

Besov class via heat semigroup on Dirichlet spaces I: Sobolev type inequalities

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Abstract

We introduce heat semigroup-based Besov classes in the general framework of Dirichlet spaces. General properties of those classes are studied and quantitative regularization estimates for the heat semigroup in this scale of spaces are obtained. As a highlight of the paper, we obtain a far reaching L^p -analogue, $p \geq 1$, of the Sobolev inequality that was proved for $p = 2$ by N. Varopoulos under the assumption of ultracontractivity for the heat semigroup. The case $p = 1$ is of special interest since it yields isoperimetric type inequalities.

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1 Introduction

Motivation

The family of Sobolev inequalities plays a major role in analysis (see for instance [52] and references therein). In the Euclidean space $\mathbb{R}^n$, $n \geq 2$, it reads

$$\|f\|_{L^q(\mathbb{R}^n)} \leq C \|\nabla f\|_{L^p(\mathbb{R}^n)}, \quad f \in C_0^\infty(\mathbb{R}^n) \quad (1)$$

where $1 \leq p < n$, $q = \frac{np}{n-p}$, and the constant C depends on n and p .

In the context of a measure space (X, μ) equipped with a symmetric Dirichlet form $\mathcal{E}$ with domain $\mathcal{F}$, N. Varopoulos proved in [56] that a heat kernel upper bound of the type $p_t(x, y) \leq \frac{C}{t^{n/2}}$, $n > 2$, implies the following Sobolev inequality

$$\|f\|_{L^q(X, \mu)} \leq C \sqrt{\mathcal{E}(f, f)}, \quad f \in \mathcal{F}, \quad (2)$$

where $q = \frac{2n}{n-2}$.

When applied to the Euclidean space equipped with the standard Dirichlet form $\mathcal{E}(f, f) = \|\nabla f\|_{L^2(\mathbb{R}^n)}^2$, Varopoulos' theorem yields the case $p = 2$ in (1). When $p \neq 2$, it is natural to look for results extending the inequalities (1) to Dirichlet spaces. In the present paper, among other results, and without using any gradient structure (e.g. [6, 38]) we extend Varopoulos' theorem to any $p \geq 1$ and prove the following:

Theorem 1.1. *Let $(X, \mu, \mathcal{E}, \mathcal{F})$ be a Dirichlet space. Let $\{P_t\}_{t \in [0, \infty)}$ denote the Markovian semigroup associated with $(X, \mu, \mathcal{E}, \mathcal{F})$. Let $p \geq 1$.*

1. *Assume that P_t admits a measurable heat kernel $p_t(x, y)$ satisfying, for some $C > 0$ and $\beta > 0$,*

$$p_t(x, y) \leq C t^{-\beta} \quad (3)$$

for $\mu \times \mu$ -a.e. $(x, y) \in X \times X$, and for each $t \in (0, +\infty)$.

2. *Assume that there exist $\alpha > 0$ and $C > 0$ such that for every $f \in L^p(X, \mu)$*

$$\|f\|_{p, \alpha} \leq C \liminf_{t \rightarrow 0} t^{-\alpha} \left(\int_X \int_X |f(x) - f(y)|^p p_t(x, y) d\mu(x) d\mu(y) \right)^{1/p}, \quad (4)$$

where

$$\|f\|_{p, \alpha} := \sup_{t > 0} t^{-\alpha} \left(\int_X \int_X |f(x) - f(y)|^p p_t(x, y) d\mu(x) d\mu(y) \right)^{1/p}. \quad (5)$$

Then, if $0 < \alpha < \beta$ and $p < \frac{\beta}{\alpha}$, there exists a constant $C_{p, \alpha, \beta} > 0$ such that for every $f \in L^p(X, \mu)$,

$$\|f\|_{L^q(X, \mu)} \leq C_{p, \alpha, \beta} \|f\|_{p, \alpha},$$

where $q = \frac{p\beta}{\beta - p\alpha}$.

Note that it is possible to verify condition (3) in many classical and fractal examples because, according to [8, 20, 45], it is equivalent to the Nash inequality

$$\|f\|_{L^2(X,\mu)}^{2+2/\beta} \leq C \mathcal{E}(f, f) \|f\|_{L^1(X,\mu)}^{2/\beta}. \quad (6)$$

In the case $p = 2$, we will prove that the condition (4) is always satisfied with $\alpha = \frac{1}{2}$ and that we have $\mathcal{E}(f, f) \simeq \|f\|_{2,1/2}^2$. As a consequence, Theorem 1.1 is indeed an extension of the Varopoulos theorem. On the other hand, when applied to the Euclidean space equipped with the standard Dirichlet form $\mathcal{E}(f, f) = \|\nabla f\|_{L^2(\mathbb{R}^n)}^2$, for every $p \geq 1$ the condition (4) is always satisfied with $\alpha = \frac{1}{2}$ and Theorem 1.1 yields all of the Sobolev inequalities (1) because in that case we have $\|f\|_{p,1/2} \simeq \|\nabla f\|_{L^p(\mathbb{R}^n)}$.

Theorem 1.1 is proved by applying the methods of [8] in the general framework of Dirichlet spaces. In a first step, we prove in Theorem 6.3 that the assumption (3) alone implies a weak Sobolev inequality

$$\sup_{s \geq 0} s \mu(\{x \in X : |f(x)| \geq s\})^{\frac{1}{q}} \leq C_{p,\alpha,\beta} \|f\|_{p,\alpha},$$

where $q = \frac{p\beta}{\beta - p\alpha}$. In a second step we prove that if the condition (4) is also assumed, then this weak Sobolev inequality implies the strong one:

$$\|f\|_{L^q(X,\mu)} \leq C_{p,\alpha,\beta} \|f\|_{p,\alpha}.$$

The case $p = 1$ is of special interest. In that case, weak Sobolev inequalities applied to indicator functions of sets are related to isoperimetric type inequalities, and by adapting a method of M. Ledoux, we obtain in Proposition 6.5 an isoperimetric type inequality with explicit constant. We note that it is well-known since the works of Maz'ja and Federer-Fleming that the case $p = 1$ in (1) is equivalent to the isoperimetric inequality in $\mathbb{R}^n$ (see [49]) and is intimately connected to the theory of BV functions. It is therefore expected that for $p = 1$ and under assumption (4) Theorem 1.1 should yield isoperimetric type inequalities and possibly also a theory of BV functions valid in very general Dirichlet spaces. Moreover, the seminorm $\|f\|_{1,\alpha}$ provides a natural notion of variation of a function in that context. In more restrictive contexts, this remark is made very precise in the papers [1] and [2], where it is moreover shown that the condition (4) is satisfied if the underlying space X satisfies a weak Bakry-Émery type curvature condition.

In view of Theorem 1.1, given a Dirichlet space $(X, \mu, \mathcal{E}, \mathcal{F})$ with semigroup $\{P_t\}_{t \in [0, \infty)}$ it is natural to thoroughly study the family of Besov type spaces

$$\mathbf{B}^{p,\alpha}(X) = \left\{ f \in L^p(X, \mu), \|f\|_{p,\alpha} := \sup_{t > 0} t^{-\alpha} \left(\int_X P_t(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p} < +\infty \right\}.$$

Structure of the paper

This paper is structured as follows. In Section 2 we introduce the notations and recall some basic facts about Dirichlet forms and their associated heat semigroups. In Section 3 we describe the basic setup of Dirichlet forms and our heat semigroup-based Besov spaces $\mathbf{B}^{p,\alpha}(X)$, and conclude with a metric characterization of these spaces under the hypothesis that the heat semigroup has a kernel with sub-Gaussian estimates; this characterization, due to Pietruska-Pałuba [51], does not play a major role in the arguments introduced in this paper, but is included here as an illustration and because of its usefulness in concrete examples.

Section 4 is devoted to obtaining fundamental properties of the Besov classes, including the Banach space property, reflexivity, interpolation properties and locality in time. We show that

certain of the Besov spaces are non-trivial, in particular by showing in Proposition 4.6 that the Besov space $\mathbf{B}^{2,1/2}(X)$ is precisely the domain $\mathcal{F}$ of the Dirichlet form, the analog of the classical Sobolev space $W^{1,2}(X)$. This is in contrast to the classical (metric-based) Besov space theory, where $B_{2,\infty}^1(\mathbb{R}^n)$ consists only of constant functions, see [17]. From the preceding one deduces by elementary convexity considerations that $\mathbf{B}^{p,1/2}(X)$ is dense in L^p if $1 \leq p \leq 2$. We also give examples that establish the range of possibilities for $\mathbf{B}^{p,1/2}(X)$ when $p > 2$: in Proposition 4.2 we describe a smooth setting in which $\mathbf{B}^{p,1/2}(X)$ contains $C_0^\infty(X)$, but in Corollary 4.13 we provide a class of Dirichlet forms for which the Besov spaces $\mathbf{B}^{p,1/2}(X)$ are trivial (contain only constant functions) when $p > 2$. Moreover, we begin to analyze the relationship between the Besov spaces $\mathbf{B}^{p,\alpha}(X)$ and the domain of the fractional powers of the generator L of the Dirichlet form, showing in particular that $(-L)^s : \mathbf{B}^{p,\alpha}(X) \rightarrow L^p$ is bounded for $0 < s < \alpha$.

In Section 5, we prove that, when $1 < p \leq 2$, the heat semigroup is always continuous as an operator $L^p(X, \mu) \rightarrow \mathbf{B}^{p,1/2}(X)$, see Theorem 5.1. It is remarkable that this is true in any Dirichlet space without any further assumption. In [1] and [2], we will see that for $p > 2$, this continuity is valid under weak Bakry-Émery type curvature conditions. We use this result to establish some refinements of our triviality and non-triviality results from Section 4 and summarize them in terms of critical exponents for density and triviality of the Besov spaces. The results of this section will play an important role in [2]. In particular, the Besov critical exponents on fractal sets will be shown to be closely related to the geometry of these sets.

In Section 6 we will consider Sobolev-type embedding theorems for the Besov classes $\mathbf{B}^{p,\alpha}(X)$. Our main assumption is that the underlying Dirichlet space admits a heat kernel $p_t(x, y)$ satisfying a global upper bound of the type $p_t(x, y) \leq ct^{-\beta}$. In Dirichlet spaces, the proof of the existence of a Sobolev inequality with $p = 2$ under this type of heat kernel estimates goes back to a celebrated work by N. Varopoulos (see Chapter 2 of [57] and the references therein). To study the case $p \neq 2$, we make use of the ideas and methods developed in [8, 52] and more recently in [12]. Those methods are general enough to apply to our setting and underline the fact that our Besov classes provide a natural framework for a general theory of BV functions and isoperimetric inequalities on arbitrary Dirichlet spaces. These outline the beginnings of a connection between our Besov classes and isoperimetric type estimates that will be further explored under various assumptions in the works [1–3]. Among many others, one of the future applications of Besov classes and isoperimetry will be to study diffusions on pattern spaces of quasicrystals [5], corresponding to a unique strongly local but not strictly local Dirichlet form for which energy measures are absolutely continuous.

In Section 7 we give some further applications of the main ideas under the assumption of either a Poincaré inequality or a log-Sobolev inequality. The idea is to replace the ultracontractivity estimate assumption of Section 6 by a supercontractive or hypercontractive one and explore the corresponding isoperimetric information. Such results may potentially be applied in infinite-dimensional situations like the Wiener space.

We conclude this introduction by noting some references from the existing literature that are closest to our work. The literature on Besov spaces is so large that it is not possible to be exhaustive, but we hope the following may be helpful to the reader. Further references and comments will be given throughout the text. For many equivalent descriptions of the Besov-Nikol'skii spaces in $\mathbb{R}^n$, including Poisson heat kernel characterizations we refer to [54]. For the classical theory of Besov spaces from the point of view of interpolation theory, we refer for example to the book of Triebel [55]. The relationship between Besov spaces and the Laplace operator or its square root in different settings has been studied from various points of view for some time; see, for example, [26] on Lie groups, [18] on spaces with polynomial upper bound on the volume and Poisson-type heat kernel bound, [41] on fractal metric spaces, and [36] on metric measure spaces with sub-Gaussian

heat kernel estimates and certain regularity assumptions. We also note that Besov spaces can be characterized via wavelet frames, see [25, 44]. Finally, our definition of Besov classes is particularly closely connected to the work of Pietruska-Paluba [51], and we learned much about the relevance of this approach to Dirichlet forms from work of Grigor'yan, Hu and Lau [34, 35].

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2 Preliminaries

In this section we introduce the notations and notions used throughout the paper and, for convenience of the reader, collect some standard definitions and known results that will be used later. The book [31] is a classical reference on the theory of Dirichlet forms and we refer to it for further details. We also refer to [15] for an exposition of the theory that does not use the hypothesis of regularity of the form. For the general theory of heat semigroups we refer for instance to [27] or [29].

2.1 Dirichlet forms

Throughout the paper, let X be a good measurable space (like a Polish space) equipped with a σ -finite measure μ . By good measurable space, we mean a measurable space for which the measure decomposition theorem holds and for which there exists a countable family generating the σ -algebra of X (see [9, page 7] for a discussion about good measurable spaces).

Let $(\mathcal{E}, \mathcal{F} = \mathbf{dom}(\mathcal{E}))$ be a densely defined closed symmetric form on $L^2(X, \mu)$. A function v on X is called a normal contraction of the function u if for almost every $x, y \in X$,

$$|v(x) - v(y)| \leq |u(x) - u(y)| \text{ and } |v(x)| \leq |u(x)|.$$

The form $\mathcal{E}$ is called a Dirichlet form if it is Markovian, that is, it has the property that if $u \in \mathcal{F}$ and v is a normal contraction of u then $v \in \mathcal{F}$ and $\mathcal{E}(v, v) \leq \mathcal{E}(u, u)$. In this paper we always assume that $\mathcal{E}$ is a Dirichlet form and refer to $(X, \mu, \mathcal{E}, \mathcal{F})$ as a Dirichlet space. Some basic properties of Dirichlet forms are collected in [31, Theorem 1.4.2]. In particular, we note that $\mathcal{F} \cap L^\infty(X, \mu)$ is an algebra and $\mathcal{F}$ is a Hilbert space with the $\mathcal{E}_1$ -norm

$$\|f\|_{\mathcal{E}_1} := \left(\|f\|_{L^2(X, \mu)}^2 + \mathcal{E}(f, f) \right)^{1/2}. \quad (7)$$

The Dirichlet space $(X, \mu, \mathcal{E}, \mathcal{F})$ is called regular if X is a locally compact topological space, μ is a Radon measure whose support is X and $(X, \mu, \mathcal{E}, \mathcal{F})$ admits a core. If we denote by $C_c(X)$ the space of continuous functions with compact support in X , we recall that a core for $(X, \mu, \mathcal{E}, \mathcal{F})$ is a subset $\mathcal{C}$ of $C_c(X) \cap \mathcal{F}$ which is dense in $C_c(X)$ in the supremum norm and dense in $\mathcal{F}$ in the $\mathcal{E}_1$ -norm.

If $\mathcal{E}$ is regular, then for every $f \in \mathcal{F} \cap L^\infty(X, \mu)$, we can define the energy measure ν_f in the sense of [14] through the formula

$$\int_X \phi d\nu_f = \mathcal{E}(f\phi, f) - \frac{1}{2}\mathcal{E}(\phi, f^2), \quad \phi \in \mathcal{F} \cap C_c(X),$$

see [24, Theorem 4.3.11]. Then ν_f can be extended to all $f \in \mathcal{F}$ by truncation, that is, for each positive integer we consider $f_n := \max\{-n, \min\{n, f\}\}$, and set ν_f to be the weak limit of the sequence of measures ν_{f_n} .

In this paper, most of the time we will not need to assume that $(X, \mu, \mathcal{E}, \mathcal{F})$ is regular, so if the regularity assumption is needed, it will be stated out explicitly.

2.2 Heat semigroup

Let $\{P_t\}_{t \in [0, \infty)}$ denote the self-adjoint semigroup on $L^2(X, \mu)$ associated with the Dirichlet space $(X, \mu, \mathcal{E}, \mathcal{F})$ and L the infinitesimal generator of $\{P_t\}_{t \in [0, \infty)}$; see [31, Section 1.4]). The semigroup $\{P_t\}_{t \in [0, \infty)}$ is referred to as the heat semigroup on $(X, \mu, \mathcal{E}, \mathcal{F})$.

The following spectral theory lemma can be found in [15, Proposition 1.2.3] or in [33, Section 4]. It shows that one can recover the Dirichlet form $\mathcal{E}$ and its domain from the semigroup $\{P_t\}_{t \in [0, \infty)}$.

Lemma 2.1. *Denoting $\langle \cdot, \cdot \rangle$ as the inner product in $L^2(X, \mu)$, for $f \in L^2(X, \mu)$ we have that*

$$0 < t \mapsto \frac{1}{t} \langle (I - P_t)f, f \rangle$$

is a decreasing function. Moreover, the limit $\lim_{t \rightarrow 0^+} \frac{1}{t} \langle (I - P_t)f, f \rangle$ exists if and only if $f \in \mathcal{F}$, in which case,

$$\mathcal{E}(f, f) = \lim_{t \rightarrow 0^+} \frac{1}{t} \langle (I - P_t)f, f \rangle. \quad (8)$$

By definition, the semigroup $\{P_t\}_{t \in [0, \infty)}$ acts on $L^2(X, \mu)$. However, it inherits from the Markovian property of the Dirichlet form the sub-Markovian property: if $0 \leq f \leq 1$ then $0 \leq P_t f \leq 1$. This fundamental property allows us to develop an L^p theory of the semigroup and from this classical theory (see for instance [27, Theorem 1.4.1 and 1.4.2]), the following properties of the semigroup $\{P_t\}_{t \in [0, \infty)}$ are known:

- The semigroup $\{P_t\}_{t \in [0, \infty)}$ maps $L^1(X, \mu) \cap L^\infty(X, \mu)$ into itself and may be extended, using the Riesz-Thorin interpolation, to a contraction semigroup on $L^p(X, \mu)$ for each $1 \leq p \leq +\infty$. We will denote that extension also by P_t . We explicitly note that the contraction property reads:

$$\|P_t f\|_{L^p(X, \mu)} \leq \|f\|_{L^p(X, \mu)}, \quad f \in L^p(X, \mu), 1 \leq p \leq \infty.$$

The semigroup $\{P_t\}_{t \in [0, \infty)}$ is said to be conservative if $P_t 1 = 1$.

In this paper, we always assume that $\{P_t\}_{t \in [0, \infty)}$ is conservative. This assumption is not overly restrictive, as it holds also for the standard Dirichlet form on the Wiener space, see [46].

- The semigroup $\{P_t\}_{t \in [0, \infty)}$ is symmetric, i.e. for $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L^p(X, \mu)$, $g \in L^q(X, \mu)$, $t \geq 0$,

$$\int_X (P_t f)(x) g(x) d\mu(x) = \int_X f(x) (P_t g)(x) d\mu(x). \quad (9)$$

- The semigroup $\{P_t\}_{t \in [0, \infty)}$ is strongly continuous on $L^p(X, \mu)$ for $1 \leq p < +\infty$, i.e.,

$$\|P_t f - f\|_{L^p(X, \mu)} \rightarrow 0, \quad \text{as } t \rightarrow 0. \quad (10)$$

- The semigroup $\{P_t\}_{t \in [0, \infty)}$ is an analytic semigroup on $L^p(X, \mu)$ for $1 < p < +\infty$. In particular, from [29, page 101], there exists a constant $C > 0$ independent of $t > 0$ (but depending on p) such that for every $f \in L^p(X, \mu)$,

$$\|LP_t f\|_{L^p(X, \mu)} \leq \frac{C}{t} \|f\|_{L^p(X, \mu)}. \quad (11)$$

Since we assume conservativeness, the semigroup $\{P_t\}_{t \in [0, \infty)}$ yields a family of heat kernel measures. Namely, from [9, Theorem 1.2.3], for every bounded or non-negative measurable function $f : X \rightarrow \mathbb{R}$,

$$P_t f(x) = \int_X f(y) p_t(x, dy), \quad t \geq 0, x \in X, \quad (12)$$

where, for each $t > 0$, $p_t(x, dy)$ is a probability kernel (that is, for every $x \in X$, $p_t(x, \cdot)$ is a probability measure on X and for every measurable set A , $x \rightarrow p_t(x, A)$ is measurable). Note that from the symmetry property of the heat semigroup, the measure defined on $X \times X$ by $\nu_t(A \times B) = \int_X 1_A P_t 1_B d\mu$ is symmetric, thus one has for every non-negative measurable function $F : X \times X \rightarrow \mathbb{R}$,

$$\int_X \int_X F(x, y) p_t(x, dy) d\mu(x) = \int_X \int_X F(x, y) p_t(y, dx) d\mu(y). \quad (13)$$

We say that the semigroup $\{P_t\}_{t \in [0, \infty)}$ admits a heat kernel if the heat kernel measures have a density with respect to μ , i.e. there exists a measurable function $p : \mathbb{R}_{>0} \times X \times X \rightarrow \mathbb{R}_{\geq 0}$, (and we denote $p(t, x, y)$ as $p_t(x, y)$ for $t > 0$ and $x, y \in X$) such that for every $t > 0, x, y \in X, f \in L^p(X, \mu)$, $1 \leq p \leq \infty$,

$$P_t f(x) = \int_X p_t(x, y) f(y) d\mu(y).$$

Many of our results do not require the existence of a heat kernel. The major exceptions are the Sobolev embeddings in Section 6. This assumption will thus be explicitly stated when needed.

The following lemma is well known. It follows from the classical Jensen's inequality applied to (12).

Lemma 2.2. *Let $\Phi : \mathbb{R} \rightarrow [0, \infty)$ be a convex function. For $1 \leq p < \infty$ and all $f \in L^p(X, \mu)$ and $t \geq 0$ we have*

$$\Phi(P_t(f)) \leq P_t(\Phi \circ f).$$

In particular, for $1 \leq p < \infty$ and all $f \in L^p(X, \mu)$ and $t \geq 0$ we have

$$|P_t(f)|^p \leq P_t(|f|^p).$$

3 Heat semigroup-based Besov spaces

Let $(X, \mu, \mathcal{E}, \mathcal{F})$ be a Dirichlet space and let $\{P_t\}_{t \in [0, \infty)}$ denote the associated heat semigroup. As already pointed out, $\{P_t\}_{t \in [0, \infty)}$ is assumed to be conservative. Our basic definition of the (heat semigroup-based) Besov seminorm is the following:

Definition 3.1. Let $p \geq 1$ and $\alpha \geq 0$. For $f \in L^p(X, \mu)$, we define the Besov seminorm:

$$\|f\|_{p,\alpha} = \sup_{t>0} t^{-\alpha} \left(\int_X P_t(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p}.$$

Observe that in terms of the heat kernel measure (12), one has:

$$\int_X P_t(|f - f(y)|^p)(y) d\mu(y) = \int_X \int_X |f(x) - f(y)|^p p_t(y, dx) d\mu(y).$$

Our goal in this paper is to study the Besov type spaces

$$\mathbf{B}^{p,\alpha}(X) = \{f \in L^p(X, \mu) : \|f\|_{p,\alpha} < +\infty\}. \quad (14)$$

The norm on $\mathbf{B}^{p,\alpha}(X)$ is defined as:

$$\|f\|_{\mathbf{B}^{p,\alpha}(X)} = \|f\|_{L^p(X, \mu)} + \|f\|_{p,\alpha}.$$

Remark 3.2. It is apparent that if v is a normal contraction of $u \in \mathbf{B}^{p,\alpha}(X)$ then $v \in \mathbf{B}^{p,\alpha}(X)$ with $\|v\|_{p,\alpha} \leq \|u\|_{p,\alpha}$ and $\|v\|_{\mathbf{B}^{p,\alpha}(X)} \leq \|u\|_{\mathbf{B}^{p,\alpha}(X)}$. This fact will be used from time to time without further comment.

One has first the following example of the standard Dirichlet form on $\mathbb{R}^n$.

Example 3.3. If $X = \mathbb{R}^n$ and $\mathcal{E}$ is the standard Dirichlet form on $\mathbb{R}^n$, that is, for $f, g \in W^{1,2}(\mathbb{R}^n)$ we have

$$\mathcal{E}(f, g) = \int_{\mathbb{R}^n} \langle \nabla f(x), \nabla g(x) \rangle dx,$$

then, for $p \geq 1$ and $\alpha \geq 0$ the class $\mathbf{B}^{p,\alpha}(X)$ coincides with the Besov-Nikol'skii class $B_{p,\infty}^{2\alpha}(\mathbb{R}^n)$ that consists of $f \in L^p(\mathbb{R}^n, dx)$ such that

$$\sup_{h \in \mathbb{R}^n, h \neq 0} \frac{\|f(\cdot + h) - f(\cdot)\|_p}{|h|^{2\alpha}} < +\infty.$$

We refer, for instance, to [7] and [54, Theorems 4 and 4*] for several equivalent descriptions of those spaces.

Comparable Besov type spaces have previously been considered in the literature in some specific settings. A most relevant reference is the paper by K. Pietruska-Paľuba [51] (see also references therein). In particular, [51] provides a metric characterization of the spaces $\mathbf{B}^{p,\alpha}(X)$ on Dirichlet spaces that admit a heat kernel with Gaussian or sub-Gaussian heat kernel estimates.

Theorem 3.4 ([51, Theorem 3.2]). Let $(X, \mu, \mathcal{E}, \mathcal{F})$ be a Dirichlet space and let d be a metric on X compatible with the topology of X . Assume that the metric space (X, d) is Ahlfors d_H -regular and that $\{P_t\}_{t \in [0, \infty)}$ admits a heat kernel $p_t(x, y)$ satisfying, for some $c_1, c_2, c_3, c_4 \in (0, \infty)$ and $d_W \in (1, \infty)$,

$$c_1 t^{-d_H/d_W} \exp\left(-c_2 \left(\frac{d(x, y)^{d_W}}{t}\right)^{\frac{1}{d_W-1}}\right) \leq p_t(x, y) \leq c_3 t^{-d_H/d_W} \exp\left(-c_4 \left(\frac{d(x, y)^{d_W}}{t}\right)^{\frac{1}{d_W-1}}\right)$$

for $\mu \times \mu$ -a.e. $(x, y) \in X \times X$ for each $t \in (0, +\infty)$. Let $p \geq 1$ and $\alpha \geq 0$. We have

$$\mathbf{B}^{p,\alpha}(X) = \left\{ f \in L^p(X, \mu) : \sup_{r>0} \frac{1}{r^{\alpha d_W + d_H/p}} \left(\iint_{\Delta_r} |f(x) - f(y)|^p d\mu(x) d\mu(y) \right)^{1/p} < \infty \right\}$$

with comparable seminorms, where for $r > 0$ the set Δ_r denotes the collection of all $(x, y) \in X \times X$ for which $d(x, y) < r$.

The further study of the spaces $\mathbf{B}^{p,\alpha}(X)$ on Dirichlet spaces that admit a heat kernel with sub-Gaussian estimates will be the object of the paper [2]. In the present paper, one of the main goals is to study the spaces $\mathbf{B}^{p,\alpha}(X)$ in the framework of a general Dirichlet space $(X, \mu, \mathcal{E}, \mathcal{F})$.

4 Properties of the heat semigroup-based Besov spaces

In this section we identify and prove some fundamental properties of the Besov spaces given in Section 3, including Banach space property, reflexivity, and non-triviality. We also show that the supremum in the definition of Besov spaces can be replaced with limit supremum; this “locality in time” property is very useful in obtaining local information about Besov functions (in particular, dimensions of boundaries of sets whose characteristic functions are in a Besov class from their norms, see [1–3]). We will also prove interpolation inequalities and pseudo-Poincaré inequalities. Those pseudo-Poincaré inequalities will play a prominent role in the study of Sobolev inequalities, see Section 6. Finally, we study the relationship between the Besov spaces and the domain of the fractional powers of the generator of the Dirichlet form.

Throughout the section, let $(X, \mu, \mathcal{E}, \mathcal{F})$ be a Dirichlet space and let $\{P_t\}_{t \in [0, \infty)}$ denote the associated Markovian semigroup.

4.1 Locality in time

The following “locality in time” property will be useful in understanding functions of bounded variation, to be studied in the second and third paper [1, 2]. It also underlines the fact that the Besov energy seminorm $\|\cdot\|_{p,\alpha}$ is a global object, since in going from the supremum in the Besov norm to limit supremum one also picks up the L^p -norm. Recall the definition of (heat semigroup-based) Besov classes from Definition 3.1.

Lemma 4.1. *Let $p \geq 1$ and $\alpha \geq 0$. Then*

$$\mathbf{B}^{p,\alpha}(X) = \left\{ f \in L^p(X, \mu) : \limsup_{t \rightarrow 0} t^{-\alpha} \left(\int_X P_t(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p} < +\infty \right\}.$$

Moreover, if $\beta > \alpha$, then $\mathbf{B}^{p,\beta}(X) \subset \mathbf{B}^{p,\alpha}(X)$. Furthermore, for $f \in \mathbf{B}^{p,\alpha}(X)$, and for every $t > 0$, we have

$$\|f\|_{p,\alpha} \leq \frac{2}{t^\alpha} \|f\|_{L^p(X, \mu)} + \sup_{s \in (0, t]} s^{-\alpha} \left(\int_X P_s(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p}.$$

Proof. The claim $\mathbf{B}^{p,\beta}(X) \subset \mathbf{B}^{p,\alpha}(X)$ for $\beta > \alpha$ is immediate.

If $f \in \mathbf{B}^{p,\alpha}(X)$, then

$$\limsup_{t \rightarrow 0} t^{-\alpha} \left(\int_X P_t(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p} \leq \|f\|_{p,\alpha}.$$

Conversely, if $\limsup_{t \rightarrow 0} t^{-\alpha} \left(\int_X P_t(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p} < +\infty$, then there is some $\varepsilon > 0$ for which

$$\sup_{t \in (0, \varepsilon]} t^{-\alpha} \left(\int_X P_t(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p} < \infty.$$

For $t > \varepsilon$, since $|f(x) - f(y)|^p \leq 2^{p-1}(|f(x)|^p + |f(y)|^p)$, the semigroup is conservative (and hence $P_t 1(x) = 1$ for all $x \in X$) and $\int_X P_t(|f|^p)(x) d\mu(x) \leq \int_X |f(x)|^p d\mu(x)$, we have

$$t^{-\alpha} \left(\int_X P_t(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p} \leq 2\varepsilon^{-\alpha} \|f\|_{L^p(X, \mu)}.$$

The last inequality stated in the lemma now follows from the above inequality, and we also have that if $\limsup_{t \rightarrow 0} t^{-\alpha} \left(\int_X P_t(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p} < \infty$, then $f \in \mathbf{B}^{p, \alpha}(X)$. This completes the proof. $\square$

Interestingly, in a large class of examples of strictly local Dirichlet forms, for $\alpha = 1/2$,

$$\limsup_{t \rightarrow 0} t^{-\alpha} \left(\int_X P_t(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p}$$

is actually a limit that can be explicitly computed:

Proposition 4.2. *Assume that X is a smooth manifold of dimension $n \geq d$. Let $L = V_0 + \sum_{i=1}^d V_i^2$ be a Hörmander's type operator on X , where the V_i 's are smooth vector fields. Let us assume that L is essentially self-adjoint on $C_0^\infty(X)$ in $L^2(X, \mu)$ for some Radon measure μ on X . Consider the Dirichlet space $(X, \mu, \mathcal{E}, \mathcal{F})$ obtained by closing the pre-Dirichlet form*

$$\mathcal{E}(f, g) = \int_X \Gamma(f, g) d\mu(x), \quad f, g \in C_0^\infty(X),$$

where $\Gamma(f, g)$ is the carré du champ operator defined by $\Gamma(f, g) = \frac{1}{2}(L(fg) - fLg - gLf) = \sum_{i=1}^d (V_i f)(V_i g)$. Assume that the associated semigroup P_t is conservative. Then, for every $p \geq 1$,

$$C_0^\infty(X) \subset \mathbf{B}^{p, 1/2}(X)$$

and one has for every $f \in C_0^\infty(X)$, and open set $A \subset X$,

$$\lim_{t \rightarrow 0} t^{-1/2} \left(\int_A P_t(|f - f(x)|^p)(x) d\mu(x) \right)^{1/p} = 2 \left(\frac{\Gamma\left(\frac{1+p}{2}\right)}{\sqrt{\pi}} \right)^{1/p} \left(\int_A \Gamma(f, f)(x)^{p/2} d\mu(x) \right)^{1/p},$$

where $\Gamma\left(\frac{1+p}{2}\right)$ denotes the Euler's gamma function.

We will prove this proposition below, after discussing its consequences.

Remark 4.3. *In the previous setting, one therefore has*

$$\left(\int_X \Gamma(f, f)(x)^{p/2} d\mu(x) \right)^{1/p} \leq C_p \|f\|_{p, 1/2} \tag{15}$$

for $f \in C_0^\infty(X)$. Moreover, if P_t satisfies the Bakry-Émery estimate $\sqrt{\Gamma(P_t f)} \leq C P_t \sqrt{\Gamma(f)}$, we will see in [1, Section 4.5] that the converse inequality to (15) holds for $p = 1$, which takes the form

$$\|f\|_{1, 1/2} \leq c \left(\int_X \Gamma(f, f)(x)^{1/2} d\mu(x) \right).$$

Remark 4.4. Proposition 4.2 indicates that at a high level of generality, one may expect the Besov spaces $\mathbf{B}^{p,1/2}(X)$, $1 \leq p < +\infty$ to be closely related to the various notions of Sobolev spaces that have been defined on metric measure spaces (see for instance [53]). While in this paper we shall only be concerned with the study of all the Besov spaces $\mathbf{B}^{p,\alpha}(X)$, the comparison between Sobolev spaces and Besov spaces will be made in [1, Section 7] in the framework of Dirichlet spaces with absolutely continuous energy measures. In the framework of [2], it will be interesting to compare our results with the recent preprint [40] on Sobolev spaces and calculus of variations on fractals. Such a comparison will be the subject of future study.

Example 4.5. If $X = \mathbb{R}^n$ and $\mathcal{E}$ is the standard Dirichlet form on $\mathbb{R}^n$, it is natural to expect that for every $p \geq 1$, and every $f \in \mathbf{B}^{p,1/2}(X)$

$$\lim_{t \rightarrow 0} t^{-1/2} \left(\int_{\mathbb{R}^n} P_t(|f - f(x)|^p)(x) dx \right)^{1/p} = 2 \left(\frac{\Gamma\left(\frac{1+p}{2}\right)}{\sqrt{\pi}} \right)^{1/p} \left(\int_{\mathbb{R}^n} |\nabla f(x)|^p dx \right)^{1/p}.$$

The case $p = 1$ is proved in [50], but we did not find it in the literature for $p > 1$, $p \neq 2$, though it seems to be closely related to [16].

Proof of Proposition 4.2. We use here a probabilistic argument. For $x \in X$, we denote by $(B_t^x)_{t \geq 0}$ the L -Brownian motion on X started from x , that is the diffusion with generator L . It can be constructed as the solution of a stochastic differential equation in Stratonovich form:

$$dB_t^x = V_0(B_t^x)dt + \sqrt{2} \sum_{i=1}^d V_i(B_t^x) \circ d\beta_t^i$$

where β is a d -dimensional Brownian motion. Let $f \in C_0^\infty(X)$. The process

$$M_t^f = f(B_t^x) - f(x) - \int_0^t Lf(B_s^x)ds$$

is a square integrable martingale that can be written

$$M_t^f = \sqrt{2} \sum_{i=1}^d \int_0^t (V_i f)(B_s^x) d\beta_s^i.$$

We have then

$$\begin{aligned} P_t(|f - f(x)|^p)(x) &= \mathbb{E}(|f(B_t^x) - f(x)|^p) \\ &= \mathbb{E} \left(\left| M_t^f + \int_0^t Lf(B_s^x)ds \right|^p \right). \end{aligned}$$

Observe now that $\frac{1}{\sqrt{t}} \int_0^t Lf(B_s^x)ds$ almost surely converges to 0 when $t \rightarrow 0$. Note also that $\frac{1}{\sqrt{t}} M_t^f$ converges in all L^p 's to the Gaussian random variable $\sqrt{2} \sum_{i=1}^d (V_i f)(x) \beta_1^i$. Since f has a compact support, one deduces that

$$\lim_{t \rightarrow 0} t^{-1/2} \left(\int_A P_t(|f - f(x)|^p)(x) d\mu(x) \right)^{1/p} = C_p \left(\int_A \Gamma(f, f)(x)^{p/2} d\mu(x) \right)^{1/p},$$

with $C_p = \sqrt{2} \mathbb{E}(|N|^p)^{1/p} = 2 \left(\frac{\Gamma(\frac{1+p}{2})}{\sqrt{\pi}} \right)^{1/p}$, where N denotes a Gaussian random variable with mean 0 and variance 1. $\square$

4.2 $\mathbf{B}^{2,1/2}(X) = \mathcal{F}$ and non-triviality of some of the spaces $\mathbf{B}^{p,\alpha}(X)$

We prove that the Besov space $\mathbf{B}^{2,1/2}(X)$ is exactly the domain $\mathcal{F}$ of the Dirichlet form. It follows that $\mathbf{B}^{2,1/2}(X)$ is dense in $L^2(X, \mu)$.

Proposition 4.6. *We have $\mathbf{B}^{2,1/2}(X) = \mathcal{F}$. Moreover, for every $f \in \mathcal{F}$, $2\mathcal{E}(f, f) = \|f\|_{2,1/2}^2$.*

Proof. Let $f \in L^2(X, \mu)$. Note that as P_t is linear and fixes constant functions (by its conservativeness), we have for $t > 0$ that

$$P_t(|f - f(y)|^2)(y) = P_t(f^2)(y) + f(y)^2 - 2f(y)P_t(f)(y).$$

Therefore,

$$\frac{1}{2t} \int_X P_t(|f - f(y)|^2)(y) d\mu(y) = \frac{1}{2t} \int_X (P_t(f^2)(y) + f(y)^2 - 2f(y)P_t(f)(y)) d\mu(y).$$

Now using the symmetry (9) and the conservativeness of $\{P_t\}_{t \in [0, \infty)}$, we have

$$\int_X P_t(f^2)(y) d\mu(y) = \int_X (P_t 1)(y) f^2(y) d\mu(y) = \int_X f^2(y) d\mu(y).$$

Therefore,

$$\begin{aligned} \frac{1}{2t} \int_X P_t(|f - f(y)|^2)(y) d\mu(y) &= \frac{1}{t} \int_X (f(y)^2 - f(y)P_t(f)(y)) d\mu(y) \\ &= \frac{1}{t} \langle (I - P_t)f, f \rangle. \end{aligned} \tag{16}$$

From Lemma 2.1 above, one sees that the right side of (16) is positive and decreasing as t increases, and has limit $\mathcal{E}(f, f)$ as $t \downarrow 0$ if and only if $f \in \mathcal{F}$. From this, we know that the limit $t \rightarrow 0^+$ of the left-hand side term above is the supremum, and the claim follows. $\square$

Proposition 4.7. *If $1 \leq q \leq p < \infty$ and $f \in \mathbf{B}^{p,\alpha}(X)$, then $|f|^{p/q} \in \mathbf{B}^{q,\alpha}(X)$ and*

$$\| |f|^{p/q} \|_{q,\alpha} \leq 2^{1/q} \left(\frac{p}{q} \right) \|f\|_{L^p(X,\mu)}^{(p/q)-1} \|f\|_{p,\alpha}. \tag{17}$$

Proof. We need only prove the seminorm estimate, as the norm estimate then follows trivially from Hölder's inequality. We use for any $a, b > 0$ such that $a \neq b$, the elementary inequality

$$\frac{|a^{p/q} - b^{p/q}|}{|a - b|} \leq \frac{p}{q} \max\{a, b\}^{\frac{p}{q}-1}.$$

Equivalently,

$$|a^{p/q} - b^{p/q}|^q \leq \left(\frac{p}{q} \right)^q \max\{a, b\}^{p-q} |a - b|^q.$$

Using this elementary inequality, one has

$$\begin{aligned} & \int_X \int_X ||f(x)|^{p/q} - |f(y)|^{p/q}|^q p_t(y, dx) d\mu(y) \\ & \leq \left(\frac{p}{q} \right)^q \int_X \int_X (|f(x)|^{p-q} + |f(y)|^{p-q}) ||f(x)| - |f(y)||^q p_t(y, dx) d\mu(y) \\ & \leq \left(\frac{p}{q} \right)^q \int_X \int_X (|f(x)|^{p-q} + |f(y)|^{p-q}) |f(x) - f(y)|^q p_t(y, dx) d\mu(y). \end{aligned} \tag{18}$$

We now observe that thanks to (13),

$$\int_X \int_X |f(x)|^{p-q} |f(x) - f(y)|^q p_t(y, dx) d\mu(y) = \int_X \int_X |f(x)|^{p-q} |f(x) - f(y)|^q p_t(x, dy) d\mu(x).$$

On the other hand,

$$\int_X \int_X |f(y)|^{p-q} |f(x) - f(y)|^q p_t(y, dx) d\mu(y) = \int_X \int_X |f(x)|^{p-q} |f(x) - f(y)|^q p_t(x, dy) d\mu(x).$$

Thus, applying Hölder's inequality and then (12) to (18), we have

$$\begin{aligned} & \int_X \int_X \left| |f(x)|^{p/q} - |f(y)|^{p/q} \right|^q p_t(y, dx) d\mu(y) \\ & \leq 2 \left(\frac{p}{q} \right)^q \int_X |f(x)|^{p-q} \left(\int_X |f(x) - f(y)|^q p_t(x, dy) \right) d\mu(x) \\ & \leq 2 \left(\frac{p}{q} \right)^q \int_X |f(x)|^{p-q} \left(\int_X |f(x) - f(y)|^p p_t(x, dy) \right)^{q/p} d\mu(x) \\ & \leq 2 \left(\frac{p}{q} \right)^q \|f\|_{L^p(X, \mu)}^{p-q} \left(\int_X \int_X |f(x) - f(y)|^p p_t(x, dy) d\mu(x) \right)^{q/p} \\ & \leq 2 \left(\frac{p}{q} \right)^q \|f\|_{L^p(X, \mu)}^{p-q} \left(\int_X P_t(|f - f(x)|^p)(x) d\mu(x) \right)^{q/p} \\ & \leq 2 \left(\frac{p}{q} \right)^q \|f\|_{L^p(X, \mu)}^{p-q} t^{\alpha q} \|f\|_{p, \alpha}^q, \end{aligned}$$

which implies (17). □

The following is an immediate consequence of Propositions 4.6 and 4.7.

Corollary 4.8. *If $1 \leq q \leq p < \infty$ and $\mathbf{B}^{p, \alpha}(X)$ is dense in $L^p(X, \mu)$, then $\mathbf{B}^{q, \alpha}(X)$ is dense in $L^q(X, \mu)$. Hence $\mathbf{B}^{p, 1/2}(X)$ is dense in $L^p(X, \mu)$ for $1 \leq p \leq 2$.*

We note that when the measure is finite a stronger statement is true:

Proposition 4.9. *Let us assume that $\mu(X) < \infty$. Then $p \geq q$ implies $\mathbf{B}^{p, \alpha}(X) \subset \mathbf{B}^{q, \alpha}(X)$ and*

$$\|f\|_{q, \alpha} \leq \mu(X)^{1/q - 1/p} \|f\|_{p, \alpha}.$$

Proof. Let $f \in \mathbf{B}^{p, \alpha}(X)$. From Lemma 2.2 and Hölder's inequality, one has

$$\int_X P_t(|f - f(y)|^q) d\mu(y) \leq \int_X P_t(|f - f(y)|^p)^{q/p} d\mu(y) \leq (\mu(X))^{1-q/p} \left(\int_X P_t(|f - f(y)|^p) d\mu(y) \right)^{q/p}.$$

□

4.3 Triviality of some of the spaces $\mathbf{B}^{p, \alpha}(X)$

As we have seen, the space $\mathbf{B}^{2, 1/2}(X)$ is dense in $L^2(X, \mu)$ since it is the domain $\mathcal{F}$ of $\mathcal{E}$ which is dense in $L^2(X)$. For other values of the parameters, it turns out that some of the spaces $\mathbf{B}^{p, \alpha}(X)$ are in general trivial.

Proposition 4.10. *Suppose that for all $f \in \mathcal{F}$ we have that f is constant whenever $\mathcal{E}(f, f) = 0$. Then, any $f \in \mathbf{B}^{p,\alpha}(X)$ with $1 \leq p \leq 2$ and $\alpha > 1/p$ is constant.*

Proof. Let $f \in \mathbf{B}^{p,\alpha}(X)$ with $1 \leq p \leq 2$. For $n \geq 0$, we set $f_n := \min\{n, \max\{-n, f\}\}$. Since $|f_n(x) - f_n(y)| \leq |f(x) - f(y)|$ for every $x, y \in X$ and therefore $P_t(|f_n - f_n(y)|^p) \leq P_t(|f - f(y)|^p)$, it is clear that $f_n \in \mathbf{B}^{p,\alpha}(X)$. Moreover,

$$P_t(|f_n - f_n(x)|^2) = P_t(|f_n - f_n(x)|^{2-p}|f_n - f_n(x)|^p) \leq 2^{2-p}\|f_n\|_{L^\infty(X,\mu)}^{2-p}P_t(|f_n - f_n(x)|^p).$$

Therefore,

$$\frac{1}{2t} \int_X P_t(|f_n - f_n(x)|^2)(x) d\mu(x) \leq 2^{1-p}t^{\alpha p-1}\|f_n\|_{L^\infty(X,\mu)}^{2-p}\|f_n\|_{p,\alpha}^p.$$

As $\alpha p > 1$, this implies that

$$\lim_{t \rightarrow 0} \frac{1}{2t} \int_X P_t(|f_n - f_n(x)|^2)(x) d\mu(x) = 0.$$

Thus $f_n \in \mathbf{B}^{2,1/2}(X)$. Hence from Lemma 2.1 and Proposition 4.6, we see that $f_n \in \mathcal{F}$ and $\mathcal{E}(f_n, f_n) = 0$. This implies that f_n is constant for every n , thus f is constant. $\square$

The following theorem says that functions in $\mathbf{B}^{p,1/2}(X) \cap \mathcal{F}$ have a property related to the carré du champ. For the definition of a regular Dirichlet space and energy measure used in the proof we refer to the preliminaries in Section 2.

Theorem 4.11. *Let $p > 2$. If $f \in \mathbf{B}^{p,1/2}(X) \cap \mathcal{F}$ then there is $\Gamma(f) \in L^1(X, \mu)$ such that for all $g \in L^\infty(X, \mu) \cap \mathcal{F}$,*

$$\int_X g \Gamma(f) d\mu = 2\mathcal{E}(gf, f) - \mathcal{E}(f^2, g). \quad (19)$$

Proof. According to [31, Theorem A.4.1(ii)], any Dirichlet space $(X, \mu, \mathcal{E}, \mathcal{F})$ satisfying our assumptions is equivalent to a regular Dirichlet space $(X', \mu', \mathcal{E}', \mathcal{F}')$ where μ' is a Radon measure. This result was first obtained in [30], and in [39, Section 6] the isomorphism is realized as a Gelfand transform. According to [31, Appendix A.4], the equivalence between the Dirichlet spaces $(X, \mu, \mathcal{E}, \mathcal{F})$ and $(X', \mu', \mathcal{E}', \mathcal{F}')$ implies the equivalence of $L^p(X, \mu)$ and $L^p(X', \mu')$ spaces and the equivalence of corresponding semigroups P_t and P'_t . The spaces $\mathbf{B}^{p,1/2}(X)$ and $\mathbf{B}^{p,1/2}(X')$ are therefore also equivalent.

Since $(X', \mu', \mathcal{E}', \mathcal{F}')$ is regular, the Radon energy measure $\nu'_{f'}$ exists for any $f' \in \mathcal{F}'$. Let $f' \in \mathbf{B}^{p,1/2}(X') \cap \mathcal{F}'$ and suppose $g' \in L^\infty(X', \mu') \cap \mathcal{F}'$ with support of finite μ' -measure. Note that then $g' \in L^{p/(p-2)}(X', \mu')$.

By Hölder's inequality from Lemma 2.2, we have

$$\begin{aligned} \frac{1}{t} \int_{X'} |g'(y)| P'_t(|f' - f'(y)|^2)(y) d\mu'(y) &\leq \frac{1}{t} \int_{X'} |g'(y)| (P'_t(|f' - f'(y)|^p)(y))^{2/p} d\mu'(y) \\ &\leq \frac{1}{t} \left(\int_{X'} P'_t(|f' - f'(y)|^p)(y) d\mu'(y) \right)^{2/p} \|g'\|_{L^{\frac{p}{p-2}}(X', \mu')} \\ &\leq \|f'\|_{p,1/2}^2 \|g'\|_{L^{\frac{p}{p-2}}(X', \mu')}. \end{aligned}$$

Observe that we have

$$\begin{aligned} \int_{X'} g'(y) P'_t(|f' - f'(y)|^2)(y) d\mu'(y) &= \langle P'_t((f')^2), g' \rangle - 2\langle P'_t f', f' g' \rangle + \langle (f')^2, g' \rangle \\ &= -\langle -P'_t((f')^2), g' \rangle - 2\langle P'_t f', f' g' \rangle + 2\langle f', f' g' \rangle - \langle (f')^2, g' \rangle \\ &= -\langle (I' - P'_t)(f')^2, g' \rangle + 2\langle (I' - P'_t)f', f' g' \rangle. \end{aligned}$$

Now using the above identity and then taking the limit $t \rightarrow 0$, we obtain

$$\begin{aligned} \|f'\|_{p,1/2}^2 \|g'\|_{L^{p/(p-2)}(X',\mu)} &\geq \lim_{t \rightarrow 0} \left(\frac{2}{t} \langle (I' - P'_t)f', f'g' \rangle - \frac{1}{t} \langle (I' - P'_t)(f')^2, g \rangle \right) \\ &= 2\mathcal{E}'(f'g', f') - \mathcal{E}'((f')^2, g') = \int_{X'} 2g' d\nu'_{f'}, \end{aligned}$$

where, as in the previous result, the limit is by Lemma 2.1 above. The final equality is from the definition of $\nu'_{f'}$ and [24, Theorem 4.3.11], see also [14].

In particular, if $E_1 \subset E_2$ are of finite μ' measure and $\mathbf{1}_{E_1} \leq g' \leq \mathbf{1}_{E_2}$ then we obtain

$$\nu'_{f'}(E_1) \leq \int_{X'} g' d\nu'_{f'} \leq \frac{1}{2} \|f'\|_{p,1/2}^2 (\mu'(E_2))^{(p-2)/p}.$$

We wish to show that $\nu'_{f'}$ -measure of a μ' -null X' -Borel set is zero. Since both μ' and $\nu'_{f'}$ are X' -Radon measures, it suffices to show this for a compact μ' -null set E_1 . For $U \supset E_1$ open with compact closure there is a continuous function h satisfying $h = 1$ on E_1 and $h = 0$ on $X' \setminus U$. Then by the regularity of $\mathcal{E}'$, we can find $k \in \mathcal{F}'$ for which $\|h - k\|_\infty < 1/3$ (see [24, Definition 1.3.10(iii)]), at which point $g' = 3((k \wedge 2/3) - 1/3) \vee 0$ satisfies the conditions of the above estimate with $E_2 = \bar{U}$, the closure of U . Then

$$\nu'_{f'}(E_1) \leq \|f'\|_{p,1/2}^2 \mu'(\bar{U})^{(p-2)/p} \leq \|f'\|_{p,1/2}^2 \mu'(V)^{(p-2)/p}$$

for any open V containing $\bar{U}$. Thus $\nu'_{f'}(E_1) \leq \inf_V \|f'\|_{p,1/2}^2 \mu'(V)^{(p-2)/p}$, with the infimum over all open sets V containing E_1 ; this is zero by the outer regularity of μ' on X' . Hence $\nu'_{f'} \ll \mu'$ with a density $\frac{\nu'_{f'}}{\mu'} \in L^1(X', \mu')$. However the equivalence of the Dirichlet forms and L^p spaces then allows us to take $\Gamma(f) \in L^1(X, \mu)$ so that for $u \in \mathcal{F} \cap L^\infty(X, \mu)$,

$$\int_X u \Gamma(f) d\mu = \int_{X'} u' \frac{\nu'_{f'}}{\mu'} d\mu' = 2\mathcal{E}'(f'u', f') - \mathcal{E}'((f')^2, g') = \int_{X'} 2u' d\nu'_{f'} = 2\mathcal{E}(fu, f) - \mathcal{E}(f^2, u). \quad \square$$

We deduce two corollaries. The first uses the definition of a carré du champ operator, see [15, Definition 4.1.2], which is that there is a map $f \mapsto \Gamma(f)$ on a $\mathcal{E}_1$ -dense subspace of $\mathcal{F} \cap L^\infty$ such that (19) holds. It shows that for $p > 2$, $\mathbf{B}^{p,1/2}(X) \cap \mathcal{F}$ is dense in $\mathcal{F}$ only in Dirichlet spaces that admit a carré du champ operator.

Corollary 4.12. *If $\mathbf{B}^{p,1/2}(X) \cap \mathcal{F}$ is dense in $\mathcal{F}$ with respect to the norm $\mathcal{E}_1$ defined in (7) for some $p > 2$, then $\mathcal{E}$ admits a carré du champ operator. In particular, (19) is true for all $f \in \mathcal{F} \cap L^\infty(X, \mu)$.*

Proof. The proof that Γ extends to represent all $f \in \mathcal{F} \cap L^\infty(X, \mu)$ is [15, Proposition 4.1.3]. $\square$

The second corollary is of interest because it is known there are spaces that admit regular Dirichlet forms for which the energy measure ν_f is singular to μ for any non-constant $f \in \mathcal{F} \cap L^\infty(X, \mu)$, see [13, 46]. Examples of such spaces include the Sierpinski gasket, see for instance [10, 11, 42, 43]. These spaces also have the property that $\mathcal{E}(f, f) = 0$ implies f is constant, so the following result says that on these spaces $\mathbf{B}^{p,1/2}(X)$ consists of constant functions when $p > 2$.

Corollary 4.13. *Suppose that for all $f \in \mathcal{F}$ we have that f is constant whenever $\mathcal{E}(f, f) = 0$. Then, if $\mathcal{E}$ is regular and the energy measure ν_f is singular to μ for any non-constant $f \in \mathcal{F}$, the space $\mathbf{B}^{p,1/2}(X)$ contains only constant functions when $p > 2$.*

Proof. Suppose that $f \in \mathbf{B}^{p,1/2}(X)$ and assume without loss of generality, by Remark 3.2, that $f \geq 0$ is bounded. Then, from Lemma 4.7 $f^{p/2} \in \mathbf{B}^{2,1/2}(X) = \mathcal{F}$. However, since f is bounded, one also has $f^{p/2} \in \mathbf{B}^{p,1/2}(X)$. Therefore, $f^{p/2} \in \mathbf{B}^{p,1/2}(X) \cap \mathcal{F}$. From the proof of Theorem 4.11 we can conclude that $\nu_{f^{p/2}} = 0$, thus $f^{p/2}$ and hence f are constants. $\square$

4.4 Banach space property and reflexivity

In this section we prove that for $p \geq 1$ and $\alpha \geq 0$, $\mathbf{B}^{p,\alpha}(X)$ is always a Banach space which is moreover reflexive if $p > 1$.

Proposition 4.14. *For $p \geq 1$ and $\alpha \geq 0$, $\mathbf{B}^{p,\alpha}(X)$ is a Banach space.*

Proof. Let f_n be a Cauchy sequence in $\mathbf{B}^{p,\alpha}(X)$. Let f be the L^p limit of f_n . From Minkowski's inequality used from the representation (12) and conservativeness of P_t , one has

$$\begin{aligned} & \left| \left(\int_X P_t(|f_n - f_n(y)|^p)(y) d\mu(y) \right)^{1/p} - \left(\int_X P_t(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p} \right| \\ & \leq \left(\int_X P_t(|(f_n - f) - (f_n(y) - f(y))|^p)(y) d\mu(y) \right)^{1/p} \\ & \leq \left(\int_X P_t(|f_n - f|^p)(y) d\mu(y) \right)^{1/p} + \left(\int_X P_t(|f_n(y) - f(y)|^p)(y) d\mu(y) \right)^{1/p} \\ & \leq 2\|f - f_n\|_{L^p(X,\mu)}. \end{aligned}$$

Therefore

$$\lim_{n \rightarrow +\infty} \left(\int_X P_t(|f_n - f_n(y)|^p)(y) d\mu(y) \right)^{1/p} = \left(\int_X P_t(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p},$$

from which we deduce that

$$\begin{aligned} \frac{1}{t^\alpha} \left(\int_X P_t(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p} &= \lim_{n \rightarrow \infty} \frac{1}{t^\alpha} \left(\int_X P_t(|f_n - f_n(y)|^p)(y) d\mu(y) \right)^{1/p} \\ &\leq \lim_{n \rightarrow \infty} \|f_n\|_{p,\alpha} < \infty. \end{aligned}$$

Therefore $f \in \mathbf{B}^{p,\alpha}(X)$ and $\|f\|_{p,\alpha} \leq \lim_{n \rightarrow +\infty} \|f_n\|_{p,\alpha}$. Similarly, for each fixed positive integer m ,

$$\|f - f_m\|_{p,\alpha} \leq \lim_{n \rightarrow +\infty} \|f_n - f_m\|_{p,\alpha}$$

and taking the limit $m \rightarrow +\infty$ together with the fact that (f_n) is Cauchy with respect to the seminorm $\|\cdot\|_{p,\alpha}$ completes the proof. $\square$

We now turn to the reflexivity of $\mathbf{B}^{p,\alpha}(X)$. The Clarkson inequalities for L^p -functions are well-known. Given them, the following equivalent norm of $\|\cdot\|_{\mathbf{B}^{p,\alpha}(X)}$ immediately verifies the Clarkson inequalities for $\mathbf{B}^{p,\alpha}(X)$ given below. The equivalent norm, still denoted by $\|\cdot\|_{\mathbf{B}^{p,\alpha}(X)}$, is given by

$$\|f\|_{\mathbf{B}^{p,\alpha}(X)} = \left(\|f\|_{L^p(X,\mu)}^p + \|f\|_{p,\alpha}^p \right)^{\frac{1}{p}}.$$

Lemma 4.15 (Clarkson type inequalities). *Let $f, g \in \mathbf{B}^{p,\alpha}(X)$, $1 < p < \infty$, and q be the Hölder conjugate of p . If $2 \leq p < \infty$, then*

$$\|(f+g)/2\|_{\mathbf{B}^{p,\alpha}(X)}^p + \|(f-g)/2\|_{\mathbf{B}^{p,\alpha}(X)}^p \leq \|f\|_{\mathbf{B}^{p,\alpha}(X)}^p + \|g\|_{\mathbf{B}^{p,\alpha}(X)}^p. \quad (20)$$

If $1 < p \leq 2$, then

$$\|(f+g)/2\|_{\mathbf{B}^{p,\alpha}(X)}^q + \|(f-g)/2\|_{\mathbf{B}^{p,\alpha}(X)}^q \leq \left(\|f\|_{\mathbf{B}^{p,\alpha}(X)}^p + \|g\|_{\mathbf{B}^{p,\alpha}(X)}^p \right)^{q-1}. \quad (21)$$

By Proposition 4.14 and by the discussion above, we know that $\mathbf{B}^{p,\alpha}(X)$ is a Banach space. By the above Clarkson inequalities, $\mathbf{B}^{p,\alpha}(X)$ is uniformly convex. These, together with the Milman-Pettis theorem, yield the following corollary.

Corollary 4.16. *For any $p > 1$ and $\alpha > 0$, $\mathbf{B}^{p,\alpha}(X)$ is a reflexive Banach space.*

4.5 Interpolation inequalities

Now we turn our attention to interpolation inequalities. This exploration is in the spirit of the classical situation, where it is known that the classical (metric) Besov classes of functions on Euclidean spaces are obtained by interpolation between the Lebesgue spaces L^p and Sobolev spaces $W^{1,p}$; see [32] for analogous results in metric setting where the measure is doubling and supports a p -Poincaré inequality. In our general setting, we have the following basic interpolation inequalities.

Proposition 4.17. *Let $\theta \in [0, 1]$, $1 \leq q, r < +\infty$ and $\beta, \gamma > 0$. Let us assume $\frac{1}{p} = \frac{\theta}{q} + \frac{1-\theta}{r}$ and $\alpha = \theta\beta + (1-\theta)\gamma$. Then, $\mathbf{B}^{q,\beta}(X) \cap \mathbf{B}^{r,\gamma}(X) \subset \mathbf{B}^{p,\alpha}(X)$ and for any $f \in \mathbf{B}^{q,\beta}(X) \cap \mathbf{B}^{r,\gamma}(X)$,*

$$\|f\|_{p,\alpha} \leq \|f\|_{q,\beta}^\theta \|f\|_{r,\gamma}^{1-\theta}.$$

Proof. Let $f \in \mathbf{B}^{q,\beta}(X) \cap \mathbf{B}^{r,\gamma}(X)$. One has for every $t > 0$

$$t^{-\alpha} \left(\int_X P_t(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p} = t^{-\theta\beta - (1-\theta)\gamma} \left(\int_X P_t(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p}.$$

Then, from Hölder's inequality

$$\begin{aligned} \int_X P_t(|f - f(y)|^p)(y) d\mu(y) &= \int_X P_t(|f - f(y)|^{p\theta + p(1-\theta)})(y) d\mu(y) \\ &\leq \left(\int_X P_t(|f - f(y)|^q)(y) d\mu(y) \right)^{\frac{p\theta}{q}} \left(\int_X P_t(|f - f(y)|^r)(y) d\mu(y) \right)^{\frac{p(1-\theta)}{r}}. \end{aligned}$$

One deduces

$$\begin{aligned} &t^{-\alpha} \left(\int_X P_t(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p} \\ &\leq t^{-\theta\beta} \left(\int_X P_t(|f - f(y)|^q)(y) d\mu(y) \right)^{\frac{\theta}{q}} t^{-(1-\theta)\gamma} \left(\int_X P_t(|f - f(y)|^r)(y) d\mu(y) \right)^{\frac{1-\theta}{r}}. \end{aligned}$$

Taking the supremum over $t > 0$ finishes the proof. $\square$

Remark 4.18. *This interpolation inequality opens the door to study the (real and complex) interpolation theory of our Besov spaces. In view of the previous interpolation inequalities, it would be natural to conjecture that $(\mathbf{B}^{q,\beta}(X), \mathbf{B}^{r,\gamma}(X))_{\theta,p} = \mathbf{B}^{p,\alpha}(X)$, where $0 < \theta < 1$, $1 < q, r < +\infty$ and α, β, γ, p are the same as in the above proposition.*

By Proposition 4.6 we know that $\mathbf{B}^{2,1/2}(X) = \mathcal{F}$. Therefore, by the above interpolation inequality from Proposition 4.17, we have the following result.

Corollary 4.19. *Let $1 < p \leq 2$ and q be its conjugate, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. Let $0 < \alpha < 1$. Then, for any $f \in \mathcal{F} \cap \mathbf{B}^{p,\alpha}(X)$ and $g \in \mathcal{F} \cap \mathbf{B}^{q,1-\alpha}(X)$, it holds that*

$$|\mathcal{E}(f, g)| \leq \|f\|_{p,\alpha} \|g\|_{q,1-\alpha}.$$

4.6 Pseudo-Poincaré inequalities and fractional powers of the generator

Our goal in this section is to relate our Besov spaces to the domain of some fractional powers of the generator of the Dirichlet form. In a very general framework, one can resort to (Hille-Yosida) spectral theory to define the fractional powers of a closed operator A on a Banach space $D(A)$ via the following formula

$$(-A)^s f = \frac{\sin \pi s}{\pi} \int_0^\infty \lambda^{s-1} (\lambda I - A)^{-1} (-A) f \, d\lambda,$$

for every $f \in D(A)$. In fact, using Bochner's subordination one can express the fractional powers of A also in terms of the heat semi-group $P_t = e^{tA}$ via the following formula, see (5) in [58, page 260],

$$(-A)^s f = -\frac{s}{\Gamma(1-s)} \int_0^\infty t^{-s-1} [P_t f - f] \, dt. \quad (22)$$

With $A = L$ where L is the generator of $\mathcal{E}$, we set, for $0 < s \leq 1$, the class $\mathcal{L}_p^s$ to be the domain of the operator $(-L)^s$ in $L^p(X, \mu)$, $1 \leq p < \infty$. In other words, $\mathcal{L}_p^s$ consists of functions from $L^p(X, \mu)$ for which there is a function $g \in L^p(X, \mu)$ such that $(-L)^s f = g$.

The following simple pseudo-Poincaré inequalities that are analogs of classical Sobolev embeddings, will later play a prominent role in Section 6 and in our three subsequent papers. In this section, we will use them to prove that the fractional operator $(-L)^s : \mathbf{B}^{p,\alpha}(X) \rightarrow L^p(X, \mu)$ is bounded, where L is the generator of the Dirichlet form $\mathcal{E}$ and $0 < s < \alpha \leq 1$.

Lemma 4.20 (Pseudo-Poincaré inequalities). *Let $p \geq 1$ and $\alpha > 0$. Then for every $f \in \mathbf{B}^{p,\alpha}(X)$, and $t \geq 0$,*

$$\|P_t f - f\|_{L^p(X, \mu)} \leq t^\alpha \|f\|_{p, \alpha}.$$

Proof. From conservativeness of the semigroup and Hölder's inequality of Lemma 2.2, we have

$$\begin{aligned} \left(\int_X |P_t f(x) - f(x)|^p d\mu(x) \right)^{1/p} &= \left(\int_X |P_t(f - f(x))|^p d\mu(x) \right)^{1/p} \\ &\leq \left(\int_X P_t(|f - f(x)|^p)(x) d\mu(x) \right)^{1/p} \leq t^\alpha \|f\|_{p, \alpha}. \quad \square \end{aligned}$$

Remark 4.21. Triebel [55] (Section 1.13.6) introduced the interpolation spaces:

$$(L^p(X, \mu), \mathcal{E})_{\alpha, \infty} = \left\{ u \in L^p(X, \mu) : \sup_{t>0} t^{-\alpha} \|P_t u - u\|_{L^p(X, \mu)} < +\infty \right\}.$$

From the previous lemma, it is therefore clear that $\mathbf{B}^{p,\alpha}(X) \subset (L^p(X, \mu), \mathcal{E})_{\alpha, \infty}$. However, it may not be true that $\mathbf{B}^{p,\alpha}(X) = (L^p(X, \mu), \mathcal{E})_{\alpha, \infty}$, even when $X = \mathbb{R}^n$, see Remark 4.5 in [50] and [54] (Theorems 4 and 4*).

The following lemma will be useful:

Lemma 4.22. *Let L be the generator of $\mathcal{E}$, and let $p > 1$, $0 < \alpha < 1$. Then, there exists a constant $C > 0$ such that for every $f \in \mathbf{B}^{p,\alpha}(X)$ and $t \geq 0$,*

$$\|LP_t f\|_{L^p(X, \mu)} \leq C \frac{\|f\|_{p, \alpha}}{t^{1-\alpha}}.$$

Proof. By the analyticity of the semigroup P_t , see (11), it follows that $\lim_{t \rightarrow +\infty} \|LP_t f\|_{L^p(X, \mu)} = 0$ for $1 < p < \infty$. Then, we have by the semigroup property of $\{P_t\}_{t \in [0, \infty)}$ that

$$\begin{aligned} \|LP_{2t}f\|_{L^p(X, \mu)} &= \left\| \sum_{k=1}^{\infty} (LP_{2^k t}f - LP_{2^{k-1}t}f) \right\|_{L^p(X, \mu)} \leq \sum_{k=1}^{\infty} \|LP_{2^k t}f - LP_{2^{k-1}t}f\|_{L^p(X, \mu)} \\ &\leq \sum_{k=1}^{\infty} \|LP_{2^{k-1}t}(P_{2^{k-1}t}f - f)\|_{L^p(X, \mu)} \\ &\leq \sum_{k=1}^{\infty} \frac{1}{2^{k-1}t} \|P_{2^{k-1}t}f - f\|_{L^p(X, \mu)} \\ &\leq C \sum_{k=1}^{\infty} \frac{(2^{k-1}t)^{\alpha}}{2^{k-1}t} \|f\|_{p, \alpha} \\ &\leq C \frac{\|f\|_{p, \alpha}}{t^{1-\alpha}}, \end{aligned}$$

where we used the analyticity of P_t in the third inequality and the pseudo-Poincaré inequality in the fourth. $\square$

One has then the following proposition:

Proposition 4.23. *Let $\alpha \in (0, 1]$, $p \geq 1$ and $0 < s < \alpha$. Then*

$$\mathbf{B}^{p, \alpha}(X) \subset \mathcal{L}_p^s,$$

and there exists a constant $C = C_{s, \alpha}$ such that for every $f \in \mathbf{B}^{p, \alpha}(X)$,

$$\|(-L)^s f\|_{L^p(X, \mu)} \leq C \|f\|_{L^p(X, \mu)}^{1-\frac{s}{\alpha}} \|f\|_{p, \alpha}^{\frac{s}{\alpha}}. \quad (23)$$

In particular, $(-L)^s : \mathbf{B}^{p, \alpha}(X) \rightarrow L^p(X, \mu)$ is bounded.

Proof. Let $f \in \mathbf{B}^{p, \alpha}(X)$. We need to prove that the integral $x \mapsto \int_0^{\infty} t^{-s-1} (P_t f(x) - f(x)) dt$ is finite for almost every $x \in X$, and therefore that $f \in \mathcal{L}_p^s$. For $\delta > 0$, one has

$$\begin{aligned} \left\| \int_0^{\infty} t^{-s-1} (P_t f - f) dt \right\|_{L^p(X, \mu)} &\leq \int_0^{\infty} t^{-s-1} \|P_t f - f\|_{L^p(X, \mu)} dt \\ &\leq \int_0^{\delta} t^{-s-1} \|P_t f - f\|_{L^p(X, \mu)} dt + \int_{\delta}^{\infty} t^{-s-1} \|P_t f - f\|_{L^p(X, \mu)} dt \\ &\leq \|f\|_{p, \alpha} \int_0^{\delta} t^{-s-1+\alpha} dt + 2\|f\|_{L^p(X, \mu)} \int_{\delta}^{\infty} t^{-s-1} dt \\ &\leq \|f\|_{p, \alpha} \frac{\delta^{\alpha-s}}{\alpha-s} + 2\|f\|_{L^p(X, \mu)} \frac{\delta^{-s}}{s}. \end{aligned}$$

Choosing $\delta = 1$ in the above shows the boundedness of $(-L)^s$. To see (23), we choose $\delta > 0$ that satisfies

$$\delta^{\alpha} = 2 \frac{\|f\|_{L^p(X, \mu)}}{\|f\|_{p, \alpha}} \frac{\alpha - s}{s}$$

so that

$$\|f\|_{p, \alpha} \frac{\delta^{\alpha-s}}{\alpha-s} = 2\|f\|_{L^p(X, \mu)} \frac{\delta^{-s}}{s}.$$

Then

$$\begin{aligned}
\frac{\Gamma(1-s)}{s} \|(-L)^s f\|_{L^p(X,\mu)} &= \left\| \int_0^\infty t^{-s-1} (P_t f - f) dt \right\|_{L^p(X,\mu)} \\
&\leq 2 \|f\|_{L^p(X,\mu)} \frac{\delta^{-s}}{s} \\
&= \frac{2^{2-s/\alpha}}{s^{1-s/\alpha}(\alpha-s)^{s/\alpha}} \|f\|_{L^p(X,\mu)}^{1-s/\alpha} \|f\|_{p,\alpha}^{s/\alpha}. \quad \square
\end{aligned}$$

5 Continuity of P_t on the Besov spaces and critical exponents

Our goal in this section is to study the continuity properties of the semigroup P_t in the Besov spaces $\mathbf{B}^{p,\alpha}(X)$ with range $1 < p \leq 2$ and parameter $\alpha = \frac{1}{2}$. As corollaries we will deduce several important properties of the Besov spaces themselves. In particular, we will obtain the non-trivial fact that for $1 < p \leq 2$, the Besov space $\mathbf{B}^{p,1/2}(X)$ contains the $L^p(X, \mu)$ domain of L .

As before, throughout the section, let $(X, \mu, \mathcal{E}, \mathcal{F})$ be a Dirichlet space and let $\{P_t\}_{t \in [0, \infty)}$ denote the associated heat semigroup.

5.1 Continuity

The main result of this section quantifies a regularization property of the heat semigroup as follows.

Theorem 5.1. *Let $1 < p \leq 2$. There exists a constant $C_p > 0$ such that for every $f \in L^p(X, \mu)$ and $t \geq 0$*

$$\|P_t f\|_{p,1/2} \leq \frac{C_p}{t^{1/2}} \|f\|_{L^p(X,\mu)}.$$

In particular $P_t : L^p(X, \mu) \rightarrow \mathbf{B}^{p,1/2}(X)$ is bounded for $t > 0$.

It is remarkable that Theorem 5.1 applies to any Dirichlet space $(X, \mu, \mathcal{E}, \mathcal{F})$. We will see in [1, 2] that the study of the continuity of the semigroup in the Besov spaces $\mathbf{B}^{p,\alpha}(X)$ with range $p > 2$ requires additional assumptions on the space (weak Bakry-Émery type curvature condition).

To prove this theorem we need the following auxiliary result. The proof of this auxiliary result is obtained from some deep ideas originally due to Nick Dungey [28] and developed further by Li Chen in [22].

Lemma 5.2. *Let $1 < p \leq 2$. There exists a constant $C_p > 0$ such that for every non-negative $f \in L^p(X, \mu)$ and $t > 0$*

$$\left(\int_X P_t(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p} \leq C_p \|f\|_{L^p(X,\mu)}^{1/2} \|P_t f - f\|_{L^p(X,\mu)}^{1/2}.$$

Proof. Let $1 < p \leq 2$ and $t > 0$ be fixed in the following proof. The constant C in the following will denote a positive constant depending only on p that may change from line to line. For $\alpha, \beta \geq 0$, set

$$\gamma_p(\alpha, \beta) := p\alpha(\alpha - \beta) - \alpha^{2-p}(\alpha^p - \beta^p)$$

and for a non-negative function $f \in L^p(X, \mu)$

$$\begin{aligned}
\Gamma_p(f)(x) &:= pf(x) \int_X (f(x) - f(y)) p_t(x, dy) - f^{2-p}(x) \int_X (f^p(x) - f^p(y)) p_t(x, dy) \\
&= \int_X \gamma_p(f(x), f(y)) p_t(x, dy).
\end{aligned}$$

Note that from [23, Lemma 3.5], one has for any $\alpha, \beta \geq 0$

$$(p-1)(\alpha - \beta)^2 \leq \gamma_p(\alpha, \beta) + \gamma_p(\beta, \alpha) \leq p(\alpha - \beta)^2$$

and that, similarly to [28, page 122], one has $\Gamma_p(f) \geq 0$. Then the same argument as in [23, Lemma 3.6] gives

$$\begin{aligned} & \int_X \int_X |f(x) - f(y)|^p p_t(x, dy) d\mu(x) \\ & \leq C \int_X \int_X (\gamma_p(f(x), f(y)) + \gamma_p(f(y), f(x)))^{p/2} p_t(x, dy) d\mu(x) \\ & \leq C \int_X \int_X \left(\gamma_p^{p/2}(f(x), f(y)) + \gamma_p^{p/2}(f(y), f(x)) \right) p_t(x, dy) d\mu(x) \\ & = C \int_X \int_X \gamma_p^{p/2}(f(x), f(y)) p_t(x, dy) d\mu(x) \\ & \leq C \int_X \left(\int_X \gamma_p(f(x), f(y)) p_t(x, dy) \right)^{p/2} d\mu(x) \\ & = C \int_X \Gamma_p^{p/2}(f)(x) d\mu(x). \end{aligned}$$

Here the fourth line follows from the symmetry property of heat kernel measure in (13). Denote $\Delta_t = I - P_t$. Then Δ_t is the generator of a strongly continuous semigroup $\{e^{-s\Delta_t}\}_{s \in [0, \infty)}$ on $L^p(X, \mu)$ given by $e^{-s\Delta_t} = \sum_{n=0}^{\infty} \frac{s^n}{n!} (P_t - I)^n$. We then follow the proof of Theorem 1 in [23] (see also Theorem 1.3 in [28]) by taking $u(s, x) = e^{-s\Delta_t} f(x)$. Note that

$$\begin{aligned} \Gamma_p(u) &= pu(u - P_t u) - u^{2-p}(u^p - P_t(u^p)) \\ &= pu\Delta_t u - u^{2-p}\Delta_t(u^p) \\ &= -pu\partial_s u - u^{2-p}\Delta_t(u^p) \\ &= -u^{2-p}(\partial_s + \Delta_t)u^p. \end{aligned}$$

Set now

$$J(s, x) = -(\partial_s + \Delta_t)u^p(s, x),$$

so that

$$\Gamma_p(u) = u^{2-p}J.$$

Note that since $u \geq 0$ and $\Gamma_p(u) \geq 0$, one has $J \geq 0$. One has then from Hölder's inequality

$$\begin{aligned} \int_X \Gamma_p^{p/2}(u) d\mu &= \int_X u^{p(2-p)/2} J^{p/2} d\mu \\ &\leq \left(\int_X u^p d\mu \right)^{\frac{2-p}{2}} \left(\int_X J d\mu \right)^{p/2}. \end{aligned}$$

Observe that $u \in L^p(X, \mu)$ and hence $u^p \in L^1(X, \mu)$. Then $\int P_t(u^p) d\mu = \int u^p P_t 1 d\mu = \int u^p d\mu$ by symmetry and the conservative property of P_t . It follows that $\int_X \Delta_t u^p d\mu = 0$. One computes then

$$\int_X J d\mu = - \int_X (\partial_s + \Delta_t) u^p(s, x) d\mu = - \int_X \partial_s(u^p) d\mu = -p \int_X u^{p-1} \partial_s u d\mu = p \int_X u^{p-1} \Delta_t u d\mu.$$

Thus, we have from Hölder's inequality

$$\int_X J d\mu \leq p \|u\|_{L^p(X,\mu)}^{p-1} \|\Delta_t u\|_{L^p(X,\mu)}.$$

From the definition of Δ_t one concludes therefore

$$\left(\int_X \int_X |u(s, x) - u(s, y)|^p p_t(x, dy) d\mu(x) \right)^{1/p} \leq C \|u(s, \cdot)\|_{L^p(X,\mu)}^{1/2} \|P_t u(s, \cdot) - u(s, \cdot)\|_{L^p(X,\mu)}^{1/2}.$$

Letting $s \rightarrow 0^+$ yields

$$\left(\int_X \int_X |f(x) - f(y)|^p p_t(x, dy) d\mu(x) \right)^{1/p} \leq C \|f\|_{L^p(X,\mu)}^{1/2} \|P_t f - f\|_{L^p(X,\mu)}^{1/2}. \quad \square$$

We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1. Let $f \in L^p(X, \mu)$. We can assume $f \geq 0$. If not, it is enough to decompose f as $f^+ - f^-$ with $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$. Let $s, t > 0$, applying Lemma 5.2 to $P_s f$, one obtains

$$\left(\int_X P_t(|P_s f - P_s f(y)|^p)(y) d\mu(y) \right)^{1/p} \leq C_p \|P_s f\|_{L^p(X,\mu)}^{1/2} \|P_{t+s} f - P_s f\|_{L^p(X,\mu)}^{1/2}.$$

Note that $\|P_s f\|_{L^p(X,\mu)} \leq \|f\|_{L^p(X,\mu)}$ and that

$$\begin{aligned} \|P_{t+s} f - P_s f\|_{L^p(X,\mu)} &= \left\| \int_0^t L P_{s+u} f du \right\|_{L^p(X,\mu)} \\ &= \left\| \int_0^t P_u L P_s f du \right\|_{L^p(X,\mu)} \\ &\leq \int_0^t \|P_u L P_s f\|_{L^p(X,\mu)} du \\ &\leq t \|L P_s f\|_{L^p(X,\mu)} \\ &\leq C \frac{t}{s} \|f\|_{L^p(X,\mu)}, \end{aligned}$$

where in the last step we used analyticity of the semigroup. One concludes

$$\left(\int_X P_t(|P_s f - P_s f(y)|^p)(y) d\mu(y) \right)^{1/p} \leq C \left(\frac{t}{s} \right)^{1/2} \|f\|_{L^p(X,\mu)}.$$

Dividing both sides by $\sqrt{t}$ and taking the supremum over $t > 0$ complete the proof. $\square$

We now collect several corollaries of Theorem 5.1. The following surprising result shows that when $p \geq 2$, the quantity

$$\sup_{t>0} t^{-1/2} \|P_t f - f\|_{L^p(X,\mu)}$$

can actually always be controlled by

$$\liminf_{t \rightarrow 0^+} t^{-1/2} \left(\int_X P_t(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p}.$$

This is another manifestation of the locality in time property of our Besov spaces (see also Section 4.1).

Proposition 5.3. *Let $2 \leq p < +\infty$. For every $f \in L^p(X, \mu)$, and $t \geq 0$,*

$$\|P_t f - f\|_{L^p(X, \mu)} \leq C_p t^{1/2} \liminf_{s \rightarrow 0} s^{-1/2} \left(\int_X P_s(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p}.$$

Proof. For $\tau \in (0, \infty)$ we set

$$\mathcal{E}_\tau(u, v) := \frac{1}{\tau} \int_X (P_\tau - I) u v d\mu. \quad (24)$$

Let $f \in L^p(X, \mu)$ and $g \in L^q(X, \mu)$ where q is the conjugate exponent of p . We note that,

$$\begin{aligned} \int_0^t \mathcal{E}_\tau(P_s f, g) ds &= \int_0^t \frac{1}{\tau} \int_X (P_{s+\tau} f - P_s f) g d\mu ds \\ &= \int_X \left(\frac{1}{\tau} \int_t^{t+\tau} P_s f ds - \frac{1}{\tau} \int_0^\tau P_s f ds \right) g d\mu \end{aligned}$$

Therefore, using the strong continuity of the semigroup in $L^p(X, \mu)$, one has for $t \geq 0$,

$$\int_X (P_t f - f) g d\mu = \lim_{\tau \rightarrow 0^+} \int_0^t \mathcal{E}_\tau(P_s f, g) ds.$$

Note now that $\mathcal{E}_\tau(P_s f, g) = \mathcal{E}_\tau(f, P_s g)$ and that from Hölder's inequality (applied as in the proof of Proposition 4.17)

$$\begin{aligned} 2|\mathcal{E}_\tau(f, P_s g)| &\leq \tau^{-1/2} \left(\int_X P_\tau(|P_s g - P_s g(y)|^q)(y) d\mu(y) \right)^{1/q} \tau^{-1/2} \left(\int_X P_\tau(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p} \\ &\leq \tau^{-1/2} \left(\int_X P_\tau(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p} \|P_s g\|_{q, 1/2} \\ &\leq C_p \tau^{-1/2} \left(\int_X P_\tau(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p} s^{-1/2} \|g\|_{L^q(X, \mu)}. \end{aligned}$$

One has therefore

$$\left| \int_X (P_t f - f) g d\mu \right| \leq C_p t^{1/2} \|g\|_{L^q(X, \mu)} \liminf_{s \rightarrow 0} s^{-1/2} \left(\int_X P_s(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p},$$

and we conclude by $L^p - L^q$ duality. $\square$

One deduces:

Corollary 5.4. *Let $2 \leq p < +\infty$ and $\alpha > 1/2$. If $f \in \mathbf{B}^{p, \alpha}(X)$ then $\mathcal{E}(f, f) = 0$.*

Proof. Indeed, for $f \in \mathbf{B}^{p, \alpha}(X)$ with $\alpha > 1/2$ one has

$$\liminf_{s \rightarrow 0} s^{-1/2} \left(\int_X P_s(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p} = 0,$$

so that for every $t \geq 0$, $P_t f = f$, and thus $\mathcal{E}(f, f) = 0$. $\square$

Our final corollary of Theorem 5.1 is as follows.

Proposition 5.5. *Let $1 < p \leq 2$. Let L be the generator of $\mathcal{E}$ and $\mathcal{L}_p$ be the domain of L in $L^p(X, \mu)$. Then*

$$\mathcal{L}_p \subset \mathbf{B}^{p,1/2}(X)$$

and for every $f \in \mathcal{L}_p$,

$$\|f\|_{p,1/2}^2 \leq C \|Lf\|_{L^p(X,\mu)} \|f\|_{L^p(X,\mu)}. \quad (25)$$

Proof. Write for $\lambda > 0$

$$R_\lambda f = (L - \lambda)^{-1} f = \int_0^\infty e^{-\lambda t} P_t f dt.$$

Consequently

$$\|R_\lambda f\|_{p,1/2} \leq \int_0^\infty e^{-\lambda t} \|P_t f\|_{p,1/2} dt \leq \int_0^\infty e^{-\lambda t} \frac{C}{t^{1/2}} \|f\|_{L^p(X,\mu)} dt \leq C \lambda^{-1/2} \|f\|_{L^p(X,\mu)}.$$

It follows that

$$\|f\|_{p,1/2} \leq C \lambda^{-1/2} \|(L - \lambda)f\|_p \leq C(\lambda^{-1/2} \|Lf\|_{L^p(X,\mu)} + \lambda^{1/2} \|f\|_{L^p(X,\mu)}).$$

Taking $\lambda = \|Lf\|_{L^p(X,\mu)} \|f\|_{L^p(X,\mu)}^{-1}$ gives the result. $\square$

5.2 Critical Besov exponents

One can summarize several of our findings about the density or the triviality of our spaces $\mathbf{B}^{p,\alpha}(X)$ by introducing the notion of Besov critical exponents. Let $p \geq 1$. For the space X we define the L^p Besov density critical exponent $\alpha_p^*(X)$ and triviality critical exponent $\alpha_p^\#(X)$ as follows:

$$\alpha_p^*(X) = \sup\{\alpha > 0 : \mathbf{B}^{p,\alpha}(X) \text{ is dense in } L^p(X, \mu)\}.$$

$$\alpha_p^\#(X) = \sup\{\alpha > 0 : \mathbf{B}^{p,\alpha}(X) \text{ contains non-constant functions}\}.$$

Evidently $\alpha_p^*(X) \leq \alpha_p^\#(X)$. Critical Besov exponents of this and similar types have appeared in several previous works [34, 35, 37]. In particular, Grigor'yan [34] points out that when Theorem 3.4 can be applied, we know $\mathbf{B}^{p,\alpha}(X)$ can be defined in a purely metric fashion and therefore the critical exponents are determined by the metric-measure structure of X and are independent of any heat kernel. He also proves the exponent $\alpha_2^*(X) = \frac{1}{2}$ if P_t is stochastically complete, see also Proposition 5.6(4) below. There does not seem to be any literature on whether $\alpha^*(X)$ and $\alpha^\#(X)$ are distinct, but note that Gu and Lau [37] gave examples of spaces and Dirichlet forms for which the Besov critical exponent for density of $\mathbf{B}^{2,\alpha}(X)$ in $C(X)$ is strictly less than $\alpha_2^\#(X)$.

Proposition 5.6. *The following are true:*

1. Both $p \mapsto \alpha_p^*(X)$ and $p \mapsto \alpha_p^\#(X)$ are non-increasing;
2. For $1 \leq p \leq 2$ we have $\alpha_p^\#(X) \geq \alpha_p^*(X) \geq \frac{1}{2}$.

If we assume that $\mathcal{E}(f, f) = 0$ implies f constant, then we have in addition

3. If $1 \leq p \leq 2$ then $\alpha_p^*(X) \leq \alpha_p^\#(X) \leq \frac{1}{p}$;
4. $\alpha_2^*(X) = \alpha_2^\#(X) = \frac{1}{2}$;

5. For $2 \leq p < \infty$ one has $\alpha_p^*(X) \leq \alpha_p^\#(X) \leq \frac{1}{2}$.

Furthermore if $\mathcal{E}$ is regular and the energy measure ν_f for each non-constant $f \in \mathcal{F}$ is singular to μ (as is the case on some fractals) we obtain

6. For $p > 2$ one has $\alpha_p^*(X) \leq \alpha_p^\#(X) < \frac{1}{2}$.

Proof.

1. This is a direct application of Proposition 4.7 and Corollary 4.8.
2. We proved in Proposition 4.6 that $\mathbf{B}^{2,1/2}(X) = \mathcal{F}$ which is dense in L^2 , which proves the result for $p = 2$. For $1 \leq p \leq 2$ we use Claim 1.
3. This follows from Proposition 4.10.
4. Combine Claim 2 and Claim 3.
5. This follows from Corollary 5.4.
6. This is Corollary 4.13. □

We now present some conjectures on the critical exponents. We state them for $\alpha_p^\#(X)$, but similar results would be expected to hold for $\alpha_p^*(X)$.

Remark 5.7. *In view of the duality given by Corollary 4.19, it is natural to conjecture that under suitable conditions one may have*

$$\alpha_p^\#(X) + \alpha_q^\#(X) = 1$$

if p and q are conjugates, i.e. satisfy $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 5.8. *We will see in [1, 2] that for local Dirichlet forms the limit*

$$\alpha_\infty^\#(X) = \lim_{p \rightarrow +\infty} \alpha_p^\#(X)$$

is closely related to a Hölder regularity property in space of the heat semigroup. If the conjecture in Remark 5.7 is true, then classical interpolation theory (see Proposition 4.17) suggests that it is reasonable to expect that for every $p \geq 1$:

$$\alpha_p^\#(X) = \frac{1}{p} + \left(1 - \frac{2}{p}\right) \alpha_\infty^\#(X).$$

Example 5.9. *For strictly local Dirichlet forms with absolutely continuous energy measures, we will see in [1] that one generically has $\alpha_p^\#(X) = \alpha_p^*(X) = \frac{1}{2}$ for every $p \geq 1$.*

Example 5.10. *On the infinite Sierpinski gaskets $\alpha_1^\#(X) = \frac{d_H}{d_W}$, where d_H is the Hausdorff dimension of X and d_W its walk dimension, see [2]. Finding the exact value of $\alpha_1^\#(X)$ is, in general, an open question; in particular it is open for Sierpinski carpets.*

6 Sobolev and isoperimetric inequalities

In this section, we are interested in sharp Sobolev type embeddings (the case $p = 1$ corresponds to isoperimetric type results) for the Besov spaces studied in this paper.

Let $(X, \mu, \mathcal{E}, \mathcal{F})$ be a Dirichlet space. Let $\{P_t\}_{t \in [0, \infty)}$ denote the Markovian semigroup associated with $(X, \mu, \mathcal{E}, \mathcal{F})$. Throughout the section, we shall assume that P_t admits a measurable heat kernel $p_t(x, y)$ satisfying, for some $C > 0$ and $\beta > 0$,

$$p_t(x, y) \leq Ct^{-\beta} \quad (26)$$

for $\mu \times \mu$ -a.e. $(x, y) \in X \times X$, and for each $t \in (0, +\infty)$. Our goal in this section is to prove for the space $\mathbf{B}^{p, \alpha}(X)$ global Sobolev embeddings with sharp exponents and one of the main results will be the following weak-type Sobolev inequality and the corresponding isoperimetric inequality:

Theorem 6.1. *Let $0 < \alpha < \beta$. Let $1 \leq p < \frac{\beta}{\alpha}$. There exists a constant $C_{p, \alpha} > 0$ such that for every $f \in \mathbf{B}^{p, \alpha}(X)$,*

$$\sup_{s \geq 0} s \mu(\{x \in X : |f(x)| \geq s\})^{\frac{1}{q}} \leq C_{p, \alpha} \|f\|_{p, \alpha},$$

where $q = \frac{p\beta}{\beta - p\alpha}$. Therefore, there exists a constant $C_{\text{iso}} > 0$, given in Proposition 6.5, such that for every subset $E \subset X$ with $\mathbf{1}_E \in \mathbf{B}^{1, \alpha}(X)$

$$\mu(E)^{\frac{\beta - \alpha}{\beta}} \leq C_{\text{iso}} \|\mathbf{1}_E\|_{1, \alpha}.$$

6.1 Weak type Sobolev inequality

We follow and adapt to our setting a general approach to Sobolev inequalities developed in [8] (see also [52]). The pseudo-Poincaré inequality proved in Lemma 4.20 plays a fundamental role here.

Lemma 6.2. *Let $1 \leq p, q < +\infty$ and $\alpha > 0$. There exists a constant $C_{q, \alpha, \beta} > 0$ such that for every $f \in \mathbf{B}^{p, \alpha}(X) \cap L^q(X, \mu)$ and $s \geq 0$,*

$$\sup_{s \geq 0} s^{1+q\frac{\alpha}{\beta}} \mu(\{x \in X : |f(x)| > s\})^{\frac{1}{p}} \leq C_{q, \alpha, \beta} \|f\|_{p, \alpha} \|f\|_{L^q(X, \mu)}^{q\frac{\alpha}{\beta}}.$$

Proof. We adapt an argument given in the proof of Theorem 9.1 in [8]. Let $f \in \mathbf{B}^{p, \alpha}(X)$ and denote

$$F(s) = \mu(\{x \in X : |f(x)| > s\}).$$

We have then

$$F(s) \leq \mu(\{x \in X : |f(x) - P_t f(x)| > s/2\}) + \mu(\{x \in X : |P_t f(x)| > s/2\}).$$

Now, from the heat kernel upper bound $p_t(x, y) \leq Ct^{-\beta}$, $t > 0$, one deduces, for $g \in L^1(X, \mu)$, that $|P_t g(x)| \leq Ct^{-\beta} \|g\|_{L^1(X, \mu)}$. Since P_t is a contraction in $L^\infty(X, \mu)$, by the Riesz-Thorin interpolation one obtains

$$|P_t f(x)| \leq \frac{C^{1/q}}{t^{\beta/q}} \|f\|_{L^q(X, \mu)}.$$

Therefore, for $s = 2 \frac{C^{1/q}}{t^{\beta/q}} \|f\|_{L^q(X, \mu)}$, one has $\mu(\{x \in X : |P_t f(x)| > s/2\}) = 0$. On the other hand, from Theorem 4.20,

$$\mu(\{x \in X : |f(x) - P_t f(x)| > s/2\}) \leq 2^p s^{-p} t^{p\alpha} \|f\|_{p, \alpha}^p.$$

We conclude that

$$F(s)^{1/p} \leq \tilde{C} s^{-1-q\frac{\alpha}{\beta}} \|f\|_{\alpha, p} \|f\|_{L^q(X, \mu)}^{q\frac{\alpha}{\beta}}. \quad \square$$

As a corollary, we are now ready to prove the weak Sobolev inequality.

Theorem 6.3. *Let $0 < \alpha < \beta$. Let $1 \leq p < \frac{\beta}{\alpha}$. There exists a constant $C_{p,\alpha,\beta} > 0$ such that for every $f \in \mathbf{B}^{p,\alpha}(X)$,*

$$\sup_{s \geq 0} s \mu(\{x \in X : |f(x)| \geq s\})^{\frac{1}{q}} \leq C_{p,\alpha,\beta} \|f\|_{p,\alpha},$$

where $q = \frac{p\beta}{\beta - p\alpha}$.

Proof. Let $f \in \mathbf{B}^{p,\alpha}(X)$ be a non-negative function. For $k \in \mathbb{Z}$, we denote

$$f_k = (f - 2^k)_+ \wedge 2^k.$$

Observe that $f_k \in L^p(X, \mu)$ and $\|f_k\|_{L^p(X, \mu)} \leq \|f\|_{L^p(X, \mu)}$. Moreover, for every $x, y \in X$, $|f_k(x) - f_k(y)| \leq |f(x) - f(y)|$ and so $\|f_k\|_{p,\alpha} \leq \|f\|_{p,\alpha}$. We also note that $f_k \in L^1(X, \mu)$, with

$$\|f_k\|_{L^1(X, \mu)} = \int_X |f_k| d\mu \leq 2^k \mu(\{x \in X : f(x) \geq 2^k\}).$$

We now use Lemma 6.2 to deduce:

$$\begin{aligned} \sup_{s \geq 0} s^{1+\frac{\alpha}{\beta}} \mu(\{x \in X : f_k(x) > s\})^{\frac{1}{p}} &\leq C_{\alpha,\beta} \|f_k\|_{p,\alpha} \|f_k\|_{L^1(X, \mu)}^{\frac{\alpha}{\beta}} \\ &\leq C_{\alpha,\beta} \|f_k\|_{p,\alpha} \left(2^k \mu(\{x \in X : f(x) \geq 2^k\})\right)^{\frac{\alpha}{\beta}}. \end{aligned}$$

In particular, by choosing $s = 2^k$ we obtain

$$2^{k\left(1+\frac{\alpha}{\beta}\right)} \mu(\{x \in X : f(x) \geq 2^{k+1}\})^{\frac{1}{p}} \leq C_{\alpha,\beta} \|f_k\|_{p,\alpha} \left(2^k \mu(\{x \in X : f(x) \geq 2^k\})\right)^{\frac{\alpha}{\beta}}.$$

Let

$$M(f) = \sup_{k \in \mathbb{Z}} 2^k \mu(\{x \in X : f(x) \geq 2^k\})^{1/q},$$

where $q = \frac{p\beta}{\beta - p\alpha}$. Using the fact that $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\beta}$ and the previous inequality we obtain:

$$2^k \mu(\{x \in X : f(x) \geq 2^{k+1}\})^{\frac{1}{p}} \leq 2^{-\frac{kq\alpha}{\beta}} C_{\alpha,\beta} \|f\|_{p,\alpha} M(f)^{\frac{q\alpha}{\beta}}.$$

and

$$2^k \mu(\{x \in X : f(x) \geq 2^{k+1}\})^{\frac{1}{q}} \leq C_{\alpha,\beta}^{\frac{p}{q}} \|f\|_{p,\alpha}^{p/q} M(f)^{\frac{p\alpha}{\beta}}.$$

Therefore

$$M(f)^{1-\frac{p\alpha}{\beta}} \leq 2C_{\alpha,\beta}^{\frac{p}{q}} \|f\|_{p,\alpha}^{p/q}.$$

One concludes

$$M(f) \leq 2^{q/p} C_{\alpha,\beta} \|f\|_{p,\alpha}.$$

This easily yields:

$$\sup_{s \geq 0} s \mu(\{x \in X : f(x) \geq s\})^{\frac{1}{q}} \leq 2^{1+q/p} C_{\alpha,\beta} \|f\|_{p,\alpha}.$$

Let now $f \in \mathbf{B}^{p,\alpha}(X)$, which is not necessarily non-negative. From the previous inequality applied to $|f|$, we deduce

$$\sup_{s \geq 0} s \mu(\{x \in X : |f(x)| \geq s\})^{\frac{1}{q}} \leq 2^{1+q/p} C_{\alpha,\beta} \|f\|_{p,\alpha} \leq 2^{1+q/p} C_{\alpha,\beta} \|f\|_{p,\alpha}. \quad \square$$

6.2 Capacitary estimates

It is well-known that Sobolev inequalities are related to capacitary estimates, see for instance [8, Section 10]. In the current subsection, we explore this relation in our case. Let $p \geq 1$ and $0 < \alpha < \beta$. For a measurable set $A \subset X$, we define its (α, p) -capacity:

$$\mathbf{Cap}_p^\alpha(A) = \inf\{\|f\|_{\alpha,p}^p : f \in \mathbf{B}^{\alpha,p}(X), \mathbf{1}_A \leq f \leq 1\}.$$

We have the following corollary:

Corollary 6.4. *Let $0 < \alpha < \beta$. Let $1 \leq p < \frac{\beta}{\alpha}$. There exists a constant $C_{p,\alpha} > 0$ such that for every measurable set $A \subset X$,*

$$\mu(A)^{1-\frac{p\alpha}{\beta}} \leq C_{p,\alpha} \mathbf{Cap}_p^\alpha(A).$$

Proof. This is an immediate corollary of Theorem 6.3. □

6.3 Isoperimetric inequalities

Let $E \subset X$ be a measurable set with finite measure. We will say that E has a finite α -perimeter if $\mathbf{1}_E \in \mathbf{B}^{1,\alpha}(X)$. In that case, we will denote

$$P_\alpha(E) = \|\mathbf{1}_E\|_{1,\alpha}.$$

Proposition 6.5. *Let $0 < \alpha < \beta$. For every $E \subset X$ with finite α -perimeter*

$$\mu(E)^{\frac{\beta-\alpha}{\beta}} \leq C_{\text{iso}} P_\alpha(E),$$

where

$$C_{\text{iso}} = \frac{C^{\frac{\alpha}{\beta}} (\alpha + \beta)^{\frac{\alpha+\beta}{\beta}}}{2\beta\alpha^{\frac{\alpha}{\beta}}}.$$

and C is the constant from (3) and (26).

Proof. We follow an argument originally due to M. Ledoux [48]. Observe, from symmetry and conservativeness of the semigroup, that

$$\begin{aligned} \|P_t \mathbf{1}_E - \mathbf{1}_E\|_{L^1(X,\mu)} &= \int_E (1 - P_t \mathbf{1}_E) d\mu + \int_{E^c} P_t \mathbf{1}_E d\mu \\ &= \int_E (1 - P_t \mathbf{1}_E) d\mu + \int_E P_t \mathbf{1}_{E^c} d\mu \\ &= 2 \left(\mu(E) - \int_E P_t \mathbf{1}_E d\mu \right) \end{aligned}$$

However, our assumption (26) on the heat kernel gives

$$\int_E P_t \mathbf{1}_E d\mu = \int_E \int_E p_t(x, y) d\mu(x) d\mu(y) \leq Ct^{-\beta} \mu(E)^2$$

Combining these equations with the pseudo-Poincaré estimate from Lemma 4.20 gives, for $t > 0$,

$$2\mu(E) \leq \|P_t \mathbf{1}_E - \mathbf{1}_E\|_{L^1(X,\mu)} + 2 \int_E P_t \mathbf{1}_E d\mu \leq t^\alpha P_\alpha(E) + 2Ct^{-\beta} \mu(E)^2.$$

If $P_\alpha(E) = 0$, then letting $t \rightarrow \infty$ we also see that $\mu(E) = 0$, and so without loss of generality we may assume that $P_\alpha(E) > 0$, in which case we also have $\mu(E) > 0$. We minimize the right-hand side by choosing t satisfying

$$t^{\alpha+\beta} = \frac{2C\beta}{\alpha} \frac{\mu(E)^2}{P_\alpha(E)}$$

and obtain, for this choice of t , that

$$2\mu(E) \leq \left(1 + \frac{\alpha}{\beta}\right) t^\alpha P_\alpha(E).$$

Raising both sides of this equality to the power $\frac{\alpha+\beta}{\beta}$ concludes the proof. $\square$

We note that in the limiting case $\alpha \rightarrow \beta$, the previous estimate for C_{iso} yields the following corollary, which is applicable on many fractal spaces, see [2, 4]:

Corollary 6.6. *For every set $E \subset X$ with finite β -perimeter and $\mu(E) > 0$,*

$$P_\beta(E) \geq \frac{1}{2C},$$

where C is the constant from (26).

6.4 Strong Sobolev inequality

We now prove strong Sobolev inequalities. This requires an additional assumption on the space.

Definition 6.7. *We say that the Dirichlet space satisfies the property $(P_{p,\alpha})$ if there exists a constant $C > 0$ such that for every $f \in \mathbf{B}^{p,\alpha}(X)$,*

$$\|f\|_{p,\alpha} \leq C \liminf_{t \rightarrow 0} t^{-\alpha} \left(\int_X \int_X |f(x) - f(y)|^p p_t(x, y) d\mu(x) d\mu(y) \right)^{1/p}.$$

Remark 6.8. *The property $(P_{p,\alpha})$ with $\alpha = 1/2$ can be seen as a stronger form of the Proposition 5.3. As shown in [1, 2], in many situations, the property $(P_{p,\alpha})$ is satisfied, provided that the space X satisfies a weak Bakry-Émery type non-negative curvature condition and that α is the L^p -Besov critical exponent of X (see Section 5.2 for the definition of Besov critical exponent).*

For instance, $(P_{p,\alpha})$ is satisfied for $p = 1, \alpha = 1/2$ for the standard Dirichlet form of $\mathbb{R}^n$. It is also satisfied for $p = 1, \alpha = 1/2$ for the standard Dirichlet form of a complete Riemannian manifold with non-negative Ricci curvature. More generally, in the framework of [1], property $(P_{p,\alpha})$ is satisfied when $p = 1, \alpha = 1/2$, see Theorem 5.2 there. In the framework of Dirichlet spaces with sub-Gaussian heat kernel estimates, see [1], the property $(P_{1,\alpha})$ is satisfied for some α (the L^1 Besov critical exponent) provided that a weak Bakry-Émery type estimate is satisfied.

Our main theorem is then the following:

Theorem 6.9. *Assume that the Dirichlet space satisfies the property $(P_{p,\alpha})$ and that β is given in (26). Let $0 < \alpha < \beta$. Let $1 \leq p < \frac{\beta}{\alpha}$. There exists a constant $C_{p,\alpha,\beta} > 0$ such that for every $f \in \mathbf{B}^{p,\alpha}(X)$,*

$$\|f\|_{L^q(X,\mu)} \leq C_{p,\alpha,\beta} \|f\|_{p,\alpha},$$

where $q = \frac{p\beta}{\beta - p\alpha}$.

Note that in the standard Euclidean setting of $\mathbb{R}^n$ the Sobolev embedding theorem holds as above with $\beta = n$. To show that the weak type inequality implies the desired Sobolev inequality, we will need another cutoff argument and the following lemma is needed.

Lemma 6.10. *For $f \in \mathbf{B}^{p,\alpha}(X)$, $f \geq 0$, denote $f_k = (f - 2^k)_+ \wedge 2^k$, $k \in \mathbb{Z}$. There exists a constant $C > 0$ such that for every $f \in \mathbf{B}^{p,\alpha}(X)$,*

$$\left(\sum_{k \in \mathbb{Z}} \|f_k\|_{p,\alpha}^p \right)^{1/p} \leq C \|f\|_{p,\alpha}.$$

Proof. By a similar type of argument, as in the the proof of Lemma 7.1 in [8], one has for some constant $C_p > 0$,

$$\sum_{k \in \mathbb{Z}} \int_X \int_X |f_k(x) - f_k(y)|^p p_t(x, y) d\mu(x) d\mu(y) \leq C_p \int_X \int_X |f(x) - f(y)|^p p_t(x, y) d\mu(x) d\mu(y).$$

As a consequence of property $(P_{p,\alpha})$,

$$\left(\sum_{k \in \mathbb{Z}} \|f_k\|_{p,\alpha}^p \right)^{1/p} \leq C'_p \|f\|_{p,\alpha}.$$

and the proof is complete. $\square$

We can now conclude the proof of Theorem 6.9.

Proof of Theorem 6.9. Let $f \in \mathbf{B}^{p,\alpha}(X)$. We can assume $f \geq 0$. As before, denote $f_k = (f - 2^k)_+ \wedge 2^k$, $k \in \mathbb{Z}$. From Lemma 6.3 applied to f_k , we see that

$$\sup_{s \geq 0} s \mu(\{x \in X : |f_k(x)| \geq s\})^{\frac{1}{q}} \leq C_{p,\alpha} \|f_k\|_{p,\alpha}.$$

In particular for $s = 2^k$, we get

$$2^k \mu(\{x \in X : f(x) \geq 2^{k+1}\})^{\frac{1}{q}} \leq C_{p,\alpha} \|f_k\|_{p,\alpha}.$$

Therefore,

$$\sum_{k \in \mathbb{Z}} 2^{kq} \mu(\{x \in X : f(x) \geq 2^{k+1}\}) \leq C_{p,\alpha}^q \sum_{k \in \mathbb{Z}} \|f_k\|_{p,\alpha}^q.$$

Since $q \geq p$, one has $\sum_{k \in \mathbb{Z}} \|f_k\|_{p,\alpha}^q \leq (\sum_{k \in \mathbb{Z}} \|f_k\|_{p,\alpha}^p)^{q/p}$. Thus, from the previous lemma

$$\sum_{k \in \mathbb{Z}} 2^{kq} \mu(\{x \in X : f(x) \geq 2^{k+1}\}) \leq C_{p,\alpha}^q \|f\|_{p,\alpha}^q.$$

Finally, we observe that

$$\begin{aligned} \sum_{k \in \mathbb{Z}} 2^{kq} \mu(\{x \in X : f(x) \geq 2^{k+1}\}) &\geq \frac{q}{2^{q+1} - 2^q} \sum_{k \in \mathbb{Z}} \int_{2^{k+1}}^{2^{k+2}} s^{q-1} \mu(\{x \in X : f(x) \geq s\}) ds \\ &\geq \frac{1}{2^{q+1} - 2^q} \|f\|_{L^q(X,\mu)}^q. \end{aligned}$$

The proof is thus complete. $\square$

6.5 Application

The Sobolev embeddings studied in this section have many applications that will be studied in great details in the papers [1–3]. We just mention here that by combining Theorem 3.4 and Theorem 6.1 one immediately obtains:

Corollary 6.11. *Let X be an Ahlfors d_H -regular space that satisfies sub-Gaussian heat kernel estimates as in Theorem 3.4. Then, one has the following weak type Besov space embedding. Let $0 < \delta < d_H$. Let $1 \leq p < \frac{d_H}{\delta}$. There exists a constant $C_{p,\delta} > 0$ such that for every $f \in \mathbf{B}^{p,\delta/d_H}(X)$,*

$$\sup_{s \geq 0} s \mu(\{x \in X, |f(x)| \geq s\})^{\frac{1}{q}} \leq C_{p,\delta} \sup_{r > 0} \frac{1}{r^{\delta + d_H/p}} \left(\iint_{\Delta_r} |f(x) - f(y)|^p d\mu(x) d\mu(y) \right)^{1/p}$$

where $q = \frac{pd_H}{d_H - p\delta}$. Furthermore, for every $0 < \delta < d_H$, there exists a constant $C_{\text{iso},\delta}$ such that for every measurable $E \subset X$, $\mu(E) < +\infty$,

$$\mu(E)^{\frac{d_H - \delta}{d_H}} \leq C_{\text{iso},\delta} \sup_{r > 0} \frac{1}{r^{\delta + d_H}} (\mu \otimes \mu) \{(x, y) \in E \times E^c : d(x, y) < r\}. \quad (27)$$

Remark 6.12. *The number δ in the previous corollary plays the role of the upper codimension of the boundary of E . This will be further commented in [2].*

Proof. From the upper sub-Gaussian estimate, one has

$$p_t(x, y) \leq Ct^{-\beta}$$

where $\beta = d_H/d_W$. Let $0 < \alpha < \beta$. Let $1 \leq p < \frac{\beta}{\alpha}$. From Theorem 6.1, there exists a constant $C_{p,\alpha} > 0$ such that for every $f \in \mathbf{B}^{p,\alpha}(X)$,

$$\sup_{s \geq 0} s \mu(\{x \in X : |f(x)| \geq s\})^{\frac{1}{q}} \leq C_{p,\alpha} \|f\|_{p,\alpha},$$

where $q = \frac{p\beta}{\beta - p\alpha}$. However, from Theorem 3.4,

$$\|f\|_{p,\alpha} \leq C \sup_{r > 0} \frac{1}{r^{\alpha d_W + d_H/p}} \left(\iint_{\Delta_r} |f(x) - f(y)|^p d\mu(x) d\mu(y) \right)^{1/p}.$$

The result follows then with $\delta = \alpha d_W$. □

7 Cheeger constant and Gaussian isoperimetry

While the previous section was devoted to Sobolev inequalities on Dirichlet spaces for which the semigroup satisfies ultracontractive estimates, the present section is devoted to situations where the Dirichlet form satisfies a Poincaré inequality or a log-Sobolev inequality.

7.1 Buser's type inequality for the Cheeger constant of a Dirichlet space

In the context of a smooth compact n -dimensional Riemannian manifold with a normalized Riemannian measure μ , Cheeger introduced in [21] the following isoperimetric constant

$$h = \inf \frac{\mathcal{H}^{n-1}(\partial A)}{\mu(A)},$$

where $\mathcal{H}^{n-1}(\partial A)$ denotes the perimeter measure of A and where the infimum runs over all open subsets A with smooth boundary ∂A such that $\mu(A) \leq \frac{1}{2}$. Cheeger's constant can be used to bound from below the first non zero eigenvalue of the manifold. Indeed, it is proved in [21] that

$$\lambda_1 \geq \frac{h^2}{4}.$$

Buser [19] then proved that if the Riemannian Ricci curvature of the manifold is non-negative, then we actually have

$$\lambda_1 \leq Ch^2,$$

where C is a universal constant depending only on the dimension. Buser's inequality was reproved by Ledoux [47] using heat semigroup techniques. Under proper assumptions, by using the tools we introduced in the present paper, Ledoux' technique can be essentially reproduced in our general framework of Dirichlet spaces.

Let $(X, \mu, \mathcal{E}, \mathcal{F})$ be a Dirichlet space such that $\mu(X) = 1$. Let $\{P_t\}_{t \in [0, \infty)}$ denote the semigroup associated with $(X, \mu, \mathcal{E}, \mathcal{F})$. The Dirichlet form $\mathcal{E}$ is said to satisfy a spectral gap inequality with spectral gap λ_1 if for every $f \in \mathcal{F}$,

$$\int_X f^2 d\mu - \left(\int_X f d\mu \right)^2 \leq \frac{1}{\lambda_1} \mathcal{E}(f, f).$$

We assume in this section that $\mathcal{E}$ satisfies a spectral gap inequality. For $\alpha \in (0, 1]$, we define the α -Cheeger's constant of X by

$$h_\alpha = \inf \frac{\|\mathbf{1}_E\|_{1, \alpha}}{\mu(E)},$$

where the infimum runs over all measurable sets E such that $\mu(E) \leq \frac{1}{2}$ and $\mathbf{1}_E \in \mathbf{B}^{1, \alpha}(X)$. We denote by λ_1 the spectral gap of $\mathcal{E}$.

Theorem 7.1. *We have $h_\alpha \geq (1 - e^{-1})\lambda_1^\alpha$.*

Proof. Let A be a set with $P_\alpha(A) := \|\mathbf{1}_A\|_{1, \alpha} < +\infty$. As shown in the proof of Proposition 6.5, we have

$$\|\mathbf{1}_A - P_t \mathbf{1}_A\|_{L^1(X, \mu)} = 2 \left(\mu(A) - \|P_{\frac{t}{2}}(\mathbf{1}_A)\|_{L^2(X, \mu)}^2 \right).$$

By the pseudo-Poincaré inequality in Lemma 4.20,

$$\|P_t \mathbf{1}_A - \mathbf{1}_A\|_{L^1(X, \mu)} \leq t^\alpha P_\alpha(A).$$

We deduce that

$$\mu(A) \leq \frac{1}{2} t^\alpha P_\alpha(A) + \|P_{\frac{t}{2}}(\mathbf{1}_A)\|_{L^2(X, \mu)}^2.$$

Now, by the spectral theorem,

$$\|P_{\frac{t}{2}}(\mathbf{1}_A)\|_{L^2(X, \mu)}^2 = \mu(A)^2 + \|P_{\frac{t}{2}}(\mathbf{1}_A - \mu(A))\|_{L^2(X, \mu)}^2 \leq \mu(A)^2 + e^{-\lambda_1 t} \|\mathbf{1}_A - \mu(A)\|_{L^2(X, \mu)}^2.$$

This yields

$$\mu(A) \leq \frac{1}{2} t^\alpha P_\alpha(A) + \mu(A)^2 + e^{-\lambda_1 t} \|\mathbf{1}_A - \mu(A)\|_{L^2(X, \mu)}^2.$$

Equivalently, one obtains

$$\frac{1}{2} t^\alpha P_\alpha(A) \geq \mu(A)(1 - \mu(A))(1 - e^{-\lambda_1 t}).$$

Therefore,

$$h_\alpha \geq \sup_{t>0} \left(\frac{1 - e^{-\lambda_1 t}}{t^\alpha} \right),$$

which completes the proof. $\square$

As already noted in [12], let us observe that it is known that the Cheeger lower bound on λ_1 may be obtained under further assumptions on the Dirichlet space $(X, d, \mathcal{E})$. Indeed, assume that $\mathcal{E}$ is strictly local with a carré du champ Γ , that Lipschitz functions are in the domain of $\mathcal{E}$ and that $\sqrt{\Gamma(f)}$ is an upper gradient in the sense that for any Lipschitz function f ,

$$\sqrt{\Gamma(f)}(x) = \limsup_{d(x,y) \rightarrow 0} \frac{|f(x) - f(y)|}{d(x,y)}.$$

In that case, if A is a closed subset of X , we define its Minkowski exterior boundary measure by

$$\mu_+(A) = \liminf_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (\mu(A_\varepsilon) - \mu(A)),$$

where $A_\varepsilon = \{x \in X, d(x, A) < \varepsilon\}$. We can then define a Cheeger's constant of X by

$$h_+ = \inf \frac{\mu_+(E)}{\mu(E)},$$

where the infimum runs over all closed sets E such that $\mu(E) \leq \frac{1}{2}$. Then, according to Theorem 8.5.2 in [9], one has

$$\lambda_1 \geq \frac{h_+^2}{4}.$$

7.2 Log-Sobolev and Gaussian isoperimetric inequalities

Let $(X, \mu, \mathcal{E}, \mathcal{F})$ be a Dirichlet space such that $\mu(X) = 1$. Let $\{P_t\}_{t \in [0, \infty)}$ denote the semigroup associated with $(X, \mu, \mathcal{E}, \mathcal{F})$. The Dirichlet form $\mathcal{E}$ is said to satisfy a log-Sobolev inequality with constant ρ_0 if for every $f \in \mathcal{F}$, $f \geq 0$,

$$\int_X f^2 \ln f^2 d\mu - \int_X f^2 d\mu \ln \int_X f^2 d\mu \leq \frac{1}{\rho_0} \mathcal{E}(f, f). \quad (28)$$

We assume in this section that $\mathcal{E}$ satisfies a log-Sobolev inequality with constant ρ_0 . We define the Gaussian isoperimetric constant of X by

$$k = \inf \frac{\|\mathbf{1}_E\|_{1,1/2}}{\mu(E) \sqrt{-\ln \mu(E)}},$$

where the infimum runs over all sets E such that $\mu(E) \leq \frac{1}{2}$ and $\mathbf{1}_E \in \mathbf{B}^{1,1/2}(X)$. Following an argument of M. Ledoux [47], we prove the following:

Theorem 7.2. *We have*

$$\rho_0 \leq C_l k^2$$

where C_l is a numerical constant.

Proof. Let A be a measurable set such that $P(A) := \|\mathbf{1}_A\|_{1,1/2} < +\infty$. By the same computations as before we have

$$\mu(A) \leq \frac{1}{2}\sqrt{t}P(A) + \|P_{\frac{t}{2}}(\mathbf{1}_A)\|_{L^2(X,\mu)}^2.$$

Now we can use the log-Sobolev constant to bound $\|P_{\frac{t}{2}}(\mathbf{1}_A)\|_2^2$. Indeed, from Gross' theorem it is well known that the logarithmic Sobolev inequality

$$\int_X f^2 \ln f^2 d\mu - \int_X f^2 d\mu \ln \int_X f^2 d\mu \leq \frac{1}{\rho_0} \mathcal{E}(f, f),$$

is equivalent to the following hypercontractivity property of the semigroup

$$\|P_t f\|_{L^q(X,\mu)} \leq \|f\|_{L^p(X,\mu)}$$

for all f in $L^p(X,\mu)$ whenever $1 < p < q < \infty$ and $e^{\rho_0 t} \geq \sqrt{\frac{q-1}{p-1}}$. Therefore, with $p(t) = 1 + e^{-2\rho_0 t} < 2$, we get

$$\begin{aligned} \sqrt{t}P(A) &\geq 2 \left(\mu(A) - \mu(A)^{\frac{2}{p(t)}} \right) \\ &\geq 2\mu(A) \left(1 - \mu(A)^{\frac{1-e^{-2\rho_0 t}}{1+e^{-2\rho_0 t}}} \right). \end{aligned}$$

By using then the computation page 956 in [47], one deduces that if A is a set which has a finite $P(A)$ and such that $0 \leq \mu(A) \leq \frac{1}{2}$, then

$$P(A) \geq \tilde{C} \sqrt{\rho_0} \mu(A) \left(\ln \frac{1}{\mu(A)} \right)^{\frac{1}{2}},$$

where $\tilde{C}$ is a numerical constant. □

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