RESEARCH ARTICLE

Maximal Cohen-Macaulay modules over a noncommutative 2-dimensional singularity

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Abstract   We study properties of graded maximal Cohen-Macaulay modules over an $\mathbb{N}$-graded locally finite, Auslander Gorenstein, and Cohen-Macaulay algebra of dimension two. As a consequence, we extend a part of the McKay correspondence in dimension two to a more general setting.

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1   Introduction

A noncommutative version of the McKay correspondence in dimension two was developed in [4–6]. One of the main ingredients was the study of the invariant subrings of connected graded, noetherian, Artin-Schelter regular algebras of global dimension two under natural actions of quantum binary polyhedral groups. The McKay quivers [6, Definition 2.9] of these quantum binary polyhedral groups are twisted versions of $\widetilde{A}\widetilde{D}\widetilde{E}$ graphs where the details can be found in [4, Proposition 7.1]. It was proved in [6, Theorem B] that the McKay quiver is isomorphic to the Gabriel quiver [6, Definition 2.8] of the smash product algebra corresponding to the action.

The noncommutative singularities (or equivalently, their associated algebras) studied in [4–6] are usually far from commutative and do not satisfy a polynomial identity. For these noncommutative singularities, we introduced the concept of a noncommutative quasi-resolution [15, Definition 0.5] which gener-

alizes Van den Bergh's noncommutative crepant resolution [22,23]. The smash product constructions used in [4–6] are examples of noncommutative quasi-resolutions. Recently, Reyes-Rogalski [17] proved that the Gabriel quivers of non-connected, $\mathbb{N}$-graded, Artin-Schelter regular algebras of global dimension two are twisted versions (which are called *pretzeled quivers* in this paper) of the $\widetilde{A}\widetilde{D}\widetilde{E}$ graphs.

Recent study of the invariant theory of (non-connected graded) preprojective algebras under finite group actions initiated by Weispfenning [24] suggests that one should extend the noncommutative McKay correspondence to a larger class of not necessarily connected, graded algebras. The aim of this paper is to supply a small piece of the puzzle in this slightly more general version of the noncommutative McKay correspondence. Let $\Bbbk$ be a base field and let MCM (resp., CM) stand for 'maximal Cohen-Macaulay' (resp., 'Cohen-Macaulay'). We summarize the main results as follows.

Theorem 1 *Let A be a noetherian $\mathbb{N}$-graded locally finite algebra of Gelfand-Kirillov dimension two. Suppose that*

   (a) *A has a balanced dualizing complex,*

   (b) *A is Auslander Gorenstein and CM, and*

   (c) *A has a noncommutative quasi-resolution B.*

Then we have the following statements.

   (1) *A is of finite Cohen-Macaulay type in the graded sense.*

   (2) *There is a one-to-one correspondence between the set of indecomposable MCM graded right A-modules up to degree shifts and isomorphisms and the set of graded simple right B-modules up to degree shifts and isomorphisms.*

   (3) *Let $\{M_1, M_2, \ldots, M_d\}$ be a complete list of the indecomposable MCM graded right A-modules up to degree shifts and isomorphisms. Then, for some choice of integers $w_1, w_2, \ldots, w_d$,*

$$C := \operatorname{End}_A\left(\bigoplus_{i=1}^d M_i(w_i) \right)$$

is an $\mathbb{N}$-graded noncommutative quasi-resolution of A. As a consequence, C is graded Morita equivalent to B.

   (4) *A is a noncommutative graded isolated singularity.*

It is natural to ask that when does A have a noncommutative quasi-resolution, one can find a partial answer in [15].

One should compare Theorem 1 with results in [9, Section 3] in the commutative case and [10, Theorem 2.5], [6, Corollary 4.5, Theorem 5.7] in the noncommutative case.

Theorem 2 *Let A satisfy the hypotheses in Theorem 1. Furthermore, assume that the noncommutative quasi-resolution B in Theorem 1 (c) is standard in the sense of Definition 5 (below). Then the following statements hold:*

(1) *the Gabriel quiver of B is a pretzeled quiver of a finite union of graphs of $\widetilde{A}\widetilde{D}\widetilde{E}$ type;*

(2) *the standard noncommutative quasi-resolution B is unique up to isomorphism.*

Theorem 2 confirms that a generalized version of the noncommutative McKay correspondence should still be within the framework of $\widetilde{A}\widetilde{D}\widetilde{E}$ diagrams. Since the Gabriel quiver $\mathscr{G}(B)$ of B is defined by using simple modules over B, by the correspondence given in Theorem 1 (3) and the uniqueness in Theorem 2 (2), $\mathscr{G}(B)$ (if it exists) is also an invariant of MCM modules over A. The proof of Theorem 2 follows from Theorem 1 and results of Reyes-Rogalski when one relates the results in [16,17] with the concept of noncommutative quasi-resolutions.

Terminology used in the above theorems will be explained in later sections. The proofs of Theorems 1 and 2 will be given in Section 3. By Lemma 30 below, Theorems 1 and 2 can be applied to fixed subrings of preprojective algebras that was recently studied by Weispfenning [24].

2   Definitions and preliminaries

Throughout, let $\Bbbk$ be a field. All algebras and modules are over $\Bbbk$. Recall that a $\Bbbk$-algebra A is $\mathbb{N}$-*graded* if $A = \oplus_{n\in\mathbb{N}}A_n$ as vector spaces with $1 \in A_0$ and $A_iA_j \subseteq A_{i+j}$ for all $i,j \in \mathbb{N}$. We say that A is *locally finite* if $\dim_{\Bbbk} A_n < \infty$ for all n. In this paper, a graded algebra usually means $\mathbb{N}$-graded. A right A-module M is $\mathbb{Z}$-graded if $M = \oplus_{n\in\mathbb{Z}}M_n$ with $M_iA_j \subseteq M_{i+j}$ for all $i,j \in \mathbb{Z}$. We write GrMod A for the category of right graded A-modules with morphisms being the degree preserving homomorphisms, and grmod A for the subcategory of finitely generated right A-modules. Other definitions such as degree shift or grading shift (w) can be found in [16,17].

2.1   Generalized Artin-Schelter regular algebras

In this subsection, we review the definition of a generalized Artin-Schelter (AS) regular algebra.

Definition 3 [16, Definition 1.4]   Let A be a locally finite graded algebra and $J := J(A)$ be the graded Jacobson radical, that is, the intersection of all graded maximal right ideals of A. Write $S = A/J$. We say that A is *generalized AS Gorenstein of dimension d* if A has graded injective dimension d and there is a graded invertible (S,S)-bimodule V such that

$$\operatorname{Ext}_A^i(S,A) \cong \begin{cases} V, & i = d, \\ 0, & i \neq d, \end{cases} \tag{2.1}$$

as (S,S)-bimodules. If further A has graded global dimension d, then A is called *generalized AS regular of dimension d*.

Definition 4 [17, Section 3]   Let A be a locally finite $\mathbb{N}$-graded $\mathbb{k}$-algebra. If the finite-dimensional semisimple algebra

$$S := A/J(A) \cong A_0/J(A_0)$$

is isomorphic to a product $\mathbb{k}^{\oplus d}$ of finitely many copies of the base field $\mathbb{k}$, then we say that A is *elementary*.

Definition 5   Let A be a graded algebra. We say that A is *standard* if A_0 is $\mathbb{k}^{\oplus d}$ for some positive integer d and A is generated by A_0 and A_1.

A very nice result proven by Reyes-Rogalski is the following theorem.

Theorem 6 [17]   *Let A be an $\mathbb{N}$-graded generalized AS regular algebra. Suppose that*

(a) *A is standard,*

(b) *A is noetherian,*

(c) *$\text{gldim } A = 2$.*

Then A is isomorphic to an algebra $A_2(Q, \tau)$ as described in [17, Definition 7.5], *where Q is a quiver whose arrows all have weight 1 and whose spectral radius $\rho(Q)$ is 2.*

Proof   We combine some results of Reyes-Rogalski. By [16, Theorem 1.5], A is twisted Calabi-Yau in the sense of [16, Definition 1.2]. By [17, p. 37], every locally finite elementary graded twisted Calabi-Yau algebra of global dimension 2 is isomorphic to an algebra $A_2(Q, \tau)$. Since A is standard, the weight of every arrow in Q is 1. Since A is noetherian, by [17, Theorem 7.8 (2)], $\rho(Q) = 2$. $\square$

It is not clear if the inverse of Theorem 6 is true, see the comments after the proof of [17, Lemma 7.6].

2.2   Noncommutative quasi-resolutions

In this subsection, we review some definitions about noncommutative quasi-resolutions from [15]. We assume that all algebras are noetherian in this subsection. First, we recall the definition of Gelfand-Kirillov dimension.

Definition 7 [11, Definition 2.1]   Let A be an algebra and M a right A-module.

(1) The *Gelfand-Kirillov dimension* (GKdim) of A is defined to be

$$\text{GKdim } A = \sup_V \left\{ \varlimsup_{n \to \infty} \log_n (\dim V^n) \mid V \subseteq A \right\},$$

where V ranges over all finite-dimensional $\mathbb{k}$-subspaces of A.

(2) The *Gelfand-Kirillov dimension* of M is defined to be

$$\text{GKdim } M = \sup_{V,W} \left\{ \varlimsup_{n \to \infty} \log_n (\dim WV^n) \mid V \subseteq A, \, W \subseteq M \right\},$$

where V and W range over all finite-dimensional $\mathbb{k}$-subspaces of A and M, respectively.

If M is a finitely generated graded module over a locally finite graded algebra A, then its GKdim can be computed by [26, (E7)]

$$\mathrm{GKdim}\, M = \varlimsup_{n \to \infty} \log_n \sum_{j \leqslant n} \dim(M_j).$$

For simplicity, we always work with the dimension function GKdim in this paper.

Definition 8 [13, Definitions 1.2, 2.1]   Let A be an algebra and M a right A-module.

(1)   The *grade number* of M is defined to be

$$j_A(M) := \inf\{i \mid \mathrm{Ext}_A^i(M, A) \neq 0\} \in \mathbb{N} \cup \{+\infty\}.$$

If no confusion can arise, we write $j(M)$ for $j_A(M)$. Note that $j_A(0) = +\infty$.

(2)   We say that M satisfies the *Auslander condition* if for any $q \geqslant 0$, $j_A(N) \geqslant q$ for all left A-submodules N of $\mathrm{Ext}_A^q(M, A)$.

(3)   We say that A is *Auslander-Gorenstein* (resp., *Auslander regular*) of dimension n if

$$\mathrm{injdim}\, A_A = \mathrm{injdim}\,{}_A A = n < \infty \quad (\text{resp., } \mathrm{gldim}\, A = n < \infty)$$

and every finitely generated left and right A-module satisfies the Auslander condition.

A graded version of an Auslander-Gorenstein (resp., Auslander regular) algebra is defined similarly.

Definition 9 [1, Definition 0.4]   Let A be a locally finite graded algebra. We say that A is *graded Cohen-Macaulay* (*graded CM*) if

$$\mathrm{GKdim}(A) = d \in \mathbb{N},$$

and

$$j(M) + \mathrm{GKdim}(M) = \mathrm{GKdim}(A)$$

for every graded finitely generated nonzero left (or right) A-module M.

Next, we specialize some definitions in [15] to the GKdim case and omit the prefix 'GKdim' in some cases.

Let A be a locally finite $\mathbb{N}$-graded algebra and n a nonnegative integer. Let $\mathrm{GrMod}_n A$ denote the full subcategory of $\mathrm{GrMod}\, A$ consisting of $\mathbb{Z}$-graded right A-modules M with $\mathrm{GKdim}(M) \leqslant n$. Since GKdim is exact over a noetherian locally finite $\mathbb{N}$-graded algebra [11, Theorem 6.14], $\mathrm{GrMod}_n A$ is a Serre subcategory of $\mathrm{GrMod}\, A$. Hence, it makes sense to define the quotient categories

$$\mathrm{QGr}_n A := \frac{\mathrm{GrMod}\, A}{\mathrm{GrMod}_n A}, \quad \mathrm{qgr}_n A := \frac{\mathrm{grmod}\, A}{\mathrm{grmod}_n A}.$$

We denote the natural and exact projection functor by

$$\pi \colon \mathrm{GrMod}\, A \to \mathrm{QGr}_n A. \tag{2.2}$$

For $M \in \mathrm{GrMod}\, A$, we write $\mathscr{M}$ for the object $\pi(M)$ in $\mathrm{QGr}_n A$. The hom-sets in the quotient category are defined by

$$\mathrm{Hom}_{\mathrm{QGr}_n A}(\mathscr{M}, \mathscr{N}) = \varinjlim \mathrm{Hom}_A(M', N'), \quad M, N \in \mathrm{GrMod}\, A, \tag{2.3}$$

where M' is a graded submodule of M such that the GKdim of M/M' is no more than n, $N' = N/T$ for some graded submodule $T \subseteq N$ with $\mathrm{GKdim}(T) \leqslant n$, and where the direct limit runs over all the pairs (M', N') with these properties. Note that π is also defined from

$$\mathrm{grmod}\, A \to \mathrm{qgr}_n A. \tag{2.4}$$

It follows from [15, (E1.10.1)] that the functor π in (2.2) has a right adjoint functor

$$\omega \colon \mathrm{QGr}_n A \to \mathrm{GrMod}\, A. \tag{2.5}$$

By [15, Proposition 2.10 (2)], if M is $(n+2)$-pure, in the sense of Definition 24 (6) below, over an Auslander Gorenstein and CM algebra A, then $\omega(\pi(M))$ agrees with the Gabber closure defined in [15, Definition 2.8].

We will use $\mathrm{qgr}_0 A$ in Section 2???, which will be denoted by $\mathrm{qgr}\, A$.

Definition 10 [15, Definition 1.5] Let $n \geqslant 0$. Let A and B be two locally finite graded algebras.

(1) Two $\mathbb{Z}$-graded right A-modules X, Y are called *n-isomorphic*, denoted by $X \cong_n Y$, if there exist a $\mathbb{Z}$-graded right A-module P and homogeneous homomorphisms of degree zero $f \colon X \to P$ and $g \colon Y \to P$ such that both the kernel and cokernel of f and g are in $\mathrm{GrMod}_n A$.

(2) Two $\mathbb{Z}$-graded (B, A)-bimodules X, Y are called *n-isomorphic*, denoted by $X \cong_n Y$, if there exist a $\mathbb{Z}$-graded (B, A)-bimodule P and homogeneous bimodule homomorphisms with degree zero $f \colon X \to P$ and $g \colon Y \to P$ such that both the kernel and cokernel of f and g are in $\mathrm{GrMod}_n A$ when viewed as graded right A-modules.

Let $\mathscr{A}$ be a category consisting of $\mathbb{N}$-graded, locally finite, noetherian algebras with finite GKdim [15, Example 3.1]. Our definition of a noncommutative quasi-resolution will be made inside the category $\mathscr{A}$.

Definition 11 [15, Definition 0.5] Let $A \in \mathscr{A}$ with

$$\mathrm{GKdim}(A) = d \geqslant 2.$$

If there are a graded Auslander-regular and CM algebra $B \in \mathscr{A}$ with $\mathrm{GKdim}(B) = d$ and two $\mathbb{Z}$-graded bimodules $_B M_A$ and $_A N_B$, finitely generated on both sides, such that

$$M \otimes_A N \cong_{d-2} B, \quad N \otimes_B M \cong_{d-2} A,$$

as $\mathbb{Z}$-graded bimodules, then the triple (B, M, N) or simply the algebra B is called a *noncommutative quasi-resolution (NQR)* of A.

If $A \in \mathscr{A}$ with GKdim $A = 2$, then by [15, Theorem 4.2, Lemma 8.2], any two NQRs of A are graded Morita equivalent, namely, up to Morita equivalent, there is a unique noncommutative quasi-resolution of A.

Lemma 12 *If A is noetherian, graded, locally finite, Auslander regular and CM, then A is generalized AS regular.*

Proof By the Auslander and CM properties, for every finite-dimensional graded right A-module M,

$$\mathrm{Ext}_A^i(M, A) = \begin{cases} 0, & i \neq d := \mathrm{gldim}\, A, \\ N, & i = d, \end{cases}$$

for some finite-dimensional graded left A-module N. By the double-Ext spectral sequence [15, (E2.13.1)], $\mathrm{Ext}_A^d(-, A)$ induces a bijection from the set of graded simple right A-modules up to isomorphism to the set of graded simple left A-modules up to isomorphism. By [16, Theorem 5.2], A is generalized AS regular. $\square$

2.3 Other concepts

We recall some other concepts that are used in the main theorems. The following definition is due to Ueyama.

Definition 13 [20, Definition 2.2] Let A be a noetherian graded algebra. We say that A is a *graded isolated singularity* if the associated noncommutative projective scheme QGr A has finite global dimension.

Ueyama gave this definition for connected graded algebras, but we consider possibly non-connected graded algebras. This concept is used in Theorem 1 (4).

We will also use some results about balanced dualizing complex over noncommutative rings introduced by Yekutieli [25]. We refer to [3,21,25] for more details. We need the following local duality formula.

Theorem 14 [21, 25] *Let A be a noetherian, graded, locally finite algebra with balanced dualizing complex R, and let M be a graded right A-module. Then*

$$R\Gamma_{\mathfrak{m}}(M)^* = \mathrm{RHom}_A(M, R),$$

where $\mathfrak{m}$ is the graded Jacobson ideal of A and $(-)^*$ denotes the graded $\mathbb{k}$-linear dual.

The following corollary is well known.

Corollary 15 *Let A be a noetherian, graded, locally finite algebra with balanced dualizing complex R. If A is generalized AS Gorenstein of injective dimension d, then R is of the form $\Omega(d)$, where Ω is a graded invertible A-bimodule.*

3 Preparations

In this section, we will give the relations of Gabriel quivers, pretzeled quivers of graphs, and noncommutative quasi-resolutions (NQRs).

3.1 Gabriel quivers

Let Q be a quiver with finitely many vertices and arrows. If we label the vertices of Q by integers from 1 to d, then the adjacency matrix of Q is a square $d \times d$-matrix over $\mathbb{N}$. It is clear that there is a one-to-one correspondence:

$$\{\text{quivers with vertices labeled } \{1, 2, \ldots, d\}\} \Leftrightarrow \{d \times d\text{-matrices over } \mathbb{N}.\} \quad (3.1)$$

For this reason, the adjacency matrix of Q is also denoted by Q if no confusion occurs. The opposite quiver of Q is obtained by changing the direction of each arrow in Q. Hence, the adjacency matrix of the opposite quiver of Q is the transpose of the adjacency matrix of Q. We denote the opposite quiver of Q by Q^{op}.

Next, we review the definition of a Gabriel quiver. Suppose that A is locally finite graded and elementary with

$$S = A/J(A) = \mathbb{k}^{\oplus d}.$$

One can lift (not necessarily uniquely) the d primitive orthogonal idempotents of S to an orthogonal family of primitive idempotents with

$$1 = e_1 + e_2 + \cdots + e_n$$

in A_0, see [12, Corollary 21.32]. Notice that

$$A = \bigoplus_{i=1}^{n} e_i A.$$

Then every $e_i A$ is a graded projective right A-module, which is an indecomposable module since e_i is a primitive idempotent. Thus, we get d distinct simple right A-modules

$$S_i := e_i S = e_i A / e_i J(A),$$

and every simple graded right module is isomorphic to a shift of one of the S_i for some i, $1 \leqslant i \leqslant d$.

Definition 16 Let A be an elementary, locally finite, graded algebra such that $A/J(A) = \mathbb{k}^{\oplus d}$. The *Gabriel quiver* $\mathscr{G}(A)$ of A is defined by

• vertices: graded simple right A-modules $S_1, S_2, \ldots, S_d$ corresponding to individual projection to $\mathbb{k}$;

• arrows: $S_i \xrightarrow{q_{ij}} S_j$ if $q_{ij} = \dim_{\mathbb{k}} \mathrm{Ext}^1_A(S_j, S_i)_{-1}$ where $\mathrm{Ext}^1_A(S_j, S_i)$ has a natural $\mathbb{Z}$-grading.

Under the identification in (3.1), $\mathscr{G}(A) = (q_{ij})_{d \times d}$, where q_{ij} is defined above.

Note that if A in the above definition is Koszul in the sense of [14, Definition 1.5], then $\operatorname{Ext}^1_A(S_j, S_i)$ is concentrated in degree -1. In this case,

$$q_{ij} = \dim_{\Bbbk} \operatorname{Ext}^1_A(S_j, S_i)_{-1} = \dim_{\Bbbk} \operatorname{Ext}^1_A(S_j, S_i).$$

Remark 17 Suppose $A_0 = \Bbbk^{\oplus d}$.

(1) Let $P_i = e_i A$. Then $\{P_1, P_2, \ldots, P_d\}$ is a complete list of indecomposable graded projective right A-modules up to degree shifts and isomorphisms.

(2) It is easy to see that

$$q_{ij} = \dim_{\Bbbk} \operatorname{Hom}_A(P_i, P_j)_1, \quad \forall \, i, j = 1, 2, \ldots, d.$$

3.2 Twists of a quiver and pretzelizations

Let $Q = (q_{ij})_{d \times d}$ be a quiver with d vertices (or a $d \times d$-matrix over $\mathbb{N}$). A *graph* is a class of special quivers Q with $q_{ij} = q_{ji}$ for all i, j. Then a quiver Q is a graph if and only if $Q = Q^{\mathrm{op}}$. A graph is also called a *symmetric quiver*. The example given in (3.3) below is a graph, and the example given in (3.4) below is not a graph in this paper.

Definition 18 Let σ be an automorphism of the vertex set $\{1, 2, \ldots, d\}$. σ is said to be an automorphism of Q if

$$q_{\sigma(i)\sigma(j)} = q_{ij}, \quad \forall \, i, j.$$

Definition 19 [2] Let $Q := (q_{ij})$ be a quiver, and let σ be an automorphism of Q. The *twist* of Q associated to σ, denoted by $^\sigma Q$, is the quiver corresponding to the matrix $P_\sigma Q \ (= Q P_\sigma)$, where P_σ is the permutation matrix associated to σ; in other words,

$$(^\sigma Q)_{ij} = q_{\sigma(i)j} = q_{i\sigma^{-1}(j)}, \quad \forall \, i, j.$$

Let G be a graph (or symmetric quiver), and let σ be an automorphism of the quiver G. Let Q be the twisted quiver $^\sigma G$. Then

$$Q^{\mathrm{op}} = {}^{\sigma^{-2}}Q,$$

which follows from the following linear algebra computation:

$$Q^{\mathrm{op}} = (P_\sigma G)^{\mathrm{op}} = (G P_\sigma)^{\mathrm{op}} = P_\sigma^{-1} G^{\mathrm{op}} = P_{\sigma^{-2}} Q = {}^{\sigma^{-2}}Q. \tag{3.2}$$
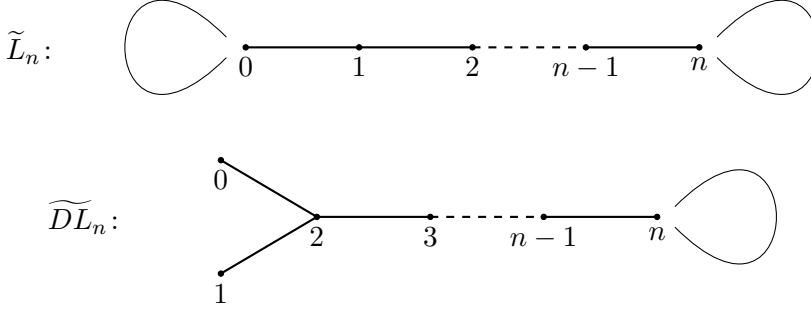
Definition 20 [2] Let G be a graph.

(1) A quiver Q is called a *pretzelization* of G or a *pretzeled* quiver of G if $Q \cup Q$ is a twisted quiver of a finite disjoint union of G. It is possible that Q itself is a twisted quiver of another finite disjoint union of copies of G. In general, a pretzelization of a graph is not a graph.

(2) We say that a graph G is of $\widetilde{AD}\widetilde{E}$ *type* if it is of type

$$\widetilde{A}_n, \ \widetilde{D}_n, \ \widetilde{L}_n, \ \widetilde{DL}_n, \ \widetilde{E}_6, \ \widetilde{E}_7, \ \widetilde{E}_8,$$

listed in [8, Theorem 2].

We assume that the readers are familiar with $\widetilde{A}\widetilde{D}\widetilde{E}$ graphs, which can be found in many papers including [8]. We only list $\widetilde{L}_n$ for $n \geqslant 0$ and $\widetilde{DL}_n$ for $n \geqslant 2$ here:

In the above, the vertex set is $\{0, 1, \ldots, n\}$. If there is an edge between vertices i and j, then both q_{ij} and q_{ji} are 1. If there is a loop at vertex i, then $q_{ii} = 1$.

A result of Smith [19] states that a simple graph G (i.e., a graph with no double edges and loops) has spectral radius 2 if and only if it is either $\widetilde{A}_n$, $\widetilde{D}_n$, $\widetilde{E}_6$, $\widetilde{E}_7$, or $\widetilde{E}_8$, see also [7, Theorem 1.3]. The following generalization is folklore for some experts; it follows from the proof of [8, Theorem 2].

Lemma 21 *Let G be a graph (i.e., a symmetric quiver) which is not necessarily simple. Then $\rho(G) = 2$ if and only if G is listed in Definition 20 (2).*

This result was also proved by Chan et al. when they were working on the project [4–6], but it was removed by the authors in the final published versions of [4–6]. Another related result in [2] is that graphs of types $\widetilde{L}_n$ and $\widetilde{DL}_n$ are pretzelizations of graphs of types $\widetilde{A}_n$ and $\widetilde{D}_n$. Note that graphs of type $\widetilde{DL}_n$ and $\widetilde{L}_1$ appeared in [4, Proposition 7.1]. A key lemma concerning pretzelization is as follows.
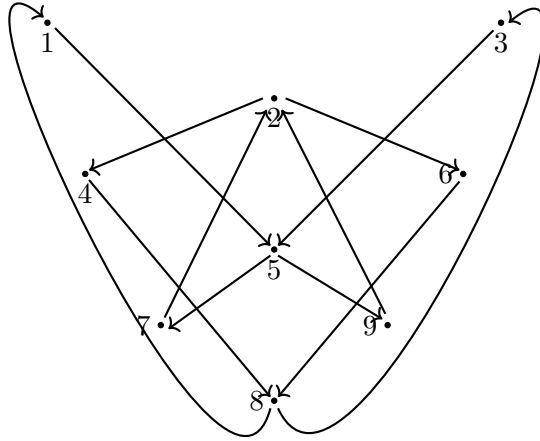
Lemma 22 [2] *Let Q be a quiver. Then Q is a pretzelization of a graph if and only if $Q^{\mathrm{op}} = {}^{\mu}Q$ for some automorphism μ of the quiver Q.*

The automorphism μ in Lemma 22 is called a *Nakayama automorphism* of the quiver Q. For example, σ^{-2} is a Nakayama automorphism of Q in (3.2).

For example, if Q is the Dynkin graph of type A_3:

$$\bullet \underset{\longleftarrow}{\overset{\longrightarrow}{}} \bullet \underset{\longleftarrow}{\overset{\longrightarrow}{}} \bullet \tag{3.3}$$

then one twist of the disconnected graph $Q \cup Q \cup Q$ (or a pretzelization of the graph Q) is the following connected pretzel-shaped quiver:

$$(3.4)$$

Proposition 23   *Let Q be a quiver such that $Q^{\mathrm{op}} = {}^{\sigma}Q$ for some automorphism σ of Q. If $\rho(Q) = 2$, then Q is a pretzelization of a graph of $\widetilde{A}\widetilde{D}\widetilde{E}$ types.*

Proof   By Lemma 22, Q is a pretzelization of a graph G. This means that $Q \cup Q = {}^{\tau}G$ for some automorphism τ of G. By a result of [2], ρ is stable under twists of quiver. Since $\rho(Q) = 2$, we have

$$\rho(G) = \rho(Q \cup Q) = \rho(Q) = 2.$$

By Lemma 21, G is of $\widetilde{A}\widetilde{D}\widetilde{E}$ type in the sense of Definition 20 (2).   □

3.3   Depth and maximal Cohen-Macaulay modules

We first collect some definitions from the literature. In the next definition we only consider graded right modules. The same definition can be made for graded left modules. Since we always consider graded algebras and graded modules, we sometimes omit the word 'graded'. The following definition of a Cohn-Macaulay module is different from the concept of a Cohen-Macaulay algebra given in Definition 9.

Definition 24   Let A be a noetherian, locally finite, graded algebra with finite GKdimension. Let $S = A/J(A)$. Let M be a nonzero finitely generated graded right A-module.

(1)   The *depth* of M is defined to be

$$\mathrm{depth}_A M := \inf\{i \mid \mathrm{Ext}^i_A(S, M) \neq 0\} \in \mathbb{N} \cup \{+\infty\}.$$

If no confusion can arise, we write $\mathrm{depth}\, M$ for $\mathrm{depth}_A M$.

(2)   We say that M is *Cohen-Macaulay* if $\mathrm{depth}\, M = \mathrm{GKdim}\, M$.

(3)   We say that M is a *maximal Cohen-Macaulay* (*MCM*) module if M is a Cohen-Macaulay and $\mathrm{GKdim}\, M = \mathrm{GKdim}\, A$.

(4)   We say that A is of *finite Cohen-Macaulay type* (in the graded sense) if there are only finite many indecomposable graded MCM modules up to degree shifts and isomorphisms.

(5) M is called *reflexive* if the natural map $M \to \operatorname{Hom}_{A^{\mathrm{op}}}(\operatorname{Hom}_A(M, A), A)$ is an isomorphism.

(6) Let n be an integer. Then M is called *n-pure* if $\operatorname{GKdim} N = n$ for every nonzero submodule $N \subseteq M$.

Note that the definition of an n-pure module in [15, Definition 2.1 (2)] is different from and related to ours. Some basic lemmas about depth can be found in [15, Section 5]. The following lemma is clear. An object $\mathscr{M}$ in qgr A is called 2-pure if $\mathscr{M} = \pi(M)$ for a 2-pure graded A-module M and there is no nonzero sub-object $\mathscr{N} \subseteq \mathscr{M}$ such that $\mathscr{N} = \pi(N)$ for some $N \in \operatorname{grmod} A$ of GKdimension 1, where π is defined in (2.2).

Lemma 25 *Let A be a noetherian, locally finite, graded algebra with $\operatorname{GKdim} A = 2$. Suppose that A is Auslander-Gorenstein and CM.*

(1) *There is a bijection between 2-pure objects in qgr A and reflexive modules in grmod A.*

(2) *The functors π and ω defined in (2.4)-(2.5) induce an equivalence between the category of 2-pure objects in qgr A and that of reflexive modules in grmod A.*

Proof We only prove (2) as (1) follows immediately from (2).

(2) For every reflexive module M, by [15, Proposition 2.14], it is 2-pure (the definition of n-pure is slightly different from the definition in this paper, but it is easily checked that this is not a problem). Then $\pi(M)$ is 2-pure in qgr A by definition. Conversely, let $\mathscr{M}$ be a 2-pure object in qgr A. Let M be any finitely generated 2-pure module such that $\mathscr{M} = \pi(M)$. Let $\widetilde{M}$ be the Gabber closure of M defined in [15, Definition 2.8]. Since two such M differ only by finite-dimensional vector spaces, $\widetilde{M}$ is independent of the choices of M. Or equivalently, $\widetilde{M}$ is only dependent on $\mathscr{M}$, which is $\omega(\mathscr{M})$. Therefore,

$$\pi\omega(\mathscr{M}) = \mathscr{M}$$

for 2-pure objects in qgr A by the fact that reflexive objects in grmod A are Gabber-closed, which is following [15, Definition 2.11] and

$$\omega\pi(M) = M$$

for reflexive objects in grmod A. The assertion follows. $\square$

Here is the main result in this subsection.

Theorem 26 *Let A be a noetherian graded locally finite algebra with $\operatorname{GKdim} 2$, and let M be a finitely generated graded A-module. Suppose that*

(a) *A has a balanced dualizing complex R, and*

(b) *A is Auslander-Gorenstein and CM.*

Then M is reflexive if and only if it is MCM in grmod A.

Proof   Since A is Auslander Gorenstein and CM, the depth of A (and A^{op}) is 2. It follows from [15, Lemma 5.6] that the depth of a nonzero reflexive module is 2. Therefore, a reflexive module is MCM as GKdim $A = 2$.

Conversely, let M be a MCM right A-module. Then $\mathrm{Ext}_A^i(S, M) = 0$ for $i = 0, 1$, where $S = A/J(A)$. This implies that

$$R^i \Gamma_{\mathfrak{m}}(M) = 0, \quad i = 0, 1.$$

By Theorem 14,
$$\mathrm{Ext}_A^{-i}(M, R) = 0, \quad i = 0, 1.$$

Since $R = \Omega[2]$, where Ω is a graded invertible A-bimodule (see Corollary 15) and $[2]$ denotes the second complex shift,

$$\mathrm{Ext}_A^j(M, A) = \mathrm{Ext}_A^j(M, \Omega) \otimes \Omega^{-1} = 0, \quad j = 1, 2.$$

By the double-Ext spectral sequence [15, (E2.13.1)], M is reflexive.   □

4   Proofs of main results

We give the proofs of the main results here.

Proof of Theorem 1   The statements apply to left and right modules. We only prove the results for right modules.

(1) By Lemma 25 (1), there is a bijection between 2-pure objects in qgr A and reflexive modules in grmod A. Similarly, there is a bijection between 2-pure objects in qgr B and reflexive modules in grmod B. By [15, Lemma 3.5], qgr A is equivalent to qgr B. Therefore, there is a bijection between reflexive modules in grmod A and those in grmod B. By Theorem 26, the reflexive modules in grmod A are exactly the MCM modules in grmod A. Since B is generalized AS regular, the reflexive modules in grmod B are precisely the projective modules in grmod B. Since B is locally finite, there are only finitely many indecomposable graded projective modules over B up to degree shifts and isomorphisms. This implies that there are only finitely many indecomposable graded MCM modules over A up to degree shifts and isomorphisms. The assertion follows by Definition 24 (4).

(2) By the proof of (1), there is a one-to-one correspondence between the set of indecomposable MCM graded right A-modules up to degree shifts and isomorphisms and the set of graded indecomposable projective right B-modules up to degree shifts and isomorphisms. The assertion follows from the fact that there is a one-to-one correspondence between the set of indecomposable graded projective right B-modules up to degree shifts and isomorphisms and the set of graded simple right B-modules up to degree shifts and isomorphisms.

(3) Let $F\colon$ qgr $B \to$ qgr A be the equivalence given in [15, Lemma 3.5]. Let (π_A, ω_A) be the adjoint pair of functors given in (2.4) and (2.5), and similar for

(π_B, ω_B). Then we have a functor

$$\Phi := \omega_A \circ F \circ \pi_B \colon \operatorname{grmod} B \to \operatorname{grmod} A, \qquad (4.1)$$

which is an equivalence of categories when restricted to the categories of reflexive modules over A and B.

Let

$$B = \bigoplus_{i=1}^{d} P_i^{\oplus u_i}, \qquad (4.2)$$

where $u_i \geqslant 1$ and $\{P_1, P_2, \ldots, P_d\}$ is a complete list of indecomposable projective right B-modules which are direct summands of B. Let

$$C = \operatorname{End}_B\Big(\bigoplus_{i=1}^{d} P_i\Big). \qquad (4.3)$$

Then C is graded Morita equivalent to B. Let $M_i = \Phi(P_i)$. Then $\{M_1, M_2, \ldots, M_d\}$ is a complete list of MCM right modules over A up to degree shifts and isomorphisms. Since Φ is an equivalence,

$$C \cong \operatorname{End}_A\Big(\bigoplus_{i=1}^{d} M_i\Big),$$

as desired.

(4) The assertion follows by Definition 13 and from the fact that qgr A is equivalent to qgr B [15, Lemma 3.5] and that the global dimension of qgr B is bounded by the global dimension of grmod B. $\qquad \square$

Recall that an algebra A is called *indecomposable* if it cannot be written as a sum of two nontrivial algebras; this is equivalent to A having no nontrivial central idempotents. Similarly, there is a definition of *graded indecomposable* algebra. By [16, Lemma 2.7], a graded algebra A is indecomposable if and only if A is graded indecomposable.

Lemma 27   *Let B_1 and B_2 be two NQRs of a noetherian graded locally finite algebra A of GKdimension two. Suppose that*

$$(B_1)_0 = \Bbbk^{\oplus d_1}, \quad (B_2)_0 = \Bbbk^{\oplus d_2},$$

and that B_1 is standard. Then $B_1 \cong B_2$ as graded algebras.

Proof   By Theorem 6, B_1 is isomorphic to $A_2(Q, \tau)$ for some quiver Q satisfying the extra conditions listed in Theorem 6. By [17, Lemma 3.4 (2)], the quiver Q agrees with the Gabriel quiver of B_1 (Definition 16). By [15, Theorem 0.6 (1)], B_1 and B_2 are graded Morita equivalent. Let

$$\Psi \colon \operatorname{grmod} B_1 \to \operatorname{grmod} B_2$$

be an equivalence of categories. Via Ψ, one sees that B_1 and B_2 have the same number of indecomposable summands and Ψ matches up these indecomposable summands as graded Morita equivalences. Therefore, without loss of generality, one can assume that both B_1 and B_2 are indecomposable.

By Theorem 1 (2), the number of graded simple right B_1-modules (up to degree shifts and isomorphisms) is the same as the number of graded simple right B_2-modules (up to degree shifts and isomorphisms). That number is $d_1 = d_2 =: d$ as

$$(B_1)_0 = \Bbbk^{\oplus d_1}, \quad (B_2)_0 = \Bbbk^{\oplus d_2}.$$

Let $\{e_i\}_{i=1}^d$ be the set of primitive idempotents of B_1 (and of B_2). Then $\{P_i := e_i B_1\}_{i=1}^d$ is a complete set of indecomposable graded projective right B_1-modules up to degree shifts and isomorphisms. Similarly, $\{R_i := e_i B_2\}_{i=1}^d$ is a complete set of indecomposable graded projective right B_2-modules up to degree shifts and isomorphisms. Since Ψ is an equivalence and the degree shifts are also equivalences, we may assume that $\Psi(P_i) = R_i(w_i)$ for some integer $w_i \geqslant 0$ with one of w_i being 0. We can further assume that all w_i are non-negative and $w_1 = 0$ after some permutation.

We claim that $w_i = 0$ for all i. If not, there is an i such that $w_i > 0$. By [17, Lemma 7.3], the quiver Q in $B_1 = A_2(Q, \tau)$ is strongly connected, that is, given any two vertices i and j in Q there is a directed path from i to j. In particular, there is a path from 1 to i in Q where $w_1 = 0$ and $w_i > 0$. Along this path, choose two vertices $a \neq b$ such that there is an arrow from a to b and $w_a = 0$ and $w_b > 0$. By the above choice,

$$\begin{aligned} \mathrm{Hom}_{B_2}(R_b(w_b), R_a(w_a))_1 &= \mathrm{Hom}_{B_2}(R_b(w_b), R_a)_1 \\ &= \mathrm{Hom}_{B_2}(R_b(w_b - 1), R_a)_0 \\ &= 0 \end{aligned}$$

as $w_b - 1 \geqslant 0$ and $a \neq b$. Applying Ψ^{-1}, we obtain

$$\mathrm{Hom}_{B_1}(P_b, P_a)_1 = 0,$$

which implies that there is no arrow from a to b in the Gabriel quiver Q. This yields a contradiction. Therefore, all $w_i = 0$ and $\Psi(P_i) = R_i$ for all i. Since Ψ is an equivalence, we have an isomorphism of algebras

$$B_1 \cong \mathrm{End}_{B_1}\left(\bigoplus_i P_i\right) \cong \mathrm{End}_{B_2}\left(\bigoplus_i R_i\right) \cong B_2,$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Theorem 2 (1) By Lemma 12, any standard NQR B is generalized AS regular. By Theorem 6, B is isomorphic to $A_2(Q, \tau)$ that is given in [17, Definition 7.5].

By Theorem 6, the arrows in the quiver Q have weight 1. Note that B is a direct sum of finitely many indecomposable algebras. Without loss of generality,

we may assume that B is indecomposable. Then Q is strongly connected by [17, Lemma 7.3]. By Theorem 6, $\rho(Q) = 2$. By [17, Theorem 1.2 (2)], there is an automorphism μ of Q such that $Q^{\mathrm{op}} = {}^{\mu}Q$. By Proposition 23, Q is a pretzelization of a graph of type $\widetilde{A}\widetilde{D}\widetilde{E}$.

By [17, Lemma 3.4] and the definition of Gabriel quiver, Q is exactly the Gabriel quiver $\mathscr{G}(B)$ of B, whence, $\mathscr{G}(B)$ is a pretzelization of a graph of type $\widetilde{A}\widetilde{D}\widetilde{E}$.

(2) Suppose that there are two standard NQRs, say B_1 and B_2, of A. The assertion follows from Lemma 27. □

Definition 28　Suppose that A satisfies the hypotheses of Theorem 2. By Theorem 2 (2), the standard NQR of A is unique up to isomorphism. In this case, the *Auslander-Reiten quiver* of A is defined to be the Gabriel quiver of the standard NQR of A.

Following Definition 28, one can speak of the Auslander-Retein quiver for MCM modules over graded algebras in Theorem 2.

Unfortunately, not every algebra A in Theorem 1 has a standard NQR, as the following example shows.

Example 29　Let $\mathrm{char}(\Bbbk) \neq 2$, and let $\Bbbk[x, y]$ be a commutative polynomial ring with $\deg x > 0$ and $\deg y > 0$. Let

$$A = \Bbbk[x, y]^{\langle \sigma \rangle},$$

where σ is the automorphism of $\Bbbk[x, y]$ of order 2 defined by

$$\sigma\colon x \to -x, \quad y \to -y.$$

Then

$$B = \Bbbk[x, y] * \langle \sigma \rangle$$

is an NQR of A by [15, Example 8.5]. It is well known by the commutative theory that A has two MCMs: A itself and the module C such that

$$A \oplus C = \Bbbk[x, y].$$

(1) If $\deg x = \deg y = 1$, then B is a standard NQR and is the preprojective algebra associated to the Dynkin graph $\widetilde{A_1}$. Let $B' = \mathrm{End}_A(A \oplus C(1))$. Then B' is another NQR of A and is isomorphic to B as ungraded algebras. As an $\mathbb{N}$-graded algebra,

$$B'_0 = \begin{pmatrix} \Bbbk & \Bbbk \oplus \Bbbk \\ 0 & \Bbbk \end{pmatrix},$$

which is not semisimple.

(2) If $\deg x > 1$ or $\deg y > 1$, then B is not standard. Note that $B_0 = \Bbbk^2$. If A has a standard NQR, say B', then by Lemma 27, $B \cong B'$ as graded algebras. This implies that B is standard, a contradiction. Therefore, A does not have a standard NQR.

(3) The uniqueness of B fails in Theorem 2 (2) if we only require $B_0 = \Bbbk^{\oplus d}$. To see this, we consider the case when $\deg x = \deg y = 2$. It is easy to see that elements in A live in degrees $4\mathbb{N}$ and elements in $C(1)$ live in degrees $4\mathbb{N} + 1$. Let $B' = \operatorname{End}_A(A \oplus C(1))$. Then B' is another NQR of A and $B'_0 = \Bbbk^2 = B_0$. But $B'_1 = \Bbbk^{\oplus 2}$ and $B_1 = 0$. Therefore, $B' \ncong B$.

The existence of standard NQRs can be proved in the following case.

Lemma 30   *Let R be a standard, noetherian, graded, locally finite Auslander regular and CM algebra with $\mathrm{GKdim} = 2$. Let H be a semisimple Hopf algebra acting on R homogeneously and inner-faithfully. Assume that the homological determinant of the H-action [18, Definition 3.7] is trivial. Let $A = R^H$. Suppose that $\Bbbk$ is algebraically closed and that the conditions in [15, Example 8.5] hold. Then A has a standard NQR. As a consequence, the Auslander-Reiten quiver of A is a pretzelization of a graph of $\widetilde{A}\widetilde{D}\widetilde{E}$ type.*

Proof   Let $B = R\#H$ as in [15, Example 8.5]. By [15, Example 8.5], B is an NQR of A. It is easy to check that B_0 is semisimple and B is generated by B_0 and B_1. Write B and C as in (4.2) and (4.3), respectively, and then C is graded Morita equivalent to B. As a consequence, C is an NQR of A. By working with the minimal projective resolution of the graded simples, one can show that C is generated by C_0 and C_1. Since $\Bbbk$ is algebraically closed, it forces that $C_0 = \Bbbk^{\oplus d}$. This means that C is standard.

The consequence follows from Theorem 2 (1) and Definition 28. $\square$

By Lemma 30, we can apply Theorems 1 and 2 to the situation where R is a preprojective algebra as studied by Weispfenning [24].

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