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A Jacobi spectral collocation method for the steady aerodynamics of porous aerofoils

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Thin aerofoil theory permits impermeable aerofoils to be analysed by decomposing their chordwise pressure jump into a series of weighted Chebyshev polynomials. However, this approach exhibits singular behaviour when the aerofoils become impermeable. In the present research, we generalise the Chebyshev expansion approach by instead choosing an expansion in terms of weighted Jacobi polynomials that can represent appropriately the singular behaviours at porous edges. Analytic expressions for the parameters of the Jacobi polynomials are derived via asymptotic analysis. The ensuing equations are collocated at the Jacobi nodes, which results in a linear system for the coefficients of the Jacobi polynomials. The approach is shown to be valid for porosity gradients that are continuous or piecewise-continuous, such as in the case of a partially porous aerofoil. A numerical validation is presented that demonstrates that the scheme converges exponentially fast.

I. Introduction

Porous aerofoils have received considerable attention over recent years due to their apparent ability to reduce acoustic emissions [1–4]. It is generally believed that porosity at the trailing-edge weakens the scattering of turbulence there and therefore reduces sound production. However, the aerodynamics of porous aerofoils have been shown to be poor in comparison to impermeable aerofoils [5–7]. Consequently, aircraft designers are faced with the difficult task of balancing the aeroacoustic advantages of porous aerofoils with the aerodynamic disadvantages.

With the goal to assess these aerodynamic effects, Hajian and Jaworski [5] developed an analytic formulation and solution for the steady aerodynamic loads on airfoils with arbitrary, realistic porosity distributions to investigate the impact of a variation in the porosity distribution. This analysis was later extended to determine the unsteady forces on an arbitrarily deforming panel with a Hölder-continuous porosity distribution [8]. An analytical expression for the non-circulatory pressure distribution was presented and evaluated for the special cases of uniform and variable-porosity panels undergoing harmonic deformations, where the effect of the panel end conditions was also investigated.

A comprehensive unsteady aerodynamic theory for lifting porous bodies is essential to predict the aeroelastic stability and aeroacoustic emissions from porous airfoils. The classical theory of Theodorsen [9] and its later extensions [10] developed closed-form expressions for the unsteady aerodynamic forces on a piecewise-continuous rigid and impermeable airfoil undergoing small-amplitude harmonic motions in a uniform incompressible flow. These analyses separated the total fluid forces or moments into circulatory and non-circulatory parts, which correspond respectively to the contribution of the unsteady shedding of vorticity into the wake and the non-lifting hydrodynamic sloshing of fluid about the airfoil [11]. These unsteady fluid forces also contribute fundamentally to the airfoil gust response problem [11, 12] and to the aerodynamic noise generation from gust encounters [13] and vortex-structure interactions [14]. Therefore, an extension of these classical models to include the effects of porosity distributions is desired. However, the singular integral equation describing the generalized aerodynamics of unsteady porous airfoils with a wake cannot be treated by conventional analysis (*e.g.*, [15]), and a different mathematical approach is required.

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There are many methods available for the numerical solutions for singular integral equations [16]. Numerical solutions in terms of orthogonal polynomials were first considered by Erdogan [17], who expressed the solution function as a series of weighted Chebyshev polynomials. However, this numerical approach was limited to particular endpoint behaviours until the generalisation to Jacobi polynomials allowed a wide class of endpoint zeros and singularities [18]. In the present research we adapt the approach of [18] to porous aerofoils, including the generalisation to aerofoils with discontinuous porosity gradients.

The expansion of the jump in surface pressure into a series of weighted Chebyshev polynomials has also been applied to aerodynamic problems for impermeable [19] and permeable [20] aerofoils. The weighted Chebyshev expansion – which is sometimes referred to as a Glauert Fourier series – is an essential feature of many reduced-order discrete-vortex models [21, 22]. These models require detailed understanding of the pressure at the leading- and trailing-edges in order to predict the vortex shedding. In particular, the leading-edge suction parameter must be accurately computed [21]. In this article we will show that the Chebyshev expansion is ill-posed for porous aerofoils, and an expansion in terms of weighted Jacobi polynomials is essential to capture the subtle behaviour at the endpoints.

The Jacobi polynomial solution technique of the present research has several powerful features. Firstly, the system is straightforward to implement, and it is no more costly to compute than the solution for impermeable aerofoils. Secondly, the solution is exact in many practical cases, or converges exponential fast in all. Thirdly, the method may be easily extended to consider multiple interacting aerofoils, and even infinite cascades of aerofoils [23, 24]. Fourthly, and this is the motivation for the present study, the method can be extended to consider the unsteady motions of porous aerofoils where the classical singular integral approach breaks down. This last point is beyond the scope of the present article and will form the basis of future work.

II. Numerical method

In this section we present our new numerical method. We begin by demonstrating why a typical Chebyshev expansion is not valid for porous aerofoils, and continue by applying our new method to a steady porous aerofoil for continuous and discontinuous porosity distributions. Analytic solutions are available for this steady problem, which enables direct comparison for our numerical solution.

Adopting the analysis given in [5], the dimensionless pressure jump p along a porous aerofoil with camber line $z(x)$ is the solution to the singular integral equation

$$2\rho U C R(x) p(x) - \frac{1}{\pi} ⨍_{-1}^{1} \frac{p(t)}{t-x} \mathrm{d}t = 4 \frac{\mathrm{d}z}{\mathrm{d}x}(x), \qquad -1 < x < 1, \tag{1}$$

where ρ, U and C represent the air density, mean flow velocity, and the porosity coefficient respectively. The solution to equation (1) is the focus of the present research. We also enforce that our solution must satisfy a Kutta condition, namely that the pressure jump must vanish at the trailing-edge. We note that analytic solutions to (1) are available [5], but we seek to develop a numerical solution that can be extended to the unsteady scenario.

A. Failure of the Chebyshev approach

We now show that an expansion in terms of Chebyshev polynomials is inappropriate for the present problem. A typical approach [19] to solve (1) for impermeable aerofoils is to write

$$p(x) = p_0 \sqrt{\frac{1-x}{1+x}} + \sqrt{1-x^2} \sum_{n=1}^{\infty} p_n U_{n-1}(x), \tag{2}$$

where U_n are the Chebyshev polynomials of the second kind and p_n are coefficients to be determined. By construction, this series satisfies the steady Kutta condition. However, we now show that this choice of series leads to spurious results at the endpoints.

By sending $x \to -1$, we obtain the following asymptotic limits

$$\rho UCR(x)p(x) \sim \rho UCR(-1)p_0\sqrt{\frac{2}{1+x}}, \tag{3}$$

$$-\frac{1}{\pi}\fint_{-1}^{1}\frac{p(t)}{t-x}\mathrm{d}t \sim \Phi^*(x), \tag{4}$$

$$4\frac{\mathrm{d}z}{\mathrm{d}x}(x) \sim 4\frac{\mathrm{d}z}{\mathrm{d}x}(-1), \tag{5}$$

where $\Phi^*(x) = o\left((1+x)^{-1/2}\right)$ according to [15, (29.8)]. Substitution of these limits into (1) results in an equation where the left hand side scales like $(1+x)^{-1/2}$ whereas the right hand side tends to a constant as $x \to -1$. Asymptotic analysis at the trailing-edge generates the same contradictions. Consequently, the Chebyshev expansion generates spurious results at both endpoints. Since the leading- and trailing-edge vortex shedding is dominated at the endpoints, it is crucial to correctly predict the pressure distribution there.

B. Steady aerofoils with continuous porosity distributions

We now adapt the Chebyshev expansion approach to derive a polynomial expansion that is uniformly valid.

1. Solution method

We speculatively write the pressure in the form

$$p(x) = \frac{(1-x)^\alpha}{(1+x)^\beta}\cdot p^*(x), \tag{6}$$

where $p^*(x)$ is a function that is Hölder continuous on $x \in [-1,1]$ and is non-zero at $x = 1$. This expression is informed by the analytic solutions for (1) [5]. Substituting (6) into our singular integral equation (1) yields

$$2\rho UCR(x)\frac{(1-x)^\alpha}{(1+x)^\beta}\cdot p^*(x) - \frac{1}{\pi}\fint_{-1}^{1}\frac{(1-t)^\alpha}{(1+t)^\beta}\cdot\frac{p^*(t)}{t-x}dt = 4\frac{\mathrm{d}z}{\mathrm{d}x}(-1). \tag{7}$$

The next step is to determine α and β, which represent the intensity of the zero/singularity at the trailing-/leading-edge. We first investigate the behaviour at the trailing edge. If we let $x \to -1$, our singular integral equation (7) becomes, at first order,

$$2\rho UCR(-1)\frac{2^\alpha}{(1+x)^\beta}p^*(-1) - \cot(\beta\pi)\cdot\frac{2^\alpha}{(1+x)^\beta}p^*(-1) = 0, \qquad \text{as } x \to -1,$$

and therefore we obtain

$$\beta = \frac{1}{\pi}\cot^{-1}\left(2\rho UCR(-1)\right). \tag{8}$$

Applying a similar approach at the trailing-edge yields, at first order,

$$2\rho UCR(1)\frac{(1-x)^\alpha}{2^\beta}p^*(1) + C_1 + \frac{(1-x)^\alpha}{2^\beta}p^*(1)\cdot\cot(\alpha\pi) = 4\frac{\mathrm{d}z}{\mathrm{d}x}(1), \qquad \text{as } x \to 1$$

where C_1 is a constant. By matching $(1-x)^\alpha$ terms, we obtain

$$\alpha = \frac{1}{\pi}\cot^{-1}\left(2\rho UCR(1)\right). \tag{9}$$

The results (8) and (9) are suggestive of the form of the pressure at the leading- and trailing-edges, respectively. Consequently, we seek an expansion of the pressure as a sequence of weighted Jacobi polynomials,

$$p(x) = p_0\frac{(1-x)^\alpha}{(1+x)^\beta} + (1-x)^\alpha(1+x)^{1-\beta}\sum_{n=1}^{\infty}p_nP_{n-1}^{\alpha,1-\beta}(x), \tag{10}$$

where $P_n^{a,b}$ represents the n^{th} Jacobi polynomial. The Jacobi polynomials are a classical family of orthogonal polynomials [25] and represent a generalisation of Chebyshev polynomials. The orthogonality relation associated with the Jacobi polynomials is

$$\int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta} P_m^{\alpha,\beta}(x) P_n^{\alpha,\beta}(x) \mathrm{d}x = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \cdot \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)n!}, \qquad \alpha,\ \beta > -1. \tag{11}$$

Generally, we will denote the weight function as $w^{a,b}(x) := (1-x)^a(1+x)^b$.

We may now substitute our Jacobi polynomials expansion (10) into our singular integral equation (1) to

$$\begin{aligned} &2\rho U C R(x)\left(p_0 w^{\alpha,-\beta}(x) + w^{\alpha,1-\beta}(x)\sum_{n=1}^{\infty} p_n P_{n-1}^{\alpha,1-\beta}(x)\right) \\ &-\frac{1}{\pi}\left(p_0 w^{\alpha,-\beta}(x) Q_0^{\alpha,-\beta}(x) + w^{\alpha,1-\beta}(x)\sum_{n=1}^{\infty} p_n Q_{n-1}^{\alpha,1-\beta}(x)\right) = 4\frac{\mathrm{d}z}{\mathrm{d}x}(x), \end{aligned} \tag{12}$$

where $Q_n^{\alpha,\beta}$ are the associated Jacobi functions of the second kind, which are expressible in terms of the hypergeometric function [26]. Equation (12) may be more conveniently represented as

$$p_0 f_0^{\alpha,-\beta}(x) + \sum_{n=1}^{\infty} p_n f_{n-1}^{\alpha,1-\beta}(x) = 4\frac{\mathrm{d}z}{\mathrm{d}x}(x), \tag{13}$$

where

$$f_n^{a,b}(x) = w^{a,b}(x)\left(\psi(x) P_n^{a,b}(x) - \frac{1}{\pi} Q_n^{a,b}(x)\right). \tag{14}$$

We now construct a linear system for the coefficients by truncating the infinite series at some N and collocating (14) at the Jacobi nodes x_j, which are the roots of the Jacobi polynomial $P_{N+1}^{\alpha,1-\beta}$. The resulting system is

$$p_0 f_0^{\alpha,-\beta}(x_j) + \sum_{n=1}^{N} p_n f_{n-1}^{\alpha,1-\beta}(x_j) = 4\frac{\mathrm{d}z}{\mathrm{d}x}(x_j), \qquad j = 1, \cdots, N+1. \tag{15}$$

In matrix form we write

$$\boldsymbol{F}\boldsymbol{p} = \boldsymbol{v}.$$

where

$$\{\boldsymbol{F}\}_{i,j} = \begin{cases} f_0^{\alpha,-\beta}(x_j), & i = 0, \\ f_{i-1}^{\alpha,1-\beta}(x_j), & i = 1, \cdots, N, \end{cases} \tag{16}$$

and

$$\{\boldsymbol{v}\}_j = 4\frac{\mathrm{d}z}{\mathrm{d}x}(x_j), \qquad j = 1, \cdots, N+1 \tag{17}$$

$$\{\boldsymbol{p}\}_i = p_i, \qquad i = 0, \cdots, N. \tag{18}$$

We now make the crucial observation that the columns (and rows) of $\boldsymbol{F}$ are linearly independent. It is straightforward to verify this: if they were linearly dependent then there is an x_j that solves $P_n^{\alpha,1-\beta}(x_j) = 0$. However, this is impossible due to the interlacing property of the zeros of orthogonal polynomials. Consequently $\boldsymbol{F}$ is invertible and we may write

$$\boldsymbol{p} = \boldsymbol{F}^{-1}\boldsymbol{v}. \tag{19}$$

2. *Expressions for aerodynamic quantities*

Compact expressions for relevant aerodynamic quantities are now available in terms of the coefficients of the Jacobi expansion. For example, by the orthogonality properties of the Jacobi polynomials (11), the lift is given by

$$c_L = \int_{-1}^{1} p(t)\mathrm{d}t = p_0 \frac{2^{\alpha-\beta+1}}{\alpha-\beta+1} \cdot \frac{\Gamma(\alpha+1)\Gamma(1-\beta)}{\Gamma(\alpha-\beta+1)} + p_1 \frac{2^{\alpha+2-\beta}}{\alpha+2-\beta} \cdot \frac{\Gamma(\alpha+1)\Gamma(2-\beta)}{\Gamma(\alpha+2-\beta)}.$$

where p_0 and p_1 are the coefficients of the weighted Jacobi polynomials (10) defined in (19). We note that, in the impermeable limit $\alpha,\ \beta \to 1/2$, we recover the usual expression [27]

$$c_L = \pi \left(p_0 + \frac{p_1}{2} \right).$$

C. Solution for discontinuous porosity distributions

We may also use the new method to find the pressure distribution for a partially porous aerofoil. These aerofoils consist of an impermeable forward section and a permeable aft section [6]. It is generally believed that this aerofoil design possess superior aerodynamic qualities compared to aerofoils that are uniformly porous [5–7]. We place the permeable-impermeable junction at $x = c$, so that the singular integral equation for a steady, partially porous aerofoil becomes

$$H(x-c)\psi(x)p(x) - \frac{1}{\pi} ⨍_{-1}^{1} \frac{p(t)}{t-x}\mathrm{d}t = 4\frac{\mathrm{d}z}{\mathrm{d}x}(x), \tag{20}$$

where $H(\cdot)$ is the heaviside function, i.e.

$$H(x-c) = \begin{cases} 0, & x < c, \\ 1, & x > c, \end{cases} \tag{21}$$

and $\psi(x) = 2\rho UCR(x)$. We first note that we require $p = 0$ at $x = c$, otherwise there would be a discontinuity in the seepage velocity through the wing. Consequently, we write the pressure in the form

$$p(x) = p^*(x)(x-c)^\gamma, \qquad p^*(x) = \frac{(1-x)^\alpha}{(1+x)^\beta} \mathrm{e}^{\mathrm{i}\pi\gamma H(c-x)} \tilde{p}(x), \tag{22}$$

where $\tilde{p}$ is piecewise Hölder continuous. Applying a similar argument to the previous section via analysis of (20) at the endpoints yields

$$\alpha = \frac{1}{\pi}\cot^{-1}\left(H(+1-c)\psi(+1)\right) = \frac{1}{\pi}\cot^{-1}\left(\psi(1)\right), \tag{23.a}$$

$$\beta = \frac{1}{\pi}\cot^{-1}\left(H(-1-c)\psi(-1)\right) = \frac{1}{2}. \tag{23.b}$$

We now consider the asymptotic behaviour as $x \to c^\pm$, where $c^\pm = c \pm 0$. In these limits, note that (22) yields

$$p^*(c^-) = \frac{(1-c)^\alpha}{(1+c)^\beta}\mathrm{e}^{\mathrm{i}\pi\gamma}\tilde{p}(c^-), \qquad p^*(c^+) = \frac{(1-c)^\alpha}{(1+c)^\beta}\tilde{p}(c^+).$$

If we substitute (22) into (20) and consider the limit $x \to c^-$ then we obtain

$$-\mathrm{i}\left(-\frac{\mathrm{e}^{-\mathrm{i}\gamma\pi}}{2\mathrm{i}\sin(\gamma\pi)}\frac{(1-c)^\alpha}{(1+c)^\beta}\tilde{p}(c^+) + \frac{\cot(\gamma\pi)}{2\mathrm{i}}\frac{(1-c)^\alpha}{(1+c)^\beta}\mathrm{e}^{\mathrm{i}\pi\gamma}\tilde{p}(c^-) \right) \cdot (x-c)^\gamma + C_0 + \Phi_0^*(x) = 4\frac{\mathrm{d}z}{\mathrm{d}x}(c), \tag{24}$$

where C_0 is a constant, $\Phi_0^*(x) = o\left((x-c)^\gamma\right)$, and we have used the asymptotic expansions found in [15].

By matching coefficients in (24), we see that the coefficient of $(x-c)^\gamma$ must vanish. Consequently, we obtain the identity

$$\tilde{p}(c^+) = \cos(\gamma\pi)\mathrm{e}^{2\mathrm{i}\pi\gamma}\tilde{p}(c^-). \tag{25}$$

If we now consider $x \longrightarrow c^+$ in (20) then we obtain

$$\psi(c^+)\frac{(1-c)^\alpha}{(1+c)^\beta}\tilde{p}(c^+)(x-c)^\gamma + \mathrm{i}\left(\frac{\mathrm{e}^{\mathrm{i}\gamma\pi}}{2\mathrm{i}\sin(\gamma\pi)}\frac{(1-c)^\alpha}{(1+c)^\beta}\mathrm{e}^{\mathrm{i}\pi\gamma}\tilde{p}(c^-) - \frac{\cot(\gamma\pi)}{2\mathrm{i}}\frac{(1-c)^\alpha}{(1+c)^\beta}\tilde{p}(c^+)\right)\cdot(x-c)^\gamma$$
$$+C_1 + \Phi_1^*(x) = 4\frac{\mathrm{d}z}{\mathrm{d}x}(c),$$

where C_1 is a constant and $\Phi_1^*(x) = o\left((x-c)^\gamma\right)$. Again, the coefficient of $(x-c)^\gamma$ must vanish, i.e.

$$\psi(c^+)\tilde{p}(c^+) - \mathrm{i}\left(\frac{\mathrm{e}^{\mathrm{i}\gamma\pi}}{2\mathrm{i}\sin(\gamma\pi)}\mathrm{e}^{\mathrm{i}\pi\gamma}\tilde{p}(c^-) - \frac{\cot(\gamma\pi)}{2\mathrm{i}}\tilde{p}(c^+)\right) = 0. \tag{26}$$

We now combine (25) and (26) to obtain

$$\gamma = \frac{1}{\pi}\arctan(\psi(c^+)) = \frac{1}{2} - \frac{1}{\pi}\arccos(\psi(c^+)).$$

In the case where R is a constant, $\psi(x) \equiv \psi_0$ and (22) becomes

$$p(x) = \left|\frac{x-c}{x+1}\right|^{1/2}\cdot\left|\frac{x-1}{x-c}\right|^\alpha\tilde{p}(x). \tag{27}$$

Note that the above expression is equivalent to [7, eq. (13)]. We now set up the collocation scheme to solve (20). First we use the substitutions

$$\begin{aligned}\tau_1(x) &= -1 + 2\frac{x+1}{1+c}, &\qquad -1 < x < c,\\ \tau_2(x) &= 1 + 2\frac{x-1}{1-c}, &\qquad c < x < 1,\end{aligned}$$

so that the integral operator becomes

$$\begin{aligned}\int_{-1}^{1}\frac{p(t)}{t-x}\mathrm{d}t &= \int_{-1}^{c}\frac{p_f(t)}{t-x}\mathrm{d}t + \int_{c}^{1}\frac{p_a(t)}{t-x}\mathrm{d}t\\ &= \frac{1+c}{2}\int_{-1}^{1}\frac{p_f\left(t(u_1)\right)}{t(u_1)-x}\mathrm{d}u_1 + \frac{1-c}{2}\int_{-1}^{1}\frac{p_a\left(t(u_2)\right)}{t(u_2)-x}\mathrm{d}u_2\\ &= \int_{-1}^{1}\frac{P_f\left(u_1\right)}{u_1-\tau_1(x)}\mathrm{d}u_1 + \int_{-1}^{1}\frac{P_a\left(u_2\right)}{u_2-\tau_2(x)}\mathrm{d}u_2,\end{aligned}$$

where $P_f(u_1) = p_f(x(u_1))$ and $P_a(u_2) = p_a(x(u_1))$.

We now consider τ_1 and τ_2 as independent variables and seek an expansion of the pressure distribution in terms of Jacobi polynomials:

$$\begin{aligned}P_f(\tau_1) &= p_0^-\frac{(1-\tau_1)^\gamma}{(1+\tau_1)^{1/2}} + (1-\tau_1)^\gamma(1+\tau_1)^{1/2}\sum_{n=1}^{\infty}p_n^-P_{n-1}^{\gamma,1/2}\left(\tau_1\right),\\ P_a(\tau_2) &= (1-\tau_2)^\gamma(1+\tau_2)^\alpha\sum_{n=1}^{\infty}p_n^+P_{n-1}^{\alpha,\gamma}\left(\tau_2\right).\end{aligned}$$

Substitution of these representations into (20) yields equations on the forward and aft parts of the aerofoils. On $-1 < x < c$ we have

$$-\left(p_0^-w^{\gamma,-1/2}(\tau_1)Q_0^{\gamma,-1/2}(\tau_1) + w^{\gamma,1/2}(\tau_1)\sum_{n=1}^{\infty}p_n^-Q_{n-1}^{\gamma,1/2}(\tau_1)\right.$$
$$\left. + w^{\alpha,\gamma}(\tau_2)\sum_{n=1}^{\infty}p_n^+Q_{n-1}^{\alpha,\gamma}(\tau_2)\right) = 4\pi\frac{\mathrm{d}z}{\mathrm{d}x}(x(\tau_1)), \tag{28}$$

and on $c < x < 1$ we have

$$\pi\psi_0 w^{\gamma,\alpha}(\tau_2)\sum_{n=1}^{\infty} p_n^+ P_{n-1}^{\alpha,\gamma}(\tau_2) - \Bigg(p_0^- w^{\gamma,-1/2}(\tau_1)Q_0^{\gamma,-1/2}(\tau_1) + w^{\gamma,1/2}(\tau_1)\sum_{n=1}^{\infty} p_n^- Q_{n-1}^{\gamma,1/2}(\tau_1)$$
$$+w^{\alpha,\gamma}(\tau_2)\sum_{n=1}^{\infty} p_n^+ Q_{n-1}^{\alpha,\gamma}(\tau_2)\Bigg) = 4\pi\frac{\mathrm{d}z}{\mathrm{d}x}(x(\tau_2)). \tag{29}$$

Similarly to the case for continuous porosity distributions, we truncate (28) and (29) at N_- and N_+ respectively. We consider (28) as a function of τ_1 and collocate at the zeros of $P_{N_-+1}^{\gamma,1/2}$. Conversely, we consider (29) as a function of τ_2 and collocate at the zeros of $P_{N^+}^{\alpha,\gamma}$. This generates a linear $(N_- + N_+ + 1) \times (N_- + N_+ + 1)$ system. By using similar reasoning to the previous section, we may show that the rows of the associated matrix are linearly independent, and the matrix is therefore invertible and the system is considered solved.

III. Validation

In this section we present numerical validation of our new method. The convergence of the scheme is observed to be exponentially fast for all tested parameters. In contrast, the Chebyshev expansion approach only to the continuous porosity solution algebraically fast and did not converge to a solution at all for the piecewise-continuous porosity problem.

A. Continuous porosity distributions

If $\tilde{p}(x)$ is expressible as a polynomial of degree n, then the proposed solution method is exact with n Jacobi polynomials. For example, consider a flat plate at angle of attack $\alpha^\star$ with constant porosity ψ_0. The exact solution is given by [5, B2] as

$$p(x) = \frac{-4\alpha^\star}{\sqrt{1+\psi_0^2}}\left(\frac{1-x}{1+x}\right)^{\frac{1}{\pi}\cot^{-1}(\psi_0)}. \tag{30}$$

Conversely, our collocation scheme (15) yields

$$p_0 f_0^{k,-k}(x_j) + \sum_{n=1}^{N} p_n f_{n-1}^{k,1-k}(x_j) = -4\alpha^\star, \qquad j = 1,\cdots,N+1.$$

Noting that $f_0^{k,-k}(x)$ is constant and the matrix M with entries $\{M\}_{n,j} = f_{n-1}^{k,1-k}(x_j)$ has linearly independent rows, we obtain

$$p_0 = \frac{-4\alpha^\star}{\sqrt{1+\psi_0}},$$
$$p_n = 0, \qquad n > 0,$$

and the analytic solution (30) is recovered.

B. Discontinuous porosity distribution

Exact solutions (not requiring Cauchy principal value integrals) are available when the porosity is constant and the aerofoil geometry is simple. The pressure jump along a parabolic aerofoil at angle of attack $\alpha^\star$ with parabolic profile $\beta^\star$ is given by [7] as

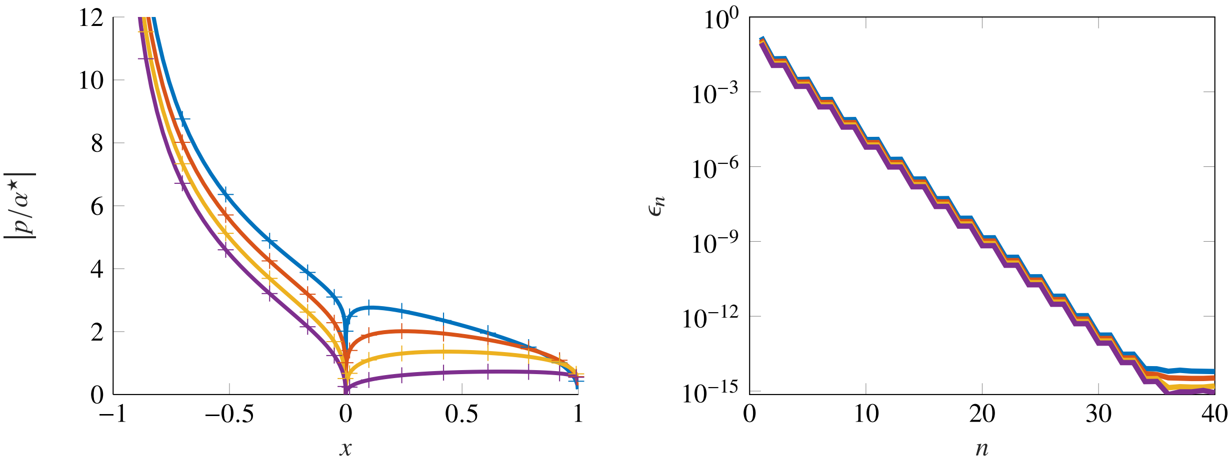
$$p(x) = \frac{2\left(\alpha^\star + \beta^\star(1 + x - (1-c)\arctan(\psi_0)/\pi)\right)}{\sqrt{1 + H(x-c)\psi_0^2}}\sqrt{\frac{1-x}{1+x}}\left|\frac{x-c}{1-x}\right|^{\arctan(\psi_0)/\pi}. \tag{31}$$

We now investigate the convergence of our method to this solution. We consider the relative $\mathcal{L}^2$ error, which is defined as

$$\epsilon_n = \sqrt{\frac{\int_{-1}^{1} |p(x) - p_n(x)|^2 \, \mathrm{d}x}{\int_{-1}^{1} |p(x)|^2 \, \mathrm{d}x}},$$

where p is the exact solution and p_n is the approximation using n Jacobi polynomials.

The error decays exponentially fast for a range of porosities, as illustrated in figures 1b and 2b. The slope of the line appears to be the same regardless of the porosity. This trend can be attributed to the fact that the porosity effect on the solution behaviour at the endpoints is accurately captured in the parameter of the Jacobi polynomials.



(a) Pressure distribution **(b) Error convergence**

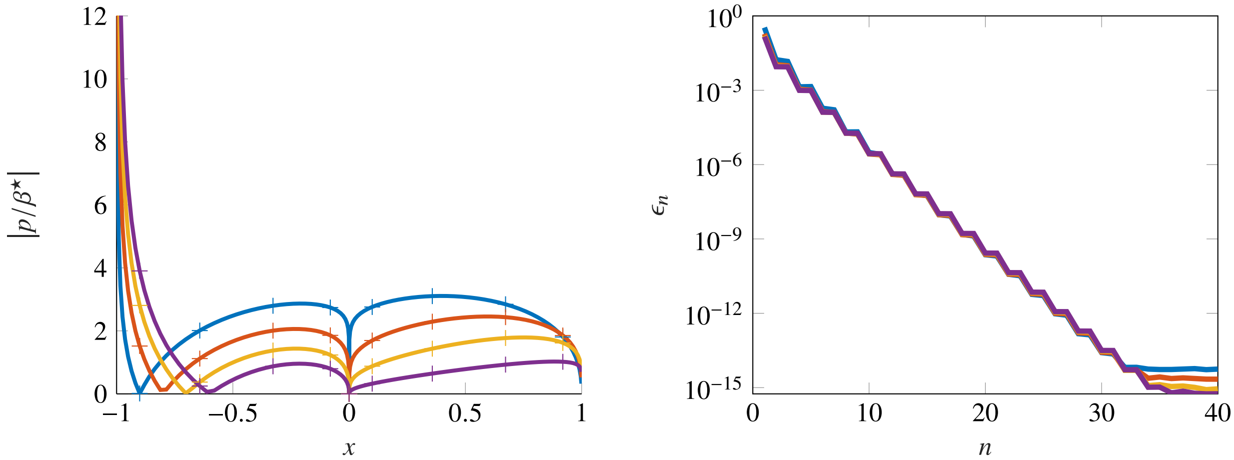
Fig. 1 The pressure distribution and error convergence for a partially porous aerofoil at angle of attack $\alpha^\star$ and permeable-impermeable junction at $c = 0$. The coloured lines correspond to $\alpha = 0.4$, $\alpha = 0.3$, $\alpha = 0.2$ and $\alpha = 0.1$, where smaller α indicates a higher porosity. Note that α is defined in (23.a). In (a) the solid line represents the exact solution (31) and the crosses (+) represent the numerical solution with 5 polynomials.

We note that the error convergence in figures 1b and 2b is not exactly linear, but instead features a "staircase" pattern. This is not a numerical error, but is rather a feature of the physical system. Recall that partially porous aerofoils consist of an impermeable forward part and a permeable aft part. The number of polynomials approximating the forward part is given by N_+ and the number of polynomials approximating the aft part is given by N_-. The variable on the x-axis in figures 1b and 2b is $n = N_+ + N_-$. When n increases from odd to even, N_+ increases by 1, whereas when n increases from even to odd, N_- increases by 1. The shallow decrease in error is associated with an improved approximation along the permeable aft section, whereas the steep decrease in error is associated with an improved approximation along the impermeable forward section. Hence there are two gradients associated with the slopes in figures 1b and 2b.

IV. Conclusions

We have developed a numerical method to solve a singular integral equation associated with the aerodynamics of porous aerofoils. The method is based on the expansion of the bound vorticity distribution as a series of weighted Jacobi polynomials. The method is a vast improvement over the traditional Chebyshev expansion because, by accurately capturing the crucial endpoint behaviour, the expansion converges exponentially fast. Consequently, very few terms – usually just a single term – are required to solve the singular integral equation to sufficient accuracy. In contrast, other research has recorded that the Chebyshev expansion requires hundreds of terms to converge to an acceptable degree of accuracy [20].

The main application of this approach in future work is to consider the unsteady motions of porous aerofoils. When a porous aerofoil is moving, the associated singular integral equation is actually of the Fredholm-Volterra type and is consequently not amenable to the typical techniques of singular integral equations. The present method is proposed to



(a) Pressure distribution **(b) Error convergence**

Fig. 2 The pressure distribution and error convergence for a partially porous aerofoil with parabolic camber $\beta^\star$ and permeable-impermeable junction at $c = 0$. The coloured lines correspond to $\alpha = 0.4$, $\alpha = 0.3$, $\alpha = 0.2$ and $\alpha = 0.1$ where smaller α indicates a higher porosity. In a) the solid line represents the exact solution (31) and the crosses (+) represent the numerical solution with 5 polynomials.

overcome these difficulties in an accurate and rapid manner. In future work the numerical scheme will be applied to extend classical aerodynamic functions of impermeable aerofoils e.g. the Theodorsen, Sears, and Küssner functions for porous aerofoils will all be considered in detail. The thin aerofoil approximation could be discarded by integrating the present approach with advanced discrete-vortex methods [21, 22] to provide more detailed insight into the effects of aerofoil porosity on unsteady aerodynamics. This method is easily adaptable to porous cascades of aerofoils, for which analytic solutions are available when the cascade stagger is small [24].

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