Singular Abreu Equations and Minimizers of Convex Functionals with a Convexity Constraint

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Abstract

We study the solvability of second boundary value problems of fourth-order equations of Abreu type arising from approximation of convex functionals whose Lagrangians depend on the gradient variable, subject to a convexity constraint. These functionals arise in different scientific disciplines such as Newton's problem of minimal resistance in physics and the monopolist's problem in economics. The right-hand sides of our Abreu-type equations are quasilinear expressions of second order; they are highly singular and a priori just measures. However, our analysis in particular shows that minimizers of the 2D Rochet-Choné model perturbed by a strictly convex lower-order term, under a convexity constraint, can be approximated in the uniform norm by solutions of the second boundary value problems of singular Abreu equations. © 2019 Wiley Periodicals, Inc.

1 Introduction

In this paper, we study the solvability and convergence properties of second boundary value problems of fourth-order equations of Abreu type arising from approximation of several convex functionals whose Lagrangians depend on the gradient variable, subject to a convexity constraint. Our analysis in particular shows that minimizers of the 2D Rochet-Choné model perturbed by a strictly convex lower-order term, under a convexity constraint, can be approximated in the uniform norm by solutions of the second boundary value problems of Abreu-type equations. An intriguing feature of our Abreu-type equations is that their right-hand sides are quasilinear expressions of second-order derivatives of a convex function. As such, they are highly singular and a priori just measures. The main results consist of Theorems 2.1, 2.3, 2.6, and 2.8 to be precisely stated in Section 2. In the following paragraphs, we motivate the problems to be studied and recall some previous results in the literature.

Let Ω_0 be a bounded, open, smooth, and convex domain in $\mathbb{R}^n$ ($n \geq 2$). Let $F(x, z, p) : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a smooth function that is convex in each of the variables $z \in \mathbb{R}$ and $p = (p_1, \ldots, p_n) \in \mathbb{R}^n$. Let φ be a convex and smooth function defined in a neighborhood of Ω_0. In several problems in different scientific disciplines such as Newton's problem of minimal resistance in physics and the monopolist's problem in economics (see, for example, [3, 6, 26]), one usually

Communications on Pure and Applied Mathematics, 0001–0036 (PREPRINT)

encounters the following variational problem with a convexity constraint:

$$\inf_{u\in \bar{S}[\varphi,\Omega_0]}\int_{\Omega_0} F(x,u(x),Du(x))dx \tag{1.1}$$

where

$$\bar{S}[\varphi,\Omega_0] = \{u:\Omega_0\to\mathbb{R} \mid u \text{ is convex, } \text{u admits a convex extension } \varphi \text{ in a neighborhood of } \Omega_0\}. \tag{1.2}$$

Due to the convexity constraint, it is in general difficult to write down a tractable Euler-Lagrange equation for the minimizers of (1.1) [5, 7, 23]. Lions [23] showed that, in the sense of distributions, the Euler-Lagrange equation for a minimizer u of (1.1) in a limiting case of the constraint (1.2) is of the form

$$\begin{aligned}&\frac{\partial F}{\partial z}(x,u(x),Du(x)) - \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\frac{\partial F}{\partial p_i}(x,u(x),Du(x))\right)\\ &= \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i\partial x_j}\mu_{ij}\end{aligned} \tag{1.3}$$

for some symmetric nonnegative matrix $\mu = (\mu_{ij})_{1\leq i,j\leq n}$ of Radon measures; see also Carlier [5] for a new proof of this result and related extensions.

The structure of the matrix μ in (1.3), to the best of the author's knowledge, is still mysterious up to now. Thus, for practical purposes such as implementing numerical schemes to find minimizers of (1.1), it is desirable to find suitably explicit approximations of μ in particular and minimizers of (1.1) in general. This has been done by Carlier and Radice [8] when the Lagrangian F does not depend on the gradient variable p; see Section 1.1 for a quick review. In this paper, we tackle the more challenging case when F depends on the gradient variable. This case is relevant to many realistic models in physics and economics such as the ones described in [3, 26].

1.1 Fourth-Order Equations of Abreu Type Approximating Convex Functionals with a Convexity Constraint

When φ is strictly convex in a neighborhood of Ω_0, the Lagrangian $F(x,z,p)$ does not depend on p, that is, $F(x,z,p) = F^0(x,z)$, and uniform convex in its second argument and $\frac{\partial}{\partial z}F^0(x,z)$ is bounded uniformly in x for each fixed z, Carlier and Radice [8] show that one can approximate the minimizer of

$$\inf_{u\in \bar{S}[\varphi,\Omega_0]}\int_{\Omega_0} F^0(x,u(x))dx \tag{1.4}$$

by solutions of second boundary value problems of fourth-order equations of Abreu type. More precisely, for each $\varepsilon > 0$, consider the following second boundary value problem for a uniformly convex function u_ε on an open euclidean ball B containing $\overline{\Omega_0}$:

$$
\begin{cases}
\varepsilon \sum_{i,j=1}^n U_\varepsilon^{ij} \frac{\partial^2 w_\varepsilon}{\partial x_i \partial x_j} = g_\varepsilon(\cdot, u_\varepsilon) & \text{in } B,\\
w_\varepsilon = (\det D^2 u_\varepsilon)^{-1} & \text{in } B,\\
u_\varepsilon = \varphi & \text{on } \partial B,\\
w_\varepsilon = \psi & \text{on } \partial B,
\end{cases}
\tag{1.5}
$$

where $\psi := (\det(D^2\varphi))^{-1}$ on ∂B,

$$
g_\varepsilon(x,u) = \begin{cases}
\frac{\partial F^0}{\partial z}(x,u), & x \in \Omega_0,\\
\frac{1}{\varepsilon}(u(x) - \varphi(x)), & x \in B \setminus \Omega_0,
\end{cases}
$$

and $U_\varepsilon = (U_\varepsilon^{ij})$ is the cofactor matrix of

$$
D^2 u_\varepsilon = \left(\frac{\partial^2 u_\varepsilon}{\partial x_i \partial x_j}\right)_{1 \le i,j \le n} \equiv ((u_\varepsilon)_{ij})
$$

of the uniformly convex function u_ε, that is,

$$
U_\varepsilon = \big(\det D^2 u_\varepsilon\big)\big(D^2 u_\varepsilon\big)^{-1}.
$$

Carlier and Radice [8, theorems 4.2 and 5.3] show that (1.5) has a unique uniformly convex solution $u_\varepsilon \in W^{4,q}(B)$ (for all $q < \infty$), which converges uniformly on $\overline{\Omega_0}$ to the unique minimizer of (1.4) when $\varepsilon \to 0$.

The first equation of (1.5) is a fourth-order equation of Abreu type. We will say a few words about this fully nonlinear, geometric equation. Let

$$
(U^{ij}) = (\det D^2 u)(D^2 u)^{-1} \quad \text{and} \quad u_{ij} := \frac{\partial^2 u}{\partial x_i \partial x_j}
$$

for any function u. The Abreu equation [1] for a uniformly convex function u

$$
\sum_{i,j=1}^n U^{ij}[(\det D^2 u)^{-1}]_{ij} = f
$$

first arises in differential geometry [1,11,12], where one would like to find a Kähler metric of constant scalar curvature. Its related and important cousin is the affine maximal surface equation [30–32] in affine geometry:

$$
\sum_{i,j=1}^n U^{ij}\Big[(\det D^2 u)^{-\frac{n+1}{n+2}}\Big]_{ij} = 0.
$$

We call (1.5) the second boundary value problem because the values of the function u_ε and its Hessian determinant $\det D^2 u_\varepsilon$ are prescribed on the boundary ∂B. This is in contrast to the first boundary value problem, where one prescribes the values of the function u_ε and its gradient Du_ε on ∂B. The fourth-order equation in (1.5) arises as the Euler-Lagrange equation of the functional

$$\int_{\Omega_0} F^0(x,u(x))dx + \frac{1}{2\varepsilon}\int_{B\setminus\Omega_0}(u-\varphi)^2\,dx - \varepsilon\int_B \log\det D^2u\,dx.$$

At the functional level, the penalization $\varepsilon\int_B \log\det D^2u\,dx$ involving the logarithm of the Hessian determinant acts as a good barrier for the convexity constraint in problems like (1.4); see also [2] for related rigorous numerical results at a discretized level. At the equation level, the results of Carlier and Radice [8] show that, when the Lagrangian F does not depend on p, the matrix μ in (1.3) is well approximated by $\varepsilon(D^2u_\varepsilon)^{-1}\equiv\varepsilon(u_\varepsilon^{ij})$ where u_ε is the solution of (1.5). To see this, we just note that the cofactor matrix U_ε of D^2u_ε is divergence-free, that is, $\sum_{j=1}^n \frac{\partial}{\partial x_j}U_\varepsilon^{ij}=0$ for all $i=1,\ldots,n$, and hence, noting that $U_\varepsilon^{ij}w_\varepsilon = u_\varepsilon^{ij}$, we can write the left-hand side of the first equation in (1.5) as

$$\varepsilon\sum_{i,j=1}^n U_\varepsilon^{ij}\frac{\partial^2 w_\varepsilon}{\partial x_i\partial x_j} = \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i\partial x_j}\left(\varepsilon U_\varepsilon^{ij}w_\varepsilon\right) = \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i\partial x_j}\varepsilon u_\varepsilon^{ij}.$$

1.2 Gradient-Dependent Lagrangians and the Rochet-Choné Model

The analysis of Carlier-Radice [8] left open the question of whether one can approximate minimizers of (1.1) by solutions of second boundary value problems of fourth-order equations of Abreu type when the Lagrangian F depends on the gradient variable p. This case is relevant to physics and economic applications. We briefly describe here the Rochet-Choné model in economics. In the Rochet-Choné model [26] of the monopolist problem in product line design where the cost of producing product q is the quadratic function $\frac{1}{2}|q|^2$, the monopolist's profit as a functional of the buyers' indirect utility function u is

$$\Phi(u)=\int_{\Omega_0}\left\{x\cdot Du(x)-\frac{1}{2}|Du(x)|^2-u(x)\right\}\gamma(x)dx.$$

Here $\Omega_0\subset\mathbb{R}^n$ is the collection of types of agents and γ is the relative frequency of different types of agents in the population. For a consumer of type $x\in\Omega_0$, the indirect utility function $u(x)$ is computed via the formula

$$u(x)=\max_{q\in Q}\{x\cdot q-p(q)\}$$

where $Q\subset\mathbb{R}^n$ is the product line and $p\colon Q\to\mathbb{R}$ is a price schedule that the monopolist needs to design to maximize her overall profit. Since u is the maximum of a family of affine functions, it is convex. Maximizing $\Phi(u)$ over convex functions

u is equivalent to minimizing the following functional J_0 over convex functions u:

$$J_0(u) = \int_{\Omega_0} F^{RC}(x, u(x), Du(x))dx \quad \text{where } F^{RC}(x, z, p) = \frac{1}{2}|p|^2\gamma(x) - x \cdot p\gamma(x) + z\gamma(x).$$

As mentioned in [16], even in this simple-looking variational problem, the convexity is not easy to handle from a numerical standpoint. Mérigot and Oudet [24] were among the first to make interesting progress in this direction. Here we analyze this problem and its generalization from an asymptotic analysis standpoint.

In this paper, we are interested in using the second boundary value problems of fourth-order equations of Abreu type to approximate minimizer(s) of the variational problem

$$\inf_{u \in \bar{S}[\varphi, \Omega_0]} J(u) \tag{1.6}$$

where

$$J(u) = \int_{\Omega_0} F(x, u(x), Du(x))dx \quad \text{with } F(x, z, p) = F^0(x, z) + F^1(x, p). \tag{1.7}$$

The choice of form of F in (1.7) simplifies some of our arguments and is clearly motivated by the analysis of the Rochet-Choné model.

Similar to the analysis of (1.5) carried out by Carlier-Radice [8], our analysis leads us to two very natural questions concerning the following second boundary value problem of a highly singular, fully nonlinear fourth-order equation of Abreu type for a uniformly convex function u:

$$\begin{cases} \sum_{i,j=1}^n U^{ij} w_{ij} = f_\delta(\cdot, u, Du, D^2u) & \text{in } \Omega, \\ w = (\det D^2u)^{-1} & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \\ w = \psi & \text{on } \partial\Omega. \end{cases} \tag{1.8}$$

Here $(U^{ij})_{1 \le i,j \le n}$ is the cofactor matrix of the Hessian matrix $D^2u = (u_{ij})$, $\delta > 0$, Ω is a bounded, open, smooth, uniformly convex domain containing $\overline{\Omega_0}$ and

$$f_\delta(x, u(x), Du(x), D^2u(x)) = \begin{cases} \frac{\partial}{\partial z} F^0(x, u(x)) - \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\frac{\partial F^1}{\partial p_i}(x, Du(x))\right) & x \in \Omega_0, \\ \frac{1}{\delta}(u(x) - \varphi(x)) & x \in \Omega \setminus \Omega_0. \end{cases} \tag{1.9}$$

Question 1.

Given φ, ψ, F^0, and F^1, can we solve the second boundary value problem (1.8)–(1.9)?

Question 2.
Are minimizers of (1.6)–(1.7) well approximated by solutions of (1.8)–(1.9) when $\delta \to 0$?

We will answer Questions 1 and 2 in the affirmative in two dimensions under suitable conditions on F^0 and F^1—see Theorems 2.1 and 2.3, respectively—via analysis of singular Abreu equations.

1.3 Singular Abreu Equations

By now, the second boundary value problem for the Abreu equation is well understood [9, 19–21, 32]. In particular, from the analysis in [20], we know that if $f \in L^q(\Omega)$ where $q > n$, then we have a unique uniformly convex $W^{4,q}(\Omega)$ solution to the second boundary value problem of a more general form of the Abreu equation:

$$\begin{cases} \sum_{i,j=1}^n U^{ij} w_{ij} = f & \text{in } \Omega, \\ w = G^{'}(\det D^2 u) & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \\ w = \psi & \text{on } \partial\Omega \end{cases} \tag{1.10}$$

where $\varphi \in W^{4,q}(\Omega)$, $\psi \in W^{2,q}(\Omega)$ with $\inf_{\partial\Omega} \psi > 0$, and G belongs to a class of concave functions that include $G(t) = \frac{t^\theta - 1}{\theta}$ where $0 < \theta < 1/n$ and $G(t) = \log t$. On the other hand, if the right-hand side f is only in $L^q(\Omega)$ with $q < n$ then solutions to (1.10) might not be in $W^{4,q}(\Omega)$.

In [8], the authors established an a priori uniform bound for solutions of (1.5), thus confirming the solvability of (1.5) in all dimensions, where g_ε is now bounded, by using the solvability results for (1.10).

The second boundary value problem of Abreu type in (1.8) has highly singular right-hand side even in the simple but nontrivial setting of $F^0(x,z) = 0$ and $F^1(x,p) = \frac{1}{2}|p|^2$. Among the simplest analogues of the first equation in (1.8) is

$$U^{ij}[(\det D^2 u)^{-1}]_{ij} = -\Delta u \quad \text{in } \Omega. \tag{1.11}$$

To the best of our knowledge, the Abreu-type equation of the form (1.11) has not appeared before in the literature. There are several serious challenges in establishing the solvability of its second boundary value problem. We highlight here two aspects among these challenges:

- (C1) It is not known a priori if we can establish the lower bound and upper bound for $\det D^2 u$. Thus, for a convex function u, Δu can be only a measure.
- (C2) Even if we can establish the positive lower bound λ_1 and upper bound λ_2 for $\det D^2 u$, that is, $\lambda_1 \leq \det D^2 u \leq \lambda_2$ in Ω, we can only deduce from the regularity results for the Monge-Ampère equation of De Philippis-Figalli-Savin [10], Schmidt [29], and Savin [27] that $\Delta u \in L^{1+\varepsilon_0}(\Omega)$

where $\varepsilon_0 = \varepsilon_0(n, \lambda_1, \lambda_2) > 0$ can be arbitrarily small. In fact, from Wang's counterexample [33] to regularity of the Monge-Ampère equations, we know that $\varepsilon_0(n, \lambda_1, \lambda_2) \to 0$ when $\lambda_2/\lambda_1 \to \infty$. In other words, the right-hand side of (1.11) has low integrability a priori, which can be less than the dimension n. Thus, the results on the solvability of the second boundary value problem of the Abreu equation in [9, 19–21, 32] do not apply to the second boundary value problems of (1.11) and (1.8).

In this paper, we are able to overcome these difficulties for both (1.8) and (1.11) in two dimensions under suitable conditions on the convex functions F^0 and F^1; see Theorem 2.1 which asserts the solvability of (1.8)–(1.9). This is done via a priori fourth-order derivative estimates and degree theory. For the a priori estimates, the structural conditions on F^0 and F^1 allow us to establish that $\lambda_1 \leq \det D^2 u \leq \lambda_2$ in Ω for some positive constants λ_1 and λ_2 and that $f_\delta(\cdot, u, Du, D^2u)$, as explained in (C2) for $-\Delta u$, belongs to $L^{1+\varepsilon_0}(\Omega)$ for some possibly small $\varepsilon_0 > 0$. We briefly explain here how we can go beyond second-order derivative estimates and why the dimension is restricted to 2.

Note that (1.8) consists of a Monge-Ampère equation for u in the form of $\det D^2 u = w^{-1}$ and a linearized Monge-Ampère equation for w in the form of $U^{ij} w_{ij} = f_\delta(\cdot, u, Du, D^2u)$ because the coefficient matrix (U^{ij}) comes from linearization of the Monge-Ampère operator:

$$U = \frac{\partial \det D^2 u}{\partial u_{ij}}.$$

For the solvability of second boundary problems such as (1.8) and (1.11), as in [9, 19–21, 32], a key ingredient is to establish global Hölder continuity of the linearized Monge-Ampère equation with right-hand side having low integrability. In our case, the integrability exponent is $1 + \varepsilon_0$ for a small $\varepsilon_0 > 0$.

To the best of our knowledge, the lowest integrability exponent q for the right-hand side of the linearized Monge-Ampère equation (with Monge-Ampère measure just bounded away from 0 and ∞) for which one can establish a global Hölder continuity estimate is $q > n/2$. This fact was proved in the author's paper with Nguyen [22]. The constraint $1 + \varepsilon_0 > n/2$ for small $\varepsilon_0 > 0$ forces n to be 2. *It is exactly for this reason that we restrict ourselves in this paper to considering the case $n = 2$.* Using the bounds on the Hessian determinant for u and the global Hölder estimates in [22], we can show that w is globally Hölder continuous. Once we have this, we can apply the global $C^{2,\alpha}$ estimates for the Monge-Ampère equation in [28, 32] to conclude that $u \in C^{2,\alpha}(\overline{\Omega})$. We update this information to $U^{ij} w_{ij} = f_\delta(\cdot, u, Du, D^2u)$ to have a second-order uniformly elliptic equation for w with global Hölder continuous coefficients and bounded right-hand side. This gives second-order derivative estimates for w. Now, fourth-order derivative estimates for u easily follows.

Under suitable conditions on F^0 and F^1 we can show that solutions to (1.8)-(1.9) converge uniformly on compact subsets of Ω to the unique minimizer of

(1.6)–(1.7); see Theorem 2.3. It implies in particular that minimizers of the 2D Rochet-Choné model perturbed by a highly convex lower-order term, under a convexity constraint, can be approximated in the uniform norm by solutions of second boundary value problems of singular Abreu equations.

Remark 1.1. Our analysis also covers the case when $w = (\det D^2 u)^{-1}$ in (1.8) is replaced by $w = (\det D^2 u)^{\theta-1}$ where $0 \leq \theta < 1/n$. We will consider these general cases in our main results.

Remark 1.2. As mentioned above, due to the possibly low integrability of the right-hand side $-\Delta u$ of (1.11), our analysis is at the moment restricted to two dimensions. However, for the solvability of the second boundary value problem, it is quite unexpected that the structure of $-\Delta u$ in two dimensions, that is, $-\Delta u = -\operatorname{trace}(U^{ij})$, allows us to replace the term $(\det D^2 u)^{-1}$ in (1.11) by $H(\det D^2 u)$ for very general functions H including $H(d) = d^{\theta-1}$ for $\theta \in [0, \infty) \setminus \{1\}$; see Theorem 2.6. Note that it is an open question if (1.10) is solvable for $f \in L^q(\Omega)$ when $q > n$ and $G^{'}(d) = d^{\theta-1}$ where $\theta \in [\frac{1}{n}, \infty)$.

Remark 1.3. It should not come as a surprise when Abreu-type equations appear in problems motivated from economics. On the one hand, in addition to [8] and this paper, Abreu-type equations also appear in the continuum Nash's bargaining problem [34]. On the other hand, the monopolist's problem can be treated in the framework of optimal transport (see, for example, [15,16]), so it is not totally unexpected to have deep connections with the Monge-Ampère equation. The interesting point here is that Abreu-type equations involve both the Monge-Ampère equation and its linearization.

NOTATION. The following notations will be used throughout the paper. Points in $\mathbb{R}^n$ will be denoted by $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ or $p = (p_1, \ldots, p_n) \in \mathbb{R}^n$. I_n is the identity $n \times n$ matrix. We use $\nu = (\nu_1, \ldots, \nu_n)$ to denote the unit outer normal vector field on $\partial\Omega$ and ν_0 on $\partial\Omega_0$. Unless otherwise stated, repeated indices are summed such as $U^{ij} w_{ij} = \sum_{i,j=1}^n U^{ij} w_{ij}$;

$$f^0(x,z) = \frac{\partial F^0(x,z)}{\partial z}; \, F^1_{p_i}(x,p) = \frac{\partial F^1(x,p)}{\partial p_i};$$

$$\nabla_p F^1(x,p) = (F^1_{p_1}(x,p), \ldots, F^1_{p_n}(x,p)); \, F^1_{p_i p_j}(x,p) = \frac{\partial^2 F^1(x,p)}{\partial p_i \partial p_j};$$

$$F^1_{p_i x_j}(x,p) = \frac{\partial^2 F^1(x,p)}{\partial p_i \partial x_j}; \operatorname{div}\big(\nabla_p F^1(x,p)\big) = \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\frac{\partial F^1(x,p)}{\partial p_i}\right).$$

We use $U = (U^{ij})_{1 \leq i,j \leq n}$ to denote the cofactor matrix of the Hessian matrix

$$D^2 u = \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right) \equiv (u_{ij})_{1 \leq i,j \leq n}$$

of a function $u \in C^2(\overline{\Omega})$. If u is uniformly convex in Ω, then we have $U = (\det D^2u)(D^2u)^{-1}$.

The rest of the paper is organized as follows. We state our main results in Section 2. In Section 3, we recall tools used in the proofs of our main theorems. In Section 4, we establish a priori estimates. Section 5 proves the main results in Theorems 2.1, 2.3, 2.6, and 2.8.

2 Statements of the Main Results

Let $\delta > 0$ and let Ω_0, Ω be open, smooth, bounded, convex domains in $\mathbb{R}^n$ such that $\Omega_0 \Subset \Omega$. We study the solvability of the following second boundary value problem of a fully nonlinear, fourth-order equation of Abreu type for a uniformly convex function u:

$$\begin{cases} \sum_{i,j=1}^n U^{ij} w_{ij} = f_\delta(\cdot, u, Du, D^2u) & \text{in } \Omega, \\ w = (\det D^2u)^{\theta-1} & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \\ w = \psi & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Here $U = (U^{ij})_{1\le i,j\le n}$ is the cofactor matrix of the Hessian matrix $D^2u = (u_{ij})$ and

$$\begin{aligned} &f_\delta(x, u(x), Du(x), D^2u(x)) \\ &\quad = \begin{cases} f^0(x, u(x)) - \operatorname{div}(\nabla_p F^1(x, Du(x))), & x \in \Omega_0, \\ \frac{1}{\delta}(u(x) - \varphi(x)), & x \in \Omega \setminus \Omega_0. \end{cases} \end{aligned} \tag{2.2}$$

We consider the following sets of assumptions for *nonnegative* constants ρ, c_0, $C_*, \bar{c}_0, \bar{C}_*$:

$$(f^0(x,z) - f^0(x,\tilde{z}))(z - \tilde{z}) \ge \rho|z - \tilde{z}|^2; \ |f^0(x,z)| \le \eta(|z|) \quad \text{for all } x \in \Omega_0 \text{ and all } z, \tilde{z} \in \mathbb{R} \tag{2.3}$$

where $\eta : [0,\infty) \to [0,\infty)$ is a continuous and increasing function;

$$0 \le F^1_{p_i p_j}(x,p) \le C_* I_n; \ \left|F^1_{p_i x_i}(x,p)\right| \le c_0|p| + C_* \quad \text{for all } x \in \Omega_0 \text{ and for each } i; \tag{2.4}$$

$$\begin{aligned} \left|F^1_{p_i}(x,p)\right| &\le \bar{c}_0|p| + \bar{C}_* && \text{for } x \in \partial\Omega_0 \text{ and for each } i; \\ |\nabla_p F^1(x,p)| &\le \eta(|p|) && \text{for all } x \in \Omega_0. \end{aligned} \tag{2.5}$$

Our first main theorem is concerned with the solvability of (2.1)–(2.2) in two dimensions.

THEOREM 2.1 (Solvability of highly singular second boundary value problems of Abreu type). *Let $n = 2$ and $0 \le \theta < 1/n$. Let $\delta > 0$ and let Ω_0, Ω be open, smooth, bounded, and convex domains in $\mathbb{R}^n$ such that $\Omega_0 \Subset \Omega$. Assume* *moreover that Ω is uniformly convex. Assume that $\varphi \in C^{3,1}(\overline{\Omega})$ and $\psi \in C^{1,1}(\overline{\Omega})$ with $\inf_{\partial\Omega} \psi > 0$. Assume that (2.3)–(2.5) are satisfied.*

(i) *If either $\min\{c_0, \bar{c}_0\}$ is sufficiently small (depending only on $\inf_{\partial\Omega} \psi$, Ω_0 and Ω), or $\min\{\rho, \frac{1}{\delta}\}$ is sufficiently large (depending only on $\min\{c_0, \bar{c}_0\}$, Ω_0, and Ω), then there is a uniformly convex solution $u \in W^{4,q}(\Omega)$ to the system (2.1)–(2.2) for all $q \in (n, \infty)$.*

(ii) *If $\bar{c}_0 = \bar{C}_* = 0$, then there is a unique uniformly convex solution $u \in W^{4,q}(\Omega)$ to the system (2.1)–(2.2) for all $q \in (n, \infty)$.*

Theorem 2.1 will be proved in Section 5. The existence proof uses a priori estimates in Theorem 4.1 and degree theory. For the a priori estimates, the technical size conditions in (i) guarantee the uniform bound for u and the L^2 bound for its gradient Du in terms of the data of the problem.

Remark 2.2. Consider the perturbed Rochet-Choné model

$$F(x, z, p) = F^0(x, z) + F^1(x, p)$$

where

$$F^0(x, z) = \gamma(x)z + \frac{\rho}{2}|z|^2, \ F^1(x, p) = \frac{1}{2}\gamma(x)|p|^2 - x \cdot p\gamma(x)$$

where $\rho \geq 0$ is a constant and γ is a Lipschitz function satisfying $0 < \gamma \leq C_1, |D\gamma| \leq C_2$ in Ω_0. Then (2.3)–(2.5) are satisfied with suitable constants c_0, $\bar{c}_0$, C_*, and $\bar{C}_*$. If $\gamma = 0$ on $\partial\Omega_0$, then $\bar{c}_0 = \bar{C}_* = 0$. More generally, if $\max_{\partial\Omega} \gamma(x)$ is small, then $\bar{c}_0$ is small. If $\|D\gamma\|_{L^\infty(\Omega_0)}$ is small, then c_0 is small.

Our second main theorem asserts the convergence of solutions to (2.1)–(2.2) in two dimensions to the unique minimizer of (1.6)–(1.7) when the Lagrangian $F(x, z, p) = F^0(x, z) + F^1(x, p)$ is highly convex in the second variable.

THEOREM 2.3 (Convergence of solutions of the approximate second boundary value problems of Abreu type to the minimizer of the convex functional). *Let $n = 2$ and $0 \leq \theta < 1/n$. Let Ω_0, Ω be open, smooth, bounded, convex domains in $\mathbb{R}^n$ such that $\Omega_0 \Subset \Omega$. Moreover, assume that Ω is uniformly convex. Assume that $\varphi \in C^{3,1}(\overline{\Omega})$ is uniformly convex with $\inf_{\Omega} \det D^2\varphi > 0$ and $\psi \in C^{1,1}(\overline{\Omega})$ with $\inf_{\partial\Omega} \psi > 0$. Assume that (2.3)–(2.5) are satisfied, and $\rho > 0$. For each $\varepsilon > 0$, consider the following second boundary value problem*:

$$\begin{cases} \varepsilon \sum_{i,j=1}^n U_\varepsilon^{ij}(w_\varepsilon)_{ij} = f_\varepsilon(\cdot, u_\varepsilon, Du_\varepsilon, D^2u_\varepsilon) & \text{in } \Omega, \\ w_\varepsilon = (\det D^2 u_\varepsilon)^{\theta-1} & \text{in } \Omega, \\ u_\varepsilon = \varphi & \text{on } \partial\Omega, \\ w_\varepsilon = \psi & \text{on } \partial\Omega. \end{cases} \tag{2.6}$$

Here $U_\varepsilon = (U_\varepsilon^{ij})_{1\le i,j\le n}$ *is the cofactor matrix of the Hessian matrix* $D^2u_\varepsilon = ((u_\varepsilon)_{ij})$ *and*

$$f_\varepsilon(x,u_\varepsilon(x),Du_\varepsilon(x),D^2u_\varepsilon(x)) = \begin{cases} f^0(x,u_\varepsilon(x)) - \operatorname{div}\big(\nabla_p F^1(x,Du_\varepsilon(x))\big), & x\in\Omega_0,\\ \frac{1}{\varepsilon}(u_\varepsilon(x)-\varphi(x)), & x\in\Omega\setminus\Omega_0.\end{cases} \tag{2.7}$$

Assume that either $\bar c_0 = \bar C_* = 0$ *or* ρ *is sufficiently large* (*depending only on* $\bar c_0 + \bar C_*$, Ω_0 *and* Ω)*. Let* u_ε *be a uniformly convex solution* $u_\varepsilon \in W^{4,q}(\Omega)$ *to the system* (2.6)–(2.7) *for all* $q\in(n,\infty)$*. Then,* u_ε *converges uniformly on compact subsets of* Ω *to the unique minimizer* $u\in\bar S[\varphi,\Omega_0]$ (*defined in* (1.2)) *of the problem* (1.6) *where* J *is defined by* (1.7)*.*

Theorem 2.3 will be proved in Section 5.

Remark 2.4. For the convergence result in Theorem 2.3, we need to establish a uniform bound for u_ε independent of ε; see Lemma 4.4. For this, the uniform convexity of φ plays an important role. On the other hand, in Theorem 2.1, we basically use the boundary value of φ on $\partial\Omega$, and therefore φ need not be uniformly convex.

Remark 2.5. Several pertinent remarks on Theorem 2.3 are in order.

(i) Theorem 2.3 is applicable to the perturbed Rochet-Choné model considered in Remark 2.2. Theorem 2.3 implies that minimizers of the 2D Rochet-Choné model perturbed by a highly convex lower-order term, under a convexity constraint, can be approximated in the uniform norm by solutions of second boundary value problems of Abreu-type equations.

(ii) The minimization problem (1.6)–(1.7) when $F^1(x,p) = \frac{1}{2}\gamma(x)|p|^2 - x\cdot p\gamma(x)$ with $\gamma(x) = 0$ on $\partial\Omega_0$ (that is, $\bar c_0 = \bar C_* = 0$) was studied by Carlier in [4].

(iii) In Theorem 2.3, when $\rho > 0$ is large and $\bar c_0 + \bar C_* > 0$, we are unable to prove the uniqueness of uniformly convex solutions u_ε to (2.6)–(2.7). Despite this lack of uniqueness, Theorem 2.3 says that we have the full convergence of all solutions u_ε to the unique minimizer $u\in\bar S[\varphi,\Omega_0]$ of the problem (1.6)–(1.7). This is surprising to us.

In Theorem 2.1, the function $F^1(x,p)$ grows at most quadratically in p. The following extension deals with a more general Lagrangian F.

THEOREM 2.6. *Let* $\Omega\subset\mathbb{R}^2$ *be an open, smooth, bounded, and uniformly convex domain. Assume that* $\varphi\in C^\infty(\bar\Omega)$ *and* $\psi\in C^\infty(\bar\Omega)$ *with* $\inf_{\partial\Omega}\psi > 0$*. Let* $F(p)\colon\mathbb{R}^2\to\mathbb{R}$ *and* $H\colon(0,\infty)\to(0,\infty)$ *be smooth. Consider the following second boundary value problem of a fourth-order equation of Abreu type for a* *uniformly convex function* u:

$$\begin{cases} \sum_{i,j=1}^2 U^{ij} w_{ij} = -\operatorname{div}(\nabla_p F(Du)) & \text{in } \Omega, \\ w = H(\det D^2 u) & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \\ w = \psi & \text{on } \partial\Omega. \end{cases} \tag{2.8}$$

(i) *Assume that* F *is convex and that* $F_{p_i p_j}(p)$ *is bounded for* p *bounded. Assume that* H *is strictly decreasing,* $H(d) \to 0$ *when* $d \to \infty$, *and* $H(d) \to \infty$ *when* $d \to 0$. *Then there exists a smooth, uniformly convex solution* $u \in C^\infty(\overline{\Omega})$ *to* (2.8). *If* $H(d) = d^{\theta-1}$ *where* $0 \le \theta < \frac{1}{2}$, *then the solution is unique.*

(ii) *Assume that* $0 \le F_{p_i p_j}(p) \le C_* I_2$. *Assume that* H *is strictly monotone and that* H^{-1} *maps compact subsets of* $(0, \infty)$ *into compact subsets of* $(0, \infty)$. *Then there exists a smooth, uniformly convex solution* $u \in C^\infty(\overline{\Omega})$ *to* (2.8).

Theorem 2.6 will be proved in Section 5.

Remark 2.7. Examples of Lagrangians F in Theorem 2.6(i) include

$$F(p) = \frac{1}{s}|p|^s \ (s \ge 2, s \text{ integer}), \text{ or } F(p) = e^{\frac{1}{2}|p|^2}.$$

The existence results in Theorem 2.1 and 2.6 can be extended to certain nonconvex Lagrangians F. To illustrate the scope of our method, we consider the case of Lagrangian

$$F(x, z, p) = \frac{1}{4}(z^2 - 1)^2 + \frac{1}{2}|p|^2$$

arising from the study of Allen-Cahn functionals. Our existence result for the singular Abreu equation with Allen-Cahn Lagrangian states as follows.

THEOREM 2.8. *Let* $\Omega \subset \mathbb{R}^2$ *be an open, smooth, bounded, and uniformly convex domain. Assume that* $\varphi \in C^\infty(\overline{\Omega})$ *and* $\psi \in C^\infty(\overline{\Omega})$ *with* $\inf_{\partial\Omega} \psi > 0$. *Then there exists a smooth, uniformly convex solution* $u \in C^\infty(\overline{\Omega})$ *to the following second boundary value problem*:

$$\begin{cases} \sum_{i,j=1}^2 U^{ij} w_{ij} = u^3 - u - \Delta u & \text{in } \Omega, \\ w = (\det D^2 u)^{-1} & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \\ w = \psi & \text{on } \partial\Omega. \end{cases} \tag{2.9}$$

Theorem 2.8 will be proved in Section 5.

Remark 2.9. It would be interesting to establish the higher-dimensional versions of Theorems 2.1, 2.3, 2.6, and 2.8.

Remark 2.10 (Universal constants). In Sections 4 and 5, we will work with a fixed exponent $q > n$, and we call a positive constant *universal* if it depends only on $n, \theta, \eta, q, \delta, c_0, \bar{c}_0, C_*, \bar{C}_*, \rho, \Omega, \Omega_0, \|\varphi\|_{W^{4,q}(\Omega)}, \|\psi\|_{W^{2,q}(\Omega)}$, and $\inf_{\partial\Omega}\psi$. We use $C, C_1, C_2, \dots$ to denote universal constants, and their values may change from line to line.

3 Tools Used in the Proofs of Main Theorems

In this section, we recall the statements of two main tools used in the proofs of our main theorems.

The first tool is the global Hölder estimates for the linearized Monge-Ampère equation with right-hand side having low integrability. These estimates were established by Nguyen and the author in [22, theorem 1.7 and lemma 1.5]. They extend in particular the previous result in [19, theorem 1.4] (see also [21, theorem 1.13]) where the case of L^n right-hand side was treated.

THEOREM 3.1 (Global Hölder estimates for the linearized Monge-Ampère equation). *Let Ω be a bounded, uniformly convex domain in $\mathbb{R}^n$ with $\partial\Omega \in C^3$. Let $\phi: \overline{\Omega} \to \mathbb{R}$, $\phi \in C^{0,1}(\overline{\Omega}) \cap C^2(\Omega)$ be a convex function satisfying*

$$0 < \lambda \leq \det D^2\phi \leq \Lambda < \infty \quad \textit{and} \quad \phi\big|_{\partial\Omega} \in C^3.$$

Denote by $(\Phi^{ij}) = (\det D^2\phi)(D^2\phi)^{-1}$ the cofactor matrix of $D^2\phi$. Let $v \in C(\overline{\Omega}) \cap W^{2,n}_{\mathrm{loc}}(\Omega)$ be the solution to the linearized Monge-Ampère equation

$$\begin{cases} \Phi^{ij} v_{ij} = f & \textit{in } \Omega, \\ v = \varphi & \textit{on } \partial\Omega, \end{cases}$$

where $\varphi \in C^\alpha(\partial\Omega)$ for some $\alpha \in (0,1)$ and $f \in L^q(\Omega)$ with $q > n/2$. Then, $v \in C^\beta(\overline{\Omega})$ with the estimate

$$\|v\|_{C^\beta(\overline{\Omega})} \leq C\left(\|\varphi\|_{C^\alpha(\partial\Omega)} + \|f\|_{L^q(\Omega)}\right)$$

where β depends only on λ, Λ, n, q, α, and C depends only on $\lambda, \Lambda, n, q, \alpha$, $\mathrm{diam}(\Omega)$*, $\|\phi\|_{C^3(\partial\Omega)}$, $\|\partial\Omega\|_{C^3}$, and the uniform convexity of Ω.*

The second tool we need is concerned with the global $W^{2,1+\varepsilon_0}$ estimates for the Monge-Ampère equation. They follow from the interior $W^{2,1+\varepsilon_0}$ estimates in De Philippis–Figalli–Savin [10] and Schmidt [29] and the global estimates in Savin [27] (see also [14, theorem 5.3]).

THEOREM 3.2 (Global $W^{2,1+\varepsilon_0}$ estimates for the Monge-Ampère equation). *Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a bounded, uniformly convex domain. Let $\phi: \overline{\Omega} \to \mathbb{R}$, $\phi \in C^{0,1}(\overline{\Omega}) \cap C^2(\Omega)$ be a convex function satisfying*

$$0 < \lambda \leq \det D^2\phi \leq \Lambda \quad \textit{in } \Omega.$$

Assume that $\varphi\big|_{\partial\Omega}$ *and* $\partial\Omega$ *are of class* C^3. *Then, there is a positive constant* $\varepsilon_0 \in (0,1)$ *depending only* n, λ, Λ, *and a positive constant* K *depending only on* n, λ, Λ, Ω, $\|\varphi\|_{C^3(\partial\Omega)}$, $\|\partial\Omega\|_{C^3}$, *and the uniform convexity of* $\partial\Omega$ *such that*

$$\|D^2\varphi\|_{L^{1+\varepsilon_0}(\Omega)} \leq K.$$

We will frequently use the following estimates for convex functions.

LEMMA 3.3. *Let* u *be a convex function on* $\overline{\Omega}$ *where* $\Omega \subset \mathbb{R}^n$ *is an open, bounded, and convex set.*

(i) *We have the following* L^∞ *estimate for* u *in terms of its boundary value and* L^2 *norm*:

$$\|u\|^2_{L^\infty(\Omega)} \leq C(n, \Omega, \max_{\partial\Omega} u) + C(n,\Omega)\int_\Omega |u|^2\, dx.$$

(ii) *If* $\Omega_0 \Subset \Omega$, *then for any* $x \in \Omega_0$, *we have*

$$|Du(x)| \leq \frac{\max_{\partial\Omega} u - u(x)}{\operatorname{dist}(x,\partial\Omega)} \leq \frac{1}{\operatorname{dist}(\Omega_0,\partial\Omega)}(\max_{\partial\Omega} u + \|u\|_{L^\infty(\Omega)}). \tag{3.1}$$

PROOF OF LEMMA 3.3.

(i) The proof is by comparison with a cone. We show that if $u \leq 0$ on $\partial\Omega$, then

$$\|u\|_{L^\infty(\Omega)} \leq \frac{n+1}{|\Omega|}\int_\Omega |u| dx. \tag{3.2}$$

Applying this inequality to the convex function $u - \max_{\partial\Omega} u$, we obtain

$$\begin{aligned}\|u\|^2_{L^\infty(\Omega)} &\leq \left(C(n,\Omega)\int_\Omega |u|dx + C(n,\Omega,\max_{\partial\Omega} u)\right)^2\\ &\leq C(n,\Omega)\int_\Omega |u|^2\, dx + C(n,\Omega,\max_{\partial\Omega} u).\end{aligned}$$

It remains to prove (3.2) when $u \leq 0$ on $\partial\Omega$. Suppose that $|u|$ attains its maximum at $x_0 \in \Omega$. Let $\hat{C}$ be the cone with base $\partial\Omega$ and vertex at $(x_0, u(x_0))$. Then (3.2) follows from the following estimates:

$$\frac{1}{n+1}\|u\|_{L^\infty(\Omega)}|\Omega| = \frac{1}{n+1}|u(x_0)||\Omega| = \text{Volume of } \hat{C} \leq \int_\Omega |u|dx.$$

(ii) The estimate (3.1) just follows from the convexity of u; see, for example [21, lemma 3.11]. □

4 A Priori Estimates for Singular Abreu Equations

In this section, $\delta > 0$ and Ω_0, Ω are open, smooth, bounded, convex domains in $\mathbb{R}^n$ such that $\Omega_0 \Subset \Omega$. We assume moreover that Ω is uniformly convex.

The main result of this section is the following global a priori estimates for the second boundary value problem (2.1)–(2.2).

THEOREM 4.1. *Let $n = 2$, $0 \le \theta < 1/n$, and $q > n$. Assume that $\varphi \in W^{4,q}(\Omega)$ and $\psi \in W^{2,q}(\Omega)$ with $\inf_{\partial\Omega} \psi > 0$. Assume that* (2.3)–(2.5) *are satisfied. Suppose that either* $\min\{c_0, \overline{c}_0\}$ *is sufficiently small* (*depending only on* $\inf_{\partial\Omega} \psi$, Ω_0, *and* Ω), *or* $\min\{\rho, \frac{1}{\delta}\}$ *is sufficiently large* (*depending only on* $\min\{c_0, \overline{c}_0\}$, Ω_0, *and* Ω). *Let u be a smooth, uniformly convex solution of the system* (2.1)–(2.2). *Then, there is a universal constant $C > 0$ such that*

$$\|u\|_{W^{4,q}(\Omega)} \le C.$$

We refer to Remark 2.10 for our convention on *universal constants*. We now give the outline of the proof of Theorem 4.1:

- We first prove the L^∞ bound for u (Lemma 4.2).
- We next prove the lower bound for the Hessian determinant $\det D^2u$ and then the upper bound for the Hessian determinant $\det D^2u$ (Lemma 4.6).
- Finally, we prove the $W^{4,q}$ estimate.

We use $\nu = (\nu_1, \ldots, \nu_n)$ to denote the unit outer normal vector field on $\partial\Omega$ and ν_0 on $\partial\Omega_0$.

For simplicity, we introduce the following size condition used in statements of several lemmas:

(SC) Either $\min\{c_0, \overline{c}_0\}$ is sufficiently small (depending only on $\inf_{\partial\Omega} \psi$, Ω_0, and Ω), or $\min\{\rho, \frac{1}{\delta}\}$ is sufficiently large (depending only on $\min\{c_0, \overline{c}_0\}$, Ω_0, and Ω).

The following lemma establishes the universal L^∞ bound for solutions to the second boundary value problem (2.1)–(2.2).

LEMMA 4.2. *Let $n \ge 2$, $0 \le \theta < 1/n$, and $q > n$. Let u be a smooth solution of the system* (2.1)–(2.2). *Assume that $\varphi \in W^{4,q}(\Omega)$ and $\psi \in W^{2,q}(\Omega)$ with $\inf_{\partial\Omega} \psi > 0$. Assume that* (2.3)–(2.5) *are satisfied. Assume that either $n \ge 3$ or that* (**SC**) *holds when $n = 2$. Then, there is a universal constant $C > 0$ such that*

$$\|u\|_{L^\infty(\Omega)} + \int_{\partial\Omega} u_\nu^n \le C.$$

In the proof of Lemma 4.2, we will use the following basic geometric construction and estimates.

LEMMA 4.3 ([20, lemma 2.1 and inequality (2.7)]). *Assume $G: (0,\infty) \to \mathbb{R}$ is a smooth, strictly increasing, and strictly concave function on* $(0,\infty)$. *Assume that $q > n \ge 2$ and $\varphi \in W^{4,q}(\Omega)$. There exist a convex function $\widetilde{u} \in W^{4,q}(\Omega)$ and constants C and $C(G)$ depending only on n, q, Ω, and $\|\varphi\|_{W^{4,q}(\Omega)}$ with the following properties*:

(i) $\widetilde{u} = \varphi$ *on* $\partial\Omega$.

(ii) $\|\widetilde{u}\|_{C^3(\overline{\Omega})} + \|\widetilde{u}\|_{W^{4,q}(\Omega)} \le C$ *and* $\det D^2\widetilde{u} \ge C^{-1} > 0$.

(iii) *Letting* $\widetilde{w} = G'(\det D^2\widetilde{u})$ *and denoting by* $(\widetilde{U}^{ij})$ *the cofactor matrix of* $(\widetilde{u}_{ij})$, *then*

$$\|\widetilde{U}^{ij}\widetilde{w}_{ij}\|_{L^q(\Omega)} \leq C(G).$$

(iv) *If* $u \in C^2(\overline{\Omega})$ *is a convex function with* $u = \varphi$ *on* $\partial\Omega$, *then for* $u_\nu^+ = \max(0, u_\nu)$, *we have*

$$\|u\|_{L^\infty(\Omega)} \leq C + C(n,\Omega)\left(\int_{\partial\Omega}(u_\nu^+)^n\right)^{1/n}.$$

PROOF OF LEMMA 4.2. The proof is similar to that for [20, lemma 2.2]. Let $u_\nu^+ = \max(0, u_\nu)$. Since u is convex with boundary value φ on $\partial\Omega$, we have

$$u_\nu \geq -\|D\varphi\|_{L^\infty(\Omega)}. \tag{4.1}$$

Our goal is reduced to showing that

$$\int_{\partial\Omega}(u_\nu^+)^n \leq C \tag{4.2}$$

because the universal L^∞ bound for u follows from Lemma 4.3(iv).

Let $G(t) = \frac{t^\theta - 1}{\theta}$ for $t > 0$ (when $\theta = 0$, we set $G(t) = \log t$). Then $G'(t) = t^{\theta-1}$ for all $t > 0$ and $w = G'(\det D^2 u)$ in Ω.

Let $\widetilde{u} \in W^{4,q}(\Omega)$ be as in Lemma 4.3. The function $\widetilde{G}(d) := G(d^n)$ on $(0,\infty)$ is strictly concave because

$$\widetilde{G}''(d) = n^2 d^{n-2}\left[G''(d^n)d^n + \left(1 - \frac{1}{n}\right)G'(d^n)\right] < 0.$$

Using this, $G' > 0$, and the concavity of the map $M \longmapsto (\det M)^{1/n}$ in the space of symmetric matrices $M \geq 0$, we obtain

$$\begin{aligned}
&\widetilde{G}((\det D^2\widetilde{u})^{1/n}) - \widetilde{G}((\det D^2 u)^{1/n})\\
&\quad\leq \widetilde{G}'((\det D^2 u)^{1/n})((\det D^2\widetilde{u})^{1/n} - (\det D^2 u)^{1/n})\\
&\quad\leq \widetilde{G}'((\det D^2 u)^{1/n})\frac{1}{n}(\det D^2 u)^{1/n-1}U^{ij}(\widetilde{u} - u)_{ij}.
\end{aligned}$$

Since $\widetilde{G}'((\det D^2 u)^{1/n}) = nG'(\det D^2 u)(\det D^2 u)^{\frac{n-1}{n}}$, we rewrite the above inequalities as

$$G(\det D^2\widetilde{u}) - G(\det D^2 u) \leq wU^{ij}(\widetilde{u} - u)_{ij}. \tag{4.3}$$

Similarly, for $\widetilde{w} = G'(\det D^2\widetilde{u})$, we have

$$G(\det D^2 u) - G(\det D^2\widetilde{u}) \leq \widetilde{w}\widetilde{U}^{ij}(u - \widetilde{u})_{ij}. \tag{4.4}$$

Adding (4.3) and (4.4), integrating by parts twice, and using the fact that (U^{ij}) is divergence free, we obtain

$$
\begin{aligned}
0 &\geq \int_{\Omega} wU^{ij}(u-\widetilde{u})_{ij} + \widetilde{w}\widetilde{U}^{ij}(\widetilde{u}-u)_{ij} \\
&= \int_{\partial\Omega} wU^{ij}(u_j-\widetilde{u}_j)\nu_i + \int_{\Omega} U^{ij}w_{ij}(u-\widetilde{u}) \\
&\quad + \int_{\partial\Omega} \widetilde{w}\widetilde{U}^{ij}(\widetilde{u}_j-u_j)\nu_i + \int_{\Omega} \widetilde{U}^{ij}\widetilde{w}_{ij}(\widetilde{u}-u) \\
&= \int_{\partial\Omega} (\psi U^{ij}-\widetilde{w}\widetilde{U}^{ij})(u_j-\widetilde{u}_j)\nu_i + \int_{\Omega} f_\delta(\cdot,u,Du,D^2u)(u-\widetilde{u}) \qquad (4.5)\\
&\quad + \int_{\Omega} \widetilde{U}^{ij}\widetilde{w}_{ij}(\widetilde{u}-u).
\end{aligned}
$$

Let us analyze the boundary terms in (4.5). Since $u-\widetilde{u}=0$ on $\partial\Omega$, we have $(u-\widetilde{u})_j=(u-\widetilde{u})_\nu\nu_j$, and hence

$$U^{ij}(u-\widetilde{u})_j\nu_i = U^{ij}\nu_j\nu_i(u-\widetilde{u})_\nu = U^{\nu\nu}(u-\widetilde{u})_\nu$$

where

$$U^{\nu\nu}=\det D^2_{x'}u$$

with $x'\perp\nu$ denoting the tangential directions along $\partial\Omega$. Therefore,

$$(\psi U^{ij}-\widetilde{w}\widetilde{U}^{ij})(u_j-\widetilde{u}_j)\nu_i = (\psi U^{\nu\nu}-\widetilde{w}\widetilde{U}^{\nu\nu})(u_\nu-\widetilde{u}_\nu). \qquad (4.6)$$

To simplify notation, we use f_δ to denote $f_\delta(\cdot,u,Du,D^2u)$ when there is no confusion.

By Lemma 4.3, the quantities $\widetilde{u}$, $\widetilde{u}_\nu$, $\widetilde{U}^{\nu\nu}$, and $\|\widetilde{U}^{ij}\widetilde{w}_{ij}\|_{L^1(\Omega)}$are universally bounded. These bounds combined with (4.5) and (4.6) give

$$
\begin{aligned}
\int_{\partial\Omega}\psi U^{\nu\nu}u_\nu &\leq C + C\|u\|_{L^\infty(\Omega)} + C\int_{\partial\Omega}(|U^{\nu\nu}|+|u_\nu|) \\
&\quad + \int_{\Omega} -f_\delta(u-\widetilde{u})dx.
\end{aligned} \qquad (4.7)
$$

On the other hand, from $u-\varphi=0$ on $\partial\Omega$, we have, with respect to a principal coordinate system at any point $y\in\partial\Omega$ (see, e.g., [17, formula (14.95) in §14.6])

$$D_{ij}(u-\varphi)=(u-\varphi)_\nu\kappa_i\delta_{ij},\quad i,j=1,\dots,n-1,$$

where $\kappa_1,\dots,\kappa_{n-1}$ denote the principal curvatures of $\partial\Omega$ at y.

Let $K=\kappa_1\cdots\kappa_{n-1}$ be the Gauss curvature of $\partial\Omega$ at $y\in\partial\Omega$. Then, at any $y\in\partial\Omega$, by noting that $\det D^2_{x'}u=\det(D_{ij}u)_{1\leq i,j\leq n-1}$ and taking the determinants of

$$D_{ij}u = u_\nu\kappa_i\delta_{ij}-\varphi_\nu\kappa_i\delta_{ij}+D_{ij}\varphi, \qquad (4.8)$$

we obtain, using also (4.1),

$$U^{\nu\nu}=K(u_\nu)^{n-1}+E \quad \text{where } |E|\leq C\big(1+|u_\nu|^{n-2}\big)=C\big(1+(u_\nu^+)^{n-2}\big). \qquad (4.9)$$

Now, it follows from (4.7), (4.9), and Lemma 4.3(iv) that

$$\begin{aligned}\int_{\partial\Omega} K\psi u_\nu^n &\leq C + C\|u\|_{L^\infty(\Omega)} + C\int_{\partial\Omega}(u_\nu^+)^{n-1} + \int_\Omega -f_\delta(u-\widetilde{u})dx \\ &\leq C + C\left(\int_{\partial\Omega}(u_\nu^+)^n\right)^{(n-1)/n} + \int_\Omega -f_\delta(u-\widetilde{u})dx.\end{aligned} \tag{4.10}$$

We will analyze the last term on the right-hand side of (4.10). By construction, $\|\widetilde{u}\|_{L^\infty(\Omega)} \leq C$ and $\|\varphi\|_{L^\infty(\Omega)} \leq C$. Thus, we have

$$-\frac{1}{\delta}(u-\varphi)(u-\widetilde{u}) \leq -\frac{|u|^2}{2\delta} + C(\delta) \tag{4.11}$$

where $C(\delta) > 0$ is a universal constant. Using (2.3), we can estimate

$$\begin{aligned}A &:= \int_{\Omega_0} -f^0(x,u)(u-\widetilde{u})dx \\ &\leq \int_{\Omega_0} -f^0(x,\widetilde{u})(u-\widetilde{u})dx - \rho\int_{\Omega_0}|u-\widetilde{u}|^2\,dx \\ &\leq \int_{\Omega_0} \eta(|\widetilde{u}|)|u-\widetilde{u}|dx - \rho\int_{\Omega_0}|u-\widetilde{u}|^2\,dx \\ &\leq C\int_{\Omega_0}|u|dx + C - \rho\int_{\Omega_0}|u-\widetilde{u}|^2\,dx.\end{aligned} \tag{4.12}$$

Step 1. *Estimate* $\int_{\Omega_0} -f_\delta(u-\widetilde{u})dx$ *by expansion.* By the convexity of u and $F^1(x,p)$ in p, we have $F^1_{p_ip_j}u_{ij} \geq 0$. Moreover, $u \leq \sup_{\partial\Omega}\varphi \leq C$ and $|\widetilde{u}| \leq C$. Thus, recalling (2.4), we find that

$$F^1_{p_ip_j}u_{ij}(u-\widetilde{u}) \leq CF^1_{p_ip_j}u_{ij} \leq CC_*\Delta u. \tag{4.13}$$

On the other hand, for any $i = 1,\ldots,n$, using (2.4) and the first inequality in (3.1), we can bound $F^1_{p_ix_i}(x,Du(x))(u-\widetilde{u})$ in Ω_0 by

$$\begin{aligned}\left|F^1_{p_ix_i}(x,Du(x))(u(x)-\widetilde{u}(x))\right| &\leq (c_0|Du(x)| + C_*)|u(x)-\widetilde{u}(x)| \\ &\leq (c_0C(\Omega_0,\Omega)|u(x)| + C)(|u(x)| + C) \\ &\leq c_0C(\Omega_0,\Omega)|u(x)|^2 + C|u(x)| + C.\end{aligned} \tag{4.14}$$

From (4.12), (4.13), and (4.14), together with the divergence theorem, we find that

$$\begin{aligned}\int_{\Omega_0} -f_\delta(u-\widetilde{u})dx &= \int_{\Omega_0}\left[-f^0(x,u(x)) + \operatorname{div}\left(\nabla_pF^1(x,Du(x))\right)\right](u-\widetilde{u})dx \\ &= A + \int_{\Omega_0}\left(F^1_{p_ix_i}(x,Du(x)) + F^1_{p_ip_j}u_{ij}\right)(u-\widetilde{u})dx \leq\end{aligned}$$

$$
\begin{aligned}
&\leq A + \int_{\Omega_0} \big(c_0 C(\Omega_0, \Omega)|u|^2 + C|u| + C\big) dx \\
&\quad + \int_{\Omega} C_* C \Delta u \, dx \\
&\leq C\|u\|_{L^\infty(\Omega)} + C + c_0 C_1(\Omega_0, \Omega) \int_{\Omega_0} |u|^2 \, dx \\
&\quad - \rho \int_{\Omega_0} |u - \widetilde{u}|^2 \, dx + C \int_{\partial\Omega} (u_\nu)^+ .
\end{aligned} \tag{4.15}
$$

Step 2. *Estimate* $\int_{\Omega_0} -f_\delta(u - \widetilde{u})dx$ *by integration by parts.* Note that $\widetilde{u}$ and $|D\widetilde{u}|$ are universally bounded. Thus, using the convexity of $F^1(x, p)$ in p together with (2.5) and (3.1), we have the following estimates in Ω_0:

$$
\begin{aligned}
-\nabla_p F^1(x, Du(x)) \cdot (Du - D\widetilde{u}) &\leq -\nabla_p F^1(x, D\widetilde{u}) \cdot (Du - D\widetilde{u}) \\
&\leq \eta(|D\widetilde{u}|)(C(\Omega_0, \Omega)|u(x)| + C) \\
&\leq C|u(x)| + C.
\end{aligned} \tag{4.16}
$$

On the other hand, also by (2.5) and (3.1), we have the following estimates on $\partial\Omega_0$:

$$
\begin{aligned}
&(u - \widetilde{u})\nabla_p F^1(x, Du(x)) \cdot \nu_0 \\
&\quad \leq (\bar{c}_0|Du| + \bar{C}_*)|u - \widetilde{u}| \\
&\quad \leq \bar{c}_0 C(\Omega_0, \Omega)\|u\|^2_{L^\infty(\Omega)} + (\bar{c}_0 + \bar{C}_*)C(\Omega_0, \Omega)\|u\|_{L^\infty(\Omega)} + C.
\end{aligned} \tag{4.17}
$$

Now, integrating by parts and using (4.12) together with (4.16)–(4.17), we obtain

$$
\begin{aligned}
&\int_{\Omega_0} -f_\delta(u - \widetilde{u}) \\
&\quad = \int_{\Omega_0} -f^0(x, u(x))(u - \widetilde{u}) + \int_{\Omega_0} \operatorname{div}\big(\nabla_p F^1(x, Du(x))\big)(u - \widetilde{u}) \\
&\quad = A + \int_{\partial\Omega_0} (u - \widetilde{u})\nabla_p F^1(x, Du(x)) \cdot \nu_0 \\
&\qquad + \int_{\Omega_0} -\nabla_p F^1(x, Du(x)) \cdot (Du(x) - D\widetilde{u}(x)) \\
&\quad \leq C + (\bar{c}_0 + \bar{C}_*)C(\Omega_0, \Omega)\|u\|_{L^\infty(\Omega)} + C \int_{\Omega_0} |u| dx \\
&\qquad - \rho \int_{\Omega_0} |u - \widetilde{u}|^2 \, dx + \bar{c}_0 C_2(\Omega_0, \Omega)\|u\|^2_{L^\infty(\Omega)}.
\end{aligned} \tag{4.18}
$$

Step 3. $n \geq 3$. From (4.18) and (4.11), and recalling Lemma 4.3(iv), we have

$$\begin{aligned}\int_{\Omega} -f_\delta(u-\widetilde{u}) &= \int_{\Omega\setminus\Omega_0} -\frac{1}{\delta}(u-\varphi)(u-\widetilde{u}) + \int_{\Omega_0} -f_\delta(u-\widetilde{u})\\ &\leq C + C|u\|_{L^\infty(\Omega)} + C\|u\|^2_{L^\infty(\Omega)}\\ &\leq C + C_2\left(\int_{\partial\Omega}(u_\nu^+)^n\right)^{2/n}.\end{aligned}$$

From $n \geq 3$, (4.10), (4.1), and the above estimates, we obtain

$$\begin{aligned}\int_{\partial\Omega} K\psi(u_\nu^+)^n &\leq C + C\left(\int_{\partial\Omega}(u_\nu^+)^n\right)^{(n-1)/n} + \int_{\Omega} -f_\delta(u-\widetilde{u})\\ &\leq C + C\left(\int_{\partial\Omega}(u_\nu^+)^n\right)^{(n-1)/n}.\end{aligned}$$

From the Hölder inequality, $n \geq 3$, and $\inf_{\partial\Omega} K\psi > 0$, we obtain $\int_{\partial\Omega}(u_\nu^+)^n \leq C$, which is (4.2).

For the rest of the proof of this lemma, we focus on the more difficult case of $n = 2$. We will now use the condition (**SC**).

Step 4. *When* $\min\{c_0, \overline{c}_0\}$ *is sufficiently small* (*depending only on* $\inf_{\partial\Omega}\psi$, Ω_0, *and* Ω). From (4.15), (4.18), and recalling (4.11), we obtain for $\widehat{c}_0 = \min\{c_0, \overline{c}_0\}$

$$\begin{aligned}\int_{\Omega} -f_\delta(u-\widetilde{u}) &\leq C\|u\|_{L^\infty(\Omega)} + C + \widehat{c}_0 C_3(\Omega_0,\Omega)\|u\|^2_{L^\infty(\Omega)}\\ &\quad + C\int_{\partial\Omega}(u_\nu)^+ + \int_{\Omega\setminus\Omega_0} -\frac{1}{\delta}(u-\varphi)(u-\widetilde{u})\\ &\leq C\|u\|_{L^\infty(\Omega)} + C + \widehat{c}_0 C_3(\Omega_0,\Omega)\|u\|^2_{L^\infty(\Omega)} + C\int_{\partial\Omega}(u_\nu)^+. \qquad (4.19)\end{aligned}$$

From (4.10), (4.19), and $n = 2$, we deduce from Lemma 4.3(iv) that

$$\begin{aligned}\int_{\partial\Omega} K\psi u_\nu^2 &\leq C + C\left(\int_{\partial\Omega}(u_\nu^+)^2\right)^{1/2} + \int_{\Omega} -f_\delta(u-\widetilde{u})\\ &\leq C\left(\int_{\partial\Omega}(u_\nu^+)^2\right)^{1/2} + C + \widehat{c}_0 C_4(\Omega_0,\Omega)\int_{\partial\Omega}(u_\nu^+)^2. \qquad (4.20)\end{aligned}$$

Suppose that $\widehat{c}_0$ is small, say

$$\widehat{c}_0 C_4(\Omega_0,\Omega) < (1/2)(\inf_{\partial\Omega}\psi)(\inf_{\partial\Omega} K).$$

Then it follows from (4.20), the Hölder inequality, and $\inf_{\partial\Omega}\psi > 0$ that

$$\int_{\partial\Omega}\left(u_\nu^+\right)^2 \leq C.$$

Thus (4.2) is proved.

Step 5. *When* $\min\{\rho, \frac{1}{\delta}\}$ *is sufficiently large* (*depending only on* $\min\{c_0, \overline{c}_0\}$, Ω_0, *and* Ω).

Case 1. $c_0 \le \overline{c}_0$. In this case, we use (4.15). Suppose that

$$\rho > 4c_0 C_1(\Omega_0, \Omega) + 1 > 0.$$

Then,

$$\begin{aligned} -\rho \int_{\Omega_0} |u - \widetilde{u}|^2\, dx &\le -\frac{\rho}{2} \int_{\Omega_0} |u|^2\, dx + C(\rho) \\ &\le -2c_0 C_1(\Omega_0, \Omega) \int_{\Omega_0} |u|^2\, dx + C. \end{aligned} \tag{4.21}$$

Combining (4.15) and (4.21) with (4.11) and Lemma 4.3(iv), we get

$$\begin{aligned} \int_{\Omega} -f_\delta (u - \widetilde{u}) &= \int_{\Omega_0} -f_\delta (u - \widetilde{u}) + \int_{\Omega \setminus \Omega_0} -\frac{1}{\delta}(u - \varphi)(u - \widetilde{u}) \\ &\le C + C\left(\int_{\partial\Omega} (u_\nu^+)^n\right)^{1/n}. \end{aligned} \tag{4.22}$$

Thus, for $n = 2$, we deduce from (4.10) and (4.22) that

$$\begin{aligned} \int_{\partial\Omega} K\psi u_\nu^2 &\le C + C\left(\int_{\partial\Omega} (u_\nu^+)^2\right)^{1/2} + \int_{\Omega} -f_\delta (u - \widetilde{u}) \\ &\le C + C\left(\int_{\partial\Omega} (u_\nu^+)^2\right)^{1/2}. \end{aligned} \tag{4.23}$$

From (4.23), the Hölder inequality, and $\inf_{\partial\Omega} \psi > 0$, we easily obtain

$$\int_{\partial\Omega} (u_\nu^+)^n \le C$$

and (4.2) follows.

Case 2. $\overline{c}_0 \le c_0$. From (4.18) and Lemma 3.3(i), and recalling (4.11), we obtain

$$\begin{aligned} &\int_{\Omega} -f_\delta (u - \widetilde{u}) \\ &\quad\le C\|u\|_{L^\infty(\Omega)} + C + \overline{c}_0 C_3(\Omega_0, \Omega)\|u\|^2_{L^\infty(\Omega)} \\ &\qquad - \rho \int_{\Omega_0} |u - \widetilde{u}|^2\, dx + \int_{\Omega \setminus \Omega_0} -\frac{1}{\delta}(u - \varphi)(u - \widetilde{u}) \\ &\quad\le C\|u\|_{L^\infty(\Omega)} + C + \overline{c}_0 C_4(\Omega_0, \Omega) \int_{\Omega} |u|^2\, dx - \frac{\rho}{2} \int_{\Omega_0} |u|^2\, dx \\ &\qquad + \int_{\Omega \setminus \Omega_0} -\frac{|u|^2}{2\delta}. \end{aligned} \tag{4.24}$$

Thus, if $\min\{\rho, \frac{1}{\delta}\} > 4\bar{c}_0 C_4(\Omega_0, \Omega)$, then (4.24) gives

$$\int_\Omega -f_\delta(u-\tilde{u}) \leq C\|u\|_{L^\infty(\Omega)} + C. \tag{4.25}$$

Now, for $n = 2$, we deduce from (4.10), (4.25), and Lemma 4.3(iv) that

$$\begin{aligned}\int_{\partial\Omega} K\psi u_\nu^2 &\leq C + C\left(\int_{\partial\Omega}(u_\nu^+)^2\right)^{1)/2} + \int_\Omega -f_\delta(u-\tilde{u})\\ &\leq C + C\left(\int_{\partial\Omega}(u_\nu^+)^2\right)^{1/2}. \qquad (4.26)\end{aligned}$$

From (4.26), the Hölder inequality, and $\inf_{\partial\Omega}\psi > 0$, we easily obtain

$$\int_{\partial\Omega}(u_\nu^+)^2 \leq C.$$

This completes the proof of (4.2) in all cases. □

In the following lemma, we establish universal a priori estimates for solutions to (2.6)–(2.7). These estimates do not depend on ε.

LEMMA 4.4. *Let $n = 2$, $q > n$, and $0 \leq \theta < 1/n$. Assume that* (2.3)–(2.5) *are satisfied and $\rho > 0$. Assume that $\varphi \in W^{4,q}(\Omega)$ with $\inf_\Omega \det D^2\varphi > 0$, and $\psi \in W^{2,q}(\Omega)$ with $\inf_{\partial\Omega}\psi > 0$. Assume that one of the following conditions holds*:

(i) $\bar{c}_0 = \bar{C}_* = 0$.

(ii) *ρ is large and ε is small* (*depending only on $\bar{c}_0 + \bar{C}_*$, Ω_0, and Ω*).

Let $u_\varepsilon \in W^{4,q}(\Omega)$ be a uniformly convex solution to the system (2.6)–(2.7). *Then, there is a universal constant $C > 0$* (*depending also on $\inf_\Omega \det D^2\varphi$ but independent of ε*) *such that*

$$\int_{\partial\Omega}\varepsilon(u_\varepsilon)_\nu^2 + \rho\int_{\Omega_0}|u_\varepsilon-\varphi|^2\,dx + \int_{\Omega\setminus\Omega_0}\frac{1}{\varepsilon}|u_\varepsilon-\varphi|^2\,dx \leq C. \tag{4.27}$$

PROOF. Let $\hat{C}_* := \bar{c}_0 + \bar{C}_*$. Let $\bar{u} = \varphi$ and $\bar{f} = \bar{U}^{ij}\bar{w}_{ij}$ where $\bar{w} = (\det D^2\bar{u})^{\theta-1}$ and $\bar{U} = (\bar{U}^{ij})$ is the cofactor matrix of $D^2\bar{u}$. Since $q > n$ and $\bar{u} \in W^{4,q}(\Omega)$ with $\inf_\Omega \det D^2\bar{u} > 0$, we have

$$\|\bar{f}\|_{L^1(\Omega)} \leq C. \tag{4.28}$$

We redo the estimates in Lemma 4.2 for u_ε where we replace $\tilde{u}$ by $\bar{u}$. First, (4.5) becomes

$$\begin{aligned}&\int_{\partial\Omega} w_\varepsilon U_\varepsilon^{ij}((u_\varepsilon)_j - \bar{u}_j)\nu_i + \int_\Omega U_\varepsilon^{ij}(w_\varepsilon)_{ij}(u_\varepsilon-\bar{u}) \qquad (4.29)\\ &\quad + \int_{\partial\Omega}\bar{w}\bar{U}^{ij}(\bar{u}_j-(u_\varepsilon)_j)\nu_i + \int_\Omega \bar{U}^{ij}\bar{w}_{ij}(\bar{u}-u_\varepsilon) \leq 0.\end{aligned}$$

To simplify notation, we use f_ε to denote $f_\varepsilon(\cdot, u_\varepsilon, Du_\varepsilon, D^2u_\varepsilon)$.

From $U_\varepsilon^{ij}(w_\varepsilon)_{ij} = \varepsilon^{-1} f_\varepsilon$, Lemma 4.3(iv), (4.28), and the uniform boundedness of $\overline{u}$, $\overline{w}$, and $D\overline{u}$, (4.29) becomes

$$\int_{\partial\Omega} (u_\varepsilon)_\nu^2 \leq C\left(\int_{\partial\Omega} (u_\varepsilon)_\nu^2\right)^{1/2} + \int_\Omega -\varepsilon^{-1} f_\varepsilon (u_\varepsilon - \overline{u}) dx + C. \tag{4.30}$$

This estimate is similar to (4.10), where now we also use $\inf_{\partial\Omega} \psi > 0$ and the uniform convexity of $\partial\Omega$ to absorb $\inf_{\partial\Omega} \psi > 0$ and the curvature of $\partial\Omega$ to the right-hand side of (4.10). From Young's inequality, we can absorb the first term on the right-hand side of (4.30) to its left-hand side. Then, multiplying both sides by ε, we get

$$\int_{\partial\Omega} \varepsilon (u_\varepsilon)_\nu^2 \leq C + C \int_\Omega -f_\varepsilon (u_\varepsilon - \overline{u}). \tag{4.31}$$

As in the estimates for A in (4.12), we have

$$\begin{aligned} A_\varepsilon &:= \int_{\Omega_0} -f^0(x, u_\varepsilon(x))(u_\varepsilon - \overline{u}) dx \\ &\leq C + C \int_{\Omega_0} |u_\varepsilon| dx - \rho \int_{\Omega_0} |u_\varepsilon - \overline{u}|^2 \, dx. \end{aligned}$$

From Lemma 3.3(i) and the inequality $|u_\varepsilon|^2 \leq 2(|u_\varepsilon - \varphi|^2 + |\varphi|^2)$, we have

$$\|u_\varepsilon\|^2_{L^\infty(\Omega)} \leq C(\Omega) \int_\Omega |u_\varepsilon|^2 \, dx + C \leq C(\Omega) \int_\Omega |u_\varepsilon - \varphi|^2 \, dx + C. \tag{4.32}$$

Applying (4.18) to u_ε, f_ε, and $\overline{u}$ and taking into account (4.32), $\rho > 0$, and $\widehat{C}_* := \overline{c}_0 + \overline{C}_*$, we have

$$\begin{aligned} &\int_{\Omega_0} -f_\varepsilon (u_\varepsilon - \overline{u}) \\ &\leq C + C \int_{\Omega_0} |u_\varepsilon| dx - \rho \int_{\Omega_0} |u_\varepsilon - \overline{u}|^2 \, dx \\ &\quad + \widehat{C}_* C_5(\Omega_0, \Omega)\big(1 + \|u_\varepsilon\|^2_{L^\infty(\Omega)}\big) \\ &\leq C - \frac{\rho}{2} \int_{\Omega_0} |u_\varepsilon - \overline{u}|^2 \, dx + \widehat{C}_* C_6(\Omega_0, \Omega) \int_\Omega |u_\varepsilon - \varphi|^2 \, dx. \end{aligned} \tag{4.33}$$

It follows from (4.31), (4.33), $\overline{u} = \varphi$ in Ω, and $f_\varepsilon = \frac{1}{\varepsilon}(u_\varepsilon - \varphi)$ on $\Omega \setminus \Omega_0$ that

$$\begin{aligned} \int_{\partial\Omega} \varepsilon (u_\varepsilon)_\nu^2 &\leq C + \int_\Omega -f_\varepsilon (u_\varepsilon - \overline{u}) dx \\ &= C + \int_{\Omega_0} -f_\varepsilon (u_\varepsilon - \overline{u}) + \int_{\Omega \setminus \Omega_0} -f_\varepsilon (u_\varepsilon - \overline{u}) dx \leq \end{aligned} \tag{4.34}$$

$$\leq C - \frac{\rho}{2}\int_{\Omega_0}|u_\varepsilon-\varphi|^2\,dx + \int_{\Omega\setminus\Omega_0} -\frac{1}{\varepsilon}(u_\varepsilon-\varphi)^2\,dx$$
$$+\hat{C}_* C_6(\Omega_0,\Omega)\int_\Omega |u_\varepsilon-\varphi|^2\,dx.$$

Case 1. $\rho > 0$ *and* $\bar{c}_0 = \bar{C}_* = 0$. In this case, $\hat{C}_* = 0$ in (4.34) and (4.27) follows from this inequality.

Case 2. ρ *is sufficiently large and* ε *is sufficiently small* (*depending only on* $\bar{c}_0 + \bar{C}_*$, Ω_0 *and* Ω). When

$$\min\left\{\frac{1}{\varepsilon},\rho\right\} > 4\hat{C}_* C_6(\Omega_0,\Omega) + 2,$$

then clearly (4.34) gives (4.27). □

We prove the uniqueness part of Theorem 2.1 in the following lemma.

LEMMA 4.5. *Let* $n \geq 2$, $q > n$, *and* $0 \leq \theta < 1/n$. *Assume that* $\varphi \in W^{4,q}(\Omega)$ *and* $\psi \in W^{2,q}(\Omega)$ *with* $\inf_{\partial\Omega}\psi > 0$. *Assume that* (2.3)–(2.5) *are satisfied with* $\bar{c}_0 = \bar{C}_* = 0$. *Then the problem* (2.1)–(2.2) *has at most one uniformly convex solution* $u \in W^{4,q}(\Omega)$.

PROOF. Suppose that $u \in W^{4,q}(\Omega)$ and $\hat{u} \in W^{4,q}(\Omega)$ are two uniformly convex solutions of (2.1)–(2.2). Let $\hat{U} = (\hat{U}^{ij})$ be the cofactor matrix of $D^2\hat{u}$ and let $\hat{w} = G'(\det D^2\hat{u})$. Here, $G(t) = \frac{t^\theta - 1}{\theta}$ for $t > 0$ (when $\theta = 0$, we set $G(t) = \log t$). We use the same notation as in the proof of Lemma 4.2. Then, we obtain as in (4.5) the estimate

$$\begin{aligned}
(4.35)\qquad 0 &\geq \int_\Omega wU^{ij}(u-\hat{u})_{ij} + \hat{w}\hat{U}^{ij}(\hat{u}-u)_{ij}\\
&= \int_{\partial\Omega}\psi(U^{ij}-\hat{U}^{ij})(u_j-\hat{u}_j)\nu_i\\
&\quad + \int_\Omega (f_\delta(\cdot,u,Du,D^2u) - f_\delta(\cdot,\hat{u},D\hat{u},D^2\hat{u}))(u-\hat{u})dx\\
&= \int_{\partial\Omega}\psi(U^{\nu\nu}-\hat{U}^{\nu\nu})(u_\nu-\hat{u}_\nu)\\
&\quad + \int_{\Omega_0}[f^0(x,u(x)) - f^0(x,\hat{u}(x))](u-\hat{u})dx\\
&\quad + \int_{\Omega_0}\operatorname{div}\big(\nabla_p F^1(x,D\hat{u}(x)) - \nabla_p F^1(x,Du(x))\big)(u-\hat{u})dx\\
&\quad + \frac{1}{\delta}\int_{\Omega\setminus\Omega_0}(u-\hat{u})^2\,dx.
\end{aligned}$$

By (2.3), the integral concerning f^0 in the above expression is nonnegative. Hence, integrating by parts and using $\bar{c}_0 = \bar{C}_* = 0$, we find that

$$
\begin{aligned}
0 \geq &\int_{\partial\Omega} \psi(U^{\nu\nu} - \widehat{U}^{\nu\nu})(u_\nu - \hat{u}_\nu) \\
&+ \int_{\Omega_0} \big(\nabla_p F^1(x, D\hat{u}(x)) - \nabla_p F^1(x, Du(x))\big)(D\hat{u} - Du)\,dx.
\end{aligned} \tag{4.36}
$$

It is clear from (4.8) that if $u_\nu > \hat{u}_\nu$, then $U^{\nu\nu} > \widehat{U}^{\nu\nu}$. Therefore, from the convexity of $F^1(x,p)$ in p, which follows from (2.4), we deduce that $u_\nu = \hat{u}_\nu$ on $\partial\Omega$ and that (4.36) must be an equality. This implies that (4.35) must be an equality. From the derivation of (4.35), using the strict concavity of G as in (4.3) and (4.4) but applied to $\det D^2 u$ and $\det D^2\hat{u}$, and the fact that (4.35) is now an equality, we deduce that $\det D^2 u = \det D^2 \hat{u}$ in Ω. Hence $u = \hat{u}$ on $\overline{\Omega}$. □

In the next lemma, we establish the universal bounds from below and above for the Hessian determinant of solutions to (2.1)–(2.2).

LEMMA 4.6. *Let* $n = 2$, $q > n$, *and* $0 \leq \theta < 1/n$. *Assume that* $\varphi \in W^{4,q}(\Omega)$ *and* $\psi \in W^{2,q}(\Omega)$ *with* $\inf_{\partial\Omega}\psi > 0$. *Assume that* (2.3)–(2.5) *are satisfied. Suppose that* (**SC**) *holds. Let u be a smooth, uniformly convex solution of the system* (2.1)–(2.2). *There is a universal constant* $C > 0$ *such that*

$$C^{-1} \leq \det D^2 u \leq C \quad \text{in } \Omega.$$

PROOF. To simplify notation, we use f_δ to denote $f_\delta(\cdot, u, Du, D^2u)$.

Step 1. *Lower bound for* $\det D^2 u$. Let $\bar{w} = w + C_*|x|^2$. Let χ_{Ω_0} be the characteristic function of Ω_0, that is, $\chi_{\Omega_0}(x) = 1$ if $x \in \Omega_0$ and $\chi_{\Omega_0}(x) = 0$ if otherwise. Then, using (2.2), (2.4), and the universal bound for u in Lemma 4.2, we find that in Ω the following hold:

$$
\begin{aligned}
U^{ij}\bar{w}_{ij} &= f_\delta + 2C_*\Delta u \\
&\geq -C - |F^1_{p_i x_i}(x, Du)|\chi_{\Omega_0}(x) - F^1_{p_i p_j}(x, Du)u_{ij} + 2C_*\Delta u \\
&\geq -C - C_*|Du|\chi_{\Omega_0} := -\hat{f}.
\end{aligned}
$$

By combining the universal bound for u in Lemma 4.2 and Lemma 3.3(ii), we obtain that $\|\hat{f}\|_{L^\infty(\Omega)} \leq C$ for a universal constant C. Thus, $\|\hat{f}\|_{L^2(\Omega)} \leq C$. In two dimensions, we have

$$\det(U^{ij}) = (\det D^2 u)^{n-1} = \det D^2 u = w^{\frac{1}{\theta - 1}},$$

where we have used the second equation of (2.1) for the last equality. Now, we apply the ABP estimate [18, theorem 2.21] to $\bar{w}$ on Ω to obtain

$$
\begin{aligned}
\|\bar{w}\|_{L^\infty(\Omega)} &\leq \sup_{\partial\Omega}\bar{w} + C_2\operatorname{diam}(\Omega)\left\|\frac{\hat{f}}{(\det U^{ij})^{1/2}}\right\|_{L^2(\Omega)} \\
&\leq \sup_{\partial\Omega}(\psi + C_*|x|^2) + C\|w^{\frac{1}{2(1-\theta)}}\hat{f}\|_{L^2(\Omega)}.
\end{aligned}
$$

Clearly, $\bar{w} \geq w > 0$. Therefore, the above estimates and $0 \leq \theta < \frac{1}{2}$ give

$$\begin{aligned}\|w\|_{L^\infty(\Omega)} &\leq C + C\|w^{\frac{1}{2(1-\theta)}}\hat{f}\|_{L^2(\Omega)}\\ &\leq C + C\|w^{\frac{1}{2(1-\theta)}}\|_{L^\infty(\Omega)}\|\hat{f}\|_{L^2(\Omega)} \leq C + C\|w\|_{L^\infty(\Omega)}^{\frac{1}{2(1-\theta)}}.\end{aligned}$$

Because $0 \leq \theta < \frac{1}{2}$, we have $\frac{1}{2(1-\theta)} < 1$. It follows that w is bounded from above. Since $\det D^2u = w^{\frac{1}{\theta-1}}$, we conclude that $\det D^2 u$ is bounded from below by a universal constant $C > 0$.

Step 2. *Upper bound for* $\det D^2u$. Since $\det D^2 u = w^{\frac{1}{\theta-1}}$ where $0 \leq \theta < \frac{1}{2}$, to prove the universal upper bound for $\det D^2u$, we only need to obtain a positive lower bound for w. For this, we use the ABP maximum principle; see [9, 31] for a slightly different argument.

First, we will use the following splitting of f_δ:

$$U^{ij}w_{ij} = f_\delta = \gamma_1\Delta u + g \quad \text{in } \Omega, \tag{4.37}$$

where

$$\gamma_1(x) = \begin{cases} -\frac{F^1_{p_ip_j}(x,Du)u_{ij}}{\Delta u}, & x \in \Omega_0,\\ 0, & x \in \Omega\setminus\Omega_0,\end{cases}$$

and

$$g(x) = \begin{cases} f^0(x,u(x)) - F^1_{p_ix_i}(x, Du(x)), & x \in \Omega_0,\\ \frac{1}{\delta}(u(x)-\varphi(x)), & x \in \Omega\setminus\Omega_0.\end{cases} \tag{4.38}$$

From (2.4), we have

$$\|\gamma_1\|_{L^\infty(\Omega)} \leq C_*. \tag{4.39}$$

We claim that

$$\|g\|_{L^\infty(\Omega)} \leq C. \tag{4.40}$$

Indeed, from (2.3) and (2.4), we easily find that for all $x \in \Omega$

$$|g(x)| \leq \begin{cases} \eta(\|u\|_{L^\infty(\Omega)}) + c_0|Du(x)| + C_*, & x \in \Omega_0,\\ \frac{1}{\delta}(\|u\|_{L^\infty(\Omega)} + \|\varphi\|_{L^\infty(\Omega)}), & x \in \Omega\setminus\Omega_0.\end{cases} \tag{4.41}$$

Now, we use the universal bound for u in Lemma 4.2 and Lemma 3.3(ii) to derive (4.40) from (4.41).

Recall from (i) that $\det D^2u \geq C_1$. Thus,

$$w \det D^2u = (\det D^2u)^\theta \geq C_1^\theta := c > 0.$$

From (4.37) and $\gamma_1 \leq 0$, we find that $f_\delta \leq g$. By (4.40), we find that $f_\delta^+ \leq |g|$ is bounded by a universal constant. Let

$$M = \frac{|f_\delta^+|_{L^\infty(\Omega)} + 1}{2c} < \infty \quad \text{and} \quad v^\varepsilon = \log(w+\varepsilon) - Mu \in W^{2,q}(\Omega)$$

where $\varepsilon > 0$. Then, in Ω, we have

$$\begin{aligned} u^{ij} v^{\varepsilon}_{ij} = u^{ij}\left(\frac{w_{ij}}{w+\varepsilon} - \frac{w_i w_j}{(w+\varepsilon)^2} - M u_{ij}\right) &\leq \frac{u^{ij} w_{ij}}{w+\varepsilon} - nM \\ &= \frac{f_\delta}{(w+\varepsilon)\det D^2 u} - 2M \\ &\leq \frac{\|f_\delta^+\|_{L^\infty(\Omega)}}{c} - 2M < 0. \end{aligned}$$

By the ABP estimate (see [18, theorem 2.21]) for $-v^\varepsilon$ in Ω, we have

$$v^\varepsilon \geq \inf_{\partial\Omega} v^\varepsilon \geq \log(\inf_{\partial\Omega} \psi) - M\varphi \geq -C \text{ in } \Omega.$$

From $v^\varepsilon = \log(w+\varepsilon) - Mu$ and the universal bound for u in Lemma 4.2, we obtain $\log(w+\varepsilon) \geq -C$. Thus, letting $\varepsilon \to 0$, we get $w \geq e^{-C}$ as desired. □

Now, we are in a position to prove Theorem 4.1.

PROOF OF THEOREM 4.1. To simplify notation, we denote $f_\delta(\cdot, u, Du, D^2u)$ by f_δ. From Lemma 4.6, we can find a universal constant $C > 0$ such that

$$C^{-1} \leq \det D^2 u \leq C \quad \text{in } \Omega. \tag{4.42}$$

From $\varphi \in W^{4,q}(\Omega)$ with $q > n$, we have $\varphi \in C^3(\overline{\Omega})$ by the Sobolev embedding theorem. By assumption, Ω is bounded, smooth, and uniformly convex. From $u = \varphi$ on $\partial\Omega$ and (4.42), we can apply the global $W^{2,1+\varepsilon_0}$ estimates for the Monge-Ampère equation in Theorem 3.2 to conclude that

$$\|D^2 u\|_{L^{1+\varepsilon_0}(\Omega)} \leq C_1 \tag{4.43}$$

for some universal constants $\varepsilon_0 > 0$ and $C_1 > 0$. Recall the following splitting of f_δ in (4.37):

$$f_\delta = \gamma_1 \Delta u + g. \tag{4.44}$$

Thus, from (4.43), (4.39), and (4.40), we find that

$$\|f_\delta\|_{L^{1+\varepsilon_0}(\Omega)} \leq C_2$$

for a universal constant $C_2 > 0$. From $\psi \in W^{2,q}(\Omega)$ with $q > n$, we have $\psi \in C^1(\overline{\Omega})$ by the Sobolev embedding theorem. Now, we apply the global Hölder estimates for the linearized Monge-Ampère equation in Theorem 3.1 to $U^{ij} w_{ij} = f_\delta$ in Ω with boundary value $w = \psi \in C^1(\partial\Omega)$ on $\partial\Omega$ to conclude that $w \in C^\alpha(\overline{\Omega})$ with

$$\|w\|_{C^\alpha(\overline{\Omega})} \leq C\big(\|\psi\|_{C^1(\partial\Omega)} + \|f_\delta\|_{L^{1+\varepsilon_0}(\Omega)}\big) \leq C_3 \tag{4.45}$$

for universal constants $\alpha \in (0,1)$ and $C_3 > 0$. Now, we note that u solves the Monge-Ampère equation

$$\det D^2 u = w^{\frac{1}{\theta-1}}$$

with the right-hand side being in $C^{\alpha}(\overline{\Omega})$ and boundary value $\varphi \in C^3(\partial\Omega)$ on $\partial\Omega$. Therefore, by the global $C^{2,\alpha}$ estimates for the Monge-Ampère equation [28, 32], we have $u \in C^{2,\alpha}(\overline{\Omega})$ with universal estimates

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \leq C_4 \quad \text{and} \quad C_4^{-1} I_2 \leq D^2 u \leq C_4 I_2. \tag{4.46}$$

As a consequence, the second-order operator $U^{ij}\partial_{ij}$ is uniformly elliptic with Hölder continuous coefficients.

Recalling (4.44) and using (4.39) together with (4.40), we obtain

$$\|f_\delta\|_{L^\infty(\Omega)} \leq C_5. \tag{4.47}$$

Thus, from the equation $U^{ij}w_{ij} = f_\delta$ with boundary value $w = \psi$ where $\psi \in W^{2,q}(\Omega)$, we conclude that $w \in W^{2,q}(\Omega)$ and therefore $u \in W^{4,q}(\Omega)$ with universal estimate

$$\|u\|_{W^{4,q}(\Omega)} \leq C_6. \qquad \square$$

5 Proofs of the Main Theorems

In this section, we prove Theorems 2.1, 2.3, 2.6, and 2.8.

PROOF OF THEOREM 2.1. The existence and uniqueness result in (ii) follows from the existence in (i) and the uniqueness result in Lemma 4.5. It remains to prove (i).

The proof of (i) uses the a priori estimates in Theorem 4.1 and degree theory as in [9, 31] (see also [21]). Since the proof is short, we include it here. Assume $q > n$.

Fix $\alpha \in (0, 1)$. For a large constant $R > 1$ to be determined, define a bounded set $D(R)$ in $C^{\alpha}(\overline{\Omega})$ as follows:

$$D(R) = \{v \in C^{\alpha}(\overline{\Omega}) \mid v \geq R^{-1},\ \|v\|_{C^{\alpha}(\overline{\Omega})} \leq R\}.$$

For $t \in [0, 1]$, we will define an operator $\Phi_t \colon D(R) \to C^{\alpha}(\overline{\Omega})$ as follows. Given $w \in D(R)$, define $u \in C^{2,\alpha}(\overline{\Omega})$ to be the unique uniformly convex solution to

$$\begin{cases} \det D^2 u = w^{\frac{1}{\theta-1}} & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

The existence of u follows from the boundary regularity result of the Monge-Ampère equation established by Trudinger and Wang [32]. Next, let $w_t \in W^{2,q}(\Omega)$ be the unique solution to the equation

$$\begin{cases} U^{ij}(w_t)_{ij} = t f_\delta(\cdot, u, Du, D^2 u) & \text{in } \Omega, \\ w_t = t\psi + (1-t) & \text{on } \partial\Omega. \end{cases} \tag{5.2}$$

Because $q > n$, w_t lies in $C^{\alpha}(\overline{\Omega})$. We define Φ_t to be the map sending w to w_t.

We note that:

(i) $\Phi_0(D(R)) = \{1\}$, and in particular, Φ_0 has a unique fixed point.

(ii) The map $[0,1]\times D(R)\to C^{\alpha}(\overline{\Omega})$ given by $(t,w)\mapsto \Phi_t(w)$ is continuous.

(iii) Φ_t is compact for each $t\in[0,1]$.

(iv) For every $t\in[0,1]$, if $w\in D(R)$ is a fixed point of Φ_t, then $w\notin \partial D(R)$.

Indeed, part (iii) follows from the standard a priori estimates for the two separate equations (5.1) and (5.2). For part (iv), let $w>0$ be a fixed point of Φ_t. Then $w\in W^{2,q}(\Omega)$ and hence $u\in W^{4,q}(\Omega)$. Next we apply Theorem 4.1 to obtain $w>R^{-1}$ and $\|w\|_{C^{\alpha}(\overline{\Omega})}<R$ for some R sufficiently large, depending only on the initial data but independent of $t\in[0,1]$.

Then the Leray-Schauder degree of Φ_t is well-defined for each t and is constant on $[0,1]$ (see [25, theorem 2.2.4], for example). Φ_0 has a fixed point and hence Φ_1 must also have a fixed point w, giving rise to a uniformly convex solution $u\in W^{4,q}(\Omega)$ of our second boundary value problem (2.1)–(2.2). □

PROOF OF THEOREM 2.3. If $\bar{c}_0=\bar{C}_*=0$, then the existence of a unique uniformly convex solution $u_\varepsilon\in W^{4,q}(\Omega)$ to the system (2.6)–(2.7) for all $q\in(n,\infty)$ follows from Theorem 2.1(ii). If ρ is sufficiently large (depending only on $\bar{c}_0+\bar{C}_*$, Ω_0 and Ω), then the existence of a uniformly convex solution $u_\varepsilon\in W^{4,q}(\Omega)$ to the system (2.6)–(2.7) for all $q\in(n,\infty)$ follows from Theorem 2.1(i); moreover, since we are interested in the limit of $\{u_\varepsilon\}$ when $\varepsilon\to 0$, we can assume that ε is sufficiently small (depending only on $\bar{c}_0+\bar{C}_*$, Ω_0, and Ω) so that Lemma 4.4 applies.

In all cases, by Lemma 4.4, there is a universal constant C independent of ε such that

$$\int_{\partial\Omega}\varepsilon(u_\varepsilon)_\nu^2+\rho\int_{\Omega_0}|u_\varepsilon-\varphi|^2\,dx+\int_{\Omega\setminus\Omega_0}\frac{1}{\varepsilon}|u_\varepsilon-\varphi|^2\,dx\le C. \tag{5.3}$$

Step 1. *A subsequence of* $\{u_\varepsilon\}$ *converges.* First, we show that, up to extraction of a subsequence, u_ε converges uniformly on compact subsets of Ω to a convex function $u\in\bar{S}[\varphi,\Omega_0]$ where $\bar{S}[\varphi,\Omega_0]$ is defined as in (1.2). Indeed, by Lemma 3.3(i), $u_\varepsilon=\varphi$ on $\partial\Omega$ and (5.3) where $\rho>0$, we have

$$\|u_\varepsilon\|^2_{L^\infty(\Omega)}\le C(n,\Omega,\max_{\partial\Omega}u_\varepsilon)+C(n,\Omega)\int_\Omega|u_\varepsilon|^2\,dx\le C$$

for a universal constant $C>0$. It follows that the sequence $\{u_\varepsilon\}$ is uniformly bounded. By Lemma 3.3(ii), $|Du_\varepsilon|$ is uniformly bounded on compact subsets of Ω. Thus, by the Arzelà-Ascoli theorem, up to extraction of a subsequence, u_ε converges uniformly on compact subsets of Ω to a convex function u. Moreover, we can assume that u_ε converges to u on $W^{1,2}(\Omega_0)$. Using (5.3), we get $u\in\bar{S}[\varphi,\Omega_0]$.

Let $G(t)=\frac{t^\theta-1}{\theta}$ for $t>0$ (when $\theta=0$, we set $G(t)=\log t$).

Next, consider the following functional J_ε over the set of convex functions v on $\overline{\Omega}$:

$$J_\varepsilon(v) = \int_{\Omega_0} [F^0(x, v(x)) + F^1(x, Dv(x))]dx + \frac{1}{2\varepsilon}\int_{\Omega\setminus\Omega_0} (v-\varphi)^2\, dx - \varepsilon\int_\Omega G(\det D^2 v)dx. \tag{5.4}$$

By the Rademacher theorem (see [13, theorem 2, p. 81]), v is differentiable a.e. By the Alexandrov theorem (see [13, theorem 1, p. 242]), v is twice differentiable a.e. and at those points of twice differentiability, we denote, with a slight abuse of notation, $D^2 v$ its Hessian matrix. Thus, the functional J_ε is well-defined with this convention.

Let $U_\varepsilon^{\nu\nu} = U_\varepsilon^{ij}\nu_i\nu_j$ be as in the proof of Lemma 4.2. Let τ be the tangential direction along $\partial\Omega$. Let K be the curvature of $\partial\Omega$. Since $u_\varepsilon = \varphi$ on $\partial\Omega$, in two dimensions, we have as in (4.8)

$$U_\varepsilon^{\nu\nu} = (u_\varepsilon)_{\tau\tau} = K(u_\varepsilon)_\nu - K\varphi_\nu + \varphi_{\tau\tau}. \tag{5.5}$$

Step 2. *Almost minimality property of* u_ε. We show that if v is a convex function in $\overline{\Omega}$ with $v = \varphi$ in a neighborhood of $\partial\Omega$, then

$$J_\varepsilon(v) - J_\varepsilon(u_\varepsilon) \geq \varepsilon\int_{\partial\Omega} \psi U_\varepsilon^{\nu\nu}\partial_\nu(u_\varepsilon - \varphi) + \int_{\partial\Omega_0} (v-u_\varepsilon)\nabla_p F^1(x, Du_\varepsilon(x))\cdot \nu_0\, dS. \tag{5.6}$$

The proof of (5.6) uses mollification to deal with general convex functions v. For $h > 0$, let

$$\Omega_h = \{x\in\Omega \mid \operatorname{dist}(x,\partial\Omega) > h\}$$

and

$$v_h(x) = h^{-n}\int_\Omega \phi\left(\frac{x-y}{h}\right)v(y)dy \quad \text{for } x\in\Omega_h$$

where $\phi \geq 0$, $\phi\in C_0^\infty(\mathbb{R}^n)$, $\operatorname{supp}\phi\subset B_1(0)$, and $\int_{\mathbb{R}^n}\phi\, dx = 1$. Clearly, $v_h\to v$ uniformly on compact subsets of Ω. Since $v = \varphi$ near $\partial\Omega$ and $\varphi\in C^{3,1}(\overline{\Omega})$ is uniformly convex in $\overline{\Omega}$, we can extend v_h to be a uniformly convex $C^3(\overline{\Omega})$ function, still denoted by v_h, such that

$$D^k v_h \to D^k v \text{ in a neighborhood of } \partial\Omega \text{ for all } k\leq 2. \tag{5.7}$$

By [30, lemma 6.3], we have

$$\lim_{h\to 0}\int_{\Omega_h} G(\det D^2 v_h) = \int_\Omega G(\det D^2 v).$$

This together with (5.7) implies that

$$\lim_{h\to 0} J_\varepsilon(v_h) = J_\varepsilon(v). \tag{5.8}$$

Now, we estimate $J_\varepsilon(v_h) - J_\varepsilon(u_\varepsilon)$.

As in the proof of Lemma 4.2, we have

$$-G(\det D^2 v_h) + G(\det D^2 u_\varepsilon) \geq w_\varepsilon U_\varepsilon^{ij}(u_\varepsilon - v_h)_{ij}.$$

Integrating by parts twice, recalling $U_\varepsilon^{ij}(w_\varepsilon)_{ij} = \varepsilon^{-1} f_\varepsilon(\cdot, u_\varepsilon, Du_\varepsilon, D^2 u_\varepsilon)$, and (4.6), we get

$$\begin{aligned}
&\int_\Omega (-G(\det D^2 v_h) + G(\det D^2 u_\varepsilon))dx \\
&\quad\geq \int_\Omega w_\varepsilon U_\varepsilon^{ij}(u_\varepsilon - v_h)_{ij} \\
&\quad= \int_\Omega \varepsilon^{-1} f_\varepsilon(\cdot, u_\varepsilon, Du_\varepsilon, D^2 u_\varepsilon)(u_\varepsilon - v_h)dx \\
&\qquad - \int_{\partial\Omega} (w_\varepsilon)_i U_\varepsilon^{ij}(u_\varepsilon - v_h)\nu_j + \int_{\partial\Omega} \psi U_\varepsilon^{\nu\nu} \partial_\nu (u_\varepsilon - v_h).
\end{aligned} \tag{5.9}$$

From the convexity of F^0, F^1, and $(v - \varphi)^2$, we have

$$\begin{aligned}
J_\varepsilon(v_h) - J_\varepsilon(u_\varepsilon) \geq &\int_{\Omega_0} \big[f^0(x, u_\varepsilon(x))(v_h - u_\varepsilon) \\
&\quad + \nabla_p F^1(x, Du_\varepsilon(x))(Dv_h - Du_\varepsilon)\big] dx \\
&+ \frac{1}{\varepsilon}\int_{\Omega\setminus\Omega_0} (u_\varepsilon - \varphi)(v_h - u_\varepsilon)dx \\
&+ \varepsilon \int_\Omega (-G(\det D^2 v_h) + G(\det D^2 u_\varepsilon))dx.
\end{aligned} \tag{5.10}$$

In view of (2.7) and (5.9), we can integrate by parts the right-hand side of (5.10) to get, after a simple cancellation,

$$\begin{aligned}
&J_\varepsilon(v_h) - J_\varepsilon(u_\varepsilon) \\
&\quad\geq -\varepsilon \int_{\partial\Omega} (w_\varepsilon)_i U_\varepsilon^{ij}(u_\varepsilon - v_h)\nu_j + \varepsilon \int_{\partial\Omega} \psi U_\varepsilon^{\nu\nu} \partial_\nu (u_\varepsilon - v_h) \\
&\qquad + \int_{\partial\Omega_0} (v_h - u_\varepsilon)\nabla_p F^1(x, Du_\varepsilon(x)) \cdot \nu_0.
\end{aligned} \tag{5.11}$$

By (5.7), the right-hand side of (5.11) tends to the right-hand side of (5.6) when $h \to 0$. On the other hand, in view of (5.8), the left-hand side of (5.11) tends to the left-hand side of (5.6) when $h \to 0$. Therefore, (5.6) is proved by letting $h \to 0$ in (5.11).

Step 3. *Minimality of* u. We show that u (in Step 1) is a minimizer of the functional J defined by (1.7) over $\overline{S}[\varphi, \Omega_0]$. For all $v \in \overline{S}[\varphi, \Omega_0]$ (extended by φ on $\Omega \setminus \Omega_0$), we use (5.6) to conclude that

$$J_\varepsilon(v_\varepsilon) - J_\varepsilon(u_\varepsilon) \geq \varepsilon \int_{\partial\Omega} \psi U_\varepsilon^{\nu\nu} \partial_\nu (u_\varepsilon - \varphi) - O(\varepsilon)$$

where

$$v_\varepsilon = (1-\varepsilon)v + \varepsilon\varphi \in \overline{S}[\varphi, \Omega_0].$$

Since $\lim_{\varepsilon\to 0} J(v_\varepsilon) = J(v)$, it follows that

$$J(v) \geq \liminf_{\varepsilon} J(u_\varepsilon) + \liminf_{\varepsilon} \varepsilon\left[\int_\Omega G\big(\det D^2 v_\varepsilon\big) - G\big(\det D^2 u_\varepsilon\big)\right] + \liminf_{\varepsilon} \varepsilon \int_{\partial\Omega} \psi U_\varepsilon^{\nu\nu}\partial_\nu(u_\varepsilon - \varphi). \tag{5.12}$$

From (5.3), we have

$$\int_{\partial\Omega} |(u_\varepsilon)_\nu| \leq C\varepsilon^{-1/2}$$

and hence, invoking (5.5), one finds that

$$\varepsilon \int_{\partial\Omega} \psi U_\varepsilon^{\nu\nu}\partial_\nu(u_\varepsilon - \varphi) \geq -C\varepsilon \int_{\partial\Omega} [1 + |(u_\varepsilon)_\nu|] \geq -C\varepsilon^{1/2}. \tag{5.13}$$

Observe from $0 \leq \theta < \frac{1}{2}$ that $G(d) \leq C(1 + d^{1/2})$ for all $d > 0$. It follows that

$$\begin{aligned}\int_\Omega G\big(\det D^2 u_\varepsilon\big) &\leq C \int_\Omega \big(1 + \big(\det D^2 u_\varepsilon\big)^{1/2}\big)dx \\ &\leq C \int_\Omega (1 + \Delta u_\varepsilon)dx \\ &= C\left(|\Omega| + \int_{\partial\Omega} (u_\varepsilon)_\nu\right) \leq C(1 + \varepsilon^{-1/2}).\end{aligned} \tag{5.14}$$

Note that $D^2 v_\varepsilon \geq \varepsilon D^2\varphi$. Therefore $\det D^2 v_\varepsilon \geq \varepsilon^2 \det D^2 \varphi \geq C_1\varepsilon^2$ in Ω for $C_1 > 0$, and hence

$$\varepsilon \int_\Omega G(\det D^2 v_\varepsilon) \geq \varepsilon \int_\Omega G(C_1\varepsilon^2) \to 0 \quad \text{when } \varepsilon \to 0. \tag{5.15}$$

From (5.12)–(5.15), we easily obtain

$$J(v) \geq \liminf_{\varepsilon} J(u_\varepsilon). \tag{5.16}$$

Since u_ε converges uniformly to u on $\overline{\Omega_0}$, by Fatou's lemma, we have

$$\liminf_{\varepsilon} \int_{\Omega_0} F^0(x, u_\varepsilon(x))dx \geq \int_{\Omega_0} F^0(x, u(x))dx. \tag{5.17}$$

From the convexity of $F^1(x, p)$ in p and from the fact that u_ε converges to u on $W^{1,2}(\Omega_0)$, by lower semicontinuity we have

$$\liminf_{\varepsilon} \int_{\Omega_0} F^1(x, Du_\varepsilon(x))dx \geq \int_{\Omega_0} F^1(x, Du(x))dx. \tag{5.18}$$

Therefore, by combining (5.16)–(5.18), we obtain

$$J(v) \geq \liminf_{\varepsilon} J(u_\varepsilon) \geq J(u) \quad \text{for all } v \in \overline{S}[\varphi, \Omega_0],$$

showing that u is a minimizer of the functional J defined by (1.7) over $\overline{S}[\varphi, \Omega_0]$.

Step 4. *Full convergence of u_ε to the unique minimizer of J.* Since $\rho > 0$, functional J defined by (1.7) over $\bar{S}[\varphi, \Omega_0]$ has a unique minimizer in $\bar{S}[\varphi, \Omega_0]$. Thus, Steps 1 and 3 actually show that the whole sequence $\{u_\varepsilon\}$ converges to the unique minimizer of J. The proof of the theorem is completed. $\square$

PROOF OF THEOREM 2.6. When $H(d) = d^{\theta-1}$ where $0 \le \theta < \frac{1}{2}$, the proof of the uniqueness of solutions in (i) is similar to that of Lemma 4.5, so we omit it. The existence proof uses a priori estimates and degree theory as in Theorem 2.1. Here, we only focus on proving the a priori estimates. The key is to obtain the positive bound from below and above for $\det D^2u$:

$$C^{-1} \le \det D^2 u \le C \quad \text{in } \Omega. \tag{5.19}$$

Once (5.19) is established, we can apply the global $W^{2,1+\varepsilon_0}(\Omega)$ estimates in Theorem 3.2 for u and argue as in the proof of Theorem 4.1 that $u \in C^{2,\alpha}(\bar{\Omega})$. A bootstrap argument concludes the proof.

It remains to prove (5.19).

(i) First, by the convexity of F and u, we have

$$\begin{aligned} U^{ij} w_{ij} &= -\operatorname{div}(\nabla_p F(Du)) = -F_{p_i p_j}(Du) u_{ij} \\ &\qquad\qquad = -\operatorname{trace}(D^2 F(Du) D^2 u) \le 0 \quad \text{in } \Omega. \end{aligned}$$

By the maximum principle, the function w attains its minimum value on $\partial\Omega$. It follows that

$$w \ge \inf_{\partial\Omega} \psi := C > 0 \quad \text{in } \Omega.$$

From the assumptions on H and $w = H(\det D^2 u)$, we deduce that

$$\det D^2 u \le C < \infty \quad \text{in } \Omega.$$

From $\det D^2 u \le C$ in Ω, $u = \varphi$ on $\partial\Omega$, and the uniform convexity of Ω, we can construct an explicit barrier to show that $|Du| \le C$ in Ω for a universal constant C. This together with the boundedness assumption on $F_{p_i p_j}(p)$ gives

$$F_{p_i p_j}(Du(x)) \le C_1 I_2 \quad \text{in } \Omega.$$

We compute, in Ω,

$$\begin{aligned} U^{ij}(w + C_1|x|^2)_{ij} &= -F_{p_i p_j}(Du) u_{ij} + 2C_1 \operatorname{trace}(U^{ij}) \\ &\ge -C_1 \operatorname{trace}(u_{ij}) + 2C_1 \Delta u = C_1 \Delta u \ge 0. \end{aligned}$$

By the maximum principle, $w(x) + C_1|x|^2$ attains it maximum value on the boundary $\partial\Omega$. Recall that $w = \psi$ on $\partial\Omega$. Thus, for all $x \in \Omega$, we have

$$w(x) \le w(x) + C_1|x|^2 \le \max_{\partial\Omega}\big(\psi + C_1|x^2|\big) \le C. \tag{5.20}$$

From this universal upper bound for w, we can use $w = H(\det D^2 u)$ and the assumptions on H to obtain $\det D^2 u \ge C^{-1} > 0$ in Ω. Therefore, (5.19) is proved.

(ii) Assume $0 \le F_{p_i p_j}(p) \le C_* I_2$. As above, using the convexity of F and u, we can prove that $w \ge C > 0$ in Ω for a universal constant $C > 0$. From

$$\begin{aligned} U^{ij}(w + C_*|x|^2)_{ij} &= -F_{p_i p_j}(Du)u_{ij} + 2C_* \operatorname{trace}(U^{ij}) \\ &\ge -C_* \operatorname{trace}(u_{ij}) + 2C_* \Delta u = C_* \Delta u \ge 0 \quad \text{in } \Omega \end{aligned}$$

and the maximum principle, we also have, as in (5.20), $w \le C_1$ in Ω. Consequently,

$$0 < C \le w \le C_1 < \infty \quad \text{in } \Omega.$$

From $w = H(\det D^2 u)$ and the fact that H^{-1} maps compact subsets of $(0, \infty)$ into compact subsets of $(0, \infty)$, we find that $C_1 \le \det D^2 u \le C_2$ in Ω. Therefore, (5.19) is also proved. □

PROOF OF THEOREM 2.8. The proof is very similar to that of Theorem 2.1. We focus here on the a priori estimates. The key point is to establish the universal bound for u as in Lemma 4.2. We do this via proving the estimate of the type (4.2). We use the same notation as in the proof of Lemma 4.2 with $\theta = 0$. The estimate (4.10) with $n = 2$ now becomes

$$\begin{aligned} \int_{\partial\Omega} K\psi u_\nu^2 \le C + C\left(\int_{\partial\Omega} (u_\nu^+)^2\right)^{1)/2} \\ + \int_\Omega [-(u^3 - u) + \Delta u](u - \widetilde{u})\,dx. \end{aligned} \tag{5.21}$$

Since $\widetilde{u}$ is universally bounded, there is a universal constant $C > 0$ such that

$$-(u^3 - u)(u - \widetilde{u}) \le C.$$

Moreover, from $u \le \sup_{\partial\Omega} \varphi \le C$ by the convexity of u, we have

$$\int_\Omega \Delta u(u - \widetilde{u}) \le C \int_{\partial\Omega} \Delta u\, dx = C \int_{\partial\Omega} u_\nu.$$

Thus (5.21) gives

$$\int_{\partial\Omega} K\psi u_\nu^2 \le C + C\left(\int_{\partial\Omega} (u_\nu^+)^2\right)^{1)/2} + C\int_{\partial\Omega} u_\nu.$$

A simple application of Young's inequality to the above inequality together with the fact that $\inf_{\partial\Omega}(K\psi) > 0$ shows that $\int_{\partial\Omega} u_\nu^2 \le C$, which is exactly what we need to prove. □

Acknowledgment. The research of the author was supported in part by the National Science Foundation under grant DMS-1764248.

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Received April 2019.