Optimal material ordering policy and allocation rule for a manufacturer making multiple products

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Abstract: Material ordering and allocation are important decisions for manufactures making multiple products, because those firms usually possess flexible production systems which can produce different products based on the same raw material. In this paper, we investigate the ordering policy (OP) and allocation rule (AR) of the raw materials for a manufacturer selling multiple products. The manufacturer's decision-making problem is analyzed under three scenarios: (1) joint decisions on OP and AR, (2) fixed AR, and (3) predetermined OP. We show that the latter two are not special cases of the first scenario, and they require different solution methods. Our objective is to derive the optimal solutions *analytically*. For the first scenario, we obtain the closed-form solution that is indeed optimal for the nonconcave profit function. For the fixed AR scenario, the products with twice-differentiable demands are studied, and the exact optimal OP for the raw material is achieved. Finally, if the OP is predetermined, we prove that the profit function is concave in AR and provide the associated optimality conditions, for which the optimal AR can be reached numerically. Different from the pervious heuristic approaches, these mathematically tractable solutions are easy to be applied by the practitioners.

Keywords: Flexible production system; multiple products; ordering policy; allocation rule; analytical solution.

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1. Introduction

With the development of modern agriculture, the agricultural companies are forced to manage their supply chains more efficiently [1]. According to a call for research in agribusiness problems [2], a particular challenge faced by the food and agribusiness sector is that how to appropriately assess and respond to the demand and supply variations, which will affect the agricultural companies' costs and profits significantly. Evidentally, in the recent years, the risk of making and selling agricultural products has shown an increasing trend. According to the report of World Economic Forum 2009, the low efficiency in the food value chain causes at least a $100 billion annual loss in the U.S. Essentially, the maximization of total profit considering the uncertainties is the ultimate goal of modern agriculture, in both developed and developing economies [3].

For agricultural companies, a commonly seen phenomenon is that a firm, who acts as a processor, orders a raw material (e.g., meat, corn, and potatoes) from its contracted suppliers (farms) and uses a flexible production system to turn the single material into multiple products. One can observe that the agricultural economists have established the analysis for the production flexibility, which is associated with the farm specialization [4]. Though, the production line provides flexibility in making different products, due to the demand uncertainty, the firm has to tackle the difficulty in making decisions on the material order quantity and the production quantities of different products. In practice, a typical example is the production system of dairy products. The system can utilize a single material, i.e., milk, to process a variety of dairy products, such as butter, nonfat dry milk, and cheese. A regular problem faced by the process factory is how to satisfy the uncertain demands of these dairy products. Similar examples can be found in the agricultural sector, e.g., cotton can be used to make multiple clothing products with different designs and colors and wheat is a major ingredient to make bread, porridge, biscuits, and doughnuts. In particular, to maximize the profit, both material procurement and production planning decisions must be made based on demand forecast and cost estimation for leftovers or unsatisfied demand. By taking the procurement and production issues into consideration, we are motivated to study the "one-material-multi-product" production system for the related agricultural companies.

In this paper, we examine the process system that manufactures multiple non-substitutable products. For example, the products such as bread, butter, nonfat dry milk, and doughnuts, are usually belong to different product categories, which have no downstream competition. The problem is modelled under a single-period multi-product setting. Under these assumptions, we mainly focus on the optimal ordering policy (OP) and/or the allocation rule (AR) of the raw materials for the agricultural companies. Specifically, we consider three scenarios for the one-material-multi-product system, i.e., (1) joint decisions on OP and AR, (2) fixed AR, and (3) predetermined OP. Note that there are three scenarios in total, because the decisions only have two parts (OP and AR).

In the first scenario, the OP and AR are considered simultaneously and the associated profit function is assumed to be twice differentiable. The main challenge to obtain the analytical solution for this scenario is that the profit function is generally not concave. Because of this, traditional concavity analysis is not applicable. However, we are able to propose an alternative approach, which derives the optimal solution in closed form for this problem; For the second scenario, we assume the AR is fixed and the firm should determine the OP accordingly. The proposed single-variable problem is concave, and the optimal OP can be achieved by traditional concavity analysis. However, when the demands are bounded, the problem cannot be separated, for which we discuss the optimal OP for the unbounded uniform demand case in detail. To model the third scenario, the OP is predetermined before making a decision on AR. In this scenario, though the profit function is concave in the AR, the exact solution method has not been reported in the literature. We first introduce a Lagrangian multiplier to relax our problem to an equivalent unconstrained version. Unlike previous research, which solves the Lagrangian relaxation problems via heuristic methods, we present a sufficient condition that can be used to obtain the optimal AR analytically. Notice that the firm may not process all of the products in one period, we also study the firm's preference in production, which may affect the profit significantly due to the unsatisfied demands. Numerical experiments are conducted to analyze the effects of some key parameters for different scenarios.

The remainder of this paper is organized as follows. Section 2 briefly reviews the related literature. A modelling framework of the proposed problem is presented in Section 3, and the analytical and numerical results for different scenarios are provided in Section 4. Finally, Section 5 concludes the work.

2. Literature Review

Our paper is closely related to two streams of research, i.e., the agriculture supply chain management and the multi-item newsvendor models.

In the first stream of research, because of the significant impacts of agricultural economy, the production and operations issues related to agriculture supply chain have attracted considerable attention from the academia. For examples, Kazaz [5] examines the production planning problems of a company that produces olive oil. In that problem, randomness exists in both demand and yield, for which the sale price and the ordering cost will be affected; Burer et al. [6] analyze the contract design problems between suppliers and retailers in the agricultural seed industry to deal with the associated tradeoffs. In particular, Sodhi and Tang [1] provide some typical examples of agriculture supply chain management, which are mainly focused on developing countries/regions (e.g., Sri Lanka, Malaysia, Afghanistan, India, Africa, Philippines, and Bangladesh). Chen et al. [3] introduce an innovative business model, named 'e-Choupals' in India, and investigate its impact on the farmers' production decisions. Under the framework of Cournot competition models, Tang et al. [7] discuss the farmers' incentives to utilize the free demand information provided by the government. They also consider the risk of demand uncertainty to derive the corresponding optimal ordering and supply decisions. Recently, Niu et al. [8] presented the contract choice issues in contract farming. They studied the agricultural companies' ordering decisions for two scenarios, i.e., with and without the farmer cooperative. For a more comprehensive review of the agribusiness decision-making problems, the readers are referred to Lowe and Preckel [2].

However, the above-mentioned studies merely consider the single-material or/and single-product problems. Differs from these articles, we propose to characterize a commonly seen flexible operation mode in the agribusiness, that is, one-material-multiple-product problem. In this problem, to improve the profitability, we analyze an agricultural company's material ordering and allocation decisions, which have been overlooked by the previous literature. Methodologywise, we also develop approaches that can be used to obtain the optimums analytically.

On the other hand, for the second related research stream, i.e., multi-item newsvendor models, the associated study can be dated back to 1960s, when Hadley and Whithin [9] introduced a Lagrange multiplier technique with a dynamic programming solution procedure to derive the optimal order quantity in a single-constraint and multi-product setting. Since then, the problem has been extensively studied by the researchers for decades. For example, the problem is extended to deal with multiple constraints by Lau and Lau [10], and an efficient approach is developed to handle the case with a large number of products. Under a budgetary constraint, when the demand uncertainty can be captured by discrete or interval scenarios, Vairaktarakis [11] proposes robust models for the multi-item newsvendor problems. Chen et al. [12] consider a risk-averse newsvendor to deliver some useful properties from a new perspective. They also compare their results with the classical risk-neutral newsvendor problems to examine the major differences. Zhang and Xu [13] further develop a multi-item model with carbon cap and trade mechanism. They derive the optimal production policy and carbon trading decisions. In the spirit of Nahmias and Schmidt [14], i.e., using simple and effective techniques, Erlebacher [15] constructs a model with a single capacity constraint. The work aims at developing easy-to-implement heuristic procedures to find the optimal order quantity. In the recent literature, considerable and intensive research has been devoted to analyzing such problems with different constraints. For example, to deal with possible production capacity shortage, Zhang and Du [16] study a multi-item newsvendor system with zero- or nonzero-lead-time production outsourcing. They characterize the structural properties of the problem and develop algorithms to solve the optimization models. By incorporating the retailer's pricing decision and supplier's quantity discount, Shi et al. [17] extend the multi-product newsvendor problem and a Lagrangian heuristic method is developed to seek the solution. Zhou et al. [18] present a Genetic-Algorithm-based approach to solve the multi-product multi-echelon model. Jarrahi and Abdul-Kader [19] measure the performance of a multi-product unreliable production line. In the model, they assume finite buffers between workstations and develop an approximation (which generalizes the processing times) to reduce the variations in the system. A detailed review of multi-product models and their extensions can be found in Choi [20] and Turken et al. [21].

While, in the aforementioned literature, multi-item newsvendor problem is considered as a natural extension of the single-product case, where the profit function is separable and concave for different products. In addition, the solutions of the constrained problems are all solved by heuristics methods. Thus far, little attention has been paid on the material-sharing (flexibility) issues in the multi-product system of agricultural companies, and most of the solution techniques rely on heuristics approaches. This motivates us to investigate an effective solution technique for such problems. Compared to the previous studies, the main contribution of our work is that we are able to deliver analytical solutions to the proposed one-material-multi-product problem, which fills a gap in the literature.

3. Problem Description

3.1. Preliminaries

We consider an agricultural firm that makes multiple products based on the same raw material (e.g., a processor firm [22]). The problem is analyzed under the newsvendor framework with a single-period make-to-stock inventory setting [23]. As a multi-product processor, before the demands are realized, the firm needs to allocate the material to process different products through a flexible production system. In particular, the demands of different products are independent and non-substitutable, because of their heterogenous properties.

Table 1: Notation

Variable	Description
N	Number of products
x	Order quantity of the raw material
w_i	Proportion of the material allocated to product i
p_i	Price of product i
c_i	Cost of product i
s_i	Salvage value of product i
b_i	Backorder cost of product i
ξ_i	Demand of product i (a random variable)
$f_i(\cdot)$	Probability density function (pdf) of ξ_i
$F_i(\cdot)$	Cumulative distribution function (CDF) of ξ_i
α_i	Overstock cost structure for product i
β_i	Shortage cost structure for product i
π	Total profit

Figure 1 depicts the operations process for the proposed one-material-multi-product production system. Suppose the unit cost of product i, denoted by c_i, includes both material ordering cost c and manufacturing cost m_i of product i, and the product is sold at a fixed price p_i. In addition, any unsold stock of product i is worth a unit salvage value s_i, and b_i is the unit back-order cost for the unsatisfied demand (assume $p_i > c_i \geq s_i \geq 0$ and $b_i \geq 0$). For the processor firm's one-material-multi-product problem, our objective is to optimize the decisions on OP and AR under possible scenarios.

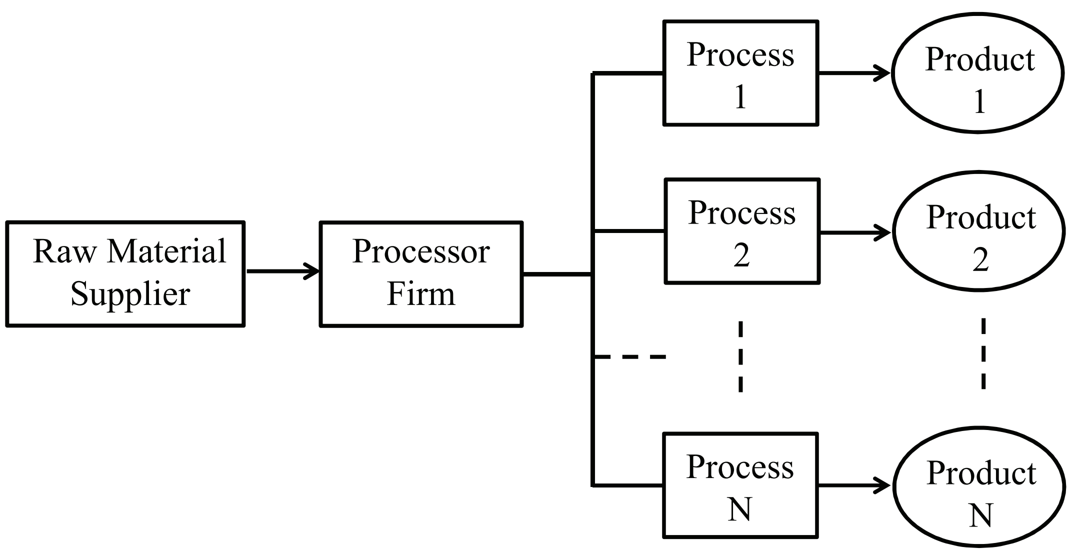

Figure 1: Flexible one-material-multi-product production process.

Because both procurement and production decisions are taken into consideration, to maximize the firm's total profit, the key decision variables are the OP and the AR of the raw material. To assist our presentation, we define $a^+ = max(a, 0)$ and $a \wedge b = \min(a, b)$ ($\forall a, b \in \mathbb{R}$ and $\mathbb{R}$ is the set of real numbers). The demand of product i is denoted by ξ_i (a random variable). Let $x \geq 0$ be the material order quantity, w_i be the proportion of the material allocated to product i ($i \in \{1, 2, \ldots, N\}$), and π be the firm's total profit. We assume the loss from turning the raw material to the finished product can be ignored, i.e., the yield rates are 100% (note that fixed yield rates will not change the solution structure for this problem). Then, given the decision variables x and w_i ($i \in \{1, 2, \ldots, N\}$), the profit function is formulated as

$$\begin{aligned}\pi &= \sum_{i=1}^{N} \left(p_i(\xi_i \wedge w_i x) + s_i(w_i x - \xi_i)^+ - b_i(\xi_i - w_i x)^+ - m_i w_i x\right) - cx \\ &= \sum_{i=1}^{N} \left(p_i(\xi_i \wedge w_i x) + s_i(w_i x - \xi_i)^+ - b_i(\xi_i - w_i x)^+ - m_i w_i x\right) - \sum_{i=1}^{N} c w_i x \\ &= \sum_{i=1}^{N} \left(p_i(\xi_i \wedge w_i x) + s_i(w_i x - \xi_i)^+ - b_i(\xi_i - w_i x)^+ - m_i w_i x - c w_i x\right) \\ &= \sum_{i=1}^{N} \left(p_i(\xi_i \wedge w_i x) + s_i(w_i x - \xi_i)^+ - b_i(\xi_i - w_i x)^+ - c_i w_i x\right),\end{aligned} \tag{1}$$

where the ith indexed term in Equation (1) shows the profit generated from product i.

Figure 2 illustrates the problem faced by the firm, that is, how to determine the optimal OP (x) and AR (w_i, $i \in \{1, 2, \ldots, N\}$) to satisfy the demands of N different products. Assuming the firm has some knowledge about the probability distribution (with CDF $F_i(\xi_i)$ and pdf $f_i(\xi_i)$) of the demand for product i, the firm's expected profit can be expressed as

$$E[\pi] = \sum_{i=1}^{N} \int_0^{\infty} \left(p_i(\xi_i \wedge w_i x) + s_i(w_i x - \xi_i)^+ - b_i(\xi_i - w_i x)^+ - c_i w_i x\right) f_i(\xi_i) d\xi_i. \tag{2}$$

The objective of the firm is to maximize the expected total profit by optimizing the associated OP and/or AR. Because the agricultural raw materials will be expired soon, we assume all of the ordered material will be consumed in one period. Then, the optimal

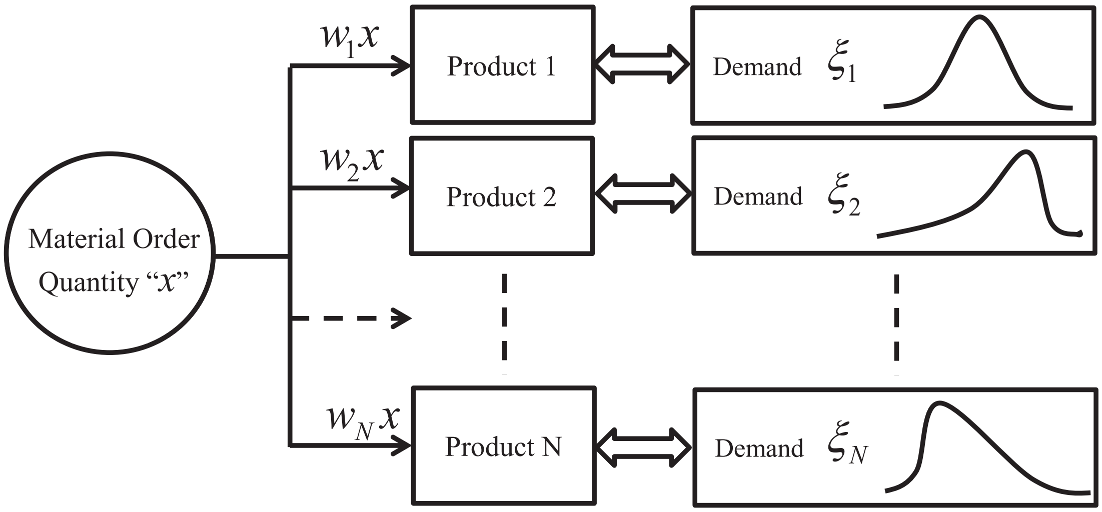

Figure 2: Order quantity and allocated proportions of raw materials for multiple products.

decisions can be obtained by solving the following optimization problem

$$\begin{aligned} \max_{x,\underline{W}} \quad & E[\pi] \\ \text{subject to:} \quad & \sum_{i=1}^{N} w_i = 1, \\ & x \geq 0, w_i \geq 0, \forall i, \end{aligned} \tag{3}$$

where $\underline{W} = (w_1, w_2, \ldots, w_N)$. It is clear that the problem reduces to a classic single-product newsvendor model with $N = 1$

$$\max_{x \geq 0} \quad E[\pi] = \int_0^\infty \left(p_1(\xi_1 \wedge x) + s_1(x - \xi_1)^+ - b_1(\xi_1 - x)^+ - c_1 x\right) f_1(\xi_1) d\xi_1. \tag{4}$$

Let $\alpha_i = p_i + b_i - s_i \geq 0$ and $\beta_i = p_i + b_i - c_i \geq 0$ be the overstock cost structure and the shortage cost structure for product i, respectively. The ratio $\frac{\beta_i}{\alpha_i}$ is referred to as the critical fractile, which is also the optimal service level for product i. For Equation (4), it is well-known that the profit function $E[\pi]$ is concave in x. Moreover, the optimal solution of Equation (4) can be achieved as $x^* = F_1^{-1}(\beta_1/\alpha_1)$ when the inverse of the distribution function $F_1(\xi_1)$ exists.

4. Optimal Solutions for Different Scenarios

In this section, we consider all possible scenarios for the decision-making process in the proposed system. Because the decisions consist of two parts from procurement and production departments, there are totally three scenarios, i.e., joint decisions on OP and AR, fixed AR, and predetermined OP (which depends on the management mode of the two departments). We discuss the optimal solutions for the three scenarios as follows.

4.1. Joint optimal solutions of OP and AR

Gotoh and Takano [24] develop a model to minimize the conditional value-at-risk for multiple products. In their model, the order quantity for each product is optimized separately, and different products are not made from the same material. Clearly, the problem is similar to the single-product case, in which the firm only needs to make the ordering decision for each product based on a concave profit function. However, when both the material order quantity x and allocation mechanism $\underline{W}$ are considered as decision variables, the problem reveals different features.

The following simple verification indicates that the profit function in Equation (2) may not necessarily be concave. For example, if $E[\pi]$ is twice differentiable, its Hessian matrix is

$$H(x, \underline{W}) = \begin{bmatrix} \frac{\partial^2 E[\pi]}{\partial x^2} & \frac{\partial^2 E[\pi]}{\partial x \partial w_1} & \frac{\partial^2 E[\pi]}{\partial x \partial w_2} & \cdots & \frac{\partial^2 E[\pi]}{\partial x \partial w_N} \\ \frac{\partial^2 E[\pi]}{\partial w_1 \partial x} & \frac{\partial^2 E[\pi]}{\partial w_1^2} & 0 & \cdots & 0 \\ \frac{\partial^2 E[\pi]}{\partial w_2 \partial x} & 0 & \frac{\partial^2 E[\pi]}{\partial w_2^2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 E[\pi]}{\partial w_N \partial x} & 0 & 0 & \cdots & \frac{\partial^2 E[\pi]}{\partial w_N^2} \end{bmatrix}. \tag{5}$$

Obviously, the structures of $E[\pi]$ and $H(x, \underline{W})$ cannot guarantee that $H(x, \underline{W})$ is negative semidefinite. As a result, the traditional concavity analysis cannot be used for this problem.

However, if we treat $w_i x$ as a single variable, the problem becomes separable. Thus, we are able to tackle the problem and derive the optimal solution in closed form. In fact, if $w_i x$ is considered as a single variable, we only need to optimize the subproblems in Equation (3) simultaneously. Suppose the demand distribution functions are invertible. Then, the following theorem gives the analytical results.

Theorem 1. *If $F_i(\xi_i)$ is invertible, $i \in \{1, 2, \ldots, N\}$, the joint optimal decisions of OP and AR are*

$$\begin{cases} x^* = \sum\limits_{i=1}^{N} F_i^{-1}(\frac{\beta_i}{\alpha_i}), \\ w_i^* = \frac{F_i^{-1}(\frac{\beta_i}{\alpha_i})}{\sum\limits_{i=1}^{N} F_i^{-1}(\frac{\beta_i}{\alpha_i})}, i = 1, 2, \ldots, N. \end{cases}$$

PROOF: Let $g_i(x, w_i) = \int_0^\infty (p_i(\xi_i \wedge w_i x) + s_i(w_i x - \xi_i)^+ - b_i(\xi_i - w_i x)^+ - c_i w_i x) f_i(\xi_i) d\xi_i$, Equation (3) can be rewritten as

$$\max \left\{ \sum_{i=1}^{N} g_i(x, w_i) : \sum_{i=1}^{N} w_i = 1, x, w_i \geq 0, \forall i \right\}.$$

If we treat $w_i x$ as a single variable, the problem becomes separable and is equivalent to

$$\sum_{i=1}^{N} \max\{g_i(x_i, w_i) : x_i, w_i \geq 0, \forall i\}. \tag{6}$$

It is easy to know that Equation (6) is a separable maximization problem for which the optimal solution can be obtained by maximizing each subproblem. Thus, the optimal solution of the ith subproblem $\max\{g(x_i, w_i) : x_i, w_i \geq 0\}$ is similar to that of Equation (4). Hence, we have the optimality condition of subproblem i as

$$w_i^* x_i^* = F_i^{-1}\left(\frac{p_i + b_i - c_i}{p_i + b_i - s_i}\right) = F_i^{-1}\left(\frac{\beta_i}{\alpha_i}\right), \forall i. \tag{7}$$

Since both w_i^* and x_i^* are positive constants, if we choose

$$\begin{cases} x_i^* = x^* = \sum_{i=1}^{N} F_i^{-1}(\frac{\beta_i}{\alpha_i}) \geq 0, \forall i, \\ w_i^* = \frac{F_i^{-1}(\frac{\beta_i}{\alpha_i})}{\sum_{i=1}^{N} F_i^{-1}(\frac{\beta_i}{\alpha_i})} \geq 0, \forall i, \end{cases} \tag{8}$$

as the optimal solutions, the optimality condition of subproblem i in Equation (7) and the corresponding equality constraint $\sum_{i=1}^{N} w_i^* = 1$ can be satisfied with $x_1^* = x_2^* =, \ldots, = x_N^* = x^*$. Hence, Equation (8) gives the optimal solution to Equation (3). This completes the proof. □

The analytical result given in Theorem 1 provides an efficient computational approach to solving the joint decision-making problem. For demonstration purposes, we assume that a processor firm orders milk from its contracted supplier and allocate it to manufacture three diary products (e.g., butter, yoghurt, and cheese), and the demand of product i follows the normal distribution $N(\mu_i, \sigma_i)$ with mean μ_i and standard deviation σ_i. Table 2 provides the

Table 2: Parameter settings for normally distributed demands

Product	Parameters (\$/lb)	Demand (lbs) $\sim N(\mu_i, \sigma_i)$
1	$p_1 = 1.5$, $b_1 = 0.3$, $s_1 = 0.15$, $c_1 = 0.5$	$\mu_1 = 900$, $\sigma_1 = 45$
2	$p_2 = 1.7$, $b_2 = 0.3$, $s_2 = 0.15$, $c_2 = 0.6$	$\mu_2 = 300$, $\sigma_2 = 11$
3	$p_3 = 1.8$, $b_3 = 0.3$, $s_3 = 0.15$, $c_3 = 0.7$	$\mu_3 = 540$, $\sigma_3 = 30$

parameters for this sample problem. According to Theorem 1, the optimal OP and AR are obtained as

$$x^* = 1800.9164\ lbs, w_1^* = 0.5197, w_2^* = 0.1708, w_3^* = 0.3095, \tag{9}$$

and the optimal expected profit is $E[\pi]^* = \$1776.3400$. The optimal solution suggests that the firm should order 1800.9164 lbs milk from the contracted supplier, and allocate $x^* w_1^* = 935.9363$ lbs, $x^* w_2^* = 307.5965$ lbs, $x^* w_3^* = 557.3836$ lbs of it to process products 1, 2, 3, respectively. One can see that the optimally allocated quantities of the milk are all close to the mean values of the demands.

Thus far, we have examined the joint decisions on the OP and AR. However, in practice, the two decisions may be separately made by different departments within the same company. Technically, we can also show that the latter two are not special cases of the scenario with joint decisions on the OP and AR. Next we provide the detailed explanations.

4.2. Optimal OP with known AR

In some cases, e.g., fresh food industry, the decisions of strategic souring and operative procurement rely on the information of production planning process [25]. Thus, in this section, we discuss the scenario when the AR is known, from the production planning decisions, before making the procurement decision. Suppose at the beginning of each period, the production department will design its production plan and make the production decision ahead of the procurement department, which results in a fixed AR for the problem. Then, the procurement department will make its ordering decision based on the given AR $\underline{W}$. Under this circumstance, the optimality conditions for all subproblems may not necessarily hold simultaneously, because the OP x is the only decision variable (i.e., $w_i x$ may not reach the optimal value for product i, $\forall i$). Indeed, the problem becomes an unconstrained newsvendor-type problem, which has been proved to be concave. However, the optimal OP will vary as the AR changes, and may be dominated by some product with higher profit. For example, if $N = 2$, the AR is that w_1 and $w_2 = 1 - w_1$ for products 1 and 2, respectively. If the profit generated from product 1 is much higher than that of product 2, the optimal solution will show a trend that pushes $w_1 x^*$ towards the optimal point of subproblem 1 while $w_2 x^*$ may be forced to leave the optima of subproblem 2. In the rest of this section, two types of demand distributions are discussed.

Type I: Unbounded demands when $E[\pi]$ is twice differentiable

Ideally, when the demand is unbounded, the production quantities will not exceed the maximum demands, i.e., $\Pr\{\xi_i \geq w_i x\} > 0$ ($i \in \{1, 2, \ldots, N\}$). For instances, normal, gamma, log-normal, and Weibull distributions can provide the possibility of infinite demand. Under this assumption, the profit function is usually twice differentiable with respective to

x. It is well known that, under newsvendor setting, the profit function is concave in the order quantity. However, our model provides a new formulation with material sharing property, for which the subproblems may not achieve optimality conditions when the demands are bounded. Thus, we would also like to examine the concavity of the unbounded case. The following lemma shows the concavity of $E[\pi]$ when the AR is pre-determined.

Lemma 1. *Given the AR for unbounded product demands, if the profit function is twice differentiable, it is concave in x.*

PROOF: Suppose the profit function is twice differentiable. Then, its first and second derivatives with respective to x are as follows.

$$\begin{cases} \frac{\partial E[\pi]}{\partial x} = \sum_{i=1}^{N} \left[-\alpha_i w_i F_i(w_i x) + \beta_i w_i\right], \\ \frac{\partial^2 E[\pi]}{\partial x^2} = \sum_{i=1}^{N} \left[-\alpha_i w_i^2 f_i(w_i x)\right]. \end{cases} \tag{10}$$

Obviously, $\frac{\partial^2 E[\pi]}{\partial x^2} \leq 0$, as $\alpha_i \geq 0$, $w_i^2 \geq 0$ and $f_i(w_i x) \geq 0$ for all i. Therefore, the profit function is concave in x. $\square$

Based on Lemma 1, the optimal order quantity x^* can be obtained by solving $\frac{\partial E[\pi]}{\partial x} = 0$, i.e.,

$$\sum_{i=1}^{N} \alpha_i w_i F_i(w_i x) - \sum_{i=1}^{N} \beta_i w_i = 0. \tag{11}$$

For general demand distributions, the equation can be evaluated numerically. For example, we consider the parameter settings given in Table 2 to study the optimal OP for a given AR $\underline{W} = (w_1, w_2, w_3)$. Based on Equation (11), we can obtain the optimal x by solving the following nonlinear equation

$$\sum_{i=1}^{3} \alpha_i w_i \int_{-\infty}^{w_i x} \frac{1}{\sigma_i \sqrt{2\pi}} e^{\frac{(\xi_i - \mu_i)^2}{2\sigma_i^2}} d\xi_i - \sum_{i=1}^{3} \beta_i w_i = 0.$$

Table 3 presents the optimal OP for different combinations of the AR. One can see that, for the AR given in Equation (9) (with a check mark in Table 3), the resulting optimal OP

Table 3: Optimal OP for the given AR

Case	Fixed $\underline{W}$ w_1	w_2	w_3	Optimal OP x^*	Optimal Profit $E[\pi]^*$
1	0.5197	0.3095	0.1708	1844.8929	1363.4090
2	0.5197	0.1708	0.3095	1800.9164	1776.3400✓
3	0.3095	0.5197	0.1708	2983.2096	1191.5776
4	0.3095	0.1708	0.5197	2701.5497	1190.3211
5	0.1708	0.5197	0.3095	1795.3993	748.0407
6	0.1708	0.3095	0.5197	1087.6032	858.6647

is exact the one given in Equation (9). In this case, the expected profit reaches its maximum value, compared to all other cases. This is because, for the joint decision scenario, we can simultaneously change two variables, which must be better than the fixed AR scenario (has only one variable). In addition, according to the parameter setting, product 1 is recognized to be the most profitable one. From the table, we can see that, at first, the optimal OP is dominated by product 1 in Cases $1-4$. When $w_1 = 0.5197$ or $w_1 = 0.3095$, it is worth manufacturing around 900 lbs (which is close to 935.9363 lbs) of product 1, and the quantities of products 2 and 3 are forced to leave 307.5965 lbs and 557.3836 lbs (the optimums of subproblems), respectively. However, if $w_1 = 0.1708$ (only a small proportion of the component is allocated to product 1), product 1 is no longer dominant. Because the firm needs to order a large amount of milk to push $w_1 x^*$ close to 935.9363 lbs, which significantly reduces the profits from the other two products (the profit generated from product 1 is not enough to cover the losses of products 2 and 3). Clearly, the predetermined AR indeed has a strong impact on the firm's procurement decision. Moreover, if a product is potentially more profitable, the optimal OP may show a favorite tendency to this product.

When the CDF of the unbounded demand of each product is strictly increasing in x, the next lemma gives the optimal OP in closed form.

Lemma 2. *If $F_i(w_i x)$ is strictly increasing ($i = 1, 2, \ldots, N$) in x and*

$$G(x) = \sum_{i=1}^{N} \alpha_i w_i F_i(w_i x), \tag{12}$$

then the inverse function of $G(x)$ exists and the optimal OP is

$$x^* = G^{-1}\left(\sum_{i=1}^{N} \beta_i w_i\right).$$

PROOF: Because $F_i(w_i x)$ is strictly increasing in x, $\alpha_i \geq 0$, and $w_i \geq 0$, it is easy to see that $G(x) = \sum\limits_{i=1}^{N} \alpha_i w_i F_i(w_i x)$ is strictly increasing in x as well. Thus, the inverse function of $G(x)$ exists. Based on $\frac{\partial E[\pi]}{\partial x} = 0$, we have

$$\sum_{i=1}^{N} [-\alpha_i w_i F_i(w_i x) + \beta_i w_i] = -G(x) + \sum_{i=1}^{N} \beta_i w_i = 0,$$

which leads to the optimal solution $x^* = G^{-1}\left(\sum_{i=1}^{N} \beta_i w_i\right)$. □

Lemma 2 provides the explicit expression for the optimal OP when the AR is predetermined. However, it is usually difficult to find the inverse function of $G(x)$ for general distributions. The following proposition addresses a special case where the optimal OP can be obtained.

Proposition 1. *If all of the demands follow the same distribution $F(\cdot)$ and the raw material is equally allocated to N products, the optimal OP is*

$$x^* = NF^{-1}\left(\sum_{i=1}^{N} \beta_i / \sum_{i=1}^{N} \alpha_i\right).$$

PROOF: Assume the AR is set as $w_1 = w_2 = \ldots = w_N = 1/N$. Suppose the demands of different products follow the same distribution $F(\cdot)$. Then, we have $F_1(w_1 x) = F_2(w_2 x) = \ldots = F_N(w_N x) = F\left(\frac{x}{N}\right)$. According to Equation (12), we have

$$G(x) = \frac{1}{N} F\left(\frac{x}{N}\right) \sum_{i=1}^{N} \alpha_i.$$

By applying Lemma 2, the optimal OQ is

$$x^* = NF_1^{-1}\left(\sum_{i=1}^{N}\beta_i / \sum_{i=1}^{N}\alpha_i\right).$$

This completes the proof. □

When the demands of different products follow the same distribution and their profitabilities are similar, e.g., yoghurts with different tastes and fruit mix, it is reasonable for the production department to apply the AU $w_i = 1/N, \forall i$. In this case, Proposition 1 can be used to determine the order quantity of the raw material. Hence, the quantity allocated to each product is simply $F_1^{-1}(\sum_{i=1}^{N}\beta_i / \sum_{i=1}^{N}\alpha_i)$. Due to the symmetric property of the demands and the AR, the optimal OP will only be affected by the critical ratio between the overall shortage cost structure $\sum_{i=1}^{N}\beta_i$ and the overall overstock cost structure $\sum_{i=1}^{N}\alpha_i$. In other words, under this situation, the firm can make the optimal procurement decision from an integrated viewpoint and does not need to consider the marginal contribution of each product to the firm's profit.

Type II: Bounded demand with uniform distribution

Practically, the demand of each product is bounded in an interval [11], e.g., $\xi_i \in [0, U_i]$, where U_i is the upper bound of the corresponding demand. One commonly used assumption is that the demand of a product follows uniform distribution, which simplifies the profit function. In this special case, we assume that the demands of all the products follow uniform distributions, i.e., $\xi_i \sim U[0, U_i]$, $\forall i$ (similar analysis can be conducted for bounded demands with the other distributions, e.g., the triangular and the truncated distributions).

First, we will further examine the expression of $E[\pi]$. Suppose ξ_i follows the uniform distribution $U[0, U_i]$, we can define an indicator set I_1 as

$$I_1 = \{i | x < V_i\},$$

where $V_i = \frac{U_i}{w_i}$ $(i = 1, 2, \ldots, N)$. Then, the profit function can be expressed as

$$\begin{aligned} E[\pi] &= \sum_{i=1}^{N} \left[\int_0^{w_i x \wedge U_i} (p_i \xi_i + s_i(w_i x - \xi_i)) \frac{1}{U_i} d\xi_i + \int_{w_i x \wedge U_i}^{U_i} (p_i w_i x - b_i(\xi_i - w_i x)) \frac{1}{U_i} d\xi_i - c_i w_i x \right] \\ &= \sum_{i \in I_1} -\frac{w_i^2 \alpha_i}{2U_i} x^2 + \left(\sum_{i \in I_1} \beta_i w_i + \sum_{i \notin I_1} (\beta_i - \alpha_i) w_i \right) x - \sum_{i \in I_1} \frac{1}{2} b_i U_i + \sum_{i \notin I_1} \frac{1}{2} (p_i - s_i) U_i. \end{aligned} \tag{13}$$

One can see that the demand of product i will not exceed U_i. However, if w_i is fixed, the optimal order quantity x^* of Equation (13) cannot guarantee $w_i x^* \leq U_i$ for all products, which reveals a different profit function structure (pay attention to the term "$w_i x \wedge U_i$") and may result in certain wastes of low-profit products. From an economic viewpoint, it is necessary to study the properties of the profit function under this circumstance.

If $E[\pi]$ is twice differentiable at $x \in [0, +\infty)$, its first two derivatives can be obtained as

$$\begin{cases} \frac{\partial E[\pi]}{\partial x} = \sum_{i \in I_1} -\frac{w_i^2 \alpha_i}{U_i} x + \sum_{i \in I_1} \beta_i w_i + \sum_{i \notin I_1} (\beta_i - \alpha_i) w_i, \\ \frac{\partial^2 E[\pi]}{\partial x^2} = \sum_{i \in I_1} -\frac{w_i^2 \alpha_i}{U_i}. \end{cases} \tag{14}$$

One can see from Equations (13) and (14), the coefficients of the quadratic function $E[\pi]$, the linear function $\frac{\partial E[\pi]}{\partial x}$, and the constant $\frac{\partial^2 E[\pi]}{\partial x^2}$ depend on the set I_1. Note that the number of elements in set I_1 decreases as x increases. Let $V_{(0)} \leq V_{(1)} \leq V_{(2)} \leq \ldots \leq V_{(N)} < V_{(N+1)}$ be the ascending sequence of points $\{0, V_1, V_2, \ldots, V_N, \infty\}$, where $V_{(0)} = 0$ and $V_{(N+1)} = \infty$. Since $E[\pi]$ is either a quadratic or a linear function in each subinterval $[V_{(i-1)}, V_{(i)})$, we can easily seek the closed-form expression for the optimal OP. For the sake of convenience, we rewrite Equation (13) as

$$E[\pi] = A_{(i)} x^2 + B_{(i)} x + C_{(i)}, x \in [V_{(i-1)}, V_{(i)}), i = 1, 2, \ldots, N+1, \tag{15}$$

where

$$\begin{aligned} A_{(i)} &= \sum_{k=i}^{N} -\frac{w_{(k)}^2 \alpha_{(k)}}{2U_{(k)}}, \quad B_{(i)} = \sum_{k=i}^{N} \beta_{(k)} w_{(k)} + \sum_{k=1}^{i-1} (\beta_{(k)} - \alpha_{(k)}) w_{(k)}, \\ C_{(i)} &= -\sum_{k=i}^{N} \frac{1}{2} b_{(k)} U_{(k)} + \sum_{k=1}^{i-1} \frac{1}{2} (p_{(k)} - s_{(k)}) U_{(k)}. \end{aligned}$$

The following theorem provides a guideline for seeking the optimal x.

Theorem 2. *If all the product demands follow uniform distributions, i.e., $\xi_i \sim U[0, U_i]$, the optimal OP belongs to the following set*

$$x^* \in \left\{0, V_{(1)}, V_{(2)}, \ldots, V_{(N)}, \frac{-B_{(1)}}{2A_{(1)}}, \frac{-B_{(2)}}{2A_{(2)}}, \ldots, \frac{-B_{(N)}}{2A_{(N)}}\right\}.$$

Proof: Based on Equation (15), we know that, for subdomain $[V_{(i-1)}, V_{(i)}]$ $(1 \le i \le N)$, the maximum value of $E[\pi]$ can be achieved either at the endpoints ($V_{(i-1)}$ and/or $V_{(i)}$) or its symmetric point $\frac{-B_{(i)}}{2A_{(i)}}$ (if $V_{(i-1)} \le \frac{-B_{(i)}}{2A_{(i)}} \le V_{(i)}$). In addition, when $x > V_{(N)}$,

$$E[\pi] = \left(\sum_{k=1}^{N} (\beta_{(k)} - \alpha_{(k)}) w_{(k)}\right) x + \sum_{k=1}^{N} \frac{1}{2} (p_{(k)} - s_{(k)}) U_{(k)}.$$

Since $(\beta_{(k)} - \alpha_{(k)}) w_{(k)} = (s_{(k)} - c_{(k)}) w_{(k)} \le 0$, the profit function is a non-increasing function in $[V_{(N)}, +\infty)$, which implies that the maximum value of $E[\pi]$ is located at $V_{(N)}$ for $x \in [V_{(N)}, +\infty)$. Hence, by comparing all the maximum points obtained in the subintervals, the global optimum for the entire interval $[0, +\infty)$ can be found and it satisfies

$$x^* \in \left\{0, V_{(1)}, V_{(2)}, \ldots, V_{(N)}, \frac{-B_{(1)}}{2A_{(1)}}, \frac{-B_{(2)}}{2A_{(2)}}, \ldots, \frac{-B_{(N)}}{2A_{(N)}}\right\}.$$

This completes the proof. $\square$

Theorem 2 suggests that the exact solution to Equation (13) can be obtained by evaluating the following simplified problem

$$\max\left\{E[\pi] : x \in \left\{0, V_{(1)}, V_{(2)}, \ldots, V_{(N)}, \frac{-B_{(1)}}{2A_{(1)}}, \frac{-B_{(2)}}{2A_{(2)}}, \ldots, \frac{-B_{(N)}}{2A_{(N)}}\right\}\right\}.$$

To illustrate the above result, we consider the problem setting given in Table 4. In this example, we assume that the firm processes three different products with uniformly distributed demands. From the problem setting, we can define $V_{(0)} = 0, V_{(1)} = V_2, V_{(2)} = V_3, V_{(3)} = V_1, V_{(4)} = \infty$. We know that $E[\pi]$ is quadratic in $[0, V_2)$, $[V_2, V_3)$, and $[V_3, V_1)$,

Table 4: Parameter settings for uniformly distributed demands

Product	Parameters (\$/lb)	Demand (lbs) $\sim U[0, U_i]$	AR
1	$p_1 = 1.5$, $b_1 = 0.3$, $s_1 = 0.15$, $c_1 = 0.5$	$U_1 = 900$	$w_1 = 0.3$
2	$p_2 = 1.7$, $b_2 = 0.3$, $s_2 = 0.15$, $c_2 = 0.6$	$U_2 = 300$	$w_2 = 0.4$
3	$p_3 = 1.8$, $b_3 = 0.3$, $s_3 = 0.15$, $c_3 = 0.7$	$U_3 = 540$	$w_3 = 0.3$
$V_2 = \frac{U_2}{w_2} = 750 < V_3 = \frac{U_3}{w_3} = 1800 < V_1 = \frac{U_1}{w_1} = 3000$			

and is linear in $[V_1, \infty)$. Solving the simplified problem, we obtain the optimal OP as $x^* = 1285.7142$ lbs, and the associated optimal profit is \$421.5. To analyze the wastes of certain products due to the dominance of the high-profit product, we only change the price of product 1 and fix the other parameters. The study is conducted on four cases shown in Figure 3. In an extreme case where product 1 is free, i.e., $p_1 = \$0$/lb, the firm only makes profit from products 2 and 3. The corresponding optimal OP ensures no "obvious" waste of each product (note: if $x^* > V_i$, we have $w_i x^* > U_i$, which implies that at least $w_i x^* - U_i$ of product i will be wasted). In this case, we have $I_1 = \{1, 2, 3\}$ without certain waste. As p_1 increases to \$1.5/lb, because the profit generated by product 1 can cover the loss in product 2, the optimal x is increased to process more product 1 while resulting in a surplus of product 2 ($I_1 = \{1, 3\}$). Once the price of product 1 reaches \$6/lb, product 1 shows a significant dominance over products 2 and 3 in terms of profit. As a result, certain amounts of products 2 and 3 are forced to be wasted ($I_1 = \{1\}$). Finally, when $p_1 \to \infty$, x^* becomes 3000 lbs which suggests the firm producing 900 lbs of product 1 (i.e., the upper bound of the demand).

Consequently, when the demands of the multi-product system are bounded and the AR decision is made prior to the OP decision, the material waste becomes an inevitable issue.

4.3. Optimal AR with predetermined OP

Besides the scenario with fixed AR, alternatively, the firm may deal with predetermined OP based on market coverage promise, that is, the procurement department makes the ordering decision before the AR decision of production department. In this subsection, we assume that $E[\pi]$ is twice differentiable and the order quantity x is known prior to planning the

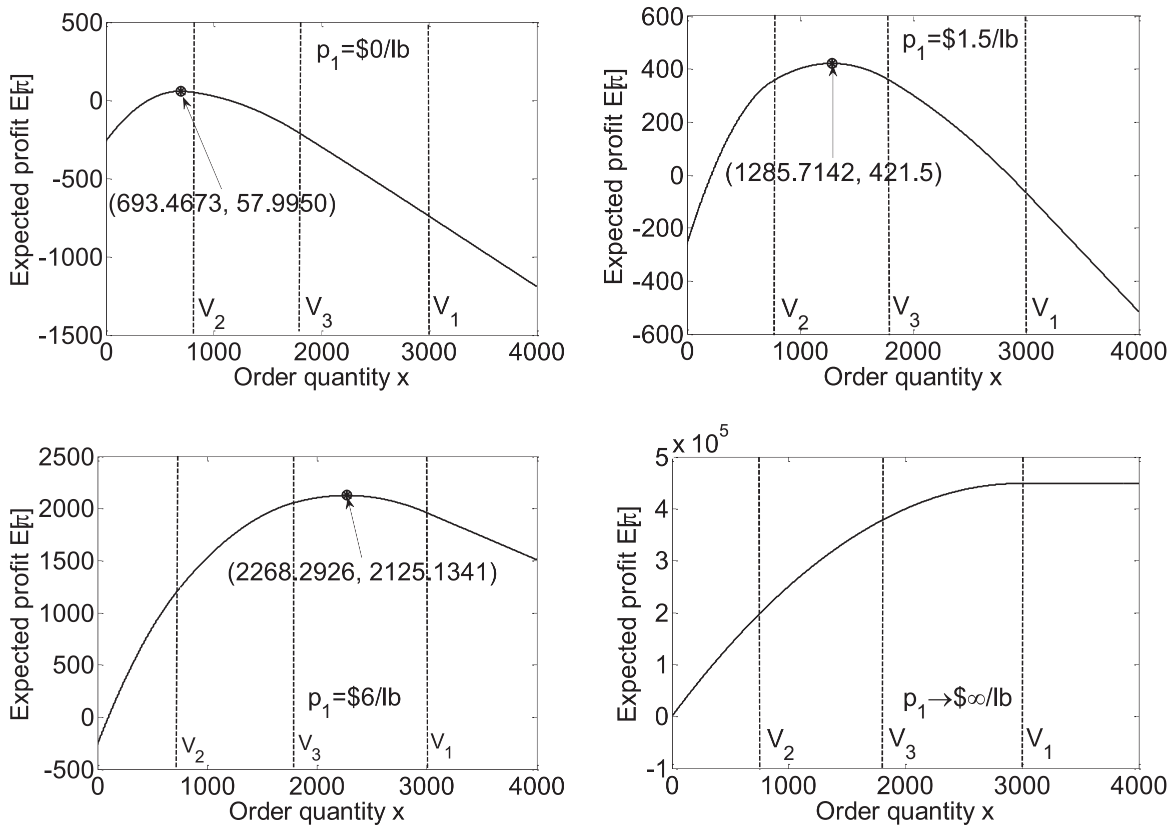

Figure 3: Profit functions with uniformly distributed demands and different p_1.

production process. Similar to the production capacity planning problem studied by Ho and Fang [26], the firm needs to allocate a favorable proportion of the raw material to different products with uncertain demands. The following lemma addresses the concavity for this problem, which allows us to develop an analytically tractable approach to determining the optimal AR.

Lemma 3. *Given the OP x, if $E[\pi]$ is twice differentiable with respective to w_i ($i \in \{1, 2, \ldots, N\}$), $E[\pi]$ is a concave function of the AR $\underline{W}$.*

Proof: The first and second derivatives of $E[\pi]$ with respective to w_i ($i \in \{1, 2, \ldots, N\}$) are

$$\begin{cases} \frac{\partial E[\pi]}{\partial w_i} = -\alpha_i x F_i(w_i x) + \beta_i x, \forall i, \\ \frac{\partial^2 E[\pi]}{\partial w_i^2} = -\alpha_i x^2 f_i(w_i x), \forall i. \end{cases} \quad (16)$$

We know that x, α_i, and $f_i(w_i x)$ are nonnegative, so the associated Hessian matrix satisfies

$$H(\underline{W}) = \begin{cases} \frac{\partial^2 E[\pi]}{\partial w_i^2} \leq 0, & \forall i, \\ \frac{\partial^2 E[\pi]}{\partial w_i \partial w_j} = 0, \forall i \neq j. \end{cases}$$

Obviously, $H(\underline{W})$ is a diagonal and negative semidefinite matrix. Therefore, $E[\pi]$ is concave in $\underline{W}$. This completes the proof. □

It is worth pointing out that the firm, depending on its preference, may not manufacture all the products in one period. Taking this into consideration, the problem becomes

$$\begin{aligned} \max_{\underline{W}} \quad & E[\pi] \\ \text{subject to:} \quad & \sum_{i=1}^{N} w_i = 1, \\ & w_i \geq 0 \ (i \in I_2), w_i = 0 \ (i \notin I_2), \end{aligned} \tag{17}$$

where $I_2 = \{i : \text{product } i \text{ is selected to process}\}$. For an extreme case, the firm only processes one product in a period, e.g., product 1, the profit function is

$$E[\pi] \ = \int_0^\infty \left(p_1(\xi_1 \wedge x) + s_1(x-\xi_1)^+ - b_1(\xi_1 - x)^+ - c_1 x\right) f_1(\xi_1) d\xi_1 - \sum_{k=2}^{N} b_k E[\xi_k].$$

Otherwise, more than one product will be produced. Let

$$L(\underline{W}, \lambda) = E[\pi] + \lambda \left(\sum_{i=1}^{N} w_i - 1 \right).$$

Then, Equation (17) can be converted to the following Lagrangian relaxation problem

$$\begin{aligned} \max_{\underline{W},\ \lambda} \quad & L(\underline{W}, \lambda) \\ \text{subject to:} \quad & w_i \geq 0 \ (i \in I_2), w_i = 0 \ (i \notin I_2), \end{aligned} \tag{18}$$

where λ is the corresponding Lagrangian multiplier. Note that Lagrangian relaxation is applied to solve constrained multi-item newsvendor problems in previous research, while the optimal solutions are usually obtained via heuristic methods (in which the optimality conditions are not explicitly derived). In this paper, to the best of our knowledge, we make the first attempt to provide an analytical solution to this type of problem. We show the equivalency of the problems given in Equations (17) and (18) in the next lemma.

Lemma 4. *For Equation (18), if a feasible solution $(w_1^*, w_2^*, \ldots, w_N^*, \lambda^*)$ satisfies the following condition*

$$\begin{cases} \frac{\partial L(\underline{W},\lambda)}{\partial w_1} = \frac{\partial L(\underline{W},\lambda)}{\partial w_2} =, \ldots, = \frac{\partial L(\underline{W},\lambda)}{\partial w_N} = \frac{\partial L(\underline{W},\lambda)}{\partial \lambda} = 0, \\ w_i \geq 0 \ (i \in I_2), w_i = 0 \ (i \notin I_2), \end{cases} \tag{19}$$

$\underline{W}^*=(w_1^*, w_2^*, \ldots, w_N^*)$ is the optimal AR of Equation (17).

Proof: For Equation (17), $E[\pi]$ is a concave function of $\underline{W}$ (see Lemma 3) and the constraint $\sum_{i=1}^{N} w_i - 1 = 0$ is a linear equation (which indicates that $\sum_{i=1}^{N} w_i - 1$ is a convex function of $\underline{W}$). According to Theorem 8 in Page 664 of Winston [27], the proof of this lemma is completed. □

As a result, one can obtain the exact optimal AR by simply solving the simultaneous linear/nonlinear equations in Equation (19), which does not rely on any approximation algorithm. However, the limitation of our approach is that we have to employ a numerical method to evaluate the optimal λ^* before computing the optimal AR. The following theorem provides the closed-form expressions for the optimal AR.

Theorem 3. *Suppose the order quantity x is predetermined and the inverse function of $F_i(\cdot)$ exists. If products $i \in I_2$ are selected, the corresponding optimal AR is*

$$\underline{W} = \begin{cases} w_i^* = \frac{1}{x} F_i^{-1}\left(\frac{\beta_i x + \lambda^*}{\alpha_i x}\right), i \in I_2, \\ w_i^* = 0, i \notin I_2. \end{cases} \tag{20}$$

Proof: If the firm selects a subset $I_2 \subset \{1, 2, \ldots, N\}$ of products to process, then the AR is restricted as $\underline{W} = \{w_i \geq 0 \ (i \in I_2), w_i = 0 \ (i \notin I_2)\}$. From Equation (19), we know the following relationships

$$\begin{cases} -\alpha_i x F_i(w_i x) + \beta_i x + \lambda = 0, i \in I_2, \\ \sum_{i=1}^{N} w_i - 1 = 0, \\ w_i \geq 0 \ (i \in I_2), w_i = 0 \ (i \notin I_2). \end{cases} \tag{21}$$

After some simple algebraic manipulation, one has $F_i(w_i x) = \frac{\beta_i x + \lambda}{\alpha_i x}$. If the inverse of $F_i(\cdot)$ exists, we have

$$w_i = \frac{1}{x} F_i^{-1} \left(\frac{\beta_i x + \lambda}{\alpha_i x} \right), i \in I_2.$$

In addition, $\frac{L(W,\lambda)}{\partial \lambda} = 0$ gives $\sum_{i=1}^{N} w_i = \sum_{i \in I_2} \frac{1}{x} F_i^{-1} \left(\frac{\beta_i}{\alpha_i} + \frac{\lambda}{\alpha_i x} \right) = 1$. Then, the value of λ^* can be determined by solving the following single-variable equation numerically

$$\sum_{i \in I_2} F_i^{-1} \left(\frac{\beta_i}{\alpha_i} + \frac{\lambda}{\alpha_i x} \right) = x.$$

Finally, the optimal AR can be found by substituting λ^* into Equation (20) and setting

$$w_i^* = 0 \ (i \notin I_2).$$

This completes the proof. □

Theorem 3 introduces an analytical approach to finding the optimal AR, which is more advance and accurate than the existing heuristics. For demonstration purposes, we first study a case where the firm manufactures all the products and the parameters and demand distributions are the same as that in Table 2. The predetermined x is changed within $\pm 50\%$ of the optimal one in Equation (9). The five cases in Table 5 ($I_2 = \{1, 2, 3\}$) illustrate the impact of the OP on the optimal AR. One can see that w_1^* increases as the value of x increases, while the profit increases from \$594.8021 to \$1776.3400 and then decreases to \$1439.9885 (note that Case 3 with check mark also provides the optimal solutions to Equation (9) and it is always better than the other fixed OP cases). At first, the amount of raw material is not enough for product 1 to generate enough profit to compensate the shortage costs of products 2 and 3. As a result, it is better for the firm to allocate more materials to the other two products. However, if more materials are ordered, the profit generated from product 1 becomes higher. Thus, the firm will allocate more materials to product 1, which shows its dominance. An interesting finding is that the ratio between w_2^* and w_3^* is around 0.56 for all these cases. The reason is that products 2 and 3 have similar profit structures, and the ratio reflects the relative magnitude of their mean demands ($\frac{\mu_1}{\mu_2} = \frac{300}{540} \approx 0.56$).

Table 5: Optimal AR based on the predetermined x and $I_2 = \{1, 2, 3\}$

Case	Predetermined OP x	Optimal $\underline{W}$ w_1^*	w_2^*	w_3^*	λ^*	Optimal Profit $E[\pi]^*$
1	800.9164	0.0344	0.3525	0.6131	-1041.1912	594.8021
2	1300.9164	0.4055	0.2170	0.3775	-1691.1914	1244.8022
3	1800.9164	0.5197	0.1708	0.3095	0.0000	1776.3400 ✓
4	2300.9164	0.6107	0.1381	0.2512	805.3206	1614.9885
5	2800.9164	0.6802	0.1134	0.2064	980.3206	1439.9885
6	3300.9164	0.7287	0.0962	0.1751	1155.3202	1264.9885
7	3800.9164	0.7643	0.0836	0.1521	1330.3199	1089.9885

Another interesting aspect is to examine the effect of the product price. To conduct the sensitivity analysis, we assume the three products have the same parameters as that of product 2 and remain their prices as ($p_1 = 1.5, p_2 = 1.6, p_3 = 1.8$). The following table shows the effect of product price on the optimal AR (we simply vary p_2 as an illustrative purpose). From Table 6, one can see that the proportion of raw material allocated to product 2 increases

Table 6: Effect of price on optimal AR with fixed optimal OP $x = 921.9238$

Case	Change of price p_2	Optimal $\underline{W}$ w_1^*	w_2^*	w_3^*	λ^*	Optimal Profit $E[\pi]^*$
1	1.4	0.3330	0.3324	0.3346	16.8684	851.5938 ✓
2	1.5	0.3328	0.3328	0.3344	8.1835	881.4095
3	1.6	0.3326	0.3332	0.3342	0.0000	911.2348 ✓
4	1.7	0.3324	0.3335	0.3340	-7.7391	941.0686
5	1.8	0.3322	0.3339	0.3339	-15.0750	970.9102
6	10.8	0.3272	0.3434	0.3294	-251.2692	3665.2679
7	100.8	0.3219	0.3531	0.3248	-517.1964	30657.1670

as p_2 increases. The reason is the profitability of product 2 is improved. Especially, when $p_2 = p_1 = 1.5$ (or $p_2 = p_3 = 1.8$), the proportions allocated to products 1 and 2 (or products 2 and 3) are the same, because they have similar profitability. Hence, the optimal AR is closely linked to the profitability of each product.

Table 7: Optimal AR based on the predetermined x and $I_2 \subset \{1,2,3\}$

Case	I_2	Predetermined OP $x\left[x_i = F_i^{-1}\left(\frac{\beta_i}{\alpha_i}\right)\right]$	Optimal $\underline{W}$				Optimal Profit $E[\pi]^*$
			w_1^*	w_2^*	w_3^*	λ^*	
1	$\{1,2,3\}$	1800.9164	0.5197	0.1708	0.3095	0.0000	1776.3400 ✓
2	$\{1,2\}$	1800.9164	0.8236	0.1764	0	630.3207	855.4463
		1800.9164-x_1	0.6736	0.3264	0	-1124.4454	630.2121
		1800.9164-x_2	0.7873	0.2127	0	522.6415	963.1256
		1800.9164-x_3	0.7526	0.2474	0	0.0000	1040.1021✓
3	$\{1,3\}$	1800.9164	0.6790	0	0.3210	630.3206	1267.2214
		1800.9164-x_1	0.4323	0	0.5677	-1124.4451	650.2887
		1800.9164-x_2	0.6268	0	0.3732	0.0000	1362.7126✓
		1800.9164-x_3	0.6051	0	0.3949	-1615.6273	1142.5312
4	$\{2,3\}$	1800.9164	0	0.6730	0.3270	810.4124	215.4312
		1800.9164-x_1	0	0.3557	0.6443	0.0000	627.8654✓
		1800.9164-x_2	0	0.6056	0.3944	671.9676	353.8760
		1800.9164-x_3	0	0.5264	0.4736	559.6266	466.2175
5	$\{1\}$	1800.9164	1	0	0	630.3208	332.6792
		1800.9164-x_1-x_2	1	0	0	-724.4936	202.4935
		1800.9164-x_1-x_3	1	0	0	-399.9515	-122.0485
		1800.9164-x_2-x_3	1	0	0	0.0000	626.4746✓
6	$\{2\}$	1800.9164	0	1	0	810.4124	-777.4124
		1800.9164-x_1-x_2	0	1	0	250.7862	-217.7862
		1800.9164-x_1-x_3	0	1	0	0.0000	-108.3725✓
		1800.9164-x_2-x_3	0	1	0	421.1814	-388.1814
7	$\{3\}$	1800.9164	0	0	1	990.5041	-459.5041
		1800.9164-x_1-x_2	0	0	1	0.0000	214.2380✓
		1800.9164-x_1-x_3	0	0	1	-430.7170	-91.2830
		1800.9164-x_2-x_3	0	0	1	514.7773	16.2227

Different from Table 5, the analysis on the firm's product selection preference is presented in Table 7. Apparently, if a product is not selected, it will not make any contribution to the total profit, and the firm may face a shortage of this product. Suppose material procurement is reasonably scheduled. In the third column of Table 7, we show the predetermined OP for different cases of I_2 (note: the profit with check mark is the maximum one under each scenario). Because each product can contribute to the total profit, the firm achieves its highest profit when all of the products are selected ($I_2 = \{1,2,3\}$). If one of the products is not processed in the period, the profit may be lowered (see Cases 2-4). Especially, when only products 2 and 3 are selected (i.e., Case 4), the profit reaches a very low level since product 1 is indeed the most profitable one. The situation gets worse if the firm only makes one product in one period. The profit may even become negative due to the significant loss in the other two products. Case 6 indicates that the profit made from product 2 is not able to cover the losses of products 1 and 3, for which the firm is unable to generate profit.

5. Conclusions

In this paper, we investigate a flexible one-material-multi-product production system for agricultural companies. We examine the ordering policy (OP) and allocation rule (AR) of the raw material for a processor firm making multiple products. Different from the previous research, which relies on heuristics methods in the solution technique, the exact optimal solutions are provided for three practical scenarios. The properties of the solutions are also discussed via numerical experiments.

For the first scenario, the firm needs to simultaneously determine the OP and the AR of the raw material. Though the profit function is nonconcave, by investigating this problem from another perspective, we are able to derive a closed-form expression for the optimal solution. The result is intuitively appealing and computationally simple.

For the scenario where the AR is fixed, two cases are discussed for seeking the optimal OP. When the demands are unbounded with general distributions, the optimal OP can be obtained numerically. If the demands are uniformly distributed, the formulation for the profit function will vary with different OP. To address this problem, we develop an easy-to-follow procedure to obtain the exact optimal solution. Since the demands are bounded, numerical results show that the optimal OP may result in material wastage due to the dominance of the most profitable product.

Finally, in the third scenario, we propose a Lagrangian relaxation approach for the problem with a predetermined OP. Differing from previous heuristic-based research, we solve the optimal AR analytically, which fills a gap in the related literature. We first investigate the change of optimal AR by varying the value of OP or product price. The results indicate that when the amount of the OP or price increases, the firm should allocate a larger proportion of raw material to the high-profit product. In addition, if two products have similar profit structures, their material allocation ratio mainly depends on their mean demands. We also consider the firm's preference in product selection when not all the products must be produced. To study the effect of the AR with preference on the total profit, we consider three products as an example to analyze several possible cases. The results illustrate that, if the firm only produces one of the products, there is a high possibility that the firm will make a negative profit due to the significant losses of the other products.

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