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Mean field dynamics of a quantum tracer particle interacting with a boson gas



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ABSTRACT

We consider the dynamics of a heavy quantum tracer particle coupled to a non-relativistic boson field in \mathbb{R}^3 . The pair interactions of the bosons are of mean-field type, with coupling strength proportional to $\frac{1}{N}$ where N is the expected particle number. Assuming that the mass of the tracer particle is proportional to N , we derive generalized Hartree equations in the limit $N \rightarrow \infty$. Moreover, we prove the global well-posedness of the associated Cauchy problem for sufficiently weak interaction potentials.

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1. Introduction of the model

We consider a heavy quantum mechanical tracer particle coupled to a field of identical scalar bosons with two-particle interactions. The Hilbert space for the quantum tracer particle (with position variable $X \in \mathbb{R}^3$) is given by $L^2(\mathbb{R}^3)$. The boson Fock space is given by

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$$\mathcal{F} = \mathbb{C} \oplus \bigoplus_{n \geq 1} \mathcal{F}_n \quad (1.1)$$

where

$$\mathcal{F}_n := (L^2(\mathbb{R}^3))^{\otimes_{sym} n} \quad (1.2)$$

is the n -particle Hilbert space. We denote the Fock vacuum by $\Omega \in \mathcal{F}$, and introduce creation- and annihilation operators satisfying the canonical commutation relations

$$[a_x, a_y^+] = \delta(x - y) \quad , \quad [a_x, a_y] = 0 \quad , \quad [a_x^*, a_y^*] = 0 \quad , \quad (1.3)$$

where $a_x \Omega = 0$ for all $x \in \mathbb{R}^3$. The Hilbert space of the coupled system is given by

$$\mathfrak{H} = L^2(\mathbb{R}^3) \otimes \mathcal{F}. \quad (1.4)$$

We will study the time evolution of this system for initial data $\Phi_0 \in \mathcal{H}$ with a large but finite expected particle number, $\langle \Phi_0, \mathbf{1} \otimes N_b \Phi_0 \rangle = N$, where

$$N_b := \int dx a_x^+ a_x \quad (1.5)$$

is the boson number operator. We assume that the bosons interact with one another via a mean field interaction potential $\frac{\lambda}{2N}v$, where $\lambda > 0$ is a coupling constant. Moreover, we assume that the mass of the heavy tracer particle is proportional to N . Accordingly, the Hamiltonian of the system is given by

$$\begin{aligned} \mathcal{H}_N := & -\frac{1}{2N} \Delta_X \otimes \mathbf{1} + \mathbf{1} \otimes T + \int dx w(X - x) \otimes a_x^+ a_x \\ & + \mathbf{1} \otimes \frac{\lambda}{2N} \int dx dy a_x^+ a_x v(x - y) a_y^+ a_y \end{aligned} \quad (1.6)$$

where

$$T := \frac{1}{2} \int dx a_x^+ (-\Delta_x a_x) \quad (1.7)$$

is the kinetic energy operator for the boson field, w is the potential energy accounting for the coupling between the tracer particle and the bosons, and $\frac{\lambda}{2N}v$ is the potential accounting for pair interactions between bosons.

This system exhibits a close formal similarity to the translation-invariant model in non-relativistic Quantum Electrodynamics (QED) describing a quantum mechanical electron coupled to the quantized electromagnetic radiation field. The framework that we will use in our analysis is strongly inspired by [20,21] and [2,3,6,7,9,8].

We define the conserved total momentum operator

$$P_{tot} = i\nabla_X \otimes \mathbf{1} + \mathbf{1} \otimes P_b \quad (1.8)$$

where

$$P_b := \int dx a_x^+ (i\nabla_x a_x) \quad (1.9)$$

is the momentum operator for the boson field. The Hamiltonian is translation invariant, $[\mathcal{H}_N, P_{tot}] = 0$. Accordingly, we consider the decomposition of \mathfrak{H} as a fiber integral

$$\mathfrak{H} = \int_{\mathbb{R}^3}^{\oplus} dk \mathfrak{H}_k \quad (1.10)$$

with respect to P_{tot} where the fiber Hilbert spaces \mathfrak{H}_k are isomorphic to \mathcal{F} , and invariant under the dynamics generated by the Hamiltonian \mathcal{H}_N .

For each fixed $k \in \mathbb{R}^3$, we consider the value Nk of the conserved total momentum P_{tot} . The restriction of \mathcal{H}_N to \mathfrak{H}_k is given by the fiber Hamiltonian

$$\begin{aligned} \mathcal{H}_N(k) &:= \frac{1}{2N} (Nk - P_b)^2 + T + \int dx w(x) a_x^+ a_x \\ &\quad + \frac{\lambda}{2N} \int dx dy a_x^+ a_x v(x - y) a_y^+ a_y \\ &=: \frac{1}{2N} (Nk - P_b)^2 + T + W_1(0) + W_2, \end{aligned} \quad (1.11)$$

where

$$W_1(y) := \int dx w(x - y) a_x^+ a_x. \quad (1.12)$$

We also introduce

$$W := W_1(0) + W_2 \quad (1.13)$$

for notational convenience. We note that here, x (under a slight abuse of notation) stands for the relative coordinate $x - X$, with origin located at $X = 0$. For a more detailed introduction to the fiber decomposition with respect to the conserved total momentum, we refer to [2].

We will in the sequel identify \mathfrak{H} with $L^2(\mathbb{R}^3, \mathcal{F})$, and omit the tensor products from the notation in (1.6). The solution of the Schrödinger equation on \mathfrak{H} has the following form.

Proposition 1.1. *Given $u \in L^2(\mathbb{R}^3)$ and $\Psi_{k,0}^{\mathcal{F}} \in \mathcal{F}$, let*

$$\Phi_{u,0}(X) := \int dk \widehat{u}(k) e^{iX \cdot (Nk - P_b)} \Psi_{k,0}^{\mathcal{F}} \in \mathfrak{H}. \quad (1.14)$$

Then,

$$\Phi_u(t, X) := \int dk \widehat{u}(k) e^{iX \cdot (Nk - P_b)} \Psi_k^{\mathcal{F}}(t) \quad (1.15)$$

solves

$$i\partial_t \Phi_u = \mathcal{H}_N \Phi_u \quad (1.16)$$

on \mathfrak{H} with initial data $\Phi_u(0, X) = \Phi_{u,0}(X) \in \mathfrak{H}$, iff $\Psi_k^{\mathcal{F}}(t) \in \mathcal{F}$ solves

$$i\partial_t \Psi_k^{\mathcal{F}}(t) = \mathcal{H}_N(k) \Psi_k^{\mathcal{F}}(t) \quad (1.17)$$

on \mathcal{F} with initial data $\Psi_k^{\mathcal{F}}(0) = \Psi_{k,0}^{\mathcal{F}} \in \mathcal{F}$.

Proof. We have

$$\begin{aligned} & \left(i\partial_t + \frac{1}{2N} \Delta_X \right) \Phi_u(t, X) \\ &= \int dk \widehat{u}(k) e^{iX \cdot (Nk - P_b)} \left(-\frac{1}{2N} (Nk - P_b)^2 \right. \\ & \quad \left. + \frac{1}{2N} (Nk - P_b)^2 + T + W \right) \Psi_k^{\mathcal{F}}(t) \\ &= \int dk \widehat{u}(k) \left(e^{-iX \cdot P_b} (T + W) e^{iX \cdot P_b} \right) e^{iX \cdot (Nk - P_b)} \Psi_k^{\mathcal{F}}(t). \end{aligned} \quad (1.18)$$

Clearly,

$$e^{-iX \cdot P_b} T e^{iX \cdot P_b} = T \quad (1.19)$$

and

$$e^{-iX \cdot P_b} a_x e^{iX \cdot P_b} = a_{x+X}, \quad e^{-iX \cdot P_b} a_x^+ e^{iX \cdot P_b} = a_{x+X}^+, \quad (1.20)$$

as one sees from

$$\begin{aligned} e^{-iX \cdot P_b} a_x^+ e^{iX \cdot P_b} &= \int dk e^{-ikx} e^{-iX \cdot P_b} a_k^+ e^{iX \cdot P_b} \\ &= \int dk e^{-ikx} a_k^+ e^{-iX \cdot (P_b + k)} e^{iX \cdot P_b} \\ &= \int dk e^{-ik(x+X)} a_k^+. \end{aligned} \quad (1.21)$$

Therefore,

$$\begin{aligned} e^{-iX \cdot P_b} W_1 e^{iX \cdot P_b} &= \int w(x) a_{x+X}^+ a_{x+X} dx \\ &= \int w(x-X) a_x^+ a_x dx = W_1(X), \end{aligned} \quad (1.22)$$

and

$$\begin{aligned} e^{-iX \cdot P_b} W_2 e^{iX \cdot P_b} &= \frac{\lambda}{2N} \int a_{x+X}^+ a_{x+X} v(x-y) a_{y+X}^+ a_{y+X} dx dy \\ &= \frac{\lambda}{2N} \int a_x^+ a_x v(x-X-(y-X)) a_y^+ a_y dx dy \\ &= W_2. \end{aligned} \quad (1.23)$$

We thus conclude that

$$\left(i\partial_t + \frac{1}{2N} \Delta_X \right) \Phi_u(t, X) = (T + W_1(X) + W_2) \Phi_u(t, X), \quad (1.24)$$

as claimed in (1.16). \square

The main results proven in this paper can be summarized as follows.

1.1. The fiber ground state on \mathcal{F}_N for large N

The fiber Hamiltonian $\mathcal{H}_N(k)$ commutes with the number operator N_b . In Section 2, we study its restriction to the N -particle Fock space \mathcal{F}_N , and prove that the fiber ground state energy,

$$\begin{aligned} E_N(k) &:= \inf_{\text{spec}} \left(H_N(k) \Big|_{\mathcal{F}_N} \right) \\ &= \inf_{\Psi_N \in \mathcal{F}_N; \|\Psi\|_{\mathcal{F}_N}=1} \left\langle \Psi_N, H_N(k) \Psi_N \right\rangle, \end{aligned} \quad (1.25)$$

satisfies the asymptotics

$$\lim_{N \rightarrow \infty} \frac{E_N(k)}{N} = \frac{k^2}{4} + \inf_{\|\phi\|_{L^2}=1} \mathcal{E}_0[\phi] \quad (1.26)$$

where

$$\begin{aligned} \mathcal{E}_0[\phi] &:= \frac{1}{2} \int dx |\nabla \phi(x)|^2 + \int dx w(x) |\phi(x)|^2 \\ &\quad + \frac{\lambda}{2} \int dx dy |\phi(x)|^2 v(x-y) |\phi(y)|^2 \end{aligned} \quad (1.27)$$

is the Hartree energy functional; see Proposition 2.1.

The problem considered here corresponds to a mean field limit; see [36] and the references quoted therein. For the much more difficult case of a dilute gas and the Gross–Pitaevskii limit, we refer to [38,40,39].

1.2. Coherent states and mean field limit as $N \rightarrow \infty$

In Section 3, we derive the mean field limit of (1.17) in the following sense. We define the Weyl operator

$$\mathcal{W}[\sqrt{N}\phi] := \exp \left(\sqrt{N} \int dx \left(\phi(x) a_x^+ - \overline{\phi(x)} a_x \right) \right), \quad (1.28)$$

and consider the solution of the Schrödinger equation (1.16) with initial data given by a coherent state of the form

$$\mathcal{W}[\sqrt{N}\phi_0] \Omega = \exp \left(-\frac{N}{2} \|\phi_0\|_2^2 \right) \exp \left(\sqrt{N} \int dx \phi_0(x) a_x^+ \right) \Omega, \quad (1.29)$$

for $\phi_0 \in H^1(\mathbb{R}^3)$.

For an arbitrary but fixed value $k \in \mathbb{R}^3$ of the conserved momentum, we assume that for some $T > 0$, $\phi_t \in L_t^\infty H_x^3([0, T] \times \mathbb{R}^3)$ is the solution of

$$i\partial_t \phi_t = - \left(k - (\phi_t, i\nabla \phi_t) \right) i\nabla \phi_t - \frac{1}{2} \Delta \phi_t + w\phi_t + \lambda(v * |\phi_t|^2) \phi_t, \quad (1.30)$$

with initial data $\phi_0 \in H_x^3(\mathbb{R}^3)$. We introduce the scalar

$$\begin{aligned} S(t, t') := N \int_{t'}^t ds \left(-\frac{1}{2} k^2 + \frac{1}{2} (\phi_s, i\nabla \phi_s)^2 \right. \\ \left. + \frac{\lambda}{2} \int |\phi_s(x)|^2 v(x-y) |\phi_s(y)|^2 dx dy \right), \end{aligned} \quad (1.31)$$

and a self-adjoint Hamiltonian quadratic in creation- and annihilation operators, of the form

$$\mathcal{H}_{mf}^{\phi_t}(k) := \mathcal{H}_{Har}^{\phi_t}(k) + \mathcal{H}_{cor}^{\phi_t} \quad (1.32)$$

where

$$\begin{aligned} \mathcal{H}_{Har}^{\phi_t}(k) := & - \left(k - (\phi_t, i\nabla \phi_t) \right) \cdot P_b + T + W_1(0) \\ & + \lambda \int |\phi_t(x)|^2 v(x-y) a_y^+ a_y dx dy \end{aligned} \quad (1.33)$$

is a generalized Hartree Hamiltonian which commutes with the particle number operator N_b , and where

$$\begin{aligned}
\mathcal{H}_{cor}^{\phi_t} &:= \frac{1}{2} \left(a^+ (i\nabla \phi_t) + a(i\nabla \phi_t) \right)^2 \\
&\quad + \lambda \int v(x-y) \phi_t(x) \overline{\phi_t(y)} a_x^+ a_y dx dy \\
&\quad + \frac{\lambda}{2} \int v(x-y) \left(\phi_t(x) \phi_t(y) a_x^+ a_y^+ + \overline{\phi_t(x) \phi_t(y)} a_y a_x \right) dx dy, \quad (1.34)
\end{aligned}$$

includes correlations which do not preserve the particle number. Then,

$$i\partial \mathcal{V}(t, s) = \mathcal{H}_{mf}^{\phi_t}(k) \mathcal{V}(t, s) \quad , \quad \mathcal{V}(s, s) = \mathbf{1}, \quad (1.35)$$

determines the unitary flow $\mathcal{V}(t, s)$ generated by $\mathcal{H}_{mf}^{\phi_t}(k)$.

Our main result in section 3 states that the limit

$$\lim_{N \rightarrow \infty} \left\| e^{-it\mathcal{H}_N(k)} \mathcal{W}[\sqrt{N}\phi_0] \Omega - e^{-iS(t,0)} \mathcal{W}[\sqrt{N}\phi_t] \mathcal{V}(t, 0) \Omega \right\|_{\mathcal{F}} = 0 \quad (1.36)$$

holds, under the assumption $v \in C^2(\mathbb{R}^3)$; see Theorem 3.1. The more technical parts of the proof are presented in Section 6. Therefore, the solution of (1.16), with initial data (1.14) characterized by a coherent state $\Psi_{k,0}^{\mathcal{F}} = \mathcal{W}[\sqrt{N}\phi_0] \Omega$, is given by

$$\Phi_u(t, X) = \int dk \widehat{u}(k) e^{iX \cdot (Nk - P_b)} e^{-iS(t,0)} \mathcal{W}[\sqrt{N}\phi_t] \mathcal{V}(t, 0) \Omega + o_N(1) \quad (1.37)$$

asymptotically, as $N \rightarrow \infty$.

The convergence in Fock space proven in our work is closely related to [32,30,31]. In our construction, the generator $\mathcal{H}_{mf}^{\phi_t}(k)$ of $\mathcal{V}(t, s)$, defined in (1.32), includes both a diagonal and an off-diagonal part, while in [32,30,31] the term analogous to $\mathcal{V}(t, s)$ has a purely off-diagonal generator. In the case considered here, the choice (1.35) with (1.32) allows us to efficiently control the operator $\frac{1}{N}(Nk - P_b)^2$ in \mathcal{H}_N which is not present in the above mentioned works. The heaviness of the mass of the tracer particle, with mass proportional to N , plays an important role because it ensures that the kinetic energies of tracer particle and boson field are comparable. The Hamiltonian $\mathcal{H}_{cor}^{\phi_t}$ comprises the $O_N(1)$ terms of the quasifree reduction of \mathcal{H}_N , in the sense of [1]. After completing this work, we noticed that a partially similar construction was used in [37] where the authors study the convergence rate in a mean field limit without tracer particle.

There is a vast literature on the derivation of time-dependent Hartree or nonlinear Schrödinger equations from an interacting boson field with mean field or Gross–Pitaevskii scaling. The first rigorous results were obtained in the pioneering papers [33] and (a few years later) [45,46]; more recently, the works [16–19], and subsequently [35] and [43], motivated much of the current activity in the research area; we refer to [10,13,14,11,12, 25–29,32,30,31,34,36,37,42].

The mean field limit of a *classical* tracer particle coupled to a nonlinear Hartree equation was derived in [15].

1.3. Analysis of the mean field equations

In sections 4 and 5, we analyze the mean field equation (1.30). In Section 4, we determine the ground state for a conserved total momentum $k \in \mathbb{R}^3$, under the assumption that the Hartree energy functional for $k = 0$ admits a minimizer; see Proposition 4.1.

In Section 5, we study dispersive solutions to (1.30). In particular, we show that (1.30) is unitarily equivalent to the nonlinear Hartree equation

$$i\partial_t \psi = -\frac{1}{2}\Delta \psi + w_\psi \psi + \lambda(v * |\psi|^2)\psi \quad , \quad \psi(t=0) = \psi_0 \equiv \phi_0 \in H_x^1(\mathbb{R}^3) \quad (1.38)$$

where

$$w_\psi(t, x) := w\left(x - X_\psi(t)\right) \quad (1.39)$$

and

$$X_\psi(t) := \int_0^t ds \left(k - (\psi, i\nabla \psi)(s) \right) \quad (1.40)$$

is the expected trajectory of the tracer particle, with respect to an average over its interactions with the boson field.

In Theorem 5.5, we prove the global well-posedness for (1.38) in the space

$$\psi \in L_t^\infty H_x^1(\mathbb{R} \times \mathbb{R}^3) \cap L_t^{\frac{10}{3}} W_x^{1, \frac{10}{3}}(\mathbb{R} \times \mathbb{R}^3), \quad (1.41)$$

under the assumption that $\|w\|_{W_x^{2, \frac{3}{2}}} < \infty$, and that $\|w\|_{W_x^{1, \frac{3}{2}}}, \|\lambda v\|_{W_x^{1, \frac{3}{2}}}$ are sufficiently small. As a corollary, we obtain global well-posedness for (1.30), in the sense stated in Theorem 5.1; in particular, $\phi(t, x) = \psi(t, x + X_\psi(t))$. We note that under less restrictive assumptions on w and λv , the problem can be controlled with methods developed in [4]. We also note that $X_\psi(t)$ can be written as $X_\psi(t) = kt - (\psi, x\psi)$, and that it satisfies the Ehrenfest dynamics

$$\begin{aligned} \partial_t^2 X_\psi(t) &= \left(\psi, \nabla(w_\psi + \lambda v * |\psi|^2)\psi \right) \\ &= \int dx (\nabla w)(x - X_\psi(t)) |\psi(x)|^2. \end{aligned} \quad (1.42)$$

The term involving v is zero because v is even, see Remark 5.3, below.

In particular, we find $\partial_t^2 X_\psi(t) = 0$ in the special case where $\phi_0 = Q_k$ is the nonlinear ground state of (1.30), with $\|Q_k\|_{L_x^2} = 1$. This is because we have $Q_k = e^{-i\frac{k}{2}x} Q_0$ where Q_0 is the rotationally symmetric minimizer of the Hartree functional (2.4), with $\|Q_0\|_{L_x^2} = 1$; see Proposition 4.1. Due to rotational symmetry of Q_0 , we find that $X_\psi(t) = \frac{k}{2}t$, and that with $\psi(t, x) = Q_k(x - \frac{k}{2}t)$, the r.h.s. of (1.42) is zero, so that $\partial_t^2 X_\psi(t) = 0$.

Clearly, (1.42) describes a *classical* tracer particle moving along the trajectory $X_\psi(t) \in \mathbb{R}^3$, coupled to a boson field described by the Hartree equation (1.38). A model of a similar type has been analyzed in [22], for which the emergence of Hamiltonian friction was established in certain cases in [23,24].

2. The fiber ground state on \mathcal{F}_N for large N

The fiber Hamiltonian $\mathcal{H}_N(k)$ commutes with the number operator N_b , and its restriction to the N -particle Fock space \mathcal{F}_N is given by

$$H_N(k) := \mathcal{H}_N(k) \Big|_{\mathcal{F}_N}. \quad (2.1)$$

In this section, we will determine the asymptotics of its ground state energy in the limit of large N ; see [5,41,44] for references. We define

$$\begin{aligned} E_N(k) &:= \inf_{\text{spec}} \left(H_N(k) \Big|_{\mathcal{F}_N} \right) \\ &= \inf_{\Psi_N \in \mathcal{F}_N; \|\Psi\|_{\mathcal{F}_N}=1} \left\langle \Psi_N, H_N(k) \Psi_N \right\rangle. \end{aligned} \quad (2.2)$$

For large N , the following asymptotics hold.

Proposition 2.1. *The ground state energy of the fiber Hamiltonian $H_N(k)$ satisfies*

$$\lim_{N \rightarrow \infty} \frac{E_N(k)}{N} = \frac{k^2}{4} + \inf_{\|\phi\|_{L^2}=1} \mathcal{E}_0[\phi] \quad (2.3)$$

where

$$\begin{aligned} \mathcal{E}_0[\phi] &:= \frac{1}{2} \int dx |\nabla \phi(x)|^2 + \int dx w(x) |\phi(x)|^2 \\ &\quad + \frac{\lambda}{2} \int dx dy |\phi(x)|^2 v(x-y) |\phi(y)|^2 \end{aligned} \quad (2.4)$$

is the Hartree energy functional.

Proof. Let

$$(\tau_{\frac{k}{2}} \Psi_N)(x_1, \dots, x_N) := \exp \left(-\frac{ik}{2} \sum_{j=1}^N x_j \right) \Psi_N(x_1, \dots, x_N). \quad (2.5)$$

Then, the kinetic energy part in (1.11) yields

$$\begin{aligned} &\left\langle \tau_{\frac{k}{2}} \Psi_N, \left(\frac{1}{2N} (Nk - P_b)^2 + T \right) \tau_{\frac{k}{2}} \Psi_N \right\rangle \\ &= \int \left(\frac{1}{2N} \left(Nk - \frac{Nk}{2} - \sum_{j=1}^N k_j \right)^2 + \frac{1}{2} \sum_{j=1}^N \left(k_j + \frac{k}{2} \right)^2 \right) |\widehat{\Psi}_N(\underline{k}_N)|^2 d\underline{k}_N \end{aligned}$$

$$\begin{aligned}
&= \int \left(\frac{Nk^2}{4} + \frac{1}{2N} \left(\sum_{j=1}^N k_j \right)^2 + \frac{1}{2} \sum_{j=1}^N k_j^2 \right) |\widehat{\Psi}_N(\underline{k}_N)|^2 d\underline{k}_N \\
&= \frac{Nk^2}{4} + \left\langle \Psi_N, \left(\frac{1}{2N} P_b^2 + T \right) \Psi_N \right\rangle,
\end{aligned} \tag{2.6}$$

where $\underline{k}_N := (k_1, \dots, k_N)$ and $d\underline{k}_N := dk_1 \cdots dk_N$, while the interaction part yields

$$\left\langle \tau_{\frac{k}{2}} \Psi_N, W \tau_{\frac{k}{2}} \Psi_N \right\rangle = \left\langle \Psi_N, W \Psi_N \right\rangle. \tag{2.7}$$

Thus, we obtain the lower bound

$$\begin{aligned}
E_N(k) &= \inf_{\Psi_N \in \mathcal{F}_N; \|\Psi_N\|_{\mathcal{F}_N}=1} \left\langle \Psi_N, H_N(k) \Psi_N \right\rangle \\
&= \inf_{\Psi_N \in \mathcal{F}_N; \|\Psi_N\|_{\mathcal{F}_N}=1} \left\langle \tau_{\frac{k}{2}} \Psi_N, H_N(k) \tau_{\frac{k}{2}} \Psi_N \right\rangle \\
&= \frac{Nk^2}{4} + \inf_{\Psi_N \in \mathcal{F}_N; \|\Psi_N\|_{\mathcal{F}_N}=1} \left\langle \Psi_N, \left(\frac{1}{2N} P_b^2 + H_N(0) \right) \Psi_N \right\rangle \\
&\geq \frac{Nk^2}{4} + E_N(0).
\end{aligned} \tag{2.8}$$

Next, we determine an upper bound.

Let

$$\Psi_{N,\phi} := \frac{1}{\sqrt{N!}} (a^+(\phi))^N \Omega \tag{2.9}$$

where $\phi \in H^1(\mathbb{R}^3)$ and $\|\phi\|_{L^2} = 1$. We choose

$$\phi = e^{-i\frac{k}{2}x} Q_0 \tag{2.10}$$

where Q_0 is the minimizer of the Hartree functional (2.4) with mass $\|Q_0\|_{L^2} = 1$. Then, we find that

$$\begin{aligned}
&\left\langle \Psi_{N,\phi}, H_N(0) \Psi_{N,\phi} \right\rangle \\
&= \frac{Nk^2}{4} + \left\langle \Psi_{N,Q_0}, \frac{1}{2N} P_b^2 \Psi_{N,Q_0} \right\rangle + \left\langle \Psi_{N,Q_0}, H_N(0) \Psi_{N,Q_0} \right\rangle \\
&= \frac{Nk^2}{4} + \left\langle \Psi_{N,Q_0}, \frac{1}{2N} P_b^2 \Psi_{N,Q_0} \right\rangle + N\mathcal{E}_0[Q_0].
\end{aligned} \tag{2.11}$$

The minimizer of $\mathcal{E}_0[\phi]$ is rotation symmetric, therefore

$$\int k |\widehat{Q}_0(k)|^2 dk = 0, \tag{2.12}$$

and we have

$$\begin{aligned}
 & \left\langle \Psi_{N,Q_0}, \frac{1}{2N} P_b^2 \Psi_{N,Q_0} \right\rangle \\
 &= \int \frac{1}{2N} \left(\sum_{j=1}^N k_j \right)^2 |\widehat{Q}_0(k_1)|^2 \cdots |\widehat{Q}_0(k_N)|^2 d\mathbf{k}_N \\
 &= \int \frac{1}{2N} \left(\sum_{j=1}^N k_j^2 \right) |\widehat{Q}_0(k_1)|^2 \cdots |\widehat{Q}_0(k_N)|^2 d\mathbf{k}_N \\
 &= \frac{1}{2} \int k^2 |\widehat{Q}_0(k)|^2 dk < \infty.
 \end{aligned} \tag{2.13}$$

Passing to the third line, we used that all terms whose integrands are proportional to $k_j \cdot k_\ell$, with $j \neq \ell$, vanish, due to (2.12). Notably, the term (2.13) is $O(\frac{1}{N})$ smaller than the other two terms on the last line of (2.11), and we conclude that

$$\frac{k^2}{4} + \frac{E_N(0)}{N} \leq \frac{E_N(k)}{N} \leq \frac{k^2}{4} + \frac{1}{2N} \int k^2 |\widehat{Q}_0(k)|^2 dk + \mathcal{E}_0[Q_0]. \tag{2.14}$$

But as was proven in [36],

$$\lim_{N \rightarrow \infty} \frac{E_N(0)}{N} = \mathcal{E}_0[Q_0]. \tag{2.15}$$

Hence, (2.14) implies that

$$\lim_{N \rightarrow \infty} \frac{E_N(k)}{N} = \frac{k^2}{4} + \mathcal{E}_0[Q_0], \tag{2.16}$$

as claimed. \square

3. Coherent states and mean field limit as $N \rightarrow \infty$

In this section, we derive the dynamical mean field limit on Fock space. In order to render the exposition more readable, we are presenting the core of the proof here, but provide the more technical parts of the proof later, in Section 6. In this paper, we are neither attempting to optimize the bounds on the convergence rates, nor the requirements on the potentials w and v .

Let $\phi \in L^2(\mathbb{R}^3)$. In this section, it will be convenient to use the notation $\phi_t(x) \equiv \phi(t, x)$. We define the Weyl operator

$$\mathcal{W}[\sqrt{N}\phi] := \exp \left(\sqrt{N} \int dx \left(\phi(x) a_x^+ - \overline{\phi(x)} a_x \right) \right) \tag{3.1}$$

We consider the solution of the Schrödinger equation (1.16) with initial data given by a coherent state of the form

$$\mathcal{W}[\sqrt{N}\phi_0]\Omega = \exp\left(-\frac{N}{2}\|\phi_0\|_2^2\right)\exp\left(\sqrt{N}\int dx\phi_0(x)a_x^+\right)\Omega, \quad (3.2)$$

for $\phi_0 \in H^1(\mathbb{R}^3)$.

Moreover, we define a time-dependent mean-field Hamiltonian which is self-adjoint and quadratic in creation- and annihilation operators, of the form

$$\mathcal{H}_{mf}^{\phi_t}(k) := \mathcal{H}_{Har}^{\phi_t}(k) + \mathcal{H}_{cor}^{\phi_t} \quad (3.3)$$

where the “diagonal term”

$$\begin{aligned} \mathcal{H}_{Har}^{\phi_t}(k) := & -\left(k - (\phi_t, i\nabla\phi_t)\right) \cdot P_b + T + W_1(0) \\ & + \lambda \int |\phi_t(x)|^2 v(x-y) a_y^+ a_y dx dy \end{aligned} \quad (3.4)$$

is the Hartree Hamiltonian which commutes with the particle number operator N_b , and where the “off-diagonal term”

$$\begin{aligned} \mathcal{H}_{cor}^{\phi_t} := & \frac{1}{2} \left(a^+(i\nabla\phi_t) + a(i\nabla\phi_t) \right)^2 \\ & + \lambda \int v(x-y) \phi_t(x) \overline{\phi_t(y)} a_x^+ a_y dx dy \end{aligned} \quad (3.5)$$

$$+ \frac{\lambda}{2} \int v(x-y) \left(\phi_t(x) \phi_t(y) a_x^+ a_y^+ + \overline{\phi_t(x) \phi_t(y)} a_y a_x \right) dx dy, \quad (3.6)$$

includes correlations which do not preserve the particle number. We obtain the unitary flow $\mathcal{V}(t, s)$,

$$i\partial_t \mathcal{V}(t, s) = \mathcal{H}_{mf}^{\phi_t}(k) \mathcal{V}(t, s) \quad , \quad \mathcal{V}(s, s) = \mathbf{1}, \quad (3.7)$$

generated by $\mathcal{H}_{mf}^{\phi_t}(k)$.

Theorem 3.1. *Let $k \in \mathbb{R}^3$. We assume that $v \in C^2(\mathbb{R}^3)$, and that for some $T > 0$, $\phi_t \in L_t^\infty H_x^3([0, T] \times \mathbb{R}^3)$ is the solution of*

$$i\partial_t \phi_t = -\left(k - (\phi_t, i\nabla\phi_t)\right) i\nabla\phi_t - \frac{1}{2} \Delta\phi_t + w\phi_t + \lambda(v * |\phi_t|^2)\phi_t, \quad (3.8)$$

with initial data $\phi_0 \in H_x^3(\mathbb{R}^3)$. Let

$$\begin{aligned} S(t, t') := & N \int_{t'}^t ds \left(-\frac{1}{2}k^2 + \frac{1}{2}(\phi_s, i\nabla\phi_s)^2 \right. \\ & \left. + \frac{\lambda}{2} \int |\phi_s(x)|^2 v(x-y) |\phi_s(y)|^2 dx dy \right). \end{aligned} \quad (3.9)$$

Then, the limit

$$\lim_{N \rightarrow \infty} \left\| e^{-it\mathcal{H}_N(k)} \mathcal{W}[\sqrt{N}\phi_0] \Omega - e^{-iS(t,0)} \mathcal{W}[\sqrt{N}\phi_t] \mathcal{V}(t,0) \Omega \right\|_{\mathcal{F}} = 0 \quad (3.10)$$

holds.

Proof. We have

$$\left\| e^{-it\mathcal{H}_N(k)} \mathcal{W}[\sqrt{N}\phi_0] \Omega - e^{-iS(t,0)} \mathcal{W}[\sqrt{N}\phi_t] \mathcal{V}(t,0) \Omega \right\|_{\mathcal{F}}^2 = 2(1 - M(t)) \quad (3.11)$$

where

$$\begin{aligned} M(t) &:= \operatorname{Re} \left\langle e^{-it\mathcal{H}_N(k)} \mathcal{W}[\sqrt{N}\phi_0] \Omega, e^{-iS(t,0)} \mathcal{W}[\sqrt{N}\phi_t] \mathcal{V}(t,0) \Omega \right\rangle \\ &= \operatorname{Re} \left\langle \Omega, \mathcal{W}^*[\sqrt{N}\phi_0] e^{it\mathcal{H}_N(k)} e^{-iS(t,0)} \mathcal{W}[\sqrt{N}\phi_t] \mathcal{V}(t,0) \Omega \right\rangle. \end{aligned} \quad (3.12)$$

One can easily verify that given (3.8), we have

$$i\partial_t \mathcal{W}[\sqrt{N}\phi_t] = [\mathcal{H}_{Har}^{\phi_t}(k), \mathcal{W}[\sqrt{N}\phi_t]]. \quad (3.13)$$

We consider the unitary flow

$$\mathcal{U}(t,s) := \mathcal{W}^*[\sqrt{N}\phi_s] e^{i(t-s)\mathcal{H}_N(k) - iS(t,s)} \mathcal{W}[\sqrt{N}\phi_t] \quad (3.14)$$

and introduce the selfadjoint operator

$$\begin{aligned} \mathcal{L}_N^{\phi_t}(k) &:= \mathcal{W}^*[\sqrt{N}\phi_t] \left(\mathcal{H}_N(k) - \partial_t S(t,0) \right) \mathcal{W}[\sqrt{N}\phi_t] \\ &\quad - \mathcal{W}^*[\sqrt{N}\phi_t] [\mathcal{H}_{Har}^{\phi_t}(k), \mathcal{W}[\sqrt{N}\phi_t]] - \mathcal{H}_{mf}^{\phi_t}(k) \\ &= \mathcal{W}^*[\sqrt{N}\phi_t] \mathcal{H}_N(k) \mathcal{W}[\sqrt{N}\phi_t] - \partial_t S(t,0) \\ &\quad - \mathcal{W}^*[\sqrt{N}\phi_t] \mathcal{H}_{Har}^{\phi_t}(k) \mathcal{W}[\sqrt{N}\phi_t] - \mathcal{H}_{cor}^{\phi_t}(k). \end{aligned} \quad (3.15)$$

Then, it is clear that

$$i\partial_t \left(\mathcal{U}_N(t,0) \mathcal{V}(t,0) \Omega \right) = -\mathcal{U}_N(t,0) \mathcal{L}_N^{\phi_t}(k) \mathcal{V}(t,0) \Omega. \quad (3.16)$$

A straightforward but somewhat lengthy calculation shows that

$$\begin{aligned} \mathcal{L}_N^{\phi_t}(k) &= \frac{1}{2\sqrt{N}} \left(P_b \cdot (a^+(i\nabla\phi_t) + a(i\nabla\phi_t)) + (a^+(i\nabla\phi_t) + a(i\nabla\phi_t)) \cdot P_b \right) \\ &\quad + \frac{1}{2N} P_b^2 \end{aligned}$$

$$\begin{aligned}
& + \frac{\lambda}{\sqrt{N}} \int v(x-y) a_x^+ \left(\overline{\phi_t(y)} a_y + \phi_t(y) a_y^+ \right) a_x dx dy \\
& + \frac{\lambda}{2N} \int v(x-y) a_x^+ a_y^+ a_y a_x dx dy.
\end{aligned} \tag{3.17}$$

This follows from Lemma 6.1.

Evidently,

$$\begin{aligned}
M(t) &= \operatorname{Re} \left\langle \Omega, \mathcal{U}_N(t, 0) \mathcal{V}(t, 0) \Omega \right\rangle \\
&= M(0) + \operatorname{Re} \int_0^t ds \partial_s M(s) \\
&= 1 - \operatorname{Re} \left\{ i \int_0^t ds \left\langle \Omega, \mathcal{U}_N(w, 0) \mathcal{L}_N^{\phi_s}(k) \mathcal{V}(s, 0) \Omega \right\rangle \right\}.
\end{aligned} \tag{3.18}$$

It follows from the unitarity of $\mathcal{U}_N(t, 0)$ that

$$\left| \left\langle \Omega, \mathcal{U}_N(t, 0) \mathcal{L}_N^{\phi_t}(k) \mathcal{V}(t, 0) \Omega \right\rangle \right| \leq \left\| \mathcal{L}_N^{\phi_t}(k) \mathcal{V}(t, 0) \Omega \right\|_{\mathcal{F}}. \tag{3.19}$$

We prove in Lemma 6.2 that

$$\left\| \mathcal{L}_N^{\phi_t}(k) \mathcal{V}(t, 0) \Omega \right\|_{\mathcal{F}} \leq C_0 \frac{e^{C_1 t}}{\sqrt{N}}, \tag{3.20}$$

for some constants C_0, C_1 depending on $\|v\|_{C^2(\mathbb{R}^3)}$ and $\|\phi_t\|_{L_t^\infty H_x^3([0, T] \times \mathbb{R}^3)}$. Hence, we find that

$$|M(t) - 1| \leq C_0 \int_0^t \frac{e^{C_1 s}}{\sqrt{N}} ds < \frac{C_0}{C_1} \frac{e^{C_1 t}}{\sqrt{N}}. \tag{3.21}$$

We therefore conclude that for any $t > 0$, the lhs of (3.11) converges to zero in the limit $N \rightarrow \infty$. \square

For the convergence to the mean field dynamics in Theorem 3.1, we required that the solution to the mean field equations obtained in (3.8),

$$i\partial_t \phi_t = - \left(k - (\phi_t, i\nabla \phi_t) \right) i\nabla \phi_t - \frac{1}{2} \Delta \phi_t + w \phi_t + \lambda (v * |\phi_t|^2) \phi_t, \tag{3.22}$$

has regularity $\phi \in L_t^\infty H_x^3([0, T] \times \mathbb{R}^3)$, for $T > 0$ (possibly $T = \infty$).

In Section 4 and Section 5, we study solutions to (3.22) under less strict assumptions on the regularity, $\phi \in L_t^\infty H_x^1$, in the following two cases. In Section 4, we assume that

the interaction potentials w and v are such that the standard Hartree equation with external potential w possesses a ground state, and we construct the ground state solution to (3.22). In Section 5, we assume that w and v do not allow for bound states. In this situation, we are considering dispersive solutions of (3.22), and prove local and global well-posedness under the assumption that $\|w\|_{W^{2,3/2}} < \infty$, and that $\|w\|_{W^{1,3/2}}$, $\|\lambda v\|_{W^{1,3/2}}$ are sufficiently small.

Remark 3.2. Global dispersive solutions to (3.22) with regularity $\phi \in L_t^\infty H_x^3(\mathbb{R} \times \mathbb{R}^3)$ can be constructed along the same lines as in our analysis in section 5, under the assumption that $\|w\|_{W^{4,3/2}} < \infty$, and that $\|w\|_{W^{3,3/2}}$, $\|\lambda v\|_{W^{3,3/2}}$ are sufficiently small. Because the arguments are straightforward, we leave this part as an exercise.

Remark 3.3. For comparison, we note that the model usually encountered in the literature, describing the boson gas without tracer particle, is given by

$$\tilde{\mathcal{H}}_N := T + \int dx w(x) a_x^+ a_x + \frac{\lambda}{2N} \int dx dy a_x^+ a_x v(x-y) a_y^+ a_y. \quad (3.23)$$

That is, no term $\frac{1}{2N}(Nk - P_b)^2$ appears here, [33,43,32].

Our constructions in this section apply to this case, too, but simplify. The mean field equation for ϕ is the standard Hartree equation,

$$i\partial_t \phi = -\Delta \phi + w\phi + \lambda(v * |\phi|^2)\phi$$

with $\phi(0) = \phi_0 \in H_x^1$.

The first term on the right hand side of (3.4) and the first term on the right hand side of (3.5) are then absent. Moreover, the terms on the first two lines on the r.h.s. of (3.17) are absent, yielding

$$\begin{aligned} \tilde{\mathcal{L}}_N^{\phi_t} &:= \frac{\lambda}{\sqrt{N}} \int v(x-y) a_x^+ \left(\overline{\phi_t(y)} a_y + \phi_t(y) a_y^+ \right) a_x dx dy \\ &\quad + \frac{\lambda}{2N} \int v(x-y) a_x^+ a_y^+ a_y a_x dx dy. \end{aligned} \quad (3.24)$$

As a consequence, a bound of the form (3.20),

$$\left\| \tilde{\mathcal{L}}_N^{\phi_t} \mathcal{V}(t, 0) \Omega \right\|_{\mathcal{F}} \leq C_0 \frac{e^{C_1 t}}{\sqrt{N}}, \quad (3.25)$$

follows from Lemma 6.3, but Lemma 6.2 does not need to be invoked. In particular, the constants C_0, C_1 depend only on $\|\phi\|_{L_t^\infty H_x^1(I \times \mathbb{R}^3)}$; that is, only H_x^1 -regularity of ϕ is required, not H_x^3 -regularity. This implies a convergence result analogous to (3.10), but only requiring that $\|\phi\|_{L_t^\infty H_x^1(I \times \mathbb{R}^3)} < \infty$. Clearly, $I = \mathbb{R}$, if the Hartree equation for ϕ is globally well-posed.

4. Ground state for the generalized Hartree functional

We introduce the generalized Hartree energy functional corresponding to an arbitrary but fixed conserved momentum $k \in \mathbb{R}^3$,

$$\begin{aligned}\mathcal{E}_k[\phi] &:= \frac{1}{N} \left\langle \Phi_{N,\phi}, \mathcal{H}_N(k) \Phi_{N,\phi} \right\rangle \\ &= \frac{1}{2} \left(k - \int dx \overline{\phi(x)} i \nabla_x \phi(x) \right)^2 + \frac{1}{2} \int dx |\nabla \phi(x)|^2 \\ &\quad + \int dx w(x) |\phi(x)|^2 + \frac{\lambda}{2} \int dx dy |\phi(x)|^2 v(x-y) |\phi(y)|^2 \\ &= \frac{1}{2} \left(k - (\phi, i \nabla \phi) \right)^2 + \mathcal{E}_0[\phi].\end{aligned}\tag{4.1}$$

Here, $\mathcal{E}_0[\phi]$ is the standard Hartree energy functional with external potential w . We note that the scaling in the model is chosen in such a manner that $\mathcal{E}_k[\phi]$ is independent of N .

Let $Q_0^{(\mu)}$ denote the minimizer of

$$\mathcal{S}_0[\phi] = \mathcal{E}_0[\phi] - \mu \|\phi\|_{L^2}^2\tag{4.2}$$

where μ is a Lagrange multiplier (the chemical potential) implementing the constraint that the L^2 -mass $\|\phi\|_{L^2}^2 = M$ is constant. Accordingly, μ depends on M , and we denote by μ_0 the value of μ for which $M = \|Q_0^{(\mu_0)}\|_{L^2}^2 = 1$. For brevity, we write $Q_0 := Q_0^{(\mu_0)}$, as we will only consider the case $M = 1$. By variation of (4.2) in ϕ , it follows that Q_0 satisfies the stationary Hartree equation

$$\mu_0 Q_0 = -\frac{1}{2} \Delta Q_0 + w Q_0 + \lambda (v * |Q_0|^2) Q_0\tag{4.3}$$

where the value of μ_0 is obtained from taking the inner product of (4.3) with Q_0 .

Likewise, for a nonzero total conserved momentum $k \in \mathbb{R}^3$, we consider the minimizer Q_k of

$$\mathcal{S}_k[\phi] := \mathcal{E}_k[\phi] - \mu \|\phi\|_{L^2}^2,\tag{4.4}$$

under the constraint condition $\|Q_k\|_{L^2}^2 = 1$, and we denote the corresponding value of the chemical potential by μ_k . By variation in ϕ , it follows that the minimizer Q_k satisfies

$$\mu_k Q_k = -\left(k - (Q_k, i \nabla Q_k)\right) i \nabla Q_k - \frac{1}{2} \Delta Q_k + w Q_k + \lambda (v * |Q_k|^2) Q_k\tag{4.5}$$

with $\|Q_k\|_{L^2} = 1$. The value of μ_k is obtained from taking the inner product of (4.5) with Q_k .

Proposition 4.1. *The vector*

$$Q_k := e^{-i\frac{k \cdot x}{2}} Q_0 \quad (4.6)$$

minimizes $\mathcal{S}_0[\phi] = \mathcal{E}_k[\phi] - \mu \|\phi\|_{L^2}^2$ with constraint $\|Q_k\|_{L^2} = 1$, and satisfies (4.5) with chemical potential

$$\mu_k = \mu_0 + \frac{k^2}{4}. \quad (4.7)$$

We remark that $\frac{k^2}{4}$ is the kinetic energy of a dressed particle, consisting of the tracer particle together with a cloud of bosons, of total mass 2.

Proof. First of all, we verify from straightforward calculation that

$$\begin{aligned} \mathcal{S}_k[e^{-i\frac{k \cdot x}{2}} \phi] &= \frac{k^2}{4} + \frac{1}{2}(\phi, i\nabla \phi)^2 + \mathcal{E}_0[\phi] - \mu \|\phi\|_{L^2}^2 \\ &\geq \frac{k^2}{4} + \mathcal{E}_0[\phi] - \mu \|\phi\|_{L^2}^2, \end{aligned} \quad (4.8)$$

noting that for the choice of frequency $\frac{k}{2}$ in the exponent of (4.6), terms linear in $i\nabla Q_0$ on the right hand side cancel. Therefore,

$$\begin{aligned} \inf_{\|\phi\|_{L^2}=1} \left\{ \mathcal{S}_k[e^{-i\frac{k \cdot x}{2}} \phi] \right\} &\geq \frac{k^2}{4} + \inf_{\|\phi\|_{L^2}=1} \left\{ \mathcal{E}_0[\phi] - \mu \|\phi\|_{L^2}^2 \right\} \\ &= \frac{k^2}{4} + \mathcal{E}_0[Q_0] - \mu_0 \|Q_0\|_{L^2}^2. \end{aligned} \quad (4.9)$$

On the other hand, because Q_0 is spherically symmetric, $(Q_0, i\nabla Q_0) = 0$. Therefore,

$$\begin{aligned} \mathcal{S}_k[e^{-i\frac{k \cdot x}{2}} Q_0] &= \frac{k^2}{4} + \frac{1}{2}(Q_0, i\nabla Q_0)^2 + \mathcal{E}_0[Q_0] - \mu \|Q_0\|_{L^2}^2 \\ &= \frac{k^2}{4} + \inf_{\|\phi\|_{L^2}=1} \left\{ \mathcal{E}_0[\phi] - \mu \|\phi\|_{L^2}^2 \right\} \end{aligned} \quad (4.10)$$

saturates the lower bound.

Furthermore, substituting $e^{-i\frac{k \cdot x}{2}} Q_0$ for Q_k in (4.5) yields

$$\mu_k Q_0 = \frac{k^2}{4} - \frac{1}{2} \Delta Q_0 + w Q_0 + \lambda(v * |Q_0|^2) Q_0 \quad (4.11)$$

where we note that all terms proportional to ∇Q_0 cancel. Comparing with (4.3), we conclude that $\mu_k = \mu_0 + \frac{k^2}{4}$, as claimed. \square

4.1. Mean field limit for the ground state

Given $\phi_t = Q_k$ for some $k \in \mathbb{R}^3$, $\forall t \in \mathbb{R}$, (3.8) is evidently solved by (4.5), and the expressions appearing in Theorem 3.1 simplify as follows. The Hamiltonian $\mathcal{H}_{mf}^{\phi_t}(k) = \mathcal{H}_{Har}^{\phi_t}(k) + \mathcal{H}_{cor}^{\phi_t}$ becomes time-independent, with

$$\mathcal{H}_{Har}^{\phi_t}(k) = T + W_1(0) + \lambda \int |Q_0(x)|^2 v(x-y) a_y^+ a_y dx dy, \quad (4.12)$$

and

$$\begin{aligned} \mathcal{H}_{cor}^{Q_k} &= \frac{1}{2} \left(a^+(i\nabla Q_k) + a(i\nabla Q_k) \right)^2 \\ &\quad + \lambda \int v(x-y) Q_k(x) \overline{Q_k(y)} a_x^+ a_y dx dy \\ &\quad + \frac{\lambda}{2} \int v(x-y) \left(Q_k(x) Q_k(y) a_x^+ a_y^+ + \overline{Q_k(x) Q_k(y)} a_y a_x \right) dx dy. \end{aligned}$$

Consequently, (3.7) simplifies to

$$\mathcal{V}(t, s) = \exp \left(-i(t-s) \mathcal{H}_{mf}^{Q_k}(k) \right), \quad (4.13)$$

and (3.9) simplifies to

$$S(t, t') = N(t-t') \left(\frac{\lambda}{2} \int |Q_0(x)|^2 v(x-y) |Q_0(y)|^2 dx dy \right), \quad (4.14)$$

inside the expression in (3.10). The nonlinear ground state Q_0 of the Hartree functional, normalized by $\|Q_0\|_{L_x^2} = 1$ is, for $w, v \in C^1(\mathbb{R}^3)$, an element of $H_x^3(\mathbb{R}^3)$, see Lemma 4.2. Accordingly, $Q_k \in H_x^3(\mathbb{R}^3)$, as required in Theorem 3.1.

Lemma 4.2. Assume that $w, v \in C^1(\mathbb{R}^3)$. Let Q_0 denote the minimizer of the Hartree functional $\mathcal{E}_0[\cdot]$, satisfying (4.3) with $\mu_0 < 0$, and $\|Q_0\|_{L_x^2} = 1$. Then, $Q_0 \in H_x^3(\mathbb{R}^3)$.

Proof. Given $\mu_0 < 0$ in (4.3), we have the identity

$$Q_0 = -(|\mu_0| - \Delta)^{-1} \left(wQ_0 + \lambda(v * |Q_0|^2)Q_0 \right) \quad (4.15)$$

where $|\mu_0| - \Delta \geq |\mu_0|$ is strictly positive. Therefore,

$$\begin{aligned} \|Q_0\|_{H_x^3} &= \left\| (1 - \Delta)^{\frac{3}{2}} (|\mu_0| - \Delta)^{-1} \left(wQ_0 + \lambda(v * |Q_0|^2)Q_0 \right) \right\|_{L_x^2} \\ &\leq |\mu_0|^{-1} \left\| wQ_0 + \lambda(v * |Q_0|^2)Q_0 \right\|_{H_x^1} \\ &\leq |\mu_0|^{-1} (\|w\|_{C^1} + \|\lambda v\|_{C^1}) \|Q_0\|_{H_x^1} < \infty, \end{aligned} \quad (4.16)$$

using that $\|\lambda(v * |Q_0|^2)\|_{C^1} \leq \lambda\|v\|_{C^1} \|Q_0\|_{L_x^2}^2$, and $\|Q_0\|_{L_x^2}^2 = 1$. \square

5. Time-dependent mean field equations

The nonlinear dispersive PDE with conserved energy functional $\mathcal{E}_k[\phi]$, for given $k \in \mathbb{R}^3$, is given by

$$i\partial_t\phi = -\left(k - j_\phi(t)\right) i\nabla\phi - \frac{1}{2}\Delta\phi + w\phi + \lambda(v * |\phi|^2)\phi, \quad (5.1)$$

where

$$j_\phi(t) := (\phi(t), i\nabla\phi(t)) \quad (5.2)$$

is the expected momentum (respectively, the current) determined by ϕ . The quantity $k - j_\phi$ is the momentum of the tracer particle, given that the bosons are in the coherent state parametrized by ϕ , and the total conserved momentum is k . We will prove the following global well-posedness result for the corresponding Cauchy problem.

Theorem 5.1. *Let $\phi_0 \in H_x^1$. Then, there exists a unique global in time mild solution to (5.1),*

$$i\partial_t\phi = -\left(k - j_\phi(t)\right) i\nabla\phi - \frac{1}{2}\Delta\phi + w\phi + \lambda(v * |\phi|^2)\phi, \quad (5.3)$$

with initial data $\phi(t=0) = \phi_0$, satisfying

$$\|\phi\|_{L_t^\infty H_x^1(\mathbb{R} \times \mathbb{R}^3)} + \|\tau_{X_\phi}\phi\|_{L_t^{\frac{10}{3}} W_x^{1, \frac{10}{3}}(\mathbb{R} \times \mathbb{R}^3)} < \infty \quad (5.4)$$

where $(\tau_{X_\phi}\phi)(t, x) := \phi(x + X_\phi(t))$, and $X_\phi(t) = \int_0^t ds(k - (\phi, i\nabla\phi)(s))$. In particular, $|\partial_t X_\phi(t)| < C\|\phi_0\|_{H_x^1}$, uniformly in $t \in \mathbb{R}$.

Proof. For any $\phi \in L_t^\infty H_x^1(\mathbb{R} \times \mathbb{R}^3)$,

$$|j_\phi(t)| \leq \|\phi\|_{L_t^\infty H_x^1(\mathbb{R} \times \mathbb{R}^3)} < C \quad (5.5)$$

is bounded, and therefore,

$$X_\phi(t) := \int_0^t ds(k - j_\phi(s)) \quad (5.6)$$

is finite for every $t \in \mathbb{R}$. Consequently, $e^{iX_\phi(t) \cdot i\nabla} : H_x^1 \rightarrow H_x^1$ is unitary for every $t \in \mathbb{R}$, and we may define

$$\psi(t, x) := e^{iX_\phi(t) \cdot i\nabla} \phi(t, x). \quad (5.7)$$

The exponential generates translations in x -space by $X_\phi(t)$, yielding

$$\psi(t, x) = \phi\left(t, x - X_\phi(t)\right) \quad (5.8)$$

as an equivalent expression. Clearly,

$$j_\phi(t) = j_\psi(t), \quad (5.9)$$

by unitarity of $e^{i \int_0^t ds(k - j_\phi(s)) \cdot x}$ on $L^2(\mathbb{R}^3)$. Therefore,

$$X_\phi(t) = X_\psi(t), \quad (5.10)$$

and

$$i\partial_t \psi = e^{iX_\phi(t) \cdot i\nabla} \left(-\frac{1}{2}\Delta + w + \lambda(v * |\phi|^2) \right) e^{-iX_\phi(t) \cdot i\nabla} \psi, \quad (5.11)$$

where we note that the first term on the r.h.s. of (5.1) has been canceled by $(i\partial_t X_\phi(t))\phi$ obtained from the time derivative. Noting that the operator $-\Delta$ is translation invariant, and

$$\begin{aligned} & \left(e^{iX_\phi(t) \cdot i\nabla} (v * |\phi|^2) e^{-iX_\phi(t) \cdot i\nabla} \right) (t, x) \\ &= \int v\left(x - X_\phi(t) - y\right) |\phi(t, y)|^2 dy \\ &= \int v(x - y) |\phi(t, y - X_\phi(t))|^2 dy \\ &= (v * |\psi|^2)(t, x), \end{aligned} \quad (5.12)$$

we find that ψ satisfies the nonlinear Hartree equation

$$i\partial_t \psi = -\frac{1}{2}\Delta \psi + w_\psi \psi + \lambda(v * |\psi|^2)\psi, \quad \psi(t=0) = \psi_0 \equiv \phi_0 \quad (5.13)$$

where

$$w_\psi(t, x) := w\left(x - X_\psi(t)\right) \quad (5.14)$$

is the potential w , translated by $X_\psi(t)$.

The claim of the theorem therefore follows from the global well-posedness of (5.13) established in Theorem 5.5, below. \square

Remark 5.2. The physical interpretation of (5.13) is as follows. The field ψ describes a self-interacting boson gas in mean field description which interacts with a point-like tracer particle traveling along a trajectory $X_\psi(t)$. The tracer particle creates an interaction

potential w which moves along $X_\psi(t)$, here denoted by w_ψ . The momentum of the tracer particle, $\partial_t X_\psi(t) = k - (\psi, i\nabla\psi)$, together with the expected momentum of the boson field, $(\psi, i\nabla\psi)$, adds up to the conserved total momentum k . We note the close similarity to the equations studied in [22] for a classical particle coupled to a boson gas.

Remark 5.3. The Ehrenfest dynamics of $X_\psi(t)$ is given by

$$\partial_t^2 X_\psi(t) = \left(\psi, \nabla(w_\psi + \lambda v * |\psi|^2)\psi \right), \quad (5.15)$$

as stated in (1.42). The term depending on v is zero because, first of all,

$$\begin{aligned} & \left(\psi, \nabla(v * |\psi|^2)\psi \right) \\ &= \int dx |\psi|^2 \nabla(v * |\psi|^2) \\ &= \int dx |\psi|^2 (v * \nabla|\psi|^2). \end{aligned} \quad (5.16)$$

On the other hand, using integration by parts,

$$\begin{aligned} & \int dx |\psi|^2 \nabla(v * |\psi|^2) \\ &= - \int dx (\nabla|\psi|^2) (v * |\psi|^2) \\ &= - \int dx (v * \nabla|\psi|^2) |\psi|^2 \end{aligned} \quad (5.17)$$

where the last line holds because v is a radial function, and thus even. Comparing (5.16) and (5.17), we find that $\int dx |\psi|^2 \nabla(v * |\psi|^2) = - \int dx |\psi|^2 \nabla(v * |\psi|^2) = 0$. Therefore,

$$\begin{aligned} \partial_t^2 X_\psi(t) &= \left(\psi, \nabla w_\psi \psi \right) \\ &= \int dx (\nabla w)(x - X_\psi(t)) |\psi(x)|^2, \end{aligned} \quad (5.18)$$

as claimed.

A key advantage of (5.13) over (5.1) is that the Cauchy problem can largely be controlled with known dispersive tools for the analysis of the Hartree equation. A main difficulty is introduced by the dependence of the potential w_ψ on ψ . Another complication arises from the fact that the energy is not conserved.

We construct local and global in time dispersive solutions under the assumption that $\|w\|_{W^{2,3/2}} < \infty$, and that $\|w\|_{W^{1,3/2}}$, $\|\lambda v\|_{W^{1,3/2}}$ are sufficiently small. These conditions

ensure, in accordance with the Birman–Schwinger principle, that neither w nor λv create bound states. We require that w has one more derivative than v , to control the dependence of w_ψ on ψ . First, we prove local well-posedness.

Theorem 5.4 (*Local well-posedness*). *Let $\psi_0 \in H^1(\mathbb{R}^3)$ with $\|\psi_0\|_{L_x^2} = 1$. Assume that $\|w\|_{W_x^{2, \frac{3}{2}}} < \infty$, and that*

$$\|w\|_{W_x^{1, \frac{3}{2}}} + 3\|\lambda v\|_{W_x^{1, \frac{3}{2}}} < 1. \quad (5.19)$$

Then, there exists a unique mild solution

$$\psi \in L_t^\infty H_x^1([0, T] \times \mathbb{R}^3) \cap L_t^{\frac{10}{3}} W_x^{1, \frac{10}{3}}([0, T] \times \mathbb{R}^3) \quad (5.20)$$

to (5.11) with initial condition $\psi(t=0) = \psi_0$, provided that $T > 0$ is sufficiently small.

Proof. We consider the map

$$\mathcal{M} : \psi \mapsto e^{it\Delta}\psi_0 + i \int_0^t ds e^{i(t-s)\Delta} \left((w_\psi \psi)(s) + \lambda((v * |\psi|^2)\psi)(s) \right), \quad (5.21)$$

where we may assume that $\|\psi\|_{L_x^2} = 1$. Clearly, using the Strichartz and Hölder inequalities as in

$$\begin{aligned} \left\| \int_0^t ds e^{i(t-s)\Delta} (w_\psi \psi)(s) \right\|_{L_t^q L_x^r} &\leq \|w_\psi \psi\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} \\ &\leq \|w_\psi \psi\|_{L_t^{\frac{10}{3}} L_x^{\frac{30}{29}}} \\ &\leq \|w_\psi\|_{L_t^\infty L_x^{\frac{3}{2}}} \|\psi\|_{L_{t,x}^{\frac{10}{3}}}, \end{aligned} \quad (5.22)$$

with (q, r) and (\tilde{q}, \tilde{r}) denoting arbitrary Strichartz admissible pairs, we find that, under inclusion of a derivative,

$$\begin{aligned} \|\mathcal{M}[\psi]\|_{L_t^q W_x^{1,r}} &\quad (5.23) \\ &\leq \|\psi_0\|_{H^1} + \|w_\psi\|_{L_t^\infty W^{1, \frac{3}{2}}} \|\psi\|_{L_t^{\frac{10}{3}} W^{1, \frac{10}{3}}} + \|\lambda v * |\psi|^2\|_{L_t^\infty W^{1, \frac{3}{2}}} \|\psi\|_{L_t^{\frac{10}{3}} W^{1, \frac{10}{3}}}. \end{aligned}$$

We use Young's inequality for convolutions in

$$\|\lambda v * |\psi|^2\|_{L_t^\infty W^{1, \frac{3}{2}}} \leq \|\lambda v\|_{W^{1, \frac{3}{2}}} \|\psi\|_{L^2}^2, \quad (5.24)$$

and observe that

$$\|w_\psi\|_{L_t^\infty W^{1, \frac{3}{2}}} \leq \sup_X \|w(\bullet - X)\|_{W^{1, \frac{3}{2}}} = \|w\|_{W^{1, \frac{3}{2}}}. \quad (5.25)$$

Therefore,

$$\|\mathcal{M}[\psi]\|_{L_t^q W_x^{1,r}} \leq \|\psi_0\|_{H^1} + \left(\|w\|_{W^{1,\frac{3}{2}}} + \|\lambda v\|_{W^{1,\frac{3}{2}}} \right) \|\psi\|_{L_t^{\frac{10}{3}} W_x^{1,\frac{10}{3}}}, \quad (5.26)$$

for any Strichartz admissible pair (q, r) . Consequently, writing $I := [0, T]$, and defining the Banach space

$$Y(I) := L_t^\infty H_x^1(I \times \mathbb{R}^3) \cap L_t^{\frac{10}{3}} W_x^{1,\frac{10}{3}}(I \times \mathbb{R}^3) \quad (5.27)$$

endowed with the norm

$$\|f\|_{Y(I)} := \|f\|_{L_t^\infty H_x^1(I \times \mathbb{R}^3)} + \|f\|_{L_t^{\frac{10}{3}} W_x^{1,\frac{10}{3}}(I \times \mathbb{R}^3)}, \quad (5.28)$$

we find

$$\|\mathcal{M}[\psi]\|_{Y(I)} \leq 2\|\psi_0\|_{H_x^1} + \left(\|w\|_{W_x^{1,\frac{3}{2}}} + \|\lambda v\|_{W_x^{1,\frac{3}{2}}} \right) \|\psi\|_{Y(I)}. \quad (5.29)$$

Assuming that $\|w\|_{W_x^{1,\frac{3}{2}}} + \|\lambda v\|_{W_x^{1,\frac{3}{2}}} < 1 - \delta$, for some $\delta \in (0, 1)$, and defining $R := 2\delta^{-1}\|\psi_0\|_{H^1}$, we find that

$$\|\mathcal{M}[\psi]\|_{Y(I)} \leq \delta R + (1 - \delta)\|\psi\|_{Y(I)}. \quad (5.30)$$

Hence, the image of the ball $B_R(0) \subset Y$ under the map \mathcal{M} is contained in itself.

Next, we prove the contractivity of \mathcal{M} . Given $\psi_1, \psi_2 \in B_R(0) \subset Y$, we have

$$\begin{aligned} \|\mathcal{M}[\psi_1] - \mathcal{M}[\psi_2]\|_{Y(I)} &\leq \|w_{\psi_1} - w_{\psi_2}\|_{L_t^\infty W_x^{1,\frac{3}{2}}} \|\psi_1\|_{Y(I)} \\ &\quad + \|w_{\psi_2}\|_{L_t^\infty W_x^{1,\frac{3}{2}}} \|\psi_1 - \psi_2\|_{Y(I)} \\ &\quad + \|\lambda v * (|\psi_1|^2 - |\psi_2|^2)\|_{L_t^\infty W_x^{1,\frac{3}{2}}} \|\psi_1\|_{Y(I)} \\ &\quad + \|\lambda v * |\psi_2|^2\|_{L_t^\infty W_x^{1,\frac{3}{2}}} \|\psi_1 - \psi_2\|_{Y(I)}. \end{aligned} \quad (5.31)$$

To control the first term on the r.h.s. of (5.31), we have

$$\|w_{\psi_1} - w_{\psi_2}\|_{L_t^\infty W_x^{1,\frac{3}{2}}} \leq \|w\|_{W_x^{2,\frac{3}{2}}} \sup_t |X_{\psi_1}(t) - X_{\psi_2}(t)|, \quad (5.32)$$

where

$$\begin{aligned} |X_{\psi_1}(t) - X_{\psi_2}(t)| &\leq \int_0^t ds |(\psi_1, i\nabla \psi_1) - (\psi_2, i\nabla \psi_2)| \\ &\leq \int_0^t ds |(i\nabla(\psi_1 - \psi_2), \psi_1) + (\psi_2, i\nabla(\psi_1 - \psi_2))| \end{aligned}$$

$$\begin{aligned}
&\leq t \|\psi_1 - \psi_2\|_{L_t^\infty H_x^1} \left(\|\psi_1\|_{L_t^\infty L_x^2} + \|\psi_2\|_{L_t^\infty L_x^2} \right) \\
&\leq 2T \|\psi_1 - \psi_2\|_{Y(I)}
\end{aligned} \tag{5.33}$$

for $t \in I = [0, T]$.

To control the term on the third line on the r.h.s. of (5.31), we use

$$\begin{aligned}
\|\lambda v * (|\psi_1|^2 - |\psi_2|^2)\|_{L_t^\infty W_x^{1, \frac{3}{2}}} &\leq \|\lambda v\|_{W_x^{1, \frac{3}{2}}} \| |\psi_1|^2 - |\psi_2|^2 \|_{L_t^\infty L_x^1} \\
&\leq \|\lambda v\|_{W_x^{1, \frac{3}{2}}} \left(\|\psi_1\|_{L_t^\infty L_x^2} + \|\psi_2\|_{L_t^\infty L_x^2} \right) \|\psi_1 - \psi_2\|_{L_t^\infty L_x^2} \\
&\leq 2\|\lambda v\|_{W_x^{1, \frac{3}{2}}} \|\psi_1 - \psi_2\|_{Y(I)}
\end{aligned} \tag{5.34}$$

where $\|\psi_1\|_{L_x^2} + \|\psi_2\|_{L_x^2} \leq \|\psi_1\|_{L_x^2} + \|\psi_2\|_{L_x^2} = 2$.

Summarizing, we have

$$\begin{aligned}
&\|\mathcal{M}[\psi_1] - \mathcal{M}[\psi_2]\|_{Y(I)} \\
&\leq 2T \|w\|_{W_x^{2, \frac{3}{2}}} \|\psi_1\|_{Y(I)} \|\psi_1 - \psi_2\|_{Y(I)} \\
&\quad + \|w\|_{W_x^{1, \frac{3}{2}}} \|\psi_1 - \psi_2\|_{Y(I)} \\
&\quad + 2\|\lambda v\|_{W_x^{1, \frac{3}{2}}} \|\psi_1 - \psi_2\|_{Y(I)} \\
&\quad + \|\lambda v\|_{W_x^{1, \frac{3}{2}}} \|\psi_1 - \psi_2\|_{Y(I)} \\
&\leq \left(2TR \|w\|_{W_x^{2, \frac{3}{2}}} + \|w\|_{W_x^{1, \frac{3}{2}}} + 3\|\lambda v\|_{W_x^{1, \frac{3}{2}}} \right) \|\psi_1 - \psi_2\|_{Y(I)}.
\end{aligned} \tag{5.35}$$

Therefore, \mathcal{M} is contractive on the ball $B_R(0) \subset Y$ if

$$2TR \|w\|_{W_x^{2, \frac{3}{2}}} + \|w\|_{W_x^{1, \frac{3}{2}}} + 3\|\lambda v\|_{W_x^{1, \frac{3}{2}}} < 1. \tag{5.36}$$

To this end, we require that

$$\|w\|_{W_x^{1, \frac{3}{2}}} + 3\|\lambda v\|_{W_x^{1, \frac{3}{2}}} < 1, \tag{5.37}$$

and that $T > 0$ is sufficiently small (depending on R). \square

We remark that the only place in the proof that requires a finite time $T > 0$ is the control of w_ψ . The Strichartz estimates employed here remain valid with \mathbb{R} instead of I . We may therefore patch together local in time solutions using a global Strichartz inequality.

Theorem 5.5 (*Global well-posedness*). *Let $\psi_0 \in H^1(\mathbb{R}^3)$ with $\|\psi_0\|_{L_x^2} = 1$. Assume that $\|w\|_{W_x^{2, \frac{3}{2}}} < \infty$, and that*

$$\|w\|_{W_x^{1,\frac{3}{2}}} + 3\|\lambda v\|_{W_x^{1,\frac{3}{2}}} < 1. \quad (5.38)$$

Then, there exists a unique global mild solution $\psi \in Y(\mathbb{R})$ to (5.11) with initial condition $\psi(t=0) = \psi_0$. In particular, it satisfies

$$\|\psi\|_{Y(\mathbb{R})} \leq 2 \left(1 - \|w\|_{W_x^{1,\frac{3}{2}}} - \|\lambda v\|_{W_x^{1,\frac{3}{2}}} \right)^{-1} \|\psi_0\|_{H_x^1}. \quad (5.39)$$

Moreover,

$$|\partial_t X_\psi(t)| \leq |k| + \left(1 - \|w\|_{W_x^{1,\frac{3}{2}}} - \|\lambda v\|_{W_x^{1,\frac{3}{2}}} \right)^{-1} \|\psi_0\|_{H_x^1}, \quad t \in \mathbb{R}; \quad (5.40)$$

that is, the momentum of the tracer particle is uniformly bounded in time.

Proof. The Strichartz estimate obtained in (5.26) holds globally in time. With $(q, r) = (\frac{10}{3}, \frac{10}{3})$, it implies

$$\|\psi\|_{L_t^{\frac{10}{3}} W_x^{1,\frac{10}{3}}(\mathbb{R} \times \mathbb{R}^3)} \leq \|\psi_0\|_{H^1} + \left(\|w\|_{W^{1,\frac{3}{2}}} + \|\lambda v\|_{W^{1,\frac{3}{2}}} \right) \|\psi\|_{L_t^{\frac{10}{3}} W_x^{1,\frac{10}{3}}(\mathbb{R} \times \mathbb{R}^3)}, \quad (5.41)$$

respectively,

$$\|\psi\|_{L_t^{\frac{10}{3}} W_x^{1,\frac{10}{3}}(\mathbb{R} \times \mathbb{R}^3)} \leq \left(1 - \|w\|_{W_x^{1,\frac{3}{2}}} - \|\lambda v\|_{W_x^{1,\frac{3}{2}}} \right)^{-1} \|\psi_0\|_{H_x^1}. \quad (5.42)$$

We use this a priori bound to control the $L^\infty H_x^1$ norm of ψ .

Let $I_j := [(j-1)T, jT]$. The estimate (5.26) with $(q, r) = (\infty, 2)$, combined with (5.42), implies that

$$\|\psi\|_{L_t^\infty H_x^1(I_1 \times \mathbb{R}^3)} \leq 2 \left(1 - \|w\|_{W_x^{1,\frac{3}{2}}} - \|\lambda v\|_{W_x^{1,\frac{3}{2}}} \right)^{-1} \|\psi_0\|_{H_x^1}, \quad (5.43)$$

and hence,

$$\|\psi(t=2T)\|_{H_x^1(I_1 \times \mathbb{R}^3)} \leq \left(1 - \|w\|_{W_x^{1,\frac{3}{2}}} - \|\lambda v\|_{W_x^{1,\frac{3}{2}}} \right)^{-1} \|\psi_0\|_{H_x^1}. \quad (5.44)$$

Applying Theorem 5.4 for I_2 with initial data $\psi(t=2T)$ yields local well-posedness on $Y(I_2)$ with the same upper bound on $\|\psi(t=3T)\|_{H_x^1(I_1 \times \mathbb{R}^3)}$ as in (5.44).

Iterating this argument for I_j , $j \in \mathbb{Z}$, we find that

$$\|\psi\|_{L_t^\infty H_x^1(\mathbb{R} \times \mathbb{R}^3)} \leq \left(1 - \|w\|_{W_x^{1,\frac{3}{2}}} - \|\lambda v\|_{W_x^{1,\frac{3}{2}}} \right)^{-1} \|\psi_0\|_{H_x^1}, \quad (5.45)$$

globally in time.

Therefore, we obtain

$$\begin{aligned} \|\psi\|_{Y(\mathbb{R})} &= \|\psi\|_{L_t^\infty H_x^1(\mathbb{R} \times \mathbb{R}^3)} + \|\psi\|_{L_t^{\frac{10}{3}} W_x^{1, \frac{10}{3}}(\mathbb{R} \times \mathbb{R}^3)} \\ &\leq 2 \left(1 - \|w\|_{W_x^{1, \frac{3}{2}}} - \|\lambda v\|_{W_x^{1, \frac{3}{2}}} \right)^{-1} \|\psi_0\|_{H_x^1}, \end{aligned} \quad (5.46)$$

as claimed in (5.39).

Finally, we note that the bound (5.45) implies that

$$\begin{aligned} |\partial_t X_\psi(t)| &= |k - (\psi, i\nabla\psi)(t)| \\ &\leq |k| + \|\psi\|_{L_t^\infty H_x^1(\mathbb{R} \times \mathbb{R}^3)} \\ &\leq |k| + \left(1 - \|w\|_{W_x^{1, \frac{3}{2}}} - \|\lambda v\|_{W_x^{1, \frac{3}{2}}} \right)^{-1} \|\psi_0\|_{H_x^1}, \quad t \in \mathbb{R}. \end{aligned} \quad (5.47)$$

Thus, the momentum of the tracer particle is uniformly bounded in time. \square

6. Proof of convergence to the mean field dynamics

6.1. Determination of $\mathcal{L}_N^{\phi_t}(k)$

In the following Lemma, we determine the explicit form of the operator (3.15). We will use the notation $\phi_t(x) \equiv \phi(t, x)$, similarly as in Section 3.

Lemma 6.1. *The selfadjoint operator $\mathcal{L}_N^{\phi_t}(k)$ in (3.15) is given by*

$$\begin{aligned} \mathcal{L}_N^{\phi_t}(k) &= \frac{1}{2\sqrt{N}} \left(P_b \cdot (a^+(i\nabla\phi_t) + a(i\nabla\phi_t)) + (a^+(i\nabla\phi_t) + a(i\nabla\phi_t)) \cdot P_b \right) \\ &\quad + \frac{1}{2N} P_b^2 \\ &\quad + \frac{\lambda}{\sqrt{N}} \int v(x-y) a_x^+ \left(\overline{\phi_t(y)} a_y + \phi_t(y) a_y^+ \right) a_x dx dy \\ &\quad + \frac{\lambda}{2N} \int v(x-y) a_x^+ a_y^+ a_y a_x dx dy. \end{aligned} \quad (6.1)$$

Proof. We recall from (3.15) that

$$\begin{aligned} \mathcal{L}_N^{\phi_t}(k) &= \mathcal{W}^*[\sqrt{N}\phi_t] \mathcal{H}_N(k) \mathcal{W}[\sqrt{N}\phi_t] - \partial_t S(t, 0) \\ &\quad - \mathcal{W}^*[\sqrt{N}\phi_t] \mathcal{H}_{Har}^{\phi_t}(k) \mathcal{W}[\sqrt{N}\phi_t] - \mathcal{H}_{cor}^{\phi_t}(k). \end{aligned} \quad (6.2)$$

The explicit expressions for the terms on the right hand side are given by

$$\begin{aligned} &\mathcal{W}^*[\sqrt{N}\phi_t] \mathcal{H}_N(k) \mathcal{W}[\sqrt{N}\phi_t] \\ &= \frac{N}{2} \left(k^2 - (\phi_t, i\nabla\phi_t)^2 \right) + \frac{N\lambda}{2} \int |\phi_t(x)|^2 v(x-y) |\phi_t(y)|^2 dx dy \end{aligned}$$

$$\begin{aligned}
 & + \mathcal{W}^*[\sqrt{N}\phi_t] \left(- \left(k - (\phi_t, i\nabla\phi_t) \right) \cdot P_b + T + W_1(0) \right) \mathcal{W}[\sqrt{N}\phi_t] \\
 & + \frac{1}{2} \left(a^+(i\nabla\phi_t) + a(i\nabla\phi_t) \right)^2 \\
 & + \frac{1}{2\sqrt{N}} \left(P_b \cdot (a^+(i\nabla\phi_t) + a(i\nabla\phi_t)) + (a^+(i\nabla\phi_t) + a(i\nabla\phi_t)) \cdot P_b \right) \\
 & + \frac{1}{2N} P_b^2 \\
 & + \lambda\sqrt{N} \int v(x-y) |\phi_t(x)|^2 \left(a_y^+ \phi_t(y) + \overline{\phi_t(y)} a_y \right) dx dy \\
 & + \lambda \int v(x-y) |\phi_t(x)|^2 a_y^+ a_y dx dy \\
 & + \lambda \int v(x-y) \phi_t(x) \overline{\phi_t(y)} a_x^+ a_y dx dy \\
 & + \frac{\lambda}{2} \int v(x-y) \left(\phi_t(x) \phi_t(y) a_x^+ a_y^+ + \overline{\phi_t(x) \phi_t(y)} a_y a_x \right) dx dy \\
 & + \frac{\lambda}{\sqrt{N}} \int v(x-y) a_x^+ \left(\overline{\phi_t(y)} a_y + \phi_t(y) a_y^+ \right) a_x dx dy \\
 & + \frac{\lambda}{2N} \int v(x-y) \left(a_x^+ a_y^+ a_y a_x \right) dx dy, \tag{6.3}
 \end{aligned}$$

and

$$\begin{aligned}
 & \mathcal{W}^*[\sqrt{N}\phi_t] \mathcal{H}_{Har}^{\phi_t}(k) \mathcal{W}[\sqrt{N}\phi_t] \\
 & = \mathcal{W}^*[\sqrt{N}\phi_t] \left(- \left(k - (\phi_t, i\nabla\phi_t) \right) \cdot P_b + T + W_1(0) \right) \mathcal{W}[\sqrt{N}\phi_t] \\
 & + N\lambda \int |\phi_t(x)|^2 v(x-y) |\phi_t(y)|^2 dx dy \\
 & + \lambda\sqrt{N} \int v(x-y) |\phi_t(x)|^2 \left(a_y^+ \phi_t(y) + \overline{\phi_t(y)} a_y \right) dx dy \\
 & + \lambda \int v(x-y) |\phi_t(x)|^2 a_y^+ a_y dx dy, \tag{6.4}
 \end{aligned}$$

and

$$\begin{aligned}
 \mathcal{H}_{cor}^{\phi_t} & = \frac{1}{2} \left(a^+(i\nabla\phi_t) + a(i\nabla\phi_t) \right)^2 \\
 & + \lambda \int v(x-y) \phi_t(x) \overline{\phi_t(y)} a_x^+ a_y dx dy \\
 & + \frac{\lambda}{2} \int v(x-y) \left(\phi_t(x) \phi_t(y) a_x^+ a_y^+ + \overline{\phi_t(x) \phi_t(y)} a_y a_x \right) dx dy, \tag{6.5}
 \end{aligned}$$

and

$$\begin{aligned} \partial_t S(t, 0) = N \Big(-\frac{1}{2}k^2 + \frac{1}{2}(\phi_t, i\nabla\phi_t)^2 \\ + \frac{\lambda}{2} \int |\phi_t(x)|^2 v(x-y) |\phi_t(y)|^2 dx dy \Big). \end{aligned} \quad (6.6)$$

We thus obtain

$$\begin{aligned} \mathcal{L}_N^{\phi_t}(k) &= \frac{1}{2\sqrt{N}} \Big(P_b \cdot (a^+(i\nabla\phi_t) + a(i\nabla\phi_t)) + (a^+(i\nabla\phi_t) + a(i\nabla\phi_t)) \cdot P_b \Big) \\ &\quad + \frac{1}{2N} P_b^2 \\ &\quad + \frac{\lambda}{\sqrt{N}} \int v(x-y) a_x^+ \Big(\overline{\phi_t(y)} a_y + \phi_t(y) a_y^+ \Big) a_x dx dy \\ &\quad + \frac{\lambda}{2N} \int v(x-y) a_x^+ a_y^+ a_y a_x dx dy, \end{aligned} \quad (6.7)$$

as claimed in (6.1).

We note that in order to obtain (6.3), we used the following. Introducing the abbreviated notations

$$\mathcal{W} := \mathcal{W}[\sqrt{N}\phi_t] \quad , \quad V := a^+(i\nabla\phi_t) + a(i\nabla\phi_t) \quad , \quad D := (\phi, i\nabla\phi), \quad (6.8)$$

it is clear that

$$\mathcal{W}^* P_b \mathcal{W} = P_b + \sqrt{N}V + ND. \quad (6.9)$$

Therefore,

$$\begin{aligned} &\frac{1}{2N} (Nk - \mathcal{W}^* P_b \mathcal{W})^2 \\ &= \frac{1}{2N} (N(k - D) - \mathcal{W}^* (P_b - ND) \mathcal{W})^2 \\ &= \frac{N}{2} (k - D)^2 - (k - D) \mathcal{W}^* (P_b - ND) \mathcal{W} + \frac{1}{2N} (P_b + \sqrt{N}V)^2 \\ &= \frac{N}{2} (k^2 - D^2) - (k - D) \mathcal{W}^* P_b \mathcal{W} + \frac{1}{2N} P_b^2 + \frac{1}{2\sqrt{N}} (P_b V + V P_b) + \frac{1}{2} V^2. \end{aligned} \quad (6.10)$$

The terms on the last line are contained in the first five lines on the rhs of (6.3). \square

6.2. Estimates on $M(t)$

In this subsection, we prove the estimate (3.20).

Lemma 6.2. *The following estimate holds,*

$$\left\| \mathcal{L}_N^{\phi_t}(k) \mathcal{V}(t, 0) \Omega \right\|_{\mathcal{F}} \leq \frac{C_0}{\sqrt{N}} e^{C_1 t}, \quad (6.11)$$

for constants C_0, C_1 depending on $\|v\|_{C^2}$ and $\|\phi_t\|_{L_t^\infty H_x^3([0, T] \times \mathbb{R}^3)}$.

Proof. Let

$$Q_b := \int dk \langle k \rangle a_k^+ a_k \quad (6.12)$$

where $\langle k \rangle^2 := 1 + k^2$. Then, for any $\Psi \in \mathcal{F}$, and $\alpha \geq 1$,

$$\| |P_b|^\alpha \Psi \|, \| N_b^\alpha \Psi \| \leq \| Q_b^\alpha \Psi \|. \quad (6.13)$$

Thus, we obtain the following bounds on the individual terms in (6.7),

$$\begin{aligned} \left\| \frac{1}{2\sqrt{N}} \left(P_b \cdot (a^+(i\nabla\phi_t) + a(i\nabla\phi_t)) + (a^+(i\nabla\phi_t) + a(i\nabla\phi_t)) \cdot P_b \right) \mathcal{V}_t \Omega \right\| \\ \leq \frac{4\|\nabla\phi_t\|_{L^2}}{\sqrt{N}} \| Q_b^{3/2} \mathcal{V}_t \Omega \| \end{aligned} \quad (6.14)$$

$$\left\| \frac{1}{2N} P_b^2 \mathcal{V}_t \Omega \right\| \leq \frac{1}{2N} \| Q_b^2 \mathcal{V}_t \Omega \| \quad (6.15)$$

$$\begin{aligned} \left\| \frac{\lambda}{\sqrt{N}} \left(\int v(x-y) a_x^+ \left(\overline{\phi_t(y)} a_y + \phi_t(y) a_y^+ \right) a_x dx dy \right) \mathcal{V}_t \Omega \right\| \\ \leq \frac{\lambda \|v\|_{L^\infty} \|\phi_t\|_{L^2}}{\sqrt{N}} \| N_b^{3/2} \mathcal{V}_t \Omega \| \end{aligned} \quad (6.16)$$

$$\left\| \frac{\lambda}{2N} \left(\int v(x-y) a_x^+ a_y^+ a_y a_x dx dy \right) \mathcal{V}_t \Omega \right\| \leq \frac{\lambda \|v\|_{L^\infty}}{2N} \| N_b^2 \mathcal{V}_t \Omega \| \quad (6.17)$$

Therefore, we obtain that

$$\begin{aligned} \left\| \mathcal{L}_N^{\phi_t}(k) \mathcal{V}(t, 0) \Omega \right\|_{\mathcal{F}} &\leq \left(\frac{4\|\nabla\phi_t\|_{L^2}}{\sqrt{N}} + \frac{1}{2N} \right) \| Q_b^2 \mathcal{V}_t \Omega \| + \frac{2\lambda \|v\|_{L^\infty}}{\sqrt{N}} \| N_b^2 \mathcal{V}_t \Omega \| \\ &\leq \frac{C_0}{\sqrt{N}} e^{C_1 t}, \end{aligned} \quad (6.18)$$

using Lemma 6.3 and Lemma 6.4. \square

Lemma 6.3. *The following estimate holds,*

$$\left\| N_b^\alpha \mathcal{V}(t, 0) \Omega \right\|_{\mathcal{F}} \leq e^{C_1 t}, \quad \alpha = 1, 2, \quad (6.19)$$

for a constant C_1 depending only on $\|\phi_0\|_{H^1}$.

Proof. We split $\mathcal{H}_{cor}^{\phi_t}$ (see (6.5) for the definition) into

$$\mathcal{H}_{cor}^{\phi_t} = \mathcal{H}_{cor,d}^{\phi_t} + \mathcal{H}_{cor,od}^{\phi_t} \quad (6.20)$$

where the diagonal part $\mathcal{H}_{cor,d}^{\phi_t}$ commutes with the number operator,

$$[\mathcal{H}_{cor,d}^{\phi_t}, N_b] = 0, \quad (6.21)$$

and where

$$\mathcal{H}_{cor,od}^{\phi_t} = \frac{1}{2} \left(a^+ (i\nabla \phi_t) a^+ (i\nabla \phi_t) + a (i\nabla \phi_t) a (i\nabla \phi_t) \right) \quad (6.22)$$

$$\begin{aligned} &+ \frac{\lambda}{2} \int v(x-y) \left(\phi_t(x) \phi_t(y) a_x^+ a_y^+ + \overline{\phi_t(x) \phi_t(y)} a_y a_x \right) dx dy \\ &=: \mathcal{H}_{cor,od}^{\phi_t,++} + \mathcal{H}_{cor,od}^{\phi_t,--} \end{aligned} \quad (6.23)$$

is the off-diagonal part.

Defining

$$N_b(t) := \mathcal{V}_t^* N_b \mathcal{V}_t, \quad (6.24)$$

where for brevity, $\mathcal{V}_t := \mathcal{V}(t, 0)$, we have, for $\alpha = 1, 2$,

$$\begin{aligned} i\partial_t N_b^\alpha(t) &= \mathcal{V}_t^* [N_b^\alpha, \mathcal{H}_{cor,od}^{\phi_t}] \mathcal{V}_t \\ &= \alpha \mathcal{V}_t^* (\mathcal{H}_{cor,od}^{\phi_t,++} + (-1)^\alpha \mathcal{H}_{cor,od}^{\phi_t,--}) \mathcal{V}_t, \end{aligned} \quad (6.25)$$

due to (6.21).

This implies that

$$\begin{aligned} \|N_b^\alpha \mathcal{V}_t \Omega\| &= \|N_b^\alpha(t) \Omega\| \\ &\leq \|N_b^\alpha(0) \Omega\| + \int_0^t ds \|\partial_s N_b(s) \Omega\| \\ &\leq \alpha \int_0^t ds \left\| \mathcal{V}_s^* (\mathcal{H}_{cor,od}^{\phi_s,++} + (-1)^\alpha \mathcal{H}_{cor,od}^{\phi_s,--}) \mathcal{V}_s \Omega \right\|. \end{aligned} \quad (6.26)$$

Writing

$$K_t(x, x') := \frac{1}{2} (i\nabla \phi_t)(x) (i\nabla \phi_t)(x') + \frac{\lambda}{2} v(x-y) \phi_t(x) \phi_t(y), \quad (6.27)$$

we have

$$\mathcal{H}_{cor,od}^{\phi_t,++} = \int dx dx' K_t(x, x') a_x^+ a_{x'}^+, \quad \mathcal{H}_{cor,od}^{\phi_t,--} = \int dx dx' \overline{K}_t(x, x') a_x a_{x'}, \quad (6.28)$$

and we will next prove that

$$\left\| \mathcal{V}_s^* \mathcal{H}_{cor,od}^{\phi_s,--} \mathcal{V}_s \Omega \right\| \leq \| K_s \|_{L^2_{x,x'}} \| N_b(s) \Omega \| \quad (6.29)$$

and

$$\begin{aligned} \left\| \mathcal{V}_s^* \mathcal{H}_{cor,od}^{\phi_s,++} \mathcal{V}_s \Omega \right\| &\leq 2 \| K_s \|_{L^2_{x,x'}} \| (1 + N_b(s)) \Omega \| \\ &\leq 2 \| K_s \|_{L^2_{x,x'}} (1 + \| N_b(s) \Omega \|). \end{aligned} \quad (6.30)$$

To prove (6.30), we note that, using $K_s(x, x') = K_s(x', x)$,

$$\begin{aligned} &\left\| \mathcal{V}_s^* \mathcal{H}_{cor,od}^{\phi_s,++} \mathcal{V}_s \Omega \right\|^2 \\ &= \left\langle \Omega, \mathcal{V}_s^* \mathcal{H}_{cor,od}^{\phi_s,--} \mathcal{H}_{cor,od}^{\phi_s,++} \mathcal{V}_s \Omega \right\rangle \\ &= \int dx dx' dy dy' \overline{K_s(x, x')} K_s(y, y') \left\langle \Omega, \mathcal{V}_s^* \left(2\delta(x - y)\delta(x' - y') \right. \right. \\ &\quad \left. \left. + 4\delta(x' - y')a_y^+ a_x + a_x^+ a_y^+ a_{y'} a_{x'} \right) \mathcal{V}_s \Omega \right\rangle \\ &=: (I) + (II) + (III). \end{aligned} \quad (6.31)$$

We have

$$(I) = 2 \| K_s \|_{L^2_{x,x'}}^2 \quad (6.32)$$

$$\begin{aligned} (II) &= 4 \int dx \left\| \int dx' K_s(x, x') a_{x'} \mathcal{V}_s \Omega \right\|^2 \\ &\leq 4 \left(\int dx \| K(x, x') \|_{L^2_{x'}}^2 \right) \| a_{x'} \mathcal{V}_s \Omega \|_{L^2_x}^2 \\ &= 4 \| K_s \|_{L^2_{x,x'}}^2 \| N_b^{\frac{1}{2}}(s) \Omega \|^2 \end{aligned} \quad (6.33)$$

$$\begin{aligned} (III) &= \left| \int dx dx' dy dy' \overline{K_s(x, x')} K_s(y, y') \left\langle a_y a_x \mathcal{V}_s \Omega, a_{y'} a_{x'} \mathcal{V}_s \Omega \right\rangle \right| \\ &\leq \left(\int dx dx' dy dy' |K_s(x, x')|^2 |K_s(y, y')|^2 \right)^{\frac{1}{2}} \\ &\quad \left(\int dx dx' dy dy' \left\| a_y a_x \mathcal{V}_s \Omega \right\|^2 \left\| a_{y'} a_{x'} \mathcal{V}_s \Omega \right\|^2 \right)^{\frac{1}{2}} \\ &= \left(\int dx dx' |K_s(x, x')|^2 \right) \end{aligned} \quad (6.34)$$

$$\begin{aligned} &\int dx dx' \left\langle \mathcal{V}_s \Omega, \left(a_x^+ a_x a_y^+ a_y - \delta(x - y) a_x^+ a_y \right) \mathcal{V}_s \Omega \right\rangle \\ &= \| K_s \|_{L^2_{x,x'}}^2 \left(\| N_b(s) \Omega \|^2 - \| N_b^{\frac{1}{2}}(s) \Omega \|^2 \right) \end{aligned} \quad (6.35)$$

so that, using $\|N_b^{\frac{1}{2}}(s)\Omega\|^2 \leq \|N_b(s)\Omega\|$ from Cauchy–Schwarz,

$$\left\| \mathcal{V}_s^* \mathcal{H}_{cor,od}^{\phi_s,++} \mathcal{V}_s \Omega \right\|^2 \leq 2 \|K_s\|_{L^2_{x,x'}}^2 \left(\|N_b(s)\Omega\| + 1 \right)^2. \quad (6.36)$$

This implies (6.30). Similarly, one arrives at (6.29).

Therefore, (6.26) implies that

$$\begin{aligned} 1 + \|N_b^\alpha \mathcal{V}_t \Omega\| &= 1 + \|N_b^\alpha(t)\Omega\| \\ &\leq 1 + \|N_b^\alpha(0)\Omega\| + \int_0^t ds \|\partial_s N_b(s)\Omega\| \\ &\leq 1 + 2\alpha \int_0^t ds \|K_s\|_{L^2_{x,x'}}^2 \left(1 + \|N_b(s)\Omega\| \right), \end{aligned} \quad (6.37)$$

and by the Gronwall inequality,

$$1 + \|N_b^\alpha \mathcal{V}_t \Omega\| \leq \exp \left(4 \int_0^t ds \|K_s\|_{L^2_{x,x'}}^2 \right), \quad (6.38)$$

for $\alpha = 1, 2$.

Finally,

$$\begin{aligned} &\|K_s\|_{L_t^\infty L_{x,x'}^2([0,T] \times \mathbb{R}^3 \times \mathbb{R}^3)} \\ &\leq \frac{1}{2} \|\nabla \phi_s\|_{L_x^2}^2 + \frac{\lambda}{2} \left(\int |\phi_s(x)|^2 |v(x-y)|^2 |\phi_s(y)|^2 dx dy \right)^{\frac{1}{2}} \\ &\leq (1 + \lambda \|v\|_{L_x^\infty}) \|\phi_t\|_{L_x^\infty H_x^1([0,T] \times \mathbb{R}^3)}. \end{aligned} \quad (6.39)$$

This implies the claim of the lemma. \square

Lemma 6.4. *Let*

$$Q_b = \int dk \langle k \rangle a_k^+ a_k \quad (6.40)$$

where $\langle k \rangle^2 = 1 + k^2$. Then, the following estimate holds,

$$\left\| Q_b^\alpha \mathcal{V}(t, 0) \Omega \right\|_{\mathcal{F}} \leq e^{C_1 t}, \quad \alpha = 1, 2, \quad (6.41)$$

for a constant C_1 depending only on $\|v\|_{C^2}$ and $\|\phi\|_{L_t^\infty H^3([0,T] \times \mathbb{R}^3)}$.

Proof. We define

$$Q_b(t) := \mathcal{V}_t^* Q_b \mathcal{V}_t, \quad (6.42)$$

where $\mathcal{V}_t := \mathcal{V}(t, 0)$ as before. For $\alpha = 1, 2$, we find

$$i\partial_t Q_b^\alpha(t) = \mathcal{V}_t^* [Q_b^\alpha, \mathcal{H}_{cor,d}^{\phi_t} + \mathcal{H}_{cor,od}^{\phi_t}] \mathcal{V}_t. \quad (6.43)$$

Similarly to (6.27) and (6.28), we define

$$J_t(x, x') := \overline{(i\nabla\phi_t)(x)}(i\nabla\phi_t)(x') + \lambda v(x - y) \overline{\phi_t(x)}\phi_t(y), \quad (6.44)$$

so that

$$\mathcal{H}_{cor,d}^{\phi_t} = \frac{1}{2} \|\nabla\phi\|_{L_x^2}^2 + \int dx dx' J_t(x, x') a_x^+ a_{x'}. \quad (6.45)$$

Since

$$\begin{aligned} [Q_b, \mathcal{H}_{cor,d}^{\phi_t} + \mathcal{H}_{cor,od}^{\phi_t}] &= \int dx dx' a_x^* a_{x'} (\langle \nabla_x \rangle - \langle \nabla_{x'} \rangle) J_t(x, x') \\ &\quad + \int dx dx' \left\{ a_x^* a_{x'} (\langle \nabla_x \rangle + \langle \nabla_{x'} \rangle) K_t(x, x') - a_x a_{x'} (\nabla_x + \nabla_{x'}) \overline{K_t(x, x')} \right\} \end{aligned} \quad (6.46)$$

Then, the estimate

$$\begin{aligned} \partial_t \|Q_b(t)\Omega\| &\leq C \left(\|(\langle \nabla_x \rangle - \langle \nabla_{x'} \rangle) J_t\|_{L_{x,x'}^2} \right. \\ &\quad \left. + \|(\langle \nabla_x \rangle + \langle \nabla_{x'} \rangle) K_t\|_{L_{x,x'}^2} \right) (\|N_b(t)\Omega\| + 1) \\ &\leq C \left(\|(\langle \nabla_x \rangle - \langle \nabla_{x'} \rangle) J_t\|_{L_{x,x'}^2} + \|(\langle \nabla_x \rangle + \langle \nabla_{x'} \rangle) K_t\|_{L_{x,x'}^2} \right) (\|Q_b(t)\Omega\| + 1) \end{aligned} \quad (6.47)$$

follows from the same arguments as the proof of (6.29), (6.30).

For $\alpha = 2$, we have

$$\partial_t \|Q_b^2 \mathcal{V}_t \Omega\| \leq \| [Q_b^2, \mathcal{H}_{cor,d}^{\phi_t} + \mathcal{H}_{cor,od}^{\phi_t}] \mathcal{V}_t \Omega \|. \quad (6.48)$$

Taking the commutator with an operator acts as a derivation, thus

$$\begin{aligned} &[Q_b^2, \mathcal{H}_{cor,d}^{\phi_t} + \mathcal{H}_{cor,od}^{\phi_t}] \\ &= Q_b [Q_b, \mathcal{H}_{cor,d}^{\phi_t} + \mathcal{H}_{cor,od}^{\phi_t}] + [Q_b, \mathcal{H}_{cor,d}^{\phi_t} + \mathcal{H}_{cor,od}^{\phi_t}] Q_b \\ &= 2[Q_b, \mathcal{H}_{cor,d}^{\phi_t} + \mathcal{H}_{cor,od}^{\phi_t}] Q_b + [Q_b, [Q_b, \mathcal{H}_{cor,d}^{\phi_t} + \mathcal{H}_{cor,od}^{\phi_t}]] \end{aligned} \quad (6.49)$$

where, similarly to (6.46),

$$\begin{aligned}
 & [Q_b, [Q_b, \mathcal{H}_{cor,d}^{\phi_t} + \mathcal{H}_{cor,od}^{\phi_t}]] \\
 &= \int dx dx' a_x^* a_{x'} (\langle \nabla_x \rangle - \langle \nabla_{x'} \rangle)^2 J_t(x, x') \\
 &+ \int dx dx' \left\{ a_x^* a_{x'}^* (\langle \nabla_x \rangle + \langle \nabla_{x'} \rangle)^2 K_t(x, x') \right. \\
 &\quad \left. + a_x a_{x'} (\langle \nabla_x \rangle + \langle \nabla_{x'} \rangle)^2 \overline{K_t(x, x')} \right\}.
 \end{aligned} \tag{6.50}$$

In analogy to (6.47), we therefore find that

$$\partial_t \|Q_b^2 \mathcal{V}_t \Omega\| \leq A_1(t) (\|Q_b^2 \mathcal{V}_t \Omega\| + 1) + A_2(t) (\|Q_b \mathcal{V}_t \Omega\| + 1) \tag{6.51}$$

where

$$A_1(t) := C \left(\|(\langle \nabla_x \rangle - \langle \nabla_{x'} \rangle) J_t\|_{L_{x,x'}^2} + \|(\langle \nabla_x \rangle + \langle \nabla_{x'} \rangle) K_t\|_{L_{x,x'}^2} \right)$$

and

$$A_2(t) := C \left(\|(\langle \nabla_x \rangle - \langle \nabla_{x'} \rangle)^2 J_t\|_{L_{x,x'}^2} + \|(\langle \nabla_x \rangle + \langle \nabla_{x'} \rangle)^2 K_t\|_{L_{x,x'}^2} \right).$$

Since $\|Q_b \mathcal{V}_t \Omega\| \leq \|Q_b^2 \mathcal{V}_t \Omega\|$, we conclude that

$$\partial_t \|Q_b^2 \mathcal{V}_t \Omega\| \leq (A_1(t) + A_2(t)) (\|Q_b^2 \mathcal{V}_t \Omega\| + 1) \tag{6.52}$$

and hence,

$$\begin{aligned}
 \|Q_b^2 \mathcal{V}_t \Omega\| &\leq \exp \left(\int_0^t ds (A_1(s) + A_2(s)) \right) \\
 &\leq \exp \left(t \|A_1 + A_2\|_{L_t^\infty([0,T])} \right),
 \end{aligned} \tag{6.53}$$

using that $\|Q_b^2 \mathcal{V}_0 \Omega\| = 0$, due to $\mathcal{V}_0 = \mathbf{1}$.

Finally, we have, for $\alpha = 1, 2$,

$$\begin{aligned}
 & \|(\langle \nabla_x \rangle - \langle \nabla_{x'} \rangle)^\alpha J_t\|_{L_t^\infty L_{x,x'}^2([0,T] \times \mathbb{R}^3 \times \mathbb{R}^3)} \\
 &+ \|(\langle \nabla_x \rangle + \langle \nabla_{x'} \rangle)^\alpha K_t\|_{L_t^\infty L_{x,x'}^2([0,T] \times \mathbb{R}^3 \times \mathbb{R}^3)} \\
 &\leq (1 + \lambda \|v\|_{C^\alpha(\mathbb{R}^3)}) \|\phi_t\|_{L_t^\infty H_x^{1+\alpha}([0,T] \times \mathbb{R}^3)}^2,
 \end{aligned} \tag{6.54}$$

as can be easily checked. \square

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