular pair (X_i', Y_i'), absorb garbage vertices from V_0 and from other sources into small ladders using vertices in $\bigcup_{i=1}^{k}(X_i' \cup Y_i')$.

Step 4. Denote the set of remaining vertices in X_i' and Y_i', after Step 3, respectively, as X_i^* and Y_i^*. Within each pair (X_i^*, Y_i^*), apply the blow-up lemma to get a ladder L^i spanning $X_i^* \cup Y_i^*$ such that its vertices in first and last, second and third, and fourth and fifth rungs are selected from a special set of vertices for connection purposes.

Step 5. Insert small ladders associated with X_i, Y_i obtained in Step 3 between the second and third rungs, or the fourth and fifth rungs of L^i from Step 4 to get a new ladder $\mathcal{L}^i$.

Step 6. Concatenating ladders $\mathcal{L}^1, \mathcal{L}^2, \ldots, \mathcal{L}^k$ using preselected vertices to get a spanning ladder of G'.

THEOREM 3.4. *Let n' be a sufficiently large even integer and G' an n'-vertex subgraph of G obtained by removing vertices in T, where $T \in \{T_1, T_2\}$ has head link x_1x_2 and tail link y_1y_2. Suppose that $\delta(G') \geq (1/2 - \beta)n'$ and $G = G[V(G') \cup V(T)]$ is in nonextremal case, then G' contains a spanning ladder with its first rung being adjacent to x_1x_2 and its last rung being adjacent to y_1y_2.*

Proof. We fix the sequence of parameters

$$0 < \varepsilon \ll d \ll \beta \ll 1$$

and specify their dependence as the proof proceeds.

Let β be the parameter defined in the two extremal cases. Then we choose $d \ll \beta$ and choose

$$\varepsilon = \frac{1}{4}\epsilon(d/2, 3, d/4)$$

following the definition of ϵ in the blow-up lemma.

Applying the regularity lemma to G' with parameters ε and d, we obtain a partition of $V(G')$ into $l+1$ clusters $V_0, V_1, \ldots, V_l$ for some $l \le M \le M(\varepsilon)$, and a spanning subgraph G'' of G' with all described properties in the regularity lemma. In particular, for all $v \in V(G')$,

$$deg_{G''}(v) > deg_{G'}(v) - (d+\varepsilon)n' \ge (1/2 - \beta - \varepsilon - d)n' \ge (1/2 - 2\beta)n' \tag{3.9}$$

provided that $\varepsilon + d \le \beta$. On the other hand,

$$e(G'') \ge e(G') - \frac{(d+\varepsilon)}{2}(n')^2 > e(G') - d(n')^2$$

by $\varepsilon < d$.

We further assume that $l = 2k$ is even; otherwise, we eliminate the last cluster V_l by moving all the vertices in this cluster to V_0. As a result, $|V_0| \le 2\varepsilon n'$, and

$$(1 - 2\varepsilon)n' \le lN = 2kN \le n', \tag{3.10}$$

where $N = |V_i|$ for $1 \le i \le l$.

For each pair i and j with $1 \le i \ne j \le l$, we write $V_i \sim V_j$ if $d(V_i, V_j) \ge d$. As in other applications of the regularity lemma, we consider the *reduced graph* G_r, whose vertex set is $\{1, 2, \ldots, l\}$ and two vertices i and j are adjacent if and only if $V_i \sim V_j$. From $\delta(G'') > (1/2 - 2\beta)n'$, we claim that $\delta(G_r) \ge (1/2 - 2\beta)l$. Suppose not, and let $i_0 \in V(G_r)$ be a vertex with $deg_{Gr}(i_0) < (1/2 - 2\beta)l$. Let V_{i_0} be the cluster in G corresponding to i_0. Then we have

$$(1/2-\beta)n'|V_{i_0}| \le |E_{G'}(V_{i_0}, V - V_{i_0})| < (1/2 - 2\beta)lN|V_{i_0}| + 2\varepsilon n'|V_{i_0}| < (1/2-\beta)n'|V_{i_0}|.$$

This gives a contradiction by $lN \le n'$ from inequality (3.10).

Let A be a cluster of G''. We say A is an (ε, d)-*cluster* if for any distinct cluster B of G'' with $d(A, B) > 0$, (A, B) is an ε-regular pair with density at least d. Let $x \in V(G'')$ be a vertex and A an (ε, d)-cluster. We say x is *typical* to A if $deg(x, A) \ge (d - \varepsilon)|A|$, and in this case, we write $x \sim A$.

Claim 3.3.2. *Each vertex from $\{x_1, x_2, y_1, y_2\}$ is typical to at least $(1/2 - 2\beta)l$ clusters in $\{V_1, \ldots, V_l\}$.*

Proof. Suppose to the contrary that there exists $x \in \{x_1, x_2, y_2, y_2\}$ such that x is typical to less than $(1/2 - 2\beta)l$ clusters in $\{V_1, \ldots, V_l\}$. Then we have $deg_{G'}(x) < (1/2 - 2\beta)lN + (d + \varepsilon)n' \le (1/2 - \beta)n'$ by $lN \le n'$ and $d + \varepsilon \le \beta$. ◻

Let $x \in V(G'')$ be a vertex. Denote by $\mathcal{V}_x$ the set of clusters to which x is typical.

Claim 3.3.3. *There exist $V_{x_1} \in \mathcal{V}_{x_1}$ and $V_{x_2} \in \mathcal{V}_{x_2}$ such that $d(V_{x_1}, V_{x_2}) \ge d$.*

Proof. We show the claim by considering two cases based on the size of $|\mathcal{V}_{x_1} \cap \mathcal{V}_{x_2}|$.

Case 1. $|\mathcal{V}_{x_1} \cap \mathcal{V}_{x_2}| \le 2\beta l$. Then we have $|\mathcal{V}_{x_1} - \mathcal{V}_{x_2}| \ge (1/2 - 4\beta)l$ and $|\mathcal{V}_{x_2} - \mathcal{V}_{x_1}| \ge (1/2 - 4\beta)l$. We conclude that there is an edge between $\mathcal{V}_{x_1} - \mathcal{V}_{x_2}$ and $\mathcal{V}_{x_2} - \mathcal{V}_{x_1}$ in G_r. Otherwise, let $\mathcal{U}$ be the union of clusters in $\mathcal{V}_{x_1} \cap \mathcal{V}_{x_2}$, $W = V_0 \cup \mathcal{U} \cup V(T)$. Let W_1 be the set of vertices contained in clusters in $\mathcal{V}_{x_1} - \mathcal{V}_{x_2}$, and let W_2 be the set of vertices contained in clusters in $\mathcal{V}_{x_1} - \mathcal{V}_{x_2}$. Then W_1 and W_2 is a partition of $V(G) - W$. Furthermore,

$$|W| \le 5\beta n, \quad e(W_1, W_2) \le (d+\varepsilon)n'|W_1| \le (d+\varepsilon)n'(1+4\beta)lN \le \beta n^2, \quad \text{and}$$
$$\delta(G[W_i]) \ge \delta(G) - 7 - |W| - (d+\varepsilon)n' \ge \delta(G) - |W| - \beta n.$$

These imply that W is an approximate vertex-cut of parameter β with size at most $5\beta n$, implying that G is in Extremal Case 1.

Case 2. $|\mathcal{V}_{x_1} \cap \mathcal{V}_{x_2}| > 2\beta l$. We may assume that $\mathcal{V}_{x_1} \cap \mathcal{V}_{x_2}$ is an independent set in G_r. Otherwise, we are done by finding an edge within $\mathcal{V}_{x_1} \cap \mathcal{V}_{x_2}$. Also we may assume that $E_{G_r}(\mathcal{V}_{x_1} \cap \mathcal{V}_{x_2}, \mathcal{V}_{x_1} - \mathcal{V}_{x_2}) = \emptyset$ and $E_{G_r}(\mathcal{V}_{x_1} \cap \mathcal{V}_{x_2}, \mathcal{V}_{x_2} - \mathcal{V}_{x_1}) = \emptyset$. Since $\delta(G_r) \geq (1/2 - 2\beta)l$ and $\delta_{G_r}(\mathcal{V}_{x_1} \cap \mathcal{V}_{x_2}, \mathcal{V}_{x_1} \cup \mathcal{V}_{x_2}) = 0$, we know that $l - |\mathcal{V}_{x_1} \cup \mathcal{V}_{x_2}| \geq (1/2 - 2\beta)l$. Hence, $|\mathcal{V}_{x_1} \cup \mathcal{V}_{x_2}| = |\mathcal{V}_{x_1}| + |\mathcal{V}_{x_2}| - |\mathcal{V}_{x_1} \cap \mathcal{V}_{x_2}| \leq (1/2 + 2\beta)l$. This gives that

$$\begin{aligned}|\mathcal{V}_{x_1} \cap \mathcal{V}_{x_2}| &\geq |\mathcal{V}_{x_1}| + |\mathcal{V}_{x_2}| - (1/2 + 2\beta)l \geq (1/2 - 2\beta)l + (1/2 - 2\beta)l - (1/2 + 2\beta)l \\ &\geq (1/2 - 6\beta)l.\end{aligned}$$

Let $\mathcal{U}$ be the union of clusters in $\mathcal{V}_{x_1} \cap \mathcal{V}_{x_2}$. Then $|\mathcal{U}| \geq (1/2 - 7\beta)n$ and $\Delta(G[\mathcal{U}]) \leq (d + \varepsilon)n' \leq \beta n$. This shows that G is in Extremal Case 2. ∎

Similarly, we have the following claim.

CLAIM 3.3.4. *There exist* $V_{y_1} \in \mathcal{V}_{y_1} - \{V_{x_1}, V_{x_2}\}$ *and* $V_{y_2} \in \mathcal{V}_{y_2} - \{V_{x_1}, V_{x_2}\}$ *such that* $d(V_{y_1}, V_{y_2}) \geq d$.

CLAIM 3.3.5. *The reduced graph* G_r *has a hamiltonian path* $X_1Y_1 \cdots X_kY_k$ *such that* $\{X_1, Y_1\} = \{V_{x_1}, V_{x_2}\}$ *and* $\{X_k, Y_k\} = \{V_{y_1}, V_{y_2}\}$.

Proof. We contract the edges $V_{x_1}V_{x_2}$ and $V_{y_1}V_{y_2}$ in G_r. Denote the two new vertices by V_x' and V_y', respectively, and denote the resulting graph by G_r'. Then we show that G_r' contains a hamiltonian (V_x', V_y')-path. This path is corresponding to a required hamiltonian path in G_r.

To show G_r' has a hamiltonian (V_x', V_y')-path, we need the following variation of a result due to Nash-Williams [22]: Let Q be a 2-connected graph of order m. If $\delta(Q) \geq \max\{(m+2)/3 + 1, \alpha(Q) + 1\}$, then Q is hamiltonian connected, where $\alpha(Q)$ is the size of a largest independent set of Q. (The result in [22] by Nash-Williams states the following: Let Q be a 2-connected graph of order m. If $\delta(Q) \geq \max\{(m+2)/3, \alpha(Q)\}$, then Q is hamiltonian.)

We claim that G_r' is $2\beta l$-connected. Otherwise, let S be a vertex-cut of G_r' with $|S| < 2\beta l$ and $\mathcal{S}$ the vertex set corresponding to S in G. Since $\delta(G_r') \geq (1/2 - 2\beta)l - 2$ and $|S| < 2\beta l$, we know that $G_r' - S$ has exactly two components. Let $W = \mathcal{S} \cup V_0 \cup V(T)$, W_1 the set of vertices contained in clusters corresponding to vertices in one component of $G_r' - S$, and $W_2 = V(G) - W_1 - W$. Then it is easy to check that $e(W_1, W_2) \leq \beta n^2$ and $\delta(G[W_i]) \geq \delta(G) - |W| - \beta n$. Hence W is an approximate vertex-cut with parameter β of size at most $5\beta n$, showing that G is in Extremal Case 1. Since $n' = Nl + |V_0| \leq (l+2)\varepsilon n'$, we have that $l \geq 1/\varepsilon - 2 \geq 1/\beta$. Hence, G_r' is 2-connected. As G is not in Extremal Case 2, $\alpha(G_r') \leq (1/2 - 7\beta)l$. By $\delta(G_r) \geq (1/2 - 2\beta)l$, we have $\delta(G_r') \geq (1/2 - 2\beta)l - 2 \geq \max\{(l+2)/3 + 1, (1/2 - 7\beta)l + 1\}$. Thus, by the result on hamiltonian connectedness given above, we know that G_r' contains a hamiltonian (V_x', V_y')-path. ∎

CLAIM 3.3.6. *For each* $1 \leq i \leq k$, *there exist* $X_i' \subseteq X_i$ *and* $Y_i' \subseteq Y_i$ *such that each of the following holds:*

(1) $|X_1'| \geq (1 - \varepsilon)|X_1| - 1$, $|Y_k'| \geq (1 - \varepsilon)|Y_k| - 1$, $|Y_1'| \geq (1 - \varepsilon)|Y_1|$, $|X_k'| \geq (1 - \varepsilon)|X_k|$, *and* $|X_i'| \geq (1 - \varepsilon)|X_i|$, $2 \leq i \leq k - 1$;
(2) (X_i', Y_i') *is* $(2\varepsilon, d - 3\varepsilon)$*-superregular with density at least* $d - \varepsilon$;
(3) $|Y_1'| = |X_1'| + 1$, $|X_k'| = |Y_k'| + 1$, *and* $|X_i'| = |Y_i'|$, $2 \leq i \leq k - 1$; *and*
(4) *for any* $A, B \in \{X_1', Y_1', \ldots, X_k', Y_k'\}$, *if* $d(A, B) > 0$, *then* (A, B) *is* 2ε*-regular with density at least* $d - \varepsilon$. *Consequently, each* A *is a* $(2\varepsilon, d - \varepsilon)$ *cluster.*

Proof. For each $1 \le i \le k$, let

$$X_i'' = \{x \in X_i \,|\, deg(x, Y_i) \ge (d-\varepsilon)N\} \quad \text{and}$$
$$Y_i'' = \{y \in Y_i \,|\, deg(y, X_i) \ge (d-\varepsilon)N\}.$$

If necessary, we either take a subset X_i' of X_i'' or take a subset Y_i' of Y_i'' such that $|Y_1'| = |X_1'|+1$, $|X_k'| = |Y_k'|+1$, and $|X_i'| = |Y_i'|$ for $2 \le i \le k-1$. Since (X_i, Y_i) is ε-regular, we have $|X_i''|, |Y_i''| \ge (1-\varepsilon)N$. This gives that $|X_1'|, |Y_k'| \ge (1-\varepsilon)N - 1$, $|Y_1'| \ge (1-\varepsilon)N$, $|X_k'| \ge (1-\varepsilon)N$, and $|X_i'| = |Y_i'| \ge (1-\varepsilon)N$ for $2 \le i \le k-1$. As a result, we have $\deg(x, Y_i') \ge (d-2\varepsilon)N$ for each $x \in X_i'$ and $deg(y, X_i') \ge (d-2\varepsilon)N - 1 \ge (d-3\varepsilon)N$ for each $y \in Y_i'$. By the slicing lemma (Lemma 3.7), (X_i', Y_i') is 2ε-regular with density at least $d-\varepsilon$. Hence (X_i', Y_i') is $(2\varepsilon, d-3\varepsilon)$-superregular for each $1 \le i \le k$. The last assertion is again an application of the slicing lemma. ∎

For $1 \le i \le k$, we call each X_i', Y_i' a *superregularized cluster (sr-cluster)* and call X_i' and Y_i' partners of each other and write $P(X_i') = Y_i'$ and $P(Y_i') = X_i'$. Denote $R = V_0 \cup (\bigcup_{i=1}^{k}((X_i \cup Y_i) - (X_i' \cup Y_i')))$. Since $|(X_i \cup Y_i) - (X_i' \cup Y_i')| \le 2\varepsilon N$ for $2 \le i \le k-1$ and $|(X_1 \cup Y_1) - (X_1' \cup Y_1')|, |(X_k \cup Y_k) - (X_k' \cup Y_k')| \le 2\varepsilon N + 1$, we have

$$|R| \le 2\varepsilon n + 2k\varepsilon N + 2 \le 3\varepsilon n'. \tag{3.11}$$

As n' is even and $|X_1'| + |Y_1'| + \cdots + |X_k'| + |Y_k'|$ is even, we know $|R|$ is even. We arbitrarily group vertices in R into $|R|/2$ pairs. Given two vertices $u, v \in R$, we define a (u,v)-chain of length $2t$ as distinct sr-clusters $A_1, B_1, \ldots, A_t, B_t$ such that $u \sim A_1 \sim B_1 \sim \ldots \sim A_t \sim B_t \sim v$ and each A_j and B_j are partners; in other words, $\{A_j, B_j\} = \{X_{i_j}', Y_{i_j}'\}$ for some $i_j \in \{1, \ldots, k\}$. Recall here $u \sim A_1$ means that $deg(u, A_1) \ge (d-3\varepsilon)|A_1|$, and $A_1 \sim B_1$ means that the two vertices corresponding to A_1 and B_1 are adjacent in G_r. We call such a chain of length $2t$ a $2t$-*chain*.

Claim 3.3.7. *For each pair (u,v) in R, we can find a (u,v)-chain of length at most 4 such that every sr-cluster is contained in at most $d^2N/5$ chains.*

Proof. Suppose we have found chains for the first $m < 2\varepsilon n'$ pairs of vertices in R such that no sr-cluster is contained in more than $d^2N/5$ chains. Let Ω be the set of all sr-clusters that are contained exactly in $d^2N/5$ chains. Then

$$\frac{d^2N}{5}|\Omega| \le 4m < 8\varepsilon n' \le 8\varepsilon\frac{2kN}{1-2\varepsilon},$$

where the last inequality follows from (3.10). Therefore,

$$|\Omega| \le \frac{80k\varepsilon}{d^2(1-2\varepsilon)} \le \frac{80l\varepsilon}{d^2} \le \beta l/2,$$

provided that $1-2\varepsilon \ge 1/2$ and $80\varepsilon \le d^2\beta/2$.

Consider now a pair (w,z) of vertices in R which does not have a chain found so far; we want to find a (w,z)-chain using sr-clusters not in Ω. Let $\mathcal{U}$ be the set of all sr-clusters to which w is typical but not in Ω, and let $\mathcal{V}$ be the set of all sr-clusters to which z is typical but not in Ω. We claim that $|\mathcal{U}|, |\mathcal{V}| \ge (1/2 - 2\beta)l$. To see this, we first observe that any vertex $x \in R$ is typical to at least $(1/2 - 3\beta/2)l$ sr-clusters. For instead,

$$\begin{aligned}(1/2-\beta)n' \le deg_{G'}(x) &< (1/2 - 3\beta/2)lN + (d-3\varepsilon)lN + 3\varepsilon n',\\ &\le (1/2 - 3\beta/2 + d)n'\\ &< (1/2-\beta)n' \text{ (provided that } d < \beta/2 \text{),}\end{aligned}$$

showing a contradiction. Since $|\Omega| \le \beta l/2$, we have $|\mathcal{U}|, |\mathcal{V}| \ge (1/2 - 2\beta)l$. Let $P(\mathcal{U})$ and $P(\mathcal{V})$ be the set of the partners of clusters in $\mathcal{U}$ and $\mathcal{V}$, respectively. By the definition of the chains, a cluster $A \in \Omega$ if and only its partner $P(A) \in \Omega$. Hence, $(P(\mathcal{U}) \cup P(\mathcal{V})) \cap \Omega = \emptyset$. Notice also that each cluster has a unique partner, and so we have $|P(\mathcal{U})| = |\mathcal{U}| \ge (1/2 - 2\beta)l$ and $|P(\mathcal{V})| = |\mathcal{V}| \ge (1/2 - 2\beta)l$.

If $E_{G_r}(P(\mathcal{U}), P(\mathcal{V})) \ne \emptyset$, then there exist two adjacent clusters $B_1 \in P(\mathcal{U})$, $A_2 \in P(\mathcal{V})$. If B_1 and A_2 are partners of each other, then $w \sim A_2 \sim B_1 \sim z$ gives a (w, z)-chain of length 2. Otherwise, assume $A_1 = P(B_1)$ and $B_2 = P(A_2)$; then $w \sim A_1 \sim B_1 \sim A_2 \sim B_2 \sim z$ gives a (w, z)-chain of length 4. Hence we assume that $E_{G_r}(P(\mathcal{U}), P(\mathcal{V})) = \emptyset$. We may assume that $P(\mathcal{U}) \cap P(\mathcal{V}) \ne \emptyset$. Otherwise, let $\mathcal{S}$ be the union of clusters contained in $V(G_r) - (P(\mathcal{U}) \cup P(\mathcal{V}))$. Then $\mathcal{S} \cup R \cup V(T)$ with $|\mathcal{S} \cup R \cup V(T)| \le 4\beta n' + 3\varepsilon n' + 7 \le 5\beta n$ (provided that $3\varepsilon + 7/n' < \beta$) is an approximate vertex-cut of G, implying that G is in Extremal Case 1. As $E_{G_r}(P(\mathcal{U}), P(\mathcal{V})) = \emptyset$, any cluster in $P(\mathcal{U}) \cap P(\mathcal{V})$ is adjacent to at least $(1/2 - 2\beta)l$ clusters in $V(G_r) - (P(\mathcal{U}) \cup P(\mathcal{V}))$ by $\delta(G_r) \ge (1/2 - 2\beta)l$. This implies that $|P(\mathcal{U}) \cup P(\mathcal{V})| \le (1/2 + 2\beta)l$, and thus $|P(\mathcal{U}) \cap P(\mathcal{V})| \ge |P(\mathcal{U})| + |P(\mathcal{V})| - |P(\mathcal{U}) \cup P(\mathcal{V})| \ge (1/2 - 6\beta)l$. Then $P(\mathcal{U}) \cap P(\mathcal{V})$ is corresponding to a subset W_1 of $V(G)$ such that $|W_1| \ge (1/2 - 6\beta)lN \ge (1/2 - 7\beta)n$ and $\Delta(G[W_1]) \le (d + \varepsilon)n' \le \beta n$. This implies that G is in Extremal Case 2, showing a contradiction. $\square$

In the following two claims, we "absorb" vertices in R into small ladders by using the chains containing the vertices. We construct the ladders in a way such that the number of vertices used by the ladders, respectively, from X_i' and Y_i', are the same. For each 2-chain $u \sim X_1' \sim Y_i' \sim v$, when we construct small ladders, the vertex u will "consume" 3 vertices from X_i' and 2 vertices from Y_i'; similarly, the vertex v will consume 3 vertices from Y_i' and 2 vertices from X_i'. Thus, every 2-chain will consume 5 vertices in total from each X_i' and Y_i' when we construct small ladders. Chains of length 4 can result in an imbalance in using vertices from X_i' and Y_i' when constructing small ladders. We explain how do we overcome this issue. Let $u \sim X_i' \sim Y_i' \sim X_j' \sim Y_j' \sim v$ be a 4-chain. When we construct small ladders, u will consume 3 vertices from X_i' and 2 vertices from Y_i', v will consume 3 vertices from Y_j' and 2 vertices from X_j'. We see that there is a one vertex difference in using vertices from X_i' and Y_i' and, respectively, from X_j' and Y_j'. The rough idea to deal with this problem is to "borrow" a vertex, say w, from X_j' and to use this vertex w as a vertex from R that is "assigned" to Y_i'. This new vertex w will consume 3 vertices from Y_i' and 2 vertices from X_i'. Thus, u and w together will consume 5 vertices in total from each X_i' and Y_i' when we construct small ladders. Furthermore, the "borrowing" of w from X_j' makes the number of usage of vertices from X_j' and Y_j' the same in this construction process corresponding to each 4-chain. We give the details on how do we work on 4-chains in the following.

By Claim 3.3.7, each vertex in R is contained in a unique chain of length at most 4. Let Z be an sr-cluster and $u \in R$ be a vertex. We say u and Z are *chain-adjacent* to each other if in the chain that contains u, Z appears next to u. For each sr-cluster $Z \in \{X_1', Y_1', \dots, X_k', Y_k'\}$, let $R(Z)$ denote the set of vertices in R that are chain-adjacent to Z. Let

$$R_4(Z) = \{u \in R(Z) \mid u \text{ is contained in a 4-chain}\},$$

and let

$$S_4(Z) = \{A \in \{X_1', Y_1', \dots, X_k', Y_k'\} \mid u, v \in R,\ u \sim Z \sim P(Z) \sim A \sim P(A) \sim v \text{ is a 4-chain}\}.$$

Obviously, by the definitions, $R(Z) - R_4(Z)$ is the set of vertices from R that are chain-adjacent to Z through 2-chains.

For $Z \in \{X_1', \dots, X_k'\}$ and for each sr-cluster $A \in S_4(Z)$, let $c(A)$ denote the number of 4-chains that contain $Z \sim P(Z) \sim A \sim P(A)$ as a sequence. For each $A \in S_4(Z)$, choose a set $R^*(A)$ consisting of $c(A)$ vertices in A such that each of them has at least $(d - 3\varepsilon)|Z| > 3d^2N/5$ neighbors in $P(Z)$. (Since $(P(Z), A)$ is 2ε-regular with density at least $d-\varepsilon$ by Claim 3.3.6(2), we know that there are at least $(1-2\varepsilon)|A|$ vertices in A with this property by Lemma 3.6.)

For each sr-cluster $Y_i' \in \{Y_1', \dots, Y_k'\}$, and each sr-cluster $X_i' \in \{X_1', \dots, X_k'\}$, let

$$R'(Y_i') = R(Y_i') \cup \left(\bigcup_{A \in S_4(X_i')} R^*(A) \right), \quad \text{and} \quad \omega(X_i') = \sum_{Z \in \{X_1', \cdots, X_k'\},\ X_i' \in S_4(Z)} c(X_i').$$

CLAIM 3.3.8. *For each $i = 1, 2, \dots, k$, each of the following holds:*
(a) $|R(X_i')| \le d^2N/5$ *and* $|R'(Y_i')| \le d^2N/5$.
(b) $|R(X_i') - R_4(X_i')| = |R(Y_i') - R_4(Y_i')|$.
(c) $\omega(X_i') = |R_4(Y_i')|$.
(d) $|R'(Y_i') - R(Y_i')| = |R_4(X_i')|$.

Proof. By Claim 3.3.7, each sr-cluster is contained in at most $d^2N/5$ chains, and a chain contains X_i' if and only if it also contains Y_i' by its definition. Since both $|R(X_i')|$ and $|R'(Y_i')|$ are bounded above by the number of chains which contain them, we have that $|R(X_i')| \le d^2N/5$ and $|R'(Y_i')| \le d^2N/5$. By the definition of 2-chains, a vertex in R is chain-adjacent to an sr-cluster A in a 2-chain if and only if there exists another vertex in R which is chain-adjacent to the partner $P(A)$ of A. Thus $|R(X_i') - R_4(X_i')| = |R(Y_i') - R_4(Y_i')|$. By the definition, if $X_i' \in S_4(Z)$ for some sr-cluster Z, then $c(X_i')$ is the number of 4-chains that contain $Y_i' \sim X_i' \sim P(Z) \sim Z$ as a sequence. All of such 4-chains is just the set of 4-chains in which Y_i' is chain-adjacent to a vertex in R. Since each vertex in R is contained in a unique chain, we then have that $\omega(X_i') = |R_4(Y_i')|$. Since each vertex in $R'(Y_i') - R(Y_i')$ is corresponding to a 4-chain in which X_i' is chain-adjacent to a vertex in R, we have that $|R'(Y_i') - R(Y_i')| = |R_4(X_i')|$. ∎

CLAIM 3.3.9. *For each $i = 1, 2, \dots, k$, there exist vertex-disjoint ladders L_x^i, L_y^i such that*
(a) $R(X_i') \subseteq V(L_x^i) \subseteq R(X_i') \cup X_i' \cup Y_i'$ *and*
(b) $R'(Y_i') \subseteq V(L_y^i) \subseteq X_i' \cup Y_i' \cup R'(Y_i')$;
(c) $|(V(L_x^i) \cup V(L_y^i)) \cap X_i'| = 4|R(X_i')| + 3|R(Y_i')| + 3|R_4(X_i')| - 2$ *and* $|(V(L_x^i) \cup V(L_y^i)) \cap Y_i'| = 4|R(Y_i')| + 4|R_4(X_i')| + 3|R(X_i')| - 2$; *and*
(d) *the vertices on the first and last rungs of each of L_x^i and L_y^i are contained in $X_i' \cup Y_i'$.*

Proof. Notice that by Claim 3.3.6, (X_i', Y_i') is 2ε-regular with density at least $d - \varepsilon$. Let $R(X_i') = \{x_1, \dots, x_r\}$. For each j, $1 \le j \le r$, since $|\Gamma(x_j, X_i')| \ge (d - 3\varepsilon)|X_i'| > 2\varepsilon|X_i'|$, by Lemma 3.6, there exists a vertex set $B_j \subseteq Y_i'$ with $|B_j| \ge (1 - 2\varepsilon)|Y_i'|$ such that for each $b_1 \in B_j$, $deg(b_1, \Gamma(x_j, X_i')) \ge (d - 3\varepsilon)|\Gamma(x_j, X_i')| > 4|R(X_i')|$. If $r \ge 2$, for $j = 1, \dots, r-1$, by Lemma 3.6, there also exists a vertex set $B_{j,j+1} \subseteq Y_i'$ with $|B_{j,j+1}| \ge (1 - 4\varepsilon)|Y_i'|$ such that for each $b_2 \in B_{j,j+1}$, we have $deg(b_2, \Gamma(x_j, X_i')) \ge (d - 3\varepsilon)|\Gamma(x_j, X_i')| > 4|R(X_i')|$ and $deg(b_2, \Gamma(x_{j+1}, X_i')) \ge (d - 3\varepsilon)|\Gamma(x_{j+1}, X_i')| > 4|R(X_i')|$. When $r \ge 2$, since $|B_j|, |B_{j,j+1}|, |B_{j+1}| \ge (d - 3\varepsilon)|Y_i'| > 2\varepsilon|Y_i'|$, there is a set $A \subseteq X_i'$ with $|A| \ge (1 - 6\varepsilon)|X_i'| \ge |R(X_i')|$ such that for each $a \in A$, $deg(a, B_j) \geq (d-3\varepsilon)|B_j|$, $deg(a, B_{j,j+1}) \geq (d-3\varepsilon)|B_{j,j+1}|$, and $deg(a, B_{j+1}) \geq (d-3\varepsilon)|B_{j+1}|$. Notice that $(d-3\varepsilon)|B_j|, (d-3\varepsilon)|B_{j,j+1}|, (d-3\varepsilon)|B_{j+1}| \geq (d-3\varepsilon)(1-4\varepsilon)|Y_i'| > 3|R(X_i')|$. Hence we can choose distinct vertices $u_1, u_2, \ldots, u_{r-1} \in A$ such that $deg(u_j, B_j), deg(u_j, B_{j,j+1}), deg(u_j, B_{j+1}) \geq 3|R(X_i')|$. Then we can choose distinct vertices $y_{23}^j \in \Gamma(u_j, B_j), z_j \in \Gamma(u_j, B_{j,j+1})$ and $y_{12}^{j+1} \in \Gamma(u_j, B_{j+1})$ for each j and choose distinct and unchosen vertices $y_{12}^1 \in B_1$ and $y_{23}^r \in B_r$. Finally, as for each vertex $b_1 \in B_j$, we have $deg(b_1, \Gamma(x_j, X_i')) > 4|R(X_i')|$, and for each vertex $b_2 \in B_{j,j+1}$, we have $deg(b_2, \Gamma(x_j, X_i')), deg(b_2, \Gamma(x_{j+1}, X_i')) > 4|R(X_i')|$, we can choose $x_{j1}, x_{j2}, x_{j3} \in \Gamma(x_j, X_i') - \{u_1, \ldots, u_{r-1}\}$ such that $y_{12}^j \in \Gamma(x_{j1}, x_{j2}, Y_i')$, $y_{23}^j \in \Gamma(x_{j2}, x_{j3}, Y_i')$, and $z_j \in \Gamma(x_{i3}, x_{i+1,1}, Y_i')$. (When $i \geq 2$, we choose all these vertices such that they are not used by existing ladders. The possibility of doing this is guaranteed by the degree conditions and the small sizes of the existing ladders.) Let L_x^i be the graph with

$$V(L_x^i) = R(X_i') \cup \{x_{i1}, x_{i2}, x_{i3}, y_{12}^i, y_{23}^i, z_i, u_i, x_{r1}, x_{r2}, x_{r3}, y_{12}^r, y_{23}^r \,|\, 1 \leq i \leq r-1\} \quad \text{and}$$

$E(L_x^i)$ consisting of the edges $x_r x_{r1}, x_r x_{r2}, x_r x_{r3}, y_{12}^r x_{r1}, y_{12}^r x_{r2}, y_{23}^r x_{r2}, y_{23}^r x_{r3}$ and the edges indicated below for each $1 \leq i \leq r-1$:

$$x_i \sim x_{i1}, x_{i2}, x_{i3};\ y_{12}^i \sim x_{i1}, x_{i2};\ y_{23}^i \sim x_{i2}, x_{i3};\ z_i \sim x_{i3}, x_{i+1,1};\ u_i \sim x_{i3}, x_{i+1,1}, z_i.$$

It is easy to check that L_x^i is a ladder spanning $R(X_i')$, $4|R(X_i')|-1$ vertices from X_i' and $3|R(X_i')|-1$ vertices from Y_i'. Similarly, we can find a ladder L_y^i spanning $R'(Y_i')$, $4|R'(Y_i')|-1$ vertices from Y_i' and $3|R'(Y_i')|-1$ vertices from X_i'. The constructions of ladders L_x^i and L_y^i verify both statements (a) and (c). Statement (b) is seen by the construction of the ladders and (d) of Claim 3.3.8, which says that $|R'(Y_i')| = |R(Y_i')| + |R_4(X_i')|$. ◻

For each $i = 1, 2, \ldots, k-1$, let $X_i^{**} = X_i' - V(\bigcup_{i=1}^k (L_x^i \cup L_y^i))$ and $Y_i^{**} = Y_i' - V(\bigcup_{i=1}^k (L_x^i \cup L_y^i))$. Using Lemma 3.6, for $i \in \{1, \ldots, k-1\}$, choose $y_i^* \in Y_i^{**}$ such that $|A_{i+1}| \geq dN/4$, where $A_{i+1} := X_{i+1}^{**} \cap \Gamma(y_i^*)$. This is possible, as (Y_i^{**}, X_{i+1}^{**}) is 4ε-regular with density at least $d-3\varepsilon$. (Apply the slicing lemma based on (Y_i', X_{i+1}').) Similarly, choose $x_{i+1}^* \in A_{i+1}$ such that $|D_i| \geq dN/4$, where $D_i := Y_i^{**} \cap \Gamma(x_{i+1}^*)$. Let $S = \{y_i^*, x_{i+1}^* \,|\, 1 \leq i \leq k-1\}$, and let $X_i^* = X_i^{**} - S$ and $Y_i^* = Y_i^{**} - S$. We have that the following holds.

Claim 3.3.10. *For each $i = 1, 2, \ldots, k$, $|X_i^*| = |Y_i^*|$ and (X_i^*, Y_i^*) is $(4\varepsilon, d/2)$-superregular.*

Proof. We show that $|X_i^*| = |Y_i^*|$ for each i, $1 \leq i \leq k$. Since $|Y_1'| = |X_1'| + 1$, $|X_k'| = |Y_k'| + 1$, and $|X_i'| = |Y_i'|$ for $2 \leq i \leq k-1$, and $|X_1^{**}| = |X_1^*|$, $|Y_k^{**}| = |Y_k^*|$, and $|X_i^{**}| = |X_i^*| - 1$, $|Y_j^{**}| = |Y_j^*| - 1$ for $2 \leq i \leq k, 1 \leq j \leq k-1$, it suffices to show that $|X_i' \cap V(\bigcup_{i=1}^k (L_x^i \cup L_y^i))| = |Y_i' \cap V(\bigcup_{i=1}^k (L_x^i \cup L_y^i))|$. This is clear by Claims 3.3.9(c) and 3.3.8, since

$$\begin{aligned}
&\left| X_i' \cap V\left(\bigcup_{i=1}^k (L_x^i \cup L_y^i)\right)\right| = 4|R(X_i')| + 3|R(Y_i')| + 3|R_4(X_i')| - 2 + \omega(X_i') \\
&= 4|R(X_i') - R_4(X_i')| + 3|R(Y_i') - R_4(Y_i')| + 7|R_4(X_i')| + 3|R_4(Y_i')| - 2 + \omega(X_i') \\
&= 7|R(X_i') - R_4(X_i')| + 7|R_4(X_i')| + 4|R_4(Y_i')| - 2,
\end{aligned}$$

and

$$\left|Y_i' \cap V\left(\bigcup_{i=1}^{k}(L_x^i \cup L_y^i)\right)\right| = 3|R(X_i')| + 4|R(Y_i')| + 4|R_4(X_i')| - 2$$
$$= 3|R(X_i') - R_4(X_i')| + 4|R(Y_i') - R_4(Y_i')| + 7|R_4(X_i')| + 4|R_4(Y_i')| - 2$$
$$= 7|R(X_i') - R_4(X_i')| + 7|R_4(X_i')| + 4|R_4(Y_i')| - 2.$$

Since $|R(X_i')|, |R'(Y_i')| \le d^2N/5$ for each i, by the first part of the argument, $|X_i' \cap V(\bigcup_{i=1}^{k}(L_x^i \cup L_y^i))| \le 4|R(X_i')| + 4|R'(Y_i')| - 2 \le 2d^2N - 2$ and $|Y_i' \cap V(\bigcup_{i=1}^{k}(L_x^i \cup L_y^i))| \le 4|R(X_i')| + 4|R'(Y_i')| - 2 \le 2d^2N - 2$. Thus $|X_i^*|, |Y_i^*| \ge (1 - \varepsilon - 2d^2)N$. As $\varepsilon, d \ll 1$, we can assume that $1 - \varepsilon - 2d^2 < 1/2$. Thus, by the slicing lemma based on the 2ε-regular pair (X_i', Y_i'), we know that (X_i^*, Y_i^*) is 4ε-regular. Recall from Claim 3.3.6 that (X_i', Y_i') is $(2\varepsilon, d - 3\varepsilon)$-superregular, we know that for each $x \in X_i^*$, $deg(x, Y_i^*) \ge (d - 3\varepsilon - 2d^2)|Y_i^*| > d|Y_i^*|/2$. Similarly, we have for each $y \in Y_i^*$, $deg(y, X_i^*) \ge d|X_i^*|/2$. Thus (X_i^*, Y_i^*) is $(4\varepsilon, d/2)$-superregular. ◻

For each $i = 1, 2, \dots, k-1$, now set $B_{i+1} := Y_{i+1}^* \cap \Gamma(x_{i+1}^*)$ and $C_i := X_i^* \cap \Gamma(y_i^*)$. Since (X_i^*, Y_i^*) is $(4\varepsilon, d/2)$-superregular, we have $|B_i|, |C_i| \ge d|X_i^*|/2 > d|X_i^*|/4$. Recall from Claim 3.3.5 that $\{X_1, Y_1\} = \{V_{x_1}, V_{x_2}\}$ and $\{X_k, Y_k\} = \{V_{y_1}, V_{y_2}\}$. We assume, w.l.o.g., that $X_1 = V_{x_1}$ and $X_k = V_{y_1}$. Let $A_1 = X_1^* \cap \Gamma(x_1)$, $B_1 = Y_1^* \cap \Gamma(x_2)$, $C_k = X_k^* \cap \Gamma(y_1)$, and $D_k = Y_k^* \cap \Gamma(y_2)$. Since $deg(x_1, X_1) \ge (d - \varepsilon)N$, we have $deg(x_1, X_1^*) \ge (d - \varepsilon - 2\varepsilon - 2d^2)N \ge d|X_1^*|/4$, and thus $|A_1| \ge d|X_1^*|/4$. Similarly, we have $|B_1|, |C_k|, |D_k| \ge d|X_1^*|/4$. For each $1 \le i \le k$, we assume that $L_x^i = a_1^ib_1^i - L_x^i - c_1^id_1^i$ and $L_y^i = a_2^ib_2^i - L_y^i - c_2^id_2^i$, where $a_j^i, c_j^i \in Y_i' \subseteq Y_i$ and $b_j^i, d_j^i \in X_i' \subseteq X_i$ for $j = 1, 2$. For $j = 1, 2$, let $A_j^i = X_i^* \cap \Gamma(a_j^i)$, $C_j^i = X_i^* \cap \Gamma(c_j^i)$, $B_j^i = Y_i^* \cap \Gamma(b_j^i)$, and $D_j^i = Y_i^* \cap \Gamma(d_j^i)$. Since (X_i', Y_i') is $(2\varepsilon, d - 3\varepsilon)$-superregular, for $j = 1, 2$, we have $|\Gamma(a_j^i, X_i')|, |\Gamma(c_j^i, X_i')| \ge (d - 3\varepsilon)|X_i'|$ and $|\Gamma(b_j^i, Y_i')|, |\Gamma(d_j^i, Y_i')| \ge (d - 3\varepsilon)|Y_i'|$. Thus, we have $|A_j^i|, |B_j^i|, |C_j^i|, |D_j^i| \ge (d - 3\varepsilon)|X_i'| - 2d^2N \ge d|X_i^*|/4 = d|Y_i^*|/4$.

We now apply the blow-up lemma on (X_i^*, Y_i^*) to find a spanning ladder L^i with its first and last rungs being contained in $A_i \times B_i$ and $C_i \times D_i$, respectively, its second and third rungs being contained in $A_1^i \times B_1^i$ and $C_1^i \times D_1^i$, respectively, and its fourth and fifth rungs being contained in $A_2^i \times B_2^i$ and $C_2^i \times D_2^i$, respectively. We can then insert L_x^i between the second and third rungs of L^i and L_y^i between the fourth and fifth rungs of L^i to obtained a ladder $\mathcal{L}^i$ spanning $X_i \cup Y_i - S$. Finally, $\mathcal{L}^1 y_1^* x_2^* \mathcal{L}^2 \cdots y_{k-1}^* x_k^* \mathcal{L}^k$ is a spanning ladder of G' with its first rung adjacent to x_1x_2 and its last rung adjacent to y_1y_2.

The proof is now complete.

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