Journal of Algebra 536 (2019) 102–148

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Noncommutative quasi-resolutions

X.-S. Qin [a], Y.-H. Wang [b], J.J. Zhang [c,*]

[a] *Shanghai Center for Mathematical Sciences, School of Mathematical Sciences, Fudan University, Shanghai 200433, China*

[b] *School of Mathematics, Shanghai Key Laboratory of Financial Information Technology, Shanghai University of Finance and Economics, Shanghai 200433, China*

[c] *Department of Mathematics, Box 354350, University of Washington, Seattle, WA 98195, USA*

A R T I C L E I N F O

Article history:
Received 23 December 2018
Available online 30 July 2019
Communicated by Louis Rowen

MSC:
primary 16E65, 16E30, 16S38, 14A22

Keywords:
Noncommutative crepant resolution (NCCR)
Noncommutative quasi-resolution (NQR)
Morita equivalent
Derived equivalent
Auslander-Gorenstein algebra
Auslander regular algebra
Cohen-Macaulay algebra
Dimension function

A B S T R A C T

The notion of a noncommutative quasi-resolution is introduced for a noncommutative noetherian algebra with singularities, even for a non-Cohen-Macaulay algebra. If A is a commutative normal Gorenstein domain, then a noncommutative quasi-resolution of A naturally produces a noncommutative crepant resolution (NCCR) of A in the sense of Van den Bergh, and vice versa. Under some mild hypotheses, we prove that

(*i*) in dimension two, all noncommutative quasi-resolutions of a given noncommutative algebra are Morita equivalent, and
(*ii*) in dimension three, all noncommutative quasi-resolutions of a given noncommutative algebra are derived equivalent.

These assertions generalize important results of Van den Bergh, Iyama-Reiten and Iyama-Wemyss in the commutative and central-finite cases.

* Corresponding author.
E-mail addresses: 13110840002@fudan.edu.cn (X.-S. Qin), yhw@mail.shufe.edu.cn (Y.-H. Wang), zhang@math.washington.edu (J.J. Zhang).

https://doi.org/10.1016/j.jalgebra.2019.07.015
0021-8693/

Introduction

A famous conjecture of Bondal-Orlov [8,9] in birational geometry states

Conjecture 0.1. *[8,9] If Y_1 and Y_2 are crepant resolutions of a scheme X, then derived categories $D^b(\mathrm{coh}(Y_1))$ and $D^b(\mathrm{coh}(Y_2))$ are equivalent.*

In dimension three (respectively, two), this conjecture was proved by Bridgeland [10] in 2002 (respectively, by Kapranov-Vasserot [24] in 2000). The conjecture is still open in higher dimensions. As was noticed by Van den Bergh [34] in the study of one-dimensional fibres and by Bridgeland-King-Reid [11] in the study of the McKay correspondence for dimension $d \leq 3$ that both $D^b(\mathrm{coh}(Y_1))$ and $D^b(\mathrm{coh}(Y_2))$ are equivalent to the derived category of certain noncommutative rings. Motivated by Conjecture 0.1 and work of [11,10,34], Van den Bergh [35] introduced the notation of a noncommutative crepant resolution (NCCR) of a commutative normal Gorenstein domain A (in the original reference, the author used the notation R). Let us recall the definition of a NCCR given in [21, Section 8] which is quite close to the original definition of Van den Bergh [35, Definition 4.1]. As usual, CM stands for Cohen-Macaulay.

Definition 0.2. Let R be a noetherian commutative CM ring and let A be a module-finite R-algebra.

(1) [4] A is called an *R-order* if A is a maximal CM R-module. An R-order A is called *non-singular* if $\operatorname{gldim} A_{\mathfrak{p}} = \operatorname{Kdim} R_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$.
(2) [21, Section 8] Let M be a finitely generated right A-module that is reflexive. We say that M gives a *noncommutative crepant resolution, or NCCR*, $B := \operatorname{End}_A(M)$ of A if
  (i) M is a height one progenerator of A (namely, $M_{\mathfrak{p}}$ is a progenerator of $A_{\mathfrak{p}}$ for any height one prime ideal $\mathfrak{p}$ of R), and
  (ii) B is a non-singular R-order.

Note that Van den Bergh's original definition of a NCCR was only for $A = R$ being Gorenstein, since these are the types of varieties which have a chance of admitting crepant resolutions and so there is a good analogy with geometry [22, after Definition 1.2]. However when A is non-Gorenstein (but CM) there are sometimes many NCCRs of A, and these are related to cluster tilting (CT) objects in the category of CM modules over A [23, Corollary 5.9]. Thus, although geometrically we are only really interested in NCCRs when A is Gorenstein, there are strong algebraic reasons to consider the more general case. In this paper we will further relax the hypotheses on A: we allow A to be non-CM and to be noncommutative in the most general sense. Van den Bergh made the following conjecture, which is an extension of Bondal-Orlov Conjecture 0.1.

Conjecture 0.3. *[35, Conjecture 4.6] If A is a normal Gorenstein domain, then all crepant resolutions of* $\operatorname{Spec}(A)$ (*commutative and noncommutative*) *are derived equivalent.*

Van den Bergh proved this conjecture for 3-dimensional terminal singularities [35, Theorem 6.6.3]. Since the existence of commutative crepant resolutions is not equivalent to the existence of noncommutative crepant resolutions in high dimension [22], one should probably break up the above conjecture into two parts: commutative crepant resolutions and noncommutative crepant resolutions. In [21, Section 8], Iyama-Reiten proved the noncommutative part of this conjecture for noncommutative algebras A as in Definition 0.2 in dimension three. Similarly in [22], Iyama-Wemyss proved Conjecture 0.3 for NCCRs for CM algebras $A(=R)$ in dimension three, therefore generalizing [21, Corollary 8.8] to algebras which do not have Gorenstein base rings.

Theorem 0.4.

(1) *[21, Corollary 8.8] Let R be a commutative normal Gorenstein domain with* $\operatorname{Kdim} R \leq 3$ *and A a module-finite R-algebra. Then all NCCRs of A are derived equivalent.*

(2) *[22, Theorem 1.5] Let A be a d-dimensional* CM *equi-codimensional commutative normal domain with a canonical module.*
   - (2a) *If $d = 2$, then all NCCRs of A are Morita equivalent.*
   - (2b) *If $d = 3$, then all NCCRs of A are derived equivalent.*

Iyama-Wemyss [22, Theorem 1.7] also gave a sufficient condition in arbitrary dimension ($d = \operatorname{Kdim} A$) to establish when any two given NCCRs of R are derived equivalent.

The study of noncommutative singularities naturally leads a question of how to deal with algebras that are not module-finite over their centers. Such questions were implicitly asked in [13,14]. Before we present our solution, we would like to discuss some major differences between the commutative and the noncommutative settings.

(•) First of all, some of previous approaches of using moduli [10] need to be re-developed in order to prove equivalences of derived categories in a general noncommutative setting. However, very little is known about the theory of general noncommutative moduli.

(•) We say an algebra is *central-finite* if it is module-finite over its center (or more precisely, a finite module over its center). When algebras are not central-finite, some homological tools fail due to the fact that localization does not work well in the noncommutative setting. Our idea is to work with global structures without going to the localization. For example, we use Auslander regular algebras instead of algebras having finite global dimension. In the commutative case, the Auslander condition is automatic. It is easy to see that Auslander regular algebras are a natural generalization of homologically homogeneous algebras which are used in Van den Bergh's definition of a NCCR.

(•) In the commutative case, the Krull dimension, denoted by Kdim (see [27, Ch. 6]), is used extensively and is implicitly assumed. It is well-known that the Krull dimension might not be a good dimension function in the noncommutative case. Sometimes Gelfand-Kirillov dimension, denoted by GKdim (see [25] and [27, Ch. 8]), is better than the Krull dimension, and at other times vice versa. In the noncommutative case, it is necessary to consider an abstract dimension function (or several different ones in the different settings).

Let ∂ be an exact symmetric dimension function in the sense of Definition 1.2 and Hypothesis 1.3(3) (or [27, Section 6.8.4]) which is defined for all right A-modules where A is an algebra. Let D be another algebra. Two right A-modules (respectively, (D, A)-bimodules) M and N are called *s-isomorphic* if there are a third right A-module (respectively, (D, A)-bimodule) P and two right A-module (respectively, (D, A)-bimodule) maps

$$f : M \to P, \quad \text{and} \quad g : N \to P$$

such that the kernel and the cokernel of f and g (viewed as right A-modules) have ∂-dimension less than or equal to s. In this case, we write $M \cong_s N$. We refer to Definition 1.5 for more details. To state our main result without going to too much detail, we give a definition of a noncommutative quasi-resolution in the following special case. Some technical details are explained in Section 3.

Definition 0.5. We fix the dimension function ∂ to be GKdim. Let A be a noetherian locally finite $\mathbb{N}$-graded algebra with $\operatorname{GKdim}(A) = d \in \mathbb{N}$. If there are a noetherian locally finite $\mathbb{N}$-graded Auslander regular CM algebra B (see Definitions 2.1 and 2.3) with $\operatorname{GKdim}(B) = d$ and two $\mathbb{Z}$-graded bimodules ${}_B M_A$ and ${}_A N_B$, finitely generated on both sides, such that

$$M \otimes_A N \cong_{d-2} B, \quad \text{and} \quad N \otimes_B M \cong_{d-2} A$$

as $\mathbb{Z}$-graded bimodules, then the triple (B, M, N) or simply the algebra B is called a *noncommutative quasi-resolution* (or *NQR* for short) of A.

An ungraded version of the above definition is given in Definition 3.2 (also see Definition 3.16 for a related definition). Note that Van den Bergh considers a normal Gorenstein noetherian commutative integral domain A. By a classical theorem of Serre, being normal is equivalent to these two conditions: for every prime ideal $\mathfrak{p} \subseteq A$ of height ≤ 1 the local ring $A_{\mathfrak{p}}$ is regular, and for every prime ideal $\mathfrak{p} \subseteq A$ of height ≥ 2 the local ring $A_{\mathfrak{p}}$ has depth ≥ 2, which is related to Definition 0.5.

By Proposition 7.5, Van den Bergh's NCCRs (or Iyama-Reiten's version, see Definition 0.2) produce naturally examples of the ungraded version of NQRs. Noncommutative examples of NQRs are given in Section 8. Our main theorem is to prove a version of Conjecture 0.3 for NQRs in dimension no more than three.

Theorem 0.6. *Fix ∂ to be* GKdim *as in the setting of Definition 0.5. Let A be a noetherian locally finite $\mathbb{N}$-graded algebra over the base field.*

(1) *Suppose* $\operatorname{GKdim}(A) = 2$. *Then all NQRs of A are Morita equivalent.*
(2) *Suppose* $\operatorname{GKdim}(A) = 3$. *Then all NQRs of A are derived equivalent.*

The proof of Theorem 0.6 is given in Section 8. A version of Theorem 0.6 holds for other dimension functions ∂ with some extra hypotheses and details are given in Theorems 4.2 and 6.6. Note that the hypotheses on ∂ (as listed in Theorem 6.6) are automatic in the commutative case or the central-finite case when $\partial = \operatorname{Kdim}$ (see Lemmas 7.1 and 7.2). Therefore Theorem 0.6, or Theorems 4.2 and 6.6 together, generalize important results of Van den Bergh [35, Theorem 6.6.3], Iyama-Reiten [21, Corollary 8.8] and Iyama-Wemyss [22, Theorem 1.5].

Inspired by the work in [21,22] and Theorem 0.6(1), we have the following question:

Question 0.7. Let B_1 and B_2 be Auslander-regular and ∂-CM algebras with $\operatorname{gldim} B_i = \partial(B_i) = 2$ for $i = 1, 2$. If B_1 and B_2 are derived equivalent, then are they Morita equivalent?

The paper is organized as follows. Sections 1 and 2 are preliminaries containing a discussion of dimension functions and homological properties. A detailed definition and basic properties of a NQR are given in Section 3. A proof of the main theorem is basically given in Sections 4, 6 and 8, while some technical material is taken care of in Section 5. The connections between NCCRs and NQRs are given in Section 7. The final Section 8 contains examples of NQRs of noncommutative algebras.

1. Dimension functions and quotient categories

Throughout let $\Bbbk$ be a field. All algebras and modules are over $\Bbbk$. We further assume that **all algebras are noetherian** in this paper.

We first briefly review background material on dimension functions and quotient categories of the module categories.

Notation 1.1. For an algebra A, we fix the following notations.

(1) $\operatorname{Mod} A$ (respectively, $\operatorname{Mod} A^{\mathrm{op}}$): the category of all right (respectively, left) A-modules.

(2) $\operatorname{mod} A$ (respectively, $\operatorname{mod} A^{\mathrm{op}}$): the full subcategory of $\operatorname{Mod} A$ (respectively, $\operatorname{Mod} A^{\mathrm{op}}$) consisting of finitely generated right (respectively, left) A-modules.
(3) $\operatorname{proj} A$ (respectively, $\operatorname{proj} A^{\mathrm{op}}$): the full subcategory of $\operatorname{mod} A$ (respectively, $\operatorname{mod} A^{\mathrm{op}}$) consisting of finitely generated projective right (respectively, left) A-modules.
(4) $\operatorname{ref} A$ (respectively, $\operatorname{ref} A^{\mathrm{op}}$): the full subcategory of $\operatorname{mod} A$ (respectively, $\operatorname{mod} A^{\mathrm{op}}$) consisting of reflexive right (respectively, left) A-modules, see Definition 2.11.
(5) $\operatorname{add}_A(M)(=\operatorname{add} M)$ for $M \in \operatorname{mod} A$: the full subcategory of $\operatorname{mod} A$ consisting of direct summands of finite direct sums of copies of M.

Usually we work with right modules. We will use the functor $\operatorname{Hom}_A(-, A_A)$ a lot, so let us mention a simple fact below. A contravariant equivalence between two categories is called a *duality*. Let A be an algebra. Then there is a duality of categories

$$\operatorname{Hom}_A(-, A_A) : \operatorname{proj} A \longrightarrow \operatorname{proj} A^{\mathrm{op}}.$$

Our proof of the main result uses quotient categories of the module categories defined via a dimension function, so we first give the following definition, which is a slight modification of the definition given in [27, Section 6.8.4]. We also refer to [5, Section 1] for a similar definition.

Definition 1.2. A function $\partial : \operatorname{Mod} A \to \mathbb{R}_{\geq 0} \cup \{\pm\infty\}$ is called a *dimension function* if,

(a) $\partial(M) = -\infty$ if and only if $M = 0$, and
(b) for all A-modules M,

$$\partial(M) \geq \max\{\partial(N), \partial(M/N)\},$$

whenever N is a submodule of M.

The ∂ is called an *exact dimension function* if, further,

(c) for all A-modules M,

$$\partial(M) = \sup\{\partial(N), \partial(M/N)\},$$

whenever N is any submodule of M, and
(d) for every direct system of submodules of M, say $\{M_i\}_{i\in I}$,

$$\partial(\bigcup_{i\in I} M_i) = \sup\{\partial(M_i) \mid i \in I\}. \tag{E1.2.1}$$

Condition (d) in Definition 1.2 is new. As a consequence of condition (d), we obtain that, for every $M \in \operatorname{Mod} A$,

$$\partial(M) := \sup\{\partial(N) \mid \text{for all finitely generated submodules } N \subseteq M\}. \tag{E1.2.2}$$

If we start with an exact dimension function ∂ defined on $\operatorname{mod} A$, then ∂ can be extended to $\operatorname{Mod} A$ by using (E1.2.2). In this case, both condition (c) and condition (d) in Definition 1.2 are automatic. In fact, natural examples of dimension functions in this paper are constructed by (E1.2.2) from an exact dimension function ∂ defined on $\operatorname{mod} A$, see Remark 1.4. One advantage of condition (d) is that, for any fixed n, every right A-module M has a maximal submodule M' with $\partial(M') \leq n$.

Similarly, one can define a dimension function on left modules. For most of the statements in this paper, we assume the following:

Hypothesis 1.3. Let A and B be algebras with dimension function ∂.

(1) ∂ is an exact dimension function defined on both right modules and left modules.
(2) $\partial(A), \partial(B) \in \mathbb{N}$.
(3) If an (A, B)-bimodule M is finitely generated as a B-module, then $\partial({}_AM) \leq \partial(M_B)$. This also holds when switching A and B. In particular, for an (A, B)-bimodule M which is finitely generated both as a left A-module and as a right B-module, we have

$$\partial({}_AM) = \partial(M_B). \tag{E1.3.1}$$

A dimension function ∂ is called *symmetric* if (E1.3.1) holds, which is identical to [40, Definition 2.20].

Note that symmetry condition (E1.3.1) resembles "symmetric derived torsion", in the sense of [36, Section 9].

Unless otherwise stated, an (A, B)-bimodule means finitely generated on both sides. Recall that A is called *central-finite* if it is a finitely generated module over its center.

Remark 1.4. Two standard choices of ∂ are the Gelfand-Kirillov dimension, denoted by GKdim, see [27, Ch. 8] and [25], and the Krull dimension, denoted by Kdim, see [27, Ch. 6]. As a convention, we define the Krull dimension of an infinitely generated module via (E1.2.2). Note that, while Kdim is defined for every noetherian ring A, it is not known whether it is always symmetric. If A is central-finite, then Kdim is symmetric [27, Corollary 6.4.13]. On the other hand, GKdim is always symmetric [25, Corollary 5.4], though it could be infinite for a nice noetherian $\Bbbk$-algebra. For a central-finite algebra A with affine center, Kdim coincides with GKdim; this is an easy consequence of the equality of the two dimensions for affine commutative algebras [25, Theorem 4.5].

We need to recall some definitions and notations introduced in [5]. From now on, we fix an exact dimension function, say ∂. We use n for a nonnegative integer. Let $\operatorname{Mod}_n A$ denote the full subcategory of $\operatorname{Mod} A$ consisting of right A-modules M with $\partial(M) \leq n$.

Since ∂ is exact, $\operatorname{Mod}_n A$ is a Serre subcategory of $\operatorname{Mod} A$. Hence it makes sense to define the quotient categories:

$$\operatorname{QMod}_n A := \frac{\operatorname{Mod} A}{\operatorname{Mod}_n A}, \quad \text{and} \quad \operatorname{qmod}_n A := \frac{\operatorname{mod} A}{\operatorname{mod}_n A},$$

which can be seen as a generalized noncommutative scheme (see [3]). Note that there are some symmetries between $\operatorname{qmod}_n A$ and $\operatorname{qmod}_n A^{\mathrm{op}}$ when A admits a nice dualizing complex, see [40, Theorem 2.15]. We denote the natural and exact projection functor by

$$\pi : \operatorname{Mod} A \longrightarrow \operatorname{QMod}_n A. \tag{E1.4.1}$$

For $M \in \operatorname{Mod} A$, we write $\mathcal{M}$ for the object $\pi(M)$ in $\operatorname{QMod}_n A$. The hom-set in the quotient category is defined by

$$\operatorname{Hom}_{\operatorname{QMod}_n A}(\mathcal{M}, \mathcal{N}) = \varinjlim \operatorname{Hom}_A(M', N') \tag{E1.4.2}$$

for M, $N \in \operatorname{Mod} A$, where M' is a submodule of M such that $\partial(M/M') \leq n$, $N' = N/T$ for some submodule $T \subseteq N$ with $\partial(T) \leq n$, and where the direct limit runs over all the pairs (M', N') with these properties.

Definition 1.5. Let $n \geq 0$. Let A and D be two algebras.

(1) Two right A-modules X, Y are called *n-isomorphic*, denoted by $X \cong_n Y$, if there exist a right A-module P and morphisms $f : X \to P$ and $g : Y \to P$ such that both the kernel and cokernel of f and g are in $\operatorname{Mod}_n A$.
(2) Two (D, A)-bimodules X, Y are called *n-isomorphic*, denoted by $X \cong_n Y$, if there exist a (D, A)-bimodule P and bimodule morphisms $f : X \to P$ and $g : Y \to P$ such that both the kernel and cokernel of f and g are in $\operatorname{Mod}_n A$ when viewed as right A-modules.

Definition 1.5(2) is useful when we consider bimodules. Most of right module statements regarding n-isomorphisms have bimodule analogues, for which we might omit the proofs if these are clear.

Remark 1.6.

(1) Since we usually consider finitely generated modules, it turns out that we are mostly talking about n-isomorphisms in $\operatorname{mod} A$. In this case, we can take $P \in \operatorname{mod} A$ in the above definition.
(2) If X is a submodule of Y in $\operatorname{mod} A$ and $Y/X \in \operatorname{mod}_n A$, then clearly $X \cong_n Y$.
(3) If

$$0 \to K \to M \to N \to C \to 0$$

is an exact sequence in $\operatorname{mod} A$ with $K, C \in \operatorname{mod}_n A$, then $M \cong_n N$ in $\operatorname{mod} A$.

The following lemma is easy and the proof is omitted.

Lemma 1.7. *Two right A-modules X and Y are n-isomorphic in* $\operatorname{mod} A$ *if and only if their images $\mathcal{X}$ and $\mathcal{Y}$ in* $\operatorname{qmod}_n A$ *are isomorphic.*

Definition 1.8. [5, Definition 1.2] Let A and B be algebras and ∂ be an exact dimension function that is defined on A-modules and B-modules. Let n and i be nonnegative integers. Suppose that ${}_A M_B$ is a bimodule.

(1) We say ∂ satisfies $\gamma_{n,i}(M)^l$ if for any $N \in \operatorname{mod}_n A$, $\operatorname{Tor}_j^A(N, M) \in \operatorname{mod}_n B$ for all $0 \leq j \leq i$.
(2) We say ∂ satisfies $\gamma_{n,i}(M)^r$ if for any $N \in \operatorname{mod}_n B^{\mathrm{op}}$, $\operatorname{Tor}_j^B(M, N) \in \operatorname{mod}_n A^{\mathrm{op}}$ for all $0 \leq j \leq i$.
(3) We say ∂ satisfies $\gamma_{n,i}(M)$ if it satisfies $\gamma_{n,i}(M)^l$ and $\gamma_{n,i}(M)^r$.
(4) We say ∂ satisfies $\gamma_{n,i}(A, B)^l$ if it satisfies $\gamma_{n,i}(M)^l$ for all ${}_A M_B$ that are finitely generated on both sides.
(5) We say ∂ satisfies $\gamma_{n,i}(A, B)^r$ if it satisfies $\gamma_{n,i}(M)^r$ for all ${}_A M_B$ that are finitely generated on both sides.
(6) We say ∂ satisfies $\gamma_{n,i}(A, B)$ if it satisfies $\gamma_{n,i}(M)$ for all ${}_A M_B$ that are finitely generated on both sides.

Note that the conditions listed in the above definition are related to some conditions in terms of symmetric (derived) torsion, generalizing the χ-condition of [3], see [38, Section 16.5].

In the most parts of this paper, we will be particularly interested in the $\gamma_{n,1}$ property.

Lemma 1.9. *[5, Lemma 1.3] Let A and B be algebras such that ∂ is an exact dimension function on A-modules and B-modules. Assume that ∂ satisfies $\gamma_{n,1}(M)^l$ for a bimodule ${}_A M_B$. Then the functor $- \otimes_A M$ induces a functor*

$$- \otimes_{\mathcal{A}} \mathcal{M} : \operatorname{QMod}_n A \longrightarrow \operatorname{QMod}_n B.$$

Since M is finitely generated on both sides, this functor restricts to:

$$- \otimes_{\mathcal{A}} \mathcal{M} : \operatorname{qmod}_n A \longrightarrow \operatorname{qmod}_n B.$$

Lemma 1.10. *Retain the hypotheses in Lemma 1.9. Suppose that X and Y are in* $\operatorname{mod} A$ *such that $X \cong_n Y$. Then $X \otimes_A M \cong_n Y \otimes_A M$ in* $\operatorname{mod} B$.

Proof. The assertion follows from Lemmas 1.7 and 1.9 or the proof of [5, Lemma 1.3]. $\square$

We will also use the right adjoint functor of π defined in (E1.4.1). Since $\mathrm{Mod}_n A$ is a Serre subcategory (or a *dense* subcategory in the sense of [29, Sect. 4.3]), every right A-module has a largest submodule in $\mathrm{Mod}_n A$ (see also (E1.2.1)). Note that $\mathrm{Mod}\, A$ is *locally small* (in the sense of [29, p. 5]) and has enough injective objects. By a well-known classical category theory result [29, Theorem 4.4.5 or Proposition 4.5.2] (which is in a different mathematical language unfortunately), there is a *section functor*, denoted by ω (we are following the notation of [3, p. 234]) such that there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Mod}\, A}(N, \omega(\mathcal{M})) \cong \mathrm{Hom}_{\mathrm{QMod}_n\, A}(\pi(N), \mathcal{M}), \tag{E1.10.1}$$

for all $M \in \mathrm{Mod}\, A$ and $\mathcal{M} \in \mathrm{QMod}_n\, A$. Given a module M, let C_M denote the filtering category of maps $M \to M'$ whose kernel and cokernel are in $\mathrm{Mod}_n A$. Then

$$\omega\pi(M) = \lim_{(M \to M') \in C_M} M'. \tag{E1.10.2}$$

By [29, Proposition 4.4.3], the unit of the adjunction $Id \to \omega\pi$ induces the natural map

$$u_M : M \to \omega\pi(M), \tag{E1.10.3}$$

which has kernel and cokernel in $\mathrm{Mod}_n A$. Further, u_M is an isomorphism if and only if M is *closed* in the sense of [29, p. 176]. By [29, Lemma 4.4.6(2)], the image of u_M, which is canonically isomorphic to M/M' where M' is the largest subobject of M in $\mathrm{Mod}_n A$, is an essential subobject in $\omega\pi(M)$. The assertions in the following lemma are known to experts.

Lemma 1.11. *Let A and B be two algebras and n be a nonnegative integer. Let M be a right A-module and M' be the largest submodule of M such that $\partial(M') \leq n$.*

(1) *$\omega\pi(M)$ is naturally isomorphic to the largest submodule of the injective hull of M/M' containing M/M' such that $X/(M/M')$ is in* $\mathrm{Mod}_n A$.
(2) *If M is a (B, A)-bimodule, then $\omega\pi(M)$ is a (B, A)-bimodule and u_M is a bimodule morphism.*

Proof. (1) Without loss of generality, we can assume that M does not contain a nonzero submodule of ∂-dimension $\leq n$, namely, $M' = 0$. Let $C(M)$ be the largest submodule $X \supseteq M$ of the injective hull of M such that X/M is in $\mathrm{Mod}_n A$. By [29, Lemma 4.4.6(2)], we have canonical injective maps

$$M \xrightarrow{u_M} \omega\pi(M) \xrightarrow{f} C(M)$$

such that the cokernel of f is in $\mathrm{Mod}_n A$. In particular, $\pi(f)$ is an isomorphism. Applying the natural transformation $Id \to \omega\pi$, we have a commutative diagram

$$
\begin{array}{ccc}
\omega\pi(M) & \xrightarrow{f} & C(M) \\
{\scriptstyle u_{\omega\pi(M)}}\downarrow & & \downarrow{\scriptstyle u_{C(M)}} \\
\omega\pi\omega\pi(M) & \xrightarrow[\omega\pi(f)]{} & \omega\pi(C(M)).
\end{array}
$$

Since $\pi\omega \cong Id$, $u_{\omega\pi(M)}$ is an isomorphism. Since $\pi(f)$ is an isomorphism, $\omega\pi(f)$ is an isomorphism. Note that both f and $u_{C(M)}$ are injective. Hence f and $u_{C(M)}$ are isomorphisms.

(2) Since M is a (B, A)-bimodule, there is an algebra map $B \to \mathrm{End}_{\mathrm{Mod}\, A}(M_A)$. Applying the functor $\omega\pi$, we obtain an algebra map

$$B \to \mathrm{End}_{\mathrm{Mod}\, A}(M_A) \to \mathrm{End}_{\mathrm{Mod}\, A}(\omega\pi(M)),$$

which means that $\omega\pi(M)$ is a (B, A)-bimodule. Since the unit of the adjunction $Id \to \omega\pi$ is a natural transformation, u_M is a bimodule morphism. $\square$

2. Preliminaries on homological properties

In this section, we review some homological properties that are needed in the definition of a noncommutative quasi-resolution.

Definition 2.1. [26, Definitions 1.2, 2.1, 2.4] Let A be an algebra and M a right A-module.

(1) The *grade number* of M is defined to be

$$j_A(M) := \inf\{i \mid \mathrm{Ext}^i_A(M, A) \neq 0\} \in \mathbb{N} \cup \{+\infty\}.$$

If no confusion can arise, we write $j(M)$ for $j_A(M)$. Note that $j_A(0) = +\infty$.

(2) A nonzero A-module M is called *n-pure* (or just *pure*) if $j_A(N) = n$ for all nonzero finitely generated submodules N of M.

(3) We say M satisfies the *Auslander condition* if for any $q \geq 0$, $j_A(N) \geq q$ for all left A-submodules N of $\mathrm{Ext}^q_A(M, A)$.

(4) We say A is *Auslander-Gorenstein* (respectively, *Auslander regular*) of dimension n if $\mathrm{injdim}\, A_A = \mathrm{injdim}\, {}_A A = n < \infty$ (respectively, $\mathrm{gldim}\, A = n < \infty$) and every finitely generated left and right A-module satisfies the Auslander condition.

Proposition 2.2. *[7, Proposition 1.8] Let A be Auslander-Gorenstein. If*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is an exact sequence of finitely generated A-modules, then

$$j(M) = \inf\{j(M'), j(M'')\}.$$

Definition 2.3. [1, Definition 0.4] Let A be an algebra with a dimension function ∂. We say A is *∂-Cohen-Macaulay* (or, ∂-CM in short) if $\partial(A) = d \in \mathbb{N}$, and

$$j(M) + \partial(M) = \partial(A)$$

for every finitely generated nonzero left (or right) A-module M. If A is GKdim-Cohen-Macaulay, namely, if $\partial = \mathrm{GKdim}$, we just say it is *Cohen-Macaulay* or CM.

There are other modified definitions of noncommutative CM algebras, in particular, using the rigid Auslander dualizing complex over the algebra A [40]. Here we are using a more classical approach in Definition 2.3.

Remark 2.4.

(1) If A is ∂-CM with $\partial(A) \in \mathbb{N}$, then $j_A(M) < \infty$ and $\partial(M) \in \mathbb{N}$ for all nonzero finitely generated A-modules M.
(2) If A is a ∂-CM algebra, then ∂ is an exact dimension [2, p. 3]. In particular, GKdim is exact on finitely generated modules over CM algebras.
(3) Let A be Auslander-Gorenstein. The *canonical dimension* of a finitely generated right (or left) A-module M is defined to be

$$\partial(M) = \mathrm{injdim}\, A - j(M), \tag{E2.4.1}$$

which was introduced in [40, Definition 2.9], in the more general setting of Auslander dualizing complexes. For infinitely generated modules, see (E1.2.2). By [26, Proposition 4.5], the canonical dimension is an exact (but not necessarily symmetric) dimension function. By (E2.4.1), A is trivially ∂-CM.

Definition 2.5. [7, Definition 1.12] Let M be a finitely generated pure right A-module, see Definition 2.1(2). A *tame and pure* extension of M is a finitely generated right A-module N such that $M \subseteq N$, N is pure and $j(N/M) \geq j(M) + 2$. Note that a tame and pure extension is always an essential extension.

The following result of Björk is called *Gabber's Maximality Principle*, see [7, Theorem 1.14].

Theorem 2.6. *[7, Theorem 1.14] Let A be an Auslander-Gorenstein algebra. Suppose that M is a finitely generated n-pure A-module. Let N be an A-module containing M such that every nonzero finitely generated submodule of N is n-pure. Then N contains a unique largest tame and pure extension of M.*

We do not assume that N is finitely generated in the above theorem. On the other hand, by definition, a tame and pure extension of M is finitely generated. We will explain the Gabber's Maximality Principle in some details in the following two lemmas. Firstly, we recall some functors. Let ∂ be the canonical dimension defined in Remark 2.4(3) when M is finitely generated and extended to $\operatorname{Mod} A$ by (E1.2.2). We fix a non-negative integer n and let $d = \operatorname{injdim} A$. If M is n-pure, then $\partial(M) = d - n$ by (E2.4.1). In the next two lemmas, M will be an n-pure right A-module. Let

$$\pi : \operatorname{Mod} A \to \operatorname{QMod}_{d-n-2} A$$

and

$$\omega : \operatorname{QMod}_{d-n-2} A \to \operatorname{Mod} A,$$

see (E1.4.1) and (E1.10.1).

The following lemma is a special case of [40, Theorem 2.19]. For the convenience of readers, we give detailed proof.

Lemma 2.7. *Let A be an Auslander-Gorenstein algebra. Suppose that M is a finitely generated n-pure A-module. Then there is an n-pure A-module $\widetilde{M}$, unique up to unique isomorphism, such that the following hold.*

(1) *$\widetilde{M}$ is a tame and pure extension of M, namely, there is a given injective morphism $g_M : M \to \widetilde{M}$,*
(2) *If N is a tame and pure extension of M, then g_M factors uniquely through the inclusion map $M \to N$.*

Further, $\widetilde{M}$ is naturally isomorphic to both $\omega\pi(M)$ and $\operatorname{Ext}^n_{A^{\mathrm{op}}}(\operatorname{Ext}^n_A(M, A), A)$ and g_M agrees with u_M in (E1.10.3) when $\widetilde{M}$ is identified with $\omega\pi(M)$.

Proof. Let ∂ be the canonical dimension defined by (E2.4.1) and $d = \operatorname{injdim} A$.

(1) Let $E(M)$ be the injective hull of M. By Lemma 1.11(1), $\omega\pi(M)$ is a largest submodule of $E(M)$ containing M such that $\omega\pi(M)/M$ has ∂-dimension at most $d - n - 2$. So every nonzero finitely generated submodule $N(\supseteq M)$ of $\omega\pi(M)$ is n-pure and $j(N/M) \geq n + 2$. Thus N is a tame and pure extension of M. By Theorem 2.6, $\omega\pi(M)$ contains a largest (and maximal) tame and pure extension, which must be $\omega\pi(M)$ itself. So $\omega\pi(M)$ satisfies (1). We now define $\widetilde{M} = \omega\pi(M)$ and g_M to be the inclusion map.

(2) Let N be a tame and pure extension of M. Then N is an essential extension of M. Let $i : M \to N$ be the inclusion map. Since $E(M)$ is injective, there is an injective map $f : N \to E(M)$ such that $f \circ i : M \to E(M)$ is the inclusion map. Since N is a tame and pure extension of M, it is easy to see that the image of f is inside $\omega\pi(M)$. Thus we have a map $f : N \to \widetilde{M} := \omega\pi(M)$ such that $g_M = f \circ i$. Finally we prove the uniqueness of this factorization. Suppose there are two maps f_1, f_2 such that $g_M = f_1 \circ i = f_2 \circ i$. Then $(f_1 - f_2) \circ i = 0$ or the $(f_1 - f_2)(M) = 0$. Then the image of $f_1 - f_2$ is a quotient module

of N/M, which has ∂-dimension strictly less than $d-n$. Since $\widetilde{M}$ is n-pure, $f_1 - f_2$ must be zero, namely, $f_1 = f_2$. This shows that uniqueness.

Part (2) can be considered as a universal property. The uniqueness of $\widetilde{M}$ follows from part (2).

For the last assertion, we let $M^{**} := \mathrm{Ext}^n_{A^{\mathrm{op}}}(\mathrm{Ext}^n_A(M, A), A)$. By [26, Lemma 2.2] and [7, Proposition 1.13], M^{**} is a tame and pure extension and there is no other tame and pure extensions properly containing M^{**}. Therefore $M^{**} \cong \omega\pi(M)$ by part (2). □

Definition 2.8. Let A be an Auslander-Gorenstein algebra. Suppose that M is a finitely generated n-pure right A-module. The map $g_M : M \to \widetilde{M}$ (or simply the module $\widetilde{M}$) in Lemma 2.7, is called a *Gabber closure* of M. By Lemma 2.7, a Gabber closure of M always exists and is unique up to a unique isomorphism. Therefore, it is no confusion to call it *the Gabber closure* of M. In this case, we write the Gabber closure as $g_M : M \to G_A(M)$ (or simply $G_A(M)$).

Suppose ∂ is an arbitrary dimension function. When A is a ∂-CM algebra, ∂ equals to the canonical dimension up to a uniform shift. Hence (E2.4.1) implies that the condition

$$j(\widetilde{M}/M) \geq j(M) + 2$$

is equivalent to

$$\partial(\widetilde{M}/M) \leq \partial(M) - 2.$$

Lemma 2.9. *Let A be an Auslander-Gorenstein algebra. Suppose that M is a finitely generated n-pure right A-module. Let N be an n-pure A-module such that*

(a) *N is an essential extension of M, and*
(b) *$j(N'/M) \geq j(M)+2$ for all finitely generated A-submodule N' of N that contains M.*

Then N is a finitely generated A-module.

Proof. In this proof, let M^{**} denote $\mathrm{Ext}^n_{A^{\mathrm{op}}}(\mathrm{Ext}^n_A(M, A), A)$. If N is not finitely generated, then there is an ascending chain of finitely generated A-submodules

$$M \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq N$$

such that $M_i \in \operatorname{mod} A$ are n-pure and $j(M_i/M) \geq j(M) + 2$ for every i. By [7, Lemma 1.15], $M_i^{**} = M^{**}$. Moreover, since every M_i are n-pure, we have

$$M \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq \cdots \subseteq M_i^{**} = M^{**}$$

for every i. Since M^{**} is finitely generated by Lemma 2.7, the ascending chain stabilizes, a contradiction. □

We collect some facts and re-statements concerning the Gabber closure.

Proposition 2.10. *Let A be an Auslander-Gorenstein algebra. Suppose that M is a finitely generated n-pure A-module.*

(1) *The Gabber closure of M, denoted by $g_M : M \to G_A(M)$ as in Definition 2.8, exists and is unique up to a unique isomorphism.*
(2) *g_M agrees with u_M in (E1.10.3) for specific choices of π and ω given before Lemma 2.7.*
(3) *$G_A(M)$ is a tame and pure extension of M. In particular, $G_A(M)$ is finitely generated over A.*
(4) *[7, Proposition 1.13] $G_A(M)$ does not have any proper tame and pure extension.*
(5) *Let N be a tame and pure extension of M. If N does not have any proper tame and pure extension, then $N \cong G_A(M)$.*
(6) *If M is a (B, A)-bimodule, then $G_A(M)$ is a (B, A)-bimodule and g_M is a morphism of (B, A)-bimodules.*

Proof. (1, 2, 3) See Lemma 2.7.

(4) Since $G_A(M)$ is identified with $\mathrm{Ext}^n_{A^{\mathrm{op}}}(\mathrm{Ext}^n_A(M, A), A)$, the assertion is exactly [7, Proposition 1.13].

(5) By Lemma 2.7(2), $G_A(M)$ is a tame and pure extension of N. Since N does not have a proper tame and pure extension, $N = G_A(M)$.

(6) Since $G_A(M)$ can be identified with $\omega\pi(M)$, the assertion follows from Lemma 1.11(2). □

For every right A-module M, let

$$M^\vee = \mathrm{Hom}_A(M, A) \tag{E2.10.1}$$

and

$$M^{\vee\vee} = \mathrm{Hom}_{A^{\mathrm{op}}}(\mathrm{Hom}_A(M, A), A). \tag{E2.10.2}$$

When $n = 0$ as in the proof of Lemma 2.9, $M^{**} = M^{\vee\vee}$. By adjunction, there is a natural map $M \longrightarrow M^{\vee\vee} := \mathrm{Hom}_{A^{\mathrm{op}}}(\mathrm{Hom}_A(M, A), A)$.

Definition 2.11. Let A be an algebra. A finitely generated right A-module M is called *reflexive* if the natural morphism $M \longrightarrow M^{\vee\vee}$ is an isomorphism. A reflexive left module is defined similarly.

It's obvious that when A is Auslander-Gorenstein, an A-module M of maximal dimension is reflexive if and only if M is its own Gabber closure. Note that the definition of a reflexive module given in [21–23] is relative to a given base commutative ring. It is clear that every projective module is reflexive, but the converse is not true. The following lemma and corollary are well-known.

Lemma 2.12. *Let A be an algebra of global dimension d. If $M \in \operatorname{mod} A$, then* $\operatorname{projdim}_{A^{\mathrm{op}}} M^\vee \le \max\{0, d-2\}$, *where* $(-)^\vee = \operatorname{Hom}_A(-, A)$.

Proof. Suppose that $\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ is a projective resolution of M such that each P_i is finitely generated. Applying $(-)^\vee$ to the above exact sequence, there is an exact sequence of left A-modules

$$0 \longrightarrow M^\vee \longrightarrow P_0^\vee \longrightarrow P_1^\vee \longrightarrow E \longrightarrow 0,$$

where $E = \operatorname{coker}(P_0^\vee \to P_1^\vee)$. Since $\operatorname{projdim}_{A^{\mathrm{op}}} E \le d$ and $P_0^\vee, P_1^\vee \in \operatorname{proj} A^{\mathrm{op}}$, we have $\operatorname{projdim}_{A^{\mathrm{op}}} M^\vee \le \max\{0, d-2\}$. □

Corollary 2.13. *Let A be an algebra of global dimension d. If $M \in \operatorname{ref} A$, then* $\operatorname{projdim}_A M \le \max\{0, d-2\}$. *In particular, if $d \le 2$, then any reflexive A-module is projective.*

Proof. Use Lemma 2.12 and the fact $M \cong \operatorname{Hom}_{A^{\mathrm{op}}}(\operatorname{Hom}_A(M, A), A)$. □

Next we recall some results about spectral sequences.

If A is noetherian with $\operatorname{injdim} A < \infty$ and M is a finitely generated A-module, then there is a convergent spectral sequence [26, Theorem 2.2(a)], see (E2.13.1) below. To simplify notation later, we use a non-standard indexing of E_2^{pq}, with our indexing, the boundary maps on the E_2-page are $d_2^{p,q} : E_2^{pq} \to E_2^{p+2,q+1}$:

$$E_2^{pq} := \operatorname{Ext}^p_{A^{\mathrm{op}}}(\operatorname{Ext}^q_A(M, A), A) \Rightarrow \mathrm{H}^{p-q}(M) := \begin{cases} 0, & \text{if } p \neq q, \\ M, & \text{if } p = q. \end{cases} \tag{E2.13.1}$$

When A is Auslander-Gorenstein with $\operatorname{injdim} A = d$, there is a canonical filtration

$$0 = F^{d+1}M \subseteq F^d M \subseteq \cdots \subseteq F^1 M \subseteq F^0 M = M \tag{E2.13.2}$$

such that $F^pM/F^{p+1}M \cong E_\infty^{pp}$. By [26, Theorem 2.2], for each p, there exists an exact sequence

$$0 \longrightarrow E_\infty^{pp} \longrightarrow E_2^{pp} \longrightarrow Q(p) \longrightarrow 0$$

with $j(Q(p)) \ge p + 2$.

We collect some facts which can be shown by using the above spectral sequences.

Proposition 2.14. *[26, Theorem 2.4] Let A be Auslander-Gorenstein and M be a nonzero finitely generated A-module. If $n = j_A(M)$, then $\operatorname{Ext}_A^n(M, A)$ is n-pure and $(E_2^{pp} =) \operatorname{Ext}_{A^{\mathrm{op}}}^p(\operatorname{Ext}_A^p(M, A), A)$ is either 0 or p-pure for every integer p.*

Proposition 2.15. *[7, Proposition 1.9] Let A be Auslander-Gorenstein. Then a finitely generated A-module M is $j(M)$-pure if and only if $E_2^{pp} = 0$ for any $p \neq j(M)$.*

Corollary 2.16. *Let A be Auslander-Gorenstein and M a nonzero reflexive A-module. Then M is 0-pure, and $E_2^{pp} = 0$ for any $p \neq 0$. As a consequence, if A is also a ∂-CM algebra, then $\partial(M) = \partial(A) = \partial(N)$ for any nonzero reflexive A-module M and any nonzero submodule N of M.*

Proof. Suppose that M is a nonzero reflexive A-module, then $j(M) = 0$ and

$$0 \neq E_2^{00} = \operatorname{Hom}_{A^{\mathrm{op}}}(\operatorname{Hom}_A(M, A), A) \cong M,$$

which is 0-pure by Proposition 2.14. By Proposition 2.15, $E_2^{pp} = 0$ for every $p \neq 0$. The remaining statement follows by the definition of ∂-CM. □

Lemma 2.17. *Let A be Auslander-Gorenstein and M a finitely generated m-pure A-module where $m = j(M)$. Then $M = F^m M \supseteq F^{m+1} M = 0$. Further, $M = E_\infty^{mm} M \subseteq E_2^{mm} M$. In particular, if M is a 0-pure module, then*

$$M \subseteq M^{\vee\vee} := \operatorname{Hom}_{A^{\mathrm{op}}}(\operatorname{Hom}_A(M, A), A).$$

Proof. If M is m-pure, then $E_2^{pp} = 0$ for every $p \neq m$. Therefore

$$E_\infty^{pp} = 0 = F^p M / F^{p+1} M$$

for $p \neq m$. Taking $p = m + 1$, we obtain that $F^{m+1} M = F^{m+2} M = \cdots = 0$, as required. □

Proposition 2.18. *Let A be an Auslander regular algebra with $\operatorname{gldim} A = 3$. If M is a nonzero reflexive A-module, then*

$$E_2^{pq} \cong \begin{cases} M, & \text{if } p = q = 0, \\ E_2^{10} \cong E_2^{31}, & \text{if } p = 1, q = 0, \\ E_2^{31} \cong E_2^{10}, & \text{if } p = 3, q = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Since A is Auslander regular, we have that the E_2 table for M looks like

$$\begin{array}{cccc} 0 & 0 & 0 & E^{33} \\ 0 & 0 & E^{22} & E^{32} \\ 0 & E^{11} & E^{21} & E^{31} \\ E^{00} & E^{10} & E^{20} & E^{30} \end{array}$$

with $E^{20} = E^{30} = 0$ by Lemma 2.12, $E^{11} = E^{22} = E^{33} = 0$ by Propositions 2.14 and 2.15, and $E^{32} = 0$ since $\operatorname{projdim} M \leq 1$ by Lemma 2.12. Hence the E_2 table reduces to

$$\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & E^{21} & E^{31} \\ E^{00} & E^{10} & 0 & 0 \end{array}$$

and so it suffices to show that $E^{21} = 0$. By (E2.13.1)-(E2.13.2), there is a canonical filtration $0 = F^4M \subseteq F^3M \subseteq F^2M \subseteq F^1M \subseteq F^0M = M$ such that $F^pM/F^{p+1}M \cong E_\infty^{pp}$. It is obvious that $F^1M = 0$. Then $E_\infty^{pp} = 0$ for every $p \neq 0$ by Proposition 2.15, $E_\infty^{00} \cong F^0M/F^1M = M$, and further, $d_2^{00} = 0$ (as M being reflexive), which implies that $E^{21} = 0$. Thus, the E_2 table for M now looks like

$$\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & E^{31} \\ E^{00} & E^{10} & 0 & 0 \end{array}$$

with $E^{10} \cong E^{31}$. The assertion follows. □

Lemma 2.19. *Let A be an Auslander Gorenstein algebra and M be a finitely generated right A-module.*

(1) *$M^\vee$ is either 0 or a finitely generated reflexive left A-module.*
(2) *If M is 0-pure, then the Gabber closure $G_A(M)$ is reflexive.*

Proof. (1) It is well-known that $M^\vee$ is a finitely generated left A-module.

Let N be the largest submodule of M such that $j(N) > 0$ or $N^\vee = 0$. Then we have a short exact sequence

$$0 \to N \to M \to M/N \to 0,$$

which gives rise to an exact sequence

$$0 \to (M/N)^\vee \to M^\vee \to N^\vee \to \cdots .$$

Since $N^\vee = 0$, we have $(M/N)^\vee \cong M^\vee$. To prove the assertion one may assume that $N = 0$, or equivalently, M is 0-pure.

By [7, Proposition 1.13] (also see Lemma 2.7), there is a short exact sequence

$$0 \to M \to M^{\vee\vee} \to M' \to 0,$$

where $j(M') \geq 2$. The above short exact sequence gives rise to an exact sequence

$$0 \to (M')^\vee \to (M^{\vee\vee})^\vee \to M^\vee \to \operatorname{Ext}^1_A(M', A) \to \cdots .$$

Since $j(M') \geq 2$, we have $(M')^\vee = \operatorname{Ext}^1_A(M', A) = 0$. Therefore $(M^{\vee\vee})^\vee$ is naturally isomorphic to $M^\vee$ as required.

(2) By Lemma 2.7, the Gabber closure of M is isomorphic to $M^{\vee\vee}$. The assertion follows from part (1). □

3. A NQR of an algebra

In this section, we introduce the notion of a noncommutative quasi-resolution (NQR), which is a further generalization of the notion of a NCCR, and then study some basic properties.

Let $\mathcal{A}$ be a category consisting of a class of noetherian $\Bbbk$-algebras such that A is in $\mathcal{A}$ if and only if A^{op} is in $\mathcal{A}$. Together with $\mathcal{A}$ we consider a special class of modules/morphisms/bimodules. Our definition of a noncommutative quasi-resolution will be made inside the category $\mathcal{A}$. Sometimes it is necessary to be specific, but in the most of cases, it is quite easy to understand what is the setting of $\mathcal{A}$. We also need to specify or fix a dimension function ∂. Here are a few examples.

Example 3.1.

(1) Let $\mathcal{A}$ be the category of $\mathbb{N}$-graded locally finite noetherian $\Bbbk$-algebras with finite GK-dimension. We only consider graded modules. An (A, B)-bimodule is a $\mathbb{Z}$-graded module that has both left graded A-module and right graded B-module structures. The dimension function ∂ is chosen to be GKdim.

(2) We might modify the category in part (1) by restricting algebras to those with balanced Auslander dualizing complexes in the sense of [40]. In this case, we might take the dimension function to be a constant shift of the canonical dimension defined in [40, Definition 2.9].

(3) Let R be a noetherian commutative algebra with finite Krull dimension. Let $\mathcal{A}$ be the category of algebras that are module-finite R-algebras. Modules are usual modules, but an (A, B)-bimodule means an R-central (A, B)-bimodule. The dimension function in this case could be the Krull dimension.

Unless otherwise stated, we retain Hypothesis 1.3 concerning the fixed dimension function ∂ for modules over A, B and B_i, a bimodule (such as M or N in most of cases) over these rings in this section (including Definitions 3.2 and 3.16) is finitely generated on both sides. As a consequence of these assumptions, $\partial(M)$ can be defined by considering M as either a left or a right module. Therefore $M \cong_n N$ is well-defined on either left or right sides for another bimodule N. If we use other rings such as D, we may not assume these hypotheses.

Here is our main definition.

Definition 3.2. Let $A \in \mathcal{A}$ be an algebra with $\partial(A) = d$. Let s be an integer between 0 and $d - 2$.

(1) If there are an Auslander regular ∂-CM algebra $B \in \mathcal{A}$ with $\partial(B) = d$ and two bimodules ${}_BM_A$ and ${}_AN_B$ such that

$$M \otimes_A N \cong_{d-2-s} B, \quad N \otimes_B M \cong_{d-2-s} A$$

as bimodules, then the triple $(B, {}_BM_A, {}_AN_B)$ is called an *s-noncommutative quasi-resolution* (*s-NQR* for short) of A.

(2) (Definition 0.5) A 0-noncommutative quasi-resolution (0-NQR) of A is called a *noncommutative quasi-resolution* (*NQR* for short) of A.

We will see that the notion of a NQR is a generalization of the notion of a NCCR in Section 7. First we prove the following lemmas.

Lemma 3.3. *Let B be a ∂-CM algebra with $\partial(B) = d$.*

(1) *If there exist B-modules M and N such that $M \cong_{d-2} N$, then $M^\vee \cong N^\vee$, where $(-)^\vee := \mathrm{Hom}_B(-, B)$.*

(2) *Let D be another algebra and supposed that M and N are (D, B)-bimodules such that $M \cong_{d-2} N$ as (D, B)-bimodules. Then $M^\vee \cong N^\vee$ as (B, D)-bimodules.*

Proof. The proofs of parts (1) and (2) are similar. We only prove part (1).

By definition, there exists a right B-module P and B-module morphisms $f : M \to P$ and $g : N \to P$ such that both the kernel and cokernel of f and g have ∂-dimension no more than $d - 2$. It suffices to show that $P^\vee \cong M^\vee$. Without loss of generality, we assume that $f : M \to N$ is a right B-morphism such that the kernel and cokernel of f have ∂-dimension no more than $d-2$. By the properties of f we have two exact sequences

$$0 \to Q \to N \to C \to 0$$

and

$$0 \to K \to M \to Q \to 0,$$

where $Q = \operatorname{Im} f$ and where C and K have ∂-dimension no more than $d-2$. We need to show that $Q^\vee \cong N^\vee$ and $M^\vee \cong Q^\vee$. The proofs of these assertions are similar, we only show the first one. Applying $\operatorname{Hom}_B(-,B)$ to the first short exact sequence, we obtain that a long exact sequence

$$0 \to \operatorname{Hom}_B(C,B) \to N^\vee \to Q^\vee \to \operatorname{Ext}^1_B(C,B) \to \cdots.$$

Since $\partial(C) + j(C) = \partial(B) = d$ and $\partial(C) \le d-2$, $j(C) \ge 2$, which means that $\operatorname{Hom}_B(C,B) = 0 = \operatorname{Ext}^1_B(C,B)$. So, $N^\vee \cong Q^\vee$. Similarly, $P^\vee \cong Q^\vee$. Therefore, $P^\vee \cong N^\vee$. $\square$

The following corollary is clear.

Corollary 3.4. *Let B be a ∂-CM algebra with $\partial(B) = d$.*

(1) *If $M, N \in \operatorname{ref} B$ such that $M \cong_{d-2} N$, then $M \cong N$.*
(2) *Let D be another algebra and supposed that M and N are (D,B)-bimodules such that $M \cong_{d-2} N$ as (D,B)-bimodules. If $M, N \in \operatorname{ref} B$, then $M \cong N$ as (B,D)-bimodules.*

Proof. We prove (2). Since $M \cong_{d-2} N$, Lemma 3.3 implies that $M^\vee \cong N^\vee$ as (B,D)-bimodules. The assertion follows by applying $(-)^\vee$ and the fact that $M, N \in \operatorname{ref} B$. $\square$

Lemma 3.5. *Let A and B be algebras and $n \in \mathbb{N}$. Suppose that there exist two bimodules ${}_BM_A$ and ${}_AN_B$ such that*

$$M \otimes_A N \cong_n B, \quad N \otimes_B M \cong_n A$$

as bimodules. If ∂ satisfies $\gamma_{n,1}(M)^l$ and $\gamma_{n,1}(N)^l$, then

$$\operatorname{qmod}_n A \cong \operatorname{qmod}_n B.$$

Proof. Let π be the natural functor $\operatorname{mod} A \longrightarrow \operatorname{qmod}_n A$ (or $\operatorname{mod} B \longrightarrow \operatorname{qmod}_n B$). Denote by $\mathcal{M} := \pi(M)$ and $\mathcal{N} := \pi(N)$. By Lemma 1.9, we have two well-defined functors

$$F(-) := - \otimes_{\mathcal{A}} \mathcal{N} : \operatorname{qmod}_n A \longrightarrow \operatorname{qmod}_n B$$

and

$$G(-) := - \otimes_{\mathcal{B}} \mathcal{M} : \operatorname{qmod}_n B \longrightarrow \operatorname{qmod}_n A.$$

By Lemma 1.9 again,

$$F \circ G(-) = - \otimes_{\mathcal{B}} \mathcal{M} \otimes_{\mathcal{A}} \mathcal{N} \cong - \otimes_{\mathcal{B}} \pi(M \otimes_A N) \cong - \otimes_{\mathcal{B}} \mathcal{B},$$

and

$$G \circ F(-) = - \otimes_{\mathcal{A}} \mathcal{N} \otimes_{\mathcal{B}} \mathcal{M} \cong - \otimes_{\mathcal{A}} \pi(N \otimes_B M) \cong - \otimes_{\mathcal{A}} \mathcal{A}.$$

Therefore F and G are equivalences, in other words, $\mathrm{qmod}_n\, A \cong \mathrm{qmod}_n\, B$. □

The equivalence $\mathrm{qmod}_n\, A \cong \mathrm{qmod}_n\, B$ in the above lemma can be considered as a noncommutative Fourier-Mukai transform between two noncommutative spaces. We refer to [20] for the classical setting.

Remark 3.6. The above lemma holds true for a NQR $(B, {}_BM_A, {}_AN_B)$ of an algebra A when ∂ satisfies $\gamma_{d-2,1}(M)^l$ and $\gamma_{d-2,1}(N)^l$.

For an n-pure (A, B)-bimodule M, the Gabber closure of ${}_AM$ is denoted by $G_{A^{\mathrm{op}}}(M)$ and the Gabber closure of M_B is denoted by $G_B(M)$. We consider both $G_{A^{\mathrm{op}}}(M)$ and $G_B(M)$ as extensions of M.

Lemma 3.7. *Let A and B be Auslander-Gorenstein and ∂-CM algebras with*

$$\partial(A) = \partial(B) = d.$$

Assume Hypothesis 1.3 holds. Let n be an integer. Let M denote an (A, B)-bimodule that is finitely generated on both sides.

(1) *Let M be n-pure on both sides. Then $G_{A^{\mathrm{op}}}(M) \cong G_B(M)$ naturally as bimodules with restriction on M being the identity.*
(2) *Let M be n-pure on both sides. Then $G_B(M) = M$ if and only if $G_{A^{\mathrm{op}}}(M) = M$.*
(3) *The (A, B)-bimodule M is reflexive on the left if and only if it is reflexive on the right.*

Proof. Without loss of generality, we can assume that ∂ is the canonical dimension.

(1) By Proposition 2.10(6), $G_B(M)$ is an (A, B)-bimodule and $g_{M_B} : M \to G_B(M)$ is a bimodule morphism. By Proposition 2.10(3), $G_B(M)$ is finitely generated on the right. We claim that

(a) g_{M_B} is an essential extension of M on the left,
(b) $G_B(M)$ is finitely generated over the left, and
(c) g_{M_B} is a tame and pure extension of M on the left.

By Hypothesis 1.3(3),

$$\partial({}_A(G_B(M)/M)) \leq \partial((G_B(M)/M)_B) \leq d-n-2. \tag{E3.7.1}$$

To prove (a) let S be a left A-submodule of $G_B(M)$ such that $S \cap M = 0$. Then S is isomorphic to a submodule of $G_B(M)/M$. As a consequence, $\partial(S) \leq d-n-2$. Let U be the largest left A-submodule of $G_B(M)$ with $\partial \leq d-n-2$. Then $U \cap M = 0$ and U is also a right B-submodule. If $U \neq 0$, it contradicts the fact that $G_B(M)$ is an essential extension of M on the right. Therefore $U = 0$ and $S = 0$. Thus Claim (a) is proven.

Claim (b) follows from Lemma 2.9 and (E3.7.1).

Claim (c) follows from Claim (b) and (E3.7.1).

Next we consider the Gabber closure of the module $N := G_B(M)$ on the left. By Proposition 2.10(6), $G_{A^{\text{op}}}(N)$ is an (A,B)-bimodule and $g_{A^{\text{op}}N} : N \to G_{A^{\text{op}}}(N)$ is a bimodule morphism. By symmetric, $g_{A^{\text{op}}N}$ has properties (a, b, c) on the right. By part (c), $g_{A^{\text{op}}N}$ is a tame and pure extension of N on the right. By Proposition 2.10(4), $N_B(:= G_B(M))$ does not have a proper tame and pure extension on the right. Therefore $G_{A^{\text{op}}}(N) = N$. This implies that ${}_AN$ does not have a proper tame and pure extension. Thus N must be $G_{A^{\text{op}}}(M)$ by Proposition 2.10(5).

(2) This is a consequence of part (1).

(3) By Lemma 2.7, M_B is reflexive if and only if M is 0-pure and $G_B(M) = M$. The assertion follows by part (2). □

Hypothesis 3.8. We are continuing to work with algebras in a given category $\mathcal{A}$ with a fixed dimension function ∂ defined for all modules over rings in $\mathcal{A}$. As indicated at the beginning of this section we assume Hypothesis 1.3 for all algebras in $\mathcal{A}$. Now we further assume that ∂ satisfies $\gamma_{d-2,1}(A,B)$ for algebras A and B in $\mathcal{A}$ with $d = \partial(A) = \partial(B)$, which covers the hypotheses in Lemmas 1.9 and 1.10.

Proposition 3.9. *Assume Hypothesis 3.8. Let B_i be Auslander-Gorenstein and ∂-CM algebras with $\partial(B_i) = d$ for $i = 1,2$. Suppose that there are bimodules ${}_{B_1}T_{B_2}$ and ${}_{B_2}\widetilde{T}_{B_1}$ (finitely generated on both sides) such that*

$$T \otimes_{B_2} \widetilde{T} \cong_{d-2} B_1 \tag{E3.9.1}$$

and

$$\widetilde{T} \otimes_{B_1} T \cong_{d-2} B_2.$$

Then there exist ${}_{B_1}U_{B_2}$ and ${}_{B_2}V_{B_1}$ (finitely generated on both sides) such that U, V are reflexive modules on both sides and

$$U \otimes_{B_2} V \cong_{d-2} B_1, \quad V \otimes_{B_1} U \cong_{d-2} B_2.$$

In other words, we can replace T and $\widetilde{T}$ with ${}_{B_1}U_{B_2}$ and ${}_{B_2}V_{B_1}$ respectively, which are reflexive modules on both sides.

In the following proof, we need to deal with multiple different rings/modules. It is convenient to fix the following notation specially when we deal with bimodules. Starting from a right B-module M, we use $M^{\vee}$ (respectively, $M^{\vee\vee}$) for $\mathrm{Hom}_B(M,B)$ (respectively, $\mathrm{Hom}_{B^{\mathrm{op}}}(\mathrm{Hom}_B(M,B),B)$). Starting from a left B-module M, we use ${}^{\vee}M$ (respectively, ${}^{\vee\vee}M$) for $\mathrm{Hom}_{B^{\mathrm{op}}}(M,B)$ (respectively, $\mathrm{Hom}_B(\mathrm{Hom}_{B^{\mathrm{op}}}(M,B),B)$). For example, for a (B_1,B_2)-bimodule M, we have

$$M^{\vee\vee} = \mathrm{Hom}_{B_2^{\mathrm{op}}}(\mathrm{Hom}_{B_2}(M,B_2),B_2)$$

and

$${}^{\vee\vee}M = \mathrm{Hom}_{B_1}(\mathrm{Hom}_{B_1^{\mathrm{op}}}(M,B_1),B_1).$$

By Lemma 2.19, for every finitely generated right B-module M, $M^{\vee\vee}$ is (either zero or) always reflexive when B is Auslander-Gorenstein.

Proof of Proposition 3.9. By Lemma 3.3(2) and (E3.9.1), we have

$${}^{\vee\vee}(T\otimes_{B_2}\widetilde{T}) \cong {}^{\vee\vee}B_1 \cong B_1,$$

as B_1-bimodules. Hence there is a composite map

$$\psi : T\otimes_{B_2}\widetilde{T} \longrightarrow {}^{\vee\vee}(T\otimes_{B_2}\widetilde{T}) \longrightarrow B_1$$

which induces the $(d-2)$-isomorphism from $T\otimes_{B_2}\widetilde{T}$ to B_1.

Define

$$\tau(T) := \{x\in T \mid xr = 0 \text{ for some regular element } r\in B_2\}.$$

By [1, Proposition 2.4(4) and Theorem 6.1], $\tau(T)$ is the maximal torsion B_2-submodule of T such that $\partial(\tau(T))$ is at most $d-1$, namely, $\tau(T)\in \mathrm{mod}_{d-1} B_2$. Since we assume that ∂ is symmetric (Hypothesis 1.3(3)), $\tau(T)\in \mathrm{mod}_{d-1} B_1^{\mathrm{op}}$.

Note that $\widetilde{T}$ is a finitely generated left B_2-module, and by the definition of $\tau(T)$, we have $\tau(T)\otimes_{B_2}\widetilde{T}\in \mathrm{mod}_{d-1} B_1^{\mathrm{op}}$. Applying $-\otimes_{B_2}\widetilde{T}$ to an exact sequence

$$0\to \tau(T)\to T\to T/\tau(T)\to 0$$

in $\mathrm{mod}\, B_2$, one has an exact sequence

$$\tau(T)\otimes_{B_2}\widetilde{T} \xrightarrow{f} T\otimes_{B_2}\widetilde{T} \to T/\tau(T)\otimes_{B_2}\widetilde{T}\to 0$$

in $\operatorname{mod} B_1$. Then

$$T/\tau(T) \otimes_{B_2} \widetilde{T} \cong (T \otimes_{B_2} \widetilde{T})/\operatorname{Im}(f)$$

in $\operatorname{mod} B_1$. Since $B_1 \in \operatorname{ref} B_1$ is a 0-pure module and $\operatorname{Im}(f) \subseteq T \otimes_{B_2} \widetilde{T}$, we have $\psi(\operatorname{Im}(f)) = 0$, whence there are well-defined morphisms

$$T \otimes_{B_2} \widetilde{T} \to T/\tau(T) \otimes_{B_2} \widetilde{T} \cong (T \otimes_{B_2} \widetilde{T})/\operatorname{Im}(f) \longrightarrow B_1 \cong_{d-2} T \otimes_{B_2} \widetilde{T}$$

such that the composition is a $(d-2)$-isomorphism. Therefore, $T/\tau(T) \otimes_{B_2} \widetilde{T} \cong_{d-2} B_1$. Now, we can replace T with $T/\tau(T)$ in (E3.9.1) and assume that $\tau(T) = 0$, namely, T is a 0-pure B_2-module (whence a 0-pure left B_1-module by the symmetry of ∂).

Let $U := G_{B_2}(T)$ be the Gabber closure of T. By Lemma 3.7, U is isomorphic to $G_{B_1^{\mathrm{op}}}(T)$ as bimodules. This implies that U is finitely generated on both sides and $T \cong_{d-2} U$ by the definition of the Gabber closure. Combining this $(d-2)$-isomorphism with (E3.9.1) and Lemma 1.10, we have

$$U \otimes_{B_2} \widetilde{T} \cong_{d-2} B_1.$$

Similarly,

$$\widetilde{T} \otimes_{B_1} V \cong_{d-2} B_2.$$

Since T is 0-pure, by Lemmas 2.19 and 3.7(3), U is reflexive on both sides. Next we take $V = G_{B_1}(\widetilde{T})$ and repeat the above argument. It is easy to see that U and V satisfy the required conditions. □

Lemma 3.10. *Let B be an Auslander-Gorenstein and ∂-CM algebra with $\partial(B) = d$ and U a nonzero reflexive B-module. Then*

(1) $\operatorname{Hom}_B(C, U) = 0$ *for any* $C \in \operatorname{mod}_{d-1} B$.
(2) $\operatorname{Ext}^1_B(K, U) = 0$ *for any* $K \in \operatorname{mod}_{d-2} B$.

Proof. (1) Let $f \in \operatorname{Hom}_B(C, U)$. Since $\operatorname{Im}(f)$ is a quotient module of C,

$$\partial(\operatorname{Im}(f)) \le \partial(C) \le d - 1.$$

By Corollary 2.16, U is 0-pure. If $\operatorname{Im}(f) \neq 0$, then $\partial(\operatorname{Im}(f)) = \partial(U) = d$, a contradiction. Therefore $\operatorname{Im}(f) = 0$, which implies that $\operatorname{Hom}_B(C, U) = 0$.

(2) If $\operatorname{Ext}^1_B(K, U) \neq 0$, there is a non-split extension

$$0 \to U \to E \to K \to 0 \tag{E3.10.1}$$

in $\operatorname{mod} B$. Let $\tau(E)$ be the maximal submodule of E such that $\partial(\tau(E)) \leq d-1$. Then there exists an induced morphism $\varphi : \tau(E) \to K$ such that $\ker \varphi \subseteq U \cap \tau(E)$. Since $\partial(U \cap \tau(E)) \leq \partial(\tau(E)) \leq d-1$ and U is 0-pure, $U \cap \tau(E) = 0$. This implies that φ is injective, whence, we can consider $\tau(E)$ as a submodule of K, and φ is not surjective (following by the fact that (E3.10.1) is non-split). Hence, we obtain a short exact sequence

$$0 \longrightarrow U \longrightarrow E/\tau(E) \longrightarrow K/\tau(E) \longrightarrow 0$$

with

$$\partial\Big(E/\tau(E)\Big/U\Big) = \partial(K/\tau(E)) \leq \partial(K) \leq d-2 = \partial(U) - 2.$$

By the definition of $\tau(E)$, $E/\tau(E)$ is 0-pure. So, $E/\tau(E)$ is a tame and pure extension of U. By hypothesis, U is a reflexive module, whence $U = G_B(U)$ by Lemma 2.7. Then, by Proposition 2.10(4), $E/\tau(E) \cong U$, or equivalently, $K/\tau(E) = 0$. This means that $K = \tau(E)$, or equivalently, the exact sequence (E3.10.1) is split, a contradiction. The assertion follows. □

Remark 3.11. The reflexivity of module U is not necessary for Lemma 3.10(1). In fact, when U is a 0-pure module, Lemma 3.10(1) is also true.

Lemma 3.12. *Let B be an Auslander-Gorenstein and ∂-*CM *algebra with $\partial(B) = d$. Suppose that $0 \neq U \in \operatorname{mod} B$ satisfies*

$$\operatorname{Hom}_B(N, U) = 0 = \operatorname{Ext}^1_B(N, U)$$

for all $N \in \operatorname{mod}_{d-2} B$.

(1) *For $M \in \operatorname{mod} B$, $\operatorname{Hom}_{\operatorname{qmod}_{d-2} B}(\mathcal{M}, \mathcal{U}) \cong \operatorname{Hom}_B(M, U)$.*
(2) *$\operatorname{End}_{\operatorname{qmod}_{d-2} B}(\mathcal{U}) \cong \operatorname{End}_B(U)$.*
(3) *In particular, if $M \in \operatorname{mod} B$ and $U \in \operatorname{ref} B$, then*

$$\operatorname{Hom}_{\operatorname{qmod}_{d-2} B}(\mathcal{M}, \mathcal{U}) = \operatorname{Hom}_B(M, U)$$

and

$$\operatorname{End}_{\operatorname{qmod}_{d-2} B}(\mathcal{U}) \cong \operatorname{End}_B(U).$$

Proof. (1) By the assumption, U does not have any nonzero B-submodule of ∂-dimension at most $d-2$. Combining with (E1.4.2), we have

$$\operatorname{Hom}_{\operatorname{qmod}_{d-2} B}(\mathcal{M}, \mathcal{U}) = \varinjlim \operatorname{Hom}_B(K, U),$$

where the limit runs over all the submodules $K \subseteq M$ such that $\partial(M/K) \leq d-2$. The functor π induces a natural morphism

$$\phi_\pi: \quad \mathrm{Hom}_B(M,U) \to \mathrm{Hom}_{\mathrm{qmod}_{d-2} B}(\mathcal{M},\mathcal{U}) := \varinjlim \mathrm{Hom}_B(K,U), \tag{E3.12.1}$$

where $K \subseteq M$ as described as above. By hypotheses,

$$\mathrm{Hom}_B(M/K,U) = 0 = \mathrm{Ext}^1_B(M/K,U).$$

Now the short exact sequence $0 \to K \to M \to M/K \to 0$ induces a long exact sequence

$$0 \to \mathrm{Hom}_B(M/K,U) \to \mathrm{Hom}_B(M,U) \to \mathrm{Hom}_B(K,U) \to \mathrm{Ext}^1_B(M/K,U) \to \cdots,$$

which implies that $\mathrm{Hom}_B(M,U) \cong \mathrm{Hom}_B(K,U)$ for all K. Thus ϕ_π in (E3.12.1) is an isomorphism. The assertion follows.

(2) Take $M = U$ in (E3.12.1), the functor π induces a morphism of algebras ϕ_π. By part (1), ϕ_π is also an isomorphism of $\Bbbk$-vector spaces. The assertion follows.

(3) If $U \in \mathrm{ref}\, B$, then by Lemma 3.10,

$$\mathrm{Hom}_B(N,U) = 0 = \mathrm{Ext}^1_B(N,U).$$

The assertion follows from parts (1, 2). □

Lemma 3.13. *Assume Hypothesis 3.8. Let ${}_{B_1}U_{B_2}$ be the module appeared in Proposition 3.9. Then it is a reflexive module on both sides such that $B_1 \cong \mathrm{End}_{B_2}(U)$ and $B_2^{\mathrm{op}} \cong \mathrm{End}_{B_1^{\mathrm{op}}}(U)$.*

Proof. By Proposition 3.9 and Lemma 3.5, ${}_{B_1}U_{B_2}$ is a reflexive module on both sides and induces the following equivalence of categories

$$F := - \otimes_{\mathcal{B}_1} \mathcal{U} : \mathrm{qmod}_{d-2}\, B_1 \longrightarrow \mathrm{qmod}_{d-2}\, B_2.$$

Since F is an equivalence functor, we obtain isomorphisms of algebras:

$$\mathrm{End}_{\mathrm{qmod}_{d-2} B_1}(\mathcal{B}_1) \cong \mathrm{End}_{\mathrm{qmod}_{d-2} B_2}(F(\mathcal{B}_1)) = \mathrm{End}_{\mathrm{qmod}_{d-2} B_2}(\mathcal{U}).$$

Now it suffices to show that

$$\mathrm{End}_{\mathrm{qmod}_{d-2} B_1}(\mathcal{B}_1) \cong B_1$$

and

$$\mathrm{End}_{\mathrm{qmod}_{d-2} B_2}(\mathcal{U}) \cong \mathrm{End}_{B_2}(U).$$

Since $B_1 \in \operatorname{ref} B_1$ and $U \in \operatorname{ref} B_2$, by Lemma 3.12(3), the above isomorphisms hold, as required.

By symmetry, $B_2^{\mathrm{op}} \cong \operatorname{End}_{B_1^{\mathrm{op}}}(U)$. □

Corollary 3.14. *Assume Hypothesis 3.8. Let A be an Auslander-Gorenstein and ∂-CM algebra with $\partial(A) = d$. Let $(B, {}_BM_A, {}_AN_B)$ be a NQR of A. Then there exists a bimodule ${}_BU_A := M^{\vee\vee}$ which is a reflexive module on both sides such that*

$$B \cong \operatorname{End}_A(U) \quad \textit{and} \quad A^{\mathrm{op}} \cong \operatorname{End}_{B^{\mathrm{op}}}(U).$$

Theorem 3.15. *Assume Hypothesis 3.8. Let A be an algebra with $\partial(A) = d$. Suppose that A has two NQRs $(B_i, {}_{B_i}(M_i)_A, {}_A(N_i)_{B_i})$ for $i = 1, 2$. Then there exists a bimodule ${}_{B_1}U_{B_2}$ which is a reflexive module on both sides such that*

$$B_1 \cong \operatorname{End}_{B_2}(U) \quad \textit{and} \quad B_2^{\mathrm{op}} \cong \operatorname{End}_{B_1^{\mathrm{op}}}(U).$$

Proof. Let $T := M_1 \otimes_A N_2$ and $\widetilde{T} := M_2 \otimes_A N_1$. Then there are isomorphisms, by Lemma 1.10,

$$T \otimes_{B_2} \widetilde{T} \cong_{d-2} B_1,$$

and

$$\widetilde{T} \otimes_{B_1} T \cong_{d-2} B_2.$$

Thus, the result follows from Lemma 3.13. □

Finally we introduce another definition, which is a bit closer to Van den Bergh's NCCR.

Definition 3.16. Let $A \in \mathcal{A}$ be an Auslander-Gorenstein algebra with $\partial(A) = d$. Let s be an integer between 0 and $d-2$.

(1) If there are an Auslander regular ∂-CM algebra $B \in \mathcal{A}$ with $\partial(B) = d$ and two bimodules ${}_BM_A$ and ${}_AN_B$ which are reflexive on both sides such that

$$M \otimes_A N \cong_{d-2-s} B, \quad N \otimes_B M \cong_{d-2-s} A$$

as bimodules, then the triple $(B, {}_BM_A, {}_AN_B)$ is called an *s-noncommutative quasi-crepant resolution* (s-*NQCR* for short) of A.

(2) A 0-noncommutative quasi-crepant resolution (0-NQCR) of A is called a *noncommutative quasi-crepant resolution* (*NQCR* for short) of A.

By definition, a NQCR of an Auslander-Gorenstein algebra A is automatic a NQR of A. Suppose A is an Auslander-Gorenstein and ∂-CM algebra. If A has a NQR, then, by Proposition 3.9, A has a NQCR. However, it is not clear to us whether an s-NQR (Definition 3.2) produces an s-NQCR when $s > 0$.

4. NQRs in dimension two

With the preparation in the last few sections, we are ready to prove a version of part (1) of the main theorem.

Lemma 4.1. *Let A be an Auslander-Gorenstein and ∂-CM algebra. Then*

$$\operatorname{injdim} A \leq \partial(A).$$

Proof. Let $d = \operatorname{injdim} A$. Then there is a right A-module M such that ${}_AN := \operatorname{Ext}^d_A(M, A) \neq 0$. By the Auslander condition, $j(N) \geq d$. Now, by the ∂-CM property,

$$\partial(A) = \partial(N) + j(N) \geq d = \operatorname{injdim} A. \quad \square$$

Theorem 4.2. *Assume Hypothesis 3.8. Suppose that $(B_i, {}_{B_i}(M_i)_A, {}_A(N_i)_{B_i})$ are two NQRs of A for $i = 1, 2$. If $\partial(A) \leq 2$, then B_1 and B_2 are Morita equivalent.*

Proof. By definition, $\partial(B_i) = \partial(A) \leq 2$. By Lemma 4.1,

$$\operatorname{gldim}(B_i) = \operatorname{injdim}(B_i) \leq \partial(B_1) \leq 2.$$

Let $T := M_1 \otimes_A N_2$ and $\widetilde{T} := M_2 \otimes_A N_1$. Then there are isomorphisms, by Lemma 1.10,

$$T \otimes_{B_2} \widetilde{T} \cong_{d-2} B_1,$$

and

$$\widetilde{T} \otimes_{B_1} T \cong_{d-2} B_2.$$

By Proposition 3.9, there exist ${}_{B_1}U_{B_2}$ and ${}_{B_2}V_{B_1}$ which are reflexive modules (and finitely generated) on both sides such that

$$U \otimes_{B_2} V \cong_{d-2} B_1 \quad \text{and} \quad V \otimes_{B_1} U \cong_{d-2} B_2.$$

Since $\operatorname{gldim}(B_i) \leq 2$, by Corollary 2.13, U and V are projective modules on both sides. Hence $U \otimes_{B_2} V$ and $V \otimes_{B_1} U$ are projective (whence reflexive) on both sides. Therefore, by Corollary 3.4, we have

$$U \otimes_{B_2} V \cong B_1 \quad \text{and} \quad V \otimes_{B_1} U \cong B_2,$$

which implies that B_1 and B_2 are Morita equivalent. This finishes the proof. $\square$

5. Depth in the noncommutative setting

The proof of part (2) of the main theorem needs some extra preparation. In particular, it uses the concept of a depth in noncommutative algebra. There are several slightly different definitions of the depth in the noncommutative setting. It is a good idea to fix some notation.

Let A be an algebra with a dimension function ∂.

Hypothesis 5.1. Let A be an algebra. Assume that $\operatorname{mod}_0 A \neq 0$, namely, there is a nonzero module $S \in \operatorname{mod}_0 A$.

Hypothesis 5.1 is sometimes quite natural, but not automatic. By abuse of notation, we can also talk about Hypothesis 5.1 for a single algebra A or for a family of algebras $\mathcal{A}$.

Definition 5.2. Let A be an algebra and ∂ be a dimension function. For an A-module $M \in \operatorname{mod} A$, define

$$\operatorname{dep}_A M = \inf\{i \mid \operatorname{Ext}^i_A(S, M) \neq 0 \text{ for some } S \in \operatorname{mod}_0 A\} \in \mathbb{N} \cup \{+\infty\}.$$

If no confusion can arise, we write $\operatorname{dep} M$ for $\operatorname{dep}_A M$. If Hypothesis 5.1 fails for A, then $\operatorname{dep}_A M = +\infty$ for every A-module M.

If $\operatorname{dep}_A M < +\infty$ for some A-module M, then Hypothesis 5.1 holds for the algebra A. One can easily prove the following depth lemma.

Lemma 5.3. *Let A be an algebra and ∂ be a dimension function. Let*

$$0 \to M' \to M \to M'' \to 0$$

be a short exact sequence of finitely generated right A-modules. Then

(1) $\operatorname{dep} M \geq \min\{\operatorname{dep} M', \operatorname{dep} M''\}$.
(2) $\operatorname{dep} M' \geq \min\{\operatorname{dep} M, \operatorname{dep} M'' + 1\}$.
(3) $\operatorname{dep} M'' \geq \min\{\operatorname{dep} M, \operatorname{dep} M' - 1\}$.

The proof of Lemma 5.3(2) is basically given in the proof of Lemma 5.5.

The following proposition resembles the "special χ condition" in [38, Definition 16.5.16].

Proposition 5.4. *Suppose that Hypothesis 5.1 holds for A. If A is a ∂-CM algebra with $\partial(A) = \partial(A^{\mathrm{op}}) = d$, then*

$$\operatorname{dep}_A A = \operatorname{dep}_{A^{\mathrm{op}}} A = d.$$

Proof. Given every nonzero $S \in \operatorname{mod}_0 A$, we have

$$j_A(S) = \partial(A) - \partial(S) = \partial(A) = d,$$

namely,

$$d = \inf\{i | \operatorname{Ext}^i_A(S, A) \neq 0\}.$$

Thus $\operatorname{dep}_A A = d$. Similarly, we have $\operatorname{dep}_{A^{\mathrm{op}}} A = d$. □

The proof of Theorem 0.6(2) also uses the following two lemmas, which were known in the local or graded setting [14, Lemma 3.15].

Lemma 5.5. *Suppose that M and N are nonzero finitely generated A-modules related by the exact sequence*

$$0 \longrightarrow M \longrightarrow P_{s-1} \longrightarrow P_{s-2} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow N \longrightarrow 0.$$

Then

$$\operatorname{dep}_A(M) \geq \min\{\operatorname{dep}_A(N) + s, \operatorname{dep}_A(P_0), \ldots, \operatorname{dep}_A(P_{s-2}), \operatorname{dep}_A(P_{s-1})\}.$$

If, further, $\operatorname{dep}_A(P_j) \geq s + \operatorname{dep}_A(N)$ *for each j, then* $\operatorname{dep}_A(M) = \operatorname{dep}_A(N) + s$.

Proof. There is nothing to be proved if Hypothesis 5.1 fails for A. So we assume that Hypothesis 5.1 holds for A for the rest of the proof. By induction on s, it suffices to show the assertion in the case of $s = 1$. For any $S \in \operatorname{mod}_0 A$, letting $P = P_0$ and applying $\operatorname{Hom}_A(S, -)$ to the short exact sequence

$$0 \to M \to P \to N \to 0,$$

we obtain a long exact sequence

$$\cdots \to \operatorname{Ext}^{i-1}_A(S, P) \to \operatorname{Ext}^{i-1}_A(S, N) \to \operatorname{Ext}^i_A(S, M) \to \operatorname{Ext}^i_A(S, P) \\ \to \operatorname{Ext}^i_A(S, N) \to \operatorname{Ext}^{i+1}_A(S, M) \to \cdots .$$

Since $\operatorname{Ext}^i_A(S, P) = 0$ for all $i < \operatorname{dep}_A P$, we get $\operatorname{Ext}^i_A(S, M) \cong \operatorname{Ext}^{i-1}_A(S, N)$ for all $i < \operatorname{dep}_A P$. The latter is equal to 0 for all $i \leq \operatorname{dep}_A N$. In other words, for every $i < \min\{\operatorname{dep}_A(N) + 1, \operatorname{dep}_A(P)\}$, we have $\operatorname{Ext}^i_A(S, M) = 0$, namely,

$$\operatorname{dep}_A(M) \geq \min\{\operatorname{dep}_A(N) + 1, \operatorname{dep}_A(P)\}.$$

This assertion also follows from Lemma 5.3(2). If $i = \operatorname{dep}_A(N) + 1 \le \operatorname{dep}_A(P)$, one has

$$0 \neq \operatorname{Ext}_A^{i-1}(S, N) \subseteq \operatorname{Ext}_A^i(S, M),$$

which implies that $\operatorname{dep}_A(M) = \operatorname{dep}_A(N) + 1$, as desired. □

Lemma 5.6. *Let A and B be algebras. Suppose that M is a finitely generated right B-module and N is an (A, B)-bimodule that is finitely generated on both sides. Then*

$$\operatorname{dep}_{A^{\mathrm{op}}}(\operatorname{Hom}_B(M, N)) \ge \min\{2, \operatorname{dep}_{A^{\mathrm{op}}}(N)\}.$$

Proof. Consider a projective resolution of the right B-module M

$$\cdots \to P_1 \to P_0 \to M \to 0,$$

where P_i is finitely generated for $i = 0, 1$. By applying $\operatorname{Hom}_B(-, N)$ to the exact sequence above, one has short exact sequences

$$0 \to \operatorname{Hom}_B(M, N) \to \operatorname{Hom}_B(P_0, N) \to C_1 \to 0, \tag{E5.6.1}$$

and

$$0 \to C_1 \to \operatorname{Hom}_B(P_1, N) \to C_2 \to 0, \tag{E5.6.2}$$

for some left A-modules C_1 and C_2. Since P_i is projective over B, $\operatorname{Hom}_B(P_i, N)$ has (left) depth at least equal to $\operatorname{dep}_{A^{\mathrm{op}}}(N)$ for $i = 0, 1$. Without loss of generality, we assume that $\operatorname{dep}_{A^{\mathrm{op}}}(N) \ge 1$. So we consider two different cases.

If $\operatorname{dep}_{A^{\mathrm{op}}}(N) = 1$, then $\operatorname{dep}_{A^{\mathrm{op}}}(\operatorname{Hom}_B(P_0, N)) \ge 1$. By (E5.6.1) and Lemma 5.5, we have $\operatorname{dep}_{A^{\mathrm{op}}}(\operatorname{Hom}_B(M, N)) \ge 1$, as desired. If $\operatorname{dep}_{A^{\mathrm{op}}}(N) \ge 2$, then $\operatorname{dep}_{A^{\mathrm{op}}}(\operatorname{Hom}_B(P_i, N)) \ge 2$ for $i = 0, 1$. Applying Lemma 5.5 to (E5.6.2) and (E5.6.1) respectively, we have $\operatorname{dep}_{A^{\mathrm{op}}}(C_1) \ge 1$ and $\operatorname{dep}_{A^{\mathrm{op}}}(\operatorname{Hom}_{B^{\mathrm{op}}}(M, N)) \ge 2$. This finishes the proof. □

Remark 5.7. The above lemma holds true for a finitely generated left B-module M and a (B, A)-bimodule N which is finitely generated on both sides, namely,

$$\operatorname{dep}_A(\operatorname{Hom}_{B^{\mathrm{op}}}(M, N)) \ge \min\{2, \operatorname{dep}_A(N)\}.$$

Corollary 5.8. *Let A be an algebra.*

1. *If* $\operatorname{dep}_{A^{\mathrm{op}}} A \ge 2$ *and* $M \in \operatorname{ref} A^{\mathrm{op}}$, *then* $\operatorname{dep}_{A^{\mathrm{op}}} M \ge 2$.
2. *If* $\operatorname{dep}_A A \ge 2$ *and* $M \in \operatorname{ref} A$, *then* $\operatorname{dep}_A M \ge 2$.

Proof. We just need to show part (1). By Lemma 5.6,

$$\begin{aligned}\operatorname{dep}_{A^{\mathrm{op}}} M &= \operatorname{dep}_{A^{\mathrm{op}}} \operatorname{Hom}_A(\operatorname{Hom}_{A^{\mathrm{op}}}(M,A),A)\\ &\geq \min\{2, \operatorname{dep}_{A^{\mathrm{op}}} A\} = 2,\end{aligned}$$

as desired. □

The following lemma is the noncommutative version of [21, Lemma 8.5].

Lemma 5.9. *Let t be a nonnegative integer and let*

$$0 \to X_t \xrightarrow{f_t} X_{t-1} \xrightarrow{f_{t-1}} \cdots \to X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 \to 0$$

be an exact sequence of finitely generated A-modules with $X_0 \in \operatorname{mod}_0 A$. If, for every $i > 0$, $\operatorname{dep}_A X_i \geq i$, then $X_0 = 0$.

Proof. The assertion is automatic if Hypothesis 5.1 fails for A. So for the rest of the proof, we assume that Hypothesis 5.1 holds for A.

Let Y_i denote $\operatorname{Im} f_i \subseteq X_{i-1}$ for $1 \leq i \leq t$. Inductively, we will show $\operatorname{dep}_A Y_i \geq i$ for all i. This is clearly true for $i = t$. Now we assume that $\operatorname{dep}_A Y_{i+1} \geq i+1$ for some i and would like to show that $\operatorname{dep}_A Y_i \geq i$. Consider the exact sequence

$$0 \longrightarrow Y_{i+1} \longrightarrow X_i \longrightarrow Y_i \longrightarrow 0$$

with the hypothesis $\operatorname{dep}_A Y_{i+1} \geq i+1$ and $\operatorname{dep} X_i \geq i$. By Lemma 5.3(3), $\operatorname{dep}_A Y_i \geq i$. This finishes the inductive step and therefore $\operatorname{dep}_A Y_i \geq i$ for all $1 \leq i \leq t$. In particular, $\operatorname{dep}_A X_0 = \operatorname{dep}_A Y_1 \geq 1$. Since $X_0 = Y_1 \in \operatorname{mod}_0 A$, the only possibility is $X_0 = 0$. □

Proposition 5.10. *Let A be an algebra and d be a positive integer. Suppose that*

$$X := 0 \to X^0 \xrightarrow{f^0} X^1 \xrightarrow{f^1} \cdots \to X^d \xrightarrow{f^d} X^{d+1} \to \cdots$$

is a complex in $\operatorname{mod} A$ satisfying the following:

(1) $\operatorname{dep} X^i \geq d - i$ *for all* $i \geq 0$;
(2) $\mathrm{H}^i = 0$ *for all* $i \geq d$, *where* H^i *denotes the* i*-th cohomology of the above complex* X;
(3) $\mathrm{H}^i \in \operatorname{mod}_0 A$ *for all* $i \geq 0$.

Then the complex X is exact.

Proof. The assertion is automatic if Hypothesis 5.1 fails for A. So for the rest of the proof, we assume that Hypothesis 5.1 holds for A.

For $i = 0$, we have an exact sequence

$$0 \to \mathrm{H}^0 \to X^0 \to X^0/\mathrm{H}^0 \to 0.$$

By Lemma 5.3(2). $\operatorname{dep} \mathrm{H}^0 \geq \min\{\operatorname{dep} X^0, \operatorname{dep}(X^0/\mathrm{H}^0) + 1\} \geq 1$. Since $\mathrm{H}^0 \in \operatorname{mod}_0 A$, we have $\mathrm{H}^0 = 0$.

Now we fix an integer $1 \leq j < d$ and assume that $\mathrm{H}^s = 0$ for all $0 \leq s \leq j-1$. Then there are two exact sequences:

$$0 \to X^0 \to \cdots \to X^j \to \operatorname{coker} f^{j-1} \to 0$$

and

$$0 \to \mathrm{H}^j \to \operatorname{coker} f^{j-1} \to X^{j+1}.$$

By using Lemma 5.3(3) repeatedly, we obtain that $\operatorname{dep}(\operatorname{coker} f^{j-1}) \geq d - j > 0$. Since $\mathrm{H}^j \in \operatorname{mod}_0 A$, the second exact sequence forces $\mathrm{H}^j = 0$. By induction, we have $\mathrm{H}^i = 0$ for all $i = 0, \cdots, d-1$ as required. $\square$

6. NQRs in dimension three

Part (2) of the main theorem concerns derived equivalences of two algebras. This can be achieved by constructing a tilting complex between them. Let Λ be an algebra. Recall that $T \in K^b(\operatorname{proj} \Lambda)$ is a *tilting complex* [31, Definition 6.5] if $\operatorname{Hom}_{D(\operatorname{Mod} \Lambda)}(T, T[i]) = 0$ for any $i \neq 0$ and the category $\operatorname{add}(T)$ generates $K^b(\operatorname{proj} \Lambda)$ as triangulated categories. Let Ω be another algebra. If there exists a tilting complex $T \in \mathrm{Kdim}^b(\operatorname{proj} \Lambda)$ such that $\Omega \cong \operatorname{End}_{D(\operatorname{Mod} \Lambda)}(T)$, then we call Λ and Ω *derived equivalent*. Rickard proved that there are other three equivalent conditions to characterize derived equivalent [31, Theorem 6.4], also see [38, Section 14.5]. If a Λ-module T is a tilting complex, then it is called a *tilting module*. Here we only need to use tilting modules, so we first recall the detailed definition of a tilting module.

Definition 6.1. [19] Let Λ be a ring. Then $T \in \operatorname{mod} \Lambda$ is called a *tilting module* if the following conditions are satisfied:

(a) $\operatorname{projdim}_\Lambda T < \infty$;
(b) $\operatorname{Ext}^i_\Lambda(T, T) = 0$ for all $i > 0$;
(c) there is an exact sequence

$$0 \longrightarrow \Lambda \longrightarrow T_0 \longrightarrow T_1 \longrightarrow \cdots \longrightarrow T_{t-1} \longrightarrow T_t \longrightarrow 0$$

with each $T_i \in \operatorname{add} T$.

Let Λ and Ω be two algebras. If there is a tilting Λ-module T such that $\Omega \cong \mathrm{End}_\Lambda(T)$, then Λ and Ω are derived equivalent, namely, there is a triangulated equivalence between $D^b(\mathrm{mod}\,\Lambda)$ and $D^b(\mathrm{mod}\,\Omega)$ [31, Theorem 6.4].

Theorem 6.2. *Let B_i be Auslander-regular and ∂-CM algebras for $i = 1, 2$. Suppose $\partial(B_i) = d \geq 3$. If there exists a (B_1, B_2)-bimodule U satisfying the following conditions:*

(1) $U \in \mathrm{ref}\, B_2$;
(2) $\mathrm{projdim}_{B_2} U \leq 1$;
(3) $\mathrm{Ext}^1_{B_2}(U, U) \in \mathrm{mod}_0\, B_1^{\mathrm{op}}$;
(4) $B_1 \cong \mathrm{End}_{B_2}(U)$;
(5) *When switching B_1 and B_2, the above conditions still hold,*

then U is a tilting B_2-module and further, B_1 and B_2 are derived equivalent.

Proof. It suffices to show that U is a tilting B_2-module as given in Definition 6.1. Below we check (a, b, c) in Definition 6.1.

(a) By hypothesis (2), $\mathrm{projdim}_{B_2} U \leq 1$, hence Definition 6.1(a) holds.

(b) By hypothesis (2), we need to prove that $\mathrm{Ext}^1_{B_2}(U, U) = 0$. If $\mathrm{mod}_0\, B_1^{\mathrm{op}}$ contains only the zero module, then hypothesis (3) implies that $\mathrm{Ext}^1_{B_2}(U, U) = 0$. Otherwise, Hypothesis 5.1 holds for left B_1-modules, which we assume for the rest of the proof.

Consider the exact sequence $0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow U \longrightarrow 0$ of B_2-modules where P_i are projective over B_2. Applying $\mathrm{Hom}_{B_2}(-, U)$, we obtain an exact sequence of B_1^{op}-modules

$$0 \to \mathrm{Hom}_{B_2}(U, U) \to \mathrm{Hom}_{B_2}(P_0, U) \to \mathrm{Hom}_{B_2}(P_2, U) \to \mathrm{Ext}^1_{B_2}(U, U) \to 0.$$

Since U is a reflexive B_1^{op}-module and $\mathrm{dep}_{B_1^{\mathrm{op}}} B_1 = d \geq 2$ (Proposition 5.4), by Lemma 5.6 and Corollary 5.8, we have

$$\mathrm{dep}_{B_1^{\mathrm{op}}}(\mathrm{Hom}_{B_2}(P_0, U)) \geq \min\{2, \mathrm{dep}_{B_1^{\mathrm{op}}} U\} = 2,$$

and

$$\mathrm{dep}_{B_1^{\mathrm{op}}}(\mathrm{Hom}_{B_2}(P_1, U)) \geq \min\{2, \mathrm{dep}_{B_1^{\mathrm{op}}} U\} = 2 \geq 1.$$

Moreover, $\mathrm{dep}_{B_1^{\mathrm{op}}}(\mathrm{Hom}_{B_2}(U, U)) = \mathrm{dep}_{B_1^{\mathrm{op}}}(B_1) = d \geq 3$, then by hypothesis (3) and Lemma 5.9, $\mathrm{Ext}^1_{B_2}(U, U) = 0$.

(c) By hypothesis (5), we have that $B_2 \cong \mathrm{End}_{B_1^{\mathrm{op}}}(U, U)$ and $\mathrm{projdim}_{B_1^{\mathrm{op}}} U \leq 1$. By the same proof as in (b), we have $\mathrm{Ext}^1_{B_1^{\mathrm{op}}}(U, U) = 0$. Let $0 \to Q_1 \to Q_0 \to U \to 0$ be a projective resolution of the B_1^{op}-module U. Applying $\mathrm{Hom}_{B_1^{\mathrm{op}}}(-, U)$ and using the fact that $\mathrm{Ext}^1_{B_1^{\mathrm{op}}}(U, U) = 0$, we obtain an exact sequence

$$0 \to B_2 \to \mathrm{Hom}_{B_1^{\mathrm{op}}}(Q_0, U) \to \mathrm{Hom}_{B_1^{\mathrm{op}}}(Q_1, U) \to 0$$

of B_2-modules, and clearly, $\mathrm{Hom}_{B_1^{\mathrm{op}}}(Q_i, U) \in \mathrm{add}_{B_2} U$ for $i = 0, 1$. Thus we proved condition (c) of a tilting module.

Thus, U is a tilting B_2-module, and consequently, B_1, B_2 are derived equivalent. □

Lemma 6.3. *Let B_1 and B_2 be algebras such that B_2 is (Auslander) Gorenstein. Suppose that ∂ satisfies $\gamma_{0,0}(B_2, B_1)^r$. Let M be a right B_2-module with* $\mathrm{projdim}\, M \leq 1$ *and U be a (B_1, B_2)-bimodule such that* $\mathrm{projdim}\, U_{B_2} < \infty$*. If $\mathrm{Ext}^1_{B_2}(M, B_2) \in \mathrm{mod}_0\, B_2^{\mathrm{op}}$, then $\mathrm{Ext}^1_{B_2}(M, U) \in \mathrm{mod}_0\, B_1^{\mathrm{op}}$.*

Proof. By [26, Section 5 (b.1)], there is an Ischebeck spectral sequence

$$\mathrm{Tor}_p^{B_2}(U, \mathrm{Ext}^q_{B_2}(M, B_2)) \Rightarrow \mathrm{Ext}^{q-p}_{B_2}(M, U).$$

Since $\mathrm{projdim}\, M \leq 1$, the E_2-page of this spectral sequence has only two nonzero columns. Therefore

$$\mathrm{Tor}_0^{B_2}(U, \mathrm{Ext}^1_{B_2}(M, B_2)) \cong \mathrm{Ext}^1_{B_2}(M, U).$$

Note that

$$\mathrm{Tor}_0^{B_2}(U, \mathrm{Ext}^1_{B_2}(M, B_2)) = U \otimes_{B_2} \mathrm{Ext}^1_{B_2}(M, B_2) \in \mathrm{mod}_0\, {B_1}^{\mathrm{op}}$$

by $\gamma_{0,0}(B_2, B_1)^r$ condition. Therefore, $\mathrm{Ext}^1_{B_2}(M, U) \in \mathrm{mod}_0\, B_1^{\mathrm{op}}$. □

Remark 6.4. If $\partial = \mathrm{GKdim}$ and B is affine over $\Bbbk$, then $M \in \mathrm{mod}_0\, B$ is equivalent to M being finite dimensional over $\Bbbk$. In this case, ∂ automatically satisfies $\gamma_{0,i}$ for all i.

Hypothesis 6.5. We assume

(1) Hypothesis 3.8 holds.
(2) $\gamma_{0,0}(A, B)$ for all $A, B \in \mathcal{A}$.

Next we prove a version of Theorem 0.6(2).

Theorem 6.6. *Assume Hypothesis 6.5. Let $A \in \mathcal{A}$ be an algebra with $\partial(A) = 3$. Suppose that $(B_i, {}_{B_i}(M_i)_A, {}_A(N_i)_{B_i})$ are two NQRs of A for $i = 1, 2$. Then B_1 and B_2 are derived equivalent.*

Proof. We need to verify the hypotheses in Theorem 6.2.

By Proposition 3.9, Theorem 3.15 and Corollary 2.13, there exists a bimodule ${}_{B_1}U_{B_2}$ which is reflexive on both sides such that $B_1 \cong \mathrm{End}_{B_2}(U)$ and $\mathrm{projdim}_{B_2} U \leq 1$. Hence hypotheses (1, 2, 4) in Theorem 6.2 hold. To show hypothesis (3) in Theorem 6.2, we follow Proposition 2.18. There are two cases that should be considered:

Case 1: $E_2^{31} = \mathrm{Ext}^3_{B_2^{\mathrm{op}}}(\mathrm{Ext}^1_{B_2}(U, B_2), B_2) = 0$. By Proposition 2.18,

$$\mathrm{Ext}^i_{B_2^{\mathrm{op}}}(\mathrm{Ext}^j_{B_2}(U, B_2), B_2) = 0$$

for all (i, j) except for $(i, j) = (0, 0)$. This implies that $U \in \mathrm{proj}\, B_2$. Then Theorem 6.2(3) holds trivially.

Case 2: $E_2^{31} \neq 0$. Then $\mathrm{Ext}^1_{B_2}(U, B_2) \neq 0$, and by Proposition 2.18,

$$j_{B_2^{\mathrm{op}}}(\mathrm{Ext}^1_{B_2}(U, B_2)) = 3.$$

Since B_2 is ∂-CM, we have

$$\partial_{B_2^{\mathrm{op}}}(\mathrm{Ext}^1_{B_2}(U, B_2)) = \partial_{B_2^{\mathrm{op}}}(B_2) - j_{B_2^{\mathrm{op}}}(\mathrm{Ext}^1_{B_2}(U, B_2)) = 3 - 3 = 0,$$

namely, $\mathrm{Ext}^1_{B_2}(U, B_2) \in \mathrm{mod}_0\, B_2^{\mathrm{op}}$. By Lemma 6.3, $\mathrm{Ext}^1_{B_2}(U, U) \in \mathrm{mod}_0\, B_1^{\mathrm{op}}$, which is Theorem 6.2(3).

Up to this point, we have proved conditions (1, 2, 3, 4) in Theorem 6.2. By symmetry, Theorem 6.2(5) holds. Therefore, by Theorem 6.2, B_1 and B_2 are derived equivalent. □

7. Connections between NQRs and NCCRs

In this section we show that Van den Bergh's noncommutative crepant resolutions (NCCRs) are in fact equivalent to noncommutative quasi-resolutions (NQRs) in the commutative or central-finite case. We use the definition given in [21, Section 8] which is slightly more general than original definition, see Definition 0.2.

Let R be a noetherian commutative domain with finite Krull dimension. Let $\mathcal{A}_{R,\mathrm{Kdim}}$ be the category of algebras that are module-finite R-algebras with ∂ being the Krull dimension (Kdim). As explained in Example 3.1(3), we need to specify modules too. As usual, one-sided modules are just usual modules, but bimodules are assumed to be R-central.

Lemma 7.1. *Retain the notation as above. Let $A, B \in \mathcal{A}_{R,\mathrm{Kdim}}$. Then Hypothesis 1.3 holds.*

Proof. By [6, Lemma 1.3], $\partial := \mathrm{Kdim}$ is exact and symmetric. Hypothesis 1.3(1) and (2) are clear. It remains to show (3). By definition all bimodules are central over R. If ${}_AM_B$ is finitely generated over B, then it is finitely generated over R as every algebra is module-finite over R. Then M is finitely generated over A. This implies that Hypothesis 1.3(3) is equivalent to the fact that ∂ is symmetric. □

We recall a definition from [6, Definition 1.1(5)]. Let A and B be two algebras. We say ∂ is $(A, B)_i$-*torsitive* if, for every (A, B)-bimodule M finitely generated on both sides and every finitely generated right A-module N, one has

$$\partial(\operatorname{Tor}_j^A(N, M)_B) \leq \partial(N_A)$$

for all $j \leq i$. Part (1) of the following lemma was proven in [6].

Lemma 7.2. *Let A and B be two algebras in $\mathcal{A}_{R,\mathrm{Kdim}}$.*

(1) *[6, Lemma 3.1] ∂ is $(A, B)_\infty$-torsitive.*
(2) *$\gamma_{k_1,k_2}(A, B)$ hold for all k_1, k_2 (see Definition 1.8).*
(3) *Hypothesis 3.8 holds.*
(4) *Hypothesis 6.5 holds.*

Proof. (2) This follows from part (1) and the definition.

(3) This follows from Lemma 7.1 and a special case of part (2).

(4) This follows from part (3) and another special case of part (2). □

For the purpose of this paper, we only need $\gamma_{0,0}(A, B)$ and $\gamma_{1,1}(A, B)$. But it is good to know that $\gamma_{k_1,k_2}(A, B)$ hold for all k_1, k_2. For the rest of this section, CM stands for "Cohen-Macaulay" in the classical sense in commutative algebra, while Kdim-CM is defined in Definition 2.3 by taking the dimension function ∂ to be the Krull dimension Kdim. By [12, p. 1435], when R is commutative and noetherian, then R is Kdim-CM if and only if R is CM and equi-codimensional. The following lemma is known.

Lemma 7.3. *Let R be a commutative d-dimensional* CM *equi-codimensional normal domain. Let A be a module-finite R-algebra and $K \in \operatorname{mod} A$. Let s be an integer between 0 and $d-2$. Then $K \in \operatorname{mod}_{d-2-s} A$ if and only if $K_{\mathfrak{p}} = 0$ for every prime ideal $\mathfrak{p}$ of R with* $\operatorname{ht}(\mathfrak{p}) \leq 1 + s$.

Proof. Since $\operatorname{Kdim} M_R = \operatorname{Kdim} M_A$, it suffices to consider the case $A = R$. By [27, Lemma 6.2.11], we can always assume that K is a critical R-module such that $\mathfrak{q} := \operatorname{Ann}_R(K) = \{x \in R | xK = 0\}$ is a prime ideal of R. In this case, K is an essential $R/\mathfrak{q}$-module.

Suppose that $K_{\mathfrak{p}} = 0$ for all prime ideals $\mathfrak{p}$ with $\operatorname{ht}(\mathfrak{p}) \leq 1+s$. If $\operatorname{Kdim}(K) \geq d-1-s$, by [12, Theorem 3.1(iv)], we have

$$\operatorname{ht}(\mathfrak{q}) = \operatorname{Kdim} R - \operatorname{Kdim} R/\mathfrak{q} = \operatorname{Kdim} R - \operatorname{Kdim} K \leq 1 + s.$$

By the definition of $\mathfrak{q}$, $K_{\mathfrak{q}} \neq 0$, which is a contradiction. Therefore $\operatorname{Kdim}(K) \leq d-2-s$, as desired.

Conversely, suppose that $\operatorname{Kdim}(K) \le d-2-s$. Then $\operatorname{Kdim} R/\mathfrak{q} \le d-2-s$. By [12, Theorem 3.1(iv)], we have $\operatorname{ht}(\mathfrak{q}) \ge 2+s$. Therefore $K_\mathfrak{p} = 0$ for all prime ideal $\mathfrak{p}$ with $\operatorname{ht}(\mathfrak{p}) \le 1+s$. □

Remark 7.4. In the papers [35,21], the commutative base ring R is a normal Gorenstein domain, which is automatically CM equi-codimensional and normal. In [22, Theorem 1.5], it is assumed that R is a CM equi-codimensional normal domain. Hence the first hypothesis of Lemma 7.3 holds.

Proposition 7.5. *Let R be a commutative noetherian* CM *equi-codimensional normal domain of dimension d. Let A be a module-finite R-algebra that is a maximal* CM *R-module. If M gives rise to a NCCR of A in the sense of Definition 0.2(2), then $(\Omega, {}_\Omega M_A, {}_A(M^\vee)_\Omega)$ is a NQR of A. In other words,*

(1) *Ω is an Auslander regular* Kdim-CM *algebra with* $\operatorname{gldim}\Omega = \operatorname{Kdim}\Omega = d$.
(2) *$M \otimes_A M^\vee \cong_{d-2} \Omega$ and $M^\vee \otimes_\Omega M \cong_{d-2} A$.*

Proof. (1) By the assumption, R is equi-codimensional, Ω is a module-finite R-algebra, and Ω is a maximal CM R-module, so, by [12, Lemma 2.8(2) and Theorem 4.8], Ω is a Kdim-CM algebra with $\operatorname{Kdim}(\Omega) = \operatorname{Kdim}(R) = d$. Moreover, Ω being a nonsingular R-order means that it is a homologically homogeneous noetherian PI ring. Then, by [33, Theorem 1.4(1)], Ω is an Auslander regular algebra with $\operatorname{gldim}\Omega = d$. The assertion follows.

(2) Let $\varphi : M^\vee \otimes_\Omega M \to A$ be the natural evaluation map. Then there is an exact sequence

$$0 \to K \to M^\vee \otimes_\Omega M \xrightarrow{\varphi} A \to C \to 0$$

with $K, C \in \operatorname{mod} A$. By definition of a NCCR, $M \in \operatorname{ref} A$ is a height one progenerator of A, we have $M_\mathfrak{p}^\vee \otimes_{\Omega_\mathfrak{p}} M_\mathfrak{p} \cong A_\mathfrak{p}$ for $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\operatorname{ht}(\mathfrak{p}) \le 1$. Therefore, $K_\mathfrak{p} = 0 = C_\mathfrak{p}$. By Lemma 7.3, $K, C \in \operatorname{mod}_{d-2} R$. Combining with the assumption that A is a module-finite R-algebra, $K, C \in \operatorname{mod}_{d-2} A$, namely, $M^\vee \otimes_\Omega M \cong_{d-2} A$.

Similarly, there is a natural map

$$\alpha : M \otimes_A M^\vee \to \operatorname{Hom}_A(M, M) =: \Omega$$

such that $\alpha(n \otimes f)(m) = nf(m)$ for all $f \in M^\vee$ and all $n, m \in M$. One can use the above argument to show that $M \otimes_A M^\vee \cong_{d-2} \Omega$, whence (2) follows. □

Conversely, a NQR is also a NCCR for Gorenstein singularities.

Proposition 7.6. *Let R be a commutative d-dimensional* CM *equi-codimensional normal domain. Let A be a module-finite R-algebra that is a maximal* CM *R-module. Suppose that*

A is Auslander-Gorenstein and Kdim-CM*, then a NQR of A, say (B, M, N), provides a NCCR B of A in the sense of Definition 0.2.*

Proof. Let (B, M, N) be a NQR of A. Then B is an Auslander regular Kdim-CM algebra of Krull dimension d, and

$$M \otimes_A N \cong_{d-2} B, \quad N \otimes_B M \cong_{d-2} A.$$

By Proposition 3.9, (M, N) can be replaced by (U, V) such that (B, U, V) is also a NQR of A and that U and V are reflexive on both sides. By Lemmas 3.13 and 7.2, $B \cong \mathrm{End}_A(U)$. Since B is Auslander regular and Kdim-CM, it is easy to check that B is a non-singular order. It remains to show that U is a height one progenerator. By Proposition 3.9, we have

$$U \otimes_A V \cong_{d-2} B \quad \text{and} \quad V \otimes_B U \cong_{d-2} A.$$

It follows from Lemma 7.3 that

$$U_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} V_{\mathfrak{p}} \cong B_{\mathfrak{p}} \quad \text{and} \quad V_{\mathfrak{p}} \otimes_{B_{\mathfrak{p}}} U_{\mathfrak{p}} \cong A_{\mathfrak{p}}$$

for every prime ideal $\mathfrak{p}$ of R with $\mathrm{ht}(\mathfrak{p}) \leq 1$. Hence U is a height one progenerator of A. Therefore U gives a NCCR of A. $\square$

By the above two propositions, NCCRs are essentially equivalent to NQRs when A is Auslander-Gorenstein. Therefore Theorem 0.4(2b) is essentially equivalent to Theorem 6.6 in this setting. In the next section, we will introduce more examples of NQRs in the noncommutative setting. One advantage of NQRs is that they can be defined for many algebras that are not Gorenstein (not even CM). In this case we do not require M or N to be reflexive. Here is an easy example.

Example 7.7. Let B be the commutative polynomial ring $\Bbbk[x_1, x_2, x_3]$ with the standard grading and let A be the subring of B generated by $B_{\geq 2}$. Then A is noetherian of Krull dimension three and the Hilbert series of A is

$$H_A(t) = \frac{1}{(1-t)^3} - 3t.$$

It is easy to see that A is not normal and that $H_A(t)$ can not be written as

$$\frac{1 + a_1 t + a_2 t^2 + \cdots + a_d t^d}{(1 - t^{n_1})(1 - t^{n_2})(1 - t^{n_3})}$$

for any nonnegative integers a_i, n_j. By [17, Ex. 21.17(b), p. 551], A is not CM. Then a NCCR of A is not defined. But A has a NQR as shown below.

Let $M = B$ as a (B, A)-bimodule and $N = B$ as an (A, B)-bimodule. It is easy to see that

$$M \otimes_A N \cong_0 B, \quad \text{and} \quad N \otimes_B M \cong_0 A.$$

Consequently, (B, M, N) is a NQR of A. As a consequence of Proposition 7.9 below, any two NQRs of A are Morita equivalent.

In addition, it is easy to check that A has an isolated singularity at the unique maximal graded ideal.

Though NQRs are weaker than NCCRs, not every algebra admits a NQR.

Example 7.8. Let A be an affine commutative Gorenstein algebra that does not admit a NCCR. Then A does not admit a NQR by Proposition 7.6. For example, A is an affine Gorenstein algebra of dimension two with non-isolated singularities, then A does not admit either a NCCR or a NQR. By [16, Theorem 1.2(2) and Example 3.5], there are isolated hypersurface singularities of (any) even dimension ≥ 4 that do not admit either a NCCR or a NQR.

Proposition 7.9. *Let A be a module-finite R-algebra with $\partial(A) = 3$. Suppose*

(a) $(B_i, {}_{B_i}(M_i)_A, {}_A(N_i)_{B_i})$ are two NQRs of A for $i = 1, 2$, and
(b) B_1 or B_2 is Azumaya.

Then B_1 and B_2 are Morita equivalent.

Proof. The hypotheses in Theorem 6.6 are automatic by Lemma 7.2. Hence B_1 and B_2 are derived equivalent. Since B_1 (or B_2) is Azumaya, by [39, Proposition 5.1], B_1 and B_2 are Morita equivalent. $\square$

8. Examples of NQRs of noncommutative algebras and comments

In this section we give some examples of NQRs of noncommutative algebras. At the end of the section we also give some comments. It turns out that, except for Example 8.6, all examples in this section have the same kind of construction, namely, by noncommutative McKay correspondence. Precisely, fixed subrings R^H, considered as noncommutative quotient singularities, have NQRs of the form $R\#H$, where R and H will be explained in details. However, by taking different R and H, we obtain many different examples.

8.1. Graded case

Let $\mathcal{A}_{gr,\mathrm{GKdim}}$ be the category of locally finite $\mathbb{N}$-graded noetherian algebras with finite Gelfand-Kirillov dimension and let $\partial = \mathrm{GKdim}$. Modules are usual $\mathbb{Z}$-graded A-modules.

In this setting, (A, B)-bimodules are assumed to be $\mathbb{Z}$-graded (A, B)-bimodules, namely, having both a left graded A-module and a right graded B-module structure with the same grading. See Example 3.1(1).

Remark 8.1. Many of basic results in ring theory and module theory have been generalized to the graded setting in the literature. For example, the graded version of some basic results in ring theory can be found in the book [28]. Using the graded version of these results one can carefully adapt the arguments to reprove all statements in Sections 1-7 in the graded setting. To save space we will use the graded version of results in Sections 1-7 without proofs.

Lemma 8.2. *Retain the notation as above concerning the category $\mathcal{A}_{gr,\mathrm{GKdim}}$.*

(1) *Hypothesis 1.3 holds.*
(2) *Let A and B be two algebras in $\mathcal{A}_{gr,\mathrm{GKdim}}$. Then ∂ is $(A, B)_\infty$-torsitive. As a consequence, $\gamma_{k_1,k_2}(A, B)$ hold for all k_1, k_2.*
(3) *Hypothesis 6.5 holds.*

Proof. (1) By [6, Lemma 1.2(1)], ∂ is exact. By [6, Lemma 1.2(4)], it is symmetric. Hypothesis 1.3(2) is [25, Lemma 5.3(b)].

(2) This is [6, Lemma 1.2(6)].

(3) This follows from parts (1) and (2) and the definition. □

From now on until Example 8.7, we are working with the category $\mathcal{A}_{gr,\mathrm{GKdim}}$.

Proof of Theorem 0.6. By Lemma 8.2(3), Hypothesis 6.5 holds.

(1) This is a (graded) consequence of Theorem 4.2.

(2) This is a consequence of Theorem 6.6 and Lemma 8.2(3). □

Proposition 8.3. *Suppose that A and B are two noetherian locally finite $\mathbb{N}$-graded algebras that satisfy the following*

(*a*) *B is an Auslander regular* CM *algebra with* $\mathrm{GKdim}(B) := d \geq 2$.
(*b*) *Let e be an idempotent in B (or in B_0) and $A = eBe$.*
(*c*) *$N := eB$ is an (A, B)-bimodule which is finitely generated on both sides.*
(*d*) *$M := Be$ is a (B, A)-bimodule which is finitely generated on both sides.*
(*e*) $\mathrm{GKdim}(B/BeB) \leq d - 2$.

Then (B, M, N) is a NQR of A.

Proof. Below is a proof in the ungraded setting which can easily adapted to the graded case. Since $\partial = \mathrm{GKdim}$, by [6, Lemma 2.2(ii)], $\partial((Ne)_A) \leq \partial(N_B)$ for every finitely generated right B-module N, which is precisely [6, Hypothesis 2.1(7)]. It is easy to verify [5, Hypothesis 2.1(1-6)]. By Lemma 8.2(2), ∂ satisfies $\gamma_{d-2,1}(eB)$, which is precisely [5, (E2.3.1)]. Therefore we can apply the proof of [5, Lemma 2.3]. By the proof of [5, Lemma 2.3], the hypothesis $\mathrm{GKdim}(B/BeB) \leq d-2$ implies that

$$M \otimes_A N \cong_{d-2} B.$$

On the other hand, it is clear that

$$N \otimes_B M = eBe = A.$$

Therefore (B, M, N) is a NQR of A. $\square$

Remark 8.4.

(1) By the above proof, Proposition 8.3 holds in the ungraded case as long as condition [5, (E2.3.1)] holds. Note that [5, (E2.3.1)] is a consequence of ∂ being $(A, B)_\infty$-torsitive for those algebras A and B in the category $\mathcal{A}$.

(2) Similar to the proof of Proposition 8.3, if we replace condition (e) by

$$\mathrm{GKdim}(B/BeB) \leq d - 2 - s,$$

then (B, M, N) is an s-NQR of A in the sense of Definition 3.2(1).

(3) If $A =: R$ is a commutative normal Gorenstein domain, then the NQRs in Proposition 8.3 might not be in the category of $\mathcal{A}_{R,\mathrm{Kdim}}$ (Section 7) as we are not required that B is R-central. Keep this in mind, it is also possible that R has a NCCR in the category $\mathcal{A}_{R,\mathrm{Kdim}}$ and a NQR not in $\mathcal{A}_{R,\mathrm{Kdim}}$ (but in a different category such as $\mathcal{A}_{gr,\mathrm{GKdim}}$).

Explicit examples of NQRs in the graded case are given next.

Example 8.5. Suppose the following hold.

(a) Let R be a noetherian connected graded (locally finite) Auslander regular CM algebra with $\mathrm{GKdim}(R) = d \geq 2$.

(b) Let H be a semisimple Hopf algebra acting on R homogeneously and inner-faithfully with integral $\int$ such that $\varepsilon(\int) = 1$.

(c) Let B be the smash product algebra $R\#H$ with $e := 1\#\int \in B$ and A be the fixed subring R^H.

(d) Suppose $\mathrm{GKdim}(B/BeB) \leq d - 2$.

By Proposition 8.3, $(B, M, N) := (B, Be, eB)$ is a NQR of A. This produces many examples of NQRs in following work. Note that the condition (d) is equivalent to that a version of Auslander's theorem holds, namely, the natural algebra morphism

$$\phi : \quad R\#H \longrightarrow \mathrm{End}_{R^H}(R)$$

is an isomorphism. In [13,14,5,6,15,18], the results state that Auslander's theorem holds instead of condition (d). Auslander's theorem is a fundamental ingredient in the study of the McKay correspondence, see [13,14]. In the following we further assume that char $\Bbbk = 0$.

(1) Let R be an Auslander regular and CM algebra of global dimension two and H act on R with trivial homological determinant. Then $A := R^H$ has a NQR [13, Theorem 0.3].
(2) Let R be a graded noetherian down-up algebra (of global dimension three) which is not $A(\alpha, -1)$ and G be a finite subgroup of $\mathrm{Aut}_{gr}(R)$. Then $A := R^G$ has a NQR [6, Theorem 0.6].
(3) Let R be a graded noetherian down-up algebra (of global dimension three) and G be a finite subgroup coacting on R with trivial homological determinant. Then $A := R^{co\ G}$ has a NQR [15, Theorem 0.1].
(4) Let $R = \Bbbk_{-1}[x_1, \cdots, x_n]$ and $\mathbb{S}_n$ act on R naturally permuting variables x_i. Then $A := R^G$ has a NQR for every nontrivial subgroup $G \subseteq \mathbb{S}_n$ [18, Theorem 2.4]. A special case was proved earlier in [5, Theorem 0.5].

Next we give an example of a NQR that does not fit into the framework of Proposition 8.3.

Example 8.6. Let q be a nonzero scalar in $\Bbbk$ that is not a root of unity. Let B be the algebra $\Bbbk\langle x, y\rangle/(yx - qxy - x^2)$, which is connected graded noetherian Auslander regular and CM of GKdim 2. Let $A := \Bbbk + By$ be the subalgebra of B as given in [32, Notation 2.1]. By [32, Theorem 2.3], A is a noetherian algebra that does not satisfy the condition χ in the sense of [3]. As a consequence, A does not admit a balanced dualizing complex in the sense of Yekutieli [37]. In other words, this algebra does not have nice properties required in noncommutative algebraic projective geometry. By [32, Corollary 2.8],

$$\mathrm{qmod}_0\, A \cong \mathrm{qmod}_0\, B.$$

This indicates that A might have a NQR. Indeed, this is the case as we show next.

Let M be the (graded) (B, A)-bimodule By and N be the (graded) (A, B)-bimodule B. One can verify that M and N are finitely generated on both sides. Note that, as a right A-module, $M \cong_0 A$ since we have an exact sequence $0 \to M \to A \to \Bbbk \to 0$. Hence, following the Hilbert series computations,

$$N \otimes_B M \cong_0 B \otimes_B By \cong By \cong_0 A$$

as A-bimodules, and

$$M \otimes_A N \cong_0 By \otimes_A B \cong_0 ByB \cong_0 B$$

as B-bimodules, where the last $\cong_0$ follows from the fact that ByB is co-finite-dimensional inside B. By definition, B is a NQR of A.

8.2. Ungraded case

All NQRs in the graded case are NQRs in the ungraded setting. Below are other ungraded examples.

Example 8.7. Let $\mathcal{A}_{ungr,\mathrm{GKdim}}$ be the category of affine $\Bbbk$-algebras with finite Gelfand-Kirillov dimension and let $\partial = \mathrm{GKdim}$. Suppose the following hold.

(a) Let R be a noetherian Auslander regular CM algebra with $\mathrm{GKdim}(R) = d \geq 2$.
(b) Let H be a semisimple Hopf algebra acting on R and inner-faithfully with integral $\int$ such that $\varepsilon(\int) = 1$.
(c) Let B be the smash product algebra $R\#H$ with $e := 1\#\int \in B$ and A be the fixed subring R^H.
(d) Suppose $\mathrm{GKdim}(B/BeB) \leq d - 2$.

If [5, (E2.3.1)] holds, by Proposition 8.3, $(B, M, N) := (B, (R\#H)e, e(R\#H))$ is a NQR of A. We have some examples in the ungraded case. Here is the first example. Again assume that char $\Bbbk = 0$. Let R be the universal enveloping algebra $U(\mathfrak{g})$ of a finite dimensional Lie algebra $\mathfrak{g}$. Suppose that $\mathfrak{g} \neq \mathfrak{g}' \ltimes \Bbbk x$ for a 1-dimensional Lie ideal $\Bbbk x \subseteq \mathfrak{g}$ and a Lie subalgebra $\mathfrak{g}' \subset \mathfrak{g}$. Then [6, Corollary 0.5] implies that R^G has a NQR for every finite group $G \subseteq \mathrm{Aut}_{Lie}(\mathfrak{g})$.

Example 8.8. Let $\mathcal{A}_{PI,\mathrm{GKdim}}$ be the category of affine $\Bbbk$-algebras that satisfy a polynomial identity and let $\partial = \mathrm{GKdim}$. In fact, $\mathrm{GKdim} = \mathrm{Kdim}$ in this case. But we do not assume that algebras are central-finite. By [6, Lemma 3.1], ∂ is $(A, B)_\infty$-torsitive for two algebras A and B in $\mathcal{A}_{ungr,PI}$. As a consequence, $\gamma_{k_1,k_2}(A, B)$ hold for all k_1, k_2, see Lemma 7.2. Then in the setting of Example 8.6, R^H has a NQR $B := R\#H$ when $\partial(B/BeB) \leq \partial(B) - 2$, or equivalently, when the Auslander's theorem holds by [6, Theorem 3.3]. Explicit examples of R and H are given in [6, Corollaries 3.4 and 3.7].

8.3. Comments on potential directions of further research

There are some further studies of NQRs in dimension two in [30, Theorem 0.2(1)] where the Gabriel quiver of a NQR is classified. It is natural to ask what we can do in dimension three.

The next question was suggested by the referee. Is it possible to remove the condition that the singular ring A (either in the commutative regime or in the noncommutative regime) is Gorenstein or CM? Instead, we just assume that A has an Auslander dualizing complex. Note that, except for the definition (Definition 3.2), we use the Gorenstein or CM property in a large part of the paper.

In future study of higher dimensional NQRs, dualizing complexes and derived categories [40,38] would play a more important role than the classical methods presented in this paper.

Acknowledgments

The authors thank Xiaowei Xu and Zhibing Zhao for their careful reading of a draft of the paper and useful suggestions and thank the referees for their very careful reading and extremely valuable comments. X.-S. Qin was partially supported by the Foundation of China Scholarship Council (Grant No. [2016]3100). Y.-H. Wang was partially supported by the Foundation of China Scholarship Council (Grant No. [2016]3009), the Foundation of Shanghai Science and Technology Committee (Grant No. 15511107300), the Natural Science Foundation of China (No. 11871071). Y.H. Wang and X.-S. Qin thank the Department of Mathematics, University of Washington for its very supportive hospitality during their visits. J.J. Zhang was partially supported by the US National Science Foundation (Grant Nos. DMS-1402863 and DMS-1700825).

References

[1] K. Ajitabh, S.P. Smith, J.J. Zhang, Auslander-Gorenstein rings, Comm. Algebra 26 (1998) 2159–2180.
[2] K. Ajitabh, S.P. Smith, J.J. Zhang, Injective resolutions of some regular rings, J. Pure Appl. Algebra 140 (1) (1999) 1–21.
[3] M. Artin, J.J. Zhang, Noncommutative projective schemes, Adv. Math. 109 (1994) 228–287.
[4] M. Auslander, Isolated singularities and existence of almost split sequences, in: Representation Theory, II, Ottawa, Ont., 1984, in: Lecture Notes in Math., vol. 1178, Springer, Berlin, 1986, pp. 194–242.
[5] Y.-H. Bao, J.-W. He, J.J. Zhang, Pertinency of Hopf actions and quotient categories of Cohen-Macaulay algebras, J. Noncommut. Geom. 13 (2) (2019) 667–710, https://doi.org/10.4171/JNCG/336.
[6] Y.-H. Bao, J.-W. He, J.J. Zhang, Noncommutative Auslander theorem, Trans. Amer. Math. Soc. 370 (12) (2018) 8613–8638.
[7] J.E. Björk, The Auslander condition on noetherian rings, in: Séminaire Dubreil-Malliavin 1987-88, in: Lecture Notes in Math., vol. 1404, Springer, Berlin, 1989, pp. 137–173.
[8] A. Bondal, D. Orlov, Derived categories of coherent sheaves, in: Proceedings of the International Congress of Mathematicians, Vol. II, Higher Ed. Press, Beijing, 2002, pp. 47–56, 2002, Beijing.
[9] A. Bondal, D. Orlov, Semi-orthogonal decompositions for algebraic varieties, available as alggeom/9506012, 1996.
[10] T. Bridgeland, Flops and derived categories, Invent. Math. 147 (3) (2002) 613–632.
[11] T. Bridgeland, A. King, M. Reid, The McKay correspondence as an equivalence of derived categories, J. Amer. Math. Soc. 14 (3) (2001) 535–554.
[12] K.A. Brown, M. MacLeod, The Cohen Macaulay property for noncommutative rings, Algebr. Represent. Theory 20 (2017) 1433–1465.
[13] K. Chan, E. Kirkman, C. Walton, J.J. Zhang, McKay correspondence for semisimple Hopf actions on regular graded algebras, I, J. Algebra 508 (2018) 512–538.
[14] K. Chan, E. Kirkman, C. Walton, J.J. Zhang, McKay correspondence for semisimple Hopf actions on regular graded algebras, II, J. Noncommut. Geom. 13 (1) (2019) 87–114.
[15] J. Chen, E. Kirkman, J.J. Zhang, Auslander's Theorem for group coactions on noetherian graded down-up algebras, Transform. Groups (2019), to appear, arXiv:1801.09020.

[16] H. Dao, Remarks on non-commutative crepant resolutions of complete intersections, Adv. Math. 224 (3) (2010) 1021–1030.
[17] D. Eisenbud, Commutative Algebra with a View Toward Algebraic Geometry, GTM, vol. 150, Springer-Verlag, New York, 1995.
[18] J. Gaddis, E. Kirkman, W.F. Moore, R. Won, Auslander's Theorem for permutation actions on noncommutative algebras, Proc. Amer. Math. Soc. 147 (5) (2019) 1881–1896.
[19] D. Happel, C.M. Ringel, Tilted algebras, Trans. Amer. Math. Soc. 274 (1982) 339–443.
[20] D. Huybrechts, Fourier-Mukai Transforms in Algebraic Geometry, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, Oxford, 2006.
[21] O. Iyama, I. Reiten, Fomin-Zelevinsky mutation and titling modules over Calabi-Yau algebras, Amer. J. Math. 130 (4) (2008) 1087–1149.
[22] O. Iyama, M. Wemyss, On the noncommutative Bondal-Orlov conjecture, J. Reine Angew. Math. 683 (2013) 119–128.
[23] O. Iyama, M. Wemyss, Maximal modifications and Auslander-Reiten duality for non-isolated singularities, Invent. Math. 197 (2014) 521–586.
[24] M. Kapranov, E. Vasserot, Kleinian singularities, derived categories and Hall algebras, Math. Ann. 316 (3) (2000) 565–576.
[25] G.R. Krause, T.H. Lenagan, Growth of Algebras and Gelfand-Kirillov Dimension, Research Notes in Mathematics, Pitman Adv. Publ. Program, vol. 116, 1985.
[26] T. Levasseur, Some properties of non-commutative regular graded rings, Glasg. Math. J. 34 (1992) 277–300.
[27] J.C. McConnell, J.C. Robson, Noncommutative Noetherian Rings, Wiley, Chichester, 1987.
[28] C. Năstăsescu, F. van Oystaeyen, Graded Ring Theory, North-Holland Mathematical Library, vol. 28, North-Holland Publishing Co., Amsterdam-New York, 1982.
[29] N. Popescu, Abelian Categories with Applications to Rings and Modules, London Mathematical Society Monographs, vol. 3, Academic Press, London-New York, 1973.
[30] X.-S. Qin, Y.-H. Wang, J.J. Zhang, Maximal Cohen-Macaulay modules over a noncommutative 2-dimensional singularity, preprint, arXiv:1906.06824, 2019.
[31] J. Rickard, Morita theory for derived categories, J. Lond. Math. Soc. (2) 39 (3) (1989) 436–456.
[32] J.T. Stafford, J.J. Zhang, Examples in non-commutative projective geometry, Math. Proc. Cambridge Philos. Soc. 116 (3) (1994) 415–433.
[33] J.T. Stafford, J.J. Zhang, Homological properties of (graded) noetherian PI rings, J. Algebra 168 (1994) 988–1026.
[34] M. Van den Bergh, Three-dimensional flops and noncommutative rings, Duke Math. J. 122 (3) (2004) 423–455.
[35] M. Van den Bergh, Non-commutative crepant resolutions, in: The Legacy of Niels Henrik Abel, Springer, Berlin, 2004, pp. 749–770.
[36] R. Vyas, A. Yekutieli, Weak proregularity, weak stability, and the noncommutative MGM equivalence, J. Algebra 513 (2018) 265–325.
[37] A. Yekutieli, Dualizing complexes over noncommutative graded algebras, J. Algebra 153 (1) (1992) 41–84.
[38] A. Yekutieli, Derived categories, arXiv:1610.09640v4, prepublication version.
[39] A. Yekutieli, J.J. Zhang, Dualizing complexes and tilting complexes over simple rings, J. Algebra 256 (2) (2002) 556–567.
[40] A. Yekutieli, J.J. Zhang, Rings with Auslander dualizing complexes, J. Algebra 213 (1) (1999) 1–51.