Particle tracking velocimetry applied to thermal counterflow in superfluid ^{4}He: Motion of the normal fluid at small heat fluxes

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This paper is concerned with thermal counterflow in superfluid ^{4}He, particularly with the motion of the normal component at small heat fluxes when this motion is believed to be laminar. Recent experiments in which this motion is traced with micron-sized particles of solid deuterium show that the motion is not spatially uniform on a scale of order the spacing of the quantized vortex lines in the superfluid component. It is argued that this lack of uniformity has its origin in the fact that the force of mutual friction, which limits the thermal conductivity, is concentrated near the cores of the vortex lines. Possible effects of this concentration of force are discussed, and it is concluded that the experimental observations can be explained only if the vortex lines are arranged randomly in space, so that the observed lack of uniformity in the normal-fluid velocity can be regarded as being due partly to spatial variations in the vortex line density. However, problems remain, in that the form of the observed velocity correlation function has still to be understood.

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I. INTRODUCTION

It has long been known that, according to the two-fluid model, heat is carried in superfluid ^{4}He by a counterflow of the two fluids, the superfluid component flowing toward the source of heat and the normal component flowing away from it [1]. Such counterflow normally takes place in a channel, and in this paper we are concerned with a channel with cross-section of order 10 mm × 10 mm. Experiments over the years (see, for example, Refs. [2–4] and references therein) have shown that in such channels the characteristics of the counterflow change as the heat flux increases. At the smallest heat fluxes the superfluid component flows without friction, while the flow of the normal fluid is limited only by its viscosity. Over a range of somewhat larger heat fluxes the superfluid component becomes turbulent, in the sense that it supports a random tangle of quantized vortex filaments; the thermal excitations constituting the normal fluid are scattered by these filaments, giving rise to a force of "mutual friction" between the two fluids. This mutual friction can modify the (laminar) velocity profile in the normal fluid, especially at larger heat fluxes. Then as the heat flux increases still further there is a transition at which large-scale turbulence is established in both fluids, the two turbulent fields being partly coupled through the mutual friction, and large-scale turbulence in the superfluid component being achieved by partial polarization of the vortex filaments.

The experiments to which we have just referred were based for the most part on observations with second sound, the attenuation of which allows measurement of vortex filament densities, and

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with He_2 excimer molecules, which trace the flow of the normal fluid. Particle tracking velocimetry (PTV), using micron-sized particles of solid hydrogen or solid deuterium as tracers, has also been used, as reviewed in a recent paper by Mastracci and Guo [5]. This paper also reports new experimental results, which include information about the motion of the tracer particles that are known to be tracking the motion of the normal fluid on length scales smaller than those easily accessible to tracking by He_2 excimer molecules (scales of order the spacing of the vortex filaments), particularly at heat fluxes at which there is a random tangle of vortex filaments and laminar flow of the normal fluid. In the present paper we discuss the interpretation of this observed motion.

At first sight it might be thought that when the flow of the normal fluid is laminar there ought to be no small-scale motion. However, this would be true only if the force of mutual friction, which is principally responsible for limiting the flow of the two fluids, were applied in a spatially uniform manner. In reality, the force is applied to the normal fluid only very close to the cores of the vortex filaments [6]; transfer to the bulk of the normal fluid takes place only through the viscosity of the normal fluid. We shall argue that the resulting perturbation to the velocity field in the normal fluid is responsible, at least in part, for the observed small-scale motion of the normal fluid.

The theoretical ideas in this paper are for the most part not new. The perturbation in the velocity field in the normal fluid resulting from motion of a single vortex was recognized in Refs. [2,6], to the extent that the effect was taken into account in a calculation of the actual magnitude of the mutual friction; but the calculations in Refs. [2,6] did not require a knowledge of the perturbation at larger distances from the lines. More recently Idowu *et al.* [7] carried out a numerical simulation of the perturbation, which we shall discuss later (see also Refs. [8,9]). It turns out, however, that, as we shall see, the probable perturbation in the normal-fluid velocity field by a single vortex is too small to account for our experimental results. We show instead that spatial inhomogeneities in the density of vortex lines are the probable cause of the observed effects, although the way in which these spatial inhomogeneities operate is quite subtle and may not be fully understood. Indeed, a fully satisfactory explanation must await a more sophisticated treatment than we are able to provide, and may involve physical effects that we have not yet fully identified. To this extent the theory that we present must be regarded as only a useful first step in understanding the experimental results that we present.

The theoretical problems underlying our experimental results will be explored in two steps. First, we shall discuss in general terms the likely perturbation in the velocity field resulting from the motion of a single rectilinear vortex. In essence this ought to involve an understanding of the way in which the moving vortex drags the normal fluid in its vicinity. We present new calculations of this dragging effect, arguing that those of Idowu *et al.* are inadequate. We conclude that dragging by the individual vortices is insufficient to account for the experimental results. Our second step is then to suggest that the experimental results can be understood only in terms of the way in which a moving disordered array of vortices perturbs the velocity of the normal fluid. As we shall see, this is in detail a difficult problem that requires new types of simulation, and we are able here to give only an introductory discussion of it.

The relevant experimental results are summarized in Sec. II; they include results already published in Ref. [5], together with some new results relating to the way that normal-fluid velocities are correlated in time. The theoretical discussion is presented in Sec. III, and a summary is given in Sec. IV. Details of the new calculation of the dragging by a single vortex are given in the Appendix.

II. EXPERIMENTAL RESULTS

The apparatus used by Maastracci and Guo was described in detail in Ref. [5]. The heat flow took place in a channel of 16-mm-square cross section. Tracks of deuterium particles were observed and analysed at various temperatures and various steady counterflow velocities, and probability distribution functions (PDFs) for the particle velocity were extracted.

In Fig. 1 we show typical PDFs for the streamwise component of the velocity (i.e., parallel to the steady heat flux). As has been observed in previous studies, this PDF exhibits two peaks at low heat fluxes [Fig. 1(a)], the peaks merging into a single peak at large heat fluxes [Fig. 1(b)]. Gaussian

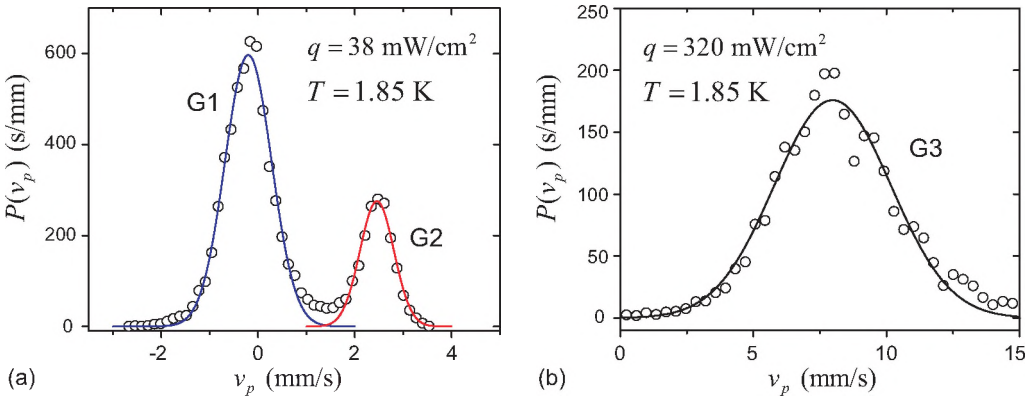

FIG. 1. The calculated streamwise particle velocity PDFs at 1.85 K for heat fluxes of 38 and 320 mW/cm^2. The solid lines are Gaussian fits to the data.

fits to the peaks lead to corresponding mean velocities, $\langle v_p \rangle$, and velocity standard deviations, $\Delta v_p = \langle (v_p - \langle v_p \rangle)^2 \rangle^{1/2}$. In the two-peak regime the value of $\langle v_p \rangle$ for the G2 peak agrees well with the expected mean velocity of the normal fluid ($q/\rho ST$), so that we can identify this peak as being due to particles that are moving, at least on average and for a significant length of time, with the normal fluid. The G1 peak is due to particles that are trapped on vortices. At higher heat fluxes, when the two peaks merge, $\langle v_p \rangle$ is observed to be equal to about half the steady normal-fluid velocity, and in this regime particles are, in rapid succession, being continuingly trapped on vortices and then released by viscous drag from the normal fluid. In this paper we are concerned only with the width (standard deviation) of the G2 peak, both for the streamwise velocity, which we have been discussing, and also for the velocity transverse to the steady normal-fluid flow.

In Fig. 2 we show these observed standard deviations as functions of the mean normal-fluid velocity, $\langle v_n \rangle$, at three temperatures. We see that the transverse velocity standard deviations are small and almost independent of $\langle v_n \rangle$, whereas the streamwise standard deviations show a marked increase. In fact, the former standard deviations are only a little larger than the effect of environmental noise. The streamwise standard deviations could well be proportional to $\langle v_n \rangle$, although the noise is too large to allow us to draw a definitive conclusion.

Our experimental results allow us to calculate not only these standard deviations, which were reported in Ref. [5], but also velocity correlation functions, which we report here for the first time,

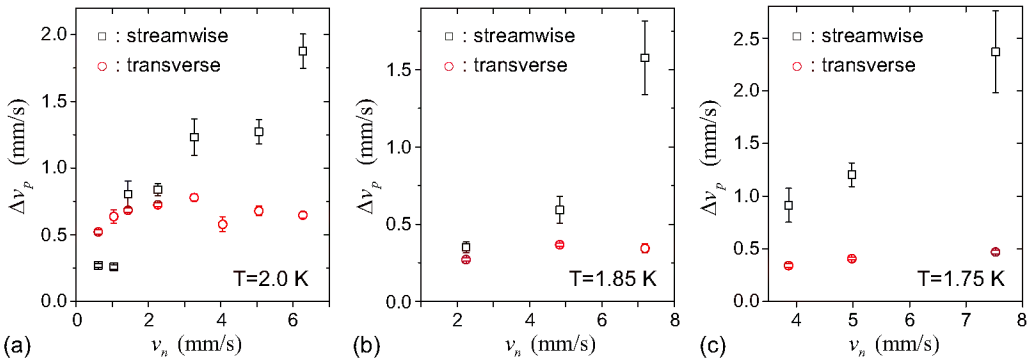

FIG. 2. Calculated streamwise and transverse velocity standard deviations for the G2 particles at three temperatures.

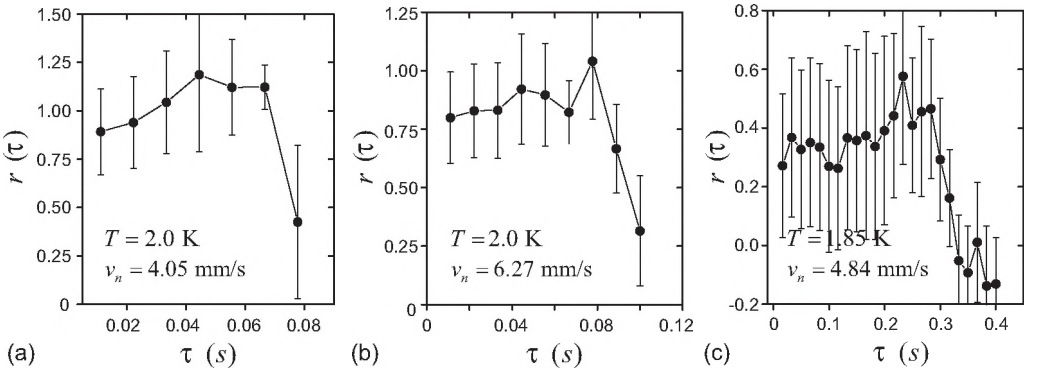

FIG. 3. Calculated streamwise velocity correlation functions for the G2 particles at different temperatures and heat fluxes.

and which are defined as

$$r(\tau) = \frac{\langle (v_p(t) - \langle v_p \rangle)(v_p(t+\tau) - \langle v_p \rangle) \rangle}{\Delta v_p^2}, \tag{1}$$

where $v_p(t)$ is the velocity of a particular particle on a track measured at time t. These correlation functions tell us the characteristic lifetimes of the velocity fluctuations, as seen by a particular particle. Our observed values of $r(\tau)$ for streamwise fluctuations under various conditions are shown in Fig. 3. Owing to our limited sample sizes, our values of $r(\tau)$ are subject to considerable uncertainty, but there is little doubt that the lifetimes of the streamwise fluctuations are roughly 100 ms at a temperature of 2 K. But the precise way in which $r(\tau)$ varies with τ is less clear: It may be more or less flat up to a point where there is an abrupt fall; or there may be a more gradual fall. Our observations of correlation functions for transverse velocities show no interesting structure, being dominated by noise. (We note that the observed $r(\tau)$ appear not always to tend to unity as τ tends to zero. This is probably due to the fact that our signals contain significant amounts of noise, for which the correlation time is very small, leading to a fall in $r(\tau)$ at unobservably small times.)

We have noted that the correlation functions are related to characteristic lifetimes of the normal-fluid fluctuations, as seen by a particular particle. We have also noted that $r(\tau)$ may be fairly flat, until τ reaches a value of τ_c, when there is a fairly sharp cutoff in the correlation. If, as we are suggesting, the fluctuations are due to local dragging of the normal fluid by vortex lines, and if we assume for the moment that the vortex lines remain at rest in the frame of reference moving with the average velocity of the superfluid, then in this frame of reference the flow of the normal fluid is stationary and the characteristic length $\zeta = (v_n - v_s)\tau_c$ is the distance in the streamwise direction over which the dragging is effective. For the case depicted in Fig. 3(b) ζ proves to be about 30 times the average vortex-line separation. The magnitude of the perturbation in v_n at distances less than ζ is of order Δv_p for the streamwise flow. If $r(\tau)$ is roughly flat, for $\tau < \tau_c$, then this magnitude does not depend strongly on distance for distances less than ζ.

We note that the measurements that we have been describing serve to measure correlations in the direction of the heat flux (i.e., longitudinal correlations). As we shall see more clearly later, it would be interesting to measure also the transverse correlations. Such measurements are possible in principle, but they require a study of the correlated motion of two particles. In our observations so far the density of tracer particles has been too small to allow such study with adequate statistical error.

In our Introduction we mentioned observations of the normal-fluid flow with He_2 excimer molecules. They involved the production of a narrow line of these molecules, perpendicular to the heat flux, and an observation of the way in which this line changes as it moves with the normal fluid.

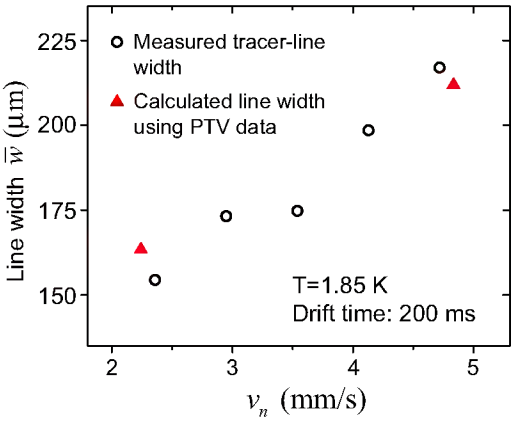

FIG. 4. The observed broadening of the excimer lines at 1.85 K with increasing normal-fluid velocity, when the flow of normal fluid is observed to be laminar. The red triangles show the broadening expected from the PTV results reported in this paper

Three types of change are possible: a broadening of the line owing to diffusion of the molecules; distortion of the line owing to turbulent or other forms of normal fluid motion on scales greater than the line width; and a further broadening owing to these forms of motion on scales less than the line width. We are arguing in this paper that when the normal-fluid flow is laminar there can still be nonuniformities in the velocity of the normal fluid on scales of order the spacing between vortex lines, these scales being less than the observed linewidths. Such nonuniformities ought therefore to contribute to the broadening of the lines of excimers, such broadening being probably proportional to the average normal-fluid velocity, v_n. Thus, if the lines are examined after they have drifted for a time $t_{\rm drift}$, we may expect the line width, $\bar{w}$, to change with v_n according to the equation $\bar{w} = \bar{w}_0 + At_{\rm drift}v_n$. where $\bar{w}_0$ is the initial width enhanced by that due to diffusion during the drift time, and where the constant of proportionality, A, is known from the experiments described earlier in this section. The results of a recent careful analysis of our earlier experiments with excimers are shown in Fig. 4, where we have plotted the observed linewidth against v_n at 1.85 K for a fixed drift time and for heat fluxes at which flow of the normal fluid is observed to be laminar. The red triangles show that the broadening expected from the above equation agrees well with experiment.

III. THEORETICAL DISCUSSION

A. Introduction

There appear to be three possible explanations of the observed fluctuations of the G2 particle velocities: that there is in fact small-scale turbulence in the normal fluid; that the velocity of G2 particles is occasionally perturbed by scattering from vortex lines; that the velocity of the normal fluid is perturbed by the frictional interactions with the vortex lines (dragging of the normal fluid).

Small-scale turbulence in the normal fluid seems possible, although it would need to be extremely anisotropic, with hardly any motion transverse to the counterflow; for reasons that will become clear later, we believe and argue that any such turbulence is not *fundamentally* responsible for the observed fluctuations. The second of the possible effects can be ruled out as follows. Scattering of a particle by a vortex line must involve a timescale of order σ/v_{G2}, where σ is the scattering cross-section and v_{G2} is the particle velocity (at this point we rule out any "scattering" that is really associated with dragging of the normal fluid [9]). This time is unlikely to be greater than about 10 μs. But we know that the velocity of a tracer particle of radius a will recover from any perturbation in a time of order $\rho a^2/3\eta_n$, where ρ is the density of the helium and η_n is the viscosity of the normal

fluid [10]; this time is of order 30 μs. These times are much smaller than the times over which the perturbed particle velocities are observed to persist.

As we suggested in our general introduction, we believe that the fluctuations in the G2 particle velocities are in fact due to dragging of the normal fluid induced by the frictional interaction with the discrete vortices. In the remaining part of this paper we shall first show that dragging by an individual vortex is too small to account for our observations; nevertheless, as far as we know, our treatment of this dragging is new and of some intrinsic interest, an earlier treatment by Idowu *et al.* [7] having been unsatisfactory. Then we shall show, with a simple model, that dragging by a disordered array of lines, such as is believed to be present in a counterflow, does lead to effects that are much larger; effects that could be, at least qualitatively, consistent with our observations. However, we are not able yet to account for the precise form of the correlation function displayed in Fig. 3, which remains a serious problem.

B. The dragging of the normal fluid by an individual quantized vortex

The existence of this dragging effect was first recognized in Refs. [2,6], to the extent that the effect was taken into account in a calculation of the actual magnitude of the mutual friction; but the calculations in Refs. [2,6] did not require a knowledge of the perturbation at larger distances from the lines. Nevertheless it was suggested that the dragging could be explained rather fully as being similar to the dragging of an infinite classical viscous fluid at low Reynolds number by a thin solid cylinder moving at right angles to its length, especially not too close to the cylinder. This view is elaborated in an Appendix, where we use it to derive expressions for the perturbed velocity field in an infinite volume of normal fluid arising from the slow movement of a rectilinear vortex at velocity U. The character of this perturbation at a distance r from the vortex depends on the value of r relative to a characteristic length $\lambda = \nu_n/2U$, where $\nu_n = \eta_n/\rho_n$ is the kinematic viscosity of the normal fluid. We have obtained expressions for the perturbation in terms of the force, f, of mutual friction per unit length of vortex in the limits of both $\lambda \gg r \gg \xi$ (the "near field") and $r \gg \lambda$ (the "far field"); the length ξ is the range over which the force of mutual friction acts near the core of the vortex. It turns out that for the experiments described in Sec. II λ is much smaller ($\sim$0.7 μm) than the distances over which the normal-fluid velocity is observed to be perturbed, so that we need focus only on the far velocity field, given by Eq. (A7).

Our experimental results, described in Sec. II, relate to thermal counterflow. For the present it is sufficient for us to assume that the vortex of interest moves with the average velocity of the superfluid component, so that we can take $U = \langle v_s\rangle - \langle v_n\rangle = (\rho/\rho_s)\langle v_n\rangle$, where ρ is the density of the helium and ρ_s is the density of the superfluid component. We note that the force of mutual friction, f, can be expressed in terms of U as $\alpha\kappa\rho_s U$ (see Ref. [11]), so that

$$f = \alpha\kappa\rho\langle v_n\rangle, \tag{2}$$

where α is a dimensionless mutual friction constant, and κ is the quantum of circulation ($2\pi\hbar/m_4$); we ignore any transverse component of the mutual friction. It follows that the far velocity field can be written

$$u_r(y,z) = -\frac{\alpha\kappa\langle v_n\rangle}{\nu}\left(\frac{\lambda}{2\pi z}\right)^{1/2}\exp\left(-\frac{y^2}{8\lambda z}\right), \tag{3}$$

where $\nu = \eta_n/\rho$, and where we have omitted the small radial flow that ensures mass continuity. We note that this equation describes a laminar wake, the width of which increases as $\sqrt{(8\lambda z)}$ and in which the velocity falls as $z^{-1/2}$.

To display the result of this calculation in a useful way, we introduce dimensionless parameters as follows: $\tilde{u}_n(\tilde{y},\tilde{z}) = u_n(y,z)/\langle v_n\rangle$, and all length scales are divided by a relevant vortex spacing ℓ, so that

$$\tilde{u}_n(\tilde{y},\tilde{z}) = -\frac{\alpha\kappa}{\nu}\left(\frac{\tilde{\lambda}}{2\pi\tilde{z}}\right)^{1/2}\exp\left(-\frac{\tilde{y}^2}{8\tilde{\lambda}\tilde{z}}\right). \tag{4}$$

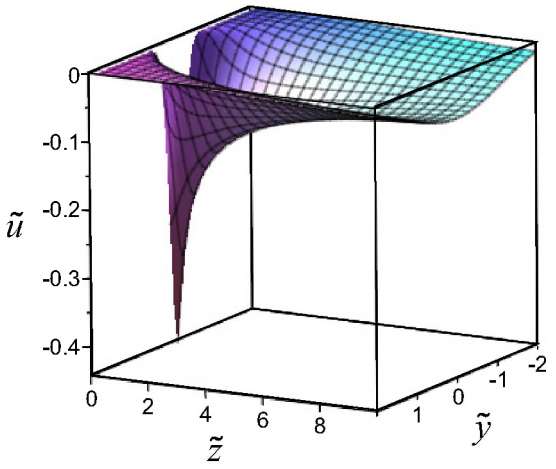

FIG. 5. The wake generated in the normal fluid by a single rectilinear vortex line.

For illustrative purposes we shall apply Eq. (4) to a counterflow at $\langle v_n \rangle = 6$ mm s^{-1}, with a temperature of 2.0 K; values of the relevant parameters (taken from our own measurements or from Ref. [12]) are then: $\ell = 46\ \mu$m; $\nu_n = 1.0 \times 10^{-8}$ m^2 s^{-1}; $\alpha = 0.279$; $\rho_n/\rho = 0.553$; $\lambda = 0.68\ \mu$m. The result is shown in Fig. 5. We note that this wake decays too rapidly with distance to account for the observations reported in Sec. II. We add that so far we have confined much of our theoretical study to temperatures close to 2 K, since it is only at these temperatures that we have satisfactory experimental results; in fact, however, the essential features of our results are not strongly temperature dependent.

At this point we must refer to work of Idowu *et al.* [7], which was also concerned with the dragging of the normal fluid by a single moving vortex. In some respects this work was more ambitious than the analysis that we have just outlined, in the sense that the motion of the vortex relative to the normal fluid was taken as due to a prescribed motion of the superfluid with a force of mutual friction between the vortex and the normal fluid. Calculations were based on a normal fluid that obeys the Navier-Stokes equation (including its nonlinear terms), with the force of mutual friction localized close to the core of the vortex. Instead of relying on analytical results, Idowu *et al.* obtained the normal-fluid velocity field from a two-dimensional direct numerical simulation, the vortex moving in a 1-mm computational square box, with periodic boundary conditions, the normal-fluid velocity field being recorded when the vortex is near the center of the box. For the most part the velocities with which the vortex moves (relative to the normal fluid) were smaller than those relevant to our present studies and in the range of roughly 0.1 to 1 mm s^{-1}. These velocities correspond at temperatures near those that we have investigated to values of λ that are significantly larger than our own; at 0.1 mm s^{-1}, $\lambda \sim 100\ \mu$m, while at the larger velocity $\lambda \sim 10\ \mu$m. This means that within the 1-mm computational box any wake would not be well developed, although it ought to be visible at the higher velocity. Nevertheless, no wake seems to have been formed in these simulations. More seriously, the characteristic length λ seemed to play no role, the form of the velocity field at all distances from the vortex being in that sense very different from that described by us. The velocity field obtained by Idowu *et al.* does seem to involve a characteristic length, as indicated in their Fig. 3, but, unlike λ, this length seems to be independent of the velocity with which the vortex moves relative to the normal fluid, except perhaps at temperature very close to the λ point. It is not clear what determines this characteristic length.

Doubts about the validity of these simulations arise for at least two reasons: the computational box was too small in relation to the spatial extent of the perturbed velocity field; and insufficient time was allowed for a steady state to be achieved. As we understand it, the velocity fields were

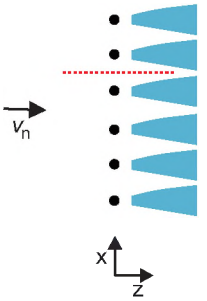

FIG. 6. Illustrating the effect of the small radial flows.

all observed at a time, t, equal to one second after the vortex was set in motion. At this time the vorticity generated by motion of the vortex can have suffered viscous diffusion to a distance of only $\delta = (2\eta_n t/\rho_n)^{1/2}$, which is equal to only 0.19 mm, and which is therefore less than the size of the computational box. This suggests that a steady state had not been reached, and that δ, rather than λ, might be the characteristic length underlying the computations.

These considerations do of course raise the question whether a steady state is reached in the experiments. It turns out that for practical purposes the steady state is reached, because the characteristic length λ is much smaller than in the simulations of Ref. [7]. Therefore, our conclusion about the perturbation of the normal fluid velocity caused by dragging by a single vortex still holds. We must therefore seek another explanation for the experimental results of Sec. II.

We know from the success of simulations of counterflow turbulence in the superfluid component that the vortex lines form at any instant a disordered array. This implies that the local density of vortex lines, and hence the corresponding local value of the mutual friction, is nonuniform. This nonuniformity will give rise to local variations in the velocity of the normal fluid, additional to those already discussed, as is observed. To see whether these local variations might be sufficiently large, we have tried to analyze the effect of a disordered array of rectilinear vortices. We shall assume that each such vortex produces a wake, as described above. After the next section, which aims to clarify how mass continuity is achieved with an array of vortices, we shall first discuss the relationship between disordered wake structures and the velocity correlation functions introduced in Sec. II; then we describe our calculated wake structures for disordered vortex arrays; and finally we calculate velocity correlation functions for these disordered vortex arrays and compare them with experiment.

C. Mass continuity in a moving vortex array

Before we describe these results with disordered arrays of vortices, we think it useful to enlarge on the effect of the small radial flows described by the second term on the right-hand side of Eq. (A7). As we have already explained, for the case of a single isolated rectilinear vortex, this term serves simply to ensure mass continuity; there is an effective flow towards the vortex in the wake, and the second term describes a consequential flow away from the vortex. However, it is less clear how mass continuity is maintained in the case of an array of vortices, as we now explain. Consider, for example, the flow of normal fluid past an infinite row of vortex lines, each line being normal to the xz plane, as shown schematically in Fig. 6, the flow being maintained by a suitable temperature gradient. In addition to the wakes (shown in blue), each vortex will produce a radial flow, away from the vortex. Components of these radial flows in the direction of x will largely cancel, but let us consider the effect of those in the direction of z along a line such as that shown broken in red. They will serve first to decelerate the flow of the normal fluid for $z < 0$ and then to accelerate it for $z > 0$. The deceleration will be canceled by the temperature gradient that is driving the flow of the normal fluid, but the acceleration for $z > 0$ serves to ensure that the total mass flow across a plane

$z =$ constant to the right of the vortices is conserved in spite of the smaller velocity existing within the wakes. Similar considerations apply to the disordered arrays that we now discuss.

D. Relations between disordered wake structures and velocity correlation functions

As we have stated, the vortices present in a thermal counterflow form a disordered array, and we must first consider how the corresponding wake structures are related to the observed velocity correlation functions described in Sec. II.

We work with cartesian coordinates (x, y, z), the average steady velocity of the normal fluid being along the $+z$ axis. We suppose that we have calculated the wake structure in the region $z > 0$ due to all the vortices, forming a disordered array, in the region $z \leqslant 0$, and we denote the corresponding velocity field in the region $z > 0$ by $v(\text{wake}, z)$. In addition there is in the region $z > 0$ the velocity field $v(2, z)$ due to the vortices in this region. Observation of the velocity correlation function serves to measure $\langle (v(\text{wake}, 0)[v(\text{wake}, z) + v(2, z)])\rangle$. In what follows we shall assume that $v(\text{wake}, 0)$ and $v(2, z)$ are uncorrelated. This assumption seems reasonable if the disorder is sufficiently strong, and if we are dealing with arrays of parallel rectilinear vortices. In practice it is unlikely to be strictly correct, especially for more realistic configurations of vortices; but it is the best we can do at present. It means that the observed correlation function serves to measure $\langle (v(\text{wake}, 0)(v(\text{wake}, z))\rangle$, which is obtained by suitable averaging of the wake structures discussed in the next section.

E. The velocity induced in the normal fluid by a disordered arrays of quantized vortex lines moving with the superfluid

Our aim in this section is to consider as best we can at this stage the perturbation in the velocity of the normal fluid caused by a disordered array of vortices, such as exists in a counterflow associated with a heat flux. We shall consider first the effect of a disordered array of parallel rectilinear vortices, aligned along the y axis in a Cartesian coordinate system, and stationary in the frame of reference moving with the average superfluid velocity.

We shall assume that the temperature and other parameters are the same as those relating to Fig. 5. Using a random number generator, we place 300 rectilinear vortices at random positions within the region defined by $-10 < z/\ell < 0$, $-15 < x/\ell < 15$; the positions of the central third of the vortices are shown in Fig. 7. We then calculate the total perturbation to the normal-fluid velocity in the region defined by $0 < z/\ell < 50$, $-5 < x/\ell < 5$, assuming that the wakes are additive. This last assumption is justified because the Oseen equation, on which our simulations are based, is linear in the velocity perturbation u. The result is shown in Fig. 8(a). The result for a different set of random positions is shown in Fig. 8(b).

We see that the randomly positioned rectilinear vortices lead to random perturbations in the normal-fluid velocity that are quite large and extend to distances that are much greater than those generated by a single vortex. Perturbations that vary slowly in the transverse direction seem to decay more slowly in the direction of flow, and can decay in distances greater than 50ℓ, although perturbations that vary more rapidly in the transverse direction tend to decay in distances of order 10ℓ. The fact that the predicted perturbations in the normal-fluid velocity extend much further than those due a single vortex suggests that we may be on the right lines to account for the experimental results of Sec. II. However, we have still to calculate and display the form of the expected correlation function that can be compared with the experimental data in Fig. 3, and we shall do this in the next Section.

As is clear from Figs. 8(a) and 8(b), the width of the region where the normal-fluid velocity is perturbed depends on the particular random arrangement of the vortices. We have already noted that, although such widths are in principle measurable, they cannot be deduced with adequate accuracy from our existing observations.

Although the large perturbations in normal-fluid velocity that we see in our simulations must arise from inhomogeneities in the density of vortex lines, we can see no simple connection with the

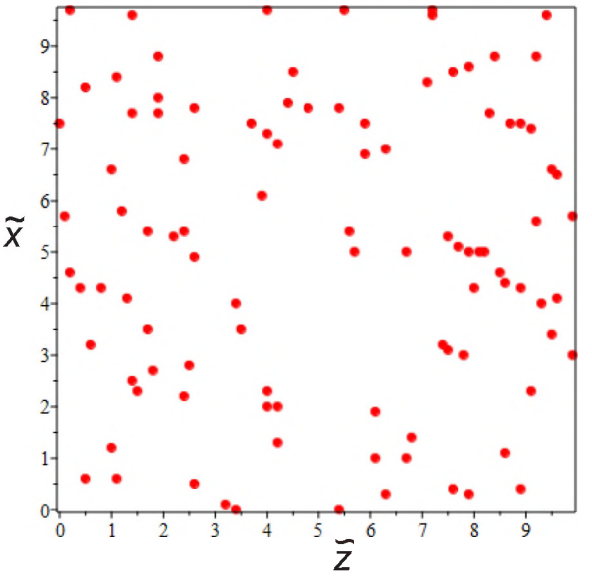

FIG. 7. Rectilinear vortices in random positions.

spectrum of vortex density fluctuations, which has been the subject of a good deal of study (see, for example, Refs. [13,14]).

We must emphasize that our theoretical analysis so far has ignored the motion of the vortex lines relative to the average superfluid velocity, and we shall attempt to address this deficiency in Sec. III(g).

Before we embark on the next section we mention briefly that we have also investigated the perturbations in the normal fluid velocity produced by a random arrangement of vortex rings at a temperature of 2 K, the density of the rings being such that the line densities are similar to

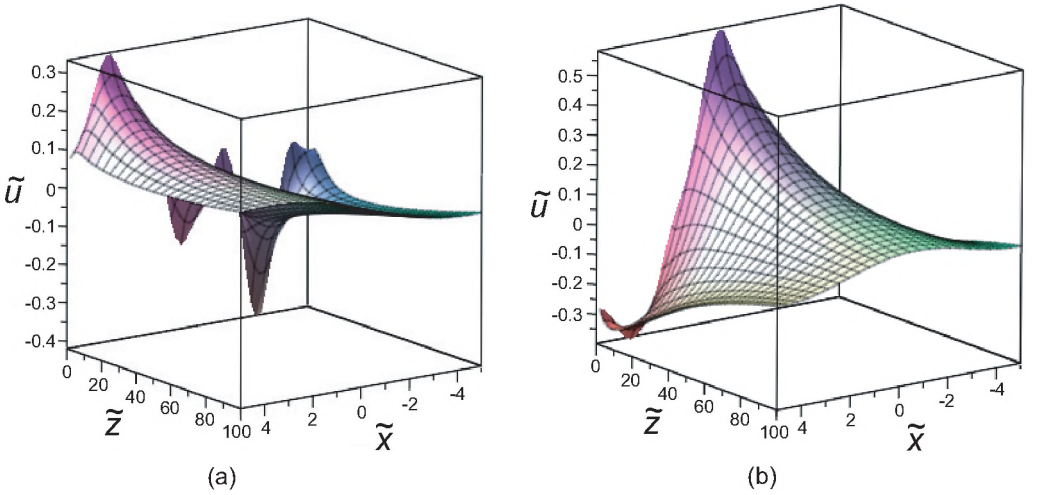

FIG. 8. (a, b) The wakes generated in the normal fluid by a randomly spaced array of rectilinear vortices; the dimensionless velocity in the wakes as a function of $\tilde{x}$ and $\tilde{z}$. Figure 8(a) relates to the distribution of vortices in Fig. 7.

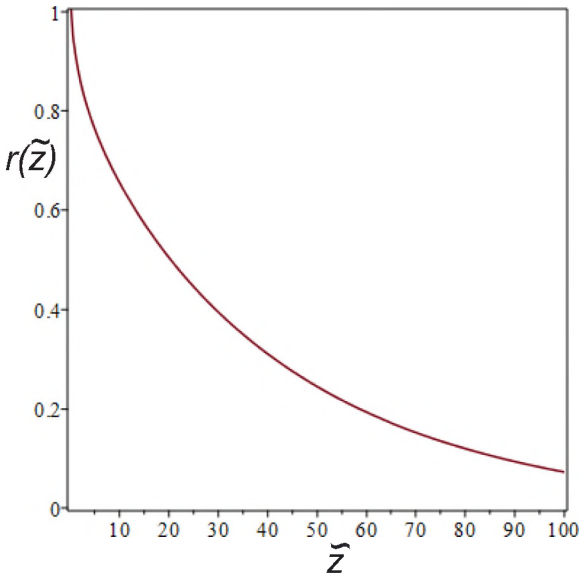

FIG. 9. The calculated correlation function, $r(\tilde{z})$, plotted against $\tilde{z}$.

those involved in our studies of random arrangements of rectilinear vortices. The results are broadly similar to those obtained with rectilinear vortices, suggesting that the results are not very sensitive to the precise arrangement of the vortices. Nevertheless, we recognize that we have not yet based our simulations on realistic vortex configurations.

F. Predicted form of the correlation function [Eq. (3)]

In accord with the argument in Sec. III(d), velocity fields of the type displayed in Fig. 8 can be used to calculate the correlation function $r(\tilde{z})$, which is equivalent to the correlation function $r(\tau)$ introduced in Sec. II if we put $\tau = \ell\tilde{z}/\langle v_n \rangle$. The result, based on sets of data of the type displayed in Fig. 8 is shown in Fig. 9.

Comparing this calculated correlation function with its experimental counterpart (Fig. 3), we see that the ranges (in time or distance) are in approximate agreement, but that the forms of the function are not. At the present time we do not know the reason for this disagreement. Our assumption that the vortices are arranged in a completely random way may be wrong, and the fact that our vortex configurations are unrealistic may also be relevant. However, it seems unlikely that our calculations are misleading to such an extent that they lead to a correlation function that is so different in form from that suggested by experiment: i.e., from a rather strange form in which there is almost complete correlation over a time during which the tracing particle has moved a distance much greater than the average spacing between the vortex lines, followed by a rather abrupt fall. But these experimental data are, as we have already emphasized, subject to considerable uncertainty, so that we are led to the conclusion that they need to be subject to further careful checking.

In spite of these reservations, there seems little doubt that disordered arrays of vortex lines in the superfluid component in a thermal counterflow must lead to substantial inhomogeneities ($\sim$30%) in the normal-fluid velocity, as is observed. It seems likely that any adequate theory of thermal counterflow ought to take account of these inhomogeneities, which has not so far been the case. Judging from the effect of a local reduction in the velocity of the normal fluid when it is associated with a normal fluid velocity profile in a channel of finite width [15], we guess that the inhomogeneities in this velocity discussed here are likely also to result in a significant inhomogeneities in the local vortex line density.

There is one apparently possible, but very speculative, explanation of the observed correlation to which we ought briefly to refer. The irregular flow of the normal fluid that underlies the correlation function displayed in Fig. 9 is essentially laminar, since it is derived from a superposition of laminar wakes. However, an effective Reynolds number for this flow, formed from the mean velocity fluctuations and the mean vortex spacing, is typically about 10, which suggests that a transition to turbulent flow might occur on scales a little larger than the vortex spacing. It might be thought that this turbulent flow would be significantly less anisotropic than the laminar flow, leading to disagreement with experiment. However, a recent theoretical study [16] has shown that this might not be the case. As we mentioned in our Introduction, it is now well-established that at heat fluxes greater than those involved in the present study there is large-scale, partly coupled, turbulence in the two fluids, in addition to the small-scale tangled motion of the vortex lines that has been the basis of our discussion here [3]. The study in Ref. [16] shows that this large-scale motion is quite anisotropic, especially on a small scale, the anisotropy arising from mutual friction because the turbulent eddies in each fluid are being continuingly pulled apart in the direction of the steady relative motion of the two fluids. The possibility exists therefore that a transition to turbulence in our case, which would involve only small-scales but would still be in the presence of a steady counterflow, would also preserve the anisotropy, especially if it were to involve partially coupled turbulence in both fluids. Such a turbulent flow might possibly lead to a correlation function similar to that observed. However, the arguments in Ref. [16] were based on continuum approximations to the flow of the superfluid component, and it is very doubtful whether they can apply on scales only a little larger than the vortex spacing. Furthermore, this small-scale turbulence could well be too strongly damped by mutual friction. It should be emphasized that, even if this small-scale turbulence were indeed to be established, it would still fundamentally have its origin in the local dragging of the normal fluid by the vortex cores.

The development of a better theory poses major problems. Such a theory must probably be based on simulations that take account of realistic vortex configurations interacting realistically with the normal fluid. Such a theory must probably be based on simulations in which both the normal fluid and the superfluid velocity fields are treated fully as coupled systems, as was done recently in a different context by Yui *et al.* [17], and must take account of the time-dependence of the vortex configurations (to which we refer in the next section). However, the spatial resolution required in the normal fluid velocity needs to be better than the vortex line spacing, ℓ, and therefore significantly better than that in the simulations of Yui *et al.* At the same time there is a clear need to improve the experimental observations, especially of the normal-fluid velocity correlation function.

G. The effect of vortex lifetimes and of the random motion of the vortices relative to the average superfluid velocity

So far we have assumed in our model calculations that the vortices are not being continually generated and destroyed in the thermal counterflow, and that they are at rest relative to the average velocity of the superfluid component.

We note first that the vortices are moving in a random fashion, and that the corresponding random vortex velocity can be obtained, at least to a reasonable approximation, from the local induction approximation as follows [11]

$$v \sim \frac{\kappa}{4\pi R} \ln\left(\frac{R}{\xi_0}\right), \tag{5}$$

where ξ_0 is the vortex core parameter, and where the mean value of $1/R^2$ is given by

$$\left\langle \left(\frac{1}{R^2}\right) \right\rangle = c_2^2 \ell^{-2}, \tag{6}$$

where c_2 is a constant equal to about 2. It follows that

$$v \sim \frac{c_2 \kappa}{\ell}. \tag{7}$$

As a result of this random vortex motion the arrangement of vortices develops in time, the characteristic time required for a significant change in this arrangement being given by

$$\tau_e = \frac{\ell^2}{c_2 \kappa}. \tag{8}$$

This random motion gives rise through mutual friction to dissipation [11,18], which is characterised by a time constant given by

$$\tau_D = \frac{2\pi}{\chi_2 \kappa L} = \frac{2\pi \ell^2}{\chi_2 \kappa}, \tag{9}$$

where χ_2 is a dimensionless parameter equal to about 3.5 at a temperature of 2 K [18]. The time constants τ_e and τ_D are similar in magnitude, and either can be taken as defining the timescale over which the arrangement of the vortices evolves. Using the numerical values set out in Sec. III(b), we find that this timescale is about 38 ms at 2 K. We emphasize that this timescale relates to vortex rearrangements on scales of order the vortex spacing; rearrangements on larger scales might well take place more slowly.

Vortex rearrangements could, to take two extreme cases, either generate wiggles on a normal-fluid wake or lead to the effective termination of the wake. The observed wakes seem to be characterized by a timescale of about 100 ms. It is not yet completely clear whether this time scale reflects the particular forms of vortex line inhomogeneities (so that, as we have assumed in earlier sections, it is not determined by random motion of the vortices), or whether it is determined by time-dependent vortex rearrangements. In any case it seems to be somewhat larger that 38 ms, although not necessarily much larger. As we have just emphasized, rearrangement on scales larger than the vortex spacing could take longer than 38 ms, and this does indeed appear to be the case. Further progress requires more sophisticated simulations than we are able to carry out, such as those of Yui *et al.*, to which we have already referred [17]. We note that wiggles on the wakes might account for the small transverse standard deviations shown in Fig. 2.

We must refer at this point to the work of Sergeev *et al.* [9]. They calculated the trajectories of micron-sized tracer particles interacting with the normal-fluid velocity field near a single moving rectilinear vortex line, expressing their results in terms of scattering cross-sections. As it turns out, this work is probably not directly relevant to the experimental results described in Sec. II, because, as we now see it, irregular motion of such tracer particles in a thermal counterflow is due more to nonuniformity in vortex density than to the dragging of normal fluid by an isolated vortex. As we have seen, the resulting irregular motion takes the form to a large extent of simply a spatial variation in the component of the drift velocity in the direction of the heat flux, without significant "scattering."

IV. CONCLUSIONS

We have presented experimental results demonstrating that, even when the flow of the normal fluid is laminar, thermal counterflow leads to nonuniformity in the velocity of the normal fluid. Calculations are presented to show that probably this nonuniformity has its origin in the fact that the force of mutual friction is concentrated near the cores of the quantized vortex lines (an idea that is not new [2,6,7]), and that the characteristics of the observed nonuniformity can be produced only by a disordered arrangement of these lines. Although these calculations are in many ways unrealistic and based sometimes on questionable assumptions, we believe that they serve to capture the essential physics and are at least qualitatively correct. Further developments will require improved experimental data and more sophisticated theoretical analysis, as we have explained.

A depressing conclusion relates to the fact that existing simulations of thermal counterflow turbulence (see, for example, Ref. [19]) do not take into account the effects that we describe and may therefore be unreliable. An interesting, if speculative, conclusion is that the inhomogeneities in the normal fluid velocity that we describe may be a precursor of the large scale coupled turbulence that is observed to set in at heat currents a little larger than those relevant to our present studies [3].

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APPENDIX

Here we set out a plausible derivation of the velocity field in the normal fluid generated by a rectilinear quantized vortex moving with steady velocity U relative to that of the normal fluid at infinity.

Let the force of mutual friction per unit length of vortex be f, and let the range of this force from the center of the vortex be ξ. We shall start by writing down the known velocity field due to a slowly-moving solid cylinder, of radius a, in a classical fluid of viscosity η_n and density ρ_n; this field can be derived only if the nonlinear term in the Navier-Stokes equation is taken into account to the extent assumed in the Oseen approximation [20]. We shall then make the plausible assumption that the required velocity field at distances from the vortex considerably greater than both a and ξ is the same as that generated by the cylinder if a is chosen to lead to same drag force f.

The velocity field due to the moving cylinder has a simple form in the limits when the distance r from the cylinder is either much less than, or much greater than, the characteristic length $\lambda = \nu_n/2U$, where $\nu_n = \eta_n/\rho_n$.

In the former case the velocity field is given in cylindrical polar coordinates by [20]

$$u_r = U\cos\theta\left(1 - \frac{C}{2}\ln\frac{r}{a} + \frac{C}{4} - \frac{Ca^2}{4r^2}\right), \quad u_\theta = -U\sin\theta\left(1 - \frac{C}{2}\ln\frac{r}{a} - \frac{C}{4} + \frac{Ca^2}{4r^2}\right), \tag{A1}$$

where the parameter C is related to the force, f, by the relation

$$f = 2\pi\eta_n C U. \tag{A2}$$

It turns out, however, that this form of the "near" velocity field cannot join a satisfactory solution for the "far" velocity field unless the parameter C is given by

$$C = \frac{2}{\ln(7.4\lambda/a)}. \tag{A3}$$

It follows from Eqs. (A1)–(A3) that

$$u_r = \frac{f\cos\theta}{4\pi\eta_n}\left(-\ln\frac{r}{7.4\lambda} + \frac{1}{2} - \frac{a^2}{4r^2}\right), \quad u_\theta = -\frac{f\sin\theta}{4\pi\eta_n}\left(-\ln\frac{r}{7.4\lambda} - \frac{1}{2} + \frac{a^2}{4r^2}\right). \tag{A4}$$

Although in the limit $r \gg a$ these expressions for the velocity field do not depend explicitly on a, we ought to require implicitly that the force f is equal to that corresponding to the mutual friction force, $\alpha\kappa\rho_s U$; thus, we ought to check that this requirement does not lead to an unacceptable value

of a. The required value of a is in fact given by

$$a = \frac{7.4\lambda}{\exp\left(4\pi\rho\nu_n/\alpha\kappa\rho_s\right)}. \tag{A5}$$

Using the numerical values for a temperature of 2 K set out in Sec. III(b), we find that $a/\lambda = 3.11 \times 10^{-4}$. The fact that this value is comparable with ξ means that it is indeed acceptable. We conclude then that the Eq. (A4) can be taken to describe the perturbation in the normal-fluid velocity field caused by a rectilinear vortex moving with velocity U relative to the normal fluid, in the near-field limit $\lambda \gg r \gg \xi, a$.

In the far field limit, $r \gg \lambda$, the velocity field produced by a moving cylinder is given by (see, for example, Mei in his Lecture Notes on Fluid Dynamics, MIT, Spring 2007)

$$u_r = -CU\left(\frac{2\pi\lambda}{r}\right)^{1/2}\exp\left(-\frac{r\theta'^2}{8\lambda}\right) + \frac{2C\lambda U}{r}, \tag{A6}$$

where $\theta' = \pi - \theta$, and the product CU is still given in terms of f by Eq. (A2). The first term on the right-hand side of Eq. (A6) describes a wake. The second term describes a small radial flow that is required to satisfy mass continuity. As in the case of the near-field, we argue that this formula ought to describe the far velocity field for the case of the moving vortex. In describing the far velocity field it will prove convenient to use Cartesian coordinates, such that the vortex is along the $x-$ axis and U is directed along the z axis, so that for a well-developed narrow wake

$$u_r(y,z) = -\frac{f}{\eta_n}\left(\frac{\lambda}{2\pi z}\right)^{1/2}\exp\left(-\frac{y^2}{8\lambda z}\right) + \frac{f\lambda}{\pi\eta_n(y^2+z^2)^{1/2}}. \tag{A7}$$

We can comment briefly on the physical interpretation of these results. In the near velocity field there is a simple diffusion of vorticity away from the core of the vortex; the characteristic logarithmic term arises straightforwardly from the two-dimensional diffusion equation in cylindrical polar coordinates. The existence of the velocity U is unimportant as long as the front of the diffusing region is moving at a speed greater than U. The velocity of this front is equal to $2\nu_n/r$. However, as soon as this velocity, in the transverse direction, becomes less than U, the effect of U becomes important and causes the front to swing round behind the moving cylinder or moving vortex, so giving rise to the wake. Therefore, the wake starts to form when $2\nu_n/r > U$; i.e., when $r > 4\lambda$.

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