INDEFINITE STEIN FILLINGS AND PIN(2)-MONOPOLE FLOER HOMOLOGY

FRANCESCO LIN

Abstract. We introduce techniques to study the topology of Stein fillings of a given contact three-manifold (Y,ξ) which are not negative definite. For example, given a spin^c rational homology sphere $(Y,\mathfrak{s})$ with $\mathfrak{s}$ self-conjugate such that the reduced monopole Floer homology group $HM_\bullet(Y,\mathfrak{s})$ has dimension one, we show that any Stein filling which is not negative definite has $b_2^+=1$ or 2, and b_2^- is determined in terms of the Frøyshov invariant. The proof of this uses Pin(2)-monopole Floer homology. More generally, we prove that analogous statements hold under certain assumptions on the contact invariant of ξ and its interaction with Pin(2)-symmetry. We also discuss consequences for finiteness questions about Stein fillings.

In the past twenty years, the topology of Stein fillings of contact three-manifolds has been a central topic of research at the intersection of symplectic geometry, gauge theory and mapping class groups (see [OS04a], [CE12] for introductions to the topic). One of the main goals is to classify the Stein fillings of a given contact three-manifold (Y,ξ), the prototypical example given by Eliashberg's celebrated result that any Stein filling of (S^3,ξ_{std}) is diffeomorphic to D^4 [Eli90]. Following that, the classification has been carried over in several other cases of interest, see for example [McD90], [Lis08], [Wen10].

In this paper, we focus on the less ambitious yet very challenging task of understanding the algebraic topology (e.g. the intersection form) of the Stein fillings of a given contact three-manifold (Y,ξ). While many classification results rely on the characterization of rational symplectic 4-manifolds [McD90], and therefore are mostly effective when dealing with negative definite Stein fillings, we will focus here on the case of indefinite fillings. We will discuss some applications toward this goal of monopole Floer homology [KM07], and in particular its Pin(2)-equivariant version defined in [Lin18]. The latter is a Floer theoretic analogue of the invariants introduced in [Man16], and can be used to provide an alternative (but formally identical) disproof of the Triangulation Conjecture.

Throughout the paper, all Floer homology groups will be taken with coefficients in $\mathbb{F}$, the field with two elements, and we will focus for simplicity on the case of a rational homology sphere Y. To give some context, recall that for a given spin^c structure $\mathfrak{s}$, if the reduced Floer homology group $HM_\bullet(Y,\mathfrak{s})$ vanishes then all Stein fillings of contact structures (Y,ξ) for which $\mathfrak{s}_\xi=\mathfrak{s}$ have $b_2^+=0$ [OS04b], [Ech18].

We focus first on the next simplest case in which $HM_\bullet(Y,\mathfrak{s})=\mathbb{F}$ has dimension one, under the additional assumption that $\mathfrak{s}$ is self-conjugate, i.e. $\mathfrak{s}=\bar{\mathfrak{s}}$ (or, equivalently, $\mathfrak{s}$ is induced by a spin structure). This in turn implies (see Section 1 below) that either

$$\widecheck{HM}_\bullet(Y,\mathfrak{s})=\mathcal{T}^+_{-2h(Y)}\oplus\mathbb{F}_{-2h(Y)}$$
$$\widecheck{HM}_\bullet(Y,\mathfrak{s})=\mathcal{T}^+_{-2h(Y)}\oplus\mathbb{F}_{-2h(Y)-1},$$

where $h(Y,\mathfrak{s})$ is the Frøyshov invariant of Y. Here $\mathcal{T}^+$ is the $\mathbb{F}[[U]]$-module $\mathbb{F}[[U,U^{-1}]/U\cdot\mathbb{F}[[U]]$ (to which we refer as the tower), and the indices determine the grading of the lowest degree element. We call these two cases *Type* I and *Type* II respectively. By Poincaré duality, if $(Y,\mathfrak{s})$ has Type I then $(-Y,\mathfrak{s})$ has Type II, and viceversa.

Remark 1. For the reader more accustomed with Heegaard Floer homology, let us point out that under the isomorphism between the theories ([Tau10],[CGH11] or [KLT11] and subsequent papers) we have $HM_\bullet(Y,\mathfrak{s})\equiv HF^{red}_*(Y,\mathfrak{s})$ and $d(Y,\mathfrak{s})=-2h(Y,\mathfrak{s})$.

We have the following.

Theorem 1. *Let (Y,ξ) with $\mathfrak{s}_\xi=\mathfrak{s}$ self-conjugate, and suppose $HM_\bullet(Y,\mathfrak{s})=\mathbb{F}$. The following hold:*

- *if $(Y,\mathfrak{s})$ has Type I, and (W,ω) is a Stein filling of (Y,ξ) which is **not** negative definite, then W has even intersection form and $b_2^+=1$ and $b_2^-=-8h(Y,\mathfrak{s})+9$.*
- *if $(Y,\mathfrak{s})$ has Type II, and (W,ω) is a Stein filling of (Y,ξ) which is **not** negative definite, then W has even intersection form and $b_2^+=2$ and $b_2^-=-8h(Y,\mathfrak{s})+10$.*

There are plenty of rational homology spheres satisfying the hypotheses of Theorem 1, including among the others many Seifert fibered spaces. Notice that when Y is an integral homology sphere, our theorem fixes the intersection form of an indefinite Stein filling completely: this follows from the classification of even indefinite unimodular lattices (Theorem 1.2.21 in [GS99]), and the fact that the self-conjugacy requirement is trivial. In particular, in the Type I case the intersection form is $H\oplus E_8^{1-h(Y)}$, while in the Type II case it is $2H\oplus E_8^{1-h(Y)}$. With this in mind, our result should be thought as a generalization of the results in [Sti02] that classify the intersection forms of Stein fillings of the Brieskorn sphere $\pm\Sigma(2,3,11)$; this has Type II (resp. Type I) and $h=\mp 1$. In particular, it is shown in [Sti02] that a non-negative definite filling of $-\Sigma(2,3,11)$ has intersection form H, while a non-negative definite filling of $\Sigma(2,3,11)$ has intersection form $2H\oplus 2E_8$. While the key input of [Sti02] is the existence of a nice embedding of $-\Sigma(2,3,11)$ inside the $K3$ surface as the boundary of the nucleus $N(2)$, our proof is purely Floer theoretic, hence it works when there are no explicit topological inputs.

Remark 2. The canonical contact structure on $\Sigma(2,3,11)$ also admits a negative definite Stein filling, namely the (deformation of the) resolution of the singularity $\{x^2+y^3+z^{11}=0\}\subset\mathbb{C}^3$. An example of indefinite Stein filling is given by the Milnor fiber of this singularity.

The Brieskorn sphere with opposite orientation $-\Sigma(2,3,7)$ has Type I with $h=0$; therefore all indefinite fillings have intersection form $H\oplus E_8$. We can generalize this example to all integral surgeries on the figure-eight knot $M(n)$ (where $-\Sigma(2,3,7)=M(-1)$), all of which have $HM_\bullet(M(n))=\mathbb{F}$ supported in a self-conjugate spin^c structure [OS03]. These manifolds are hyperbolic for $|n|\geqslant 5$ [Mar16]. Contact structures on these manifolds, and their fillability properties, have been thoroughly studied in [CM19]. For example, the authors construct for $n=-1,\dots,-9$ a Stein fillable contact structure ξ on $M(n)$ for which $\mathfrak{s}_\xi$ is self-conjugate (as it has non-vanishing reduced contact invariant). For negative n, $(Y,\mathfrak{s}_\xi)$ has Type I and $h=-\frac{n+1}{8}$, so that our main result implies that any indefinite Stein filling of $(M(n),\xi)$ has $b_2^+=1$ and $b_2^-=10+n$. We also refer the reader to [Bal08] for more examples obtained by analyzing genus one, one boundary component open books where the result applies.

Our first theorem has consequences regarding finiteness questions for Stein fillings. In particular, it was conjectured (Conjecture 12.3.16 of [OS04a]) that given a contact manifold (Y, ξ), the Euler characteristic of its Stein fillings can only attain finitely many values. While this was later shown to be false [BVHM15], it is still of great interest to understand for which classes of contact three-manifolds this finiteness condition holds. On our end, we have the following.

Corollary 2. *Given a contact rational homology sphere (Y, ξ), suppose $\mathfrak{s}_\xi = \mathfrak{s}$ is self-conjugate and $HM_\bullet(Y, \mathfrak{s})$ has dimension one. Then the set of Euler characteristics of the Stein fillings of (Y, ξ) is finite.*

Remark 3. While the examples in [BVHM15] are not rational homology spheres, it was pointed out by Stipsicz that their examples can be modified to show that the boundedness property does not hold for this restricted class. To see this, one just need to attach a fixed Stein cobordism from the contact manifolds considered in [BVHM15] to a rational homology sphere; this is easily constructed via surgery.

Even though the statement of Theorem 1 itself does not involve Pin(2)-monopole Floer homology [Lin18], its proof relies on it. Indeed, Pin(2)-symmetry has been already fruitfully exploited to study the topology of closed symplectic 4-manifolds [Bau08], [Li06], and our result can be thought as an analogue of those in the case of Stein fillings. The key observation is that the bar version of Pin(2)-monopole Floer homology $\overline{HS}_\bullet$, unlike its standard counterpart $\overline{HM}_\bullet$, is non-trivial for cobordisms with small positive b_2^+ [Lin17]; this in turn is a version for cobordisms of the classical Donaldson Theorems B and C [DK90].

More generally, we can provide obstructions on the intersection form of the indefinite Stein fillings of (Y, ξ) in terms of the interaction with Pin(2)-monopole Floer homology of the contact invariant $\mathbf{c}(\xi) \in \widecheck{HM}_\bullet(-Y, \mathfrak{s}_\xi)$ defined in [KMOS07]; as the precise statements require some additional background and terminology (which we review in Section 1), we postpone them to Section 3. Let us point out that the obstructions we provide are applicable in concrete situations where the reduced Floer homology $\widecheck{HM}_\bullet(-Y, \mathfrak{s}_\xi)$ has arbitrarily large dimension.

Theorem 3. *There exists a sequence (Y_i, ξ_i), $i \geqslant 1$, of contact rational homology three-spheres such that:*

- *any indefinite Stein filling of (Y_i, ξ_i) has intersection form $2H \oplus E_8$;*
- *the dimension of $HM_\bullet(-Y_i, \mathfrak{s}_{\xi_i})$ tends to infinity; if fact, the image of the contact element $j_*\mathbf{c}(\xi) \in HM_\bullet(-Y_i, \mathfrak{s}_{\xi_i})$ in the reduced group is a non-zero element in the image of U^{2i}.*

As a consequence, for each $i \geqslant 1$ the set of the Euler characteristics of Stein fillings of (Y_i, ξ_i) is finite.

Remark 4. We emphasize in the statement of the result that $j_*\mathbf{c}(\xi)$ is non-zero, as when $j_*\mathbf{c}(\xi) = 0$ all Stein fillings are negative definite [OS04b], [Ech18].

Acknowledgements. The author would like to thank András Stipsicz for sharing his expertise on the subject. This work was partially funded by NSF grant DMS-1807242.

1. Pin(2)-monopole Floer homology

We will review some background on the Pin(2)-monopole Floer homology groups defined in [Lin18]. We will assume the reader to be familiar with the formal properties of usual monopole Floer homology [KM07]; for a gentle introduction, we refer the reader to [Lin16]. Consider $(Y,\mathfrak{s})$ a rational homology sphere equipped with a self-conjugate spin^c structure. The Seiberg-Witten equations then admit a Pin(2)-symmetry, where

$$\mathrm{Pin}(2) = S^1 \cup j\cdot S^1 \subset \mathbb{H}.$$

In particular, the action of j induces an involution $\jmath$ on the moduli spaces of configurations $\mathcal{B}(Y,\mathfrak{s})$. One can then exploit this extra symmetry to provide refined Floer theoretic invariants

$$\cdots \xrightarrow{i_*} \widecheck{HS}_\bullet(Y,\mathfrak{s}) \xrightarrow{j_*} \widehat{HS}_\bullet(Y,\mathfrak{s}) \xrightarrow{p_*} \overline{HS}_\bullet(Y,\mathfrak{s}) \xrightarrow{i_*} \dots \tag{1}$$

which are read respectively *H-S-to*, *H-S-from* and *H-S-bar*. These are also absolutely graded modules over the graded ring

$$\mathcal{R} = \mathbb{F}[[V]][Q]/(Q^3)$$

where V and Q have degree respectively -4 and -1. We will be mostly interested in the $\widehat{HS}_\bullet(Y,\mathfrak{s})$ version. This invariant fits in the Gysin exact sequence

$$\begin{tikzcd}
\widehat{HS}_\bullet(Y,\mathfrak{s}) \arrow[rr, "\cdot Q"] & & \widehat{HS}_\bullet(Y,\mathfrak{s}) \arrow[dl, "\iota"] \\
& \widehat{HM}_\bullet(Y,\mathfrak{s}) \arrow[ul, "\pi_*"] &
\end{tikzcd}$$

where π_* and ι have degree 0. This is an exact sequence of $\mathcal{R}$-modules, where the $\mathcal{R}$-action on $\widehat{HM}_\bullet(Y,\mathfrak{s})$ is obtained by sending Q to 0 and V to U^2. In the case $Y = S^3$, we have $\widehat{HS}_\bullet(S^3) \equiv \mathcal{R}\langle -1\rangle$ and this fits together with $\widehat{HM}_\bullet(S^3) = \mathbb{F}[[U]]\langle -1\rangle$ in gradings between $-4k$ and $-4k-3$ for $k \geqslant 0$ as follows:

$$\begin{tikzcd}
\cdot & \cdot & \cdot \\
\mathbb{F} \arrow[r] & \mathbb{F} & \mathbb{F} \arrow[lld] \\
\mathbb{F} & \cdot & \mathbb{F} \arrow[lld] \\
\mathbb{F} & \mathbb{F} \arrow[r] & \mathbb{F}
\end{tikzcd}$$

Here the three columns represent (left to right) $\widehat{HS}_\bullet$, $\widehat{HM}_\bullet$ and $\widehat{HS}_\bullet$ respectively, and the horizontal maps are ι and π_*.

In general, we always have for a rational homology sphere that, up to grading shift,

$$\overline{HS}_\bullet(Y,\mathfrak{s}) \equiv \mathbb{F}[[V,V^{-1}]][Q]/(Q^3), \tag{2}$$

and that p_* is an isomorphism in degrees low enough and zero in degrees high enough.

Using the Gysin exact sequence, one can show the following [HKL17].

Theorem 4. *[Corollary 2 of [HKL17]] Suppose $\mathfrak{s}$ is self-conjugate, and $HM_\bullet(Y,\mathfrak{s}) = 1$. Then the extra $\mathbb{F}$ summand is in degree either $-2h(Y,\mathfrak{s})$ or $-2h(Y,\mathfrak{s})-1$.*

Accordingly, we have that one of the two alternatives

$$\widehat{HM}_\bullet(Y,\mathfrak{s}) = \mathbb{F}[[U]]_{-2h(Y,\mathfrak{s})-1} \oplus \mathbb{F}_{-2h(Y,\mathfrak{s})}$$
$$\widehat{HM}_\bullet(Y,\mathfrak{s}) = \mathbb{F}[[U]]_{-2h(Y,\mathfrak{s})-1} \oplus \mathbb{F}_{-2h(Y,\mathfrak{s})-1}$$

holds, corresponding to our definitions of Type I and II. When $(Y,\mathfrak{s})$ has Type I, we have the identification as $\mathbb{F}[[V]]$-modules

$$\widehat{HS}_\bullet(Y,\mathfrak{s}) = \mathbb{F}[[V]]\langle -2h-3\rangle \oplus \mathbb{F}[[V]]\langle -2h\rangle \oplus \mathbb{F}[[V]]\langle -2h-1\rangle$$

and the action of Q sends each column injectively into the next one. The corresponding high degree part of the Gysin exact sequence looks like

$$\begin{tikzcd}
\mathbb{F} \arrow[r] & \mathbb{F} & \mathbb{F} \arrow[dll] \\
\mathbb{F} & \mathbb{F} \arrow[r] & \mathbb{F} \\
\cdot & \cdot & \cdot \\
\mathbb{F} \arrow[r] & \mathbb{F} & \mathbb{F} \arrow[dll] \\
\mathbb{F} & \cdot & \mathbb{F} \arrow[dll] \\
\mathbb{F} & \mathbb{F} \arrow[r] & \mathbb{F}
\end{tikzcd}$$

When $(Y,\mathfrak{s})$ has Type II, we have the identification as $\mathbb{F}[[V]]$-modules

$$\widehat{HS}_\bullet(Y,\mathfrak{s}) = \mathbb{F}[[V]]\langle -2h-3\rangle \oplus \mathbb{F}[[V]]\langle -2h-4\rangle \oplus \mathbb{F}[[V]]\langle -2h-1\rangle$$

and the action of Q sends each column injectively into the next one. The corresponding high degree part of the Gysin exact sequence looks like

$$\begin{tikzcd}
\mathbb{F} \arrow[r] & \mathbb{F}\oplus\mathbb{F} \arrow[r] \arrow[dd, dotted, bend left] & \mathbb{F} \\
\cdot & \cdot & \cdot \\
\mathbb{F} \arrow[r] & \mathbb{F} \arrow[dd, dotted, bend left] & \mathbb{F} \arrow[dll] \\
\mathbb{F} & \cdot & \mathbb{F} \arrow[dll] \\
\mathbb{F} & \mathbb{F} \arrow[r] & \mathbb{F}
\end{tikzcd}$$

Here, in the middle column we have highlighted the U-action on the $\mathbb{F}[[U]]$-summand using dotted arrows. We record the following observations.

Lemma 5. *Suppose $HM_\bullet(Y,\mathfrak{s}) = \mathbb{F}$. Then the map p_* in* Pin(2)*-monopole Floer homology is injective. Furthermore, the generator of $HM_\bullet(Y,\mathfrak{s})$ is in the image of ι.*

A cobordism $(W,\mathfrak{s})$ with $\mathfrak{s} = \bar{\mathfrak{s}}$ induces a map on the corresponding Pin(2)-monopole Floer homology groups; this fits in a natural commutative diagram with the Gysin exact sequence and the map induced on the usual Floer homology groups. The key observation underlying our results is the following.

Proposition 6. *Given two rational homology spheres $(Y_0, \mathfrak{s}_0)$ and $(Y_1, \mathfrak{s}_1)$ equipped with self-conjugate spin^c structures, and consider a cobordism $(W, \mathfrak{s})$ with $\mathfrak{s}$ self-conjugate and $b_1 = 0$ between them. Consider the induced map*

$$\overline{HS}_\bullet(W, \mathfrak{s}) : \overline{HS}_\bullet(Y_0, \mathfrak{s}_0) \to \overline{HS}_\bullet(Y_1, \mathfrak{s}_1),$$

where we can identify both groups up to grading shift as in Equation (2). Then:

- *if $b_2^+ = 0$, $\overline{HS}_\bullet(W, \mathfrak{s})$ acts as multiplication by V^k for some k (so it is an isomorphism);*
- *if $b_2^+ = 1$, $\overline{HS}_\bullet(W, \mathfrak{s})$ acts as multiplication by QV^k for some k;*
- *if $b_2^+ = 2$, $\overline{HS}_\bullet(W, \mathfrak{s})$ acts as multiplication by Q^2V^k for some k;*
- *if $b_2^+ \geqslant 3$, $\overline{HS}_\bullet(W, \mathfrak{s})$ vanishes.*

In all cases, the value of k is determined by the grading shift, which is given by the formula $\frac{1}{4}(5b_2^+ - b_2^-)$.

Graphically, the cases $b_2^+ = 0, 1, 2$ look like

$$\begin{tikzcd} \mathbb{F} \arrow[r] & \mathbb{F} \\ \mathbb{F} \arrow[r] & \mathbb{F} \\ \mathbb{F} \arrow[r] & \mathbb{F} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{F} \arrow[dr] & \mathbb{F} \\ \mathbb{F} \arrow[dr] & \mathbb{F} \\ \mathbb{F} & \mathbb{F} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{F} \arrow[ddr] & \mathbb{F} \\ \mathbb{F} & \mathbb{F} \\ \mathbb{F} & \mathbb{F} \end{tikzcd}$$

after a suitable grading shift.

Proof. The first three bullets were proved in [Lin18] and [Lin17]. Let us recall the basic idea behind their proof. In the case $b_2^+ = 0$, the relevant reducible moduli spaces are transversely cut out and consist of a copy of $\mathbb{C}P^1$. In the case $b_2^+ = 1$, before adding perturbations in the blow-up, the moduli space of solutions is again $\mathbb{C}P^1$ but in this case the linearization has one dimensional cokernel; adding an extra perturbation in the blow-up corresponds to choosing a generic equivariant map

$$s : \mathbb{C}P^1 \to \mathbb{R}$$

where the action is the antipodal on the source and $x \mapsto -x$ on the target. The zero set $s^{-1}(0)$ is a generator of the one dimensional homology of $\mathbb{C}P^1/\jmath = \mathbb{R}P^2$, which corresponds to the claimed computation of multiplication by Q. Notice that a choice of such an equivariant section is equivalent to choosing a section of the tautological line bundle over $\mathbb{R}P^2$. Similarly, when $b_2^+ = 2$ we are dealing with equivariant maps $\mathbb{C}P^1 \to \mathbb{R}^2$; here the zero set consists of $4N + 2$ points, which again descends to a generator of the zero dimensional homology of $\mathbb{R}P^2$.

Now, if $b_2^+ = 4k + 3$, the map is zero for grading reasons. Let us focus on the case $b_2^+ = 4k + 2$. In this case, before adding perturbations the relevant moduli space of reducible solutions is a copy of $\mathbb{C}P^{2k+1}$, and the cokernel has real dimension $4k + 2$. The perturbations in the blow-up correspond to equivariant maps

$$\mathbb{C}P^{2k+1} \to \mathbb{R}^{4k+2},$$

where the action on $\mathbb{C}P^{2k+1}$ is given by

$$[z_1 : w_1 : \cdots : z_{k+1} : w_{k+1}] \mapsto [-\bar{w}_1 : \bar{z}_1 : \cdots : -\bar{w}_{k+1} : \bar{z}_{k+1}]$$

and is again $x \mapsto -x$ on the target. We are interested in the zero locus of such map. On the other hand, when $k \geqslant 1$ any such map is homotopic to a map with no zeroes, because there is an equivariant map

$$\mathbb{C}P^{2k+1} \to \mathbb{R}^{3k+3}$$

which is never zero. This is obtained by taking $k+1$-copies of the standard squaring map

$$\mathbb{C}^2 \to \mathbb{R}^3$$
$$(\alpha, \beta) \mapsto \left(\frac{1}{2}(|\alpha|^2 - |\beta|^2), \mathrm{Re}(\alpha, \bar{\beta}), \mathrm{Im}(\alpha\bar{\beta})\right),$$

restricting to the unit sphere in $\mathbb{C}^{2k+2}$, and modding out by the S^1-action. This implies that the induced map vanishes. Finally, the cases $b_2^+ = 4k$ or $4k+1$ follow from this by stabilizing with an equivariant map with values in $\mathbb{R}^2$ and $\mathbb{R}$ respectively. □

2. Proofs

We begin by discussing some results about Floer theoretic properties of Stein cobordisms; these are essentially obtained by rephrasing the results in [Sti02] in the language of monopole Floer homology. The key observation here is that the assumption that the Stein filling is *not* negative definite allows us to apply the gluing properties of the Seiberg-Witten invariants; we refer the reader to Chapter 3 of [KM07] for a thorough discussion of the formal picture, and briefly discuss here what we will need.

Suppose we are given a closed oriented 4-manifold X given as the union $X = X_1 \cup_Y X_2$ along a hypersurface Y. We will assume that $b_2^+(X_i) \geqslant 1$ and Y is a rational homology sphere. The latter assumption assures that a spin^c structure $\mathfrak{s}_X$ on X is determined by the two restrictions $\mathfrak{s}_i = \mathfrak{s}_X|_{X_i}$. We will write $\mathfrak{s}_X = \mathfrak{s}_1 \cup_Y \mathfrak{s}_2$, and denote by $\mathfrak{s}$ the restriction to Y. As $b_2^+(X_i) \geqslant 1$, we obtain relative invariants

$$\psi_{(X_1,\mathfrak{s}_1)} \in HM_\bullet(Y, \mathfrak{s})$$
$$\psi_{(X_2,\mathfrak{s}_2)} \in HM_\bullet(-Y, \mathfrak{s})$$

where we can identify the last group with $HM^\bullet(Y, \mathfrak{s})$ using Poincaré duality. The pairing theorem for the Seiberg-Witten invariants $\mathfrak{m}(X, \mathfrak{s})$ says that

$$\mathfrak{m}(X, \mathfrak{s}_1 \cup_Y \mathfrak{s}_2) = \langle \psi_{(X_1,\mathfrak{s}_1)}, \psi_{(X_2,\mathfrak{s}_2)} \rangle \qquad (\mathrm{mod}\ 2)$$

where

$$\langle \cdot, \cdot \rangle : HM_\bullet(Y, \mathfrak{s}) \otimes HM^\bullet(Y, \mathfrak{s}) \to \mathbb{F}$$

is the duality pairing. The relative invariants $\psi_{(X_i,\mathfrak{s}_i)}$ can be interpreted in terms of cobordism maps; for example $\psi_{(X_1,\mathfrak{s}_1)}$ is the image under

$$\widehat{HM}_\bullet(X_1 \backslash B^4, \mathfrak{s}_1) \to \widehat{HM}_\bullet(S^3) \to \widehat{HM}_\bullet(Y, \mathfrak{s})$$

of $1 \in \mathbb{F}[[U]] = \widehat{HM}_\bullet(S^3)$; under our assumption $b_2^+(X_1) \geqslant 1$, the image lies in $HM_\bullet(Y, \mathfrak{s}) = \ker(p_*) \subset \widehat{HM}_\bullet(Y, \mathfrak{s})$. With this in mind, we have the following.

Lemma 7. *Suppose (W, ω) is a Stein filling of (Y, ξ), and Y is rational homology sphere. Suppose that furthermore $b_2^+(W) > 0$, $\mathbf{s}_\xi = \mathbf{s}$ is self-conjugate and $HM_\bullet(Y, \mathfrak{s}) = \mathbb{F}$. Denoting by $\mathfrak{s}_\omega$ the canonical spin^c structure of ω, we have:*

- *the map induced by $(W \backslash B^4, \mathfrak{s}_\omega)$ from $\widehat{HM}_\bullet(S^3) = \mathbb{F}[[U]]$ to $\widehat{HM}_\bullet(Y, \mathfrak{s})$ maps 1 to the generator of $HM_\bullet(Y, \mathfrak{s}) \subset \widehat{HM}_\bullet(Y, \mathfrak{s})$.*
- *$\mathfrak{s}_\omega$ is self-conjugate.*

Proof. The key input is that any Stein domain W can be embedded in Kähler surface X which is minimal of general type [LM97]; in fact, we can assume $b_2^+(X\backslash W) \geqslant 1$ and that $X\backslash W$ is not spin [Sti02]. To see why the last condition can be achieved, one can do surgery on W to create a new Stein domain W' containing a surface with odd self-intersection (hence not spin), and then apply the Lisca-Matic construction to W'. Now, we know that X has basic classes exactly the canonical classes $\pm K_X$, with Seiberg-Witten invariant ± 1 [Mor96]. The pairing theorem therefore implies that

$$\psi_{(W,\mathfrak{s}_\omega)} \in HM_\bullet(Y,\mathfrak{s}) = \mathbb{F}$$

is a non-zero element, and the first bullet follows.

Suppose now that $\mathfrak{s}_\omega \neq \bar{\mathfrak{s}}_\omega$. Denoting by $\mathfrak{s}'$ the canonical spin^c structure on $X\backslash W$, we have

$$K_X = \mathfrak{s}_\omega \cup_Y \mathfrak{s}' \quad \text{and} \quad -K_X = \bar{\mathfrak{s}}_\omega \cup_Y \bar{\mathfrak{s}}'.$$

Furthemore, as $X\backslash W$ is not spin, $\mathfrak{s}' \neq \bar{\mathfrak{s}}'$. Now, conjugation acts trivially on $HM(Y,\mathfrak{s}) = \mathbb{F}$, so by the pairing theorem $\bar{\mathfrak{s}}_\omega \cup_Y \mathfrak{s}'$ and $\mathfrak{s}_\omega \cup_Y \bar{\mathfrak{s}}'$ are also basic classes, which is a contradiction as $\pm K_X$ are the only basic classes. □

Lemma 8. *Suppose (W,ω) is a Stein filling of (Y,ξ), and Y is rational homology sphere. Then $b_1(W) = 0$.*

Proof. Topologically, any Stein domain is obtained from B^4 by attaching only 1 and 2-handles [OS04a]; in particular, the map $\pi_1(Y) \to \pi_1(W)$ is surjective, and the result follows. □

Proof of Theorem 1. Let us focus on the case of Type I, as the case of Type II is identical. Denote by $\mathbf{x}$ the generator of $HM_\bullet(Y,\mathfrak{s}) = \mathbb{F}$. By the previous lemma, the spin^c structure $\mathfrak{s}_\omega$ on W induced by ω is self-conjugate; in particular W is spin, hence it has even intersection form. We have the commutative diagram

$$\begin{tikzcd}
\widehat{HS}_\bullet(S^3) \arrow[r, "\iota_{S^3}"] \arrow[d, "\widehat{HS}_\bullet(W\backslash B^4,\mathfrak{s}_\omega)"'] & \widehat{HM}_\bullet(S^3) \arrow[d, "\widehat{HM}_\bullet(W\backslash B^4,\mathfrak{s}_\omega)"] \\
\widehat{HS}_\bullet(Y,\mathfrak{s}) \arrow[r, "\iota_Y"] & \widehat{HM}_\bullet(Y,\mathfrak{s})
\end{tikzcd}$$

By Lemma 7, $\widehat{HM}_\bullet(W\backslash B^4)$ maps 1 to $\mathbf{x}$; by commutativity of the diagram, we see that $\widehat{HS}_\bullet(W\backslash B^4)$ maps $1 \in \mathcal{R}$ to the top degree element of $\widehat{HS}_\bullet(Y,\mathfrak{s})$ (cfr. Lemma 5). Now, we have also the commutative diagram

$$\begin{tikzcd}
\widehat{HS}_\bullet(S^3) \arrow[r, "p_*"] \arrow[d, "\widehat{HS}_\bullet(W\backslash B^4,\mathfrak{s}_\omega)"'] & \overline{HS}_\bullet(S^3) \arrow[d, "\overline{HS}_\bullet(W\backslash B^4,\mathfrak{s}_\omega)"] \\
\widehat{HS}_\bullet(Y,\mathfrak{s}) \arrow[r, "p_*"] & \overline{HS}_\bullet(Y,\mathfrak{s})
\end{tikzcd}$$

As both horizontal maps p_* are injective, we see that $\overline{HS}_\bullet(W\backslash B^4)$ is given by multiplication by an element of the form QV^k; therefore Proposition 6 implies that $b_2^+ = 1$. Finally, as $\mathbf{x}$ has degree $-2h$ and 1 has degree -1, the map has grading shift $-2h(Y,\mathfrak{s}) + 1$, so we also readily obtain the formula for b_2^-. □

Proof of Corollary 2. Given a contact three-manifold (Y, ξ), there is a constant C (depending only on (Y, ξ)) such that for any Stein filling W we have $3\sigma(W)+2\chi(W) \geqslant C$ (Theorem 12.3.14 in [OS04a]). As in our case $b_1(W) = 0$, the latter inequality reads is $5b_2^+(W) \geqslant b_2^-(W) + C$. In particular an upper bound on $b_2^+(W)$ of fillings, as provided by Theorem 1, implies an upper bound $b_2^-(W)$, hence on the Euler characteristic. □

3. Generalizations involving the contact invariant

As mentioned in the introduction, our main result can be generalized in terms of how the contact invariant $\mathbf{c}(\xi) \in \widecheck{HM}_\bullet(-Y, \mathfrak{s}_\xi)$ defined in [KMOS07] interacts with Pin(2)-monopole Floer homology. Here we follow the notation of [Ech18], and we refer the reader to that paper for a pleasant introduction to the subject. For our purposes, we will only need the fact that the contact invariant of (S^3, ξ_{std}) is the bottom of the tower in $\widecheck{HM}_\bullet(-S^3)$, and the following naturality property.

Theorem 9 (Theorem 1 of [Ech18])**.** *Let (W, ω) be a strong symplectic cobordism between (Y, ξ) and (Y', ξ'), and let $\mathfrak{s}_\omega$ the corresponding spin^c structure on W. We have*

$$\mathbf{c}(\xi) = \widecheck{HM}_\bullet(W^\dagger, \mathfrak{s}_\omega)\mathbf{c}(\xi').$$

where $W^\dagger$ denotes the reversed cobordism from $-Y'$ to $-Y$,

In what follows, for a self-conjugate spin^c structure $\mathfrak{s}$ we will work as it is customary with the dual groups

$$\widecheck{HM}_\bullet(-Y, \mathfrak{s}) \equiv \widehat{HM}^\bullet(Y, \mathfrak{s})$$
$$\widecheck{HS}_\bullet(-Y, \mathfrak{s}) \equiv \widehat{HS}^\bullet(Y, \mathfrak{s}).$$

In this setup, recall that there are natural $\mathbb{F}[[V]]$-submodules of $\widecheck{HS}_\bullet(-Y, \mathfrak{s})$ defined as follows. Consider the map

$$i_* : \overline{HS}_\bullet(-Y, \mathfrak{s}) \to \widecheck{HS}_\bullet(-Y, \mathfrak{s}).$$

Here the first group can be identified with $\mathbb{F}[[V, V^{-1}][Q]/(Q^3)$, and the map is an isomorphism in degrees high enough and zero in degrees low enough. We then define the α, β and γ-towers to be respectively the images

$$i_*(Q^2 \cdot \mathbb{F}[[V, V^{-1}]), \quad i_*(Q \cdot \mathbb{F}[[V, V^{-1}]), \quad i_*(\mathbb{F}[[V, V^{-1}]).$$

Notice that the chain involution $\jmath$ of Pin(2)-monopole Floer homology induces an involution at the homology level $\jmath_*$ on $\overline{HM}_\bullet(-Y, \mathfrak{s})$ and $\widecheck{HM}_\bullet(-Y, \mathfrak{s})$; as i_* is equivariant with respect to these actions, $\jmath_*$ also induces an involution on $HM_\bullet(-Y, \mathfrak{s}) = \ker p_* = \mathrm{im} j_*$. As it will not cause confusion, in what follows we will also denote the image of $\mathbf{c}(\xi)$ in $HM_\bullet(-Y, \mathfrak{s})$ again by $\mathbf{c}(\xi)$.

Theorem 10. *Consider a contact rational homology sphere (Y, ξ) with $\mathfrak{s}_\xi = \mathfrak{s}$ self-conjugate. Assume that the contact invariant $\mathbf{c}(\xi) \in HM_\bullet(-Y, \mathfrak{s})$ is invariant under the action of $\jmath_*$, and denote its grading by $g(\xi)$. Consider*

$$\pi_*\mathbf{c}(\xi) \in \widecheck{HS}_\bullet(-Y, \mathfrak{s}).$$

*Suppose (W, ω) is a Stein filling which is **not** negative definite. Then:*

- *if $\pi_*\mathbf{c}(\xi)$ is a non-zero element of the β-tower, W has even intersection form, $b_2^+ = 1$ and $b_2^- = 5 - 4g(\xi)$.*
- *if $\pi_*\mathbf{c}(\xi)$ is a non-zero element of the γ-tower, W has even intersection form, $b_2^+ = 2$ and $b_2^- = 10 - 4g(\xi)$.*

This is a genuine generalization of Theorem 1; this is because, as the discussion below will show, under the assumptions of Theorem 1 the contact invariant $\mathbf{c}(\xi)$ is the generator of $HM_\bullet(-Y,\mathfrak{s}) = \mathbb{F}$. This is of course invariant under the action of $\jmath_*$, and furthermore when $(Y,\mathfrak{s})$ has Type I (resp. Type II), π_* maps it to the β (resp. γ) tower.

Remark 5. We have omitted in the statement the case $\pi_*\mathbf{c}(\xi)$ belongs to the α-tower; this is because, as the proof below would show, W would be negative definite.

As a consequence, we have the following finiteness result.

Corollary 11. *Suppose that a contact rational homology sphere (Y,ξ) with $\mathfrak{s}_\xi = \mathfrak{s}$ self-conjugate satisfies the following:*

- $\mathbf{c}(\xi) \in HM_\bullet(-Y,\mathfrak{s})$ *is invariant under the action of $\jmath_*$;*
- $\pi_*\mathbf{c}(\xi)$ *is a non-zero element in $i_*\left(\overline{HS}_\bullet(-Y,\mathfrak{s})\right)$.*

Then the set of Euler characteristics of Stein fillings of (Y,ξ) is finite.

We describe now a concrete example to which our result applies; up to reindexing, this family is the one in the statement of Theorem 3. We look for $n \geqslant 1$ at the Stein fillable contact structure ξ_n on the Brieskorn homology sphere $Y_n = \Sigma(2, 2n+1, 4n+3)$ constructed in [Kar14] by Legendrian surgery on a certain stabilized Legendrian $(2, 2n+1)$-torus knot. Provided $n > 1$, this manifold has $HM_\bullet$ of dimension strictly larger than 1. Let us focus on the case $n = 4k+1$ for $k \geqslant 1$. Following [Kar14], we see that for some $\mathbb{F}[[U]]$-module J,

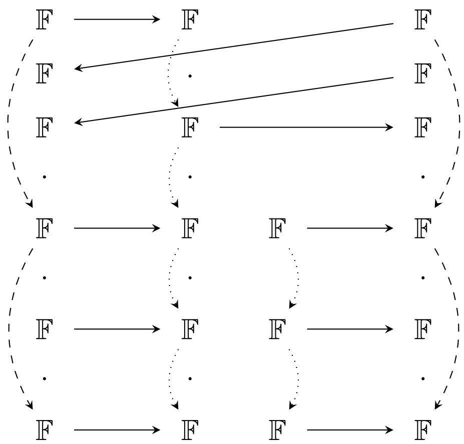
$$\widecheck{HM}_\bullet(-Y_{4k+1}) = \mathcal{T}_0^+ \oplus (\mathbb{F}[[U]]/U^{(2k+1)})_0 \oplus J^{\oplus 2}.$$

The computation of the Heegaard Floer contact invariant provided in [Kar14], together with its identification with the one in monopole Floer homology (see [Tau10], [CGH11] and subsequent papers), implies that $\mathbf{c}(\xi_{4k+1})$ is either $(0, U^{2k}, 0, 0)$ or $(1, U^{2k}, 0, 0)$ (i.e. either the bottom element of the second summand, or its sum with the bottom of the tower).

In this example, both the involution $\jmath_*$ and the map π_* are explicitly computable (see [Sto15], [Dai18] for a more general discussion of Seifert-fibered spaces):

- the action of $\jmath_*$ fixes $\mathcal{T}_0^+$, switches the two copies of J, and sends $(0, U^l, 0, 0)$ to $(U^{l-2k}, U^l, 0, 0)$. Consequently, the action of $\jmath_*$ on $HM_\bullet(-Y,\mathfrak{s})$ fixes the summand $\mathbb{F}[[U]]/U^{(2k+1)}$ and switches the two copies of J;
- π_* maps the contact invariant $\mathbf{c}(\xi_{4k+1})$ to the bottom of the γ-tower.

Below we depict graphically (omitting the $J^{\oplus 2}$ summand, which is irrelevant for our purposes) the Gysin exact sequence in the case in which $k = 1$.



In the picture, the relevant part of $\widecheck{HM}_\bullet$ is represented by the two middle columns, with the dotted arrows denoting the U-action. The dashed arrows represent the V-action on the γ-tower of $\widecheck{HS}_\bullet$. Our result then implies that any Stein filling of (Y_{4k+1}, ξ_{4k+1}) which is not negative definite is spin with $b_2^+ = 2$ and $b_2^- = 10$, hence has intersection form $2H \oplus E_8$.

As in the proof Theorem 1, an important step is to show the following.

Lemma 12. *Suppose $\mathbf{c}(\xi) \in HM_\bullet(-Y, \mathfrak{s})$ is $\jmath_*$ invariant. Then for any Stein filling (W, ω) of (Y, ξ) with $b_2^+ > 0$, $\mathfrak{s}_\omega$ is self-conjugate.*

Given this lemma (which we will prove later) we are ready to prove Theorem 10.

Proof of Theorem 10. Recall that the contact invariant of (S^3, ξ_{std}) is the bottom of the tower of $\widecheck{HM}_\bullet(-S^3, \xi_{std})$. By removing a standard ball around a point, we get a strong symplectic cobordism between (S^3, ξ_{std}) and (Y, ξ), so that by Theorem 9 the map $\widecheck{HM}_\bullet((W \backslash B^4)^\dagger, \mathfrak{s}_\omega)$ maps $\mathbf{c}(\xi)$ to the bottom of the tower of $\widecheck{HM}_\bullet(-S^3)$. Suppose W is not negative definite, so that by Lemma 12 the induced spin^c structure $\mathfrak{s}_\omega$ is self-conjugate. Then we get a commutative diagram of the form

$$\begin{tikzcd}
\widecheck{HM}_\bullet(-Y, \mathfrak{s}) \arrow[r, "\pi_{-Y}"] \arrow[d, "\widecheck{HM}_\bullet((W\backslash B^4)^\dagger, \mathfrak{s}_\omega)"'] & \widecheck{HS}_\bullet(-Y, \mathfrak{s}) \arrow[d, "\widecheck{HS}_\bullet((W\backslash B^4)^\dagger, \mathfrak{s}_\omega)"] \\
\widecheck{HM}_\bullet(-S^3) \arrow[r, "\pi_{-S^3}"] & \widecheck{HS}_\bullet(-S^3)
\end{tikzcd}$$

Here π_{-S^3} sends the bottom of the tower to the bottom of the α-tower. Focusing now on the first bullet of the theorem, this implies that $\widecheck{HS}_\bullet((W \backslash B^4)^\dagger, \mathfrak{s}_\omega)$ sends an element of the β-tower (i.e. $\pi_* \mathbf{c}(\xi)$) to the bottom of the α-tower; as both these elements are in the image of the respective map i_*, looking at the commutative diagram

$$
\begin{tikzcd}
\overline{HS}_{\bullet}(-Y,\mathfrak{s}) \arrow[r, "i_*"] \arrow[d, "\overline{HS}_{\bullet}((W\backslash B^4)^{\dagger},\mathfrak{s}_{\omega})"'] & \widecheck{HS}_{\bullet}(-Y,\mathfrak{s}) \arrow[d, "\widecheck{HS}_{\bullet}((W\backslash B^4)^{\dagger},\mathfrak{s}_{\omega})"] \\
\overline{HS}_{\bullet}(-S^3) \arrow[r, "i_*"] & \widecheck{HS}_{\bullet}(-S^3)
\end{tikzcd}
$$

we obtain from Proposition 6 that $b_2^+ = 1$, and furthermore b_2^- can be understood in terms of the grading shift. Finally, both the second bullet of the theorem and the observation in Remark 5 follow in the same way. $\square$

We conclude by proving Lemma 12, which is essentially a refinement of Lemma 7.

Proof of Lemma 12. Consider again the embedding $W \hookrightarrow X$ into a minimal Kähler surface of general type, with $V = X\backslash W$ not spin and $b_2^+(V) > 0$. This embedding is Kähler, so in particular V is a strong *concave* filling of (Y,ξ). This implies that $\mathbf{c}(\xi)$ is in fact the relative invariant of $(V^{\dagger},\mathfrak{s}_{\omega})$, i.e.

$$\mathbf{c}(\xi) = \overrightarrow{HM}_{\bullet}(V^{\dagger}\backslash B^4,\mathfrak{s}_{\omega})(1) \in HM_{\bullet}(-Y,\mathfrak{s}).$$

While this is not explicitly stated in [Ech18], the gluing results in the *dilating the cone* approach of discussed in his Section 2 applies directly to show a correspondence between moduli spaces on

$$(\mathbb{R}^+ \times \{-Y\}) \cup ([1,\infty) \times Y),$$

which are used to define the contact invariant $\mathbf{c}(\xi)$, and on

$$(\mathbb{R}^+ \times \{-Y\}) \cup V$$

where in the latter case we use a symplectic form for which the cone is dilated enough. In the latter case all solutions are irreducible, so they define a chain representing the relative invariant of the cobordism $V^{\dagger}$.

The lemma now follows as Lemma 7, using the fact that X has exactly two basic classes $\pm K_X$, the Seiberg-Witten invariants relative of to $\mathfrak{s}_1 \cup_Y \mathfrak{s}_2$ is the coefficient of $1 \in \widecheck{HM}_{\bullet}(S^3)$ in

$$\widecheck{HM}_{\bullet}((W\backslash B^4)^{\dagger},\mathfrak{s}_1)) \circ \overrightarrow{HM}_{\bullet}((V\backslash B^4)^{\dagger},\mathfrak{s}_2)(1),$$

and the fact that $\mathbf{c}(\xi)$ is invariant under the conjugation action (so that we also have $\mathbf{c}(\xi) = \overrightarrow{HM}_{\bullet}(V^{\dagger}\backslash B^4,\bar{\mathfrak{s}}_{\omega})(1)$). $\square$

Remark 6. A similar identification of the contact invariant as a relative invariant is proved in the setting of Heegaard Floer homology in [Pla04]; there the author discusses the specific case in which V is explicitly constructed in terms of the open book decomposition of (Y,ξ).

References

[Bal08] John A. Baldwin. Heegaard Floer homology and genus one, one-boundary component open books. *J. Topol.*, 1(4):963–992, 2008.

[Bau08] Stefan Bauer. Almost complex 4-manifolds with vanishing first Chern class. *J. Differential Geom.*, 79(1):25–32, 2008.

[BVHM15] R. İnanç Baykur and Jeremy Van Horn-Morris. Families of contact 3-manifolds with arbitrarily large Stein fillings. *J. Differential Geom.*, 101(3):423–465, 2015. With an appendix by Samuel Lisi and Chris Wendl.

[CE12] Kai Cieliebak and Yakov Eliashberg. *From Stein to Weinstein and back*, volume 59 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2012. Symplectic geometry of affine complex manifolds.

[CGH11] Vincent Colin, Paolo Ghiggini, and Ko Honda. Equivalence of Heegaard Floer homology and embedded contact homology via open book decompositions. *Proc. Natl. Acad. Sci. USA*, 108(20):8100–8105, 2011.

[CM19] James Conway and Hyunki Min. Classification of tight contact structures on surgeries on the figure-eight knot. *preprint*, 2019.

[Dai18] Irving Dai. On the Pin(2)-equivariant monopole Floer homology of plumbed 3-manifolds. *Michigan Math. J.*, 67(2):423–447, 2018.

[DK90] S. K. Donaldson and P. B. Kronheimer. *The geometry of four-manifolds*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1990. Oxford Science Publications.

[Ech18] Mariano Echeverria. Naturality of the contact invariant in monopole Floer homology under strong symplectic cobordisms. *preprint*, 2018.

[Eli90] Yakov Eliashberg. Filling by holomorphic discs and its applications. In *Geometry of low-dimensional manifolds, 2 (Durham, 1989)*, volume 151 of *London Math. Soc. Lecture Note Ser.*, pages 45–67. Cambridge Univ. Press, Cambridge, 1990.

[GS99] Robert E. Gompf and András I. Stipsicz. *4-manifolds and Kirby calculus*, volume 20 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1999.

[HKL17] Jonathan Hanselman, Cagatay Kutluhan, and Tye Lidman. A remark on the geography problem in Heegaard Floer homology. *preprint*, 2017.

[Kar14] ÇağrıKarakurt. Contact structures on plumbed 3-manifolds. *Kyoto J. Math.*, 54(2):271–294, 2014.

[KLT11] Cagatay Kutluhan, Yi-Jen Lee, and Clifford Taubes. HF=HM I : Heegaard Floer homology and Seiberg–Witten Floer homology. *preprint*, 2011.

[KM07] Peter Kronheimer and Tomasz Mrowka. *Monopoles and three-manifolds*, volume 10 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2007.

[KMOS07] P. Kronheimer, T. Mrowka, P. Ozsváth, and Z. Szabó. Monopoles and lens space surgeries. *Ann. of Math. (2)*, 165(2):457–546, 2007.

[Li06] Tian-Jun Li. Symplectic 4-manifolds with Kodaira dimension zero. *J. Differential Geom.*, 74(2):321–352, 2006.

[Lin16] Francesco Lin. Lectures on monopole Floer homology. In *Proceedings of the Gökova Geometry-Topology Conference 2015*, pages 39–80. Gökova Geometry/Topology Conference (GGT), Gökova, 2016.

[Lin17] Francesco Lin. The surgery exact triangle in Pin(2)-monopole Floer homology. *Algebr. Geom. Topol.*, 17(5):2915–2960, 2017.

[Lin18] Francesco Lin. A Morse-Bott approach to monopole Floer homology and the triangulation conjecture. *Mem. Amer. Math. Soc.*, 255(1221):v+162, 2018.

[Lis08] Paolo Lisca. On symplectic fillings of lens spaces. *Trans. Amer. Math. Soc.*, 360(2):765–799, 2008.

[LM97] P. Lisca and G. Matić. Tight contact structures and Seiberg-Witten invariants. *Invent. Math.*, 129(3):509–525, 1997.

[Man16] Ciprian Manolescu. Pin(2)-equivariant Seiberg-Witten Floer homology and the triangulation conjecture. *J. Amer. Math. Soc.*, 29(1):147–176, 2016.

[Mar16] Bruno Martelli. An introduction to Geometric Topology. *arXiv:math/1610.02592*, 2016.

[McD90] Dusa McDuff. The structure of rational and ruled symplectic 4-manifolds. *J. Amer. Math. Soc.*, 3(3):679–712, 1990.

[Mor96] John W. Morgan. *The Seiberg-Witten equations and applications to the topology of smooth four-manifolds*, volume 44 of *Mathematical Notes*. Princeton University Press, Princeton, NJ, 1996.

[OS03] Peter Ozsváth and Zoltán Szabó. Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary. *Adv. Math.*, 173(2):179–261, 2003.

[OS04a] Burak Ozbagci and András I. Stipsicz. *Surgery on contact 3-manifolds and Stein surfaces*, volume 13 of *Bolyai Society Mathematical Studies*. Springer-Verlag, Berlin; János Bolyai Mathematical Society, Budapest, 2004.

[OS04b] Peter Ozsváth and Zoltán Szabó. Holomorphic disks and genus bounds. *Geom. Topol.*, 8:311–334, 2004.

[Pla04] Olga Plamenevskaya. Contact structures with distinct Heegaard Floer invariants. *Math. Res. Lett.*, 11(4):547–561, 2004.

[Sti02] András I. Stipsicz. Gauge theory and Stein fillings of certain 3-manifolds. *Turkish J. Math.*, 26(1):115–130, 2002.

[Sto15] Matthew Stoffregen. Pin(2)-equivariant seiberg-witten floer homology of seifert fibrations. *preprint*, 2015.

[Tau10] Clifford Henry Taubes. Embedded contact homology and Seiberg-Witten Floer cohomology I. *Geom. Topol.*, 14(5):2497–2581, 2010.

[Wen10] Chris Wendl. Strongly fillable contact manifolds and J-holomorphic foliations. *Duke Math. J.*, 151(3):337–384, 2010.

Department of Mathematics, Columbia University
E-mail address: `flin@math.columbia.edu`