COLENGTH, MULTIPLICITY, AND IDEAL CLOSURE OPERATIONS

LINQUAN MA, PHAM HUNG QUY, AND ILYA SMIRNOV

Dedicated to Professor Bernd Ulrich on the occasion of his 65th birthday

Abstract. In a formally unmixed Noetherian local ring, if the colength and multiplicity of an integrally closed ideal agree, then R is regular. We deduce this using the relationship between multiplicity and various ideal closure operations.

1. Introduction

Let $(R, \mathfrak{m})$ be a Noetherian local ring, I be an $\mathfrak{m}$-primary ideal, and M be a finitely generated R-module of dimension d. The Hilbert–Samuel multiplicity of M with respect to I is defined as

$$\mathrm{e}(I, M) = \lim_{n\to\infty} \frac{d!\ell(M/I^n M)}{n^d}.$$

We simplify our notation by letting $\mathrm{e}(I) := \mathrm{e}(I, R)$ and $\mathrm{e}(R) := \mathrm{e}(\mathfrak{m})$. The importance of the Hilbert–Samuel multiplicity in the study of singularities comes from Nagata's fundamental theorem: a Noetherian local ring $(R, \mathfrak{m})$ is regular if and only if it is formally unmixed and $\mathrm{e}(R) = 1$. An ideal-theoretic concept naturally associated to multiplicity is integral closure. Under mild assumptions on R, for a pair of ideals $J \subseteq I$ we have equality $\mathrm{e}(I) = \mathrm{e}(J)$ if and only if $I \subseteq \overline{J}$.

In this short note, we further the relationship between multiplicity and integral closure by showing that in a formally equidimensional ring $\mathrm{e}(I) \geq \ell(R/\overline{I})$ and characterizing that in a formally unmixed ring the equality holds for *some* parameter ideal if and only if R is regular. The latter is a vast generalization of Nagata's theorem: we view his statement as $\mathrm{e}(\mathfrak{m}) = \ell(R/\mathfrak{m})$. These results are obtained by investigating the relationship between multiplicity and various closure operations of parameter ideals. Let J be an ideal generated by a system of parameters of R. We have the following containments of ideal closure operations under mild assumptions:

$$J \subseteq J^{\lim} \subseteq J^*(\text{in characteristic } p > 0) \subseteq \overline{J}.$$

The equalities between the multiplicity and the colength of these closures encode special properties of R (again, under mild assumptions of R):

(1) $\mathrm{e}(J) = \ell(R/J)$ for all (or some) J if and only if R is Cohen–Macaulay;
(2) $\mathrm{e}(J) = \ell(R/J^{\lim})$ for all (or some) J if and only if R is Cohen–Macaulay (Le–Nguyen [5], Theorem 9);
(3) $\mathrm{e}(J) = \ell(R/J^*)$ for all (or some) J if and only if R is F-rational (Goto–Nakamura [10], Corollary 10);
(4) $\mathrm{e}(J) = \ell(R/\overline{J})$ for some J if and only if R is regular (Corollary 12).

We remark that, our main contribution, Corollary 12, also follows from the main result of [21], *if $(R, \mathfrak{m})$ is an excellent normal domain with an algebraically closed residue field.*[1] The point is that, under these assumptions of R, $\mathrm{e}(I) = \ell(R/I)$ for an integrally closed $\mathfrak{m}$-primary ideal I implies $\mathrm{e}(\mathfrak{m}) = 1$ by [21, Theorem 2.1] (using Theorem 6), and hence R is regular by Nagata's theorem. However, we do not see how to extend this approach to get the full version of Corollary 12.

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2. Colength and multiplicity

The goal of this section is to prove Theorem 6. This theorem can be also deduced from the methods in the next section. But we give an elementary approach here that avoids the use of limit closure and big Cohen-Macaulay algebras.

We recall that a Noetherian local ring $(R, \mathfrak{m})$ is equidimensional (resp., unmixed) if $\dim R/P = \dim R$ for every minimal (resp., associated) prime P of R. In other words, R is unmixed if it is equidimensional and (S_1). We say that a Noetherian local ring R is formally equidimensional (resp., unmixed) if $\widehat{R}$ is equidimensional (resp., unmixed). For an ideal $I \subseteq R$ and an element $x \in R$ we use $I : x^\infty$ to denote $\cup_n (I : x^n)$.

Definition 1. Let $x_1, \ldots, x_t$ be a sequence of elements in a Noetherian local ring R. We define $(x_1, \ldots, x_t)^\infty$ inductively as follows:

(1) $(x_1)^\infty = (x_1) + 0 : x_1^\infty$ if $t = 1$
(2) $(x_1, \ldots, x_t)^\infty = (x_t) + (x_1, \ldots, x_{t-1})^\infty : x_t^\infty$ if $t > 1$.

Example 2. The reader should be warned that this is not a closure operation on ideals, and the result may depend on the order of elements. Consider $R = k[[x^4, x^3y, xy^3, y^4]]$. Then x^4, y^4 form a system of parameters, but

$$(x^4, y^4)^\infty = (x^4, x^6y^2, y^4) \neq (x^4, x^2y^6, y^4) = (y^4, x^4)^\infty.$$

We record the following properties.

Lemma 3. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d. For any sequence $x_1, \ldots, x_d$, $(x_1, \ldots, x_d)^\infty$ is either $\mathfrak{m}$-primary or the unit ideal.*

Proof. If $0 : x_1^\infty$ is a proper ideal, i.e., $x_1 \notin \sqrt{(0)}$, then x_1 is a regular element modulo $0 : x_1^\infty$. Hence $\dim R/(x_1)^\infty < d$ and we are done by induction. □

[1] As pointed out in [17, Lemma 2.1], Watanabe's result in [21] can be generalized to complete local domain with an algebraically closed residue field.

Lemma 4. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d and $x_1, \dots, x_d$ be a system of parameters. Then* $\mathrm{e}((x_1, \dots, x_d)) = \ell(R/(x_1, \dots, x_d)^{\infty})$.

Proof. This is clear if $d = 1$, since $R/(0 : x_1^{\infty})$ is Cohen–Macaulay and $\dim(0 : x_1^{\infty}) = 0$. If $d > 1$, then by the associativity and additivity formula for multiplicity (see [19]),

$$\begin{aligned}\mathrm{e}((x_1, \dots, x_d), R) &= \sum_P \mathrm{e}((x_2, \dots, x_d), R/P)\, \mathrm{e}(x_1, R_P) = \sum_P \mathrm{e}((x_2, \dots, x_d), R/P)\ell(R_P/(x_1)^{\infty}) \\ &= \mathrm{e}((x_2, \dots, x_d), R/(x_1)^{\infty}),\end{aligned}$$

where P ranges over all minimal primes of (x_1) such that $\dim R/P = \dim R/(x_1)$. Therefore we are done by induction on d. □

Remark 5. Let $(R, \mathfrak{m})$ be a Noetherian local ring and let $S = \widehat{R(t)} := \widehat{R[t]_{\mathfrak{m}[t]}}$. We note that S is complete, has an infinite residue field, and is a faithfully flat R-algebra such that $\mathfrak{m}_R S$ is the maximal ideal of S. It follows that $\ell_R(R/I) = \ell_S(S/IS)$ for every $\mathfrak{m}$-primary ideal and, thus, $\mathrm{e}(I) = \mathrm{e}(IS)$. Moreover, if I is integrally closed in R then IS is integrally closed in S. This follows from [18, Lemma 8.4.2 (9)], which allows us to pass to $R(t)$, and the fact that there is one-to-one correspondence between $\mathfrak{m}$-primary ideals in R and $\widehat{R}$, so if $I\widehat{R}$ is a reduction of a larger ideal, then I is a reduction too.

Theorem 6. *Let $(R, \mathfrak{m})$ be a formally equidimensional Noetherian local ring. Then for every $\mathfrak{m}$-primary integrally closed ideal I we have* $\mathrm{e}(I) \geq \ell(R/I)$.

Proof. We may pass from R to $R(t)$ without changing the colength and the integral closedness of I. Thus we assume that R has an infinite residue field. Let $(x_1, \dots, x_d)$ be a minimal reduction of I. By Lemma 4, it is enough to show that $(x_1, \dots, x_d)^{\infty} \subseteq I$. This is a consequence of colon-capturing ([20], [18, Theorem 5.4.1]). Namely, it is clear that $(x_1)^{\infty} = (x_1) + 0 : x_1^{\infty} \subseteq \overline{(x_1)}$, and for $i > 1$ we can use induction to see that

$$(x_1, \dots, x_i)^{\infty} = (x_i) + (x_1, \dots, x_{i-1})^{\infty} : x_i^{\infty} \subseteq (x_i) + \overline{(x_1, \dots, x_{i-1})} : x_i^{\infty} \subseteq \overline{(x_1, \dots, x_i)}. \quad \square$$

Example 7. The equidimensionality assumption in Theorem 6 is necessary. Let $R = k[[x, y, z]]/(xy, xz)$ and consider the ideal (x^n, y, z). One can check that this ideal is integrally closed, has multiplicity 1, and colength n.

3. Limit closure, integral closure, and the main result

In this section we study a relation between multiplicity and the colength of limit closure, and we prove our main result. As a byproduct of our methods, we also recover some results in [5] and [10].

Definition 8. Let $(R, \mathfrak{m})$ be a Noetherian local ring and let $x_1, \dots, x_d$ be a system of parameters of R. The limit closure of $(x_1, \dots, x_d)$ in R is defined as

$$(x_1, \dots, x_d)^{\lim_R} = \bigcup_{n \geq 0} (x_1^{n+1}, \dots, x_d^{n+1}) :_R (x_1 \cdots x_d)^n.$$

We will write $(x_1, \dots, x_d)^{\lim}$ if R is clear from the context.

We note that $(x_1,\ldots,x_d)^{\lim}/(x_1,\ldots,x_d)$ is the kernel of the natural map $R/(x_1,\ldots,x_d)\to \mathrm{H}^d_{\mathfrak{m}}(R)$: since $\mathrm{H}^d_{\mathfrak{m}}(R)=\varinjlim_n \frac{R}{(x_1^n,\ldots,x_d^n)}$ with connection map multiplication by $x_1\cdots x_d$, $\overline{z}\in R/(x_1,\ldots,x_d)$ maps to 0 in $\mathrm{H}^d_{\mathfrak{m}}(R)$ if and only if $z(x_1\cdots x_d)^n\in(x_1^{n+1},\ldots,x_d^{n+1})$ for some n, that is, $z\in(x_1,\ldots,x_d)^{\lim}$. In particular, limit closure of an ideal generated by a system of parameters is independent of the choice of the generators. In general, limit closure is hard to study: Hochster's monomial conjecture/theorem simply says that $(x_1,\ldots,x_d)^{\lim}$ is not the unit ideal. This was proved by Hochster in the equal characteristic case [12] and was proved by André in mixed characteristic [1].

The next theorem is a crucial ingredient towards proving our main result. It follows from [5, Theorem 3.1]. But we provide a different and simpler proof.

Theorem 9. *Let $(R,\mathfrak{m})$ be a Noetherian local ring of dimension d. Then for every system of parameters $x_1,\ldots,x_d$, we have*

$$\mathrm{e}((x_1,\ldots,x_d))\geq \ell(R/(x_1,\ldots,x_d)^{\lim}).$$

Moreover, if R is unmixed and is a homomorphic image of a Cohen–Macaulay ring, then the equality holds for one (equivalently, all) system of parameters if and only if R is Cohen–Macaulay.

Proof. The first assertion is well-known (for example, see [4, Lemma 2.3]). The point is that, by Lech's formula [19], $\mathrm{e}((x_1,\ldots,x_d))=\lim_{n\to\infty}\frac{\ell(R/(x_1^n,\ldots,x_d^n))}{n^d}$. We can filter $R/(x_1^n,\ldots,x_d^n)$ by n^d ideals generated by monomials in $x_1,\ldots,x_d$, and it is easy to check that each factor maps onto $R/(x_1,\ldots,x_d)^{\lim}$.[2]

Now we prove the second assertion. We may assume the residue field of R is infinite. We proceed by induction on d. If $d=1$ the assertion is obvious. If $d=2$, the statement follows from [6, Theorem 1.5].[3] Now we assume $d\geq 3$, it follows from [7, Proposition 4.16] that if $z\in(x_1,\ldots,x_d)$ is general, then $R':=R/zR$ is equidimensional and (S_1) on the punctured spectrum. Let $S=R'/\mathrm{H}^0_{\mathfrak{m}}(R')$. We know that S is unmixed. Since $\mathrm{H}^0_{\mathfrak{m}}(R')$ has finite length and z is a general element in $(x_1,\ldots,x_d)$, we have

$$\mathrm{e}((x_1,\ldots,x_d),S)=\mathrm{e}((x_1,\ldots,x_d),R')=\mathrm{e}((x_1,\ldots,x_d)).$$

Replacing $x_1,\ldots,x_{d-1}$ if necessary, we may assume that $x_1,\ldots,x_{d-1},z$ form a system of parameters of R, and thus $x_1,\ldots,x_{d-1}$ form a system of parameters on R' and S. By [6,

[2]For example, if $d=1$, the we have a filtration $(x_1^n)\subseteq(x_1^{n-1})\subseteq\cdots\subseteq(x_1)\subseteq R$, the i-th factor $(x_1^i)/(x_1^{i+1})\cong R/(x_1^{i+1}:x_1^i)$, since $(x_1^{i+1}:x_1^i)\subseteq(x_1)^{\lim}$ by definition, $(x_1^i)/(x_1^{i+1})\twoheadrightarrow R/(x_1)^{\lim}$. In the general case, each factor looks like $(J,x_1^{n_1}\cdots x_d^{n_d})/J\cong R/(J:x_1^{n_1}\cdots x_d^{n_d})$ where J is an $\mathfrak{m}$-primary ideal generated by monomials $x_1^{j_1}\cdots x_d^{j_d}$ in $x_1,\ldots,x_d$ such that $j_i>n_i$ for some i, i.e., at least one exponent is bigger than that appearing in $x_1^{n_1}\cdots x_d^{n_d}$. Now for every $y\in J:(x_1^{n_1}\cdots x_d^{n_d})$, we have $yx_1^{n_1}\cdots x_d^{n_d}=\sum a_{j_1\ldots j_d}x_1^{j_1}\cdots x_d^{j_d}$. Pick n that is larger than all n_i and multiply this equation by $x_1^{n-n_1}\cdots x_d^{n-n_d}$ we get $y(x_1\cdots x_d)^n=\sum a_{j_1\ldots j_d}x_1^{j_1+n-n_1}\cdots x_d^{j_d+n-n_d}\in(x_1^{n+1},\ldots,x_d^{n+1})$ by the assumptions on j_i. Hence $y\in(x_1,\ldots,x_d)^{\lim}$ and thus $R/(J:x_1^{n_1}\cdots x_d^{n_d})\twoheadrightarrow R/(x_1,\ldots,x_d)^{\lim}$.

[3]Note that the "unmixed" assumption in [6, Theorem 1.5] means formally unmixed in our context, and if R is a homomorphic image of a Cohen–Macaulay ring, then R is unmixed implies R is formally unmixed [2, Theorem 2.1.15].

Theorem 1.2 and Proposition 2.7] , we know that $(x_1,\ldots,x_{d-1})^{\lim_{R'}}S = (x_1,\ldots,x_{d-1})^{\lim_S}$. Moreover, if $r \in (x_1^{n+1},\ldots,x_{d-1}^{n+1},z) : (x_1\cdots x_{d-1})^n$, then

$$r(x_1\cdots x_{d-1}z)^n \subseteq z^n(x_1^{n+1},\ldots,x_{d-1}^{n+1},z) \subseteq (x_1^{n+1},\ldots,x_{d-1}^{n+1},z^{n+1}).$$

This implies that the pre-image of $(x_1,\ldots,x_{d-1})^{\lim_{R'}}$ in R is contained in $(x_1,\ldots,x_d)^{\lim_R}$. Thus we have

$$\begin{aligned}\mathrm{e}((x_1,\ldots,x_d),S) &\geq \ell(S/(x_1,\ldots,x_{d-1})^{\lim_S}) = \ell(R'/(x_1,\ldots,x_{d-1})^{\lim_{R'}})\\ &\geq \ell(R/(x_1,\ldots,x_d)^{\lim_R}) = \mathrm{e}((x_1,\ldots,x_d)) = \mathrm{e}((x_1,\ldots,x_d),S)\end{aligned}$$

and so we must have equalities all over. Therefore S is Cohen–Macaulay by the induction hypothesis, and it follows that $\mathrm{H}^i_{\mathfrak{m}}(R') \cong \mathrm{H}^i_{\mathfrak{m}}(S) = 0$ for $0 < i < \dim S = \dim R'$.

Finally, since R is unmixed, z is a regular element, so the sequence

$$0 \to R \xrightarrow{\times z} R \to R' = R/zR \to 0$$

is exact and induces the exact sequence

$$0 \to \mathrm{H}^0_{\mathfrak{m}}(R/zR) \to \mathrm{H}^1_{\mathfrak{m}}(R) \xrightarrow{\times z} \mathrm{H}^1_{\mathfrak{m}}(R) \to 0.$$

Because R is unmixed, $\mathrm{H}^1_{\mathfrak{m}}(R)$ has finite length. The sequence above then implies that $\mathrm{H}^0_{\mathfrak{m}}(R/zR) = 0$. Thus R/zR is Cohen–Macaulay, so R is Cohen–Macaulay. □

Using limit closure we recover the main result of [10, Theorem 1.2], see also [3, Corollary 1.9 and Remark 1.10].

Corollary 10. *Let $(R,\mathfrak{m})$ be an equidimensional Noetherian local ring of characteristic $p > 0$ which is a homomorphic image of a Cohen–Macaulay ring. Then for any system of parameters $x_1,\ldots,x_d$ of R we have $\mathrm{e}((x_1,\ldots,x_d)) \geq \ell(R/(x_1,\ldots,x_d)^*)$. Moreover, if, in addition, R is unmixed, then the equality holds for one (equivalently, all) system of parameters if and only if R is F-rational.*

Proof. The first assertion follows from Theorem 9 and colon-capturing: $(x_1,\ldots,x_d)^{\lim} \subseteq (x_1,\ldots,x_d)^*$, see [16, Theorem 2.3 and Remark 5.4]. If R is unmixed and equality holds, then by Theorem 9, R is Cohen–Macaulay and thus $\mathrm{e}((x_1,\ldots,x_d)) = \ell(R/(x_1,\ldots,x_d))$. Hence $(x_1,\ldots,x_d) = (x_1,\ldots,x_d)^*$, so R is F-rational by [8, Proposition 2.2]. □

We next show that limit closure is contained in the integral closure in all characteristics using the existence of big Cohen–Macaulay algebras.

Theorem 11. *Let $(R,\mathfrak{m})$ be a formally equidimensional Noetherian local ring, then for every system of parameters $x_1,\ldots,x_d$ we have*

$$(x_1,\ldots,x_d)^{\lim} \subseteq \overline{(x_1,\ldots,x_d)}.$$

Proof. We may assume that R is complete. To check whether an element is in the integral closure, it is enough to check this modulo every minimal prime of R. Since R is equidimensional, $x_1,\ldots,x_d$ is still a system of parameters modulo every minimal prime of R. So if r is in $(x_1,\ldots,x_d)^{\lim}$, then this is also true modulo every minimal prime of R. Therefore we reduce to the case that R is a complete local domain.

Now let B be a big Cohen–Macaulay R-algebra, whose existence follows from [14] and [15] in equal characteristic, and from [1] (see also [11]) in mixed characteristic. If $r \in (x_1, \ldots, x_d)^{\lim}$, then $r \in (x_1^t, \ldots, x_d^t) :_R (x_1 \cdots x_d)^{t-1}$ for some t. It follows that

$$r \in \left((x_1^t, \ldots, x_d^t) :_B (x_1 \cdots x_d)^{t-1}\right) \cap R = (x_1, \ldots, x_d)B \cap R,$$

since $x_1, \ldots, x_d$ is a regular sequence on B.

Thus it is enough to prove that $(x_1, \ldots, x_d)B \cap R$ is contained in $\overline{(x_1, \ldots, x_d)}$. In fact, $JB \cap R$ is contained in $\overline{J}$ for every ideal J of R: since R is a complete local domain and B is a big Cohen–Macaulay algebra, B is a solid R-algebra in the sense of [13, Corollary 10.6], thus $JB \cap R$ is contained in the solid closure of J, but solid closure is always contained in the integral closure by [13, Theorem 5.10]. □

Corollary 12. *Let $(R, \mathfrak{m})$ be a Noetherian local ring that is formally equidimensional. Then for every $\mathfrak{m}$-primary integrally closed ideal I, we have $\mathrm{e}(I) \geq \ell(R/I)$. Moreover, if, in addition, R is formally unmixed and equality holds for some I, then R is regular.*

Proof. We may assume that R is complete with an infinite residue field by Remark 5. Let $(x_1, \ldots, x_d)$ be a minimal reduction of I. By Theorem 9 and Theorem 11,

$$\mathrm{e}(I) = \mathrm{e}((x_1, \ldots, x_d)) \geq \ell(R/(x_1, \ldots, x_d)^{\lim}) \geq \ell(R/I).$$

Now if R is formally unmixed and $\mathrm{e}(I) = \ell(R/I)$, then $\mathrm{e}((x_1, \ldots, x_d)) = \ell(R/(x_1, \ldots, x_d)^{\lim})$ so by Theorem 9, R is Cohen–Macaulay. But then $\mathrm{e}((x_1, \ldots, x_d)) = \ell(R/(x_1, \ldots, x_d))$ and hence $I = (x_1, \ldots, x_d)$, so R is regular by [9, Corollary 2.5]. □

We would like to note that [9, Theorem 1.1] shows that an integrally closed $\mathfrak{m}$-primary parameter ideal in a regular ring $(R, \mathfrak{m})$ has the form $x_1^n, x_2, \ldots, x_d$ where $x_1, \ldots, x_d$ are minimal generators of $\mathfrak{m}$.

Example 13. One might ask that, in a formally unmixed Noetherian local ring $(R, \mathfrak{m})$, whether $(x_1, \ldots, x_d)^{\lim} = \overline{(x_1, \ldots, x_d)}$ for a system of parameters already implies R is regular. However this is not true in general: Let $R = k[[a, b, c, d]]/(a, b) \cap (c, d)$. Then R is complete, unmixed, has dimension 2, with $\mathrm{e}(R) = 2$ and $\mathrm{H}^1_{\mathfrak{m}}(R) \cong k$. Let (x, y) be a minimal reduction of $\mathfrak{m}$. It follows from [6, Theorem 1.5] that $\ell(R/(x, y)^{\lim}) = 1$, so $(x, y)^{\lim} = \mathfrak{m} = \overline{(x, y)}$.

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Department of Mathematics, Purdue University, West Lafayette, IN 47907 USA
Email address: ma326@purdue.edu

Department of Mathematics, FPT University, and Thang Long Institute of Mathematics and Applied Sciences, Hanoi, Vietnam
Email address: quyph@fe.edu.vn

Department of Mathematics, Stockholm University, S-10691, Stockholm, Sweden
Email address: smirnov@math.su.se