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On the Bi-Lipschitz Geometry of Lamplighter Graphs

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Abstract
In this article we start a systematic study of the bi-Lipschitz geometry of lamplighter graphs. We prove that lamplighter graphs over trees bi-Lipschitzly embed into Hamming cubes with distortion at most 6. It follows that lamplighter graphs over countable trees bi-Lipschitzly embed into ℓ_1. We study the metric behaviour of the operation of taking the lamplighter graph over the vertex-coalescence of two graphs. Based on this analysis, we provide metric characterisations of superreflexivity in terms of lamplighter graphs over star graphs or rose graphs. Finally, we show that the presence of a clique in a graph implies the presence of a Hamming cube in the lamplighter graph over it. An application is a characterisation, in terms of a sequence of graphs with uniformly bounded degree, of the notion of trivial Bourgain–Milman–Wolfson type for arbitrary metric spaces, similar to Ostrovskii's characterisation previously obtained in Ostrovskii (C. R. Acad. Bulgare Sci. **64**(6), 775–784 (2011)).

Keywords Lamplighter graphs · Wreath products · Embeddings of graphs into ℓ_1 and other Banach spaces

Mathematics Subject Classification 05C05 · 05C12 · 46B85

1 Introduction

Wreath products of groups provide a wealth of fundamental examples with various algebraic, spectral and geometric properties. Given two groups Γ_1 and Γ_2, we denote by $\Gamma_2^{(\Gamma_1)}$ the set of all functions $f\colon \Gamma_1 \to \Gamma_2$ with finite support, i.e., with $\{x \in$

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$\Gamma_1 : f(x) \neq e_{\Gamma_2}\}$ finite, where e_{Γ_2} is the identity element of Γ_2. This is a group with pointwise multiplication. We let $\lambda\colon \Gamma_1 \to \mathrm{Aut}\left(\Gamma_2^{(\Gamma_1)}\right)$ denote the left-regular representation given by $\lambda(x)(f) = f^x$, where $f^x(y) = f(x^{-1}y)$. The (*restricted*) *wreath product* $\Gamma_2 \wr \Gamma_1$ *of* Γ_2 *with* Γ_1 is then defined as the semi-direct product $\Gamma_2^{(\Gamma_1)} \rtimes_\lambda \Gamma_1$. It is the group of all pairs (f, x), where $f \in \Gamma_2^{(\Gamma_1)}$ and $x \in \Gamma_1$, equipped with the product $(f, x) \cdot (g, y) = (fg^x, xy)$. When $\Gamma_2 = \mathbb{Z}_2$ (the cyclic group of order 2), the wreath product $\mathbb{Z}_2 \wr \Gamma_1$ is commonly referred to as *the lamplighter group of* Γ_1. We shall often identify $\mathbb{Z}_2^{(\Gamma_1)}$ with the set of all finite subsets of Γ_1. Under this identification, pointwise product becomes symmetric difference, and hence the group operation of $\mathbb{Z}_2 \wr \Gamma_1$ is given by $(A, x) \cdot (B, y) = (A \vartriangle xB, xy)$, where $xB = \{xb : b \in B\}$.

The group $\mathbb{Z}_2 \wr \mathbb{Z}$ is an example of an amenable group with exponential growth. Random walks on wreath product groups have been extensively studied and are well known to exhibit interesting behaviours. In an influential article [11], Kaĭmanovich and Vershik showed that $\mathbb{Z}_2 \wr \mathbb{Z}$ is an example of a group of exponential growth for which the simple random walk on the Cayley graph has zero speed. The variety of geometric features of wreath products of groups has also come to play an important role, sometimes quite unexpectedly, in metric geometry. For instance, the geometry of $\mathbb{Z} \wr \mathbb{Z}$ is closely related to the extension of Lipschitz maps [18], and is also used in distinguishing bi-Lipschitz invariants, namely Enflo type and edge Markov type [17].

In geometric group theory, the theory of compression exponents has undergone a detailed study, in particular the behaviour of compression exponents under taking wreath products. Compression exponents were introduced by Guentner and Kaminker in order to measure how well an infinite, finitely generated group that does not admit a bi-Lipschitz embedding into a certain metric space, can be faithfully represented in it. A deep result of Naor and Peres states that the ℓ_1-compression of a lamplighter group over a group with at least quadratic growth is 1. This result includes the case of the planar lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}^2$. However, it is not known whether $\mathbb{Z}_2 \wr \mathbb{Z}^2$ bi-Lipschitzly embeds into ℓ_1. This challenging problem was raised by Naor and Peres in [18]. Understanding the ℓ_1-embeddability of graphs is motivated by its profound connections with the design of efficient algorithms for some NP-hard problems (see [6, Chap. 10], [8, Chapters 8 and 43], and [16]). Very little is known about the bi-Lipschitz embeddability of lamplighter groups into Banach spaces. The Euclidean distortion of $\mathbb{Z}_2 \wr \mathbb{Z}_k$ is of the order $\sqrt{\log k}$. The lower bound was proved in [12] and the upper bound in [1]. It was shown in [17] that $\mathbb{Z}_2 \wr \mathbb{Z}_k$ bi-Lipschitzly embeds into ℓ_1 with some distortion independent of k (and thus so does $\mathbb{Z}_2 \wr \mathbb{Z}$). In [21], it was proved that a Banach space is superreflexive if and only if it does not contain bi-Lipschitz copies of $\mathbb{Z}_2 \wr \mathbb{Z}_k$ (for every $k \in \mathbb{N}$ and with uniformly bounded distortions). In [5], Cornulier et al. proved that for a finitely generated group Γ and for a finitely generated free group F, the equivariant L_1-compression of $\Gamma \wr \mathrm{F}$ is equal to that of Γ. It follows from this that $\mathbb{Z}_2 \wr \mathrm{F}$ bi-Lipschitzly embeds into ℓ_1.

Working with groups might be restrictive because relatively few graphs can be realised as Cayley graphs of groups. In this paper we consider the most general graph-theoretic setting and we will be concerned with the metric geometry of lamplighter graphs. We anticipate that working in this more flexible framework will be fruitful to

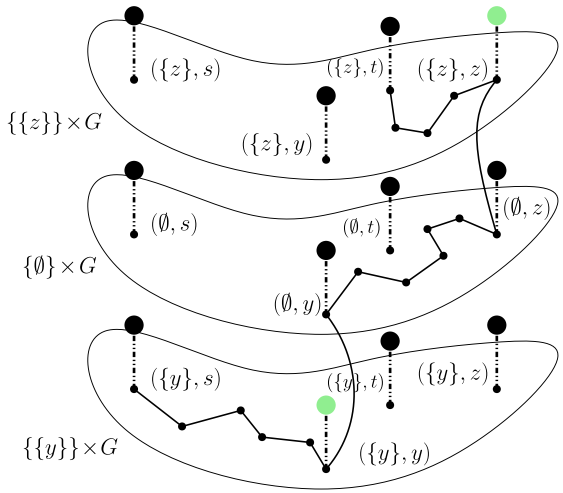

Fig. 1 Horizontal moves within fibers and vertical moves between fibers of the lamplighter graph

construct new graphs with subtle geometric properties. Moreover, lamplighter graphs are generalisations of the wreath product construction in group theory and our results apply to lamplighter groups as well. Indeed, in the context of graph theory it is possible to define a notion of the wreath product of two graphs that is compatible with the wreath product construction in group theory in the sense that the wreath product of two Cayley graphs of groups is the Cayley graph of the wreath product of the two groups for a well-chosen set of generators (*cf.* [7]). For practical purposes which will be explained in the next section, we chose to work with the *walk/switch model* of the lamplighter graph over a graph G, simply denoted $\mathrm{La}(G)$. Specifically, $\mathrm{La}(G)$ is the graph whose vertex set consists of all pairs (A, x) where A is a finite subset of the vertex set of G, and x is a vertex of G. Vertices (A, x) and (B, y) of $\mathrm{La}(G)$ are joined by an edge if and only if *either* $A = B$ and xy is an edge in G *or* $x = y$ and $A \triangle B = \{x\}$. A well-known description of this graph is as follows. Assume there is a lamp attached to each vertex of G and a lamplighter is able to walk along edges of G and switch lights on and off. A vertex (A, x) corresponds to the lamplighter standing at vertex x of G with A being the set of lamps that are currently lit. The lamplighter can make one of two types of moves: he can either move to a neighbouring vertex of G without changing the configuration of lamps that are lit, or he can change the state of the lamp at vertex x and stay at vertex x. We will refer to these as *horizontal* and *vertical* moves, respectively (see Fig. 1).

Other models with different available moves can also be considered, such as the move-and-switch/move model or the like. Note that just as different finite generating sets of a group lead to bi-Lipschitzly equivalent Cayley graphs, it is also easy to verify whether two models of lamplighter graphs are bi-Lipschitzly equivalent. Here we are talking about graphs as metric spaces with the geodesic distance. We will recall this and other standard graph-theoretic notions in Sect. 2.

Our first main result is about lamplighter graphs over arbitrary trees.

Theorem 1.1 *Let T be a (non-empty) tree. Then there is a set I such that* $\mathrm{La}(T)$ *bi-Lipschitzly embeds into the Hamming cube* H_I. *More precisely, there exists a map* $f\colon \mathrm{La}(T) \to \mathrm{H}_I$ *such that*

$$\frac{1}{2} \cdot d_{\mathrm{La}(T)}(x, y) \leqslant d_{\mathrm{H}}(f(x), f(y)) \leqslant 3 \cdot d_{\mathrm{La}(T)}(x, y) \tag{1}$$

for all $x, y \in \mathrm{La}(T)$. *Moreover, if T is finite or countable, then I can also be chosen to be finite or countable, respectively.*

It follows from Theorem 1.1 that the lamplighter graph over a countable tree bi-Lipschitzly embeds into ℓ_1. In particular, this applies to the lamplighter group of a finitely generated free group; as we mentioned in the introduction, this result also follows from more general results by Cornulier et al. [5]. Unlike [5], which relies on geometric group-theoretic arguments, our approach is based on elementary metric techniques.

Our second main result is a technical structural result (Theorem 4.2) which relates the geometry of the lamplighter graph over the vertex-coalescence of two graphs with the geometry of the coalesced components. By combining this structural result together with several embedding results which are discussed in Sect. 6, we extend the metric characterisations of superreflexivity in terms of lamplighter groups of [21] to characterisations in terms of lamplighter graphs over graphs that are built by coalescing several copies of elementary graphs such as cycles or paths. In order to state our next result, we recall some basic definitions from metric geometry. Let (M, d_M) and (N, d_N) be two metric spaces. A map $f\colon M \to N$ is called a *bi-Lipschitz embedding* if there exist $s > 0$ and $D \geqslant 1$ such that for all $u, v \in M$,

$$s \cdot d_M(u, v) \leqslant d_N(f(u), f(v)) \leqslant D \cdot s \cdot d_M(u, v). \tag{2}$$

The *distortion* $\mathrm{dist}(f)$ of a bi-Lipschitz embedding f is given by

$$\mathrm{dist}(f) = \sup_{u \neq v} \frac{d_N(f(u), f(v))}{d_M(u, v)} \cdot \sup_{u \neq v} \frac{d_M(u, v)}{d_N(f(u), f(v))}.$$

As usual,

$$c_N(M) = \inf \left\{ \mathrm{dist}(f) \mid f\colon M \to N \text{ is a bi-Lipschitz embedding} \right\}$$

denotes the *N-distortion of M*. If there is no bi-Lipschitz embedding from M into N, then we set $c_N(M) = \infty$. A sequence $(M_k)_{k\in\mathbb{N}}$ of metric spaces is said to *equi-bi-Lipschitzly embed* into a metric space N if $\sup_{k\in\mathbb{N}} c_N(M_k) < \infty$.

Denote by $\mathrm{St}_{n,k}$ the *star graph* with n branches of length k, and by $\mathrm{Ro}_{n,k}$ the *rose graph* whose n leaves are k-cycles (see Fig. 2; definitions will be given in Sect. 4).

While these graphs can be easily embedded into every finite-dimensional Banach space of a sufficiently large dimension, it is far from being the case for lamplighter graphs over them.

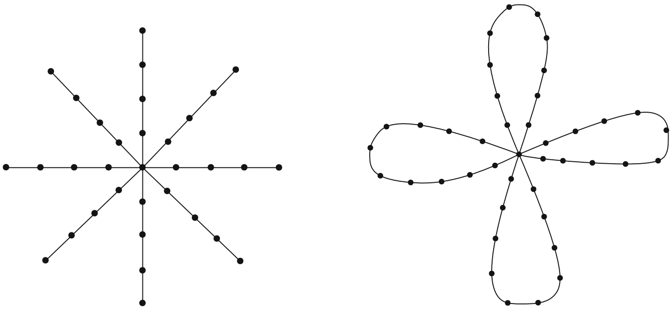

Fig. 2 The star graph $\mathrm{St}_{8,4}$ and the rose graph $\mathrm{Ro}_{4,11}$

Theorem 1.2 *Let Y be a Banach space and $n \in \mathbb{N}$. The following assertions are equivalent.*

(i) *Y is superreflexive*;
(ii) $\sup_{k\in\mathbb{N}} c_Y(\mathrm{La}(\mathrm{St}_{n,k})) = \infty$;
(iii) $\sup_{k\in\mathbb{N}} c_Y(\mathrm{La}(\mathrm{Ro}_{n,k})) = \infty$.

If $(M_k)_{k\in\mathbb{N}}$ and $(N_k)_{k\in\mathbb{N}}$ are sequences of metric spaces, we say that $(M_k)_{k\in\mathbb{N}}$ *equi-bi-Lipschitzly embeds into* $(N_k)_{k\in\mathbb{N}}$, or $(N_k)_{k\in\mathbb{N}}$ *equi-bi-Lipschitzly contains* $(M_k)_{k\in\mathbb{N}}$, if $\sup_k \inf_\ell c_{N_\ell}(M_k) < \infty$, or equivalently, if there is a $C > 0$ such that for all $k \in \mathbb{N}$ there exists $\ell \in \mathbb{N}$ such that M_k bi-Lipschitzly embeds into N_ℓ with distortion at most C. We say that $(M_k)_{k\in\mathbb{N}}$ and $(N_k)_{k\in\mathbb{N}}$ are *Lipschitz-comparable* if $(M_k)_{k\in\mathbb{N}}$ equi-bi-Lipschitzly embeds into $(N_k)_{k\in\mathbb{N}}$ and $(N_k)_{k\in\mathbb{N}}$ equi-bi-Lipschitzly embeds into $(M_k)_{k\in\mathbb{N}}$. In Sect. 6 we prove that the lamplighter graph over K_k, the complete graph with k vertices, contains a bi-Lipschitz copy of the k-dimensional Hamming cube H_k with distortion independent of k. Together with Theorem 1.1, it follows that the geometry of lamplighter graphs over complete graphs or over binary trees is essentially the same as the geometry of the Hamming cubes.

Theorem 1.3 *The sequences $(\mathrm{La}(\mathrm{K}_k))_{k\in\mathbb{N}}$, $(\mathrm{La}(\mathrm{B}_k))_{k\in\mathbb{N}}$ and $(\mathrm{H}_k)_{k\in\mathbb{N}}$ are pairwise Lipschitz-comparable.*

Theorem 1.3 has an important consequence regarding characterisations of the notion of trivial Bourgain–Milman–Wolfson type [4] (BMW-type in short). In 1986, Bourgain, Milman, and Wolfson showed that a metric space Y has trivial BMW-type if and only if $\sup_{k\in\mathbb{N}} c_Y(\mathrm{H}_k) < \infty$. This result is a nonlinear analogue of the Maurey–Pisier theorem for trivial type. The Hamming cube H_k is a k-regular graph and thus $(\mathrm{H}_k)_{k\in\mathbb{N}}$ is a sequence of graphs with unbounded degree. The notion of BMW-type comes from the local theory of Banach spaces and a natural question is whether trivial BMW-type can be characterised as above using a sequence of graphs $(G_k)_{k\in\mathbb{N}}$ with uniformly bounded degree. For Banach spaces, Ostrovskii [19] answered this question positively and it is not difficult to see that the sequence of graphs with maximum degree 3 from [19] is actually Lipschitz-comparable to the sequence of Hamming cubes and thus also settles the question for arbitrary metric spaces. Since every graph in the sequence $(\mathrm{La}(\mathrm{B}_k))_{k\in\mathbb{N}}$ has maximum degree 4, Theorem 1.3 also provides a sought after sequence $(G_k)_{k\in\mathbb{N}}$.

Corollary 1.4 *Let (Y, d_Y) be a metric space. Then*

$$Y \text{ has trivial BMW-type if and only if } \sup_{k\in\mathbb{N}} c_Y(\mathrm{La}(\mathrm{B}_k)) < \infty.$$

Note that the similar question for the nonlinear analogue of the Maurey–Pisier theorem for trivial cotype, obtained by Mendel and Naor [15], has a simple solution for arbitrary metric spaces (see [19] for a proof and a discussion of these questions).

2 Preliminaries on Lamplighter Graphs

We shall use standard graph theory terminology as can be found in [2]. In particular, a graph G is a pair (V, E) where $V = V(G)$ is an arbitrary set (the set of vertices) and $E = E(G)$ is the set of edges, i.e., a set consisting of some unordered pairs of distinct vertices. (So edges are not directed and there are no multiple edges or loops.) We shall often write $x \in G$ instead of $x \in V$ for a vertex x. The edge connecting distinct vertices x and y is simply denoted by xy (which is the same as yx). A *walk in G* is a finite sequence $w = (x_0, x_1, \dots, x_n)$ of vertices of G with $n \geqslant 0$ such that $x_{i-1}x_i$ is an edge of G for all $1 \leqslant i \leqslant n$. We call w a *walk from $x = x_0$ to $y = x_n$* and call n the *length of w*. If w has no repetition of vertices other than the first and last vertices, i.e., if $x_i \neq x_j$ whenever $1 < j-i < n$, then w is called a *path* (*from x to y*). If w is a walk and $x_r = x_s$ for some r, s with $1 < s-r < n$, then $(x_0, \dots, x_{r-1}, x_s, x_{s+1}, \dots, x_n)$ is a strictly shorter walk from x to y. It follows that if w' is a subsequence of w of minimal length such that w' is a walk from x to y, then w' is in fact a path. We say that the graph G is *connected* if any two vertices are connected in G by a walk (or, equivalently, by a path).

A connected graph G becomes a metric space in a natural way. For vertices x and y of G, we denote by $d_G(x, y)$ (or sometimes simply by $d(x, y)$) the length of a shortest path in G (called a *geodesic*) from x to y. It is easy to verify that d_G is a metric. An important example for us are *Hamming cubes*. For an arbitrary set I, the Hamming cube H_I has vertex set $\{0,1\}^{(I)}$ consisting of all functions $\varepsilon\colon I \to \{0,1\}$ with finite support, i.e., the set $\{i \in I : \varepsilon_i = 1\}$ is a finite subset of I. Two vertices ε and δ are joined by an edge if and only if they differ in exactly one coordinate, i.e., there is a unique $i \in I$ with $\varepsilon_i \neq \delta_i$. The graph distance on H_I, denoted d_H and referred to as the *Hamming metric*, is the ℓ_1-metric given by

$$d_\mathrm{H}(\varepsilon, \delta) = \sum_{i\in I} |\varepsilon_i - \delta_i|.$$

We shall often identify $\{0,1\}^{(I)}$ with the set of all finite subsets of I. Under this identification, the Hamming metric becomes the symmetric difference metric given by $d_\mathrm{H}(A, B) = |A \vartriangle B|$ for finite subsets A and B of I.

2.1 A Closed Formula for the Lamplighter Graph Metric

Let us recall the definition of the lamplighter graph $\mathrm{La}(G)$ of a graph G. The vertex set of $\mathrm{La}(G)$ consists of all pairs (A, x) with $x \in G$ and A a finite subset of G. Two vertices (A, x) and (B, y) are joined by an edge in $\mathrm{La}(G)$ if and only if *either* $A = B$ and xy is an edge in G *or* $x = y$ and $A \triangle B = \{x\}$, and these correspond, respectively, to horizontal and vertical moves by the lamplighter.

It is clear that if $\mathrm{La}(G)$ is connected, then so is G. Indeed, the horizontal moves in a path in $\mathrm{La}(G)$ from $(\emptyset, x)$ to $(\emptyset, y)$ correspond to a path from x to y in G. The converse also holds and its proof yields a formula for the graph metric of $\mathrm{La}(G)$ given in Proposition 2.1 below. Computing the distance in $\mathrm{La}(G)$ boils down to the problem of finding a shortest walk in G from a vertex x to another vertex y that visits all vertices in a given subset C of G. This is a well-known and famous problem, the *travelling salesman problem* for G. We shall denote by $\mathrm{tsp}_G(x, C, y)$ the length of a solution to this problem, i.e., the least $n \geqslant 0$ for which there is a walk $(x_0, x_1, \dots, x_n)$ from $x = x_0$ to $y = x_n$ such that $C \subset \{x_0, x_1, \dots, x_n\}$.

Proposition 2.1 *Let G be a connected graph. Then the lamplighter graph* $\mathrm{La}(G)$ *is also connected with graph metric given by*

$$d_{\mathrm{La}(G)}\big((A, x), (B, y)\big) = \mathrm{tsp}_G(x, A \triangle B, y) + |A \triangle B|. \tag{3}$$

Proof Let us fix vertices (A, x) and (B, y). The lamplighter clearly needs at least $|A \triangle B|$ vertical moves in getting from (A, x) to (B, y) in order to switch all lamps in $A \setminus B$ off and to lit all lamps in $B \setminus A$. As the lamplighter can only alter the state of the lamp at the vertex he is currently at, his horizontal moves must visit all vertices in $A \triangle B$ while travelling from x to y. Thus, the right-hand side in the expression above is a lower bound for the distance. It is easy to see that this lower bound is attained. Indeed, let $n = \mathrm{tsp}_G(x, A \triangle B, y)$ and let $w = (x_0, x_1, \dots, x_n)$ be a walk in G from $x = x_0$ to $y = x_n$ such that $A \triangle B \subset \{x_0, x_1, \dots, x_n\}$. Let $m = |A \triangle B|$ and let $0 \leqslant i_1 < i_2 < \dots < i_m \leqslant n$ be such that $A \triangle B = \{x_{i_1}, \dots, x_{i_m}\}$. Set $i_{m+1} = n$. Now consider the following path of length $m + n$ in $\mathrm{La}(G)$ from (A, x) to (B, y). Start with horizontal moves (A, x_i), $0 \leqslant i \leqslant i_1$, from (A, x) to (A, x_{i_1}). Having reached the vertex $\big(A \triangle \{x_{i_1}, \dots, x_{i_{j-1}}\}, x_{i_j}\big)$ for some $1 \leqslant j \leqslant m$, make the vertical move $\big(A \triangle \{x_{i_1}, \dots, x_{i_j}\}, x_{i_j}\big)$ followed by horizontal moves $\big(A \triangle \{x_{i_1}, \dots, x_{i_j}\}, x_i\big)$ for $i_j < i \leqslant i_{j+1}$. These moves end at the vertex $\big(A \triangle \{x_{i_1}, \dots, x_{i_j}\}, x_{i_{j+1}}\big)$ which becomes (B, y) when $j = m$. □

In general, the Travelling Salesman Problem is NP-hard. However, for some graphs it is possible to find explicit algorithms. We present one such algorithm for trees in Sect. 3.1. This is essentially a pre-order traversal algorithm which also accounts for backtracking.

2.2 A First Example: The Lamplighter Graph over a Path

It was shown, amongst other things, in [23] that the group $\mathbb{Z}_q \wr \mathbb{Z}$ (with a specific generating set) embeds bi-Lipschitzly with distortion at most 4 in the Cartesian product of two infinite $(q+1)$-regular trees. Their proof is based on a lengthy and tricky computation of the word-length of certain group elements. We provide a simpler proof of the finite analogue for the case $q = 2$ in the purely graph-theoretic context. We replace $\mathbb{Z}$ with P_k, the path of length k, which has vertices $v_0, v_1, \dots, v_k$ and edges $v_{i-1}v_i$, $1 \leqslant i \leqslant k$, and replace the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ with the lamplighter graph $\mathrm{La}(\mathrm{P}_k)$. After the proof we explain how our argument extends to the infinite case, and hence shows the result for $\mathbb{Z}_2 \wr \mathbb{Z}$ just mentioned.

We first describe *the binary tree* B_k *of height* k, and introduce some notation. The vertex set of B_k is $\bigcup_{i=0}^k \{0,1\}^i$. Let $\delta = (\delta_1, \dots, \delta_m)$ and $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ be vertices of B_k. We write $\delta \prec \varepsilon$ if $m < n$ and $\delta_i = \varepsilon_i$ for $1 \leqslant i \leqslant m$. Then $\delta\varepsilon$ is an edge of B_k if and only if $|m - n| = 1$ and either $\delta \prec \varepsilon$ or $\varepsilon \prec \delta$. We will write $\delta \preccurlyeq \varepsilon$ if $\delta = \varepsilon$ or $\delta \prec \varepsilon$. We define the length of δ to be $|\delta| = m$. If $m \geqslant 1$, we let $\delta' = (\delta_1, \dots, \delta_{m-1})$. Note that if $|\delta| \leqslant |\varepsilon|$, then $\delta\varepsilon$ is an edge of B_k if and only if $\delta = \varepsilon'$. The unique vertex of length zero will be denoted by $\emptyset$ and the graph metric by d_{B} regardless of the value of k.

Given graphs G and H, the *Cartesian product graph* $G \square H$ of G and H has vertex set $V(G) \times V(H)$, and vertices (x, y) and (v, z) are joined by an edge if and only if *either* $x = v$ and yz is an edge in H *or* $y = z$ and xv is an edge in G. Observe that the graph metric on the Cartesian product is given by

$$d_\square\big((x, y), (v, z)\big) = d_G(x, v) + d_H(y, z).$$

Note that the Hamming cube $\mathrm{H}_n = \mathrm{H}_{\{1,\dots,n\}}$ is the n-fold Cartesian product graph $\mathrm{P}_1 \square \dots \square \mathrm{P}_1$.

Proposition 2.2 *Let* $k \in \mathbb{N}$. *There exists a map* $f\colon \mathrm{La}(\mathrm{P}_k) \to \mathrm{B}_{k+1} \square \mathrm{B}_{k+1}$ *such that for all* $x, y \in \mathrm{La}(\mathrm{P}_k)$ *we have*

$$\frac{2}{3} \cdot d_{\mathrm{La}(\mathrm{P}_k)}(x, y) \leqslant d_\square(f(x), f(y)) \leqslant 2 \cdot d_{\mathrm{La}(\mathrm{P}_k)}(x, y).$$

Proof For $(A, v_m) \in \mathrm{La}(\mathrm{P}_k)$ let $f(A, v_m) = \big((\varepsilon_i^A)_{i=1}^m, (\varepsilon_{k+1-i}^A)_{i=0}^{k-m}\big)$ where

$$\varepsilon_i^A = \begin{cases} 1 & \text{if } v_{i-1} \in A, \\ 0 & \text{if } v_{i-1} \notin A. \end{cases}$$

Let $(A, v_m), (B, v_n) \in \mathrm{La}(\mathrm{P}_k)$ and assume without loss of generality that $m \leqslant n$. If $A \triangle B = \emptyset$, then $A = B$, $\varepsilon^A = \varepsilon^B$ and

$$\begin{aligned} &d_\square\big(f(A, v_m), f(B, v_n)\big) \\ &\quad = d_{\mathrm{B}}\big((\varepsilon_i^A)_{i=1}^m, (\varepsilon_i^B)_{i=1}^n\big) + d_{\mathrm{B}}\big((\varepsilon_{k+1-i}^A)_{i=0}^{k-m}, (\varepsilon_{k+1-i}^B)_{i=0}^{k-n}\big) \\ &\quad = n - m + (k - m) - (k - n) = 2 \cdot d_{\mathrm{La}(\mathrm{P}_k)}\big((A, v_m), (B, v_n)\big). \end{aligned} \tag{4}$$

If $v_m = v_n$ and $A \triangle B = \{v_m\}$, then $\varepsilon_i^A = \varepsilon_i^B$ if and only if $i \neq m+1$, and hence

$$
\begin{aligned}
& d_{\square}\big(f(A, v_m), f(B, v_n)\big) \\
& \quad = d_{\mathrm{B}}\big((\varepsilon_i^A)_{i=1}^m, (\varepsilon_i^B)_{i=1}^m\big) + d_{\mathrm{B}}\big((\varepsilon_{k+1-i}^A)_{i=0}^{k-m}, (\varepsilon_{k+1-i}^B)_{i=0}^{k-m}\big) = 0 + 2 = 2.
\end{aligned} \tag{5}
$$

Since $d_{\mathrm{La}(\mathrm{P}_k)}$ is a graph metric, it is sufficient to estimate the Lipschitz constant on adjacent vertices, and it follows from (4) and (5) that f is 2-Lipschitz.

Assume now that $A \triangle B \neq \emptyset$. Set $\ell = \min\{i : v_{i-1} \in A \triangle B\} = \min\{i : \varepsilon_i^A \neq \varepsilon_i^B\}$ and $r = \max\{i : v_{i-1} \in A \triangle B\} = \max\{i : \varepsilon_i^A \neq \varepsilon_i^B\}$. From the definition of ℓ and r it follows that

$$
\begin{aligned}
d_{\mathrm{B}}\big((\varepsilon_i^A)_{i=1}^m, (\varepsilon_i^B)_{i=1}^n\big) &= \begin{cases} n - m & \text{if } \ell > m, \\ m - (\ell - 1) + n - (\ell - 1) & \text{if } \ell \leqslant m, \end{cases} \\
&= \begin{cases} n - m & \text{if } \ell > m, \\ m + n + 2 - 2\ell & \text{if } \ell \leqslant m, \end{cases}
\end{aligned} \tag{6}
$$

and

$$
\begin{aligned}
& d_{\mathrm{B}}\big((\varepsilon_{k+1-i}^A)_{i=0}^{k-m}, (\varepsilon_{k+1-i}^B)_{i=0}^{k-n}\big) \\
& \quad = \begin{cases} (n+1) - (m+1) & \text{if } r \leqslant n, \\ (r+1) - (m+1) + (r+1) - (n+1) & \text{if } r > n, \end{cases} \\
& \quad = \begin{cases} n - m & \text{if } r \leqslant n, \\ 2r - m - n & \text{if } r > n. \end{cases}
\end{aligned} \tag{7}
$$

Obtaining a lower bound on $d_{\square}\big(f(A, v_m), f(B, v_n)\big)$ using (6) and (7) naturally splits into four cases. In all cases we will use the estimate $|A \triangle B| \leqslant r - \ell + 1$.

Case 1: $\ell \leqslant m$ and $r \leqslant n$. In this case $\mathrm{tsp}_{\mathrm{P}_k}(v_m, A \triangle B, v_n) = m + n + 2 - 2\ell$ as the salesman moves from v_m to $v_{\ell-1}$ and then to v_n. We then get

$$
\begin{aligned}
& d_{\mathrm{La}(\mathrm{P}_k)}\big((A, v_m), (B, v_n)\big) \\
& \quad \leqslant m + n + 2 - 2\ell + r - \ell + 1 = r + m + n - 3(\ell - 1) \\
& \quad \leqslant 3(n - \ell + 1) = \frac{3}{2} \cdot d_{\square}\big(f(A, v_m), f(B, v_n)\big).
\end{aligned}
$$

Case 2: $\ell \leqslant m$ and $r > n$. In this case $\mathrm{tsp}_{\mathrm{P}_k}(v_m, A \triangle B, v_n) = m - n + 2r - 2\ell$ as the salesman moves from v_m to $v_{\ell-1}$, then to v_{r-1} and finally to v_n. Thus,

$$
\begin{aligned}
& d_{\mathrm{La}(\mathrm{P}_k)}\big((A, v_m), (B, v_n)\big) \\
& \quad \leqslant m - n + 2r - 2\ell + r - \ell + 1 = m - n - 2 + 3(r - \ell + 1) \\
& \quad \leqslant 3(r - \ell + 1) = \frac{3}{2} \cdot d_{\square}\big(f(A, v_m), f(B, v_n)\big).
\end{aligned}
$$

Case 3: $\ell > m$ and $r \leqslant n$. Then $\mathrm{tsp}_{\mathrm{P}_k}(v_m, A \triangle B, v_n) = n - m$ as the optimal walk for the salesman is from v_m to v_n. Therefore,

$$\begin{aligned} &d_{\mathrm{La}(\mathrm{P}_k)}\big((A, v_m), (B, v_n)\big) \\ &\quad \leqslant n - m + r - \ell + 1 \leqslant n + r - 2m \\ &\quad \leqslant 2(n - m) = d_{\square}\big(f(A, v_m), f(B, v_n)\big). \end{aligned}$$

Case 4: $\ell > m$ and $r > n$. In this range $\mathrm{tsp}_{\mathrm{P}_k}(v_m, A \triangle B, v_n) = 2r - 2 - m - n$ as the salesman moves from v_m to v_{r-1} and to v_n. Using (6) and (7) for the last time, we get

$$\begin{aligned} &d_{\mathrm{La}(\mathrm{P}_k)}\big((A, v_m), (B, v_n)\big) \\ &\quad \leqslant 2r - 2 - m - n + r - \ell + 1 = 3r - 2 - m - n - (\ell - 1) \\ &\quad \leqslant 3r - 3m = \frac{3}{2} \cdot d_{\square}\big(f(A, v_m), f(B, v_n)\big). \end{aligned}$$

□

Remark Let Z denote the double-infinite path. This graph has vertex set $\mathbb{Z}$ and edges between consecutive integers. Let T_3 be the 3-regular (infinite) tree. A description of T_3 is as follows. For $n \in \mathbb{Z}$ denote by $\mathbb{Z}_{\leqslant n}$ the initial segment $\{m \in \mathbb{Z} : m \leqslant n\}$ of $\mathbb{Z}$. Then T_3 has vertex set

$$\big\{\varepsilon\colon \mathbb{Z}_{\leqslant n} \to \{0, 1\} \mid n \in \mathbb{Z},\ \varepsilon \text{ has finite support}\big\}$$

and vertices $(\delta_i)_{i=-\infty}^{m}$ and $(\varepsilon_i)_{i=-\infty}^{n}$ with $m \leqslant n$ are joined by an edge if and only if $n = m + 1$ and $\delta_i = \varepsilon_i$ for all $i \leqslant m$. An almost identical argument as the one used in the proof above shows that the map $f\colon \mathrm{La}(\mathrm{Z}) \to \mathrm{T}_3 \square \mathrm{T}_3$ defined by $f(A, n) = \big((\varepsilon_i^A)_{i=-\infty}^{n}, (\varepsilon_{-i}^A)_{i=-\infty}^{-n-1}\big)$ has distortion at most 3, where ε^A denotes the indicator function of the finite subset A of $\mathbb{Z}$. It is clear that $\mathrm{La}(\mathrm{Z})$ is isometric to $\mathbb{Z}_2 \wr \mathbb{Z}$ with respect to a suitable set of generators.

2.3 Lamplighter Graphs vs Lamplighter Groups

We conclude this section by making precise the connection between lamplighter graphs and lamplighter groups. As previously mentioned, the lamplighter group of a group Γ is the (restricted) wreath product $\mathbb{Z}_2 \wr \Gamma$. This can be thought of as the set of all pairs (A, x) with A a finite subset of Γ and $x \in \Gamma$, with multiplication defined by

$$(A, x) \cdot (B, y) = (A \triangle xB, xy),$$

where $xB = \{xb : b \in B\}$. Now assume that Γ is generated by $S \subset \Gamma$. We assume that the identity $e \notin S$ and that $x^{-1} \in S$ whenever $x \in S$. The (*right*) *Cayley graph* $\mathrm{Cay}(\Gamma, S)$ of Γ with respect to S has vertex set Γ, and $x, y \in \Gamma$ are joined by an edge if and only if $y^{-1}x \in S$. Since S generates Γ, it follows that $\mathrm{Cay}(\Gamma, S)$ is connected. It is easy to verify that

$$S' = \{(\emptyset, s) : s \in S\} \cup \{(\{e\}, e)\}$$

generates $\mathbb{Z}_2 \wr \Gamma$. Moreover, the Cayley graph Cay $(\mathbb{Z}_2 \wr \Gamma, S')$ is the lamplighter graph La $(\mathrm{Cay}(\Gamma, S))$.

Remark It is possible to define the wreath product of graphs which generalises the notion of wreath product of groups. Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two graphs, and let v_0 be a distinguished point in V_H. A function $f\colon V_G \to V_H$ is called finitely supported if $f(v) = v_0$ for all but finitely many $v \in V_G$. The *wreath product* $H \wr G$ *of* H *with* G is the graph with vertex set

$$V_H^{(V_G)} \times V_G = \big\{(f, v) \mid f\colon V_G \to V_H \text{ finitely supported}, \ v \in V_G\big\},$$

and two vertices (f, x) and (g, y) are connected by an edge if and only if *either* $f = g$ and xy is an edge in G *or* $x = y$, $f(v) = g(v)$ for every $v \in G \setminus \{x\}$, and $f(x)g(x)$ is an edge in H. As in the special case above, it is easy to verify that if G and H are Cayley graphs of groups Γ and Δ, respectively, then $H \wr G$ is the Cayley graph of $\Delta \wr \Gamma$ with respect to a suitable generating set.

In this paper we are concerned with lamplighter graphs. It is not too hard to verify that some of our results extend fairly easily to more general wreath products. As wreath products with $\mathbb{Z}_2$ are of greatest interest, we prefer to state and prove our results only for such products. One justification for concentrating on lamplighter graphs instead of more general wreath products is as follows. Any finite graph H is Lipschitz isomorphic to the complete graph S on the vertex set of H with distortion D depending on H. This naturally induces a Lipschitz isomorphism between $H \wr G$ and $S \wr G$ with distortion at most D for any graph G. An argument similar to [18, Lem. 2.1] shows that the bi-Lipschitz embeddability into L_p of $S \wr G$ and La(G) are the same up to universal constants.

3 Embeddability of Lamplighter Graphs over Trees into Hamming Cubes

3.1 The Travelling Salesman Problem for Trees

A *tree* is a connected acyclic graph, i.e., a connected graph in which there is no path $(x_0, \dots, x_n)$ with $n \geqslant 3$ and $x_0 = x_n$. Equivalently, a tree is a graph such that for any two vertices x and y there is a unique path from x to y. E.g. every binary tree is a tree.

We now fix a tree T for the rest of this section. For vertices $x, y \in T$ we denote by $p(x, y)$ the unique path in T from x to y. If $p(x, y) = (x_0, x_1, \dots, x_n)$, then we let $p_i(x, y) = x_i$ for $0 \leqslant i \leqslant n$, and we also let $[x, y] = \{x_{i-1}x_i : 1 \leqslant i \leqslant n\}$ be the set of edges on the path $p(x, y)$. By definition of a path, every edge in $[x, y]$ occurs exactly once, and so $|[x, y]| = d_T(x, y)$. It is also clear that if $p(x, y) = (x_0, x_1, \dots, x_n)$, then $p(y, x) = (x_n, x_{n-1}, \dots, x_1, x_0)$, and hence $[x, y] = [y, x]$.

For $x \in T$ and for $A \subset T$ we let $[x, A] = \bigcup_{a \in A} [x, a]$. Note that if A is finite, then so is $[x, A]$. We are now ready to provide a closed formula for the Travelling Salesman Problem on a tree.

Theorem 3.1 *For $x, y \in T$ and a finite $A \subset T$, we have*

$$\mathrm{tsp}_T(x, A, y) = 2\big|[x, A] \setminus [x, y]\big| + |[x, y]|.$$

We begin the proof with a couple of simple lemmas.

Lemma 3.2 *Let x, y, a be vertices of T. Then $[x, a] \setminus [x, y] = [y, a] \setminus [x, y]$.*

Proof We may assume that x, y, a are pairwise distinct, otherwise the result is clear. Let $p(x, y) = (x_0, x_1, \dots, x_m)$ and $p(x, a) = (y_0, y_1, \dots, y_n)$. Then $x_0 = y_0 = x$, $x_m = y$ and $y_n = a$. Choose i maximal with $0 \leqslant i \leqslant \min(m, n)$ such that $x_j = y_j$ for $0 \leqslant j \leqslant i$. Then

$$w = (y_n, y_{n-1}, \dots, y_{i+1}, y_i = x_i, x_{i+1}, x_{i+2}, \dots, x_m)$$

is a walk from a to y. We show that w is in fact a path. If it is not, then we must have $i < \min(m, n)$ and $x_k = y_\ell$ for some k, ℓ with $i + 1 \leqslant k \leqslant m$ and $i + 1 \leqslant \ell \leqslant n$. Choosing k minimal, we obtain a cycle

$$p = (x_i, x_{i+1}, \dots, x_k = y_\ell, y_{\ell-1}, \dots, y_{i+1}, y_i)$$

in T. Indeed, $x_i = y_i$ and there is no other repetition of vertices by minimality of k. Thus, p is a path from x_i to y_i. Moreover, since $x_{i+1} \neq y_{i+1}$, either $k > i + 1$ or $\ell > i + 1$, and hence the length $(k - i) + (\ell - i)$ of p is at least 3. This contradiction completes the proof that w is a path, and so $p(a, y) = w$.

Now let $e \in [x, a] \setminus [x, y]$. Since $e \in [x, a]$, we have $e = y_{j-1}y_j$ for some $1 \leqslant j \leqslant n$, and since $e \notin [x, y]$, we must have $i < j$. It follows that e is also on the path $w = p(a, y)$, i.e., that $e \in [y, a]$. The inclusion $[x, a] \setminus [x, y] \subset [y, a] \setminus [x, y]$ follows, and the reverse inclusion holds by symmetry in x, y. □

The next lemma shows that any walk from x to y must travel through every edge in the unique path from x to y.

Lemma 3.3 *Let $x, y \in T$ and $w = (w_0, w_1, \dots, w_n)$ be a walk from x to y. Then for every $e \in [x, y]$ there exists $1 \leqslant j \leqslant n$ such that $e = w_{j-1}w_j$.*

Proof Let $p(x, y) = (x_0, x_1, \dots, x_m)$. Then $e = x_{i-1}x_i$ for some $1 \leqslant i \leqslant m$. We observed at the start of the previous section that in any graph, every walk between vertices contains a subsequence which is a path between the same vertices. It follows that $p(x, y)$ is a subsequence of w. Hence there is a maximal j, $1 \leqslant j \leqslant n$, such that $w_{j-1} = x_{i-1}$. If $w_j = x_i$, then we are done. So let us assume $w_j \neq x_i$. Then $w_j \notin \{x_k : i \leqslant k \leqslant m\}$ since otherwise we obtain a cycle in T. It follows that

$$p(w_j, y) = (w_j, x_{i-1}, x_i, \dots, x_m),$$

which therefore must be a subsequence of the walk $(w_j, w_{j+1}, \dots, w_n)$. In particular, $x_{i-1} = w_{k-1}$ for some $j < k \leqslant n$, which contradicts the maximality of j. □

We are now ready to prove the lower bound for the Travelling Salesman Problem in T.

Proposition 3.4 *For $x, y \in T$ and a finite $A \subset T$, we have*

$$\mathrm{tsp}_T(x, A, y) \geqslant 2\big|[x, A] \setminus [x, y]\big| + |[x, y]|.$$

Proof Let $n = \mathrm{tsp}_T(x, A, y)$ and let $w = (w_0, w_1, \ldots, w_n)$ be a walk from x to y such that $A \subset \{w_0, w_1, \ldots, w_n\}$. By Lemma 3.3, for every $e \in [x, y]$, there is at least one $j \in \{1, 2, \ldots, n\}$ such that $e = w_{j-1}w_j$.

Now assume $e \in [x, A] \setminus [x, y]$. Then $e \in [x, a] \setminus [x, y]$ for some $a \in A$. Choose i with $0 \leqslant i \leqslant n$ and $a = w_i$. Then $(w_0, w_1, \ldots, w_i)$ is a walk from x to a, and hence by Lemma 3.3, $e = w_{j-1}w_j$ for some $1 \leqslant j \leqslant i$. On the other hand, by Lemma 3.2 we also have $e \in [a, y]$. Since $(w_i, w_{i+1}, \ldots, w_n)$ is a walk from a to y, it follows that $e = w_{k-1}w_k$ for some $i < k \leqslant n$. Since $j \neq k$, it follows that every edge in $[x, A] \setminus [x, y]$ appears at least twice in the walk w. The result follows. □

We next introduce some more notation. For $x \in T$ we denote by N_x the set of *neighbours of* x given by $N_x = \{y \in T : xy \in E(T)\}$. For $y \in N_x$ we let $T_{x,y} = \{z \in T : p_1(x, z) = y\}$, and for $y \in N_x$ and $A \subset T$ we let $A_{x,y} = A \cap T_{x,y}$. We now establish some simple properties.

Lemma 3.5 *Fix $x \in T$ and $A \subset T$. We have then the following.*

(i) $T = \{x\} \cup \bigcup_{y \in N_x} T_{x,y}$.
(ii) *For $y \in N_x$ and $z \in T_{x,y}$, we have $[x, z] = \{xy\} \cup [y, z]$. Moreover, the endvertices of an edge in $[x, z]$ lie in $\{x\} \cup T_{x,y}$.*
(iii) $A \setminus \{x\} = \bigcup_{y \in N_x} A_{x,y}$ *and* $[x, A] = \bigcup_{y \in N_x} [x, A_{x,y}]$.
(iv) *Given $y \in N_x$, if $A_{x,y} \neq \emptyset$, then $[x, A_{x,y}] = \{xy\} \cup [y, A_{x,y}]$.*

Furthermore, all unions above are disjoint unions.

Proof Given a vertex $z \neq x$, let $p(x, z) = (x_0, x_1, \ldots, x_n)$. Then $n \geqslant 1$, $y = x_1$ is a neighbour of $x_0 = x$, and $z \in T_{x,y}$. It is clear that $x \notin T_{x,y}$ for any $y \in N_x$. Moreover, it is immediate from definition that $T_{x,y} \cap T_{x,z} = \emptyset$ for distinct neighbours y, z of x. Thus, (i) follows.

To see (ii), let $p(x, z) = (x_0, x_1, \ldots, x_n)$. Then $n \geqslant 1$, $x_0 = x$, $x_1 = y$ and $p(y, z) = (x_1, \ldots, x_n)$. Hence $[x, z] = \{xy\} \cup [y, z]$, and $xy \notin [y, z]$ since the vertices $x_0, x_1, \ldots, x_n$ are pairwise distinct. For $1 \leqslant i \leqslant n$, we have $p(x, x_i) = (x_0, x_1, \ldots, x_i)$, and hence $p_1(x, x_i) = y$. This implies that $\{x_0, x_1, \ldots, x_n\} \subset \{x\} \cup T_{x,y}$ and the second part of (ii) follows.

It follows from (i) that $A \setminus \{x\} = \bigcup_{y \in N_x} A_{x,y}$ and that this is a disjoint union. The second part of (iii) now follows:

$$[x, A] = \bigcup_{a \in A} [x, a] = \bigcup_{y \in N_x} \bigcup_{a \in A_{x,y}} [x, a] = \bigcup_{y \in N_x} [x, A_{x,y}],$$

and moreover, since the endvertices of an edge in $[x, A_{x,y}]$ lie in $\{x\} \cup T_{x,y}$, it follows that the sets $[x, A_{x,y}]$, $y \in N_x$, are pairwise disjoint.

Finally, we establish (iv). If $A_{x,y} \neq \emptyset$, then from (ii) it follows that

$$[x, A_{x,y}] = \bigcup_{a \in A_{x,y}} [x, a] = \bigcup_{a \in A_{x,y}} \{xy\} \cup [y, a] = \{xy\} \cup [y, A_{x,y}]$$

and the union is a disjoint union. □

We next prove Theorem 3.1 in a special case.

Theorem 3.6 *For $x \in T$ and a finite $A \subset T$, we have* $\mathrm{tsp}_T(x, A, x) = 2|[x, A]|$.

Proof We may assume $A \neq \emptyset$, otherwise the result is clear. Let us define $h = h(x, A) = \max_{a \in A} d(x, a)$. We construct by recursion on h a walk w from x to x of length $n = 2|[x, A]|$ visiting all vertices in A. Together with Proposition 3.4, this will complete the proof.

If $h = 0$, then $A = \{x\}$ and the result is clear. Let us now assume that $h \geqslant 1$. Set $N = N_x$ and $A_y = A_{x,y}$ for each $y \in N$. Let $M = \{y \in N : A_y \neq \emptyset\}$. Fix $y \in M$. For every $a \in A_y$, we have $p(x, a) = (x, p(y, a))$, and thus, $h(y, A_y) \leqslant h - 1$. By recursion, there is a walk $w^{(y)}$ from y to y of length $2|[y, A_y]|$ visiting all vertices of A_y. Now since A is finite, so is M, which we can then enumerate as $y_1, y_2, \ldots, y_k$. Then

$$w = \left(x, w^{(y_1)}, x, w^{(y_2)}, x, \ldots, x, w^{(y_k)}, x\right)$$

is a walk from x to x visiting all vertices in $\bigcup_{y \in M} A_y$. It follows from Lemma 3.5 that w visits all vertices in A and has length

$$2k + \sum_{y \in M} 2|[y, A_y]| = \sum_{y \in M} 2\left|\{xy\} \cup [y, A_y]\right| = \sum_{y \in M} 2|[x, A_y]| = 2|[x, A]|. \qquad \square$$

We are finally ready to complete the proof of our main result.

Proof We proceed by induction on $d_T(x, y)$ and construct a walk from x to y of length $2\left|[x, A] \setminus [x, y]\right| + |[x, y]|$ visiting all vertices of A. Together with Proposition 3.4, this will complete the proof.

When $d_T(x, y) = 0$, the result follows from Theorem 3.6. Now assume $d_T(x, y) \geqslant 1$ and set $x_1 = p_1(x, y)$. Let $N = N_x$ and $A_y = A_{x,y}$ for all $y \in N$. Set $A_0 = \bigcup_{z \in N, z \neq x_1} A_z$ and $A_1 = A_{x_1}$. From Lemma 3.5 we have the following:

$$A \setminus \{x\} = \bigcup_{z \in N_x} A_z = A_0 \cup A_1,$$

$$[x, A_1] = \{xx_1\} \cup [x_1, A_1] \quad \text{if } A_1 \neq \emptyset,$$

$$[x, y] = \{xx_1\} \cup [x_1, y],$$

$$[x, A] = \bigcup_{z \in N_x} [x, A_z] = [x, A_0] \cup [x, A_1],$$

$$[x, A_0] \cap [x, y] = \emptyset,$$

and moreover, all unions are disjoint unions. From this we obtain

$$\begin{aligned}[x_1, A_1] \setminus [x_1, y] &= [x, A_1] \setminus [x, y] \quad \text{and} \\ [x, A] \setminus [x, y] &= [x, A_0] \cup \big([x, A_1] \setminus [x, y]\big). \end{aligned} \tag{8}$$

By Theorem 3.6, there is a walk $w^{(0)}$ from x to x of length $\ell_0 = 2|[x, A_0]|$ visiting all vertices in A_0. By induction hypothesis, there is a walk $w^{(1)}$ from x_1 to y of length $\ell_1 = 2|[x_1, A_1] \setminus [x_1, y]| + |[x_1, y]|$ visiting all vertices in A_1. It follows that $w = (w^{(0)}, w^{(1)})$ is a walk from x to y visiting all vertices in $\{x\} \cup A_0 \cup A_1 = \{x\} \cup A$. Let ℓ be the length of w. Then from (8) we obtain

$$\begin{aligned}\ell = \ell_0 + \ell_1 + 1 &= 2|[x, A_0]| + 2\big|[x_1, A_1] \setminus [x_1, y]\big| + |[x_1, y]| + 1 \\ &= 2|[x, A_0]| + 2\big|[x, A_1] \setminus [x, y]\big| + |[x, y]| = 2\big|[x, A] \setminus [x, y]\big| + |[x, y]|,\end{aligned}$$

as required. □

Remark It is not hard to see that the proof of Theorem 3.1 yields an efficient algorithm for finding optimal walks in T for the Travelling Salesman Problem.

3.2 Embeddability into Hamming Cubes

We will show that the lamplighter graph over a tree bi-Lipschitzly embeds into a Hamming cube, and thus prove Theorem 1.1. Let us fix a tree T. By Proposition 2.1 and Theorem 3.1, the graph metric in the lamplighter graph $\mathrm{La}(T)$ is given by

$$d_{\mathrm{La}(T)}\big((A, x), (B, y)\big) = 2\big|[x, A \triangle B] \setminus [x, y]\big| + |[x, y]| + |A \triangle B|. \tag{9}$$

For $C \subset T$ let $[C] = \bigcup_{x,y \in C} [x, y]$ be the minimal set of edges needed to travel between different vertices of C. Define

$$I = \big\{(e, C) : e \in E(T),\ \emptyset \neq C \subset T,\ C \text{ finite},\ e \notin [C]\big\}.$$

For $A \subset T$, $x \in T$ and $e \in E(T)$, let $A_{x,e} = \{a \in A : e \in [x, a]\}$. We now define a map into the Hamming cube H_I whose role is to capture the first of the three summands in the right-hand side of (9).

Lemma 3.7 *Define* $f\colon \mathrm{La}(T) \to \mathrm{H}_I$ *as follows. For* $(A, x) \in \mathrm{La}(T)$ *and* $i \in I$ *we let*

$$f(A, x)_i = 1 \iff \exists\, e \in E(T)\ A_{x,e} \neq \emptyset \text{ and } i = (e, A_{x,e}).$$

Then for vertices (A, x) *and* (B, y) *of* $\mathrm{La}(T)$ *we have*

$$\big|[x, A \triangle B] \setminus [x, y]\big| \leqslant d_{\mathrm{H}}\big(f(A, x), f(B, y)\big) \leqslant 2\big|[x, A \triangle B] \setminus [x, y]\big| + 2|[x, y]|.$$

Proof We first check that f is well-defined, i.e., that $f(A, x)$ has finite support. Given $i = (e, C) \in I$, if $f(A, x)_i = 1$, then $C = A_{x,e} \neq \emptyset$, and hence $e \in [x, A]$. It follows that the support of $f(A, x)$ has at most (in fact, exactly) $|[x, A]|$ elements. Since A is finite, so is $[x, A]$, and hence $f(A, x)$ is finitely supported.

We now turn to the inequalities. Given $i = (e, C) \in I$, we have

$$f(A, x)_i \neq f(B, y)_i \iff A_{x,e} \neq B_{y,e} \text{ and } \textit{either } C = A_{x,e} \textit{ or } C = B_{y,e}.$$

Thus, setting $E = \{e \in E(T) : A_{x,e} \neq B_{y,e} \text{ and } \textit{either } A_{x,e} \neq \emptyset \textit{ or } B_{y,e} \neq \emptyset\}$, we have

$$|E| \leqslant d_{\mathrm{H}}\big(f(A, x), f(B, y)\big) \leqslant 2|E|. \tag{10}$$

To estimate $|E|$, let us first consider an edge $e \in E \setminus [x, y]$. By definition of E, there is a vertex $c \in A_{x,e} \triangle B_{y,e}$. Hence, using Lemma 3.2, we have $e \in [x, c] \setminus [x, y] = [y, c] \setminus [x, y]$. It follows that $c \in A \triangle B$, and thus $e \in [x, A \triangle B] \setminus [x, y]$. This shows the upper bound

$$|E| \leqslant \big|[x, A \triangle B] \setminus [x, y]\big| + |[x, y]|. \tag{11}$$

Next consider $e \in [x, A \triangle B] \setminus [x, y]$. Then, using Lemma 3.2 again, we have some $c \in A \triangle B$ such that $e \in [x, c] \setminus [x, y] = [y, c] \setminus [x, y]$. It follows that $c \in A_{x,e} \triangle B_{y,e}$, and hence $e \in E$. This yields the lower bound

$$\big|[x, A \triangle B] \setminus [x, y]\big| \leqslant |E|. \tag{12}$$

Combining the inequalities (10), (11) and (12) completes the proof of the lemma. □

After some definitions, we will state and prove the main result of this section, which then immediately yields Theorem 1.1. Note that for disjoint sets J and K, the product $\mathrm{H}_J \Box \mathrm{H}_K$ is the Hamming cube $\mathrm{H}_{J \cup K}$. In the next result we identify the vertices of a Hamming cube H_J with finite subsets of J.

Theorem 3.8 *Let T be a (non-empty) tree. Let $f\colon \mathrm{La}(T) \to \mathrm{H}_I$ be the map from Lemma 3.7. Fix $x_0 \in T$ and define $F\colon \mathrm{La}(T) \to \mathrm{H}_I \Box \mathrm{H}_{E(T)} \Box \mathrm{H}_T$ by*

$$F(A, x) = \big(f(A, x), [x_0, x], A\big).$$

Then F is a bi-Lipschitz embedding with distortion at most 6.

Proof Fix two vertices (A, x) and (B, y) in $\mathrm{La}(T)$. Then

$$d_{\Box}\big(F(A, x), F(B, y)\big) = d_{\mathrm{H}}\big(f(A, x), f(B, y)\big) + \big|[x_0, x] \triangle [x_0, y]\big| + |A \triangle B|.$$

We first estimate the middle term. Let $p(x_0, x) = (x_0, x_1, \ldots, x_m)$ and $p(x_0, y) = (y_0, y_1, \ldots, y_n)$. As in the proof of Lemma 3.2, if $i \leqslant \min(m, n)$ is maximal such that $x_j = y_j$ for $0 \leqslant j \leqslant i$, then

$$p(y, x) = (y_n, y_{n-1}, \dots, y_{i+1}, y_i = x_i, x_{i+1}, \dots, x_m).$$

It follows at once that

$$[x_0, x] \triangle [x_0, y] = [x, y].$$

Hence, using (9) and Lemma 3.7, we deduce that

$$\begin{aligned} d_{\mathrm{La}(T)}\big((A, x), (B, y)\big) &= 2\big|[x, A \triangle B] \setminus [x, y]\big| + |[x, y]| + |A \triangle B| \\ &\leqslant 2 \cdot d_{\mathrm{H}}\big(f(A, x), f(B, y)\big) + \big|[x_0, x] \triangle [x_0, y]\big| + |A \triangle B| \\ &\leqslant 2 \cdot d_{\square}\big(F(A, x), F(B, y)\big) \end{aligned}$$

and that

$$\begin{aligned} d_{\square}\big(F(A, x), F(B, y)\big) &\leqslant 2\big|[x, A \triangle B] \setminus [x, y]\big| + 3|[x, y]| + |A \triangle B| \\ &\leqslant 3 \cdot d_{\mathrm{La}(T)}\big((A, x), (B, y)\big). \end{aligned}$$

□

4 Lamplighter Graph over the Vertex-Coalescence of Two Graphs

The procedure that consists of gluing two graphs at a common vertex is known as *vertex-coalescence* or, simply, *coalescence* of two graphs. Consider two *pointed graphs* $G_1 = (V_1, E_1, v_1)$ and $G_2 = (V_2, E_2, v_2)$, i.e., graphs G_i, $i = 1, 2$, with vertex set V_i, edge set E_i, and a specified vertex $v_i \in V_i$. We define *the vertex-coalescence* $G_1 * G_2$ *of* G_1 *and* G_2 by first taking the disjoint union of G_1 and G_2 followed by identifying the vertices v_1 and v_2. Formally, $G_1 * G_2$ has vertex set

$$V = \big\{(x, i) : x \in V_i \setminus \{v_i\},\ i = 1, 2\big\} \cup \{v_0\}$$

where $v_0 \notin V_1 \times \{1\} \cup V_2 \times \{2\}$, and edge set

$$\begin{aligned} E = &\big\{((x, i), (y, i)) : x, y \in V_i \setminus \{v_i\},\ xy \in E_i,\ i = 1, 2\big\} \\ &\cup \big\{((x, i), v_0) : x \in V_i \setminus \{v_i\},\ xv_i \in E_i,\ i = 1, 2\big\}. \end{aligned}$$

This formal definition is rather cumbersome. In practice, we shall either assume after relabeling that $V_1 \cap V_2 = \{v_0\}$ and $v_0 = v_1 = v_2$, in which case we can simply take $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$, or, particularly in the case of gluing several copies of the same pointed graph together, we shall refer to vertices of the original graph as being in the k^{th} copy in the coalesced graph for $k = 1, 2, \dots$ Note that if G_1 and G_2 are connected, then so is $G_1 * G_2$ with the graph metric given by

$$d_{G_1*G_2}(u,v)=\begin{cases} d_{G_i}(u,v) & \text{if } u,v\in G_i,\\ d_{G_1}(u,v_0)+d_{G_2}(v_0,v) & \text{if } u\in G_1, v\in G_2.\end{cases} \tag{13}$$

In the next lemma we record a relationship between the Travelling Salesman Problem on the coalescence graph with the ones on its components. The proof is elementary and left to the reader.

Lemma 4.1 *Let $G_1 * G_2$ be the coalescence of two connected pointed graphs G_1 and G_2 at a common vertex v_0. Let $x, y \in G_1 * G_2$ and $C \subset G_1 * G_2$ with C finite. If there exists $i \in \{1,2\}$ such that $x, y \in G_i$ and $C \subset G_i$, then*

$$\mathrm{tsp}_{G_1*G_2}(x,C,y)=\mathrm{tsp}_{G_i}(x,C,y). \tag{14}$$

If $x \in G_1$, $y \in G_2$ and $C = C_1 \cup C_2$ with $C_i \subset G_i$ for $i = 1, 2$, then

$$\mathrm{tsp}_{G_1*G_2}(x,C,y)=\mathrm{tsp}_{G_1}(x,C_1,v_0)+\mathrm{tsp}_{G_2}(v_0,C_2,y). \tag{15}$$

If $x \in G_1$, $y \in G_1$ and $C \cap G_2 \neq \emptyset$, then

$$\begin{aligned}&\mathrm{tsp}_{G_1*G_2}(x,C,y)\\ &\quad=\min\big\{\mathrm{tsp}_{G_1}(x,C',v_0)+\mathrm{tsp}_{G_2}(v_0,C\cap V_2,v_0)+\mathrm{tsp}_{G_1}(v_0,C'',y)\big\},\end{aligned} \tag{16}$$

where the minimum is taken over all sets $C', C'' \subset V_1$ with $C' \cup C'' = C \cap V_1$.

The purpose of the next theorem is to establish a metric connection between the lamplighter graph over the coalescence of two graphs with the lamplighter graphs over its components. To do this, we need to make use of clover graphs. Given $n \in \mathbb{N}$ and a pointed graph $G = (V, E, v_0)$, *the clover graph* $\mathrm{Clo}(G, n)$ is obtained by coalescing n copies of G at v_0 in an obvious inductive fashion.

Theorem 4.2 *Let $G_1 * G_2$ be the coalescence of two finite, connected pointed graphs G_1 and G_2 at a common vertex v_0. Then there exists a map*

$$f\colon\ \mathrm{La}(G_1*G_2)\to \mathrm{La}(G_1)\square\,\mathrm{La}(G_2)\square\,\mathrm{Clo}\big(G_1,2^{|G_2|}\big)\square\,\mathrm{Clo}\big(G_2,2^{|G_1|}\big)$$

such that

$$d_{\mathrm{La}(G_1*G_2)}(u,v)\leqslant d_{\square}\big(f(u),f(v)\big)\leqslant 2\cdot d_{\mathrm{La}(G_1*G_2)}(u,v) \tag{17}$$

for all $u, v \in \mathrm{La}(G_1 * G_2)$.

Proof Observe that $2^{|G_2|}$ is the number of subsets of G_2, and thus we can index the $2^{|G_2|}$ copies of G_1 in $\mathrm{Clo}\big(G_1, 2^{|G_2|}\big)$ by the collection of all subsets of G_2. For $x \in G_1$ and $S \subset G_2$ we denote by $\iota_S(x)$ the element x considered in the copy of G_1 in $\mathrm{Clo}\big(G_1, 2^{|G_2|}\big)$ that is indexed by S. We proceed in a similar way for $\mathrm{Clo}\big(G_2, 2^{|G_1|}\big)$. We define the function

$$f\colon\ \mathrm{La}(G_1*G_2)\to \mathrm{La}(G_1)\square\,\mathrm{La}(G_2)\square\,\mathrm{Clo}\big(G_1,2^{|G_2|}\big)\square\,\mathrm{Clo}\big(G_2,2^{|G_1|}\big)$$

as follows. Given a vertex (A, x) of $\mathrm{La}(G_1 * G_2)$, we let $A_i = A \cap G_i$ for $i = 1, 2$, and set

$$f(A, x) = \begin{cases} \big((A_1, x), (A_2, v_0), \iota_{A_2}(x), v_0\big) & \text{if } x \in G_1, \\ \big((A_1, v_0), (A_2, x), v_0, \iota_{A_1}(x)\big) & \text{if } x \in G_2. \end{cases} \tag{18}$$

To establish (17), we fix vertices (A, x) and (B, y) in $\mathrm{La}(G_1 * G_2)$, we let $A_i = A \cap G_i$ and $B_i = B \cap G_i$ for $i = 1, 2$, and consider several cases. To make the notation less crowded, we will at times drop subscripts in the graph distance.

Case 1: $x, y \in G_1$ and $A \triangle B \subset G_1$. In this case we have $A \triangle B = A_1 \triangle B_1$ and $A_2 = B_2$. It follows that

$$\begin{aligned} &d_\square\big(f(A, x), f(B, y)\big) \\ &\quad = d\big((A_1, x), (B_1, y)\big) + d\big((A_2, v_0), (B_2, v_0)\big) + d\big(\iota_{A_2}(x), \iota_{B_2}(y)\big) \\ &\qquad + d(v_0, v_0) \\ &\quad = \mathrm{tsp}_{G_1}(x, A_1 \triangle B_1, y) + |A_1 \triangle B_1| + \mathrm{tsp}_{G_2}(v_0, A_2 \triangle B_2, v_0) + |A_2 \triangle B_2| \\ &\qquad + d\big(\iota_{A_2}(x), \iota_{B_2}(y)\big) \qquad \text{(by (3))} \\ &\quad = \mathrm{tsp}_{G_1 * G_2}(x, A \triangle B, y) + |A \triangle B| + d_{G_1}(x, y) \qquad \text{(by (14) and (13))} \\ &\quad = d_{\mathrm{La}(G_1 * G_2)}\big((A, x), (B, y)\big) + d_{G_1 * G_2}(x, y) \qquad \text{(by (3) and (13))}, \end{aligned}$$

which implies (17) since

$$0 \leqslant d_{G_1 * G_2}(x, y) \leqslant \mathrm{tsp}_{G_1 * G_2}(x, A \triangle B, y) \leqslant d_{\mathrm{La}(G_1 * G_2)}\big((A, x), (B, y)\big).$$

Case 2: $x \in G_1$ and $y \in G_2$. Then

$$\begin{aligned} &d_\square\big(f(A, x), f(B, y)\big) \\ &\quad = d\big((A_1, x), (B_1, v_0)\big) + d\big((A_2, v_0), (B_2, y)\big) + d\big(\iota_{A_2}(x), v_0\big) + d\big(v_0, \iota_{B_1}(y)\big) \\ &\quad = \mathrm{tsp}_{G_1}(x, A_1 \triangle B_1, v_0) + |A_1 \triangle B_1| + \mathrm{tsp}_{G_2}(v_0, A_2 \triangle B_2, y) + |A_2 \triangle B_2| \\ &\qquad + d_{G_1}(x, v_0) + d_{G_2}(v_0, y) \qquad \text{(by (3) and (13))} \\ &\quad = \mathrm{tsp}_{G_1 * G_2}(x, A \triangle B, y) + |A \triangle B| + d_{G_1}(x, v_0) + d_{G_2}(v_0, y) \qquad \text{(by (15))} \\ &\quad = d_{\mathrm{La}(G_1 * G_2)}\big((A, x), (B, y)\big) + d_{G_1 * G_2}(x, y) \qquad \text{(by (3) and (13))} \end{aligned}$$

and we are again done as in the previous case.

Case 3: $x, y \in G_1$ and $A \triangle B \cap G_2 \neq \emptyset$. Then $A_2 \triangle B_2 \neq \emptyset$, and thus $A_2 \neq B_2$. Therefore,

$$\begin{aligned} &d_\square\big(f(A, x), f(B, y)\big) \\ &\quad = d\big((A_1, x), (B_1, y)\big) + d\big((A_2, v_0), (B_2, v_0)\big) + d\big(\iota_{A_2}(x), \iota_{B_2}(y)\big) + d(v_0, v_0) \\ &\quad = \mathrm{tsp}_{G_1}(x, A_1 \triangle B_1, y) + |A_1 \triangle B_1| + \mathrm{tsp}_{G_2}(v_0, A_2 \triangle B_2, v_0) + |A_2 \triangle B_2| \end{aligned}$$

$$+ d_{G_1}(x, v_0) + d_{G_1}(v_0, y) \qquad \text{(by (3) and (13))}$$
$$= \operatorname{tsp}_{G_1}(x, A_1 \triangle B_1, y) + \operatorname{tsp}_{G_2}(v_0, A_2 \triangle B_2, v_0) + d_{G_1}(x, v_0) + d_{G_1}(v_0, y) + |A \triangle B|.$$

For any decomposition $C \cup D$ of $A_1 \triangle B_1$ one has

$$\operatorname{tsp}_{G_1}(x, A_1 \triangle B_1, y) \leqslant \operatorname{tsp}_{G_1}(x, C, v_0) + \operatorname{tsp}_{G_1}(v_0, D, y),$$
$$d_{G_1}(x, v_0) \leqslant \operatorname{tsp}_{G_1}(x, C, v_0),$$
$$d_{G_1}(v_0, y) \leqslant \operatorname{tsp}_{G_1}(v_0, D, y),$$

and hence

$$\begin{aligned}
&d_{\square}\big(f(A, x), f(B, y)\big)\\
&\quad\leqslant 2 \cdot \operatorname{tsp}_{G_1}(x, C, v_0) + \operatorname{tsp}_{G_2}(v_0, A_2 \triangle B_2, v_0) + 2 \cdot \operatorname{tsp}_{G_1}(v_0, D, y) + |A \triangle B|\\
&\quad\leqslant 2\big(\operatorname{tsp}_{G_1}(x, C, v_0) + \operatorname{tsp}_{G_2}(v_0, A_2 \triangle B_2, v_0) + \operatorname{tsp}_{G_1}(v_0, D, y)\big) + |A \triangle B|\\
&\quad= 2 \cdot \operatorname{tsp}_{G_1 * G_2}(x, A \triangle B, y) + |A \triangle B| \qquad \text{(for some choice of } C, D \text{ by (16))}\\
&\quad\leqslant 2 \cdot d_{\mathrm{La}(G_1 * G_2)}\big((A, x), (B, y)\big). \qquad \text{(by (3))}
\end{aligned}$$

For the lower bound, assume without loss of generality that $d_{G_1}(x, v_0) \geqslant d_{G_1}(v_0, y)$. Since $\operatorname{tsp}_{G_1}(x, A_1 \triangle B_1, y) + d_{G_1}(v_0, y) \geqslant \operatorname{tsp}_{G_1}(x, A_1 \triangle B_1, v_0)$, it follows that

$$\begin{aligned}
&d_{\square}\big(f(A, x), f(B, y)\big)\\
&\quad\geqslant \operatorname{tsp}_{G_1}(x, A_1 \triangle B_1, v_0) + \operatorname{tsp}_{G_2}(v_0, A_2 \triangle B_2, v_0) + d_{G_1}(x, v_0) + |A \triangle B|\\
&\quad\geqslant \operatorname{tsp}_{G_1}(x, A_1 \triangle B_1, v_0) + \operatorname{tsp}_{G_2}(v_0, A_2 \triangle B_2, v_0) + \operatorname{tsp}_{G_1}(v_0, \emptyset, y)\\
&\qquad + |A \triangle B|\\
&\quad\geqslant \operatorname{tsp}_{G_1 * G_2}(x, A \triangle B, y) + |A \triangle B| \qquad \text{(by (16))}\\
&\quad= d_{\mathrm{La}(G_1 * G_2)}\big((A, x), (B, y)\big) \qquad \text{(by (3))}
\end{aligned}$$
□

We will now illustrate the utility of Theorem 4.2 with two applications. The first result (Proposition 4.5) is concerned with embeddings of lamplighter graphs over star graphs into non-superreflexive Banach spaces. Given $k, n \in \mathbb{N}$, we define the *star graph* $\mathrm{St}_{n,k}$ to be the clover graph $\mathrm{Clo}(\mathrm{P}_k, n)$ obtained by coalescing n copies of a path of length k at an endvertex (see Fig. 2).

Lemma 4.3 *Let $k, n \in \mathbb{N}$ and let $(E, \|\cdot\|)$ be an n-dimensional Banach space. Then* $\mathrm{St}_{n,k}$ *bi-Lipschitzly embeds into E with distortion at most* 2.

Proof Let $(e_i)_{i=1}^n$ be an Auerbach basis for E. By this we mean that $\|e_i\| = 1$ for $i = 1, \ldots, n$ and that there are functionals $(e_i^*)_{i=1}^n$ in the dual of E which are also normalised and such that $e_i^*(e_j) = \delta_{ij}$ for all $i, j = 1, \ldots, n$. Define $f\colon \mathrm{St}_{n,k} \to E$ by $f(x) = d_{\mathrm{P}_k}(v_0, x)e_i$ if x belongs to the i^{th} copy of P_k in $\mathrm{St}_{n,k}$. Here v_0 is the endvertex of P_k at which the n copies of P_k are coalesced. If, for some i, both x and y belong to the i^{th} copy of P_k in $\mathrm{St}_{n,k}$, then

$$\|f(x) - f(y)\| = |d_{\mathrm{P}_k}(v_0, x) - d_{\mathrm{P}_k}(v_0, y)| \cdot \|e_i\| = d_{\mathrm{P}_k}(x, y) = d_{\mathrm{St}_{n,k}}(x, y).$$

If, for some $i \neq j$, we have that x belongs to the i^{th} copy and y to the j^{th} copy of P_k in $\mathrm{St}_{n,k}$, then

$$\begin{aligned}\|f(x) - f(y)\| = \|d_{\mathrm{P}_k}(v_0, x)e_i - d_{\mathrm{P}_k}(v_0, y)e_j\| &\geqslant \max\{d_{\mathrm{P}_k}(v_0, x), d_{\mathrm{P}_k}(v_0, y)\}\\ &\geqslant \frac{1}{2}\big(d_{\mathrm{P}_k}(v_0, x) + d_{\mathrm{P}_k}(v_0, y)\big) = \frac{1}{2}\cdot d_{\mathrm{St}_{n,k}}(x, y).\end{aligned}$$

On the other hand, f is clearly 1-Lipschitz by the triangle inequality. □

The following lemma says that under certain conditions one can embed a finite product of metric spaces into a Banach space if the metric spaces are themselves embeddable in a particular fashion. Similar arguments have already been used in metric geometry (*cf.* [20, Thm. 1.7]) and their proofs simply rely on basic functional analytic principles. We provide a proof for the convenience of the reader unfamiliar with those.

Lemma 4.4 *Let $M_1, \ldots, M_n$ be metric spaces, and let Y be an infinite-dimensional Banach space. Assume that there exist positive real numbers $D_1, \ldots, D_n$ such that for every $i = 1, \ldots, n$ and for every finite-codimensional subspace Z of Y, there is a bi-Lipschitz embedding $\varphi_{i,Z}$ of M_i of distortion at most D_i into a finite-dimensional subspace of Z. Then for every $\varepsilon > 0$, the product $M = \prod_{i=1}^n M_i$ equipped with the ℓ_1-metric bi-Lipschitzly embeds into Y with distortion at most $(2+\varepsilon)n\max_{1\leqslant i\leqslant n} D_i$.*

Proof We begin with a basic result from the geometry of Banach spaces. Given $\delta > 0$ and a finite-dimensional subspace E of Y, there is a finite-codimensional subspace Z of Y such that $\|x + z\| \geqslant (1-\delta)\|x\|$ for all $x \in E$ and $z \in Z$. Indeed, choose a δ-net $x_1, \ldots, x_K$ in the unit sphere of E together with norming functionals $x_1^*, \ldots, x_K^*$ in Y^*. Set $Z = \bigcap_{i=1}^K \ker x_i^*$. Given $x \in E$ and $z \in Z$, assuming as we may that $\|x\| = 1$, choose $i \in \{1, \ldots, K\}$ such that $\|x - x_i\| \leqslant \delta$. Then we have

$$\|x + z\| \geqslant \|x_i + z\| - \delta \geqslant |x_i^*(x_i + z)| - \delta = 1 - \delta,$$

as required.

Let us now turn to the statement of the lemma. Firstly, after scaling, we may assume that for every $i = 1, \ldots, n$ and for every finite-codimensional subspace Z of Y we have

$$d_{M_i}(u, v) \leqslant \|\varphi_{i,Z}(u) - \varphi_{i,Z}(v)\| \leqslant D_i \cdot d_{M_i}(u, v) \qquad \text{for all } u, v \in M_i.$$

Fix $\varepsilon > 0$ and choose $\delta > 0$ satisfying $2(1-\delta)^{-n} < 2+\varepsilon$. We will recursively construct finite-codimensional subspaces $Z_1, \ldots, Z_n$ of Y together with finite-dimensional subspaces E_i of Z_i as follows. At the j^{th} step, having chosen $E_i \subset Z_i$ for $1 \leqslant i < j$, we choose a finite-codimensional subspace Z_j of Y such that $\|x+z\| \geqslant (1-\delta)\|x\|$ for all $x \in E_1 + \cdots + E_{j-1}$ and for all $z \in Z_j$. We then choose a finite-dimensional subspace E_j of Z_j containing $\varphi_{j,Z_j}(M_j)$. This completes the recursive construction, which has the following consequence. Given $x_i \in E_i$ for $i = 1, \ldots, n$, we have

$$(1-\delta)^{n-m} \cdot \Big\| \sum_{i=1}^{m} x_i \Big\| \leqslant \Big\| \sum_{i=1}^{n} x_i \Big\|.$$

for each $m = 1, \ldots, n$. It follows by the triangle inequality and by the choice of δ that

$$\max_{1 \leqslant m \leqslant n} \|x_m\| \leqslant (2+\varepsilon) \Big\| \sum_{i=1}^{n} x_i \Big\|. \tag{19}$$

We now define $\varphi\colon M \to Y$ by $\varphi(\mathbf{u}) = \sum_{i=1}^{n} \varphi_{i,Z_i}(u_i)$ for $\mathbf{u} = (u_1, \ldots, u_n)$ in the product space $M = \prod_{i=1}^{n} M_i$. We claim that φ is bi-Lipschitz with distortion at most $(2+\varepsilon)Dn$ where $D = \max_{1 \leqslant i \leqslant n} D_i$. Let us fix $\mathbf{u} = (u_i)_{i=1}^{n}$ and $\mathbf{v} = (v_i)_{i=1}^{n}$ in M. On the one hand, the triangle inequality yields

$$\|\varphi(\mathbf{u}) - \varphi(\mathbf{v})\| \leqslant \sum_{i=1}^{n} \|\varphi_{i,Z_i}(u_i) - \varphi_{i,Z_i}(v_i)\| \leqslant \sum_{i=1}^{n} D_i \cdot d_{M_i}(u_i, v_i) \leqslant D \cdot d_M(\mathbf{u}, \mathbf{v}).$$

On the other hand, using (19) we obtain the following lower bound.

$$\begin{aligned}(2+\varepsilon) \cdot \|\varphi(\mathbf{u}) - \varphi(\mathbf{v})\| &\geqslant \max_{1 \leqslant i \leqslant n} \|\varphi_{i,Z_i}(u_i) - \varphi_{i,Z_i}(v_i)\| \\ &\geqslant \max_{1 \leqslant i \leqslant n} d_{M_i}(u_i, v_i) \geqslant \frac{1}{n} \cdot d_M(\mathbf{u}, \mathbf{v}).\end{aligned}$$

Thus, φ has distortion at most $(2+\varepsilon)Dn$, as claimed. □

Using Theorem 4.2 we can show that for fixed $n \in \mathbb{N}$, the sequence $(\mathrm{La}(\mathrm{St}_{n,k}))_{k \in \mathbb{N}}$ of lamplighter graphs equi-bi-Lipschiztly embeds into any non-superreflexive Banach space.

Proposition 4.5 *Let Y be a non-superreflexive Banach space. For all $n \in \mathbb{N}$, there exist $C(n) \in (0, \infty)$ and maps $f_{n,k}\colon \mathrm{La}(\mathrm{St}_{n,k}) \to Y$ such that*

$$d_{\mathrm{La}(\mathrm{St}_{n,k})}(x, y) \leqslant \|f_{n,k}(x) - f_{n,k}(y)\|_Y \leqslant C(n) \cdot d_{\mathrm{La}(\mathrm{St}_{n,k})}(x, y)$$

for all $k \in \mathbb{N}$ and for all $x, y \in \mathrm{La}(\mathrm{St}_{n,k})$.

Proof It is sufficient to prove the proposition for each $n \in \{2^i : i \in \mathbb{N}\}$. Observe that $\mathrm{St}_{2^i,k} = \mathrm{St}_{2^{i-1},k} * \mathrm{St}_{2^{i-1},k}$ and $|\mathrm{St}_{2^{i-1},k}| = k \cdot 2^{i-1} + 1$. Set $\alpha_{i,k} = 2^{k \cdot 2^{i-1}+1}$. Applying Theorem 4.2, $\mathrm{La}(\mathrm{St}_{2^i,k})$ bi-Lipschitzly embeds with distortion at most 2 into

$$\mathrm{La}(\mathrm{St}_{2^{i-1},k}) \square \mathrm{La}(\mathrm{St}_{2^{i-1},k}) \square \mathrm{Clo}(\mathrm{St}_{2^{i-1},k}, \alpha_{i,k}) \square \mathrm{Clo}(\mathrm{St}_{2^{i-1},k}, \alpha_{i,k}).$$

Now observe that $\mathrm{Clo}(\mathrm{St}_{r,s}, t) = \mathrm{St}_{rt,s}$ for any $r, s, t \in \mathbb{N}$. If we apply Theorem 4.2 another $i-1$ times, we obtain that $\mathrm{La}(\mathrm{St}_{2^i,k})$ bi-Lipschitzly embeds with distortion at most 2^i into the Cartesian product of $4 \cdot 2^{i-1} + 2 \cdot 2^{i-2} + \cdots + 2 \cdot 2 + 2 = 3 \cdot 2^i - 2$ graphs each of which is either $\mathrm{La}(\mathrm{P}_k)$ or a graph of the form $\mathrm{St}_{r,k}$ for some $r \in \mathbb{N}$. Note that all these graphs admit bi-Lipschitz embeddings into every finite-codimensional subspace of any non-superreflexive Banach space. Indeed, by Bourgain's metric characterisation of superreflexivity [3], for every $\varepsilon > 0$, every binary tree of finite height admits a bi-Lipschitz embedding into every finite-codimensional subspace of any non-superreflexive Banach space with distortion at most $1 + \varepsilon$. Therefore the conclusion follows by combining this result with Lemma 4.3, Proposition 2.2, and Lemma 4.4. □

Let C_k denote the k-cycle, i.e., the cycle of length k with vertices $v_1, \ldots, v_k$ and edges $v_{i-1}v_i$ for $i = 1, \ldots, k$, where we set $v_0 = v_k$. Given $k, n \in \mathbb{N}$, we define the *rose graph* $\mathrm{Ro}_{n,k}$ to be the clover graph $\mathrm{Clo}(\mathrm{C}_k, n)$ obtained by coalescing n copies of C_k at v_0. Using Theorem 4.2 together with the main result from [21], we can show that for fixed $n \in \mathbb{N}$ the sequence $(\mathrm{La}(\mathrm{Ro}_{n,k}))_{k \in \mathbb{N}}$ of lamplighter graphs equi-bi-Lipschiztly embeds into any non-superreflexive Banach space. First we need to prove that $\mathrm{Ro}_{n,k}$ can be well embedded into Euclidean spaces.

Lemma 4.6 *Let $n \in \mathbb{N}$. There exist maps $g_{n,k}\colon \mathrm{Ro}_{n,k} \to \ell_2^{2n}$ such that*

$$\frac{1}{\sqrt{2}} \cdot d_{\mathrm{Ro}_{n,k}}(x, y) \leqslant \|g_{n,k}(x) - g_{n,k}(y)\| \leqslant \frac{\pi}{2} \cdot d_{\mathrm{Ro}_{n,k}}(x, y)$$

for all $k \in \mathbb{N}$ and for all $x, y \in \mathrm{Ro}_{n,k}$.

Proof It was proved in [13] that the natural embedding of the k-cycle onto the vertices of the regular k-gon in $\mathbb{R}^2$ is optimal and has distortion exactly $\frac{k}{2} \sin \frac{\pi}{k} \leqslant \frac{\pi}{2}$. Therefore, there exist maps $\varphi_k\colon \mathrm{C}_k \to \ell_2^2$ with $\varphi_k(v_0) = 0$ and such that

$$d_{\mathrm{C}_k}(x, y) \leqslant \|\varphi_k(x) - \varphi_k(y)\|_2 \leqslant \frac{\pi}{2} \cdot d_{\mathrm{C}_k}(x, y).$$

Let $E_i = \ell_2^2$ for all $i \in \mathbb{N}$ and define

$$\begin{aligned} g_{n,k}\colon \mathrm{Ro}_{n,k} &\to (E_1 \oplus \cdots \oplus E_{i-1} \oplus E_i \oplus E_{i+1} \oplus \cdots \oplus E_n)_{\ell_2} = \ell_2^{2n}, \\ x &\mapsto (0, \ldots, 0, \varphi_k(x), 0, \ldots, 0) \qquad \text{if } x \text{ belongs to the } i^{\text{th}} \text{ copy of } \mathrm{C}_k. \end{aligned}$$

Observe that if x and y belong to the same copy of C_k in $\mathrm{Ro}_{n,k}$, then one has

$$\|g_{n,k}(x) - g_{n,k}(y)\|_2 = \|\varphi_k(x) - \varphi_k(y)\|_2.$$

Otherwise

$$\begin{aligned}\|g_{n,k}(x)-g_{n,k}(y)\|_2 &= \sqrt{\|\varphi_k(x)\|_2^2+\|\varphi_k(y)\|_2^2} \leqslant \|\varphi_k(x)\|_2+\|\varphi_k(y)\|_2\\ &\leqslant \frac{\pi}{2}\cdot d_{\mathrm{C}_k}(x,v_0)+\frac{\pi}{2}\cdot d_{\mathrm{C}_k}(y,v_0)=\frac{\pi}{2}\cdot d_{\mathrm{Ro}_{n,k}}(x,y)\end{aligned}$$

and

$$\begin{aligned}\|g_{n,k}(x)-g_{n,k}(y)\|_2 &\geqslant \frac{1}{\sqrt{2}}\big(\|\varphi_k(x)\|_2+\|\varphi_k(y)\|_2\big)\\ &\geqslant \frac{1}{\sqrt{2}}\big(d_{\mathrm{C}_k}(x,v_0)+d_{\mathrm{C}_k}(y,v_0)\big)=\frac{1}{\sqrt{2}}\cdot d_{\mathrm{Ro}_{n,k}}(x,y). \qquad \square\end{aligned}$$

Proposition 4.7 *Let Y be a non-superreflexive Banach space. For all $n\in\mathbb{N}$ there exist $D(n)\in(0,\infty)$ and maps $f_{n,k}\colon \mathrm{La}(\mathrm{Ro}_{n,k})\to Y$ such that*

$$d_{\mathrm{La}(\mathrm{Ro}_{n,k})}(x,y)\leqslant \|f_{n,k}(x)-f_{n,k}(y)\|_Y\leqslant D(n)\cdot d_{\mathrm{La}(\mathrm{Ro}_{n,k})}(x,y)$$

for all $k\in\mathbb{N}$ and for all $x,y\in\mathrm{La}(\mathrm{Ro}_{n,k})$.

Proof It is sufficient to prove the proposition for each $n\in\{2^i: i\in\mathbb{N}\}$. Observe that $\mathrm{Ro}_{2^i,k}=\mathrm{Ro}_{2^{i-1},k}*\mathrm{Ro}_{2^{i-1},k}$ and $|\mathrm{Ro}_{2^{i-1},k}|=(k-1)2^{i-1}+1$. Set $\beta_{i,k}=2^{(k-1)2^{i-1}+1}$. Applying Theorem 4.2, $\mathrm{La}(\mathrm{Ro}_{2^i,k})$ bi-Lipschitzly embeds with distortion at most 2 into

$$\mathrm{La}(\mathrm{Ro}_{2^{i-1},k})\square\mathrm{La}(\mathrm{Ro}_{2^{i-1},k})\square\mathrm{Clo}(\mathrm{Ro}_{2^{i-1},k},\beta_{i,k})\square\mathrm{Clo}(\mathrm{Ro}_{2^{i-1},k},\beta_{i,k}).$$

Now observe that $\mathrm{Clo}(\mathrm{Ro}_{r,s},t)=\mathrm{Ro}_{rt,s}$ for any $r,s,t\in\mathbb{N}$. If we apply Theorem 4.2 another $i-1$ times, we obtain that $\mathrm{La}(\mathrm{Ro}_{2^i,k})$ bi-Lipschitzly embeds with distortion at most 2^i into the Cartesian product of $4\cdot2^{i-1}+2\cdot2^{i-2}+\dots+2\cdot2+2=3\cdot2^i-2$ graphs each of which is either $\mathrm{La}(\mathrm{C}_k)$ or a graph of the form $\mathrm{Ro}_{r,k}$ for some $r\in\mathbb{N}$. Note that all these graphs admit bi-Lipschitz embeddings into every finite-codimensional subspace of any non-superreflexive Banach space. Indeed, it was proved in [21] that $\mathrm{La}(\mathrm{C}_k)$ can be embedded into a product of 8 trees, and hence one can again use Bourgain's metric characterisation of superreflexivity [3]. The conclusion follows by appealing to Lemma 4.6, Dvoretzky's theorem, and Lemma 4.4. $\square$

Remark By carefully keeping track of the distortions of embeddings in the proofs of Propositions 4.5 and 4.7, one obtains order n^2 upper bounds on $C(n)$ and $D(n)$.

Remark At this point we have established the implications "(ii) ⇒ (i)" and "(iii) ⇒ (i)" in Theorem 1.2. The remaining implications will be shown in Sect. 6.

5 Induced Maps Between Lamplighter Graphs

A map $f\colon G \to H$ between two graphs induces a map $\bar{f}\colon \mathrm{La}(G) \to \mathrm{La}(H)$ defined by $\bar{f}(A, x) = (f(A), f(x))$, where $f(A) = \{f(y) : y \in A\}$. Moreover, if G and H are connected and for some $a, b \in [0, \infty]$ we have

$$a \cdot d_G(x, y) \leqslant d_H(f(x), f(y)) \leqslant b \cdot d_G(x, y)$$

for all $x, y \in G$, then it is easy to see (*cf.* Remark following Lemma 5.1 below) that

$$a' \cdot d_{\mathrm{La}(G)}(u, v) \leqslant d_{\mathrm{La}(H)}(\bar{f}(u), \bar{f}(v)) \leqslant b' \cdot d_{\mathrm{La}(G)}(u, v)$$

for all $u, v \in \mathrm{La}(G)$, where $a' = \min\{1, a\}$ and $b' = \max\{1, b\}$. Of course, this result is only interesting if $a > 0$, $b < \infty$ or both, i.e., if f is co-Lipschitz, Lipschitz or bi-Lipschitz, respectively. In particular, if (G, d_G) bi-Lipschitzly embeds into (H, d_H), then $\mathrm{La}(G)$ bi-Lipschitzly embeds into $\mathrm{La}(H)$. Observe that if f is injective, then $b \geqslant 1$, and if in addition $0 < a \leqslant 1$, then $\mathrm{dist}(\bar{f}) \leqslant \mathrm{dist}(f)$. However, there are bi-Lipschitz embeddings of interest where $a \to \infty$ with b/a bounded. In this case, b'/a' gets arbitrarily large. For this reason, we will consider more complicated induced maps in Lemmas 5.1 and 5.2 below.

Lemma 5.1 *Let $f\colon G \to H$ be a map between connected graphs G and H, and let $a, b \in [0, \infty]$ be given so that*

$$a \cdot d_G(x, y) \leqslant d_H(f(x), f(y)) \leqslant b \cdot d_G(x, y) \qquad \text{for all } x, y \in G.$$

Then for every $m \in \{0\} \cup \{1, \dots, \lceil a/2 \rceil - 1\}$, there is a map $\bar{f}_m\colon \mathrm{La}(G) \to \mathrm{La}(H)$ induced by f and m such that $\bar{f}_0 = \bar{f}$ and

$$a' \cdot d_{\mathrm{La}(G)}(u, v) \leqslant d_{\mathrm{La}(H)}(\bar{f}_m(u), \bar{f}_m(v)) \leqslant b' \cdot d_{\mathrm{La}(G)}(u, v)$$

for all $u, v \in \mathrm{La}(G)$, where $a' = \min\{a, m+1\}$ and $b' = \max\{b, 3m+1\}$.

Remark Assume that $f\colon G \to H$ is a bi-Lipschitz embedding, and thus that $b \geqslant 1$ and $a > 0$. The observations made before the statement of the lemma follow immediately by taking $m = 0$. On the other hand, choosing $m = \lceil a/2 \rceil - 1$, and by considering the fractions $\frac{b}{a}$, $\frac{b}{m+1}$, $\frac{3m+1}{a}$ and $\frac{3m+1}{m+1}$, it is easy to see that $\mathrm{dist}(\bar{f}_m) \leqslant b'/a' \leqslant 3b/a$. Hence we obtain the universal bound

$$\frac{\mathrm{dist}\big(\bar{f}_{\lceil a/2 \rceil - 1}\big)}{\mathrm{dist}(f)} \leqslant 3.$$

Proof If $m = 0$ we define $\bar{f}_0$ to be the natural induced map $\bar{f}$, and the conclusion holds with $a' = \min\{a, 1\}$ and $b' = \max\{b, 1\}$. Indeed, let $x_0, x_1, \dots, x_n$ be vertices of G, and let $y_i = f(x_i)$ for $0 \leqslant i \leqslant n$. Consider a walk w of length ℓ in G from x_0 to x_n visiting $x_0, x_1, \dots, x_n$ in this order. For each $i = 1, \dots, n$, let w_i be the section of w from x_{i-1} to x_i, and let ℓ_i be the length of w_i. Then $d_H(f(x_{i-1}), f(x_i)) \leqslant b \cdot d_G(x_{i-1}, x_i) \leqslant b \cdot \ell_i$ for each $i = 1, \ldots, n$. Hence there is a walk in H of length at most $b \cdot \ell$ from y_0 to y_n visiting $y_0, y_1, \ldots, y_n$ in this order. An essentially identical argument shows that if there is a walk in H of length ℓ from y_0 to y_n visiting $y_0, y_1, \ldots, y_n$ in this order, then there is a walk in G of length at most ℓ/a from x_0 to x_n visiting $x_0, x_1, \ldots, x_n$ in this order. It follows that

$$a \cdot \mathrm{tsp}_G(x, C, y) \leqslant \mathrm{tsp}_H\big(f(x), f(C), f(y)\big) \leqslant b \cdot \mathrm{tsp}_G(x, C, y) \tag{20}$$

for all $x, y \in G$ and for all finite $C \subset G$. Now fix vertices (A, x) and (B, y) of $\mathrm{La}(G)$. Observe that $f(A) \triangle f(B) \subset f(A \triangle B)$ and $|f(A \triangle B)| \leqslant |A \triangle B|$, and moreover equality holds when $a > 0$. Hence, using Proposition 2.1 and (20), we have

$$\begin{aligned} d_{\mathrm{La}(H)}\big(f(A,x), f(B,y)\big) &= \mathrm{tsp}_H\big(f(x), f(A) \triangle f(B), f(y)\big) + |f(A) \triangle f(B)| \\ &\leqslant b \cdot \mathrm{tsp}_G(x, A \triangle B, y) + |A \triangle B| \leqslant b' \cdot d_{\mathrm{La}(G)}\big((A,x),(B,y)\big) \end{aligned}$$

and if $a > 0$, then

$$\begin{aligned} d_{\mathrm{La}(H)}\big(f(A,x), f(B,y)\big) &= \mathrm{tsp}_H\big(f(x), f(A \triangle B), f(y)\big) + |f(A \triangle B)| \\ &\geqslant a \cdot \mathrm{tsp}_G(x, A \triangle B, y) + |A \triangle B| \geqslant a' \cdot d_{\mathrm{La}(G)}\big((A,x),(B,y)\big). \end{aligned}$$

Assume now that $1 \leqslant m \leqslant \lceil a/2 \rceil - 1$ and in particular that $a > 0$. Set $a' = \min(a, m+1)$ and $b' = \max(b, 3m+1)$. Without loss of generality we will assume that G has at least two vertices. For every vertex y in the image of f, choose a path $(u_0, u_1, u_2, \ldots, u_m)$ in H starting at $u_0 = y$, and set $W_y = \{u_0, u_1, \ldots, u_m\}$. This can always be done by picking another vertex $z \in f(G)$ and using $d_H(y, z) \geqslant a$ which in turn follows from the assumptions on f. It is easy to see that $\mathrm{tsp}_H(y, W_y, y) = 2m$ since the unique optimal walk for the salesman is the path from u_0 to u_m and back. Note also that the sets W_y, $y \in f(G)$, are pairwise disjoint, since the vertices in $f(G)$ are a-separated and $2m < a$. For a finite set $C \subset f(G)$ we put $W_C = \bigcup_{y \in C} W_y$. Finally, we define $\bar{f}_m \colon \mathrm{La}(G) \to \mathrm{La}(H)$ by letting $\bar{f}_m(A, x) = \big(W_{f(A)}, f(x)\big)$.

Given vertices y, z and a finite subset C in the image of f, we now obtain estimates on $\mathrm{tsp}_H(y, W_C, z)$. Since $C \subset W_C$, we immediately obtain $\mathrm{tsp}_H(y, W_C, z) \geqslant \mathrm{tsp}_H(y, C, z)$. On the other hand, consider the following walk. Start with a walk w in H of length $\mathrm{tsp}_H(y, C, z)$ from y to z visiting all vertices of C, and each time w visits a vertex $u \in C$, insert a walk of length $2m = \mathrm{tsp}_H(u, W_u, u)$ starting and ending at u and visiting all vertices in W_u. The resulting walk from y to z visits all the vertices in W_C and has length $\mathrm{tsp}_H(y, C, z) + 2m|C|$. Therefore, we have

$$\mathrm{tsp}_H(y, C, z) \leqslant \mathrm{tsp}_H(y, W_C, z) \leqslant \mathrm{tsp}_H(y, C, z) + 2m|C|. \tag{21}$$

Let us now fix vertices (A, x) and (B, y) in $\mathrm{La}(G)$. Observe that $W_{f(A)} \triangle W_{f(B)} = W_{f(A) \triangle f(B)}$ and $f(A) \triangle f(B) = f(A \triangle B)$. Combining Proposition 2.1, (21) and (20), we obtain

$$d_{\mathrm{La}(H)}\big(\bar{f}_m(A,x), \bar{f}_m(B,y)\big) = \mathrm{tsp}_H\big(f(x), W_{f(A \triangle B)}, f(y)\big) + \big|W_{f(A \triangle B)}\big|$$

$$\begin{aligned}
&\leqslant \mathrm{tsp}_H\big(f(x), f(A \triangle B), f(y)\big) + 2m \cdot |f(A \triangle B)| + (m+1) \cdot |f(A \triangle B)| \\
&\leqslant b \cdot \mathrm{tsp}_G(x, A \triangle B, y) + (3m+1) \cdot |A \triangle B| \\
&\leqslant b' \cdot d_{\mathrm{La}(G)}\big((A, x), (B, y)\big),
\end{aligned}$$

and

$$\begin{aligned}
d_{\mathrm{La}(H)}\big(\bar{f}_m(A, x), \bar{f}_m(B, y)\big) &\geqslant \mathrm{tsp}_H\big(f(x), f(A \triangle B), f(y)\big) + \big|W_{f(A\triangle B)}\big| \\
&\geqslant a \cdot \mathrm{tsp}_G(x, A \triangle B, y) + (m+1) \cdot |f(A \triangle B)| \\
&\geqslant a' \cdot d_{\mathrm{La}(G)}\big((A, x), (B, y)\big). \qquad \square
\end{aligned}$$

In the last lemma of this section we consider a more sophisticated construction in order to improve the bound on the distortion. This construction is of a slightly different nature since it provides an embedding with a higher degree of faithfulness at the expense that we need to consider the lamplighter graph over a slightly bigger graph that contains the original graph H under scrutiny. In some specific situations (see Proposition 6.4), this turns out not to be an issue and Lemma 5.2 can be efficiently used to significantly improve the distortion.

Let us fix a map $f\colon\ G \to H$ between two graphs. Let $Q = (V, E, v_0)$ be a pointed graph and W be a finite subset of V with $v_0 \in W$. Let $\widetilde{H}$ be the graph obtained by coalescing H with $|f(G)|$ copies of Q as follows. For each vertex y in the image of f, we coalesce to H at the vertex y the copy of Q that corresponds to y. This leads to a map $\tilde{f}\colon\ \mathrm{La}(G) \to \mathrm{La}(\widetilde{H})$ induced by f, Q and W and defined as follows. For $y \in f(G)$ we let W_y denote the set W considered in the copy of Q that corresponds to y, and for a finite subset C of $f(G)$ we let $W_C = \bigcup_{y\in C} W_y$. Finally, for a vertex (A, x) of $\mathrm{La}(G)$ define $\tilde{f}(A, x) = \big(W_{f(A)}, f(x)\big)$.

Lemma 5.2 *Let $f\colon\ G \to H$ be a map between connected graphs. Let $Q = (V, E, v_0)$ be a connected pointed graph and W be a finite subset of V with $v_0 \in V$. Let $\widetilde{H}$ and $\tilde{f}\colon\ \mathrm{La}(G) \to \mathrm{La}(\widetilde{H})$ be the map defined above. Assume that there exist $a, b \in [0, \infty]$ such that*

$$a \cdot d_G(x, y) \leqslant d_H(f(x), f(y)) \leqslant b \cdot d_G(x, y)$$

for all $x, y \in G$. Then it follows for all $u, v \in \mathrm{La}(G)$ that

$$a' \cdot d_{\mathrm{La}(G)}(u, v) \leqslant d_{\mathrm{La}(\widetilde{H})}\big(\tilde{f}(u), \tilde{f}(v)\big) \leqslant b' \cdot d_{\mathrm{La}(G)}(u, v),$$

where $a' = \min\{a, c\}$, $b' = \max\{b, c\}$ and $c = \mathrm{tsp}_Q(v_0, W, v_0) + |W|$.

Remark This result can be very versatile. Assume that $a, b \in \mathbb{N}$ and $b - a \geqslant 2$. Assume further that $|V| \geqslant b$. Then W can be chosen so that $a \leqslant c \leqslant b$, and hence $\mathrm{dist}(\tilde{f}) \leqslant b/a$. Indeed, given a finite $W \subset V$, replacing W by $W \cup \{q\}$ for some $q \in V \setminus W$ that is joined to a vertex in W, the value of c increases by at most 3. Hence, starting with $W = \{v_0\}$ and adding one vertex at a time, we eventually arrive at a set W for which $a \leqslant c \leqslant b$ holds.

Proof Given vertices y, z and a finite subset C in the image of f, an optimal solution for computing $\mathrm{tsp}_{\widetilde{H}}(y, W_C, z)$ can be obtained as follows. Start with a walk w in H of length $\mathrm{tsp}_H(y, C, z)$ from y to z visiting all vertices of C, and each time w visits a vertex $u \in C$ insert a walk of length $\mathrm{tsp}_Q(v_0, W, v_0)$ in the copy of Q corresponding to u that starts and ends at v_0 and visits all vertices in W. The resulting walk is easily seen to be optimal, and hence yields the formula

$$\mathrm{tsp}_{\widetilde{H}}(y, W_C, z) = \mathrm{tsp}_H(y, C, z) + |C| \cdot \mathrm{tsp}_Q(v_0, W, v_0). \tag{22}$$

Let us now fix vertices (A, x) and (B, y) in $\mathrm{La}(G)$. Observing that $W_{f(A)} \triangle W_{f(B)} = W_{f(A) \triangle f(B)}$, and combining Proposition 2.1 with (22), we obtain

$$\begin{aligned} d_{\mathrm{La}(\widetilde{H})}\big(\tilde{f}(A, x), \tilde{f}(B, y)\big) &= \mathrm{tsp}_{\widetilde{H}}\big(f(x), W_{f(A)\triangle f(B)}, f(y)\big) + \big|W_{f(A)\triangle f(B)}\big| \\ &= \mathrm{tsp}_H\big(f(x), f(A) \triangle f(B), f(y)\big) + \big(\mathrm{tsp}_Q(v_0, W, v_0) + |W|\big) \cdot |f(A) \triangle f(B)|. \end{aligned}$$

As before, we have $f(A) \triangle f(B) \subset f(A \triangle B)$ and $|f(A \triangle B)| \leqslant |A \triangle B|$, and moreover equality holds when $a > 0$. Hence, by (20), which is still valid in this context, we obtain

$$\begin{aligned} d_{\mathrm{La}(\widetilde{H})}\big(\tilde{f}(A, x), \tilde{f}(B, y)\big) &\leqslant b \cdot \mathrm{tsp}_G(x, A \triangle B, y) + c \cdot |A \triangle B| \\ &\leqslant b' \cdot d_{\mathrm{La}(G)}\big((A, x), (B, y)\big), \end{aligned}$$

and if $a > 0$, then

$$\begin{aligned} d_{\mathrm{La}(\widetilde{H})}\big(\tilde{f}(A, x), \tilde{f}(B, y)\big) &\geqslant a \cdot \mathrm{tsp}_G(x, A \triangle B, y) + c \cdot |f(A \triangle B)| \\ &\geqslant a' \cdot d_{\mathrm{La}(G)}\big((A, x), (B, y)\big). \end{aligned}$$

6 Binary Trees and Hamming Cubes in Lamplighter Graphs

It was observed in [14] that the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ contains a bi-Lipschitz copy of the infinite binary tree. We provide a simple proof of the finite version of this fact.

Lemma 6.1 *Let $k \in \mathbb{N}$. Then B_k bi-Lipschitzly embeds with distortion at most 2 into* $\mathrm{La}(\mathrm{P}_k)$.

Proof Let $v_0, \dots, v_k$ be the vertices of P_k with edges $v_{i-1}v_i$ for $1 \leqslant i \leqslant k$. For any $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \mathrm{B}_k$, let $A_\varepsilon = \{v_{s-1} : \varepsilon_s = 1\}$, and define $f\colon \mathrm{B}_k \to \mathrm{La}(\mathrm{P}_k)$ by setting $f(\varepsilon) = (A_\varepsilon, v_{|\varepsilon|})$. We show that f is a bi-Lipschitz embedding with distortion at most 2.

Let us fix $\delta, \varepsilon \in \mathrm{B}_k$ and assume without loss of generality that $|\delta| \leqslant |\varepsilon|$. Then by Proposition 2.1 we have

$$d_{\mathrm{La}(\mathrm{P}_k)}(f(\delta), f(\varepsilon)) = \mathrm{tsp}_{\mathrm{P}_k}\big(v_{|\delta|}, A_\delta \triangle A_\varepsilon, v_{|\varepsilon|}\big) + |A_\delta \triangle A_\varepsilon|. \tag{23}$$

Assume that δ and ε are adjacent, and thus $\varepsilon = (\delta, \delta_{m+1})$, where $m = |\delta|$. If $\delta_{m+1} = 0$, then $A_\delta \triangle A_\varepsilon = \emptyset$, otherwise $A_\delta \triangle A_\varepsilon = \{v_m\}$. Therefore,

$$d_{\mathrm{La}(\mathrm{P}_k)}(f(\delta), f(\varepsilon)) = \mathrm{tsp}_{\mathrm{P}_k}\big(v_m, A_\delta \triangle A_\varepsilon, v_{m+1}\big) + |A_\delta \triangle A_\varepsilon| \leqslant 1 + 1 = 2,$$

and thus f is 2-Lipschitz.

We now derive the lower bound. Let $\delta \wedge \varepsilon$ denote the last common ancestor of δ and ε. Thus, $\delta \wedge \varepsilon = (\delta_1, \dots, \delta_r)$, where $r = \max\{i \geqslant 0 : \delta_j = \varepsilon_j \text{ for } 1 \leqslant j \leqslant i\}$. Since by definition we have $A_\delta \triangle A_\varepsilon \subset \{v_{|\delta\wedge\varepsilon|}, \dots, v_{|\varepsilon|-1}\}$, an optimal solution for computing $\mathrm{tsp}_{\mathrm{P}_k}\big(v_{|\delta|}, A_\delta \triangle A_\varepsilon, v_{|\varepsilon|}\big)$ starts at $v_{|\delta|}$, then travels to $v_{|\delta\wedge\varepsilon|}$, and finally travels to $v_{|\varepsilon|}$. Thus,

$$\begin{aligned}\mathrm{tsp}_{\mathrm{P}_k}\big(v_{|\delta|}, A_\delta \triangle A_\varepsilon, v_{|\varepsilon|}\big) &= d_{\mathrm{P}_k}\big(v_{|\delta|}, v_{|\delta\wedge\varepsilon|}\big) + d_{\mathrm{P}_k}\big(v_{|\delta\wedge\varepsilon|}, v_{|\varepsilon|}\big)\\ &= (|\delta| - |\delta\wedge\varepsilon|) + (|\varepsilon| - |\delta\wedge\varepsilon|) = d_{\mathrm{B}}(\delta, \varepsilon).\end{aligned}$$

It follows that

$$d_{\mathrm{La}(\mathrm{P}_k)}(f(\delta), f(\varepsilon)) \geqslant d_{\mathrm{B}}(\delta, \varepsilon).$$ □

It is clear that a similar argument as in the proof above shows that the lamplighter graph $\mathrm{La}(\mathrm{P}_\infty)$ over the infinite path P_∞ contains a bi-Lipschitz copy of the infinite binary tree. Since $\mathrm{La}(\mathrm{P}_\infty)$ and $\mathbb{Z}_2 \wr \mathbb{Z}$ are isometric (with suitable choice of generators for $\mathbb{Z}_2 \wr \mathbb{Z}$), the observation from [14] can be recovered. Lemma 6.1 also provides the final result we need to complete the proof of Theorem 1.2.

Proof The implications "(ii) ⇒ (i)" and "(iii) ⇒ (i)" follow from Propositions 4.5 and 4.7, respectively. To establish the reverse implications, fix $n \in \mathbb{N}$. Observe that for each $k \in \mathbb{N}$ the graphs $\mathrm{St}_{n,k}$ and $\mathrm{Ro}_{n,2k}$ contain isometric copies of P_k, and hence by combining Lemmas 5.1 and 6.1, the binary tree B_k bi-Lipschitzly embeds with distortion at most 2 into the lamplighter graphs $\mathrm{La}(\mathrm{St}_{n,k})$ and $\mathrm{La}(\mathrm{Ro}_{n,2k})$. The implications "(i) ⇒ (ii)" and "(i) ⇒ (iii)" now follow from Bourgain's metric characterisation of superreflexivity [3]. □

We now turn to the embeddability of Hamming cubes into lamplighter graphs. Here K_n, for $n \in \mathbb{N}$, denotes the complete graph on n vertices.

Lemma 6.2 *Let $k, m \in \mathbb{N}$. Then H_k bi-Lipschitzly embeds into $\mathrm{La}(\mathrm{K}_{km})$ with distortion at most $1 + \frac{1}{2m}$.*

Proof Recall that H_k can be thought of as the set of all subsets of $\{1, \dots, k\}$ and that under this identification the Hamming metric becomes the symmetric difference metric. Let us now partition the vertex set of K_{km} into k sets $V_1, \dots, V_k$ each of size m, and let us also fix a vertex v_0 of K_{km}. Define $f\colon \mathrm{H}_k \to \mathrm{La}(\mathrm{K}_{km})$ by setting $f(I) = (V_I, v_0)$, where $V_I = \bigcup_{i\in I} V_i$.

To estimate the distortion of f, let us fix distinct elements $I, J \in \mathrm{H}_k$. Note that $V_I \triangle V_J = V_{I\triangle J}$, and hence

$$|V_I \triangle V_J| = m|I \triangle J| = m \cdot d_{\mathrm{H}}(I, J).$$

It follows that

$$\mathrm{tsp}_{\mathrm{K}_{km}}(v_0, V_I \triangle V_J, v_0) = \begin{cases} m \cdot d_{\mathrm{H}}(I, J) & \text{if } v_0 \in V_I \triangle V_J, \\ m \cdot d_{\mathrm{H}}(I, J) + 1 & \text{if } v_0 \notin V_I \triangle V_J. \end{cases}$$

Combining the above with Proposition 2.1 yields

$$2m \cdot d_{\mathrm{H}}(I, J) \leqslant d_{\mathrm{La}(\mathrm{K}_{km})}(f(I), f(J)) \leqslant 2m \cdot d_{\mathrm{H}}(I, J) + 1 \leqslant (2m+1) \cdot d_{\mathrm{H}}(I, J). \qquad \square$$

Remark Lemma 6.2 shows in particular that for every $k \in \mathbb{N}$, there is a bi-Lipschitz embedding of H_k into $\mathrm{La}(\mathrm{K}_k)$ of distortion at most $\frac{3}{2}$, and that for every $k \in \mathbb{N}$ and $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that H_k bi-Lipschitzly embeds into $\mathrm{La}(\mathrm{K}_n)$ with distortion at most $1 + \varepsilon$, and moreover n can be chosen to be $\frac{k}{2\varepsilon}$.

At this point, we need one more ingredient to prove Theorem 1.3, which is the following well-known fact.

Lemma 6.3 *Let $k \in \mathbb{N}$ and $\varepsilon > 0$. Then K_k embeds with distortion at most $1 + \varepsilon$ into B_n whenever $n \geqslant \frac{1+\varepsilon}{\varepsilon} \log_2 k$.*

Proof Choose $s, t \in \mathbb{N}$ such that $2^s \geqslant k$ and $\frac{s+t}{t+1} < 1 + \varepsilon$. We show that $n = s + t$ works. By a *leaf* of the binary tree B_s of height s, we mean a vertex ε with $|\varepsilon| = s$. The binary tree B_n of height $n = s + t$ can be considered as being constructed by coalescing 2^s copies of the binary tree B_t to the leaves of the binary tree B_s as follows. For each leaf ε of B_s, we coalesce a copy of B_t at $\emptyset$, its root, to B_s at ε.

Pick k leaves $\ell_1, \dots, \ell_k$ of B_n, one from each of k different copies of B_t. Let $v_1, \dots, v_k$ be the vertices of K_k, and define $f\colon \mathrm{K}_k \to \mathrm{B}_n$ by $f(v_i) = \ell_i, i = 1, \dots, k$. We then have

$$2t + 2 \leqslant d_{\mathrm{B}}(f(v_i), f(v_j)) \leqslant \mathrm{diam}(\mathrm{B}_{s+t}) = 2(s+t)$$

for all $i \neq j$. Thus, f has distortion at most $\frac{s+t}{t+1}$, which in turn is at most $1 + \varepsilon$ by the choice of s and t. $\square$

Proof It follows from Theorem 1.1 that $\mathrm{La}(\mathrm{B}_k)$ embeds with distortion at most 6 into a finite Hamming cube. In turn, by Lemma 6.2, the Hamming cube H_k embeds into $\mathrm{La}(\mathrm{K}_k)$ with distortion at most $\frac{3}{2}$. It remains to show that $(\mathrm{La}(\mathrm{K}_k))_{k \in \mathbb{N}}$ equi-bi-Lipschitzly embeds into $(\mathrm{La}(\mathrm{B}_k))_{k \in \mathbb{N}}$, but this is true due to Lemma 6.3 combined with Lemma 5.1. $\square$

The equi-bi-Lipschitz embeddability of $(\mathrm{La}(\mathrm{K}_k))_{k \in \mathbb{N}}$ into $(\mathrm{La}(\mathrm{B}_k))_{k \in \mathbb{N}}$ can be made quantitatively more precise using Lemma 5.2.

Proposition 6.4 *Let $k \in \mathbb{N}$ and $\varepsilon > 0$. Then $\mathrm{La}(\mathrm{K}_k)$ embeds with distortion at most $1 + \varepsilon$ into $\mathrm{La}(\mathrm{B}_N)$ whenever $N > n + \log_2 n + 1$ and $n \geqslant \frac{1+\varepsilon}{\varepsilon} \log_2 k$.*

Proof For $k = 1, 2$, it is clear that K_k embeds isometrically into B_k, and hence the same holds for the corresponding lamplighter graphs. We now assume that $k \geqslant 3$ and follow the notation from the proof of Lemma 6.3. We have $s \geqslant 2$, and hence $2(s+t) - (2t+2) = 2s - 2 \geqslant 2$. Choose $r \in \mathbb{N}$ with $2^r > 2(s+t)$, and let Q be the pointed graph $(\mathrm{B}_r, \emptyset)$. It follows from Lemma 5.2 and the subsequent remark that there is a subset W of the vertices of Q such that the map $\tilde{f}\colon\ \mathrm{La}(\mathrm{K}_k) \to \mathrm{La}(\widetilde{\mathrm{B}}_n)$, induced by f, Q and W, has distortion at most $\frac{s+t}{t+1} < 1 + \varepsilon$. Finally observe that, since the image of f is contained in the set of leaves of B_n, it follows that $\widetilde{\mathrm{B}}_n$ isometrically embeds into B_{n+r}, which in turn implies that $\mathrm{La}(\widetilde{\mathrm{B}}_n)$ isometrically embeds into $\mathrm{La}(\mathrm{B}_{n+r})$. □

7 Conclusions

Assume that a sequence $(G_k)_{k\in\mathbb{N}}$ of graphs equi-bi-Lipschitzly contains $(K_k)_{k\in\mathbb{N}}$. It follows then from Theorem 1.3 and Lemma 5.1, together with the remark thereafter, that the sequence $(\mathrm{H}_k)_{k\in\mathbb{N}}$ of Hamming cubes equi-bi-Lipschitzly embeds into $(\mathrm{La}(G_k))_{k\in\mathbb{N}}$. We do not know if the converse holds.

Problem 7.1 *Given a sequence $(G_k)_{k\in\mathbb{N}}$ of graphs, if the Hamming cubes $(\mathrm{H}_k)_{k\in\mathbb{N}}$ equi-bi-Lipschitzly embed into $(\mathrm{La}(G_k))_{k\in\mathbb{N}}$, does it follow that $(\mathrm{K}_k)_{k\in\mathbb{N}}$ equi-bi-Lipschitzly embeds into $(G_k)_{k\in\mathbb{N}}$?*

The following tree might be a counterexample to Problem 7.1.

Example We construct a tree which can be seen as a "binary tree with variable-size legs" as follows. Given $k \in \mathbb{N}$ and $\bar{\ell} = (\ell_1, \ell_2, \ldots, \ell_k) \in \mathbb{N}^k$, replace each edge on the j^{th} level of the binary tree of length k by a path of length ℓ_j, where by an edge on the j^{th} level we mean an edge such that the distance from its farthest endpoint to the root is j. Denote by $\mathrm{B}_{\bar{\ell}}$ the new tree, of length $\ell = \sum_{i=1}^k \ell_i$, thus obtained. The tree $\mathrm{B}_{\bar{\ell}}$, with $\bar{\ell} = (4, 2, 1)$ is the tree of length $\ell = 7$ depicted in the illustration below (Fig. 3).

If we choose for every $k \in \mathbb{N}$, the sequence $\bar{\ell}^{(k)} = \big(\ell_1^{(k)}, \ell_2^{(k)}, \ldots, \ell_k^{(k)}\big)$ so that $\ell_1^{(k)}$ is chosen large enough compared to $\ell_2^{(k)}$, $\ell_2^{(k)}$ is chosen large enough compared to $\ell_3^{(k)}$, *etc.* it is not hard, but cumbersome, to prove that the sequence $\big(\mathrm{B}_{\bar{\ell}^{(k)}}\big)_{k\in\mathbb{N}}$ does not equi-bi-Lipschitzly contain $(\mathrm{K}_k)_{k\in\mathbb{N}}$. So for this example to become a counterexample to Problem 7.1, we need a positive answer to the following question.

Problem 7.2 *Let $\big(\mathrm{B}_{\bar{\ell}^{(k)}}\big)_{k\in\mathbb{N}}$ be constructed as in the description above. Does $(\mathrm{H}_k)_{k\in\mathbb{N}}$ equi-bi-Lipschitzly embed into $\big(\mathrm{La}(\mathrm{B}_{\bar{\ell}^{(k)}})\big)_{k\in\mathbb{N}}$?*

Any counterexample to Problem 7.1 would be a counterexample to the following problem.

Problem 7.3 *If $(G_k)_{k\in\mathbb{N}}$ is a sequence of graphs which does not equi-bi-Lipschitzly contain $(\mathrm{K}_k)_{k\in\mathbb{N}}$, and if X is a non-reflexive Banach space, does it follow that the sequence $(\mathrm{La}(G_k))_{k\in\mathbb{N}}$ equi-bi-Lipschitzly embeds into X?*

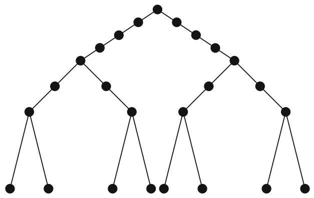

Fig. 3 $\mathrm{B}_{\bar{\ell}}$, with $\bar{\ell} = (4, 2, 1)$

Indeed, there are non-reflexive Banach spaces X with non-trivial type (*cf.* [9,10] or [22]). By the observation at the beginning of this section, these spaces cannot equi-bi-Lipschitzly contain sequences of graphs which equi-bi-Lipschitzly contain $(\mathrm{H}_k)_{k\in\mathbb{N}}$.

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References

1. Austin, T., Naor, A., Valette, A.: The Euclidean distortion of the lamplighter group. Discrete Comput. Geom. **44**(1), 55–74 (2010)
2. Bollobás, B.: Modern Graph Theory. Graduate Texts in Mathematics, vol. 184. Springer, New York (1998)
3. Bourgain, J.: The metrical interpretation of superreflexivity in Banach spaces. Isr. J. Math. **56**(2), 222–230 (1986)
4. Bourgain, J., Milman, V., Wolfson, H.: On type of metric spaces. Trans. Am. Math. Soc. **294**(1), 295–317 (1986)
5. Cornulier, Y., Stalder, Y., Valette, A.: Proper actions of wreath products and generalizations. Trans. Am. Math. Soc. **364**(6), 3159–3184 (2012)
6. Deza, M.M., Laurent, M.: Geometry of Cuts and Metrics. Algorithms and Combinatorics, vol. 15. Springer, Berlin (1997)
7. Donno, A.: Generalized wreath products of graphs and groups. Graphs Comb. **31**(4), 915–926 (2015)
8. Goodman, J.E., O'Rourke, J., Tóth, C.D. (eds.): Handbook of Discrete and Computational Geometry. Discrete Mathematics and its Applications (Boca Raton). CRC Press, Boca Raton (2018)
9. James, R.C.: A nonreflexive Banach space that is uniformly nonoctahedral. Isr. J. Math. **18**, 145–155 (1974)
10. James, R.C.: Nonreflexive spaces of type 2. Isr. J. Math. **30**(1–2), 1–13 (1978)
11. Kaĭmanovich, V.A., Vershik, A.M.: Random walks on discrete groups: boundary and entropy. Ann. Probab. **11**(3), 457–490 (1983)
12. Lee, J.R., Naor, A., Peres, Y.: Trees and Markov convexity. Geom. Funct. Anal. **18**(5), 1609–1659 (2009)
13. Linial, N., Magen, A.: Least-distortion Euclidean embeddings of graphs: products of cycles and expanders. J. Comb. Theory Ser. B **79**(2), 157–171 (2000)
14. Lyons, R., Pemantle, R., Peres, Y.: Random walks on the lamplighter group. Ann. Probab. **24**, 1993–2006 (1996)
15. Mendel, M., Naor, A.: Metric cotype. Ann. Math. **168**, 247–298 (2008)
16. Naor, A.: L_1 embeddings of the Heisenberg group and fast estimation of graph isoperimetry. In: Proceedings of the International Congress of Mathematicians, vol. 3, pp. 1549–1575. Hindustan Book Agency, New Delhi (2010)
17. Naor, A., Peres, Y.: Embeddings of discrete groups and the speed of random walks. Int. Math. Res. Not. IMRN **2008**, # rnn076 (2008)

18. Naor, A., Peres, Y.: L_p compression, traveling salesmen, and stable walks. Duke Math. J. **157**(1), 53–108 (2011)
19. Ostrovskii, M.I.: On metric characterizations of some classes of Banach spaces. C. R. Acad. Bulgare Sci. **64**(6), 775–784 (2011)
20. Ostrovskii, M.I.: Metric characterizations of superreflexivity in terms of word hyperbolic groups and finite graphs. Anal. Geom. Metric Spaces **2**(1), 154–168 (2014)
21. Ostrovskii, M.I., Randrianantoanina, B.: A characterization of superreflexivity through embeddings of lamplighter groups. Proc. Am. Math. Soc. **147**(11), 4745–4755 (2019)
22. Pisier, G., Xu, Q.H.: Random series in the real interpolation spaces between the spaces v_p. In: Geometrical Aspects of Functional Analysis (1985/86). Lecture Notes in Math., vol. 1267, pp. 185–209. Springer, Berlin (1987)
23. Stein, M., Taback, J.: Metric properties of Diestel–Leader groups. Mich. Math. J. **62**(2), 365–386 (2013)

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