A SECOND ORDER, LINEAR, UNCONDITIONALLY STABLE, CRANK-NICOLSON-LEAPFROG SCHEME FOR PHASE FIELD MODELS OF TWO-PHASE INCOMPRESSIBLE FLOWS

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Abstract. In this article we propose a second order, linear, unconditionally stable, implicit-explicit scheme based on the Crank-Nicolson-Leapfrog discretization and the artificial compression method for solving phase field models of two-phase incompressible flows. We show that the scheme is unconditionally long-time stable. Numerical examples are provided to demonstrate the accuracy and long-time stability.

Key words. phase field models; Cahn-Hilliard-Navier-Stokes; finite element method; artificial compression; unconditional stability

1. Introduction.

Multiphase flows are ubiquitous in science and in industrial applications. Due to moving boundaries and possible phase changes, it is challenging to model and perform numerical simulations on. multiphase flow phenomena. In recent years phase field (diffuse interface) models have become an increasingly popular approach in the study of multiphase flows. Various phase field fluid models have been proposed, including the Cahn-Hilliard-Navier-Stokes system (model H) for two-phase flows of matched densities [11, 8, 19], the Cahn-Hilliard-Hele-Shaw (Darcy) model for two-phase flows in a Hele-Shaw cell or porous media [17, 18], the Cahn-Hilliard-Stokes-Darcy system for multiphase flows in karst geometry [9], the quasi-incompressible Cahn-Hilliard fluid models for two-phase flows of variable density [20], the incompressible diffuse interface model for two-phase flows of different densities [1], among many others, cf. [2, 15] for general reviews.

In a diffuse interface fluid model the sharp interface between two immiscible fluids is replaced by a diffusive interface of finite thickness over which field variables such as order parameter, pressure vary continuously. The major obstacle for numerical simulations of diffuse interface models is the stiffness associated with the diffusive interface (steep transition over thin layers). It is of utmost importance to design high-order, unconditional stable numerical schemes for solving these Cahn-Hilliard fluid models. There are two popular approaches. The first approach is the celebrated convex-concave splitting which treats the convex part of the potential function implicitly and concave part explicitly, cf. [21, 10] for Crank-Nicolson schemes, and [26] for a second order BDF scheme, see also [25] for a stabilized linear second order convex-splitting scheme. The second approach is the Lagrange multiplier approach [7] and its recent generalization: the Invariant Energy Quadratization method (IEQ) [30, 6] and the Scalar Auxiliary Variable approach (SAV) [22]. In these methods an auxiliary variable (Lagrange multiplier) is introduced to rewrite the energy functional leading to an expanded gradient flow which is suitable for the design of high-order unconditionally stable linear schemes.

While current literature focuses on either the Crank-Nicolson or the BDF2 temporal discretization, this article contributes to a second order, linear, unconditionally stable scheme based on the Crank-Nicolson-Leapfrog discretization (CNLF) and the

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artificial compression method (AC) for solving phase field fluid models. We follow closely the CNLF scheme proposed and analyzed in [16, 12, 13, 14], and the AC scheme studied in [5] for the Navier-Stokes equations which is in turn inspired by the classical fractional step methods [24, 23] and the pressure projection method [4]. As a particular example we present the scheme for solving the Cahn-Hilliard-Navier-Stokes system. The discretization of the Cahn-Hiliard equation is performed in the framework of the IEQ approach with stabilization to enhance stability. The computation of the velocity is decoupled from that of the pressure at the expense of the coupling among the velocity components (hence increased condition number) due to the grad-div stabilization, and no boundary condition is needed in the update of pressure thanks to the application of the AC method, cf. [5]. We establish the unconditional long-time stability of the scheme. Numerical resulted to presented to gauge the accuracy and stability of the proposed algorithm. We emphasize that the method can be readily generalized to the other phase field fluid models such as the Allen-Cahn-Navier-Stokes equations, the Cahn-Hilliard-Hele-Shaw system etc. It is also possible to design similar algorithm (CNLF+AC) for solving phase field fluid models of variable densities, following the approach in [6]. This will be pursued in a separate work.

The rest of the article is divided into two sections. In Sec. 2 we recall the Cahn-Hilliard-Navier-Stokes system, present the semi-discrete numerical scheme, and establish the unconditional long-time stability. In Sec. 3 we provide numerical examples to verify the accuracy and long-time stability of the algorithm.

2. The model and the numerical scheme.

2.1. The model. As an example we present our numerical algorithm for solving the Cahn-Hilliard-Navier-Stokes system that models two-phase flows of matched density. Similar schemes can be constructed for other phase field fluid models. We consider a mixture of two immiscible, incompressible fluids in a bounded Lipschitz domain Ω in R^d $(d = 2, 3)$ with matched density assumed to be unity for simplicity of presentation. Introducing a phase function ϕ such that

$$\phi(x,t) \approx \begin{cases} 1, & \text{for fluid 1,} \\ -1, & \text{for fluid 2,} \end{cases} \tag{2.1}$$

we adopt the Ginzburg-Landau type free energy associated with the binary system

$$W(\phi, \nabla\phi) = \int_\Omega \lambda \left(\frac{1}{2} |\nabla\phi|^2 + F(\phi) \right) dx. \tag{2.2}$$

In (2.2) the first term contributes to the hydrophilic type (tendency of mixing) of interactions between the materials while the second part, the double-well bulk energy $F(\phi) = \frac{1}{4\eta^2}(\phi^2 - 1)^2$, represents the hydrophobic type (tendency of separation) of interactions. As the consequence of the competition between the two types of interactions, the equilibrium configuration will include a diffusive interface with thickness proportional to the parameter η.

The governing equations are the following Cahn-Hilliard-Navier-Stokes (CHNS) system:

$$\begin{cases} \phi_t + \nabla \cdot (\phi \mathbf{u}) = M\Delta\mu, \\ \mu = \lambda(-\Delta\phi + f(\phi)), \\ \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \nu\Delta\mathbf{u} = -\phi\nabla\mu, \\ \nabla \cdot u = 0, \end{cases} \tag{2.3}$$

equipped with the boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \quad \partial_n \phi|_{\partial\Omega} = 0, \quad \partial_n \mu|_{\partial\Omega} = 0.$$

Assuming no external forcing other than gravity, the CHNS system satisfies an energy law, i.e.

$$\frac{d}{dt}\left\{W(\phi,\nabla\phi) + \int_\Omega \frac{1}{2}|\mathbf{u}|^2 dx\right\} = -\int_\Omega M|\nabla\mu|^2 + \nu|\nabla\mathbf{u}|^2 dx. \tag{2.4}$$

2.2. The algorithm. In this section we introduce the algorithm in the semi-discrete form and establish its unconditional long-time stability. Throughout the $L^2(\Omega)$ norm of scalars, vectors, and tensors will be denoted by $\|\cdot\|$ with the usual L^2 inner product denoted by $(\cdot,\cdot)$. Let $q = \frac{1}{\eta^2}(\phi^2-1)$ and thus $f(\phi) = \phi q$. Taking derivative of q with respect to gives $q_t = \frac{2}{\eta^2}\phi\phi_t$, which can discretized by the Crank-Nicolson-Leapfrog scheme as follows.

$$\frac{q^{n+1}-q^{n-1}}{2\Delta t} = \frac{2}{\eta^2}\phi^n\frac{\phi^{n+1}-\phi^{n-1}}{2\Delta t}.$$

We then propose a similar second order, partitioned, linear, Crank-Nicolson Leap-Frog artificial compression method for the CHNS system given by

ALGORITHM 2.1. *Given $\mathbf{u}^{n-1}$, $\mathbf{u}^n$, p^{n-1}, p^n, ϕ^{n-1},ϕ^n, find $\mathbf{u}^{n+1}$, p^{n+1}, ϕ^{n+1} and μ^n satisfying*

$$\frac{\phi^{n+1}-\phi^{n-1}}{2\Delta t} + \nabla\cdot\left(\phi^n\frac{\mathbf{u}^{n+1}+\mathbf{u}^{n-1}}{2}\right) - M\Delta\mu^n = 0, \tag{2.5}$$

$$\mu^n = \lambda\left(-\Delta\frac{\phi^{n+1}+\phi^{n-1}}{2} + \gamma(\phi^{n+1}-2\phi^n+\phi^{n-1}) + \phi^n\frac{q^{n+1}+q^{n-1}}{2}\right), \tag{2.6}$$

$$q^{n+1}-q^{n-1} = \frac{2}{\eta^2}\phi^n\left(\phi^{n+1}-\phi^{n-1}\right), \tag{2.7}$$

$$\begin{aligned}&\frac{\mathbf{u}^{n+1}-\mathbf{u}^{n-1}}{2\Delta t} - \beta\Delta t^{-1}\nabla\nabla\cdot(\mathbf{u}^{n+1}-\mathbf{u}^{n-1}) + \mathbf{u}^n\cdot\nabla\left(\frac{\mathbf{u}^{n+1}+\mathbf{u}^{n-1}}{2}\right)\\&+\frac{1}{2}(\nabla\cdot\mathbf{u}^n)\left(\frac{\mathbf{u}^{n+1}+\mathbf{u}^{n-1}}{2}\right) - \nu\Delta\left(\frac{\mathbf{u}^{n+1}+\mathbf{u}^{n-1}}{2}\right) + \nabla p^n + \phi^n\nabla\mu^n = 0,\end{aligned} \tag{2.8}$$

$$\alpha\Delta t\left(p^{n+1}-p^{n-1}\right) + \nabla\cdot\mathbf{u}^n = 0, \tag{2.9}$$

$$\mathbf{u}^{n+1}|_{\partial\Omega} = 0, \quad \nabla\phi^{n+1}\cdot n|_{\partial\Omega} = 0, \quad \nabla\mu^n\cdot n|_{\partial\Omega} = 0. \tag{2.10}$$

The Eqs. (2.5)–(2.7) represent a Leap-frog time marching scheme for the Cahn-Hilliard equation in which the nonlinear potential is treated by the the Invariant Energy Quadratization method (IEQ) [27, 30, 3, 28, 31, 29] with stabilization. Note that the recently developed scalar auxiliary variable (SAV) approach [22] is equally applicable here. Eqs. (2.8)–(2.9) is the Leap-frog artificial compression method for solving Navier-Stokes equations [5].

THEOREM 2.2. *Taking α and β such that $\alpha\beta \geq \frac{1}{4}$, then the scheme* (2.5)–(2.10) *is unconditionally long-time stable in the sense that for any $N \geq 2$*

$$2\Delta t\sum_{n=1}^{N-1}\|\nabla\mu^n\|^2 + \frac{\lambda}{2}\left(\|\nabla\phi^N\|^2 + \|\nabla\phi^{N-1}\|^2\right) + \frac{\lambda\eta^2}{4}\left(\|q^N\|^2 + \|q^{N-1}\|^2\right)$$

$$
\begin{aligned}
&+ \Delta t \sum_{n=1}^{N-1} 2\nu \| \frac{\nabla \mathbf{u}^{n+1} + \nabla \mathbf{u}^{n-1}}{2} \|^2 + \frac{1}{2} \left(\|\mathbf{u}^N\|^2 + \|\mathbf{u}^{N-1}\|^2 \right) \\
&+ \frac{\lambda\gamma}{2} ||\phi^N - \phi^{N-1}||^2 + \frac{\lambda\gamma}{2} \sum_{n=1}^{N-1} ||\phi^{n+1} - 2\phi^n + \phi^{n-1}||^2 \\
\le\ & \frac{\lambda}{2} \left(\|\nabla\phi^1\|^2 + \|\nabla\phi^0\|^2 \right) + \frac{\lambda\eta^2}{4} \left(\|q^1\|^2 + \|q^0\|^2 \right) + \frac{1}{2} \left(\|\mathbf{u}^1\|^2 + 2\beta\|\nabla\cdot\mathbf{u}^1\|^2 \right) \\
&+ \frac{1}{2} \left(\|\mathbf{u}^0\|^2 + 2\beta\|\nabla\cdot\mathbf{u}^0\|^2 \right) + \alpha\Delta t^2 \left(\|p^1\|^2 + \|p^0\|^2 \right) + \frac{\lambda\gamma}{4} ||\phi^1 - \phi^0||^2 \\
&+ \Delta t (p^1, \nabla\cdot\mathbf{u}^0) - \Delta t (p^0, \nabla\cdot\mathbf{u}^1).
\end{aligned}
$$

Proof. Taking the inner product of (2.5) with μ^n and multiply through by $2\Delta t$ gives

$$
\left(\phi^{n+1} - \phi^{n-1}, \mu^n\right) - 2\Delta t \left(\phi^n \left(\frac{\mathbf{u}^{n+1} + \mathbf{u}^{n-1}}{2} \right), \nabla\mu^n \right) + 2\Delta t M \|\nabla\mu^n\|^2 = 0. \tag{2.11}
$$

Taking the inner product of (2.6) with $\phi^{n+1} - \phi^{n-1}$ gives

$$
\begin{aligned}
&- \left(\phi^{n+1} - \phi^{n-1}, \mu^n\right) + \lambda \left(\frac{1}{2} \|\nabla\phi^{n+1}\|^2 - \frac{1}{2} \|\nabla\phi^{n-1}\|^2 \right) \\
&+ \frac{\lambda\gamma}{2} \left(||\phi^{n+1} - \phi^n||^2 - ||\phi^n - \phi^{n-1}||^2 + ||\phi^{n+1} - 2\phi^n + \phi^{n-1}||^2 \right) \\
&+ \lambda \left(\phi^n \frac{q^{n+1} + q^{n-1}}{2}, \phi^{n+1} - \phi^{n-1} \right) = 0.
\end{aligned} \tag{2.12}
$$

Taking the inner product of (2.7) with $\lambda\eta^2 \frac{q^{n+1}+q^{n-1}}{4}$ gives

$$
\frac{\lambda\eta^2}{4} \left(\|q^{n+1}\|^2 - \|q^{n-1}\|^2 \right) = \lambda \left(\phi^n \left(\phi^{n+1} - \phi^{n-1} \right), \frac{q^{n+1} + q^{n-1}}{2} \right). \tag{2.13}
$$

Summing up (2.11), (2.12) and (2.13) yields

$$
\begin{aligned}
&- 2\Delta t \left(\phi^n \left(\frac{\mathbf{u}^{n+1} + \mathbf{u}^{n-1}}{2} \right), \nabla\mu^n \right) + 2\Delta t M \|\nabla\mu^n\|^2 \\
&+ \frac{\lambda}{2} \left(\|\nabla\phi^{n+1}\|^2 - \|\nabla\phi^{n-1}\|^2 \right) + \frac{\lambda\eta^2}{4} \left(\|q^{n+1}\|^2 - \|q^{n-1}\|^2 \right) \\
&+ \frac{\lambda\gamma}{2} \left(||\phi^{n+1} - \phi^n||^2 - ||\phi^n - \phi^{n-1}||^2 + ||\phi^{n+1} - 2\phi^n + \phi^{n-1}||^2 \right) = 0.
\end{aligned} \tag{2.14}
$$

Taking inner product of (2.8) with $\Delta t(\mathbf{u}^{n+1} + \mathbf{u}^{n-1})$ gives

$$
\frac{1}{2} \left(\|\mathbf{u}^{n+1}\|^2 + 2\beta\|\nabla\cdot\mathbf{u}^{n+1}\|^2 \right) - \frac{1}{2} \left(\|\mathbf{u}^{n-1}\|^2 + 2\beta\|\nabla\cdot\mathbf{u}^{n-1}\|^2 \right) \tag{2.15}
$$

$$
\begin{aligned}
&+ 2\Delta t \nu \| \frac{\nabla\mathbf{u}^{n+1} + \nabla\mathbf{u}^{n-1}}{2} \|^2 - \Delta t \left(p^n, \nabla\mathbf{u}^{n+1} + \nabla\mathbf{u}^{n-1} \right) \\
&+ \Delta t \left(\phi^n \nabla\mu^n, \mathbf{u}^{n+1} + \mathbf{u}^{n-1} \right) = 0.
\end{aligned} \tag{2.16}
$$

Taking inner product of (2.9) with $p^{n+1}+p^{n-1}$ and multiplying through by Δt

$$\alpha\Delta t^2\left(\|p^{n+1}\|^2-\|p^{n-1}\|^2\right)+\Delta t\left(\nabla\cdot u^n,p^{n+1}+p^{n-1}\right)=0. \tag{2.17}$$

Adding (2.14), (2.15) and (2.17) yields

$$\begin{aligned}
&2\Delta tM\|\nabla\mu^n\|^2+\frac{\lambda}{2}\left(\|\nabla\phi^{n+1}\|^2+\|\nabla\phi^n\|^2\right)-\frac{\lambda}{2}\left(\|\nabla\phi^n\|^2+\|\nabla\phi^{n-1}\|^2\right) \qquad (2.18)\\
&+\frac{\lambda\eta^2}{4}\left(\|q^{n+1}\|^2-\|q^n\|^2\right)+\frac{\lambda\eta^2}{4}\left(\|q^n\|^2-\|q^{n-1}\|^2\right)+2\Delta t\nu\|\frac{\nabla\mathbf{u}^{n+1}+\nabla\mathbf{u}^{n-1}}{2}\|^2\\
&+\frac{1}{2}\left(\|\mathbf{u}^{n+1}\|^2+2\beta\|\nabla\cdot\mathbf{u}^{n+1}\|^2\right)-\frac{1}{2}\left(\|\mathbf{u}^n\|^2+2\beta\|\nabla\cdot\mathbf{u}^n\|^2\right)\\
&+\frac{1}{2}\left(\|\mathbf{u}^n\|^2+2\beta\|\nabla\cdot\mathbf{u}^n\|^2\right)-\frac{1}{2}\left(\|\mathbf{u}^{n-1}\|^2+2\beta\|\nabla\cdot\mathbf{u}^{n-1}\|^2\right)\\
&+\alpha\Delta t^2\left(\|p^{n+1}\|^2+\|p^n\|^2\right)-\alpha\Delta t^2\left(\|p^n\|^2+\|p^{n-1}\|^2\right)\\
&+\frac{\lambda\gamma}{2}\left(||\phi^{n+1}-\phi^n||^2-||\phi^n-\phi^{n-1}||^2+||\phi^{n+1}-2\phi^n+\phi^{n-1}||^2\right)\\
&+\Delta t\left(\nabla\cdot u^n,p^{n+1}+p^{n-1}\right)-\Delta t\left(p^n,\nabla u^{n+1}+\nabla u^{n-1}\right)=0.
\end{aligned}$$

The last two terms of (2.18) can be rewritten as

$$\begin{aligned}
&\left(\nabla\cdot\mathbf{u}^n,p^{n+1}+p^{n-1}\right)-\left(p^n,\nabla\mathbf{u}^{n+1}+\nabla\mathbf{u}^{n-1}\right) \qquad (2.19)\\
&=\left[(p^{n+1},\nabla\cdot\mathbf{u}^n)-(p^n,\nabla\cdot\mathbf{u}^{n-1})\right]-\left[(p^n,\nabla\cdot\mathbf{u}^{n+1})-(p^{n-1},\nabla\cdot\mathbf{u}^n)\right].
\end{aligned}$$

Then summing up (2.18) from $n=1$ to $n=N-1$ gives

$$\begin{aligned}
&2\Delta t\sum_{n=1}^{N-1}M\|\nabla\mu^n\|^2+\frac{\lambda}{2}\left(\|\nabla\phi^N\|^2+\|\nabla\phi^{N-1}\|^2\right)+\frac{\lambda\eta^2}{4}\left(\|q^N\|^2+\|q^{N-1}\|^2\right)\\
&+\Delta t\sum_{n=1}^{N-1}2\nu\|\frac{\nabla\mathbf{u}^{n+1}+\nabla\mathbf{u}^{n-1}}{2}\|^2+\frac{1}{2}\left(\|\mathbf{u}^N\|^2+2\beta\|\nabla\cdot\mathbf{u}^N\|^2\right)\\
&+\frac{1}{2}\left(\|\mathbf{u}^{N-1}\|^2+2\beta\|\nabla\cdot\mathbf{u}^{N-1}\|^2\right)+\alpha\Delta t^2\left(\|p^N\|^2+\|p^{N-1}\|^2\right) \qquad (2.20)\\
&+\frac{\lambda\gamma}{2}||\phi^N-\phi^{N-1}||^2+\frac{\lambda\gamma}{2}\sum_{n=1}^{N-1}||\phi^{n+1}-2\phi^n+\phi^{n-1}||^2\\
&+\Delta t(p^N,\nabla\cdot\mathbf{u}^{N-1})-\Delta t(p^{N-1},\nabla\cdot\mathbf{u}^N)\\
&=\frac{\lambda}{2}\left(\|\nabla\phi^1\|^2+\|\nabla\phi^0\|^2\right)+\frac{\lambda\eta^2}{4}\left(\|q^1\|^2+\|q^0\|^2\right)+\frac{1}{2}\left(\|\mathbf{u}^1\|^2+2\beta\|\nabla\cdot\mathbf{u}^1\|^2\right)\\
&+\frac{1}{2}\left(\|\mathbf{u}^0\|^2+2\beta\|\nabla\cdot\mathbf{u}^0\|^2\right)+\alpha\Delta t^2\left(\|p^1\|^2+\|p^0\|^2\right)+\frac{\lambda\gamma}{2}||\phi^1-\phi^0||^2\\
&+\Delta t(p^1,\nabla\cdot\mathbf{u}^0)-\Delta t(p^0,\nabla\cdot\mathbf{u}^1).
\end{aligned}$$

The last two terms on the left hand side of (2.20) can be bounded as

$$\begin{aligned}
&\Delta t(p^N,\nabla\cdot\mathbf{u}^{N-1})-\Delta t(p^{N-1},\nabla\cdot\mathbf{u}^N) \qquad (2.21)\\
&\le\beta\|\nabla\cdot\mathbf{u}^{N-1}\|^2+\frac{1}{4\beta}\Delta t^2\|p^N\|^2+\beta\|\nabla\cdot\mathbf{u}^N\|^2+\frac{1}{4\beta}\Delta t^2\|p^{N-1}\|^2.
\end{aligned}$$

So if $\alpha \geq \frac{1}{4\beta}$, (2.20) reduces to

$$\begin{aligned} &2\Delta t \sum_{n=1}^{N-1} \|\nabla \mu^n\|^2 + \frac{\lambda}{2}\left(\|\nabla \phi^N\|^2 + \|\nabla \phi^{N-1}\|^2\right) + \frac{\lambda\eta^2}{4}\left(\|q^N\|^2 + \|q^{N-1}\|^2\right) \\ &\quad + \Delta t \sum_{n=1}^{N-1} 2\nu \|\frac{\nabla \mathbf{u}^{n+1} + \nabla \mathbf{u}^{n-1}}{2}\|^2 + \frac{1}{2}\left(\|\mathbf{u}^N\|^2 + \|\mathbf{u}^{N-1}\|^2\right) \\ &\quad + \frac{\lambda\gamma}{2}||\phi^N - \phi^{N-1}||^2 + \frac{\lambda\gamma}{2}\sum_{n=1}^{N-1}||\phi^{n+1} - 2\phi^n + \phi^{n-1}||^2 \\ &\leq \frac{\lambda}{2}\left(\|\nabla\phi^1\|^2 + \|\nabla\phi^0\|^2\right) + \frac{\lambda\eta^2}{4}\left(\|q^1\|^2 + \|q^0\|^2\right) + \frac{1}{2}\left(\|\mathbf{u}^1\|^2 + 2\beta\|\nabla\cdot\mathbf{u}^1\|^2\right) \\ &\quad + \frac{1}{2}\left(\|\mathbf{u}^0\|^2 + 2\beta\|\nabla\cdot\mathbf{u}^0\|^2\right) + \alpha\Delta t^2\left(\|p^1\|^2 + \|p^0\|^2\right) + \frac{\lambda\gamma}{2}||\phi^1 - \phi^0||^2 \\ &\quad + \Delta t(p^1, \nabla\cdot\mathbf{u}^0) - \Delta t(p^0, \nabla\cdot\mathbf{u}^1). \end{aligned} \tag{2.22}$$

This completes the proof. □

Despite the unconditional stability established in Theorem 2.2, it is well-known that the Leapfrog time marching scheme suffers from nonphysical oscillations, cf. [5] and references therein. Standard time filters can be applied to control these oscillations. We given an example here using the Robert-Asselin filter. Introducing

$$h^n(\varphi) = \frac{\kappa}{2}(\varphi^{n+1} - 2\varphi^n + \varphi^{n-1}),$$

at each time step one post-processes

$$\phi^n = \phi^n + h^n(\phi), \quad \mu^n = \mu^n + h^n(\mu), \quad \mathbf{u}^n = \mathbf{u}^n + h^n(\mathbf{u}), \quad p^n = p^n + h^n(p).$$

Here the coefficient κ is typically chose in the range $(0, 0.2]$ in the geophysical fluid dynamics applications.

3. Numerical experiments. In this section we numerically demonstrate the accuracy and stability of the Leapfrog artificial compression method for solving the CHNS system.

3.1. Convergence test. We first verify the temporal convergence of the scheme (2.5)–(2.10) by method of manufactured solutions. The computational domain is $[0, 1] \times [0, 1]$, and the parameters are taken to be $M = \eta = \lambda = 1$, $\nu = 0.1$. We also set the stabilization constants to be unity, i.e. $\gamma = \alpha = \beta = 1$. The manufactured solutions are as follows

$$\begin{aligned} \phi &= \cos(t)\cos(\pi x)\cos(\pi y), \\ \mu &= \sin(t)\cos(\pi x)\cos(\pi y), \\ u &= -(\sin(\pi x))^2\sin(2\pi y)\cos(t), \\ v &= (\sin(\pi y))^2\sin(2\pi x)\cos(t), \\ p &= \cos(t)(xy - 0.25), \end{aligned}$$

where $\mathbf{u} := (u, v)$ is the velocity field.

In the numerical experiment we compute the solution up to the final time $T = 1$, and we allocate 160 grid points in each direction. $P2 - P2$ finite element pair is used for ϕ and μ, and Taylor-Hood finite elements are employed for $\mathbf{u}$ and p. From the consistency error and the stability bound in Theorem 2.2, one would expect second order convergence rate for ϕ, mu and $\mathbf{u}$, and first order convergence rate for p. We use the true solution to initialize the first two steps of the computation. Fig. 3.1 demonstrates the log-log plot of the error L^2 norm for $\mathbf{u}$, p, and ϕ as a function of time step Δt. Second order temporal accuracy is observed for ϕ and $\mathbf{u}$, and first order accuracy is roughly observed for pressure p with some error oscillation. Similar error behaviour is also present for the scheme with filters even though the error tends to be smaller. We remark here that selective application of filters such as only on velocities did not improve pressure accuracy.

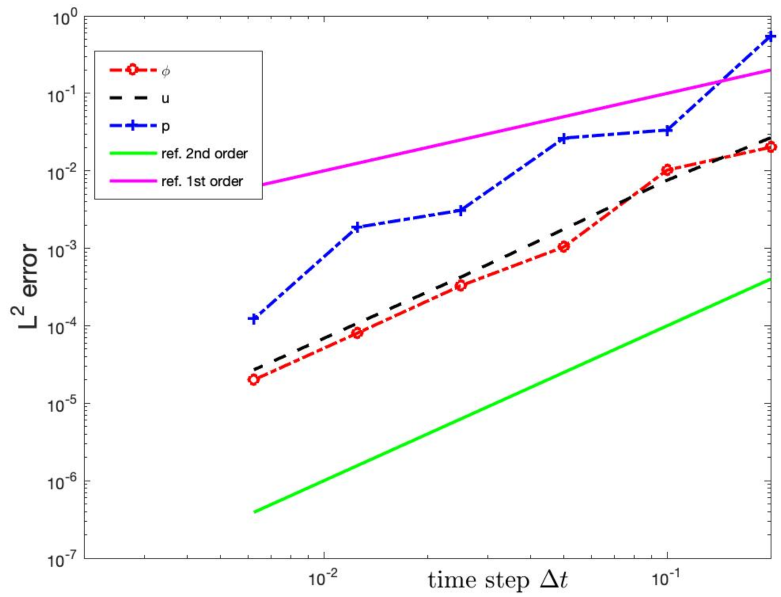

FIG. 3.1. *Log-Log plot of the error in L^2 norm for $\mathbf{u}$, p, and ϕ as a function of time step Δt. The solid green line is the reference line of 2nd order $e = 0.01\Delta t^2$, and the solid red line is the reference line of first order $e = \Delta t$. The final time is $T = 1$. $h = 1/160$. P2–P2 is used for ϕ and μ, P2–P1 is used for $\mathbf{u}$ and p. The other parameters are set to be unity.*

3.2. Stability test. In order to verify the long-time stability, we run the classical test of spinodal decomposition of a binary fluid for a long time, and show that the energy law (2.4) is preserved by our scheme. The initial velocity is given by

$$u = -(\sin(\pi x))^2 \sin(2\pi y), \qquad v = (\sin(\pi y))^2 \sin(2\pi x).$$

The phase field variable ϕ takes initially a random field of values $\phi_0 = \bar{\phi} + r(x, y)$ with an average composition $\bar{\phi} = -0.05$ and random $r \in [-0.05, 0.05]$ which represents a uniform mixture. The parameters are $\epsilon = 0.01$, $M = 10^{-3}$, $\nu = 0.1$, $\lambda = 10^{-3}$. We take uniform step-size $\delta t = 0.05$ and $h = 0.01$. Under the Cahn-Hilliard dynamics the mixture first undergoes a quick phase separation process in which phases of same composition quickly cluster, then followed by the slow process of coarsening where larger droplets grow at the expense of smaller ones so as to minimize the surface area. Fig. 3.2 shows the monotonic evolution of the energy in which the initial rapid decline of the energy corresponds to the stage of spinodal decomposition, and the ensuing coarsening stage is slow.

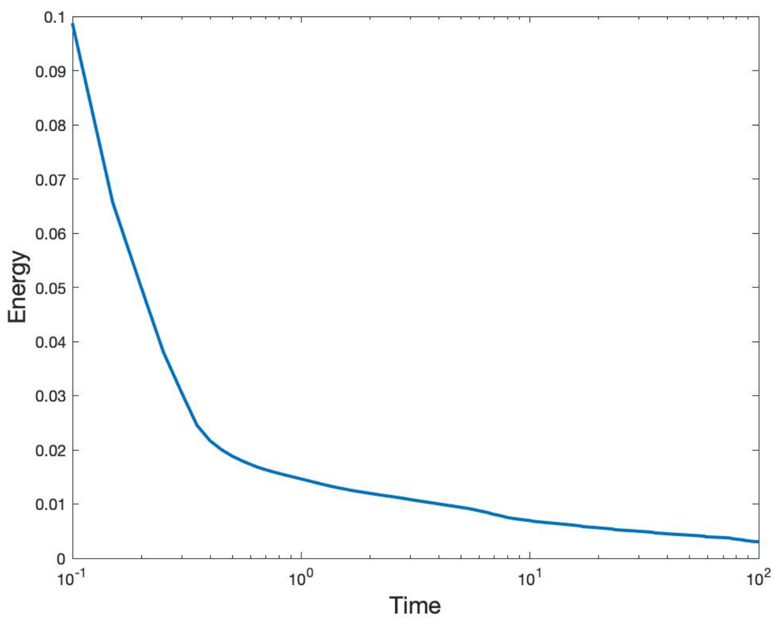

FIG. 3.2. *Stability test: energy as a function of time.* $\Delta t = 0.05$*,* $h = 0.01$*,* $\epsilon = 0.01$*,* $\nu = 0.1$*.*

4. Conclusion. This article contributes to a second order, linear, unconditionally stable scheme based on the Crank-Nicolson-Leapfrog discretization and the artificial compression method for solving phase field fluid models. As an example we present our algorithm for Cahn-Hilliard-Navier-Stokes equations, and we establish the unconditional long-time stability. Numerical examples are presented to verify the accuracy and long-time stability. Similar algorithms can be constructed for other phase field fluid models.

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