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Zariski cancellation problem for non-domain noncommutative algebras

O. Lezama[1] · Y.-H. Wang[2] · J. J. Zhang[3]

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Abstract
We study Zariski cancellation problem for noncommutative algebras that are not necessarily domains.

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Introduction

This paper can be considered as a sequel to [5], where Bell–Zhang studied Zariski Cancellation Problem for noncommutative domains (in particular, for several families of Artin–Schelter regular algebras [2]). Many mathematicians have been making significant contributions to this research direction and related topics, see Brown–Yakimov [8], Ceken–Palmieri–Wang–Zhang [9,10], Chan–Young–Zhang [11,12], Gaddis [17], Gaddis–Kirkman–Moore [18], Gaddis–Won–Yee [19], Levitt–Yakimov [25], Lü–Mao–Zhang [27], Nguyen–Trampel–Yakimov [32], Tang [37,38] and others.

Throughout let k be a base commutative domain and everything is over k. If k is a field, and then it is denoted by $\Bbbk$ instead. For example we assume that k is a field $\Bbbk$ in Theorems 0.4 and 0.6. Recall that an algebra A is called *cancellative* if for any algebra B, an algebra isomorphism $A[t] \cong B[s]$ implies that $A \cong B$. Here t and s are independent central variables. In the commutative case, the famous Zariski Cancellation Problem (abbreviated as ZCP) asks

✉ J. J. Zhang
zhang@math.washington.edu

O. Lezama
jolezamas@unal.edu.co

Y.-H. Wang
yhw@mail.shufe.edu.cn

1 Departamento de Matemáticas, Universidad Nacional de Colombia, Bogotá, Colombia

2 School of Mathematics, Shanghai Key Laboratory of Financial Information Technology, Shanghai University of Finance and Economics, Shanghai 200433, China

3 Department of Mathematics, University of Washington, Box 354350, Seattle, WA 98195, USA

if the commutative polynomial ring $\Bbbk[x_1, \ldots, x_n]$ is cancellative for all $n \geq 1$. It is well-known that $\Bbbk[x_1]$ is cancellative by a result of Abhyankar–Eakin–Heinzer [1], while $\Bbbk[x_1, x_2]$ is cancellative by Fujita [16] and Miyanishi–Sugie [30] in characteristic zero and by Russell [34] in positive characteristic. The original ZCP (for $n \geq 3$) was open for many years. In 2014 Gupta [20,21] settled the ZCP negatively in positive characteristic for $n \geq 3$. The ZCP in characteristic zero remains open for $n \geq 3$.

The ZCP is related to the Automorphism Problem, Characterization Problem, Linearization Problem, Embedding Problem, and Jacobian Conjecture, see a discussion in [5,23]. In the noncommutative setting, the recent research suggests that it is closely related to the Automorphism Problem of noncommutative algebras, but it is unclear how to formulate the noncommutative version of the Characterization Problem, Linearization Problem, Embedding Problem and so on.

Some general methods were introduced in [5,27] to attack the ZCP for noncommutative domains. One effective method is to use the discriminant to control the cancellation property [5]. To extend the results in [5], it is natural to ask if the idea of the discriminant can be applied to non-domain noncommutative algebras.

The first aim of this paper is to introduce several new concepts and new methods to handle the ZCP for noncommutative algebras that are not domain. We study *retractable* and *detectable* properties of algebras and relate these properties to the discriminant computation. Then we generalize some results in [5] to the non-domain case. Results concerning the retractable and the detectable properties (and the rigidity introduced in [5]) can be summarized in the following diagram

$$\begin{array}{ccc}
\text{LND-rigid} & \longrightarrow & \text{LND}_Z\text{-rigid} \\
\downarrow & & \downarrow \\
\text{retractable} & \longrightarrow & Z\text{-retractable} \\
 & & \downarrow \\
 & & \text{detectable} \\
 & & \downarrow \\
 & & \text{cancellative.}
\end{array}$$

The second aim of this paper is to further understand which classes of noncommutative algebras are cancellative. A proposition in [5, Proposition 1.3] states that, if the center of an algebra A is the base field $\Bbbk$, then A is cancellative. This suggests

Question 0.1 *If the center of A is finite dimensional over $\Bbbk$, is A cancellative?*

We partially answer the above question.

Theorem 0.2 *Suppose A is strongly Hopfian (Definition 3.4) and the center of A is artinian. Then A is cancellative.*

Question 0.1 is still open for non-Hopfian algebras. Note that left (or right) noetherian algebras and locally finite $\mathbb{N}$-graded algebras are strongly Hopfian (Lemma 3.5). So Theorem 0.2 covers a large class of algebras. An immediate consequence of this theorem is the following corollary.

Corollary 0.3 *If A is left (or right) artinian, then A is cancellative. For example, every finite dimensional algebra over a base field $\Bbbk$ is cancellative.*

For non-artinian (and non-noetherian) algebras we have the following.

Theorem 0.4 *For every finite quiver Q, the path algebra $\Bbbk Q$ is cancellative.*

We refer to [24] for the definition and basic properties of the Gelfand–Kirillov dimension (or GKdimension for short). In this paper an affine algebra means that it is finitely generated over the base ring as an algebra. It is proved in [5, Theorem 0.5] that, if A is an affine domain of GKdimension two over an algebraically closed field of characteristic zero and if A is not commutative, then A is cancellative. In contrast, noncommutative affine prime (non-domain) algebras of GKdimension two need not be cancellative [Example 1.3(5)]. Further there are affine commutative domains of GKdimension two that are not cancellative, see [14] or Example 1.3(2). By a result of Abhyankar–Eakin–Heinzer [1, Theorem 3.3], every affine commutative domain of GKdimension one is cancellative. These results suggest the following question.

Question 0.5 *Is every affine prime $\Bbbk$-algebra of GKdimension one cancellative?*

We answer this question in the following case.

Theorem 0.6 *Let $\Bbbk$ be algebraically closed. Then every affine prime $\Bbbk$-algebra of GKdimension one is cancellative.*

Our third aim is to introduce some basic questions. For example, Theorem 0.6 leads naturally to the following two questions.

Question 0.7 *Let $\Bbbk$ be algebraically closed of characteristic zero and let A be an affine prime $\Bbbk$-algebra of GKdimension one. What can we say about the (outer) automorphism group of A?*

Question 0.8 *Let A be an algebra of global dimension one (respectively, Krull dimension one). Is then A is cancellative?*

A few other questions are listed in Sect. 5.

The paper is organized as follows. Section 1 contains definitions, known examples and preliminaries. In Sects. 2 and 3, we introduce retractable and detectable properties. In Sect. 4, we prove the Theorems 0.2, 0.4 and 0.6. In Sect. 5, some comments and questions are given.

1 Definitions and preliminaries

We recall some definitions, known examples and basic properties. First we copy the definition in [5, Definition 1.1].

Definition 1.1 Let A be an algebra.

(1) We call A *cancellative* if any algebra isomorphism $\phi : A[t] \cong B[s]$ for an algebra B implies that $A \cong B$.
(2) We call A *strongly cancellative* if, for each $n \geq 1$, any algebra isomorphism

$$A[t_1, \ldots, t_n] \cong B[s_1, \ldots, s_n]$$

for an algebra B implies that $A \cong B$.

In [1, p. 311], the *strongly cancellative* property is called the *invariant* property. In this paper we will study various versions of the cancellative property.

Definition 1.2 Let A be an algebra and let $\mathrm{LND}(A)$ be the collection of locally nilpotent $\Bbbk$-derivations of A.

(1) [28] The *Makar–Limanov invariant* of A is defined to be

$$ML(A) = \bigcap_{\delta \in \mathrm{LND}(A)} \ker(\delta).$$

(2) [5] A is called LND-rigid if $ML(A) = A$, or equivalently, $\mathrm{LND}(A) = \{0\}$.
(3) [5] A is called *strongly* LND-*rigid* if $ML(A[t_1, \ldots, t_n]) = A$ for all $n \geq 1$.
(4) The *Makar–Limanov center* of A is defined to be

$$ML_Z(A) = ML(A) \cap Z(A)$$

where $Z(A)$ denotes the center of A.
(5) We say that A is LND_Z-rigid if $ML_Z(A[t]) = Z(A)$, and *strongly* LND_Z *-rigid* if $ML_Z(A[t_1, \ldots, t_n]) = Z(A)$ for all $n \geq 1$.

Note that we use $A[t]$ (instead of A) in the definition of LND_Z-rigidity [Definition 1.2(5)] which is slightly different from the LND-rigidity in Definition 1.2(2). Next we give some known examples. Recall that if an affine domain (containing $\mathbb{Q}$) of finite GKdimension is LND-rigid, then it is cancellative [5, Theorem 0.4].

Example 1.3 Let $\Bbbk$ be an algebraically closed field of characteristic zero.

(1) Let A be an affine commutative domain of GKdimension one. If A is not isomorphic to $\Bbbk[x]$, then A is LND-rigid [13, Lemma 2.3]. In fact, it follows from [13, Main Theorem, p.6] that A is strongly LND-rigid. An earlier result of [1, Theorem 3.3 and Corollary 3.4] states that every affine commutative domain of GKdimension one is cancellative.
(2) Not every affine commutative domain of GKdimension two is cancellative. The following example is due to Danielewski [14]. Let $n \geq 1$ and let B_n be the coordinate ring of the surface $x^n y = z^2 - 1$ over $\Bbbk := \mathbb{C}$. Then $B_i \not\cong B_j$ if $i \neq j$, but $B_i[t] \cong B_j[s]$ for all $i, j \geq 1$, see [15,39] for more details.
(3) The famous Zariski cancellation problem (ZCP) asks if $\Bbbk[x_1, \ldots, x_n]$ is cancellative. This is still open for any $n \geq 3$ when char $\Bbbk = 0$. Also see Example 1.6 when char $\Bbbk > 0$.
(4) Here is an easy way of producing noncommutative algebras that are not cancellative. Starting with algebras B_i in part (2). Let A be a noncommutative algebra. If we can verify that $B_1 \otimes A \not\cong B_2 \otimes A$ (this is the case when the center of A is $\mathbb{C}$), then $B_1 \otimes A$ is not cancellative. For example, if A is the nth Weyl algebra, $B_1 \otimes A$ is not cancellative.
(5) As a consequence of part (4), by taking A to be the matrix algebra $M_n(\mathbb{C})$ for some $n > 1$, $M_n(B_1)$ is a noncommutative affine prime $\mathbb{C}$-algebra of GKdimension two that is not cancellative. See Theorem 0.6 for a related result.

One focus of this paper is to show that some classes of noncommutative algebras are cancellative. The following two lemmas contain elementary facts.

Lemma 1.4 *Let n be a positive integer. Let A and B be two algebras such that $A[t_1, \ldots, t_n] \cong B[s_1, \ldots, s_n]$. Then the following hold.*

(1) *A is commutative if and only if B is.*

(2) *A is left (or right) noetherian if and only if B is.*
(3) *Assume that A is left noetherian. Then,*

$$\operatorname{Kdim} A = \operatorname{Kdim} B.$$

Here Kdim *denotes the left Krull dimension.*
(4) *A is left (or right) artinian if and only if B is.*
(5) GKdim $A =$ GKdim B.
(6) *A is a field if and only if B is.*
(7) *A is a division ring if and only if B is.*
(8) *A is a finite direct sum of fields if and only if B is.*
(9) *A is a finite direct sum of division rings if and only if B is.*
(10) *A is local left (or right) artinian if and only if B is.*

For any algebra A, let $CI(A)$ denote the set of central idempotents of A. There are two trivial central idempotents $0, 1 \in CI(A)$. We say A is *indecomposable* if A is not isomorphic to $A_1 \oplus A_2$. It is clear that A is indecomposable if and only if $CI(A) = \{0, 1\}$. The next lemma is clear.

Lemma 1.5 *Let A be an algebra and let $Z(A)$ be the center of A.*

(1) $CI(A) = CI(Z(A))$.
(2) *The following are equivalent.*
    (a) $CI(A)$ *is finite.*
    (b) $|CI(A)| = 2^n$ *for some integer n.*
    (c) $A = A_1 \oplus A_2 \oplus \cdots \oplus A_n$ *where each A_i is an indecomposable algebra.*
(3) *If A is $\mathbb{N}$-graded, then* $CI(A) = CI(A_0)$.
(4) $CI(A[t_1, \ldots, t_n]) = CI(A)$.

To use the method of reduction modulo p, we need consider the case when the base field has positive characteristic. We start with the following example.

Example 1.6 [20,21] Let $\Bbbk$ be a base field of positive characteristic. Then $\Bbbk[x_1, \ldots, x_n]$ is not cancellative for every $n \geq 3$. This is a counterexample to the ZCP in positive characteristic.
Similar to Example 1.3(4), one can construct noncommutative algebras over a field of positive characteristic that are not cancellative.

When char $\Bbbk > 0$, the derivations in Definition 1.2 need to be replaced by higher derivations, which we now recall.

Definition 1.7 Let A be an algebra.

(1) [22] A *higher derivation* (or *Hasse–Schmidt derivation*) on A is a sequence of k-linear endomorphisms $\partial := \{\partial_i\}_{i=0}^{\infty}$ such that:

$$\partial_0 = id_A, \quad \text{and} \quad \partial_n(ab) = \sum_{i=0}^{n} \partial_i(a)\partial_{n-i}(b)$$

for all $a, b \in A$ and all $n \geq 0$. The collection of higher derivations is denoted by $\operatorname{Der}^H(A)$.
(2) A higher derivation is called *locally nilpotent* if
    (a) given every $a \in A$ there exists $n \geq 1$ such that $\partial_i(a) = 0$ for all $i \geq n$,

(b) the map

$$G_{\partial,t} : A[t] \to A[t]$$

defined by

$$a \mapsto \sum_{i=0}^{\infty} \partial_i(a)t^i, \quad t \mapsto t, \quad \text{for all } a \in A,$$

is an algebra automorphism of $A[t]$.

(3) The collection of locally nilpotent higher derivations is denoted by $\mathrm{LND}^H(A)$.
(4) For every $\partial \in \mathrm{Der}^H(A)$, the kernel of ∂ is defined to be

$$\ker \partial = \bigcap_{i \geq 1} \ker \partial_i.$$

Definition 1.8 Let A be an algebra.

(1) [5,28] The *Makar–Limanov*^{H} *invariant* of A is defined to be

$$ML^H(A) = \bigcap_{\delta \in \mathrm{LND}^H(A)} \ker(\delta). \tag{E1.8.1}$$

(2) [5] We say that A is LND^H*-rigid* if $ML^H(A) = A$, or $\mathrm{LND}^H(A) = \{0\}$.
(3) [5] A is called *strongly* LND^H*-rigid* if $ML^H(A[t_1, \ldots, t_n]) = A$, for all $n \geq 0$.
(4) The *Makar–Limanov*^{H} *center* of A is defined to be

$$ML^H_Z(A) = ML^H(A) \cap Z(A).$$

(5) A is called *strongly* LND^H_Z*-rigid* if $ML^H_Z(A[t_1, \ldots, t_n]) = Z(A)$, for all $n \geq 0$.

In [5, Theorem 3.3], the strong LND-rigidity (LND^H-rigidity) is the key to proving that several classes of algebras are cancellative. However, if A is not a domain (which is the case we are considering in the present paper), A is rarely LND-rigid (respectively, LND^H-rigid).

Lemma 1.9 *Let A be an algebra.*

(1) *If A has a non-central nilpotent element, then A is not* LND*-rigid.*
(2) *If A is a prime algebra that is not a domain, then every nilpotent element is not central. As a consequence, A is not* LND*-rigid.*
(3) *Let A be prime. If A is* LND*-rigid, then A is a domain.*

Proof (1) Let x be a non-central nilpotent element. Then $ad_x : a \mapsto xa - ax$ is a nonzero LND. So A is not LND-rigid.
(2) Since A is prime, $Z(A)$ is a domain. So every nilpotent element is not in $Z(A)$. Since A is not a domain, there are $0 \neq x, y \in A$ such that $xy = 0$. Since A is prime, $yAx \neq 0$. Let $f = yax \neq 0$ for some $a \in A$. Then $f^2 = 0$. By part (1) and the assertion we just proved, A is not LND-rigid.
(3) This follows from part (2).

□

By Lemma 1.9, all prime algebras that are not domains are not LND-rigid. However, in the next section, we will show that many non-domain prime algebras are LND_Z-rigid.

2 LND$_Z$-rigidity controls retractability

In this and the next sections we study two properties that are closely related to the cancellative property. Retractable property, see the next definition, was studied in a paper of Abhyankar–Eakin–Heinzer [1]. In [1, p. 311] it was called *strongly invariant* property. But we changed the term *invariant* to *strongly cancellative*, see Definition 1.1(2), and the term *strongly invariant* to *strongly retractable*, see the next.

Definition 2.1 Let A be an algebra.

(1) A is called *retractable* if, for any algebra B, any algebra isomorphism $\phi : A[t] \cong B[s]$ implies that $\phi(A) = B$.
(2) [1, p. 311] A is called *strongly retractable* if, for any algebra B and integer $n \geq 1$, any algebra isomorphism $\phi : A[t_1, \ldots, t_n] \cong B[s_1, \ldots, s_n]$ implies that $\phi(A) = B$.

Example 2.2 Let A be an affine commutative domain of GKdimension one over a field $\Bbbk$. By a result of [1, Theorem 3.3 and Corollary 3.4], if A is not $\Bbbk'[x]$ for some field extension $\Bbbk' \supseteq \Bbbk$, then it is strongly retractable; if $A = \Bbbk'[x]$ for some field extension $\Bbbk' \supseteq \Bbbk$, then it is strongly cancellative (but not retractable).

If A is retractable, then every algebra automorphism $\phi : A[t] \to A[t]$ restricts to an automorphism $\phi\mid_A : A \to A$. This implies that there is a retraction of the natural embedding

$$\mathrm{Aut}(A) \to \mathrm{Aut}(A[t]).$$

As a consequence, $\mathrm{Aut}(A[t])$ is a semidirect product of $\mathrm{Aut}(A) \ltimes N$ for some normal subgroup $N \subseteq \mathrm{Aut}(A[t])$.

Every strongly retractable algebra is retractable, and hence obviously, cancellative. For every fixed integer $n \geq 1$, $A[t_1, \ldots, t_n]$ may be abbreviated as $A[\underline{t}]$ and $B[s_1, \ldots, s_n]$ as $B[\underline{s}]$.

Lemma 2.3 *If A is a finite direct sum of division rings, then A is strongly retractable.*

Proof Suppose that $\phi : A[\underline{t}] \to B[\underline{s}]$ is an algebra isomorphism. Similar to Lemma 1.4(9), B is a direct sum of division algebras. Let $A^{\times}$ denote the set of invertible elements in A. In this case, one can show that $(A[\underline{t}])^{\times} = A^{\times}$ (the same is true for B). Let $A = D_1 \oplus \cdots \oplus D_r$ for some $r \geq 1$, where each D_i is a division ring. For every $a \in A$, write

$$a = (d_1, \ldots, d_r) = (d_1, 1, \ldots, 1)(1, 0, \ldots, 0) + \cdots + (1, \ldots, 1, d_r)(0, \ldots, 0, 1),$$

with $d_i \in D_i$, $1 \leq i \leq r$. Observe that $(1, \ldots 1, d_i, 1, \ldots, 1)$ is either invertible or idempotent, and $(0, \ldots, 0, 1, 0, \ldots, 0)$ is idempotent, hence A is generated by $(A[t])^{\times}$ and $CI(A[t])$. The same properties hold for B. Since ϕ maps $(A[\underline{t}])^{\times}$ to $(B[\underline{s}])^{\times}$ and $CI(A[\underline{t}])$ to $CI(B[\underline{s}])$, we have $\phi(A) \subseteq B$. By symmetry, $\phi(A) = B$. □

By [5] the Makar–Limanov invariants and LND-rigidity control cancellation, actually they control retractable property. The following result is basically [5, Theorem 3.3].

Lemma 2.4 *Suppose A is an affine domain of finite GK-dimension.*

(1) *If* $\mathrm{ML}(A[t]) = A$ *or* $\mathrm{ML}^H(A[t]) = A$*, then A is retractable.*
(2) *If A is strongly* LND*-rigid or strong* LND^H*-rigid, then A is strongly retractable.*

Proof See the proof of [5, Theorem 3.3]. □

The above lemma provides a lot of examples that are strongly retractable. We refer to papers [5,9,10,12,25] for many examples that are strongly LND-rigid or strongly LND^H-rigid.

For the rest of this section we want to deal with non-domain case.

Definition 2.5 Let A be an algebra.

(1) We call A *Z-retractable* if, for any algebra B, any algebra isomorphism $\phi : A[t] \cong B[s]$ implies that $\phi(Z(A)) = Z(B)$.
(2) We call A *strongly Z-retractable* if, for any algebra B and integer $n \geq 1$, any algebra isomorphism $\phi : A[t_1, \ldots, t_n] \cong B[s_1, \ldots, s_n]$ implies that $\phi(Z(A)) = Z(B)$.

It is clear that (strongly) retractable algebras are (strongly) Z-retractable. Copying the proof of [5, Theorem 3.3], we also have the following, which is similar to Lemma 2.4.

Lemma 2.6 *Let A be an algebra such that the center $Z(A)$ is an affine domain.*

(1) *Suppose $ML_Z(A[t]) = Z(A)$ or $ML_Z^H(A[t]) = Z(A)$. Then A is Z-retractable.*
(2) *Suppose that A is strongly* LND_Z*-rigid (or strongly* LND_Z^H*-rigid). Then A is strongly Z-retractable.*

By [5, Section 5], effectiveness of the discriminant controls LND^H-rigidity. Since we will not compute explicitly discriminants in this paper, to save space, we refer to [5,9,10] for the definition of the discriminant. But we will use the idea of "effectiveness" in later sections, we now recall its definition next. An algebra is called *PI* if it satisfies a polynomial identity.

Definition 2.7 [5, Definition 5.1] Let A be a domain and suppose that $Y = \bigoplus_{i=1}^n kx_i$ generates A as an algebra. An element $f \in A$ is called *effective* if the following conditions hold.

(1) There is an $\mathbb{N}$-filtration $\{F_i A\}_{\geq i}$ on A such that the associated graded ring gr A is a domain (one possible filtration is the trivial filtration $F_0 A = A$). With this filtration we define the degree of elements in A, denoted by $\deg_A$.
(2) For every testing $\mathbb{N}$-filtered PI algebra T with gr T being an $\mathbb{N}$-graded domain and for every testing subset $\{y_1, \ldots, y_n\} \subset T$ satisfying

    (a) it is linearly independent in the quotient k-module $T/k1_T$, and
    (b) $\deg y_i \geq \deg x_i$ for all i and $\deg y_{i_0} > \deg x_{i_0}$ for some i_0,

    there is a presentation of f of the form $f(x_1, \ldots, x_n)$ in the free algebra $k\langle x_1, \ldots, x_n\rangle$, such that either $f(y_1, \ldots, y_n)$ is zero or $\deg_T f(y_1, \ldots, y_n) > \deg_A f$.

The following is a special case of [5, Lemma 5.3(6)].

Example 2.8 [5, Lemma 5.3(6)] Let $A = \Bbbk[x]$. Every non-unit element $f \in A$ is effective in A. To prove this we write $f = \sum_{i=0}^n a_i x^i$ where $a_i \in \Bbbk$ and $a_n \neq 0$. Without loss of generality we might assume that $a_n = 1$. Consider A as a filtered algebra generated by $Y = \Bbbk x$ by defining $\deg x = 1$. Clearly gr $A \cong A$ is a domain. Let T be any testing filtered algebra and let $\{y\}$ be a testing subset of T. If $\deg y > 1 = \deg x$, then $\deg f(y) = n \deg y > n = \deg f$. Therefore f is effective.

More complicated examples of effective elements are given in [5, Section 5]. There is another concept, called "dominating", see [5, Definition 4.5] or [9, Definition 2.1(2)], that is similar to effective. Both of these properties control LND$_Z^H$-rigidity. The following result is similar to [5, Theorem 5.2].

Theorem 2.9 *Let A be a PI prime algebra such that the discriminant (or w-discriminant for some w) over its center Z is effective (respectively, dominating) in Z. Suppose Z is an affine domain, then A is strongly LND_Z^H-rigid.*

Proof Since the proofs for the "effective" case and the "dominating" case are very similar, we only prove the "effective" case. We also copy the proof of [5, Theorem 5.2].

Let f be the discriminant of A over Z, $d(A/C_A)$ (or w-discriminant $d_w(A/C_A)$). Suppose Z is generated by $\{x_1, \dots, x_n\}$ as in Definition 2.7 (as f is effective).

Let $R = k[t_1, \dots, t_d][t]$. By a prime algebra version of [5, Lemma 4.6(2)],

$$d_w(A \otimes R/C_A \otimes R) = d_w(A/C_A) = f,$$

which is effective by hypothesis. Let $\partial \in \mathrm{LND}^H(A[t_1, \dots, t_d])$. By definition, $G := G_{\partial,t} \in \mathrm{Aut}_{k[t]}(A[t_1, \dots, t_d][t])$. For each j,

$$G(x_j) = x_j + \sum_{i \geq 1} t^i \partial_i(x_j).$$

Then G is also an automorphism of $Z[t_1, \dots, t_d][t]$. We take the test algebra T to be $Z[t_1, \dots, t_d][t]$ where the filtration on T is induced by the filtration on Z together with $\deg t_s = 1$ for all $s = 1, \dots, d$ and $\deg t = \alpha$ where α is larger than $\deg \partial_i(x_j)$ for all $j = 1, \dots, n$ and all $i \geq 1$. Now set $y_j = G(x_j) \in T$. By the choice of α, we have that

(a) $\deg y_j \geq \deg x_j$, and that
(b) $\deg y_j = \deg x_j$ if and only if $y_j = x_j$.

If $G(x_j) \neq x_j$ for some j, by effectiveness as in Definition 2.7, $\deg f(y_1, \dots, y_n) > \deg f$. So $f(y_1, \dots, y_n) \neq_{A^\times} f$. But $f(y_1, \dots, y_n) = G(f) =_{Z^\times} f$ by [5, Lemma 4.4(4)], a contradiction. Therefore $G(x_j) = x_j$ for all j. As a consequence, $\partial_i(x_j) = 0$ for all $i \geq 1$, or equivalently, $x_j \in \ker \partial$. Since Z is generated by x_j's, $Z \subset \ker \partial$. Thus $Z \subseteq ML_Z^H(A[t_1, \dots, t_d])$. It is clear that $Z \supseteq ML_Z^H(A[t_1, \dots, t_d])$, see [5, Example 2.4]. Therefore $Z = ML_Z^H(A[t_1, \dots, t_d])$ as required. □

Corollary 2.10 *Let A be a prime PI algebra over a field $\Bbbk$ such that the center of A is $\Bbbk[x]$. If the discriminant of A over $Z(A)$ is a nonunit in $Z(A)$, then A is LND_Z^H-rigid.*

Proof This follows from Example 2.8 and Theorem 2.9. □

3 Detectability and cancellation

If B is a subring of C and $f_1, \dots, f_m$ are elements of C, then the subring generated by B and the set $\{f_1, \dots, f_m\}$ is denoted by $B\{f_1, \dots, f_m\}$.

Definition 3.1 Let A be an algebra.

(1) We call A *detectable* if, for any algebra B, an algebra isomorphism $\phi : A[t] \cong B[s]$ implies that $B[s] = B\{\phi(t)\}$, or equivalently, $s \in B\{\phi(t)\}$.
(2) We call A *strongly detectable* if, for any algebra B and any integer $n \geq 1$, an algebra isomorphism

$$\phi : A[t_1, \dots, t_n] \cong B[s_1, \dots, s_n]$$

implies that $B[s_1, \dots, s_n] = B\{\phi(t_1), \dots, \phi(t_n)\}$, or equivalently, for each $i = 1, \dots, n$, $s_i \in B\{\phi(t_1), \dots, \phi(t_n)\}$.

In the above definition, we do not assume that $\phi(t) = s$. Every strongly detectable algebra is detectable. The polynomial ring $\Bbbk[x]$ is cancellative, but not detectable.

Lemma 3.2 *If A is Z-retractable (respectively, strongly Z-retractable), then it is detectable (respectively, strongly detectable).*

Proof We only show the "strongly" version. The proof of the non-"strongly" version is similar.

Suppose that A is strongly Z-retractable. Let B be any algebra such that $\phi : A[\underline{t}] \to B[\underline{s}]$ is an isomorphism. Since A is strongly Z-retractable, ϕ restricts to an isomorphism $\phi\mid_{Z(A)}: Z(A) \to Z(B)$. Write $f_i := \phi(t_i)$ for $i = 1, \ldots, n$. Then

$$\begin{aligned} Z(B)\{f_1, \ldots, f_n\} &= \phi(Z(A))\{\phi(t_1), \ldots, \phi(t_n)\} \\ &= \phi(Z(A)\{t_1, \ldots, t_n\}) = \phi(Z(A)[\underline{t}]) \\ &= \phi(Z(A[\underline{t}])) = Z(B[\underline{s}]) \\ &= Z(B)[\underline{s}]. \end{aligned}$$

Then, for every i, $s_i \in Z(B)[\underline{s}] = Z(B)\{f_1, \ldots, f_n\} \subseteq B\{f_1, \ldots, f_n\}$ as desired. □

Retractability is stronger than detectability. The next example shows that these two properties are not equivalent.

Example 3.3 Let $A = \Bbbk[x, y]/(x^2 = y^2 = xy = 0)$. By Theorem 4.1 in the next section, A is strongly detectable. But A is not retractable as we show next. Define an isomorphism $\phi : A[t] \to A[s]$ where $\phi(x) = x$, $\phi(t) = s$ and $\phi(y) = y + sx$. Clearly $\phi(A) \neq A$, so A is not retractable.

Next we show that detectability implies cancellation under some mild conditions. We first recall a definition.

Definition 3.4 Let A be a k-algebra.

(1) We say A is *Hopfian* if every k-algebra epimorphism from A to itself is an automorphism.
(2) We say A is *strongly Hopfian* if $A[t_1, \ldots, t_n]$ is Hopfian for every $n \geq 0$.

In this definition, Hopfian property is dependent on the base ring k. Hopfian algebras have been studied by several authors [6,26,29,33]. Every left noetherian algebra is strongly Hopfian, see the next lemma. Some non-noetherian examples can be constructed using Lemma 3.5(2,3). An $\mathbb{N}$-graded k-algebra $A = \bigoplus_{i=0}^{\infty} A_i$ is called *locally finite* if each homogeneous component A_i is a finitely generated k-module.

Lemma 3.5 *The following algebras are strongly Hopfian.*

(1) *Left or right noetherian algebras.*
(2) *Finitely generated locally finite $\mathbb{N}$-graded k-algebras that are k-flat.*
(3) *Prime affine $\Bbbk$-algebras satisfying a polynomial identity.*

Proof (1) Let A be left noetherian, then A is Hopfian. Since $A[t_1, \ldots, t_n]$ is also left noetherian, A is strongly Hopfian.

(2) If A is an affine locally finite $\mathbb{N}$-graded algebra, so is $A[t]$. Thus it suffices to show that A is Hopfian. Replacing A by its localization $A \otimes_k \Bbbk$ where $\Bbbk$ is the fraction field of k, we might assume that k is a field $\Bbbk$. Note that the k-flatness insures that $A \otimes_k \Bbbk$ being $\Bbbk$-Hopfian implies that A is k-Hopfian.

Next we assume that $k = \Bbbk$ which is a field. In this case the assertion is a special case of [7, Theorem 7, p.77], which is a consequence of a result of Mal'tsev (or Malcev) [29]. We include a detailed proof here for the convenience of the reader.

We prove that A is $\Bbbk$-Hopfian by contradiction. Suppose that $\phi : A \to A$ is a surjective endomorphism of A that has a nonzero kernel. Fix $0 \neq r \in \ker\phi$. Let $r \in \oplus_{i=0}^{s} A$, for some integer $s \geq 0$, and write $r = \sum_{i=0}^{s} r_i$ where some $r_i \in A_i$ are nonzero. Choose a $\Bbbk$-linear basis $\mathbf{b} := \{b_j\}_{j=1}^{d}$ of $\oplus_{i=0}^{s} A$ of homogeneous elements so that $\mathbf{b}$ contains all nonzero r_is.

We use R to denote subrings of $\Bbbk$ of the special form $\mathbb{Z}\{f_1, \ldots, f_w\}$ when char $\Bbbk = 0$ or of the special form $\mathbb{F}_p\{f_1, \ldots, f_w\}$ when char $\Bbbk = p > 0$. Since A is finitely generated, A is generated by a finite set of homogeneous generators, say $\mathbf{g} := \{g_t\}_{t=1}^{d}$. We may assume that $\mathbf{g}$ is $\Bbbk$-linearly independent and contains $\mathbf{b}$. Then there is a subring $R \subseteq \Bbbk$ of the form specified as above such that ϕ restricts a surjective R-algebra endomorphism of A_R, where A_R denotes the R-subalgebra $R\{g_1, \ldots, g_d\}$ of A generated by $\mathbf{g}$. Adding only finitely many new f_i to R if necessary we can assume that every product of any two generators $g_{t_1}, g_{t_2} \in \mathbf{g}$ has coefficients in R in terms of the basis $\mathbf{b}$ if such a product has degree no more than s. By the choice of $\mathbf{b}$ and $\mathbf{g}$, it is clear that $r \in A_R$ and that $\oplus_{i=0}^{s}(A_R)_i$ is a finitely generated free R-module with R-basis $\mathbf{b}$. By the construction of R, every simple factor ring F of R is a finite field. The induced map $\phi_R \otimes F : A_R \otimes_R F \to A_R \otimes_R F$ is still a surjective F-algebra endomorphism. Since F is finite and $A_R \otimes_R F$ is locally finite over F, $A_R \otimes_R F$ is residually finite in the sense that the ideals of finite index in ring have a trivial intersection. By [26, Theorem 3], $A_R \otimes_R F$ is Hopfian (and then F-Hopfian). This yields a contradiction because $\phi_R \otimes_R F$ is a surjective endomorphism such that $\phi_R \otimes_R F(r \otimes_R 1) = 0$, but $r \otimes_R 1 \neq 0$. Therefore the assertion follows.

(3) This is basically [6, Corollary 2.3].

□

We need Hopfian algebras in the next lemma.

Lemma 3.6 *Suppose A is strongly Hopfian.*

(1) *If A is detectable, then A is cancellative.*
(2) *If A is strongly detectable, then A is strongly cancellative.*

Proof We only prove (2).

Let $\phi : A[\underline{t}] \to B[\underline{s}]$ be an isomorphism. Let $f_i = \phi(t_i)$ for $i = 1, \ldots, n$. Then f_i are central elements in $B[\underline{s}]$. Thus $B\{f_1, \ldots, f_n\}$ is a homomorphic image of $B[s_1, \ldots, s_n]$ by sending $s_i \mapsto f_i$. Suppose A is strongly detectable. Then $B\{f_1, \ldots, f_n\} = B[\underline{s}]$. Then we have algebra homomorphisms

$$B[\underline{s}] \xrightarrow{\pi} B\{f_1, \ldots, f_n\} \xrightarrow{=} B[\underline{s}] \cong A[\underline{t}]. \tag{E3.6.1}$$

Since A is strongly Hopfian, $A[\underline{t}]$ and then $B[\underline{s}]$ are Hopfian. Now (E3.6.1) implies that π is an isomorphism. As a consequence, $B\{f_1, \ldots, f_n\} = B[f_1, \ldots, f_n]$ considering f_i as central indeterminants in $B[f_1, \ldots, f_n]$. Now we have

$$A \cong A[\underline{t}]/(t_i) \xrightarrow{\overline{\phi}} B[f_1, \ldots, f_n]/(f_i) \cong B.$$

Therefore A is strongly cancellative. □

Combining these results we have the following diagram for an algebra.

$$\begin{array}{ccc}
 & & \text{[Discriminant being] effective} \\
 & & \text{Remark 3.7(5)} \not\Uparrow \big\downarrow \text{Theorem 2.9} \\
\text{(strongly) LND-rigid} & \xrightarrow[\not\Leftarrow \text{Remark 3.7(4)}]{\text{easy}} & \text{(strongly) LND}_Z\text{-rigid} \\
\text{Lemma 2.4} \big\downarrow \Uparrow \text{Remark 3.7(6)} & & \Uparrow \text{Remark 3.7(7)} \big\downarrow \text{Lemma 2.6} \\
\text{(strongly) retractable} & \xrightarrow[\not\Leftarrow \text{Remark 3.7(3)}]{\text{easy}} & \text{(strongly) } Z\text{-retractable} \\
 & & \text{Remark 3.7(2)} \not\Uparrow \big\downarrow \text{Lemma 3.2} \\
 & & \text{(strongly) detectable} \\
 & & \text{Remark 3.7(1)} \not\Uparrow \big\downarrow \text{Lemma 3.6} \\
 & & \text{(strongly) cancellative}
\end{array}$$

If char $\Bbbk > 0$, one should replace LND by LND^H in the above diagram.

Remark 3.7 The following are easy.

(1) $\Bbbk[x]$ is (strongly) cancellative, but not detectable.
(2) The algebra in Example 3.3 is strongly detectable, but not retractable (or Z-retractable).
(3) Let $Z = \Bbbk[x^{\pm 1}]$ and $A = M_2(Z)$. Then the center of A is Z. By Example 1.3(1), Z is strongly retractable. Therefore A is strongly Z-retractable. Consider the conjugation automorphism $\phi : A[t] \to A[t]$ determined by

$$\phi : a \longrightarrow \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} a \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}, \qquad \forall a \in A.$$

It is easy to see that $\phi(A) \neq A$. Therefore A is not retractable.
(4) One can also check that the algebra A in part (3) is strongly LND_Z-rigid, but not LND-rigid by Lemma 1.9.
Here is another example. Let A be an affine PI prime such that the discriminant over its affine center is effective, for example, A is in Example 5.1. By Theorem 2.9, A is strongly LND_Z-rigid. If A is not a domain [Example 5.1], then it is not LND-rigid by Lemma 1.9.
(5) Let $A = \Bbbk[x^{\pm 1}]$. Since A is a commutative domain, the discriminant of A over its center is trivial, whence, not effective. But A is strongly LND-rigid and strongly LND_Z-rigid by Example 1.3(1).
(6) The strong LND-rigidity (when char $\Bbbk = 0$) and the strong LND^H-rigidity follow from the strong retractability without any hypotheses as given in Lemma 2.4. For example, if A is not strongly LND-rigidity and if char $\Bbbk = 0$, then there is a locally nilpotent derivation of ∂ of $A[t_1, \dots, t_n]$, for some $n \geq 1$, such that $\ker \partial \not\supseteq A$. Consider the automorphism

$$\exp(\partial, t) : A[t_1, \dots, t_n][t] \to A[t_1, \dots, t_n][t]$$

determined by

$$\exp(\partial, t) : a \mapsto \sum_{i=0}^{\infty} \frac{t^i}{i!} \partial^i(a), \quad t \mapsto t,$$

for all $a \in A[t_1, \dots, t_n]$. Then $\exp(\partial, t)(A) \neq A$. Therefore A is not strongly retractable. If ∂ is a higher derivation, one needs use the automorphism similar to the one in Definition 1.7(2).

(7) The strong LND_Z-rigidity (when char $\Bbbk = 0$) and the strong LND_Z^H-rigidity follow from the strong Z-retractability without any hypotheses as given in Lemma 2.6. The argument in part (6) can be used with some small modification.

One advantage of considering the detectable property is the following.

Lemma 3.8 *Let A be an algebra with center $Z(A)$. If $Z(A)$ is (strongly) detectable, so is A.*

Proof Suppose B is an algebra such that $\phi : A[\underline{t}] \cong B[\underline{s}]$. Taking the center, we have $\phi : Z(A)[\underline{t}] \cong Z(B)[\underline{s}]$. Since $Z(A)$ is strongly detectable, $s_i \in Z(B)\{\phi(t_j)\}$ for all i. Thus $s_i \in B\{\phi(t_j)\}$ for all i. This means that A is strongly detectable. □

Another property concerning detectable property is the following.

Lemma 3.9 *Let A be an algebra and J be the prime radical of A that is nilpotent. If A/J is (strongly) detectable, so is A.*

Proof Suppose B is any algebra such that $\phi : A[\underline{t}] \cong B[\underline{s}]$. Since $J^n = 0$ for some n, the prime radical of $A[\underline{t}]$ is $J[\underline{t}]$ which is also nilpotent. This implies that the prime radical of $B[\underline{s}]$ is nilpotent and of the form $J(B)[\underline{s}]$ where $J(B)$ is the prime radical of B. Modulo the prime radical, we obtain that

$$\phi' : \quad (A/J)[\underline{t}] \cong (B/J(B))[\underline{s}]$$

where $\underline{t}$ and $\underline{s}$ still denote the images of $\underline{t}$ and $\underline{s}$ in appropriate algebras respectively. Since A/J is strongly detectable, $s_i \in (B/J(B))\{f_1, \dots, f_n\}$ for all i, which means that $s_i \in B\{f_1, \dots, f_n\}$ modulo $J(B)[\underline{s}]$. Here $f_i = \phi(t_i)$ for all i. Another way of saying this is, for every i,

$$f_i = s_i + \sum b_{d_1,\dots,d_n} s_1^{d_1} \cdots s_n^{d_n} \in B\{f_1, \dots, f_n\}$$

for $b_{d_1,\dots,d_n} \in J(B)$. The point is that $s_i = f_i$ modulo $J(B)[\underline{s}]$. Now we re-write s_i as a polynomial in f_j with coefficients in $J(B)$, starting with,

$$\begin{aligned} s_i &= f_i - \sum b_{d_1,\dots,d_n} s_1^{d_1} \cdots s_n^{d_n} \\ &= f_i - \sum b_{d_1,\dots,d_n} f_1^{d_1} \cdots f_n^{d_n} + \sum b'_{d'_1,\dots,d'_n} s_1^{d'_1} \cdots s_n^{d'_n} \end{aligned}$$

where $b'_{d'_1,\dots,d'_n}$ are in $J(B)^2[\underline{s}]$. This means that s_i equals a polynomial of $f_1, \dots, f_n$ modulo $J(B)^2[\underline{s}]$. By induction, s_i equals a polynomial of $f_1, \dots, f_n$ modulo $J(B)^p[\underline{s}]$ for every $p \geq 1$. Since $J(B)$ is nilpotent, s_i is a polynomial of $f_1, \dots, f_n$ (with coefficients in B) when taking $p \gg 0$. Therefore $s_i \in B\{f_1, \dots, f_n\}$ as required. □

We also mention an easy consequence of Lemma 3.8 and [1, Theorem 3.3].

Proposition 3.10 *If the center of A is an affine domain of GKdimension one that is not isomorphic to $\Bbbk'[x]$ for some field extension $\Bbbk' \supseteq \Bbbk$, then A is strongly detectable.*

Proof By [1, Theorem 3.3], $Z(A)$ is strongly retractable. By Lemma 3.2, $Z(A)$ is strongly detectable. The assertion follows from Lemma 3.8. □

Lemma 3.11 *Let $\Bbbk'$ is a field extension of $\Bbbk$. If $A \otimes_{\Bbbk} \Bbbk'$ is (strongly) detectable as an algebra over $\Bbbk'$, then A is (strongly) detectable as an algebra over $\Bbbk$.*

Proof We only show the "strongly" version. Suppose that B is an algebra over $\Bbbk$ such that $\phi : A[\underline{t}] \cong B[\underline{s}]$ is a $\Bbbk$-algebra isomorphism. Let C be the $\Bbbk$-subalgebra of $B\{\phi(t_i)\}_{i=1}^{n}$. Then $C \subseteq B[\underline{s}]$. We claim that $C = B[\underline{s}]$.

For any $\Bbbk$-module M, let $M_{\Bbbk'}$ denote $M \otimes_{\Bbbk} \Bbbk'$. Consider the isomorphism $\phi_{\Bbbk'} : A_{\Bbbk'}[\underline{t}] \cong B_{\Bbbk'}[\underline{s}]$. By hypothesis, $A_{\Bbbk'}$ is a (strongly) detectable algebra over $\Bbbk'$. Then $B_{\Bbbk'}\{\phi(t_i)\}_{i=1}^{n}$ is $B_{\Bbbk'}[\underline{s}] = (B[\underline{s}])_{\Bbbk'}$. Now one can easily show that $C_{\Bbbk'} = B_{\Bbbk'}\{\phi(t_i)\}_{i=1}^{n}$. Thus $C_{\Bbbk'} = (B[\underline{s}])_{\Bbbk'}$, consequently, $C = B[\underline{s}]$. Therefore A is strongly detectable as an algebra over $\Bbbk$. □

4 Applications to the cancellation problem

In this section we will use the results proven in the previous sections to show some classes of algebras are cancellative. We first prove Corollary 0.3.

Theorem 4.1 *If A is left (or right) artinian, then A is strongly detectable. As a consequence, A is strongly cancellative.*

Proof Since A is artinian, it is noetherian. By Lemma 3.5(1), it is strongly Hopfian. Now the strongly cancellative property is a consequence of the strongly detectable property by Lemma 3.6(2). It remains to show that A is strongly detectable.

Let J be the Jacobson radical of A. Then J is also the prime radical of A and J is nilpotent. By Lemma 3.9, it suffices to show that $A' := A/J$ is strongly detectable. Since A' is a finite direct sum of matrix algebras over division rings, $Z(A')$ is a finite direct sum of fields. By Lemmas 2.3 and 3.2, $Z(A')$ is strongly retractable, and then strongly detectable. By Lemma 3.8, A' is strongly detectable as required. □

Theorem 4.2 *Let A be an algebra with center Z. Suppose J is the prime radical of Z such that* (a) *J is nilpotent and* (b) *Z/J is a finite direct sum of fields.*

(1) *A is strongly detectable.*
(2) *If further A is strongly Hopfian, then A is strongly cancellative.*

Theorem 0.2 is an immediate consequence of the above theorem.

Proof of Theorem 4.2 (1) Similar to the proof of Theorem 4.1, Z is strongly detectable. By Lemma 3.8 A is strongly detectable.

(2) Follows from Lemma 3.6 and part (1).

□

The following is an immediate consequence of the above theorem and Lemma 3.5(1).

Corollary 4.3 *Let A be a left noetherian algebra such that its center is artinian. Then A is strongly detectable and strongly cancellative.*

Next we will prove Theorem 0.4. We start with the following lemma. We refer to [3] for basic definitions of quivers and their path algebra. Let C_n be the cyclic quiver with n vertices and n arrow connecting these vertices in one oriented direction. In representation theory of finite dimensional algebras, quiver C_n is also called *type* $\widetilde{A}_{n-1}$. Let $0, 1, \ldots, n-1$ be the vertices of C_n, and $a_i : i \to i+1$ (in $\mathbb{Z}/(n)$) be the arrows in C_n. Then $w := \sum_{i=0}^{n-1} a_i a_{i+1} \cdots a_{i+n}$ is a central element in $\Bbbk C_n$.

Lemma 4.4 *Let Q be a finite quiver that is connected and let $\Bbbk Q$ be its path algebra. Then*

$$Z(\Bbbk Q) = \begin{cases} \Bbbk & \text{if } Q \text{ has no arrows,} \\ \Bbbk[x] & \text{if } Q = C_1 \text{ or equivalently } \Bbbk Q = \Bbbk[x], \\ \Bbbk[w] & \text{if } Q = C_n \text{ for } n \geq 2, \\ \Bbbk & \text{otherwise.} \end{cases}$$

Proof Since $\Bbbk Q$ is $\mathbb{N}$-graded, its center $Z(\Bbbk Q)$ is also $\mathbb{N}$-graded.

If Q has one vertex without arrow, $\Bbbk Q = \Bbbk$ and the center is $\Bbbk$. If $Q = C_1$, then $\Bbbk Q = \Bbbk[x]$ and the center is $\Bbbk[x]$. If Q has one vertex with more than one arrows, then $\Bbbk Q$ is a free algebra and in this case, its center is $\Bbbk$.

For the rest of the proof we suppose that Q has at least two vertices. Let $\{e_i\}_{i=1}^{v}$ be the vertices in Q where $v \geq 2$.

First of all, every central element of degree 0 is in the base field $\Bbbk$. Let f be a nonzero central element of degree $d > 0$. Write f as $\sum c a_{s_1} a_{s_2} \cdots a_{s_d}$ where $c \in \Bbbk$ and a_{s_i} are arrows in Q. Since f is central, $e_i f = f e_i$. Thus,

$$f = \left(\sum_{i=1}^{v} e_i\right) f = \sum_{i=1}^{v} e_i f = \sum_{i=1}^{v} e_i f e_i = \sum_{i=1}^{v} f_i$$

where $f_i = e_i f e_i$. Since $f \neq 0$, $f_i \neq 0$ for some i. Without loss of generality, we may assume that $f_1 \neq 0$.

If a is an arrow from 1 to i with $i \neq 1$, then

$$0 \neq f_1 a = f a = a f = a f_i.$$

So $f_i \neq 0$. Similarly, if a is an arrow from $i \neq 1$ to 1, then $f_i a = a f_1 \neq 0$. Since Q is connected, every vertex is connected with the vertex 1. It follows by induction that $f_i \neq 0$. Further, for every arrow b from i to j, we have

$$f_i b = b f_j \neq 0. \tag{E4.4.1}$$

Write $f_i = \sum c a_{s_1} a_{s_2} \cdots a_{s_d}$ with some $c \in \Bbbk$. Then equation (E4.4.1) implies that

$$a_{s_1} = b \tag{E4.4.2}$$

for all possible arrow b from i to j and all possible nonzero terms $c a_{s_1} a_{s_2} \cdots a_{s_d}$ in f_i. Equation (E4.4.2) implies that

(1) b is unique, and
(2) a_{s_1} is unique.

This means that, for every i, there is a unique j, denoted by $\sigma(i)$, such that there is a unique arrow from i to $\sigma(i)$. Since Q is connected with v vertices, $\sigma(i) \neq i$ for all i. This characterizes the quiver C_n. Now its routine to check that the center of $\Bbbk C_n$ is as described.

The above paragraph shows that, if Q is not C_n, then there is no nonzero central element of positive degree. Therefore, $Z(\Bbbk Q) = \Bbbk$. This finishes the proof. □

Lemma 4.5 *Let $Q = C_n$ for $n \geq 2$.*

(1) *$\Bbbk Q$ is a prime algebra of GKdimension 1 that is not Azumaya.*
(2) *$\Bbbk Q$ is strongly detectable and strongly cancellative.*

Proof (1) By a direct computation.

(2) If $\Bbbk$ is algebraically closed, this is a special case of Theorem 4.12(2), which we will prove later. If $\Bbbk$ is not algebraically closed, let $\Bbbk'$ be the closure of $\Bbbk$. By Theorem 4.12(2), $\Bbbk' Q$ is strongly detectable over $\Bbbk'$. By Lemma 3.11, $\Bbbk Q$ is strongly detectable over $\Bbbk$, and then strongly cancellative by Lemmas 3.5(2) and 3.6(2). □

We need an elementary lemma. Two idempotents e_1 and e_2 in an algebra A are *counital* if $e_1 + e_2 = 1$.

Lemma 4.6 *Let A and B be two algebras.*

(1) *If A and B are (strongly) cancellative, so is $A \oplus B$.*
(2) *If A and B are (strongly) retractable, so is $A \oplus B$.*
(3) *If A and B are (strongly) detectable, so is $A \oplus B$.*

Proof We only prove part (1). The proof of parts (2, 3) is similar.

Suppose $\phi : (A \oplus B)[\underline{t}] \cong C[\underline{s}]$ is an algebra isomorphism, and let e_1 and e_2 be two counital idempotents corresponding to the decomposition $A \oplus B$. Let $f_i = \phi(e_i)$ for $i = 1, 2$. By Lemma 1.5(4), f_1 and f_2 are two counital idempotents of C. Thus $C = C_1 \oplus C_2$ and $A[\underline{t}] \cong C_1[\underline{s}]$ and $B[\underline{t}] \cong C_2[\underline{s}]$. Since A and B are strongly cancellative, $A \cong C_1$ and $B \cong C_2$, which implies that $A \oplus B \cong C_1 \oplus C_2 = C$. The proof of non-"strongly" version is similar. □

The next is Theorem 0.4.

Theorem 4.7 *Let Q be a finite quiver and let A be the path algebra $\Bbbk Q$. Then A is strongly cancellative. If further Q has no connected component being C_1, then A is strongly detectable.*

Proof By Lemma 4.6, we may assume that Q is connected.

If $Q = C_1$, then $A = \Bbbk[x]$ and the assertion follows from [1, Theorem 3.3 and Corollary 3.4].

If $Q = C_n$, then this is Lemma 4.5(2).

If $Q \neq C_n$ for any $n \geq 1$, then by Lemma 4.4, the center of A is $\Bbbk$. By Theorem 4.2(1), A is strongly detectable. Since A is $\mathbb{N}$-graded and locally finite, it is strongly Hopfian by Lemma 3.5(2). By Theorem 4.2(2), A is strongly cancellative. This completes the proof. □

For the rest of this section we consider affine prime algebras of GKdimension one and prove Theorem 0.6. For simplicity, we assume that $\Bbbk$ is algebraically closed. It is not immediately clear to us if this assumption can be removed. However, this assumption definitely makes some of argument and even language easier.

Let A be an affine prime algebra of GKdimension one. By a result of Small–Warfield [36], A is a finitely generated module over its affine center. As a consequence, A is noetherian.

Let R be a commutative algebra, an R-algebra A is called *Azumaya* if A is a finitely generated faithful projective R-module and the canonical morphism

$$A \otimes_R A^{op} \to \mathrm{End}_R(A) \qquad \text{(E4.7.1)}$$

is an isomorphism. Note that the Brauer group of a commutative algebra R, denoted by $Br(R)$, is the set of Morita-type-equivalence classes of Azumaya algebras over R, in other words, $Br(R)$ classifies Azumaya algebras over R up to an equivalence relation [4]. See [35] for some discussion about the Brauer group. Another way of defining an Azumaya algebra is the following (when $\Bbbk$ is algebraically closed). We refer to [8, Introduction] for a discussion.

Definition 4.8 [8, Introduction] Suppose that $\Bbbk$ is algebraically closed. Let A be an affine prime $\Bbbk$-algebra which is a finitely generated module over its affine center $Z(A)$. Let n be the PI-degree of A, which is also the maximal possible $\Bbbk$-dimension of irreducible A-modules.

(1) The *Azumaya locus* of A, denoted by $\mathcal{A}(A)$, is the dense open subset of Maxspec $Z(A)$ which parametrizes the irreducible A-modules of maximal $\Bbbk$-dimension. In other word, $\mathfrak{m} \in \mathcal{A}(A)$ if and only if $\mathfrak{m}A$ is the annihilator in A of an irreducible A-module V with $\dim V = n$, if and only if $A/\mathfrak{m}A \cong M_n(\Bbbk)$.
(2) If $\mathcal{A}(A) = \text{Maxspec } Z(A)$, A is called *Azumaya*.

There are several other equivalent definitions of an Azumaya algebra, but we will only use Definition 4.8(2).

For affine prime algebras of GKdimension one, we have the following.

Lemma 4.9 *Let A be an affine prime algebra of GKdimension one over an algebraically closed field $\Bbbk$. Then one of the following holds.*

(1) $Z(A) \ncong \Bbbk[x]$.
(2) $Z(A) \cong \Bbbk[x]$ *and A is not Azumaya.*
(3) $Z(A) \cong \Bbbk[x]$ *and A is Azumaya. In this case, A is a matrix algebra over* $\Bbbk[x]$.

Proof The only non-trivial assertion is the second statement in part (3).

Now we consider $R = \Bbbk[x]$. By [4, Theorem 7.5], $\Bbbk[x]$ has trivial Brauer group when $\Bbbk$ is algebraically closed. Since every projective $\Bbbk[x]$-module is free, $\text{Hom}_{\Bbbk[x]}(E, E)$ is a matrix algebra over $\Bbbk[x]$ for every finitely generated projective $\Bbbk[x]$-module E. By definition [4, Section 5], every Azumaya algebra A over $\Bbbk[x]$ is Morita equivalent to $\Bbbk[x]$. Again, by the fact every finitely generated projective $\Bbbk[x]$-module is free, A is a matrix algebra over $\Bbbk[x]$. □

In fact, if char $\Bbbk = 0$, by [4, Proposition 7.7], every Azumaya algebra over $\Bbbk[x_1, \ldots, x_n]$ is a matrix algebra over $\Bbbk[x_1, \ldots, x_n]$. But this is not true when char $\Bbbk > 0$ and $n \geq 2$, see Remark 5.2.

Proposition 4.10 *Suppose that $Z(A) = \Bbbk[x]$ and that A is not Azumaya.*

(1) *If* char $\Bbbk = 0$, *then A is strongly* LND_Z^H*-rigid.*
(2) *In general, A is Z-retractable.*

As a consequence, A is strongly detectable and strongly cancellative.

Proof The consequence follows from Lemmas 2.6, 3.2 and 3.6.

(1) Thanks to a wonderful result of Brown-Yakimov [8, Main Theorem], the discriminant of A over $Z(A)$ is nonunit (this happens if and only if A is not Azumaya). Since $Z(A) = \Bbbk[x]$, by Corollary 2.10, A is strongly LND_Z^H-rigid.
(2) If char $\Bbbk$ is positive, a trace on A might not be representation theoretic in the sense of [8, Section 2.1]. In this case, we can not use [8, Main Theorem]. So we need a slightly different approach. Let C be a PI prime algebra that is finitely generated over its center $Z(C)$. Let $\mathcal{N}(C)$ denote the complement of the Azumaya locus of C and let $N(C)$ be the ideal of $Z(C)$ that corresponds to $\mathcal{N}(C)$. Then $N(C)$ is a nonzero proper ideal of $Z(C)$ if C is not Azumaya.

Let D be the polynomial extension

$$D := C[t_1, \dots, t_n] \cong C \otimes \Bbbk[t_1, \dots, t_n]$$

for some $n \geq 1$. Then D is a finitely generated module over its center $Z(D) = Z(C)[t_1, \dots, t_n]$. Using Definition 4.8, one can routinely show that

$$N(C[t_1, \dots, t_n]) = N(C)[t_1, \dots, t_n]. \tag{E4.10.1}$$

Now let A be the algebra in the proposition and let B be an algebra such that $\phi : A[t_1, \dots, t_n] \cong B[s_1, \dots, s_n]$. We claim that $\phi(Z(A)) = Z(B)$. First of all, ϕ induces an isomorphism $Z(A)[t_1, \dots, t_n] \cong Z(B)[s_1, \dots, s_n]$. Since $Z(A) = \Bbbk[x]$ is strongly cancellative, $Z(B) = \Bbbk[y]$. Since A is not Azumaya, there is a $f \in \Bbbk[x] \backslash \Bbbk$ such that $N(A) = f \Bbbk[x]$. Similarly, $N(B) = g\Bbbk[y]$ for some $g \in \Bbbk[y] \setminus \Bbbk$. It follows from (E4.10.1) that

$$\begin{aligned} g\Bbbk[y, s_1, \dots, s_n] &= N(B)[s_1, \dots, s_n] \\ &= N(B[s_1, \dots, s_n]) \\ &\cong N(A[t_1, \dots, t_n]) \qquad \text{(via } \phi^{-1}) \\ &= f\Bbbk[x, t_1, \dots, t_n]. \end{aligned}$$

Thus ϕ maps f to a scalar multiple of g. Define

$$W_f(A[t_1, \dots, t_n]) = \{w \in Z(A[t_1, \dots, t_n]) \mid \exists v \in Z(A[t_1, \dots, t_n]), s.t.\ wv = f\}.$$

Recall that $Z(A[t_1, \dots, t_n]) = \Bbbk[x, t_1, \dots, t_n]$ and $f \in Z(A) = \Bbbk[x]$ is not a scalar. Then $W_f(A[t_1, \dots, t_n])$ is a subset of $\Bbbk[x]$ containing at least an element of form $a + x$ for some $a \in \Bbbk$. Similarly, $W_g(B[t_1, \dots, t_n])$ is a subset of $\Bbbk[y]$ containing an element of form $a' + y$ for some $a' \in \Bbbk$. Since ϕ maps $W_f(A[t_1, \dots, t_n])$ to $W_g(B[t_1, \dots, t_n])$ and since $Z(A)$ (respectively, $Z(B)$) is generated by $a + x$ (respectively, $a' + y$), we have $\phi(Z(A)) = Z(B)$ as required. □

Lemma 4.11 *The matrix algebra $A := M_n(\Bbbk[x])$ is strongly cancellative.*

Proof If $n = 1$, this is [1, Theorem 3.3 and Corollary 3.4]. Now assume that $n > 1$. Suppose B is an algebra such that $\phi : A[\underline{t}] \cong B[\underline{s}]$ is an isomorphism. Then $Z(A)[\underline{t}] \cong Z(B)[\underline{s}]$. Since $Z(A) = \Bbbk[x]$, by [1, Theorem 3.3 and Corollary 3.4], $Z(B) \cong \Bbbk[x]$. It is clear that B is prime of GKdimension one. If B is not Azumaya over $Z(B)$, then B is strongly cancellative by Proposition 4.10. As a consequence, $A \cong B$. If B is Azumaya over $Z(B)$, then B is a matrix algebra over $\Bbbk[x]$ by Lemma 4.9(3), say $B = M_{n'}(\Bbbk[x])$. Now algebra isomorphisms

$$M_n(\Bbbk[x, t_1, \dots, t_n]) \cong A[\underline{t}] \cong B[\underline{s}] \cong M_{n'}(\Bbbk[x, s_1, \dots, s_n])$$

implies that $n = n'$. Therefore $A \cong B$ as desired. □

Next we are ready to prove Theorem 0.6.

Theorem 4.12 *Let A be an affine prime algebra of GKdimension one over an algebraically closed field.*

(1) *A is strongly cancellative.*
(2) *If either $Z(A) \neq \Bbbk[x]$ or A is not Azumaya, then A is strongly detectable.*

Proof By Lemma 4.9, there are three cases to consider.

Case 1: $Z(A) \ncong \Bbbk[x]$. By [1, Theorem 3.3 and Corollary 3.4], $Z(A)$ is strongly retractable. By Lemma 3.2, A is strongly detectable, and by Lemma 3.6(2), A is strongly cancellative.

Case 2: $Z(A) \cong \Bbbk[x]$ and A is not Azumaya. The assertion follows from Proposition 4.10.

Case 3: $Z(A) \cong \Bbbk[x]$ and A is Azumaya. The assertion follows from Lemma 4.9(3) and 4.11. □

5 Examples, remarks and questions

First of all, it is quite easy to produce a large family of prime, but non-domain, PI rings that are strongly cancellative.

Example 5.1 Let R be an affine commutative domain and let f be a product of a set of generating elements of R. Let

$$A = \begin{pmatrix} R & fR \\ R & R \end{pmatrix}.$$

Then the discriminant of A over its center R is $-f^2$. It easy to see that $-f^2$ is a effective element in R. So A is strongly LND_Z^H-rigid [Theorem 2.9]. As a consequence, A is strongly Z-retractable, detectable and cancellative.

Recall that the Brauer group of a commutative domain A is denoted by $Br(A)$.

Remark 5.2 Let $\Bbbk$ be an algebraically closed field of characteristic zero and let A be a commutative affine regular domain over $\Bbbk$. By [4, Proposition 7.7], $Br(A)$ is naturally isomorphic to $Br(A[t])$. Consequently, $Br(\Bbbk[x_1, \dots, x_n])$ is trivial for all $n \geq 0$. If $A[t] \cong B[s]$ for some algebra B, then B is also a commutative affine regular domain. Therefore

$$Br(A) \cong Br(A[t]) \cong Br(B[s]) \cong Br(B).$$

We are wondering if there are other connections between the cancellative property of A and the structure of $Br(A)$. For example, what kind of structure of $Br(A)$ forces A to be cancellative?

If $\Bbbk$ is an algebraically closed field of positive characteristic, the situation is different. First of all, $Br(\Bbbk[x]) = Br(\Bbbk) = \{1\}$ by [4, Theorem 7.5]. By [31, Lemma 6.1], the nth Weyl algebra is a nontrivial Azumaya algebra over its center and that the center of the nth Weyl algebra is isomorphic to the polynomial ring of $2n$ variables. This implies that $Br(\Bbbk[x_1, \dots, x_{2n}])$ is non-trivial for $n \geq 1$. Since $Br(A)$ is a subgroup of $Br(A[t])$, $Br(\Bbbk[x_1, \dots, x_m])$ is non-trivial for all $m \geq 2$. It is known that, by [34], $\Bbbk[x_1, x_2]$ is cancellative, while, by [20], $\Bbbk[x_1, x_2, x_3]$ is not.

The study of connections between the cancellative property of A and other invariants such as the Picard group or Grothendieck group of A might also be interesting.

Remark 5.3 It would be interesting to know if retractable (respectively, detectable, cancellative) property is preserved under some usual constructions. More precisely, suppose A has property $\mathcal{P}$, where $\mathcal{P}$ stands for the *retractable*, *detectable*, or *cancellative* property.

(a) Under what hypotheses, does a localization AS^{-1} have $\mathcal{P}$?
(b) Under what hypotheses, does the skew polynomial extension $A[t, \sigma, \delta]$ have $\mathcal{P}$?

(c) Under what hypotheses, does the matrix algebra $M_n(A)$ have $\mathcal{P}$?

We are also wondering the following.

(d) Suppose A is filtered with associated graded ring gr A. If gr A has property $\mathcal{P}$ (where $\mathcal{P}$ stands for the *retractable*, *detectable*, or *cancellative* property), under what hypotheses, does A have $\mathcal{P}$?
(e) Let $J(A)$ be the Jacobson radical of A. Suppose $A/J(A)$ has $\mathcal{P}$, under what hypotheses, does A have $\mathcal{P}$?

The study of noncommutative Cancellation Problem has just started a couple of years ago. One can ask if the cancellative property holds for any interesting class of noncommutative algebras. Here are some examples.

Question 5.4 *Prove or disprove the following classes of algebras are cancellative.*

(1) *Preprojective algebras.*
(2) *The Weyl algebras in positive characteristic. Note that the Weyl algebras in characteristic zero are cancellative by [5, Proposition 1.3].*
(3) *The Sklyanin algebras of dimension* ≥ 3*, or the PI Sklyanin algebras of dimension* ≥ 3.

Several other questions have already been stated in the introduction.

Finally we have the following remark from the referee report. We thank the referee for allowing us to include it in this paper.

Remark 5.5 (Referee) Question 0.7 is vague, but one can say a bit about the outer automorphism group. If A is an affine prime algebra of GKdimension one, then, by Small–Warfield theorem [36], A is a finite module over its center Z, which is also affine by the Artin–Tate lemma. Then Z is an affine domain of Krull dimension one. Now let $X = \mathrm{Spec}(Z)$ and let Y be the normalization of its projective closure. Then every automorphism of X lifts to an automorphism of Y. Now let G be the group of automorphisms of A that are the identity on Z. Then G induces an automorphism of the quotient algebra $Q(A)$ of A, which is a central simple algebra, and this automorphism fixes the center of $Q(A)$. Then every element σ of G is inner; that is, there is some $u \in Q(A)$ such that σ is the conjugation by u. Since $Q(A) = (Z \setminus \{0\})^{-1}A$, we can clear denominators and assume that $u \in A$. Since σ is an automorphism of A, one sees that u is in fact a normal element in A and therefore σ is an inner automorphism of A. The above argument shows that there is an injective group homomorphism from the outer automorphism group of A to the automorphism group $\mathrm{Aut}(Z)$, given by the restriction

$$g \mapsto g|_Z.$$

As said before, $\mathrm{Aut}(X)(= \mathrm{Aut}(Z))$ lifts to the automorphism group of Y but there is more: these automorphisms must share common periodic points, and permute the points at infinity. If Y has genus ≥ 2, then the automorphism group of Y is finite; if Y has genus 1 then it is an elliptic curve, but since the automorphisms permute a finite (nonempty) set of points, some finite-index subgroup has a fixed point, which means that these can be seen as group automorphisms of an elliptic curve and so they are again finite. Finally if Y has genus zero, then X is rational and $Z \subseteq \Bbbk(t)$. In this case, if X is not smooth, then the automorphisms preserve the singular points of X and the points at infinity, so a finite-index subgroup of the automorphism group of Z induces a subgroup of $\mathrm{Aut}(Y)$ where $Y \cong \mathbb{P}^1$, in which after a change of coordinates every element fixes 0 and 1. Now the automorphisms that fix 0 and

1 are just those of the form $t \mapsto ct$, with $c \in \Bbbk^{\times}$. So the automorphism group of X has a finite-index subgroup that is a subgroup of a torus. Otherwise X is smooth. So if we have two or more points at infinity, the same argument applies, so we may assume that X is smooth, has one point at infinity, so $Z \cong \Bbbk[t]$ and so every automorphism is of the form $t \mapsto at + b$ for $a \in \Bbbk^{\times}$ and $b \in \Bbbk$. Combining these cases, the outer automorphism group of A always has a finite-index subgroup that is isomorphic to a subgroup of the affine linear group. Of course, taking A commutative shows we can get the torus, the affine linear group, etc. as outer automorphism groups. Presumably, one can also say that the outer automorphism group is an algebraic group, by studying how automorphisms of the center lift. In this case, it has dimension at most 2 and it has connected component of the identity that is solvable and when one mods out by the unipotent radical one always has $\Bbbk^{\times}$ or $\{1\}$.

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