TRANSFER IDEALS AND TORSION IN THE MORAVA E-THEORY OF ABELIAN GROUPS

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ABSTRACT. Let A be a finite abelian p-group of rank at least 2. We show that $E^0(BA)/I_{tr}$, the quotient of the Morava E-cohomology of A by the ideal generated by the image of the transfers along all proper subgroups, contains p-torsion. The proof makes use of transchromatic character theory.

1. INTRODUCTION

The close relationship between the Morava E-cohomology of spaces and algebraic geometry has played an important role in problems related to power operations for Morava E-theory, character theory for Morava E-theory, and understanding H_∞-ring maps between H_∞-ring spectra and E. Let $\mathbb{G}$ be the universal deformation formal group associated to E and write I_{tr} for the transfer ideal. A fundamental result of Strickland in [Str98] shows that $E^0(B\Sigma_{p^k})/I_{tr}$ corepresents the formal scheme parametrizing subgroup schemes of order p^k in $\mathbb{G}$. For instance, this theorem plays a vital role in Rezk's study [Rez09, Rez12] of the Dyer–Lashof algebra for Morava E-theory. One of the key ingredients in Strickland's proof is the fact that $E^0(B\Sigma_{p^k})/I_{tr}$ is a free E^0-module.

Let $\text{Level}(A^*, \mathbb{G})$ be the formal scheme of A^*-level structures in $\mathbb{G}$, where A^* is the Pontryagin dual of a finite abelian p-group A. In [Dri74], it is shown that the ring of functions on $\text{Level}(A^*, \mathbb{G})$ is a finitely generated free E^0-module. Proposition 7.5 in [AHS04] implies that there is a close relationship between $E^0(BA)/I_{tr}$ and the ring of functions on $\text{Level}(A^*, \mathbb{G})$. In fact, if we let $(E^0(BA)/I_{tr})^{\text{free}}$ be the image of $E^0(BA)/I_{tr}$ in $\mathbb{Q} \otimes E^0(BA)/I_{tr}$, then there is a canonical isomorphism

$$\text{Spf}((E^0(BA)/I_{tr})^{\text{free}}) \cong \text{Level}(A^*, \mathbb{G}).$$

For a transitive abelian subgroup $A \subseteq \Sigma_{p^k}$, Ando, Hopkins, and Strickland use this isomorphism to give an algebro-geometric description of the power operation

$$E^0 \xrightarrow{P_{p^k}/I_{tr}} E^0(B\Sigma_{p^k})/I_{tr} \to E^0(BA)/I_{tr} \to (E^0(BA)/I_{tr})^{\text{free}}.$$

This motivates the question of whether $E^0(BA)/I_{tr}$ contains p-torsion, i.e., if

$$(E^0(BA)/I_{tr})^{\text{free}} \ncong E^0(BA)/I_{tr}.$$

These two rings are isomorphic when A is cyclic. In contrast, the main result of this note implies that they are distinct when the rank of A is greater than 1.

Theorem (Theorem 3.1)**.** *Let A be a finite abelian p-group with* $\text{rank}(A) \geq 2$*. Let $I_{tr} \subset E^0(BA)$ be the ideal generated by the image of transfers along all proper subgroups of A. The E^0-algebra $E^0(BA)/I_{tr}$ contains p-torsion.*

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In particular, this answers a question raised in [HS20, Remark 5.3]. It also fits into the wider context of the existence of torsion in the Morava E-cohomology of finite groups and its algebro-geometric interpretation. A finite group G is called good (at the given height and prime) if $E^*(BG)$ is finitely free and concentrated in even degrees, in which case $\mathrm{Spf}(E^*(BG))$ is a well-behaved algebro-geometric object [Str99]. However, Kriz [Kri97] showed that not all finite groups are good when the height of E is greater than 1, see also [KL00, Sch11] for further investigations in this direction. One might hope that our proof sheds further light on these questions. However, it crucially uses the transchromatic character maps of [BS16] to first reduce to a question about p-adic K-theory. We then use classical character theory to show that $K_p^0(BA)/I_{tr}$ is non-zero and contains torsion when $\mathrm{rank}(A) \geq 2$. Since $K_p^*(BG)$ is a even and torsion free for all finite groups, our methods cannot be adapted to gain further understanding of good groups. A direct computational proof of the main result at height 2 and for small groups might be possible and potentially illuminating, but we will not pursue this here.

2. p-adic K-theory of finite abelian p-groups and torsion

Fix a prime p and let K_p denote p-adic K-theory. Given a finite p-group G, there is a canonical isomorphism

$$\mathbb{Z}_p \otimes R(G) \xrightarrow{\cong} K_p^0(BG),$$

where $R(G)$ is the representation ring of G. It follows that there is an isomorphism

$$K_p^0(B\mathbb{Z}/p^k) \cong \mathbb{Z}_p[x]/(x^{p^k} - 1) \cong \mathbb{Z}_p[x]/([p^k](x)),$$

where $[p^k](x)$ is the p^k-series for the multiplicative formal group law. Since $R(G \times H) \cong R(G) \otimes R(H)$ for finite groups G and H, the isomorphisms above give an explicit description of $K_p^0(BA)$ for any finite abelian p-groups A.

Character theory for p-adic K-theory was developed by Adams in [Ada78, Section 2] and generalized to all Morava E-theories by Hopkins, Kuhn, and Ravenel in [HKR00]. Let $D = \mathrm{colim}_{i \geq 0} \mathbb{Z}_p(\zeta_{p^i})$, the ring obtained from $\mathbb{Z}_p$ by adjoining all of the p^ith roots of unity as i varies. Let $C_0 = \mathbb{Q} \otimes D$. For a finite group G, the character map is a canonical inclusion of $\mathbb{Z}_p$-algebras

$$K_p^0(BG) \hookrightarrow Cl_p(G, C_0),$$

where $Cl_p(G, C_0)$ is the ring of C_0-valued functions on the set of conjugacy classes of p-power order elements in G. By construction, the character map factors through $Cl_p(G, D)$, the ring of D-valued functions on the set of conjugacy classes of p-power order elements in G, giving

$$K_p^0(BG) \hookrightarrow Cl_p(G, D) \hookrightarrow Cl_p(G, C_0). \tag{2.1}$$

The ring D is local with maximal ideal $(p) \subset D$.

Given $H \subseteq G$, there is a transfer map $K_p^0(BH) \to K_p^0(BG)$, which is a map of modules for the $K_p^0(BG)$-module structure coming from restriction. This transfer map is compatible with the transfer map on class functions

$$\mathrm{Tr}_H^G \colon Cl_p(H, C_0) \to Cl_p(G, C_0)$$

given by

$$\mathrm{Tr}_H^G(f)([g]) = \frac{1}{|H|} \sum_{k \in G, kgk^{-1} \in H} f(kgk^{-1}),$$

see for example [HKR00, Theorem D]. Given a finite abelian p-group A and a subgroup $A' \subseteq A$, the transfer map $K_p^0(BA') \to K_p^0(BA)$ is determined by the image of $1 \in K_p^0(BA')$ as the restriction $K_p^0(BA) \to K_p^0(BA')$ is surjective. Applying the formula above, we have

$$\mathrm{Tr}_{A'}^{A}(1)(a) = \begin{cases} |A/A'| & \text{if } a \in A' \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

We define $I_{tr} \subset K_p^0(BA)$ to be the ideal generated by the image of the transfer maps along all proper subgroups of A. We will abuse notation and denote the ideal generated by transfers along all proper subgroups in $Cl_p(A, C_0)$ by I_{tr} as well. In fact, since the class functions in (2.2) take values in the integers, we may also consider the ideal I_{tr} in $Cl_p(A, D)$.

Lemma 2.3. *Assume A is a finite abelian p-group, then the commutative ring $K_p^0(BA)/I_{tr}$ is non-zero.*

Proof. This is clear when A is cyclic. Now we assume that $\mathrm{rank}(A) \geq 2$. Since there is a map of commutative rings

$$K_p^0(BA)/I_{tr} \to Cl_p(A, D)/I_{tr} \cong \left(\prod_{a \in A} D\right)/I_{tr},$$

it suffices to prove that the target is non-zero. Consider the factor of $\prod_{a \in A} D$ corresponding to $0 \in A$. By the formula in (2.2), the projection of I_{tr} to that factor is contained in the ideal $(p) \subset D$. □

Lemma 2.4. *The commutative ring $K_p^0(B(\mathbb{Z}/p)^{\times 2})/I_{tr}$ is an $\mathbb{F}_p$-algebra.*

Proof. By Lemma 2.3, it suffices to show that $(p) \subseteq I_{tr}$. Let $V_1, \ldots, V_{p+1}$ be the maximal abelian subgroups of $A = (\mathbb{Z}/p)^{\times 2}$, then

$$p = \left(\sum_{i=1}^{p+1} \mathrm{Tr}_{V_i}^{A}(1)\right) - \mathrm{Tr}_0^A(1).$$

This can be checked in the ring of class functions. The value of the class function

$$\sum_{i=1}^{p+1} \mathrm{Tr}_{V_i}^{A}(1)$$

is p on any non-zero element of A and $p^2 + p$ on $0 \in A$. □

Proposition 2.5. *Assume A is a finite abelian p-group with* $\mathrm{rank}(A) \geq 2$*, then the commutative ring $K_p^0(BA)/I_{tr}$ is an $\mathbb{F}_p$-algebra.*

Proof. Fix a surjection $\rho\colon A \twoheadrightarrow (\mathbb{Z}/p)^{\times 2}$. Since ρ is surjective, given any proper subgroup $H \subset (\mathbb{Z}/p)^{\times 2}$, $\rho^{-1}(H)$ is a proper subgroup of A. Thus the map ρ induces a map of commutative rings

$$K_p^0(B(\mathbb{Z}/p)^{\times 2})/I_{tr} \to K_p^0(BA)/I_{tr}.$$

It suffices to prove that $K_p^0(B(\mathbb{Z}/p)^{\times 2})/I_{tr}$ is an $\mathbb{F}_p$-algebra, which is the content of Lemma 2.4. □

3. Morava E-theory of finite abelian p-groups and torsion

Let E be a height n Morava E-theory at the prime p and let A be a finite abelian p-group. Just as before, we define the transfer ideal $I_{tr} \subset E^0(BA)$ to be the ideal generated by the image of the transfer maps along proper subgroups of A.

Morava E-theory admits a variety of versions of character theory (e.g., [HKR00], [GS99], [Sta13], [Sta15], [BS16]). In [BS16, Section 5] the authors constructed a commutative ring $\bar{C}_1$, which depends on n and p, that is a flat E^0-algebra and a faithfully flat $\mathbb{Z}_p$-algebra with the property that

$$\bar{C}_1 \otimes_{E^0} E^0(BA) \cong \prod_{A^{\times n-1}} \bar{C}_1 \otimes_{\mathbb{Z}_p} K_p^0(BA).$$

In fact, the isomorphism above can be extended to all finite groups but we will not need that generality here. Write $\mathcal{L}$ for the free loop space functor. Since $\mathcal{L}^{n-1}BA \simeq A^{\times n-1} \times BA$, we have an isomorphism

$$\prod_{A^{\times n-1}} \bar{C}_1 \otimes_{\mathbb{Z}_p} K_p^0(BA) \cong \bar{C}_1 \otimes_{\mathbb{Z}_p} K_p^0(\mathcal{L}^{n-1}BA).$$

The isomorphism is compatible with transfers in the sense that there is a commutative diagram of abelian groups

$$\begin{array}{ccc} E^0(BA') & \longrightarrow & \bar{C}_1 \otimes_{\mathbb{Z}_p} K_p^0(\mathcal{L}^{n-1}BA') \\ \downarrow{\scriptstyle \mathrm{Tr}} & & \downarrow{\scriptstyle \bar{C}_1 \otimes_{\mathbb{Z}_p} \mathrm{Tr}} \\ E^0(BA) & \longrightarrow & \bar{C}_1 \otimes_{\mathbb{Z}_p} K_p^0(\mathcal{L}^{n-1}BA), \end{array}$$

where the transfer on the right is along the finite cover $\mathcal{L}^{n-1}BA' \to \mathcal{L}^{n-1}BA$. The commutativity of this diagram is due to the fact that it is induced by a map of cohomology theories. This is explained in Proposition 4.14 in [BS17].

We say that a commutative ring R contains torsion if the underlying $\mathbb{Z}$-module contains torsion. For a flat map $R \to S$ of commutative rings, an R-module M contains p-torsion if $S \otimes_R M$ contains p-torsion. This follows from the fact that if $p\colon M \to M$ is injective then $p\colon S \otimes_R M \to S \otimes_R M$ is injective.

Theorem 3.1. *Assume that A is an abelian p-group with* $\mathrm{rank}(A) \geq 2$, *then the commutative ring $E^0(BA)/I_{tr}$ contains p-torsion.*

Proof. It follows from Theorem 6.9 in [BS16] that

$$\bar{C}_1 \otimes_{E^0} E^0(BA)/I_{tr} \cong \prod_{(a_i)\in A^{\times n-1}} \bar{C}_1 \otimes_{\mathbb{Z}_p} K_p^0(BA)/I_{tr}^{(a_i)}, \tag{3.1}$$

where $I_{tr}^{(a_i)}$ is a certain transfer ideal depending on the tuple (a_i). The transfer ideals of the right-hand side of (3.1) were studied in [HS20]. The ideal $I_{tr}^{(0)} \subset K_p^0(BA)$ is the ideal generated by transfers from all proper subgroups of A. Proposition 2.5 implies that $K_p^0(BA)/I_{tr}^{(0)}$ has torsion. Since $\bar{C}_1$ is faithfully flat as a $\mathbb{Z}_p$-algebra, the ring

$$\prod_{(a_i)\in A^{\times n-1}} \bar{C}_1 \otimes_{\mathbb{Z}_p} K_p^0(BA)/I_{tr}^{(a_i)}$$

contains torsion, hence so does $\bar{C}_1 \otimes_{E^0} E^0(BA)/I_{tr}$. Since $\bar{C}_1$ is a flat E^0-algebra, it follows that $E^0(BA)/I_{tr}$ contains torsion as well. □

We conclude with a short comparison of Theorem 3.1 with the existing literature. In [Str98], Strickland proves that $E^0(B\Sigma_{p^k})/I_{tr}$ is a free E^0-module, where I_{tr} is the ideal generated by transfers along $\Sigma_i \times \Sigma_j \subset \Sigma_{p^k}$ with $i + j = p^k$ and $i, j > 0$. Of course, Σ_{p^k} is only abelian when $p = 2$ and $k = 1$, in which case it is cyclic. Further, given a transitive abelian subgroup $A \subseteq \Sigma_{p^k}$, there is an induced map of commutative rings

$$E^0(B\Sigma_{p^k})/I_{tr} \to E^0(BA)/I_{tr}.$$

However, due to the direction of the map, Theorem 3.1 does not have any consequences on the existence torsion in $E^0(B\Sigma_{p^k})/I_{tr}$.

In [SS15], the main result of [Str98] is generalized to groups of the form $A \wr \Sigma_{p^k}$. The transfer ideal $I_{tr} \subset E^0(BA \wr \Sigma_{p^k})$ considered there is generated by transfers along $A \wr (\Sigma_i \times \Sigma_j) \subseteq A \wr \Sigma_{p^k}$ with $i + j = p^k$ and $i, j > 0$. They show that $E^0(BA \wr \Sigma_{p^k})/I_{tr}$ is a free E^0-module. In [Nel18], Nelson proves that $E^0(B\Sigma_{p^k} \wr \Sigma_{p^l})/I_{tr}$ is free, where I_{tr} is generated by transfers along the subgroups $(\Sigma_i \times \Sigma_j) \wr \Sigma_{p^l} \subset \Sigma_{p^k} \wr \Sigma_{p^l}$, where $i + j = p^k$ and $i, j > 0$ and along the subgroups $\Sigma_{p^k} \wr (\Sigma_i \times \Sigma_j) \subset \Sigma_{p^k} \wr \Sigma_{p^l}$, where $i + j = p^l$ and $i, j > 0$. In both of these cases, these results do not contradict our main result.

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