LARGE TOURNAMENT GAMES

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We consider a stochastic tournament game in which each player is rewarded based on her rank in terms of the completion time of her own task and is subject to cost of effort. When players are homogeneous and the rewards are purely rank dependent, the equilibrium has a surprisingly explicit characterization, which allows us to conduct comparative statics and obtain explicit solution to several optimal reward design problems. In the general case when the players are heterogenous and payoffs are not purely rank dependent, we prove the existence, uniqueness and stability of the Nash equilibrium of the associated mean field game, and the existence of an approximate Nash equilibrium of the finite-player game.

CONTENTS

1. Introduction.   To put our mathematical problem of a large population tournament into a real life context, consider the pre-Google time when a lot of players were competing to build a successful internet search engine. This is a game in which the future reward depends mostly on which player would come up first with a superior product, and not so much on the actual date of the invention. This is the type of tournament games we mostly focus on in this paper: many players with the rewards depending only on the ranking of the times needed to complete a task, in which the progress is gradual rather than sudden.

We formulate the large population tournament as a mean field game—a stochastic game with infinitely many small players interacting through their aggregate distribution, introduced by Lasry and Lions (2006a, 2006b, 2007), and Huang, Caines and Malhamé (2007), Huang, Malhamé and Caines (2006). The advantage of the mean field formulation is that the equilibrium has an appealing decentralized structure: each player bases her decisions on her own state variable and a deterministic measure of completion time distribution that is obtained from the solution of a fixed-point problem. One can then use the mean field game solution to construct an approximate Nash equilibrium of the finite-player game. Apart from our application to the analysis of tournaments, mean field games have been applied to macroeconomics by Achdou et al. (2014), analysis of bank runs by Carmona, Delarue and Lacker (2017), systemic risk by Carmona, Fouque and Sun (2015) and analysis of queueing systems with strategic servers by Bayraktar, Budhiraja and Cohen (2019). We also refer to the works of Guéant, Lasry and Lions (2011), Bensoussan, Frehse and Yam (2013) and Carmona and Delarue (2018) for further reading.

What sets our set-up apart from the above is the rank-based feature of our problem: players are rewarded according to how their *completion times* are ranked. The completion time is the hitting time of the player's own diffusion process, whose drift she controls by costly effort. Mean field games with rank-based features are generally difficult to analyze due to the lack of a priori regularity of the rank function that depends on the equilibrium. Except for the terminal position ranking game of Bayraktar and Zhang (2016) and the one-stage Poisson game of Nutz and Zhang (2017), one can usually only hope for abstract existence under the weak formulation as in Carmona and Lacker (2015), Example 5.9. Hence, one of the *contributions* of our work is the construction of a solvable mean field model, which is rare outside the classical linear-quadratic framework. Moreover, even in the general setup without an explicit solution, we are still able to handle reward

functions that are discontinuous in the rank variable. A key ingredient to our analysis is recognizing that the optimally controlled state density is a distortion of the density of Brownian motion by a factor related to the Cole–Hopf transformation of the value function. Particle systems with rank-based interaction, but no strategic actions, have been widely studied in the literature; see, for example, Shkolnikov (2012), and the references therein. A recent paper by Nadtochiy and Shkolnikov (2019) considers particles interacting through hitting times.

In economics, the analysis of tournaments goes back to the seminal paper of Lazear and Rosen (1981), which has inspired many papers that followed, including Akerlof and Holden (2012), Balafoutas et al. (2017), Olszewski and Siegel (2017) and Fang, Noe and Strack (2018), among the more recent ones. Most of these works focus on finitely many players or static models. Recently, there has been active research on models in which the randomness is driven by Poisson arrivals, and the player's effort affects the probability of a breakthrough; see, for example, Bimpikis, Ehsani and Mostagir (2019), Halac, Kartik and Liu (2017) and Nutz and Zhang (2017). Those models are aimed more at applications to R&D with sudden innovation breakthroughs, whereas our model is more appropriate for the cases in which the progress is incremental.

In our model, when the players are homogeneous in their starting point and efficiency (i.e., cost of effort), and the reward function depends only on the ranks and the given deadline, we find that the equilibrium has a semi-explicit formula. This enables us to analyze some interesting comparative statics. In particular, we find that the aggregate welfare may be increasing in the cost of effort. This indicates, for example, that for an economy in which building a start-up is a relatively complex endeavor, the complexity may not be such a bad thing—if it was less complex, too many entrepreneurs may put in an inefficiently high effort. We also find that, when the total pie is sufficiently large, high inequality in the rewards has a demoralizing effect for many players, a phenomenon emphasized in Fang, Noe and Strack (2018). However, in our model, when the pie is sufficiently small, the higher prize inequality improves the welfare and the average effort. This is because then, in our benchmark case in which the total reward amount is fixed, a higher percentage of players give up, and it is worthwhile for the players who do not give up to exert higher effort. In the example of internet search engines (or social media sites, or computer operating systems) the total pie is large, but the rewards are uneven, suggesting that there is loss of efficiency in that many players get discouraged from applying effort. This logic lends some support to having government subsidies, for example, for R&D in renewable energy, if those subsidies make the rewards more even, assuming the profits to be made are substantial.

Having a semiexplicit equilibrium also allows us to tackle the problem of *mechanism design*, that is, designing the game rewards to optimize a social welfare criterion, an area that has not been widely explored in the mean field game literature due to the lack of tractability. Our works adds to literature of a principal

and many agents considered by Elie, Mastrolia and Possamaï (2019) and Nutz and Zhang (2017).

We also consider an extension of the benchmark model where the total reward, the "pie," may depend on the population completion rate, as in institutions in which the wealth created by production increases with the completion rate of individual tasks. It turns out that, with a completion rate-dependent pie, there may be multiple equilibria all of which can be characterized in a semiexplicit manner. Furthermore, by interpolating between a pure contribution game and a pure competition game, we are able to draw a *bifurcation diagram of the equilibria*.

While we do not have an explicit characterization of equilibrium when the players are heterogeneous or when the reward function is not purely rank-based, we prove its existence using Schauder or Brouwer's fixed-point theorems, and uniqueness under an additional monotonicity condition. We also show the stability of the fixed point, which enables us to approximate the mean field equilibrium by the solution to a finite dimensional system of nonlinear equations. Finally, we construct an approximate equilibrium for a game with a large finite number of players from the mean field equilibrium.

The rest of the paper is structured as follows: we present the model in Section 2; the results for homogeneous players, including comparative statics, optimal reward design and extension to the completion rate-dependent pie in Section 3; general existence, uniqueness, stability and the approximation to finite-player games in Section 4; and a numerical example with heterogeneous players in Section 5; some minor proofs are provided in the Appendix.

2. Model setup.
We consider a game with a large number of independent players, in which each player can exert effort to move her project forward, and is rewarded based on the time needed to complete the project and/or the ranking of that time relative to other players. The completion time is modeled as the first time that her project value or progress process reach a target level normalized to level zero. The tournament ends when a given deadline $T \in (0, \infty]$ is reached. The total amount of the reward, or the "pie," is fixed for now, although we extend in Section 3.5 some of our results to a pie that depends on the *terminal completion rate*, that is, the percentage of players who manage to meet the deadline. The reader can have in mind a population of technology firms competing to build a new app/website where the reward depends on the relative time of completing a superior product (as in the search engines example from the Introduction). More generally, our setup might include cases in which the players derive utility from their ranking relative to their peers, even if the ranking is not rewarded by a monetary payment.

The reward a player gets for finishing at time t with rank r is given by a bounded function $R(t, r) : \mathbb{R}_+ \times [0, 1] \mapsto \mathbb{R}$. We assume R is decreasing in both variables. Throughout the paper, increasing and decreasing are understood in the weak sense. When $T < \infty$, we also impose $R(t, r) \equiv R_\infty$ for all $t > T$ so that everyone who

fails to complete her project by the deadline receives the same minimum partici-
pation reward or incompletion penalty. Denote the set of reward functions by $\mathcal{R}$.
We start by assuming an infinite population of players, and later study how well it
approximates a finite population in the limit.

2.1. *A single player's problem.* The optimization problem of a single player,
in an infinite population of players, is given as follows. Denote by $\mu \in \mathcal{P}(\mathbb{R}_+)$ the
distribution of the completion times of the population, and by F_μ its cumulative
distribution function (c.d.f.). The rank of the single player who finishes at time t is
given by $F_\mu(t)$, as this is the fraction of players who finish before or at the same
time as her. In our game, μ will be nonatomic, hence whether the rank is measured
using $F_\mu(t)$ or $F_\mu(t-)$ does not matter.

Let $R_\mu(t) := R(t, F_\mu(t))$, which is then a bounded and decreasing function of t.
We assume the player is risk-neutral and has quadratic cost, and that her state pro-
cess X, representing her distance to completion, follows the stochastic differential
equation (SDE):

$$(2.1) \qquad dX_s = -a_s\, ds + \sigma\, dB_s,$$

where B is a Brownian motion. The player's effort process a is *admissible* if it
is nonnegative, progressively measurable with respect to the filtration of B (the
filtration can be larger; see Remark 2.1), and yields a unique strong solution of
(2.1) up to the first passage time to level zero, and if that time is nonatomic. The
nonatomic condition is for technical reasons, and can be removed if R_μ is lower
semicontinuous for all $\mu \in \mathcal{P}(\mathbb{R}_+)$, which will be the case in our examples. The
optimization problem of the single player has the value function

$$(2.2) \qquad v(t, x) = v(t, x; \mu, c) := \sup_a E\left[R_\mu(\tau) - \int_t^{\tau \wedge T} ca_s^2\, ds \,\middle|\, X_t = x \right],$$

where $c > 0$ is the cost parameter and $\tau = \inf\{s \geq t : X_s = 0\}$ is the completion
time.

While we assume no discounting for tractability, it can be interpreted as the case
in which the cost of effort and the reward increase exactly at the same rate at which
the discount factor decreases. For example, the salaries in a certain profession in-
crease over time, as does the cost of education and job searching. If those increases
are exactly offset by the players' discount factor, the value function is as above.
We can also consider the case in which the reward may decrease with the interval
in which completion occurs; see Remark 3.2.

For any $y \neq 0$, denote by τ_y° the first passage time of a Brownian motion to
level y. Its density is well known and given by (see, e.g., page 197 of Karatzas and
Shreve (1991))

$$(2.3) \qquad f_{\tau_y^\circ}(s) = \frac{|y|}{s\sqrt{2\pi s}} \exp\left(-\frac{y^2}{2s} \right), \quad s > 0.$$

Also introduce

$$u(t, x) = u(t, x; \mu, c)$$

(2.4)
$$:= E\left[\exp\left(\frac{R_\mu(t + \tau^\circ_{x/\sigma})}{2c\sigma^2}\right)\right] = E\left[\exp\left(\frac{R_\mu(t + \frac{x^2}{\sigma^2}\tau^\circ_1)}{2c\sigma^2}\right)\right].$$

We have the following result.

PROPOSITION 2.1. *The value function $v(t, x)$ of the single player's problem is given by*

$$v(t, x) = 2c\sigma^2 \ln(u(t, x)).$$

There exists an optimal (Markovian/feedback) action a^* that attains $v(t, x)$, given by

$$a^*(t, x) = -\frac{1}{2c}v_x(t, x)1_{\{x>0\}} = -\sigma^2\frac{u_x(t, x)}{u(t, x)}1_{\{x>0\}}.$$

The corresponding completion time $\tau^* = \tau^{*,t,x}$ is nonatomic and has probability density function (p.d.f.)

(2.5)
$$f_{\tau^*}(s) = \frac{u(s, 0)}{u(t, x)}f_{\tau^\circ_{x/\sigma}}(s - t), \quad s \geq t.$$

PROOF. Let u be defined by (2.4), $v := 2c\sigma^2 \ln u$ and $a^* := -v_x/(2c)1_{\{x>0\}} = -\sigma^2(u_x/u)1_{\{x>0\}}$.

Step 1. Show that v is a solution to the Hamilton–Jacobi–Bellman (HJB) equation:

(2.6)
$$v_t + \sup_a\left\{-av_x + \frac{1}{2}\sigma^2v_{xx} - ca^2\right\} = 0,$$

with boundary condition $v(t, 0) = R_\mu(t)$ for $0 \leq t \leq T$, and when $T < \infty$, also the terminal condition $v(T, x) = R_\infty = R_\mu(T+)$ for $x > 0$.

By the first-order condition, (2.6) is equivalent to

$$v_t + \frac{1}{4c}(v_x)^2 + \frac{1}{2}\sigma^2v_{xx} = 0,$$

where the supremum is attained by a^* pointwise. To show v is a solution to the above equation with the desired boundary and terminal conditions, it is equivalent to show its Cole–Hopf transformation $u = e^{(2c\sigma^2)^{-1}v}$ satisfies

(2.7)
$$u_t + \frac{1}{2}\sigma^2u_{xx} = 0,$$

with

$$u(t, 0) = \exp\left(\frac{R_\mu(t)}{2c\sigma^2}\right), \qquad u(T, x) = \exp\left(\frac{R_\infty}{2c\sigma^2}\right) \quad \text{if } T < \infty.$$

But this is immediate from the Feynman–Kac theorem.

Step 2. Show that the SDE (2.1) controlled by $a^* \geq 0$ has a unique strong solution $X^*(= X^{*,t,x})$ up to $\tau^*(= \tau^{*,t,x})$, the first passage time of X^* to level zero.

By the monotonicity and boundedness of R_μ, u is decreasing in both t and x, and

$$(2.8) \qquad \exp\left(\frac{R_\infty}{2c\sigma^2}\right) \leq u(t, x) \leq \exp\left(\frac{R(0, 0)}{2c\sigma^2}\right).$$

Since τ_1° has a smooth density, even if R_μ is not continuous, it is not hard to see that $u \in C^{1,2}(\mathbb{R}_+ \times (0, \infty))$. In fact, direct differentiation of (2.4) yields

$$u_x(t, x) = \frac{1}{x} E\left[\exp\left(\frac{R_\mu(t + \frac{x^2}{\sigma^2}\tau_1^\circ)}{2c\sigma^2}\right)\left(1 - \frac{1}{\tau_1^\circ}\right)\right],$$

and

$$u_{xx}(t, x) = \frac{1}{x^2} E\left[\exp\left(\frac{R_\mu(t + \frac{x^2}{\sigma^2}\tau_1^\circ)}{2c\sigma^2}\right)\left(\frac{1}{(\tau_1^\circ)^2} - \frac{3}{\tau_1^\circ}\right)\right].$$

It can be shown using the reflection principle that $1/\tau_1^\circ$ has finite moments; in particular, $E[1/\tau_1^\circ] = 1$ and $E[(1/\tau_1^\circ)^2] = 3$. So u_x and u_{xx} are bounded on any region away from $x = 0$. It follows that a^* is bounded, and Lipschitz continuous in x on any compact subset of $(0, \infty)$. Standard SDE theory implies the existence of a unique strong solution up to τ^*.

Step 3. Show that τ^* is nonatomic and has the desired p.d.f. (2.5).

We employ a change of probability measure argument. Let B° be a Brownian motion under some probability measure $\mathbb{P}^\circ$. Define $X_s := x + \sigma(B_s^\circ - B_t^\circ)$ and $\tau := \inf\{s \geq t : X_s = 0\}$. Let u be the function given by (2.4). Define

$$Z_s := \frac{u(s \wedge \tau, X_{s \wedge \tau})}{u(t, x)}.$$

We have $Z_t = 1$ and that the paths of Z are $\mathbb{P}^\circ$-a.s. continuous. To see the latter, observe that if $R_\mu(s)$ is continuous at $s = \tau(\omega)$, then $\lim_{s \to \tau(\omega)} Z_s = \exp(\frac{R_\mu(\tau(\omega))}{2c\sigma^2})/u(t, x) = Z_{\tau(\omega)}$. Since R_μ is decreasing, it has at most countably many points of discontinuity. Since τ is nonatomic under $\mathbb{P}^\circ$, with $\mathbb{P}^\circ$-probability one, $R_\mu(s)$ is continuous at $s = \tau$, and thus, $s \mapsto Z_s$ is continuous at $s = \tau$. Path continuity before time τ is trivial. By Itô's lemma,

$$dZ_s = \begin{cases} \dfrac{\sigma u_x(s, X_s)}{u(t, x)} dB_s^\circ & t \leq s < \tau, \\ 0 & s \geq \tau, \end{cases}$$

where the drift of Z prior to time τ is killed using (2.7). Since Z is bounded, it is a $\mathbb{P}^\circ$-martingale. Hence we can define a probability measure $\mathbb{P}$ via $d\mathbb{P}/d\mathbb{P}^\circ = Z_\infty = Z_\tau$. Since Z_τ is strictly positive, $\mathbb{P}$ is equivalent to $\mathbb{P}^\circ$.

Next, we rewrite the dynamics of Z as $dZ_s = Z_s \, dY_s$ where

$$dY_s = \begin{cases} \dfrac{\sigma u_x}{u}(s, X_s) \, dB_s^{\circ} & t \le s < \tau. \\ 0 & s \ge \tau. \end{cases}$$

We see that Z is the stochastic exponential of Y. (We could set $Z_s \equiv 1$ and $dY_s \equiv 0$ for $s < t$ if necessary.) The Girsanov theorem implies that

$$B_s := B_s^{\circ} - 1_{\{s \ge t\}} \int_t^{s \wedge \tau} \frac{\sigma u_x}{u}(r, X_r) \, dr$$

is a $\mathbb{P}$-Brownian motion. Replacing B° by B in the dynamics of X, we obtain

$$dX_s = \sigma^2 \frac{u_x}{u}(s, X_s) 1_{\{s < \tau\}} \, ds + \sigma \, dB_s = -a^*(s, X_s) \, ds + \sigma \, dB_s, \quad s \ge t.$$

That is, X has the same distribution under $\mathbb{P}$ as the process X^* under the candidate optimal effort a^*. Under $\mathbb{P}$, we have

$$\mathbb{P}(\tau \in ds) = \frac{u(s, 0)}{u(t, x)} \mathbb{P}^{\circ}(\tau \in ds), \quad s \ge t,$$

from which we conclude that τ (and hence τ^*) is nonatomic and has the desired p.d.f.

Step 4. Verify that v is indeed the value function and that a^* is indeed the optimal effort.

The verification argument is standard except for an extra localization step to take care of the potential singularity of v_x near $x = 0$. Specifically, fixing an arbitrary admissible control a and the associated state process X, we apply Itô's lemma to $v(s, X_s)$ from time t to $\tau_\varepsilon \wedge T$, where $\varepsilon > 0$ and $\tau_\varepsilon := \{s \ge t : X_s = \varepsilon\}$, and use the HJB equation (2.6). This leads to

$$(2.9) \qquad v(t, x) \ge E v(\tau_\varepsilon \wedge T, X_{\tau_\varepsilon \wedge T}) - E \int_t^{\tau_\varepsilon \wedge T} c a_s^2 \, ds.$$

Now, as $\varepsilon \searrow 0$, since τ_ε is increasing and bounded by $\inf\{s \ge t : x + \sigma(B_s - B_t) = 0\}$, it has a limit $\tau_0 \le \tau := \inf\{s \ge t : X_s = 0\}$. We claim that $\tau_0 = \tau$. Indeed, if $X_{\tau_0} > 0$, then by path continuity, $\varepsilon = X_{\tau_\varepsilon} > \delta$ for some $\delta > 0$, for all ε sufficiently small, which is a contradiction.

By monotone convergence theorem, we have that $E \int_t^{\tau_\varepsilon \wedge T} c a_s^2 \, ds$ converges to $E \int_t^{\tau \wedge T} c a_s^2 \, ds$ as $\varepsilon \searrow 0$. To obtain the limit of $E v(\tau_\varepsilon \wedge T, X_{\tau_\varepsilon \wedge T})$, note that $\tau_\varepsilon \wedge T \to \tau \wedge T$ and $X_{\tau_\varepsilon \wedge T} \to X_{\tau \wedge T}$. If $\tau > T$, then $X_{\tau \wedge T} = X_T > 0$, and $v(\tau_\varepsilon \wedge T, X_{\tau_\varepsilon \wedge T}) \to v(T, X_T) = R_\infty = R_\mu(\tau)$ by the continuity of v in $\mathbb{R}_+ \times (0, \infty)$. If $\tau \le T$, then $X_{\tau \wedge T} = X_\tau = 0$, and $v(\tau_\varepsilon \wedge T, X_{\tau_\varepsilon \wedge T}) \to v(\tau, 0) = R_\mu(\tau)$ if τ is a continuity point of R_μ. Since R_μ has at most countably many points of discontinuity (by monotonicity) and τ is nonatomic by admissibility, we have, after combining the two cases, that $v(\tau_\varepsilon \wedge T, X_{\tau_\varepsilon \wedge T})$ converges to $R_\mu(\tau)$ a.s. Bounded

convergence theorem then implies $Ev(\tau_\varepsilon \wedge T, X_{\tau_\varepsilon \wedge T}) \to ER_\mu(\tau)$. Note that if R_μ is lower semicontinuous, then we can directly obtain $\liminf_\varepsilon v(\tau_\varepsilon \wedge T, X_{\tau_\varepsilon \wedge T}) \geq R_\mu(\tau)$ without the nonatomic property of τ. So, letting $\varepsilon \searrow 0$ in (2.9) yields

$$v(t, x) \geq E\left[R_\mu(\tau) - \int_t^{\tau \wedge T} ca_s^2 \, ds\right].$$

Since a is an arbitrary admissible control, taking supremum over a leads to the conclusion that v dominates the value function.

Replacing a by a^* which is admissible by Steps 2 and 3, all inequalities above become equalities, which implies that

$$v(t, x) = E\left[R_\mu(\tau^*) - \int_t^{\tau^* \wedge T} c\big(a^*(s, X_s^*)\big)^2 \, ds\right]$$

is dominated by the value function. Putting everything together, we conclude the verification proof. $\square$

REMARK 2.1. From the the verification step of the proof of Proposition 2.1, it is not hard to see that the feedback control a^* remains optimal if admissible controls are progressively measurable with respect to a filtration larger than the one generated by the driving Brownian motion B, as long as B remains a Brownian motion for that filtration. Later we will use this fact in the construction of an approximate Nash equilibrium for the N-player game in which each player observes not only her private state, but also the states of her competitors.

Having computed the best response of a single player to a given distribution μ, we are now ready to study the Nash equilibrium of this infinite population game. Supposing every player uses the optimal feedback control $a^* = a^*(\cdot; \mu, c)$ given by Proposition 2.1, we obtain a new population completion time distribution. If this new distribution is equal to μ, then we say μ is an equilibrium (completion time distribution). We first consider a case in which equilibrium can be fully characterized in Section 3, and then discuss the general case in Section 4.

3. Homogeneous players. In this section, we assume a unit mass of homogeneous players with independent Brownian motions driving their state processes. By homogeneous, we mean the players all start playing at the same time $t = 0$ and the same distance $x_0 > 0$ from the goal, and they all have the same cost parameter c. Moreover, we specialize to the purely rank-based reward functions of the form

$$(3.1) \qquad R(t, r) = 1_{\{t \leq T\}} H(r) + 1_{\{t > T\}} R_\infty,$$

where $H \geq R_\infty$ is a bounded decreasing function. That is, the only time dependence is through the deadline T. We will show below that there always exists a unique equilibrium in semiexplicit form. The proof is based on considering equation (2.5) as a fixed-point equation for the p.d.f. of the population completion time.

3.1. *Explicit characterization and properties of the equilibrium.* Denote by T_r^μ the rth quantile of a completion time distribution μ, and by $N(x)$ the c.d.f. of the standard normal distribution. We have the following results.

THEOREM 3.1.   *Suppose that $R \in \mathcal{R}$ is of the form* (3.1).

(i) *For $T < \infty$, the unique equilibrium completion time distribution μ has a quantile function given by*

$$(3.2) \qquad T_r^\mu = F_{\tau_{x_0/\sigma}^\circ}^{-1}\left(\frac{1 - F_{\tau_{x_0/\sigma}^\circ}(T)}{1 - F_\mu(T)} \int_0^r \exp\left(\frac{R_\infty - H(z)}{2c\sigma^2}\right) dz\right),$$

$$r \in [0, F_\mu(T)],$$

where

$$(3.3) \qquad F_{\tau_{x_0/\sigma}^\circ}(t) = 2\left(1 - N\left(\frac{x_0}{\sigma\sqrt{t}}\right)\right),$$

and the equilibrium terminal completion rate $F_\mu(T) \in (0, 1)$ is the unique solution of

$$(3.4) \qquad F_{\tau_{x_0/\sigma}^\circ}(T) = \frac{1 - F_{\tau_{x_0/\sigma}^\circ}(T)}{1 - F_\mu(T)} \int_0^{F_\mu(T)} \exp\left(\frac{R_\infty - H(z)}{2c\sigma^2}\right) dz.$$

Moreover, the value of the game is given by

$$(3.5) \qquad V = v(0, x_0; \mu, c) = R_\infty + 2c\sigma^2 \ln\left(\frac{1 - F_{\tau_{x_0/\sigma}^\circ}(T)}{1 - F_\mu(T)}\right).$$

(ii) *For $T = \infty$, the unique equilibrium completion time distribution μ has a quantile function given by*

$$(3.6) \qquad T_r^\mu = F_{\tau_{x_0/\sigma}^\circ}^{-1}\left(\frac{\int_0^r \exp(-\frac{H(z)}{2c\sigma^2}) dz}{\int_0^1 \exp(-\frac{H(z)}{2c\sigma^2}) dz}\right).$$

Moreover, the value of the game is given by

$$(3.7) \qquad V_\infty = v(0, x_0; \mu, c) = -2c\sigma^2 \ln\left(\int_0^1 \exp\left(-\frac{H(z)}{2c\sigma^2}\right) dz\right).$$

PROOF.   (i) Suppose $T < \infty$. By (2.5), the fixed-point equation (in terms of the p.d.f. of the completion time distribution) is

$$f_\mu(t) = \frac{u(t, 0; \mu, c)}{u(0, x_0; \mu, c)} f_{\tau_{x_0/\sigma}^\circ}(t), \quad t \in [0, T].$$

Denote $y(r) = F_{\tau^\circ_{x_0/\sigma}}(T_r^\mu)$. Since any fixed point μ has positive density on $[0, T]$, $y(r)$ is differentiable on $[0, F_\mu(T)]$ with

$$
\begin{aligned}
y'(r) &= \frac{f_{\tau^\circ_{x_0/\sigma}}(T_r^\mu)}{f_\mu(T_r^\mu)} = \frac{u(0, x_0; \mu, c)}{u(T_r^\mu, 0; \mu, c)} \\
&= \frac{\int_0^T \exp(\frac{H(F_\mu(s))}{2c\sigma^2})\, dF_{\tau^\circ_{x_0/\sigma}}(s) + \exp(\frac{R_\infty}{2c\sigma^2})(1 - F_{\tau^\circ_{x_0/\sigma}}(T))}{\exp(\frac{H(F_\mu(T_r^\mu))}{2c\sigma^2})} \\
&= \frac{\int_0^{F_\mu(T)} \exp(\frac{H(z)}{2c\sigma^2})y'(z)\, dz + \exp(\frac{R_\infty}{2c\sigma^2})(1 - F_{\tau^\circ_{x_0/\sigma}}(T))}{\exp(\frac{H(r)}{2c\sigma^2})}.
\end{aligned}
$$

Observe that $\exp(\frac{H(r)}{2c\sigma^2})y'(r)$ is independent of r, hence is constant for $r \in [0, F_\mu(T)]$. Let C be this constant. We have

$$
C = C F_\mu(T) + \exp\left(\frac{R_\infty}{2c\sigma^2}\right)(1 - F_{\tau^\circ_{x_0/\sigma}}(T)).
$$

Since $F_{\tau^\circ_{x_0/\sigma}}(T) < 1$, the above equation implies $F_\mu(T) < 1$ and

$$
\exp\left(\frac{H(r)}{2c\sigma^2}\right)y'(r) \equiv C = \frac{1 - F_{\tau^\circ_{x_0/\sigma}}(T)}{1 - F_\mu(T)}\exp\left(\frac{R_\infty}{2c\sigma^2}\right), \quad r \in [0, F_\mu(T)].
$$

It follows that

$$
y(r) = \frac{1 - F_{\tau^\circ_{x_0/\sigma}}(T)}{1 - F_\mu(T)}\int_0^r \exp\left(\frac{R_\infty - H(z)}{2c\sigma^2}\right) dz, \quad r \in [0, F_\mu(T)].
$$

Setting $r = F_\mu(T)$ yields equation (3.4) for $F_\mu(T)$. The existence and uniqueness of solution follows from the fact that

$$
\tag{3.8} \phi(r) := \frac{1}{1 - r}\int_0^r \exp\left(\frac{R_\infty - H(z)}{2c\sigma^2}\right) dz
$$

is a continuous, strictly increasing function on $[0, 1)$ satisfying $\phi(0) = 0$ and $\lim_{r \to 1}\phi(r) = \infty$. Finally, we have $v(0, x_0; \mu, c) = 2c\sigma^2 \ln u(0, x_0; \mu, c)$ and

$$
\begin{aligned}
u(0, x_0; \mu, c) &= C F_\mu(T) + \exp\left(\frac{R_\infty}{2c\sigma^2}\right)(1 - F_{\tau^\circ_{x_0/\sigma}}(T)) \\
&= \frac{1 - F_{\tau^\circ_{x_0/\sigma}}(T)}{1 - F_\mu(T)}\exp\left(\frac{R_\infty}{2c\sigma^2}\right)
\end{aligned}
$$

from which (3.5) follows.

(ii) Suppose $T = \infty$. First, note that any possible fixed-point μ should have strictly increasing c.d.f. which satisfies $F_\mu(\infty) = 1$ since the effort is nonnegative

and the first passage time of a Brownian motion is almost surely finite. Let $y(r)$ be defined as above. A similar calculation shows that

$$\exp\left(\frac{H(r)}{2c\sigma^2}\right)y'(r) = \int_0^1 \exp\left(\frac{H(z)}{2c\sigma^2}\right)y'(z)\,dz$$

is constant (denoted by C) and

$$F_{\tau_{x_0/\sigma}^\circ}(T_r^\mu) = y(r) = C\int_0^r \exp\left(-\frac{H(z)}{2c\sigma^2}\right)dz.$$

We can then use $y(1) = 1$ to find that $C = (\int_0^1 \exp(-\frac{H(z)}{2c\sigma^2})\,dz)^{-1}$. Having obtained $y(r)$, we have $T_r^\mu = F_{\tau_{x_0/\sigma}^\circ}^{-1}(y(r))$. The equilibrium value is $v(0, x_0; \mu, c) = 2c\sigma^2 \ln u(0, x_0; \mu, c) = 2c\sigma^2 \ln C$. $\square$

REMARK 3.1. Observe that the finite horizon equilibrium converges to the infinite horizon equilibrium as $T \to \infty$. To see this, use (3.4) to rewrite the finite horizon equilibrium quantile function and value of the game as

$$T_r^\mu(T) = F_{\tau_{x_0/\sigma}^\circ}^{-1}\left(\frac{\int_0^r \exp(\frac{R_\infty - H(z)}{2c\sigma^2})\,dz}{\int_0^{F_\mu(T)} \exp(\frac{R_\infty - H(z)}{2c\sigma^2})\,dz} F_{\tau_{x_0/\sigma}^\circ}(T)\right), \quad r \in [0, F_\mu(T)]$$

and

$$V(T) = R_\infty + 2c\sigma^2 \ln\left(\frac{F_{\tau_{x_0/\sigma}^\circ}(T)}{\int_0^{F_\mu(T)} \exp(\frac{R_\infty - H(z)}{2c\sigma^2})\,dz}\right)$$

$$= 2c\sigma^2 \ln\left(\frac{F_{\tau_{x_0/\sigma}^\circ}(T)}{\int_0^{F_\mu(T)} \exp(\frac{-H(z)}{2c\sigma^2})\,dz}\right).$$

As $T \to \infty$, $F_{\tau_{x_0/\sigma}^\circ}(T)/(1 - F_{\tau_{x_0/\sigma}^\circ}(T)) \to \infty$. For (3.4) to hold, we must have $\lim_{T \to \infty} F_\mu(T) = 1$. It follows that the finite horizon equilibrium $T_r^\mu(T)$ and $V(T)$ converge to their infinite horizon counterparts as $T \to \infty$, where the convergence is pointwise for the quantile function.

REMARK 3.2. More generally, if $R \in \mathcal{R}$ is of the form

$$R(t, r) = \sum_{k=1}^n \mathbf{1}_{\{T_{k-1} < t \leq T_k\}} \delta_k H(r) + \mathbf{1}_{\{t > T\}} R_\infty,$$

where $0 = T_0 < T_1 < \cdots < T_n = T < \infty$, and δ_k is the discounting factor on $[T_{k-1}, T_k]$, then with $\alpha_k := F_{\tau_{x_0/\sigma}^\circ}(T_k)$ and $\beta_k := F_\mu(T_k)$, the equilibrium quantile function is given by

$$T_r^\mu = F_{\tau_{x_0/\sigma}^\circ}^{-1}\left(\frac{1 - \alpha_n}{1 - \beta_n} \sum_{k=1}^n \int_{\beta_{k-1} \wedge r}^{\beta_k \wedge r} \exp\left(\frac{R_\infty - \delta_k H(z)}{2c\sigma^2}\right)dz\right), \quad r \in [0, \beta_n],$$

where $\beta_1, \ldots, \beta_n$ can be found by solving the following nonlinear system of equations:

$$\alpha_k - \alpha_{k-1} = \frac{1-\alpha_n}{1-\beta_n} \int_{\beta_{k-1}}^{\beta_k} \exp\left(\frac{R_\infty - \delta_k H(z)}{2c\sigma^2}\right) dz, \quad k = 1, \ldots, n.$$

The game value in this case is $V = R_\infty + 2c\sigma^2 \ln(\frac{1-\alpha_n}{1-\beta_n})$. Similarly, a semi-explicit formula can be derived for the case $T = \infty$.

The semiexplicit solution allows us to obtain the following comparative statics analytically. The proof is based on elementary calculus and can be found in the longer ArXiv version arXiv:1811.00076.

PROPOSITION 3.1. *Suppose that $R \in \mathcal{R}$ is of the form* (3.1) *with H continuous. Then the terminal equilibrium completion rate $\beta := F_\mu(T)$ and the game value V (written as V_∞ when $T = \infty$) have the following properties*:

(i) *When $T < \infty$, β is increasing in T and decreasing in x_0 and c, where the monotonicity in T and x_0 are strict. Moreover,*

$$\lim_{T \to 0} \beta = 0, \qquad \lim_{T \to \infty} \beta = 1, \qquad \lim_{x_0 \to 0} \beta = 1, \qquad \lim_{x_0 \to \infty} \beta = 0,$$

$$\lim_{c \to 0} \beta = 1, \qquad \lim_{c \to \infty} \beta = F_{\tau_{x_0/\sigma}^\circ}(T).$$

(ii) *When $T < \infty$, V is increasing in T and decreasing in x_0. Moreover,*

$$\lim_{T \to 0} V = R_\infty, \qquad \lim_{T \to \infty} V = V_\infty, \qquad \lim_{x_0 \to 0} V = V_\infty, \qquad \lim_{x_0 \to \infty} V = R_\infty.$$

(iii) *V_∞ is independent of x_0, and increasing in c with*

$$\lim_{c \to 0} V_\infty = H(1-) = \lim_{r \nearrow 1} H(r), \qquad \lim_{c \to \infty} V_\infty = \int_0^1 H(r)\, dr.$$

(iv) *If $H_1 \geq H_2$, then $V_\infty(H_1) \geq V_\infty(H_2)$. When $T < \infty$ and $R_i = 1_{\{t \leq T\}} H_i + 1_{\{t > T\}} R_{i,\infty}$, if $H_1 - R_{1,\infty} \geq H_2 - R_{2,\infty}$, then $\beta(R_1) \geq \beta(R_2)$ and $V(R_1) - R_{1,\infty} \geq V(R_2) - R_{2,\infty}$.*

The results in items (i), (ii) and (iv) are intuitive and not surprising. Somewhat surprising, at first sight, might be the fact stated in (iii) that the value V_∞ of the infinite horizon game is increasing in the cost parameter c. Let us consider the limiting cases. When the cost goes to zero, all the players will apply very high effort and the value will converge to the value $H(1)$ (if H is left continuous at 1) of the lowest ranked player. On the other hand, when the cost goes to infinity, they will apply very small effort and the project values be driven by pure noise, which results in an aggregate gain of $\int_0^\infty H(F_\mu(t))\, dF_\mu(t) = \int_0^1 H(r)\, dr$, where the equilibrium

measure μ is distributed as $\tau^{\circ}_{x_0/\sigma}$ in the limit. Since H is decreasing, the averaged value is higher than $H(1)$.

In effect, low cost incentivizes players to apply too much effort in competing with each other, without resulting in good ranking. It is a rat race without winners. Define the (total) welfare as the aggregate game value of all players. When the population is homogeneous and have unit mass, the terms "welfare" and "game value" are used as synonyms. For the aggregate welfare, it does not matter which of the players finish early, and competing too hard with each other to reach the goal sooner decreases the welfare. Thus, in this game it is beneficial if the players are discouraged from working too hard by having a high cost of effort. For example, if building a start-up was too easy, too many entrepreneurs may apply too high effort. However, with a finite deadline, as we will see in the next subsection, the effect of the cost on the value may be increasing or decreasing. Moreover, if the pie was not fixed, but for example, if it grew with the population completion rate and the speed of completion, then lower cost may lead to higher value.

We end the theoretical analysis with a result on the expected total effort, needed later below for further comparative statics.

PROPOSITION 3.2. *Let $T < \infty$. Suppose that $R \in \mathcal{R}$ is of the form* (3.1). *Let μ be the unique equilibrium completion time distribution given by Theorem* 3.1(*i*), *and $a^*_t := a^*(t, X_t; \mu, c)$, X, τ be the associated equilibrium action, state process and completion time, respectively. Then the expected total effort in equilibrium is given by*

$$(3.9) \qquad E \int_0^{\tau \wedge T} a^*_t \, dt = \frac{x_0(F_\mu(T) - F_{\tau^{\circ}_{x_0/\sigma}}(T))}{1 - F_{\tau^{\circ}_{x_0/\sigma}}(T)}.$$

PROOF. From $X_{\tau \wedge T} = x_0 - \int_0^{\tau \wedge T} a^*_t \, dt + \sigma B_{\tau \wedge T}$, we get $E \int_0^{\tau \wedge T} a^*_t \, dt = x_0 - EX_{\tau \wedge T}$. To compute $EX_{\tau \wedge T}$, we make use of the change of measure introduced in the proof of Proposition 2.1. Let $\mathbb{P}^{\circ}$ and Z_∞ as in step 3 of the proof of Proposition 2.1 with $(t, x) = (0, x_0)$. We have

$$EX_{\tau \wedge T} = E[1_{\{\tau > T\}} X_T] = E^{\mathbb{P}^{\circ}}[Z_\infty 1_{\{\tau > T\}} X_T]$$

$$= E^{\mathbb{P}^{\circ}}\left[\frac{u(\tau, X_\tau; \mu, c)}{u(0, x_0; \mu, c)} 1_{\{\tau > T\}} X_T\right] = E^{\mathbb{P}^{\circ}}\left[\frac{\exp(\frac{R_\infty}{2c\sigma^2})}{u(0, x_0; \mu, c)} 1_{\{\tau > T\}} X_T\right]$$

$$= \frac{\exp(\frac{R_\infty}{2c\sigma^2})}{u(0, x_0; \mu, c)} E^{\mathbb{P}^{\circ}} X_{\tau \wedge T}.$$

Since X is a martingale under $\mathbb{P}^{\circ}$, we have $E^{\mathbb{P}^{\circ}} X_{\tau \wedge T} = x_0$. By Theorem 3.1(i),

$$u(0, x_0; \mu, c) = \exp\left(\frac{v(0, x_0; \mu, c)}{2c\sigma^2}\right) = \frac{1 - F_{\tau^{\circ}_{x_0/\sigma}}(T)}{1 - F_\mu(T)} \exp\left(\frac{R_\infty}{2c\sigma^2}\right).$$

Putting all pieces together, we get (3.9). $\square$

We see that the expected total effort increases with the completion rate $F_\mu(T)$ when x_0, σ, T are fixed. Since $F_\mu(T)$ decreases with c, so does the expected total effort.

3.2. *The finite deadline*: *Numerical comparative statics*. We now perform a numerical study of the equilibrium in games with a finite deadline. The benchmark choice of model inputs are $T = 1$, $\sigma = 0.25$, $x_0 = 1$, $c = 1$ and $R_\infty = 0$. Whenever we vary one parameter, we keep the other parameters fixed.

Figures 1 and 2 illustrate some features of the equilibrium with a smooth reward function $H(r) = 6(1-r)^2$, and a step reward function $H(r) = 5 \cdot 1_{[0,0.25)}(r) + 2 \cdot 1_{[0.25,0.5)}(r) + 1_{[0.5,1]}(r)$, respectively. We can see the following:
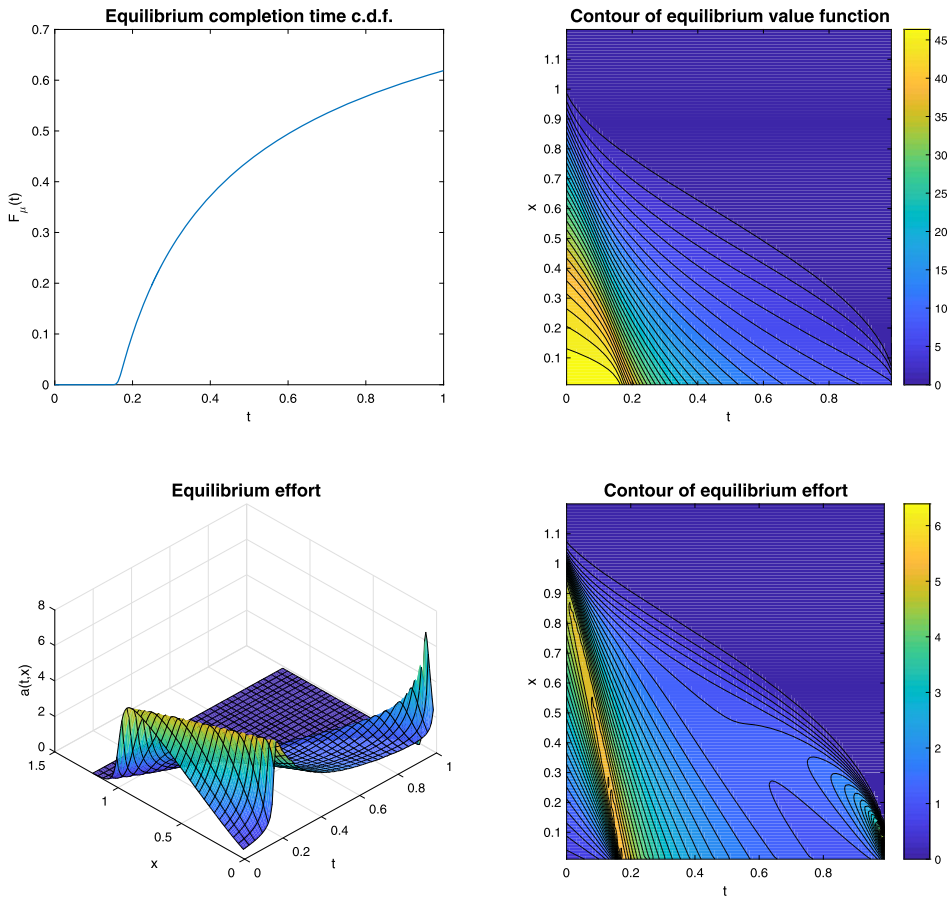

FIG. 1. *Equilibrium completion time distribution*, *value function and effort function with* $H(r) = 6(1-r)^2$.
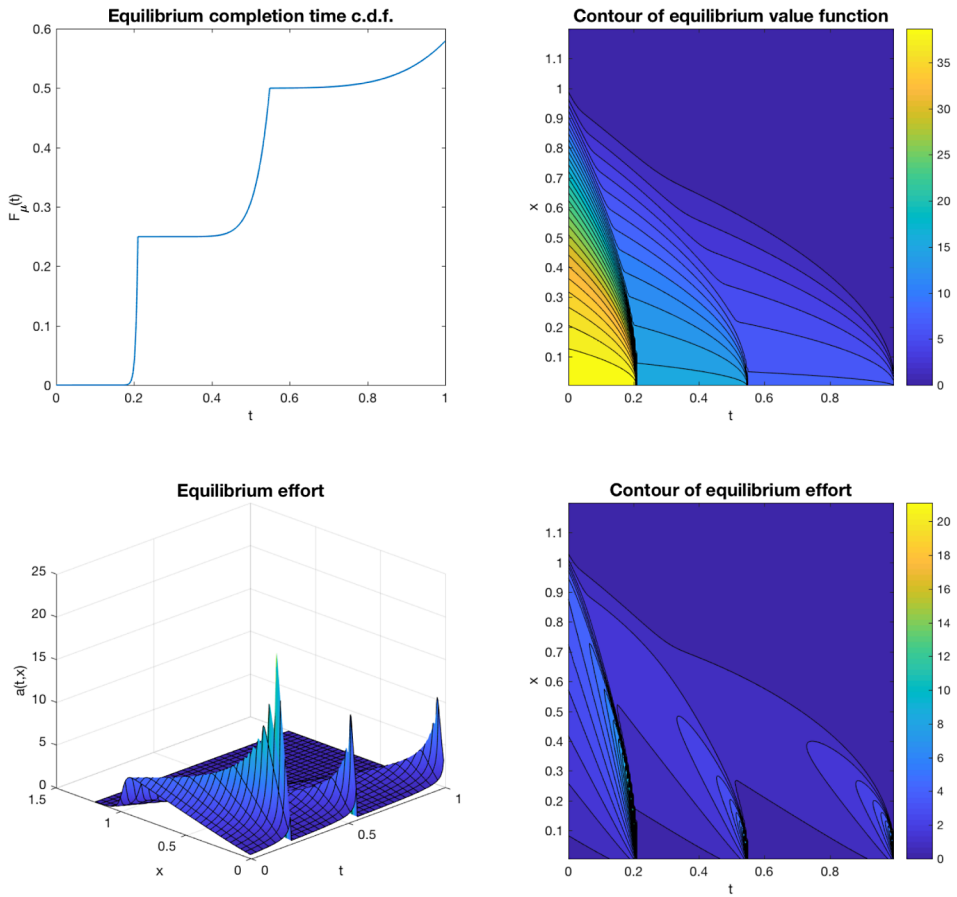
FIG. 2. *Equilibrium completion time distribution, value function and effort function with* $H(r) = 5 \cdot 1_{[0,0.25)}(r) + 2 \cdot 1_{[0.25,0.5)}(r) + 1_{[0.5,1]}(r)$.

- For a large subset of time-location pairs (t, x), the effort is low. The effort is high initially when there are a lot of players around the same level of progress, and close to the cutoff dates that distinguish between different rank rewards (the only such cutoff date being the deadline in the case of a smooth reward). Thus in Figure 2, there is, for example, a lot of completion in the short interval before F_μ hits 0.25 because 0.25 is the cutoff point for a higher reward. Once that point is crossed, the players apply low effort until getting close to the next cutoff point, so that the completion rate increases very slowly in the period between a cutoff point and close to the next cutoff point.

- Moreover, the effort is very low (but not zero) when it is hard to complete the task due to distant location or high cost, or when the reward too small. In those cases (figures not shown), we find that the players maintain a low effort, relying

TABLE 1
Equilibrium quartiles, completion rates and game values under varying deadline

T	1st quartile	median	3rd quartile	$F_\mu(T)$	V
0.5	0.281	–	–	44.9%	0.074
1	0.285	0.613	–	61.9%	0.121
2	0.289	0.630	–	73.6%	0.166
5	0.293	0.649	2.424	83.4%	0.215
10	0.295	0.658	2.545	88.1%	0.237
100	0.296	0.666	2.661	96.0%	0.256
∞	0.296	0.667	2.667	100%	0.257

on randomness of the project to bring them closer to completion, which happens with low probability.

- The c.d.f of the completion time for early times is close to, but not exactly equal to zero, because some players finish early by sheer randomness, even if they apply low effort. After those, there is a whole bunch of players who apply the optimal strategy and, when the reward is piecewise constant, finish about the same time. This is where the first sudden increase in the c.d.f shows up.

In the remaining part of this section, we fix $H(r) = 6(1 - r)^2$ except in Section 3.2.2 where we analyze the dependence on H.

3.2.1. *Dependence on the tournament horizon.* Table 1 shows that as the deadline increases, both the equilibrium terminal completion rate and the game value increase (as proved in Proposition 3.1). However, with a longer deadline it takes longer to reach a fixed percentage of completion. On the other hand, we also see that the quantile functions are not too sensitive to changes in the deadline, except near the discontinuity $r = F_\mu(T)$. This suggests that competition alone is often good enough to drive the progress; the external deadline provides a little extra, but not significant motivation.

3.2.2. *Dependence on the reward function.* Next, we focus on the two-parameter family:

$$H(r) = K(1 + p)(1 - r)^p,$$

where $K = \int_0^1 H(r)\, dr$ represents the total reward budget, and p determines the convexity of the reward function. A large p means that most of the reward is given to highly ranked players. In many examples, including population income in US, the prize money decreases in a very convex manner, with high "earners" earning a very large chunk of the pie.

We see from the top left panel of Figure 3 that, with a moderate value $K = 2$, as p increases, the peak of $f_\mu = F'_\mu$ shifts to the left, meaning most of the
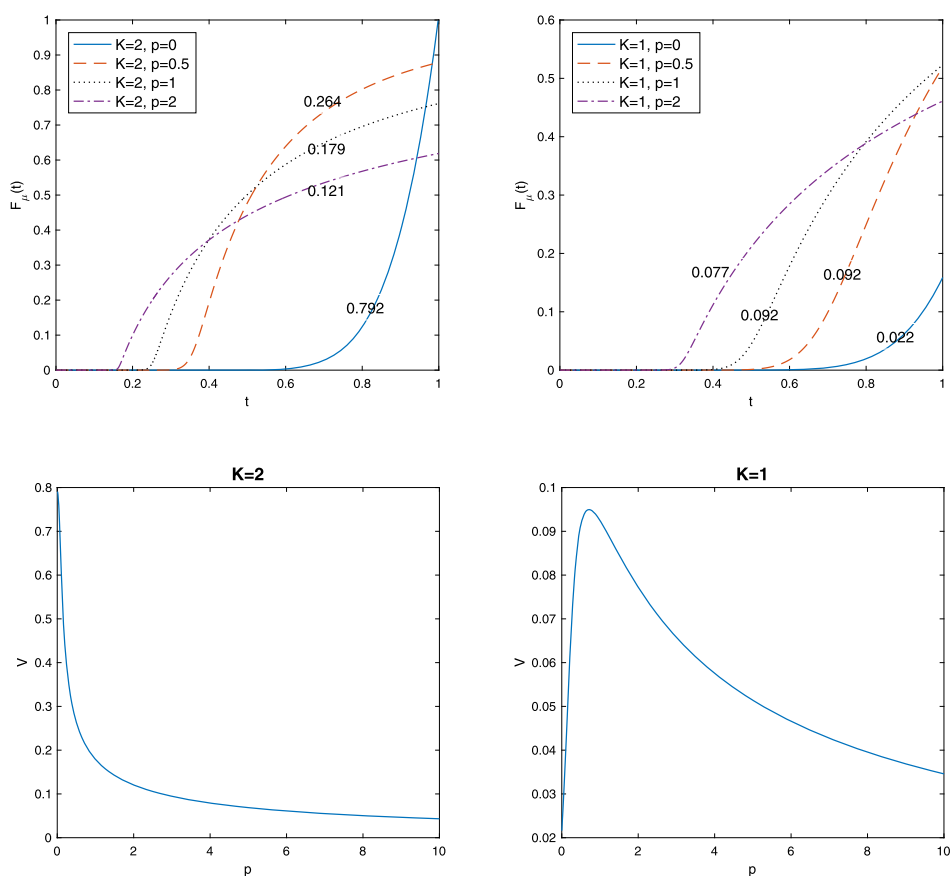
FIG. 3. *Equilibrium completion time distribution and game value under different total reward K and convexity index p. The numbers on the curves in the top panels represent their corresponding game values.*

players finish earlier. On the other hand, this is at the expense of a lower population completion rate, since the laggards, knowing that the reward drops quickly once the leaders have occupied the high ranks, put in less effort and give up more easily. The lower completion rate also leads to a lower game value; see the bottom left panel of Figure 3. Thus, when the pie is not too small, the higher the convexity of the reward function, the lower the welfare. That is, shifting the rewards more to the highly ranked players decreases the welfare. However, the monotonicity of the welfare in p is only true when there is sufficient benefit for finishing early. The top right panel of Figure 3 shows that when the reward K is small (similar behavior can be observed when x_0 or c is large), both the population terminal completion rate and the game value are no longer decreasing in p. A more complete picture is

shown in the bottom right panel of Figure 3. We only plot the game value, since it moves in the same direction as the completion rate.

We focus now on the following implications of our results: since the expected total effort increases with the (terminal) completion rate when we fix x_0, σ, T (see Proposition 3.2), the expected effort will be lower as we increase p in the top left panel of Figure 3; that is, the expected total effort decreases with the level of competition (more unequal rewards). Thus, when the total pie is large enough, the competitiveness resulting from inequality in rewards has a demoralizing effect, as in Fang, Noe and Strack (2018). However, the top right panel of Figure 3 shows that when the total pie K is small, the completion rate, and hence the effort level may go up with more unequal payment. This is because there is another effect that decreases competitiveness—a higher percentage of players gives up, and if the rewards are more uneven it is still worthwhile for the players who do not give up to exert higher effort, bringing up the aggregate completion rate. That is, the players who do not give up are less discouraged by the prize inequality, because they compete within a smaller group. If we consider, for example, the competition of building internet search engines, or means of renewable energy as a large tournament race, since the total pie is large and the rewards are uneven, our results suggest that government subsidies for R&D that make the rewards more even among different players may be efficiency improving, by making the players less discouraged.

3.2.3. *Dependence on the cost parameter.*   Figure 4 illustrates the dependence on the cost parameter c. As c increases, the welfare value first experiences an increase before it starts to decrease. The intuition is the same as in the case of $T = \infty$ (in which case the value is always increasing in cost): the aggregate welfare does not depend on the relative ranks of the players, while, with low cost, the
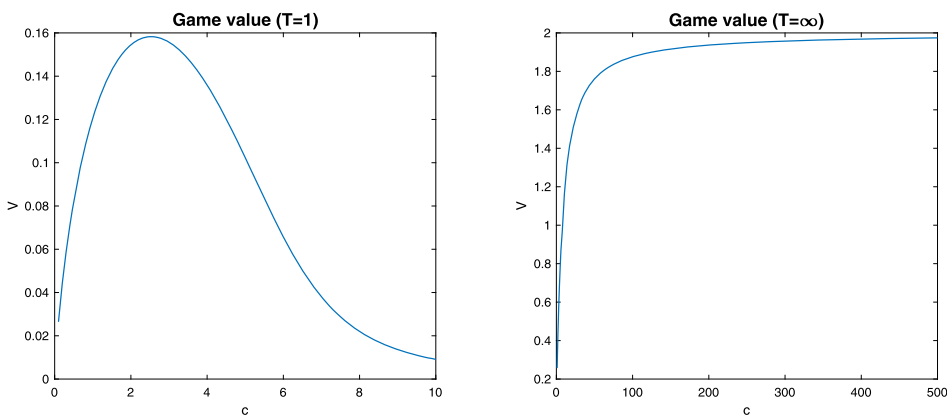

FIG. 4.   *Game value against the cost parameter.*

players compete for those ranks "too hard" against each other, raising the realized cost of effort and bringing down the welfare value. Thus, higher cost c can be beneficial, by discouraging the players to work too hard. However, as c continues to increase, the possibility of failing to finish by the deadline starts to offset the benefit of a reduced effort, and the value starts to decrease. We note that with Poisson type uncertainty (sudden breakthroughs), in the infinite horizon model of Nutz and Zhang (2017), the game value is independent of c.

3.3. *Reverse engineering*: *Realizing a target distribution.*   Theorem 3.1 allows us to obtain, for a given reward function, the corresponding equilibrium distribution. The opposite problem is also important: if we want to achieve a certain equilibrium distribution, how do we go about it? The following two theorems, for $T = \infty$ and $T < \infty$ respectively, do the following:

   (i) identify which distributions μ are realized in equilibrium and by which reward function;
   (ii) identify conditions under which a distribution μ can be realized by an expected budget not higher than a given budget K.

These results are also helpful in allowing us to convert an optimization over reward functions to an equivalent one over the set of feasible equilibrium distributions. The latter problem is easier in some cases; see Section 3.4.3 for an example.

Let $\mathcal{H}_0$ be the set of bounded, decreasing reward functions from $[0, 1]$ to $[R_\infty, \infty)$, and

$$\mathcal{E} : \mathcal{H}_0 \to \mathcal{P}(\mathbb{R}_+) \quad \text{for } T = \infty,$$

$$\mathcal{E}_T : \mathcal{H}_0 \to \mathcal{P}(\mathbb{T}_0) \quad \text{for } T < \infty,$$

be the mappings from H to the equilibrium completion time distribution μ, where $\mathbb{T}_0 = [0, T] \cup \{\infty\}$. When $T = \infty$, the purpose of introducing R_∞ is to facilitate the study of optimal reward design in Section 3.4, where a lower bound on the reward function is necessary for the design problem to be well-posed. Observe that $\mathcal{E}$ is translation invariant, that is, $\mathcal{E}(H + C) = \mathcal{E}(H)$ for any $C \in \mathbb{R}$ such that $H + C \geq R_\infty$. We first show that $\mathcal{E}$ and $\mathcal{E}_T$ are one-to-one mappings up to a.e. equivalence, and in the infinite horizon case, also up to an additive constant.

LEMMA 3.1.   *Let* $H_1, H_2 \in \mathcal{H}_0$.

   (i) *Suppose* $\mathcal{E}(H_1) = \mathcal{E}(H_2) = \mu$, *then* $H_1 = H_2 + C$ *a.e. for some constant* C.
   (ii) *Suppose* $\mathcal{E}_T(H_1) = \mathcal{E}_T(H_2) = \mu$, *then* $H_1 = H_2$ *a.e. on* $[0, F_\mu(T)]$.

PROOF.   We only prove (ii); case (i) is similar. By (3.2),

$$F_{\tau^\circ_{x_0/\sigma}}(t) = \frac{1 - F_{\tau^\circ_{x_0/\sigma}}(T)}{1 - F_\mu(T)} \int_0^{F_\mu(t)} \exp\left(\frac{R_\infty - H_i(z)}{2c\sigma^2}\right) dz, \quad t \in [0, T], i = 1, 2.$$

Differentiating both sides and rearranging terms yields

$$H_i(F_\mu(t)) = 2c\sigma^2 \ln\left(\frac{f_\mu(t)}{f_{\tau^\circ_{x_0/\sigma}}(t)}\right) + 2c\sigma^2 \ln\left(\frac{1 - F_{\tau^\circ_{x_0/\sigma}}(T)}{1 - F_\mu(T)}\right) + R_\infty \quad \text{a.e.}$$

It follows that

$$\int_0^{F_\mu(T)} |H_1(r) - H_2(r)| \, dr = \int_0^T |H_1(F_\mu(t)) - H_2(F_\mu(t))| f_\mu(t) \, dt = 0,$$

and consequently, $H_1 = H_2$ a.e. on $[0, F_\mu(T)]$. $\square$

Here and in the sequel, we identify a reward function H with its equivalence class under a.e. relation.

Denote by $\mathcal{P}^+(\mathbb{R}_+)$ (resp., $\mathcal{P}^+(\mathbb{T}_0)$) the set of probability distributions on $\mathbb{R}_+$ (resp., $\mathbb{T}_0$) that have strictly positive density on $\mathbb{R}_+$ (resp., $[0, T]$). The key quantity is the normalized density

$$\zeta_\mu := f_\mu / f_{\tau^\circ_{x_0/\sigma}},$$

and if $T < \infty$, also the normalized incompletion rate

$$\delta_\mu := \frac{1 - F_\mu(T)}{1 - F_{\tau^\circ_{x_0/\sigma}}(T)}.$$

THEOREM 3.2. *Fix $K \geq R_\infty$. For $\mu \in \mathcal{P}^+(\mathbb{R}_+)$, define*

$$H_\mu(r) := 2c\sigma^2 \ln \zeta_\mu(F_\mu^{-1}(r)).$$

We have:

(i) $\mu \in \mathcal{E}(\mathcal{H}_0)$ *if and only if* $\mu \in \mathcal{P}^+(\mathbb{R}_+)$ *and* $\ln \zeta_\mu$ *is bounded and decreasing. Moreover*,

$$\mathcal{E}^{-1}(\mu) = \{H_\mu + C : C \geq R_\infty - 2c\sigma^2 \ln(\inf \zeta_\mu)\}, \quad \mu \in \mathcal{E}(\mathcal{H}_0).$$

(ii) *For* $\mu \in \mathcal{E}(\mathcal{H}_0)$, $\int_0^1 (H_\mu(r) + C) \, dr \leq K$ *if and only if* $C \leq K - 2c\sigma^2 \times \int_0^\infty \ln \zeta_\mu(t) \, d\mu(t)$.

PROOF. (i) Necessity holds by Proposition 2.1. For sufficiency, given any $\mu \in \mathcal{P}^+(\mathbb{R}_+)$ such that $\ln \zeta_\mu$ is bounded and decreasing, H_μ is also bounded and decreasing. Since $F_\mu^{-1} : [0, 1] \to \mathbb{R}_+$ is bijective, $H_\mu + C \geq R_\infty$ if and only if $C \geq R_\infty - 2c\sigma^2 \ln(\inf \zeta_\mu)$. It is straightforward to check that μ satisfies (3.6) with $H = H_\mu$. Hence, $\mathcal{E}(H_\mu + C) = \mathcal{E}(H_\mu) = \mu$ by the translation invariance of $\mathcal{E}$. By Lemma 3.1(i), any admissible reward scheme realizing μ differs from H_μ by a constant. (ii) follows from straightforward calculation. $\square$

THEOREM 3.3.   *Fix $T < \infty$ and $K \geq R_\infty$. For $\mu \in \mathcal{P}^+(\mathbb{T}_0)$, define*

$$\widetilde{H}_\mu(r) := R_\infty + 2c\sigma^2 1_{[0, F_\mu(T)]}(r)\{\ln \zeta_\mu(F_\mu^{-1}(r)) - \ln \delta_\mu\}.$$

We have:

(i) $\mu \in \mathcal{E}_T(\mathcal{H}_0)$ *if and only if $\mu \in \mathcal{P}^+(\mathbb{T}_0)$, $\ln \zeta_\mu$ is bounded and decreasing, and $\inf_{t \in [0, T]} \zeta_\mu(t) \geq \delta_\mu$. Moreover,*

$$\mathcal{E}_T^{-1}(\mu) = \{H \in \mathcal{H}_0 : H|_{[0, F_\mu(T)]} = \widetilde{H}_\mu|_{[0, F_\mu(T)]}\}, \quad \mu \in \mathcal{E}_T(\mathcal{H}_0).$$

(ii) *For $\mu \in \mathcal{E}_T(\mathcal{H}_0)$, $\int_0^1 \widetilde{H}_\mu(r)\, dr = \int_0^{F_\mu(T)} \widetilde{H}_\mu(r)\, dr + (1 - F_\mu(T))R_\infty \leq K$ if and only if*

$$\int_0^T \ln \zeta_\mu(t)\, d\mu(t) \leq \frac{K - R_\infty}{2c\sigma^2} + F_\mu(T) \ln \delta_\mu.$$

PROOF.   (i) Let $\mu \in \mathcal{E}_T(\mathcal{H}_0)$. By Proposition 2.1, $\mu \in \mathcal{P}^+(\mathbb{T}_0)$ and $\ln \zeta_\mu$ is bounded and decreasing. Moreover, since $H \geq R_\infty$, we have that for $t \in [0, T]$,

$$\zeta_\mu(t) = \frac{f_\mu(t)}{f_{\tau_{x_0/\sigma}^\circ}(t)} = \frac{u(t, 0; \mu, c)}{u(0, x_0; \mu, c)} = \delta_\mu \exp\left(\frac{H(F_\mu(t)) - R_\infty}{2c\sigma^2}\right) \geq \delta_\mu.$$

Conversely, given any $\mu \in \mathcal{P}^+(\mathbb{T})$ such that $\ln \zeta_\mu$ is bounded and decreasing, and $\inf_{t \in [0, T]} \zeta_\mu(t) \geq \delta_\mu$, we have $\widetilde{H}_\mu \in \mathcal{H}_0$. It is straightforward to check that μ satisfies (3.2) with $H = \widetilde{H}_\mu$, that is, $\mathcal{E}_T(\widetilde{H}_\mu) = \mu$. By Lemma 3.1(ii), any admissible reward scheme realizing μ must agree with $\widetilde{H}_\mu$ on $[0, F_\mu(T)]$. On the other hand, since the reward after rank $F_\mu(T)$ is irrelevant in determining the individual's best response, any $H \in \mathcal{H}_0$ which agrees with $\widetilde{H}_\mu$ on $[0, F_\mu(T)]$ satisfies $\mathcal{E}_T(H) = \mathcal{E}_T(\widetilde{H}_\mu) = \mu$; (ii) follows from straightforward calculation.   □

3.4. *Optimal reward design.*   The semiexplicit characterization of the equilibrium allows us to further study the optimal reward design problem for a principal or social planner. We continue to consider only the reward functions of the form (3.1) with $H \in \mathcal{H}_0$, in which case there exists a unique equilibrium with completion time distribution denoted by $\mathcal{E}(H)$ or $\mathcal{E}_T(H)$.

We consider three different optimization criteria: minimizing time to achieve a given population completion rate (Section 3.4.1), maximizing welfare (Section 3.4.2) and maximizing net profit (Section 3.4.3). To preview the results: for the first two criteria, the optimal reward is a two-step function—the same reward for all sufficiently high ranks, and the same minimum guaranteed payment R_∞ for the low ranks. This is different from the one-stage Poisson game of Nutz and Zhang (2017) where the quantile-minimizing reward scheme is concave for high ranks. For the third problem where for a given profit function g, we maximize the expected profit $Eg(\tau)$ minus the cost of reward, with τ drawn from the infinite

horizon equilibrium distribution, the optimal reward for finishing at time t is a linear transformation of $g(t)$ for t lower than a bonus deadline t_b^*, and R_∞ otherwise.

Here, we only highlight the proof of the third problem where we rely on the result from reverse engineering. All other proofs are provided in the Appendix.

3.4.1. *Minimizing the time to achieve a given completion rate.* We fix a deadline $T \in (0, \infty]$, a total reward budget K, a target completion rate $\alpha \in (0, 1)$ and a minimum participation reward $R_\infty \leq K$, and look for reward function $H(r) \geq R_\infty$ that minimizes the time it takes α fraction of the population to complete their projects in equilibrium. More precisely, the feasible set of reward functions is

$$\mathcal{H} := \left\{ H \in \mathcal{H}_0 : \int_0^1 H(r)\, dr \leq K \right\} \quad \text{for } T = \infty,$$

and, with $\beta(H) := F_{\mathcal{E}_T(H)}(T)$, $T < \infty$,

$$\mathcal{H}_T^\alpha := \left\{ H \in \mathcal{H}_0 : \beta(H) \geq \alpha, \int_0^{\beta(H)} H(r)\, dr + (1 - \beta(H)) R_\infty \leq K \right\}.$$

For $H \in \mathcal{H}$, let $T_\alpha(H) := T_\alpha^{\mathcal{E}(H)}$ be the α-quantile of $\mathcal{E}(H)$. We wish to find $T_\alpha^* = \inf_{H \in \mathcal{H}} T_\alpha(H)$, and identify the minimizer H^*, if it exists. Similarly, for $H \in \mathcal{H}_T$, let $T_\alpha(H; T) := T_\alpha^{\mathcal{E}_T(H)}$ be the α-quantile of $\mathcal{E}_T(H)$. We will look for the optimizer of $T_\alpha^*(T) = \inf_{H \in \mathcal{H}_T^\alpha} T_\alpha(H; T)$.

REMARK 3.3. For the finite horizon problem, there are two different ways to impose the budget constraint: (i) to require $\int_0^1 H(r)\, dr \leq K$, as in the definition of $\mathcal{H}$; in this case, the budget may not be fully utilized due to a portion of the players failing to complete by time T, but the advantage of such a constraint is that the total reward does not go over K even out of equilibrium; (ii) to bound the total reward only in equilibrium: $\int_0^{\beta(H)} H(r)\, dr + (1 - \beta(H)) R_\infty$, as in the definition of $\mathcal{H}_T^\alpha$; such a constraint is weaker, but it might be violated if the population does not end up in equilibrium. It turns out, as we will see in Theorem 3.5, that the two constraints result in the same optimal value and optimal reward function.

The following two theorems present the optimal reward functions and the corresponding equilibria.

THEOREM 3.4. *Let $T = \infty$. Then $\inf_{H \in \mathcal{H}} T_\alpha(H)$ is uniquely attained (up to a.e. equivalence) by the uniform scheme with cutoff rank α:*

$$H^*(r) = R_\infty + \frac{K - R_\infty}{\alpha} 1_{[0,\alpha]}(r).$$

The minimal time is

$$T_\alpha^* = T_\alpha(H^*) = F_{\tau_{x_0/\sigma}^\circ}^{-1} \left(\frac{\alpha}{\alpha + (1 - \alpha) \exp(\frac{K - R_\infty}{2\alpha c \sigma^2})} \right).$$

The c.d.f. of $\mu = \mathcal{E}(H^*)$ is given by

$$F_\mu(t) = \begin{cases} \left(\alpha + (1-\alpha)\exp\left(\dfrac{K - R_\infty}{2\alpha c\sigma^2}\right)\right) F_{\tau^\circ_{x_0/\sigma}}(t) & \text{if } t \leq T^*_\alpha, \\[3mm] F_{\tau^\circ_{x_0/\sigma}}(t) + \alpha\left(1 - F_{\tau^\circ_{x_0/\sigma}}(t)\right)\left(1 - \exp\left(\dfrac{R_\infty - K}{2\alpha c\sigma^2}\right)\right) & \text{if } t > T^*_\alpha. \end{cases}$$

The equilibrium value attained by each player is

$$V = R_\infty - 2c\sigma^2 \ln\left(\alpha \exp\left(\frac{R_\infty - K}{2\alpha c\sigma^2}\right) + 1 - \alpha\right),$$

and the equilibrium effort, in feedback form, is

$$a(t, x) = \frac{2[\exp(\frac{K - R_\infty}{2\alpha c\sigma^2}) - 1]N'(\frac{x}{\sigma\sqrt{T^*_\alpha - t}})\frac{\sigma}{\sqrt{T^*_\alpha - t}}}{1 + 2[\exp(\frac{K - R_\infty}{2\alpha c\sigma^2}) - 1][1 - N(\frac{x}{\sigma\sqrt{T^*_\alpha - t}})]},$$

where N and N' are the c.d.f. and p.d.f. of the standard normal distribution.

THEOREM 3.5. *Let the deadline $T < \infty$, the minimum participation reward R_∞, the total reward budget $K \geq R_\infty$ and the target completion rate $\alpha \in (0, 1)$ be given. We have $T^*_\alpha(T) = \inf_{H \in \mathcal{H}^\alpha_T} T_\alpha(H; T) = \inf_{H \in \mathcal{H}} T_\alpha(H; T)$. Let T^*_α be the optimal time given by Theorem* 3.4.

- *If $T < T^*_\alpha$, then $\mathcal{H}^\alpha_T = \varnothing$ and $T^*_\alpha(T) = \infty$.*
- *If $T \geq T^*_\alpha$, then $T^*_\alpha(T) = T^*_\alpha$ is uniquely attained (up to a.e. equivalence) by the uniform scheme with cutoff rank α:*

$$H^*(r) = R_\infty + \frac{K - R_\infty}{\alpha} 1_{[0,\alpha]}(r).$$

The c.d.f. of $\mu = \mathcal{E}_T(H^*)$ on $[0, T]$, the equilibrium value V attained by each player and the equilibrium effort $a(t, x)$, $t \in [0, T]$ in feedback form have the same expression as those of Theorem 3.4.

- *The minimum budget needed to ensure that α fraction of players finish by time T is*

$$K_{\min} = R_\infty + 2\alpha c\sigma^2 \ln\left(\frac{\alpha}{1 - \alpha} \cdot \frac{1 - F_{\tau^\circ_{x_0/\sigma}}(T)}{F_{\tau^\circ_{x_0/\sigma}}(T)}\right).$$

The reward scheme which achieves the goal with budget $K_{\min}$ is given by

$$H(r) = R_\infty + \frac{K_{\min} - R_\infty}{\alpha} 1_{[0,\alpha]}(r).$$

- *Under the given budget K, the maximum equilibrium completion rate $\alpha_{\max}$ attainable at time T is the unique solution of*

$$C_T := \frac{F_{\tau^\circ_{x_0/\sigma}}(T)}{1 - F_{\tau^\circ_{x_0/\sigma}}(T)} = \frac{\alpha_{\max}}{1 - \alpha_{\max}} \exp\left(\frac{R_\infty - K}{2\alpha_{\max}c\sigma^2}\right),$$

that is, $\alpha_{\max} = (1 + \frac{2c\sigma^2}{K - R_\infty} W(C_T^{-1}))^{-1}$, where W is the Lambert-W function. The reward scheme which yields $\alpha_{\max}$ is given by

$$H(r) = R_\infty + \frac{K - R_\infty}{\alpha_{\max}} 1_{[0, \alpha_{\max}]}(r).$$

3.4.2. *Maximizing welfare.* We now find the reward scheme that maximizes the total welfare, that is, the aggregate game value of all players, again in the homogeneous case.

THEOREM 3.6. *Fix the participation reward R_∞ and the reward budget $K \geq R_\infty$. Then*:

(i) *When $T = \infty$, the maximum welfare $\sup_{H \in \mathcal{H}} V_\infty(H) = K$ is uniquely attained (up to a.e. equivalence) by the uniform scheme $H^*(r) \equiv K$ (thus, with zero effort by everyone).*

(ii) *When $T < \infty$, the maximum welfare*

$$\sup_{H \in \mathcal{H}_T^0} V(H) = R_\infty + 2c\sigma^2 \ln\left(\frac{1 - F_{\tau_{x_0/\sigma}^\circ}(T)}{1 - \alpha}\right)$$

is uniquely attained (up to a.e. equivalence) by the uniform scheme with cutoff rank α:

$$H^*(r) = R_\infty + \frac{K - R_\infty}{\alpha} 1_{[0, \alpha]}(r),$$

where α is the maximum attainable completion rate by time T, given in Theorem 3.5. *Moreover, H^* also maximizes the expected total effort given by Proposition* 3.2.

3.4.3. *Maximizing net profit.* We now suppose that each project completed at time t generates a profit of $g(t) = g(t; x_0)$ for a principal. We assume that g is continuous, bounded and decreasing, and that $g \not\equiv g(\infty)$. The principal wants to maximize the expected net profit $E[g(\tau) - R_\mu(\tau)]$, $\tau \sim \mu$, subject to the participation constraint $R \geq R_\infty$. We only consider the case $T = \infty$.

THEOREM 3.7. *Suppose $T = \infty$. A reward scheme $H^* \in \mathcal{H}_0$ is optimal if and only if*

$$H^*(r) = R_\infty + g\left(F_{\mu^*}^{-1}(r) \wedge t_b^*\right) - g\left(t_b^*\right),$$

where the "bonus" deadline t_b^* is given by $t_b^* = \inf\{z \geq 0 : g(z) = g(z^*)\}$ for some

$$z^* = \arg\max_{z \in [0, \infty)} \frac{\int_0^\infty g(t \vee z) f_{\tau_{x_0/\sigma}^\circ}(t) \exp(\frac{g(t \wedge z)}{2c\sigma^2}) \, dt}{\int_0^\infty f_{\tau_{x_0/\sigma}^\circ}(s) \exp(\frac{g(s \wedge z)}{2c\sigma^2}) \, ds},$$

and where the associated equilibrium distribution μ^* *has p.d.f.*

$$f_{\mu^*}(t) = \frac{f_{\tau^\circ_{x_0/\sigma}}(t) \exp(\frac{g(t \wedge t^*_b)}{2c\sigma^2})}{\int_0^\infty f_{\tau^\circ_{x_0/\sigma}}(s) \exp(\frac{g(s \wedge t^*_b)}{2c\sigma^2}) \, ds}.$$

The maximum net profit is

$$U = \frac{\int_0^\infty g(t \vee t^*_b) f_{\tau^\circ_{x_0/\sigma}}(t) \exp(\frac{g(t \wedge t^*_b)}{2c\sigma^2}) \, dt}{\int_0^\infty f_{\tau^\circ_{x_0/\sigma}}(s) \exp(\frac{g(s \wedge t^*_b)}{2c\sigma^2}) \, ds} - R_\infty.$$

Note that in equilibrium the players receive $H^*(F_{\mu^*}(t)) = R_\infty + g(t \wedge t^*_b) - g(t^*_b)$ for finishing at time t. That is, the optimal reward for finishing at time t is a linear transformation of $g(t)$ for those who finish before the bonus deadline t^*_b; otherwise it is the minimum guaranteed payment. Thus, it is optimal for the principal to align the players' interest with her own.

PROOF OF THEOREM 3.7. By Theorem 3.2, we only need to optimize over $\mu \in \mathcal{E}(\mathcal{H}_0)$, which can be realized by the reward $H_\mu + C$ for any constant $C \geq R_\infty - 2c\sigma^2 \ln(\inf f_\mu / f_{\tau^\circ_{x_0/\sigma}})$. It is clear that among the translations of H_μ, the principal should choose the smallest C which meets the reservation reward constraint, namely, $C_\mu := R_\infty - 2c\sigma^2 \ln(\inf f_\mu / f_{\tau^\circ_{x_0/\sigma}})$. Hence we can rewrite the optimization problem as

$$U = \sup_{f_\mu} \int_0^\infty \left[g(t) - 2c\sigma^2 \ln\left(\frac{f_\mu(t)}{f_{\tau^\circ_{x_0/\sigma}}(t)}\right) - C_\mu \right] f_\mu(t) \, dt.$$

The proof consists of two steps: (i) Fix $b = \inf f_\mu / f_{\tau^\circ_{x_0/\sigma}} \in (0, 1]$ and solve the constraint optimization problem:

$$U(b) := \sup_{f_\mu \geq b f_{\tau^\circ_{x_0/\sigma}}} \int_0^\infty \left[g(t) - 2c\sigma^2 \ln\left(\frac{f_\mu(t)}{b f_{\tau^\circ_{x_0/\sigma}}(t)}\right) \right] f_\mu(t) \, dt - R_\infty$$

subject to the additional integral constraint

$$\int_0^\infty f_\mu(t) \, dt = 1.$$

In addition, we also need to $f_\mu / f_{\tau^\circ_{x_0/\sigma}}$ to be bounded and decreasing in order to obtain a bounded, decreasing reward function H_μ. This can be verified after we find the optimizer. (ii) Maximize $U(b)$ over $b \in (0, 1]$.

We begin with the first step. Define

$$L(f_\mu, \lambda; b) := \int_0^\infty \left[g(t) - 2c\sigma^2 \ln\left(\frac{f_\mu(t)}{b f_{\tau^\circ_{x_0/\sigma}}(t)}\right) \right] f_\mu(t) \, dt$$

$$- \lambda \left(\int_0^\infty f_\mu(t) \, dt - 1 \right).$$

For each fixed λ, the integrand, being a concave function of f_μ, attains a pointwise maximum on $[bf_{\tau_{x_0/\sigma}^\circ}(t), \infty)$ at

(3.10)
$$f_{\mu(b)}(t) = bf_{\tau_{x_0/\sigma}^\circ}(t) \exp\left(\left[\frac{g(t) - \lambda}{2c\sigma^2} - 1\right]^+\right).$$

We then find $\lambda = \lambda_b$ via

(3.11) $1 = \int_0^\infty f_{\mu(b)}(s)\,ds = \int_0^\infty bf_{\tau_{x_0/\sigma}^\circ}(s) \exp\left(\left[\frac{g(s) - \lambda}{2c\sigma^2} - 1\right]^+\right) ds =: \phi(\lambda).$

Note that the above equation has a unique solution $\lambda_b \in (-\infty, g(0) - 2c\sigma^2]$ since ϕ is continuous, strictly decreasing and satisfies $\phi(g(0) - 2c\sigma^2) = b \leq 1$ and $\lim_{\lambda \to -\infty} \phi(\lambda) = \infty$. Since any $\lambda \geq g(0) - 2c\sigma^2$ (which is only possible when $b = 1$) leads to the same $f_{\mu(b)}$, we may assume without loss of generality that $\lambda_b \leq g(0) - 2c\sigma^2$. It is clear that for each $b \in (0, 1]$, $f_{\mu(b)}/f_{\tau_{x_0/\sigma}^\circ}$ is bounded and decreasing. So the associated reward scheme $H_{\mu(b)} + C_{\mu(b)} \in \mathcal{H}_0$.

Next, consider the function $\psi : [0, \infty] \to (0, 1]$ defined by

$$\psi(z) := \left(\int_0^\infty f_{\tau_{x_0/\sigma}^\circ}(t) \exp\left(\frac{g(t \wedge z) - g(z)}{2c\sigma^2}\right) dt\right)^{-1}.$$

It can be shown that ψ is decreasing with $\psi(0) = 1$, and strictly decreasing on any interval where g is strictly decreasing. This implies $\psi(z_1) = \psi(z_2)$ if and only if $g(z_1) = g(z_2)$. Let $b_0 := \psi(\infty) \in (0, 1)$. It is not hard to see from (3.11) that $b \geq b_0$ if and only if $g(\infty) \leq 2c\sigma^2 + \lambda_b$. We consider two cases for b.

Case 1. If $0 < b < b_0$, or equivalently, $g(\infty) > 2c\sigma^2 + \lambda_b$, the positive part in (3.10) and (3.11) can be removed, leading to

$$f_{\mu(b)}(t) = \frac{f_{\tau_{x_0/\sigma}^\circ}(t) \exp(\frac{g(t)-\lambda_b}{2c\sigma^2} - 1)}{\int_0^\infty f_{\tau_{x_0/\sigma}^\circ}(s) \exp(\frac{g(s)-\lambda_b}{2c\sigma^2} - 1)\,ds} = b_0 f_{\tau_{x_0/\sigma}^\circ}(t) \exp\left(\frac{g(t) - g(\infty)}{2c\sigma^2}\right).$$

In this case,

$$
\begin{aligned}
U(b) &= \int_0^\infty \left[g(t) - 2c\sigma^2 \ln\left(\frac{f_{\mu(b)}(t)}{bf_{\tau_{x_0/\sigma}^\circ}(t)}\right)\right] f_{\mu(b)}(t)\,dt - R_\infty \\
&= \int_0^\infty \left[g(\infty) + 2c\sigma^2 \ln\frac{b}{b_0}\right] b_0 f_{\tau_{x_0/\sigma}^\circ}(t) \exp\left(\frac{g(t) - g(\infty)}{2c\sigma^2}\right) dt - R_\infty \\
&< \int_0^\infty g(\infty) b_0 f_{\tau_{x_0/\sigma}^\circ}(t) \exp\left(\frac{g(t) - g(\infty)}{2c\sigma^2}\right) dt - R_\infty \\
&= \frac{\int_0^\infty g(\infty) f_{\tau_{x_0/\sigma}^\circ}(t) \exp(\frac{g(t)}{2c\sigma^2})\,dt}{\int_0^\infty f_{\tau_{x_0/\sigma}^\circ}(s) \exp(\frac{g(s)}{2c\sigma^2})\,ds} - R_\infty =: U_0.
\end{aligned}
$$

Case 2. If $b_0 \leq b \leq 1$, or equivalently, $g(\infty) \leq 2c\sigma^2 + \lambda_b$, let

$$t_b := \inf\{t \geq 0 : g(t) \leq 2c\sigma^2 + \lambda_b\} \in [0, \infty]$$

be the point after which the constraint $f_{\mu(b)} \geq b f_{\tau_{x_0/\sigma}^\circ}$ will be binding. By the continuity of g and that $g(0) - \lambda_b \geq 2c\sigma^2$, we must have $g(t_b) = 2c\sigma^2 + \lambda_b$. It follows that

$$\left[\frac{g(s) - \lambda_b}{2c\sigma^2} - 1\right]^+ = \left[\frac{g(s) - g(t_b)}{2c\sigma^2}\right]^+ = \frac{g(s \wedge t_b) - g(t_b)}{2c\sigma^2}.$$

Thus, we are again able to get rid of the positive part in (3.10) and (3.11), and get

$$(3.12) \qquad f_{\mu(b)}(t) = b f_{\tau_{x_0/\sigma}^\circ}(t) \exp\left(\frac{g(t \wedge t_b) - g(t_b)}{2c\sigma^2}\right),$$

and

$$(3.13) \qquad b = \left(\int_0^\infty f_{\tau_{x_0/\sigma}^\circ}(s) \exp\left(\frac{g(s \wedge t_b) - g(t_b)}{2c\sigma^2}\right) ds\right)^{-1} = \psi(t_b).$$

Recall that ψ has the property that $\psi(z_1) = \psi(z_2)$ if and only if $g(z_1) = g(z_2)$. So if z is another solution of $\psi(z) = b$, then we must have $g(z) = g(t_b)$, and since t_b is the first time g hits $2c\sigma^2 + \lambda_b$, we must have $z \geq t_b$. In other words, t_b can be characterized as the smallest solution of $\psi(z) = b$. Alternatively, if $b = \psi(z_0)$, then $t_b = \inf\{z \geq 0 : g(z) = g(z_0)\}$, independent of the choice of z_0.

Using (3.12)–(3.13), we obtain

$$U(b) = \int_0^\infty \left[g(t) - 2c\sigma^2 \ln\left(\frac{f_{\mu(b)}(t)}{b f_{\tau_{x_0/\sigma}^\circ}(t)}\right)\right] f_{\mu(b)}(t)\, dt - R_\infty$$

$$= \frac{\int_0^\infty g(t \vee t_b) f_{\tau_{x_0/\sigma}^\circ}(t) \exp(\frac{g(t \wedge t_b)}{2c\sigma^2})\, dt}{\int_0^\infty f_{\tau_{x_0/\sigma}^\circ}(s) \exp(\frac{g(s \wedge t_b)}{2c\sigma^2})\, ds} - R_\infty, \quad b \in [b_0, 1].$$

Observe that setting $t_b = \infty$ in the above expression yields U_0.

Combining the two cases, we see that $U(b) < \widetilde{U}(\infty)$ if $b \in (0, b_0)$ and $U(b) = \widetilde{U}(t_b)$ if $b \in [b_0, 1]$, where we introduce the auxiliary objective function

$$\widetilde{U}(z) := \frac{\int_0^\infty g(t \vee z) f_{\tau_{x_0/\sigma}^\circ}(t) \exp(\frac{g(t \wedge z)}{2c\sigma^2})\, dt}{\int_0^\infty f_{\tau_{x_0/\sigma}^\circ}(s) \exp(\frac{g(s \wedge z)}{2c\sigma^2})\, ds} - R_\infty.$$

Clearly, $\sup_{b \in (0, 1]} U(b) \leq \sup_{z \in [0, \infty]} \widetilde{U}(z)$. Since $\widetilde{U}(z)$ is continuous and $\widetilde{U}(0) > g(\infty) - R_\infty = \widetilde{U}(\infty)$, $\sup_{z \in [0, \infty]} \widetilde{U}(z)$ is attained by some $z^* < \infty$ which can be found numerically.

CLAIM. *Maximizing $U(b)$ is equivalent to maximizing $\widetilde{U}(z)$ in the following sense*:

(i) $\sup_{b\in(0,1]} U(b) = \sup_{z\in[0,\infty]} \widetilde{U}(z)$;

(ii) $b^* \in \arg\max_{b\in(0,1]} U(b)$ if and only if $b^* = \psi(z^*)$ for some $z^* \in \arg\max_{z\in[0,\infty]} \widetilde{U}(z)$.

The proof of the claim relies on the observation that if $b = \psi(z) \geq b_0$, then $U(b) = \widetilde{U}(t_b) \geq \widetilde{U}(z)$. Indeed, $b = \psi(z)$ implies that $z \geq t_b$ and $g(z) = g(t_b)$ as we have argued before, which further yields $g(t \wedge z) = g(t \wedge t_b)$. Consequently,

$$U(b) = \widetilde{U}(t_b) = \frac{\int_0^\infty g(t \vee t_b) f_{\tau_{x_0/\sigma}^\circ}(t) \exp(\frac{g(t\wedge z)}{2c\sigma^2}) \, dt}{\int_0^\infty f_{\tau_{x_0/\sigma}^\circ}(s) \exp(\frac{g(s\wedge z)}{2c\sigma^2}) \, ds} - R_\infty$$

$$\geq \frac{\int_0^\infty g(t \vee z) f_{\tau_{x_0/\sigma}^\circ}(t) \exp(\frac{g(t\wedge z)}{2c\sigma^2}) \, dt}{\int_0^\infty f_{\tau_{x_0/\sigma}^\circ}(s) \exp(\frac{g(s\wedge z)}{2c\sigma^2}) \, ds} - R_\infty = \widetilde{U}(z).$$

The above observation implies that if z^* is optimal for $\widetilde{U}$, then we have $\sup_{b\in(0,1]} U(b) \geq U(\psi(z^*)) \geq \widetilde{U}(z^*) = \sup_{z\in[0,\infty]} \widetilde{U}(z) \geq \sup_{b\in(0,1]} U(b)$. This proves the first claim and the "if" part of the second claim. For the "only if" part of the second claim, suppose b^* is optimal for U, then $U(b^*) = \sup_{b\in(0,1]} U(b) = \sup_{z\in[0,\infty]} \widetilde{U}(z) \geq \widetilde{U}(\infty)$. Since $U(b) < \widetilde{U}(\infty)$ for all $b \in (0, b_0)$, we must have $b^* \in [b_0, 1]$. It follows that $b^* = \psi(t_{b^*})$ and $\widetilde{U}(t_{b^*}) = U(b^*) = \sup_{b\in(0,1]} U(b) = \sup_{z\in[0,\infty]} \widetilde{U}(z)$. The latter implies t_{b^*} is optimal for $\widetilde{U}$.

In view of the claim, we can find the optimal value and all optimizers of $\sup_{b\in(0,1]} U(b)$ by solving the auxiliary problem. For each optimizer z^* for the auxiliary problem, we get an optimal $b^* = \psi(z^*)$. We then recover $f_{\mu^*} = f_{\mu(b^*)}$ by (3.12) and H_{μ^*} by Theorem 3.2. The optimal reward is given by $H^*(r) = H_{\mu^*}(r) + R_\infty - 2c\sigma^2 \ln b^* = R_\infty + g(F_{\mu^*}^{-1}(r) \wedge t_{b^*}) - g(t_{b^*})$. $\square$

3.5. Extension: Completion rate-dependent pie and multiple equilibria.
Consider a reward pie that depends on the aggregate completion rate by time $T < \infty$. (That rate is 100% when $T = \infty$.) Specifically, we assume the reward for finishing at time t rank r, and when the population completion rate (by time T) is β, is given by

$$(3.14) \qquad R(t, r, \beta) = 1_{\{t \leq T\}} H(r, \beta) + 1_{\{t > T\}} R_\infty(\beta),$$

where $H : [0, 1] \times [F_{\tau_{x_0/\sigma}^\circ}(T), 1] \mapsto \mathbb{R}$ and $R_\infty : [F_{\tau_{x_0/\sigma}^\circ}(T), 1] \mapsto \mathbb{R}$ are continuous in β. That is, the individual reward for finishing at time τ equals $R(\tau, F_\mu(\tau), F_\mu(T))$ where μ is the completion time distribution of the population. What we have in mind is an organization or a society in which the overall wealth coming from individual projects is higher when more individuals complete their goals. The role of r and β are different in that the former measures individual performance in relative terms, and the latter measures population performance.

Moreover, the game is competitive in r and collaborative in β. Similar to the case of a fixed pie, the equilibrium also admits a semiexplicit characterization.

THEOREM 3.8.   *Let $T < \infty$. Suppose the reward function is of the form* (3.14). *Then there exist at least one equilibrium, and a distribution μ is an equilibrium completion time distribution if and only if for $r \in [0, F_\mu(T)]$,*

$$(3.15) \quad T_r^\mu = F_{\tau^\circ_{x_0/\sigma}}^{-1}\left(\frac{1 - F_{\tau^\circ_{x_0/\sigma}}(T)}{1 - F_\mu(T)} \int_0^r \exp\left(\frac{R_\infty(F_\mu(T)) - H(z, F_\mu(T))}{2c\sigma^2} \right) dz \right),$$

where

$$F_{\tau^\circ_{x_0/\sigma}}(t) = 2\left(1 - N\left(\frac{x_0}{\sigma\sqrt{t}} \right) \right).$$

Moreover, the equilibrium terminal completion rate $F_\mu(T) \in (0, 1)$ is a solution of

$$(3.16) \quad \begin{aligned} &\frac{F_{\tau^\circ_{x_0/\sigma}}(T)}{1 - F_{\tau^\circ_{x_0/\sigma}}(T)} \\ &= \frac{1}{1 - F_\mu(T)} \int_0^{F_\mu(T)} \exp\left(\frac{R_\infty(F_\mu(T)) - H(z, F_\mu(T))}{2c\sigma^2} \right) dz. \end{aligned}$$

The associated value and expected total effort of the game are given by

$$(3.17) \quad V(\mu) = R_\infty\big(F_\mu(T)\big) + 2c\sigma^2 \ln\left(\frac{1 - F_{\tau^\circ_{x_0/\sigma}}(T)}{1 - F_\mu(T)} \right)$$

and (3.9), *respectively.*

The proof is similar to the case of a fixed pie, so we omitted it here. A sketch proof can be found in arXiv:1811.00076. Note that we do not necessarily have uniqueness under conditions of Theorem 3.8. Equation (3.16) may have multiple solutions, leading to multiple equilibria. For example, when $H(r, \beta) = R_\infty(\beta) + \beta$, one can check that equation (3.16) may have one, two or three solutions, depending on the value of T, x_0, c, σ (see Figure 5). Among the equilibria, the dominant one (i.e., the one with the highest game value) corresponds to the largest solution of (3.16). In this example, we see that by limiting the size of the projects or imposing a long enough deadline, the "bad" equilibria can be avoided.

When reward $R(t, r, \beta)$ is independent of rank r and increasing in completion rate β, we have a contribution game in which, by gradually injecting rank dependence into the game, one can analyze the effect of competition on the completion rate and the game value.

Consider, for example,

$$R(t, r, \beta) = \Pi(\beta)\big[\gamma + 1_{\{t \leq T\}}(1 - \gamma)H_\varepsilon(r)\big]$$
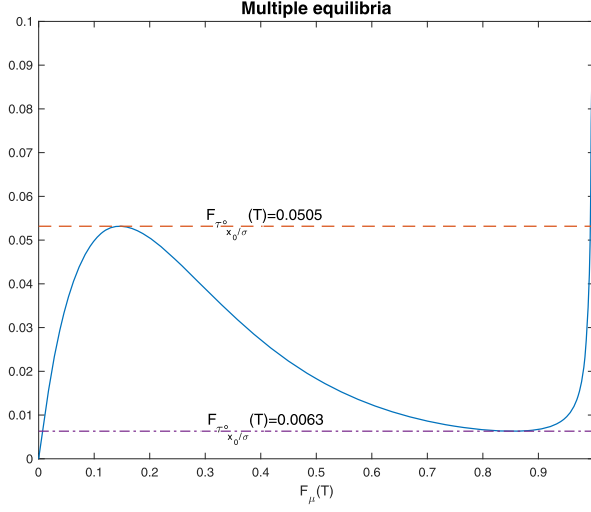
FIG. 5. *The blue solid curve plots the right-hand side of (3.16) as a function of $F_\mu(T)$ with $c = 1$, $\sigma = 0.25$ and $H(r, \beta) = R_\infty(\beta) + \beta$. The two horizontal lines plot the left-hand side of (3.16) in the two critical cases. We see that there is a unique equilibrium when $F_{\tau_{x_0/\sigma}^\circ} < 0.0063$ or $F_{\tau_{x_0/\sigma}^\circ} > 0.0505$, two equilibria when $F_{\tau_{x_0/\sigma}^\circ} = 0.0063$ or 0.0505, and three equilibria when $0.0063 < F_{\tau_{x_0/\sigma}^\circ} < 0.0505$.*

where $\Pi(\beta) := K(1 + \beta)$ represents the size of the pie, $\gamma \in [0, 1]$ is the fraction of the pie that is used to reward participation, and $H_\epsilon(r) = 1 + \epsilon(1 - 2r)$ specifies how the remaining $(1 - \gamma)$ fraction of the pie is shared among players who finish by the deadline and are ranked. A larger ϵ corresponds to a larger degree of competition (or inequality) among ranked players. When $\epsilon = 0$, we have a pure contribution game. Figure 6 plots the equilibrium completion rate and game value against ϵ for $\gamma = 0.5$. We see that the degree of inequality in the rewards has a positive effect on the equilibrium completion rate and the game value when the pie (i.e., K) is small, an adverse effect when the pie is large, and a mixed effect when the size of the pie is moderate. The intuition is the same as in the case of a fixed pie.

4. General case: Existence, uniqueness, stability and ϵ-Nash equilibrium.
In this section, we work with general reward functions $R \in \mathcal{R}$, and no longer assume players are homogeneous. We introduce heterogeneity by assuming that a tournament is characterized by the initial condition (t_0, π, m), where $t_0 < T$ is the starting time, π is the proportion of players that have already finished by time t_0, and $m \in \mathcal{P}(\mathbb{R}_{++} \times [\underline{c}, \overline{c}])$, with $0 < \underline{c} \leq \overline{c} < \infty$, is the joint distribution of the time-t_0 location, denoted by ξ, and the cost of effort c of the population still playing. Here for simplicity, we do not consider inhomogeneity in $\sigma \in [\underline{\sigma}, \overline{\sigma}]$ which could be easily incorporated by taking m to be the joint distribution of (ξ, c, σ). We assume that we are in the nondegenerate case $\pi < 1$.
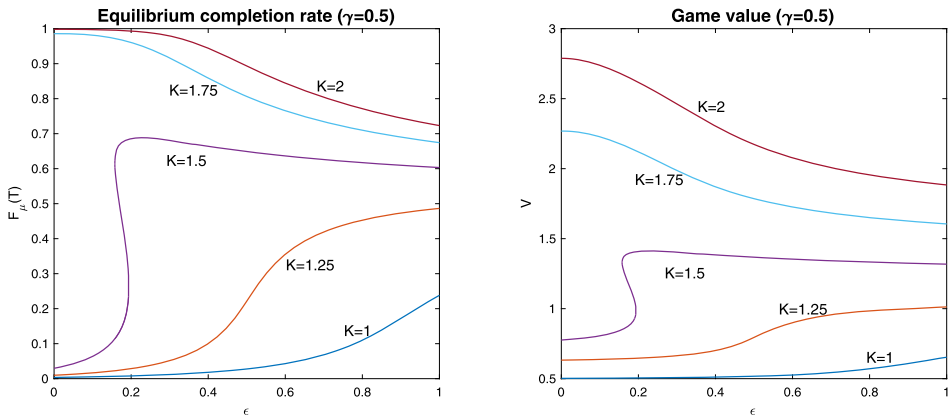
FIG. 6. $F_\mu(T)$ and V against ϵ. Parameters are set to $x_0 = c = T = 1$, $\sigma = 0.25$ and $\gamma = 0.5$. When $K = 1.5$, there is an interval of ϵ where the mapping $\epsilon \mapsto F_\mu(T)$ is multivalued, meaning the game has multiple equilibria.

4.1. *Existence of a fixed point.*   Given an initial condition (t_0, π, m), for a representative player we introduce a binary random variable $\theta \in \{0, 1\}$ with $P(\theta = 1) = \pi$ and $P(\theta = 0) = 1 - \pi$. We interpret θ as the player's game completion status at time t_0; $\theta = 1$ means game is completed and $\theta = 0$ means game is in progress. If $\theta = 0$, we further pick a time-t_0 location ξ and cost of effort coefficient c randomly from the joint distribution m. The initial randomizations (θ, ξ, c) are assumed to be independent of the Brownian motions driving the state process.

For a given starting time t_0, fix a $[t_0, \infty)$-supported completion time distribution μ of the population with $\mu(\{t_0\}) = \pi$, that is, μ is a modification of the completion time distribution by moving all the mass before time t_0 to time t_0. Since the probability for finishing exactly at time t_0 is zero, such a modification does not affect the player's optimization problem.

The existence proof will be based on a fixed-point argument. For that purpose, we now define the best response mapping $\Phi^{t_0, \pi, m}$ which maps μ to the distribution $\mathcal{L}(\tau_\mu)$ of the optimal completion time τ_μ. Assume for the moment that $T < \infty$. If $\theta = 1$, we set $\tau_\mu := t_0$. If $\theta = 0$, we solve the player's optimization problem with initial randomization (ξ, c) and using payoff function $R_\mu(t) = R(t, F_\mu(t))$. Under the convention that $\inf \varnothing = \infty$, we set $\tau_\mu := \inf\{t \in [t_0, T] : X_t^\mu = 0\} \wedge \Delta$, where Δ is a fixed number in (T, ∞) corresponding to incompletion, and

$$dX_t^\mu = -a^*(t, X_t^\mu; \mu, c)\, dt + \sigma\, dB_t, \quad X_{t_0}^\mu = \xi.$$

By Proposition 2.1, we know that for $t \in (t_0, T]$, the density of the optimal completion time is given by

$$\mathbb{P}(\tau_\mu \in dt | \theta = 0, \xi, c) = \frac{u(t, 0; \mu, c)}{u(t_0, \xi; \mu, c)} f_{\tau_{\xi/\sigma}^\circ}(t - t_0).$$

It follows that

$$(4.1) \qquad \mathbb{P}(\tau_\mu = t_0) = \pi, \qquad \mathbb{P}(\tau_\mu = \Delta) = 1 - \mathbb{P}(\tau_\mu \le T),$$

$$\mathbb{P}(\tau_\mu \le t) = \pi + (1 - \pi)E\left[\int_{t_0}^t \frac{u(s, 0; \mu, c)}{u(t_0, \xi; \mu, c)} f_{\tau_{\xi/\sigma}^\circ}(s - t_0)\, ds\right],$$

(4.2)
$$t \in [t_0, T].$$

In terms of the p.d.f., we have

$$(4.3) \qquad \mathbb{P}(\tau_\mu \in dt) = (1 - \pi)E\left[\frac{u(t, 0; \mu, c)}{u(t_0, \xi; \mu, c)} f_{\tau_{\xi/\sigma}^\circ}(t - t_0)\right], \quad t \in (t_0, T].$$

Let $\mathbb{T} := [t_0, T] \cup \{\Delta\}$. Our goal is to find a fixed point of $\Phi^{t_0, \pi, m} : \mu \mapsto \mathcal{L}(\tau_\mu)$ in the space $\mathcal{P}(\mathbb{T})$ of probability measures on $\mathbb{T}$, which is compact in the topology of weak convergence We remark that $\mathcal{P}(\mathbb{T})$ can be embedded in the vector space $\mathcal{M}(\mathbb{T})$ of finite signed measures on $\mathbb{T}$, endowed with the Kantorovich–Rubinstein norm which coincides with the 1-Wasserstein metric on $\mathcal{P}(\mathbb{T})$ and metrizes the topology of weak convergence on $\mathcal{P}(\mathbb{T})$; see pages 119–120 and Corollary 6.13 of Villani (2009). When $T = \infty$, we can define $\Phi^{t_0, \pi, m}$ in a similar fashion with $\Delta := \infty$. In this case, additional care is needed to ensure the compactness of $\mathcal{P}([t_0, \infty])$; see the proof of Theorem 4.1 for details.

The main ingredient of fixed-point theorems is the continuity of the best response mapping which, in view of (4.2), amounts to the continuity of $u(t, x; \mu, c)$ with respect to μ. We can guarantee this by restricting ourselves to reward functions $R \in \mathcal{R}$ that are continuous in the rank variable, or more generally, of the form $R(t, r) = G(t, H(r))$ for some functions $G : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}$ and $H : [0, 1] \mapsto \mathbb{R}$ such that $y \mapsto G(t, y)$ is continuous for each t, and H is monotone. In this way, the (potential) discontinuity in rank is decoupled from time variable. To fix notation, let

$$\mathcal{R}_D := \big\{R \in \mathcal{R} : R(t, r) = G\big(t, H(r)\big),$$

$$y \mapsto G(t, y) \text{ is continuous for each } t, H \text{ is monotone}\big\},$$

where D stands for "decoupled" or "decomposable."

There is another case that is sufficient for existence of a fixed point. Consider piecewise constant H of the form

$$(4.4) \quad H(r) = \sum_{k=1}^d R_k 1_{[r_{k-1}, r_k)}(r) + R_{d+1} 1_{[r_d, 1]}, \quad 0 = r_0 < r_1 < \cdots < r_d < 1,$$

and define

$$\mathcal{R}_S := \big\{R \in \mathcal{R} : R(t, r) = G\big(t, H(r)\big), H \text{ is of the form } (4.4)\big\},$$

where S stands for "step function." The nice thing about $R \in \mathcal{R}_S$ is that R_μ depends on μ only through its quantiles $(T_1^\mu, \ldots, T_d^\mu)$; the shape of μ between each

neighboring T_k^μ's becomes irrelevant. In other words, each μ can be encoded to d numbers. As a consequence, we have a fixed-point problem in a d-dimensional space instead of the infinite dimensional space of measures.

We now state our general existence theorem.

THEOREM 4.1. *Assume $R \in \mathcal{R}_D \cup \mathcal{R}_S$. Then $\Phi^{t_0, \pi, m}$ has a fixed point. Thus, an equilibrium exists.*

PROOF. Assume first $T < \infty$. We consider $\mathcal{R}_D$ and $\mathcal{R}_S$, separately.

(a) Assume $R \in \mathcal{R}_D$. In order for F_μ not to hit the (potential) discontinuity of H on a nonnegligible subset of $[t_0, T]$, we need to make sure that it is not flat there. Observe from (4.3) that any fixed-point μ of $\Phi^{t_0, \pi, m}$, if it exists, must have a strictly increasing c.d.f. on $[t_0, T]$. This motivates us to define the following nonempty, convex subset of $\mathcal{P}(\mathbb{T})$:

$$(4.5) \qquad \mathcal{D}_{\underline{f}} := \left\{ \mu \in \mathcal{P}(\mathbb{T}) : \mu([a, b]) \geq \int_a^b \underline{f}(t) \, dt \ \forall a, b \in [t_0, T] \right\},$$

where

$$(4.6) \qquad \underline{f}(t) := (1 - \pi) \exp\left(\frac{R_\infty - R(0, 0)}{2\bar{c}\sigma^2} \right) E f_{\tau_{\xi/\sigma}^\circ}(t - t_0) > 0.$$

We claim then that:

    (i) $\Phi^{t_0, \pi, m}(\mathcal{P}(\mathbb{T})) \subseteq \mathcal{D}_{\underline{f}}$.
   (ii) $\mathcal{D}_{\underline{f}}$ is closed under weak convergence.
  (iii) Any $\mu \in \mathcal{D}_{\underline{f}}$ has strictly increasing c.d.f. on $[t_0, T]$.

(i) follows from (4.3) and (2.8). (ii) holds because μ_n converges to μ weakly if and only if $\limsup_n \mu_n(C) \leq \mu(C)$ for any closed sets $C \subseteq \mathbb{T}$. (iii) is obvious.

(ii) implies that $\mathcal{D}_{\underline{f}}$ is a closed subset of the compact space $\mathcal{P}(\mathbb{T})$, hence is also compact. To apply Schauder's fixed-point theorem in $\mathcal{M}(\mathbb{T})$, it remains to show that $\Phi^{t_0, \pi, m}$ is continuous on $\mathcal{D}_{\underline{f}}$. For metrizable spaces, continuity is equivalent to sequential continuity.

Let $\{\mu_k\} \subseteq \mathcal{D}_{\underline{f}}$ converge to μ weakly. We wish to show $\mathcal{L}(\tau_{\mu_k})$ converges to $\mathcal{L}(\tau_\mu)$ weakly. It suffices to show that $E\varphi(\tau_{\mu_k}) \to E\varphi(\tau_\mu)$ for any $\varphi \in C_b(\mathbb{T})$. From (4.1)–(4.3), we see that

$$E\varphi(\tau_{\mu_k}) = \varphi(t_0)\pi + (1 - \pi) E\left[\int_{t_0}^T \varphi(t) \frac{u(t, 0; \mu_k, c)}{u(t_0, \xi; \mu_k, c)} f_{\tau_{\sigma/\xi}^\circ}(t - t_0) \, dt \right]$$

$$+ \varphi(\Delta)(1 - \pi)\left(1 - E\left[\int_{t_0}^T \frac{u(t, 0; \mu_k, c)}{u(t_0, \xi; \mu_k, c)} f_{\tau_{\sigma/\xi}^\circ}(t - t_0) \, dt \right] \right).$$

Since u is bounded by a constant independent of μ_k, to show the expected values converge, we only need to show that for every $x \geq 0$ and realization of c, $u(t, x; \mu_k, c)$ converges to $u(t, x; \mu, c)$ for a.e. $t \in (t_0, T]$.

Weak convergence of μ_k to μ implies $F_{\mu_k}(t)$ converges to $F_\mu(t)$ at every point of continuity of F_μ. Since $F_\mu(t)$ and $H(r)$ are monotone, they have at most countably many points of discontinuity, denoted by $t_1, t_2, \ldots$ and $r_1, r_2, \ldots$, respectively. Now, since $\mu \in \mathcal{D}_f$, F_μ is strictly increasing on $[t_0, T]$ by item (iii) above. It follows that each r_i can only be attained by $F_\mu(t)$ for at most one $t \in [t_0, T]$, denoted by $\tilde{t}_i$. For any t not belonging to the countable set $\{t_i, \tilde{t}_i\}_{i=1,2,\ldots}$, we have that $F_{\mu_k}(t)$ converges to $F_\mu(t)$, that $H(F_{\mu_k}(t))$ converges to $H(F_\mu(t))$ and that $R_{\mu_k}(t) = G(t, H(F_{\mu_k}(t)))$ converges to $R_\mu(t) = G(t, H(F_\mu(t)))$. The bounded convergence theorem applied to (2.4) then yields the pointwise convergence of $u(t, x; \mu_k, c)$ to $u(t, x; \mu, c)$.

(b) Assume $R \in \mathcal{R}_S$. Let $T_k^\mu = T_{r_k}^\mu$ be the (r_k)-quantile of μ for $k = 1, \ldots, d$, and set $T_0^\mu = 0$, $T_{d+1}^\mu = \Delta$. Exploiting the piecewise constant structure of H, we have

$$(4.7) \qquad R_\mu(t) = \sum_{k=1}^{d+1} G(t, R_k) 1_{[T_{k-1}^\mu, T_k^\mu]}(t), \quad t \in [t_0, T].$$

Since R_μ depends on μ only through its quantiles $(T_1^\mu, \ldots, T_d^\mu)$, as pointed out earlier, the fixed-point problem can be reduced to a finite dimensional one in the space

$$\mathcal{Q} := \{(T_1, \ldots, T_d) \in \widetilde{\mathbb{T}}^d : T_1 \le T_2 \le \cdots \le T_d\},$$

where $\widetilde{\mathbb{T}} := [t_0, \Delta]$ is the convex hull of $\mathbb{T}$.

We will use the notation $R_{\mathbf{q}}(t)$, $u(t, x; \mathbf{q}, c)$ and $\tau_{\mathbf{q}}$ instead of $R_\mu(t)$, $u(t, x; \mu, c)$ and τ_μ, where $\mathbf{q} = (T_1, \ldots, T_d)$ is a vector of quantiles. The best response mapping is reformulated as

$$\Theta : \mathbf{q} \mapsto \left(T_1^{\mathcal{L}(\tau_{\mathbf{q}})}, \ldots, T_d^{\mathcal{L}(\tau_{\mathbf{q}})}\right).$$

Any fixed-point $\mathbf{q}$ of Θ induces a fixed-point $\mathcal{L}(\tau_{\mathbf{q}})$ of $\Phi^{t_0, \pi, m}$. Note that $\mathcal{Q}$ is a convex, compact set which is mapped into itself under Θ. Once we show Θ is continuous on $\mathcal{Q}$, we can use Brouwer's fixed-point theorem to conclude that Θ has a fixed point. Continuity is proved as follows.

Let $\mathbf{q}_n = (T_1^n, \ldots, T_d^n) \to \mathbf{q} = (T_1, \ldots, T_d)$ in $\mathcal{Q}$. It is easy to see from (4.7) that $R_{\mathbf{q}_n}(t)$ converges to $R_{\mathbf{q}}(t)$ if $t \notin \{T_1, \ldots, T_d\}$. Since $\tau_{x/\sigma}^\circ$ is nonatomic, bounded convergence theorem implies for any $(t, x) \in [0, T] \times \mathbb{R}_{++}$ and any (positive) realization of c,

$$u(t, x; \mathbf{q}_n, c) \to u(t, x; \mathbf{q}, c) \quad \text{as } n \to \infty.$$

Once we have the pointwise convergence of u, we can then use (4.3) to obtain weak convergence of $\mathcal{L}(\tau_{\mathbf{q}_n})$ to $\mathcal{L}(\tau_{\mathbf{q}})$. To show $T_k^{\mathcal{L}(\tau_{\mathbf{q}_n})} \to T_k^{\mathcal{L}(\tau_{\mathbf{q}})}$, we consider three cases of $k \in \{1, \ldots, d\}$.

Case (i). If $r_k \le \pi$, then by our construction, $T_k^{\mathcal{L}(\tau_{\mathbf{q}_n})} = T_k^{\mathcal{L}(\tau_{\mathbf{q}})} = t_0$.

Case (ii). If $\pi < r_k \leq F_{\tau_{\mathbf{q}}}(T)$, then we observe from (4.3) that $\tau_{\mathbf{q}}$ has strictly positive density in (t_0, T). So the quantile function of $\mathcal{L}(\tau_{\mathbf{q}})$ is continuous in $(\pi, F_{\tau_{\mathbf{q}}}(T)]$. Weak convergence of $\mathcal{L}(\tau_{\mathbf{q}_n})$ to $\mathcal{L}(\tau_{\mathbf{q}})$ then implies $T_k^{\mathcal{L}(\tau_{\mathbf{q}_n})} \to T_k^{\mathcal{L}(\tau_{\mathbf{q}})}$.

Case (iii). If $r_k > F_{\tau_{\mathbf{q}}}(T)$, then $T_k^{\mathcal{L}(\tau_{\mathbf{q}})} = \Delta$. By weak convergence of $\mathcal{L}(\tau_{\mathbf{q}_n})$ to $\mathcal{L}(\tau_{\mathbf{q}})$ and the continuity of $F_{\tau_{\mathbf{q}}}$ at time T, we have that $F_{\tau_{\mathbf{q}_n}}(T) \to F_{\tau_{\mathbf{q}}}(T)$. This implies $r_k > F_{\tau_{\mathbf{q}_n}}(T)$ and hence $T_k^{\mathcal{L}(\tau_{\mathbf{q}_n})} = \Delta$ for n sufficiently large.

Finally, we consider the case $T = \infty$. For the case $R \in \mathcal{R}_D$, let us view $[t_0, \infty]$ as the one-point compactification of $[t_0, \infty)$ which is homeomorphic to $[0, 1]$. Then $\mathcal{P}([t_0, \infty])$ is again a compact, convex subset of the space of finite signed measures $\mathcal{M}([t_0, \infty])$ endowed with the Kantorovich–Rubinstein norm. It can be shown that weak convergence of μ_k to μ in $\mathcal{P}([t_0, \infty])$ still implies the convergence of F_{μ_k} to F_μ at the continuity points of F_μ (here ∞ is a continuity point of F_μ if and only if $\mu(\{\infty\}) = 0$), from which we can obtain $u(t, x; \mu, c) \to u(t, x; \mu, c)$ pointwise on $[0, \infty) \times \mathbb{R}_+$, and $E\varphi(\tau_{\mu_k}) \to E\varphi(\tau_\mu)$ for any $\varphi \in C_b([t_0, \infty])$. Thus, the proof from the case $T < \infty$ remains valid.

For the case $R \in \mathcal{R}_S$, we identify $\mathcal{Q} = \{(T_1, \ldots, T_d) \in [t_0, \infty]^d : T_1 \leq T_2 \leq \cdots \leq T_d\}$ with the convex, compact set $\mathcal{Y} := \{(y_1, \ldots, y_d) \in [0, 1]^d : y_1 \leq y_2 \leq \cdots \leq y_d\}$ under some homeomorphism h. Since the extended real line is not a vector space, Schauder's fixed-point theorem does not directly apply to $\mathcal{Q}$. Instead, we look for a fixed point of $\Xi := h \circ \Theta \circ h^{-1}$ in $\mathcal{Y}$, where Θ is the original best response mapping. Any fixed-point $\mathbf{y}$ of Ξ induces a fixed-point $\mathbf{q} = h^{-1}(\mathbf{y})$ of Θ. Since h is a homeomorphism, Ξ is continuous if and only if Θ is continuous. The proof then proceeds similarly as before. □

4.2. *Uniqueness of the fixed point.* Denote by $\mathcal{AC}_{t_0}^T$ the set of measures $\mu \in \mathcal{P}(\mathbb{R}_+)$ with c.d.f. F_μ that is absolutely continuous on $[t_0, T] \cap [t_0, \infty)$. The following monotonicity condition, which we will show is sufficient for uniqueness, is in the same spirit as Lasry and Lions (2007). The proofs in this section are slight variations of those in Bayraktar and Zhang (2016), Section 3.2. So we only provide a sketch here. Interested readers can refer to arXiv:1811.00076 for details.

ASSUMPTION 4.1. For any $\mu, \mu' \in \mathcal{AC}_{t_0}^T$ such that $F_\mu(t_0) = F_{\mu'}(t_0)$, we have

$$\int_{t_0}^T (R_\mu - R_{\mu'})(t) \, d(\mu - \mu')(t) \leq 0.$$

PROPOSITION 4.1. *Assumption 4.1 is satisfied if the function*

$$h(t, x, y) := \begin{cases} \dfrac{R(t, x) - R(t, y)}{x - y} & t \in [t_0, T] \cap [t_0, \infty), 0 \leq x \neq y \leq 1, \\ \dfrac{\partial}{\partial x} R(t, x) & t \in [t_0, T] \cap [t_0, \infty), 0 \leq x = y \leq 1 \end{cases}$$

is well-defined, nonpositive, increasing in t, x, y, *and the functions* $R(t, F_\mu(t))$, $h(t, F_\mu(t), F_{\mu'}(t))$ *are absolutely continuous on* $[t_0, T] \cap [t_0, \infty)$ *for any* $\mu, \mu' \in AC_{t_0}^T$. *In particular, Assumption* 4.1 *is satisfied for reward functions of the form* $R(t, r) = 1_{\{t \leq T\}} H(r) + 1_{\{t > T\}} R_\infty$, *where* H *is a convex, decreasing,* $C^2[0, 1]$ *function, such that* $H(r) \geq R_\infty$ *for all* r.

PROOF. Using integration by parts for absolutely continuous functions, one can show that

$$2 \int_{t_0}^T (R_\mu - R_{\mu'})(t) \, d(\mu - \mu')(t) = \left. (F_\mu(t) - F_{\mu'}(t))^2 h(t, F_\mu(t), F_{\mu'}(t)) \right|_{t=t_0}^T$$
$$- \int_{t_0}^T (F_\mu - F_{\mu'})^2(t) \, dh(t, F_\mu(t), F_{\mu'}(t)).$$

The boundary term is nonpositive since $F_\mu(t_0) = F_{\mu'}(t_0)$ and $h \leq 0$. The remaining term is also nonpositive since $h(t, F_\mu(t), F_{\mu'}(t))$ is increasing. □

THEOREM 4.2. *Under Assumption* 4.1, $\Phi^{t_0, \pi, m}$ *has at most one fixed point.*

PROOF. Suppose μ and μ' are two fixed points of $\Phi^{t_0, \pi, m}$. Write $v(t, x; c) := v(t, x; \mu, c)$ and $v'(t, x; c) := v(t, x; \mu', c)$. Let (θ, ξ, c) be a randomization according to the initial condition (t_0, π, m), and let X^μ, $X^{\mu'}$, τ_μ, $\tau_{\mu'}$ be constructed as in Section 4.1. By Assumption 4.1, Itô's lemma, the PDE satisfied by v and v', and that $\mu = \Phi^{t_0, \pi, m}(\mu) = \mathcal{L}(\tau_\mu)$, $R_\mu(t_0) = R(t_0, \pi) = R_{\mu'}(t_0)$ and $R_\mu(t) = R_{\mu'}(t) = R_\infty$ for any $t > T$, one can show that

$$0 \geq \int_{t_0}^T [R_\mu(t) - R_{\mu'}(t)] \, d(\mu - \mu')(t)$$
$$= E\left[\frac{1_{\{\theta=0\}}}{4c} \left\{ \int_{t_0}^{\tau_\mu \wedge T} (v_x - v'_x)^2(t, X_t^\mu; c) \, dt \right. \right.$$
$$\left. \left. + \int_{t_0}^{\tau_{\mu'} \wedge T} (v'_x - v_x)^2(t, X_t^{\mu'}; c) \, dt \right\} \right]$$
$$\geq 0.$$

So on the set $\{\theta = 0\}$, we must have

$$v_x(t, X_t^\mu; c) 1_{\{t_0 \leq t < \tau_\mu \wedge T\}} = v'_x(t, X_t^\mu; c) 1_{\{t_0 \leq t < \tau_\mu \wedge T\}} P \times dt\text{-a.e.}$$

By the uniqueness of the solution of the SDE (2.1), X^μ and $X^{\mu'}$ are indistinguishable up to time $\tau_\mu \wedge T$. Thus, on the set $\{\tau_\mu \leq T\}$, we have $\tau_\mu = \tau_{\mu'}$ a.s. On the set $\{\tau_\mu > T\}$, we also have $\tau_{\mu'} > T$, and consequently, $\tau_\mu = \tau_{\mu'} = \Delta$ by our construction. It remains to note that $\tau_\mu = \tau_{\mu'}$ a.s. implies $\mu = \mathcal{L}(\tau_\mu) = \mathcal{L}(\tau_{\mu'}) = \mu'$. □

4.3. *Stability of the fixed point.* How stable or sensitive is the equilibrium with respect to changes in the reward function? This question is not only of interest in its own, but also provides the basis for numerical computation of the mean field equilibrium (see the beginning of Section 5). Specifically, the stability result guarantees that the sequence of equilibria of discretized (mean field) games, if converging, will converge weakly to an equilibrium of the original game.

Introduce a Lipschitz condition:

$$\mathcal{R}_{DL} := \{ R \in \mathcal{R}_D : \exists L_G \in \mathbb{R} \text{ s.t.}$$

$$\left| G(t, y_1) - G(t, y_2) \right| \le L_G |y_1 - y_2| \, \forall t \in \mathbb{R}_+, y_1, y_2 \in \mathbb{R} \}.$$

We then have the following stability (continuity) result.

THEOREM 4.3. *Suppose* $\{ R_n(t, r) = G(t, H_n(r)) \} \subseteq \mathcal{R}_{DL}$ *is a sequence of reward functions which converges to* $R(t, r) = G(t, H(r)) \in \mathcal{R}_{DL}$ *in the following sense*:

$$\| H_n - H \|_{L^1[0,1]} \to 0 \quad \text{as } n \to \infty.$$

Let μ_n *be a fixed point of the best response mapping* $\Phi_n^{t_0, \pi, m}$ *associated with* R_n (*given by Theorem* 4.1). *Then* μ_n *has a subsequence which converges weakly to a fixed-point* μ *of the best response mapping* $\Phi^{t_0, \pi, m}$ *associated with* R.

PROOF. Assume first $T < \infty$. First, we construct a weak limit. Since each μ_n belongs to the compact space $\mathcal{P}(\mathbb{T})$, we can extract a subsequence of μ_n which converges weakly to some μ. With a slight abuse of notation, we still denote the convergent subsequence by μ_n.

Next, from part (a) of the the proof of Theorem 4.1, we know $\Phi^{t_0, \pi, m}$ is continuous on $\mathcal{D}_{\underline{f}}$. It follows that $\tilde{\mu}_n := \Phi^{t_0, \pi, m}(\mu_n)$ converges weakly to $\Phi^{t_0, \pi, m}(\mu)$. The main task is to show the the c.d.f.s of $\mu_n = \Phi_n^{t_0, \pi, m}(\mu_n)$ and of $\tilde{\mu}_n = \Phi^{t_0, \pi, m}(\mu_n)$ are uniformly close for n large. More precisely, we claim that

$$(4.8) \qquad \| F_{\mu_n} - F_{\tilde{\mu}_n} \|_\infty \le C \| H_n - H \|_{L^1[0,1]}$$

for some constant C independent of n. Observe that by our definition of the best response mapping, both F_{μ_n} and $F_{\tilde{\mu}_n}$ stay constant after time T until time Δ where they jump to one. So to bound $|F_{\mu_n}(t) - F_{\tilde{\mu}_n}(t)|$, it suffices to restrict our attention to $t \in [t_0, T]$.

Denote by u_n the transformed value function associated with the reward function R_n. Using (2.4), the Lipschitz continuity of $G(t, \cdot)$, the assumption that $G(t, x) = R_\infty$ for all $t > T$, (4.3) and (2.8), we can obtain the following two estimates:

$$(4.9) \qquad E \left| u_n(t_0, \xi; \mu_n, c) - u(t_0, \xi; \mu_n, c) \right| \le C_1 \| H_n - H \|_{L^1[0,1]},$$

$$(4.10) \quad E \int_{t_0}^{T} \left| u_n(t, 0; \mu_n, c) - u(t, 0; \mu_n, c) \right| f_{\tau_{\xi/\sigma}^{\circ}}(t - t_0) \, dt$$

$$\leq C_1 \| H_n - H \|_{L^1[0,1]},$$

where

$$C_1 := \frac{L_G}{2\underline{c}\sigma^2(1 - \pi)} \exp\left(\frac{R(0,0)}{2\underline{c}\sigma^2} + \frac{R(0,0) - R_\infty}{2\overline{c}\sigma^2} \right).$$

Then we use (4.2), (2.8), (4.9) and (4.10) to deduce that

$$\sup_{t \in [t_0, T]} \left| F_{\mu_n}(t) - F_{\tilde{\mu}_n}(t) \right|$$

$$\leq C_1(1 - \pi) \left\{ \exp\left(\frac{R(0,0)}{2\underline{c}\sigma^2} - \frac{R_\infty}{\overline{c}\sigma^2} \right) + \exp\left(\frac{-R_\infty}{2\overline{c}\sigma^2} \right) \right\} \| H_n - H \|_{L^1[0,1]},$$

which proves (4.8).

Finally, we show the weak limit μ is a fixed point of $\Phi^{t_0, \pi, m}$. Since $\tilde{\mu}_n$ converges to $\Phi^{t_0, \pi, m}(\mu)$ weakly and $|F_{\mu_n} - F_{\tilde{\mu}_n}|$ converges to zero uniformly, we know μ_n also converges to $\Phi^{t_0, \pi, m}(\mu)$ weakly. By uniqueness of the weak limit, we must have $\Phi^{t_0, \pi, m}(\mu) = \mu$.

For $T = \infty$, we use the compactification of $\mathcal{P}([t_0, \infty])$ as in the proof of Theorem 4.1. $\quad\square$

4.4. *Approximate equilibrium of the N-player game.* We want to see when we can approximate a game with finitely many players with the game with infinitely many players. Our results will be derived for the reward functions that are Hölder continuous in the rank variable. For $\alpha \in (0, 1]$, define

$$\mathcal{R}_{C^\alpha} := \big\{ R \in \mathcal{R} : \exists L_\alpha \in \mathbb{R} \text{ s.t.}$$

$$\left| R(t, r_1) - R(t, r_2) \right| \leq L_\alpha |r_1 - r_2|^\alpha \forall t \in \mathbb{R}_+, r_1, r_2 \in \mathbb{R} \big\}$$

$$\subseteq \mathcal{R}_D.$$

For simplicity of the presentation, we consider only a freshly started tournament, that is, $t_0 = 0$, and $\pi = 0$.

The N-player system is given by

$$X_{i,t}^{a_i} = x_i - \int_0^t a_{i,s} \, ds + \sigma B_{i,t}, \quad i = 1, \ldots, N,$$

where $B_1, \ldots, B_N$ are independent Brownian motions, and $\mathbf{a} = (a_1, \ldots, a_N)$ is a vector of admissible actions, meaning that each a_i is nonnegative and progressively measurable with respect to the filtration of $B_1, \ldots, B_N$, and yields a unique strong solution $X_i^{a_i}$ up to $\tau_i^{a_i} := \inf\{t \geq 0 : X_{i,t}^{a_i} = 0\}$, and if $\tau_i^{a_i}$ is nonatomic.

Let $\bar{\mu}^{N,\mathbf{a}} = \frac{1}{N} \sum_{i=1}^{N} \delta_{\tau_i^{a_i}}$ be the empirical distribution of the completion time.

DEFINITION 4.1. An admissible action vector $\mathbf{a} = (a_1, \ldots, a_N)$ is called an ϵ-Nash equilibrium action of the N-player game if for any $i \in \{1, \ldots, N\}$ and any admissible control β, we have

$$E\left[R_{\bar{\mu}^N, \mathbf{a}}\left(\tau_i^{a_i}\right) - \int_0^{\tau_i^{a_i} \wedge T} c_i a_{i,t}^2 \, dt\right] + \epsilon \geq E\left[R_{\bar{\mu}^N, \mathbf{a}_\beta^i}\left(\tau_i^\beta\right) - \int_0^{\tau_i^\beta \wedge T} c_i \beta_t^2 \, dt\right],$$

where $\mathbf{a}_\beta^i = (a_1, \ldots, a_{i-1}, \beta, a_{i+1}, \ldots, a_N)$.

We have the following approximation result.

THEOREM 4.4. *Assume* $R \in \mathcal{R}_{C^\alpha}$ *for some* $\alpha \in (0, 1]$ *and* (x_i, c_i), $i = 1, \ldots, N$ *are i.i.d. samples from* m. *For any fixed-point* μ *of* $\Phi^{0,0,m}$, *the vector* $\bar{\mathbf{a}} = (\bar{a}_1, \ldots, \bar{a}_N)$ *with*

$$\bar{a}_{i,t} := -(2c_i)^{-1} v_x(t, X_{i,t}^{\bar{a}_i}; \mu, c_i) 1_{\{t < \tau_i^{\bar{a}_i}\}}, \quad i = 1, \ldots, N$$

is an $O(N^{-\alpha/2})$-*Nash equilibrium action of the* N-*player game, as* $N \to \infty$.

PROOF. Let μ and $\bar{a}_{i,t}$ be defined as in the theorem statement. To simplify notation, we omit the superscript of any state process X_i and its first passage time τ_i if it is controlled by the optimal Markovian feedback strategy $-(2c_i)^{-1} v_x(t, x; \mu, c_i)$. For a given (deterministic) initial position x and cost c, let

$$V(x, c) := v(0, x; \mu, c) = E\left[R_\mu\left(\tau^{x,c}\right) - \int_0^{\tau^{x,c} \wedge T} \frac{1}{4c} v_x^2(t, X_t^{x,c}; \mu, c) \, dt\right]$$

be the value of the game with infinitely many players, where

$$dX_t^{x,c} = -\frac{v_x(t, X_t^{x,c}; \mu, c)}{2c} 1_{\{t < \tau^{x,c}\}} \, dt + \sigma \, dB_t, \quad X_0^{x,c} = x$$

and $\tau^{x,c}$ is the first passage time of $X^{x,c}$ to level zero with distribution $q(\cdot; x, c)$. Since $X_i|_{(x_i, c_i)}$ is an identical copy of X^{x_i, c_i}, we have

$$V(x_i, c_i) = E\left[R_\mu(\tau_i) - \int_0^{\tau_i \wedge T} c_i \bar{a}_{i,t}^2 \, dt \Big| x_i, c_i\right].$$

It is also easy to see that the τ_i's are i.i.d. random variables with distribution $m \otimes q = \mu$. Let

$$\bar{\mu}^N := \frac{1}{N} \sum_{i=1}^N \delta_{\tau_i}$$

be their empirical distribution, and

$$J_i^N := E\left[R_{\bar{\mu}^N}(\tau_i) - \int_0^{\tau_i \wedge T} c_i \bar{a}_{i,t}^2 \, dt \Big| x_i, c_i\right]$$

be the (random) net gain of player i in an N-player game when everybody uses the candidate approximate Nash equilibrium $\bar{\mathbf{a}}$. We first show that J_i^N and $V(x_i, c_i)$ are close.

Rewrite $\bar{\mu}^N$ as $\bar{\mu}^N = \frac{1}{N}\delta_{\tau_i} + \frac{N-1}{N}\bar{\mu}_{-i}^N$ where

$$\bar{\mu}_{-i}^N := \frac{1}{N-1}\sum_{j\neq i}\delta_{\tau_j}$$

is the empirical distribution of $(N-1)$ i.i.d. samples from the distribution μ. By Dvoretzky–Kiefer–Wolfowitz inequality,

$$P\big(\|F_{\bar{\mu}_{-i}^N} - F_\mu\|_\infty > z\big) \leq 2e^{-2(N-1)z^2} \quad \forall z > 0,$$

which implies that

$$E\big[\|F_{\bar{\mu}_{-i}^N} - F_\mu\|_\infty\big] = \int_0^\infty P\big(\|F_{\bar{\mu}_{-i}^N} - F_\mu\|_\infty > z\big)\,dz$$

$$\leq \int_0^\infty 2e^{-2(N-1)z^2}\,dz = \frac{2}{(N-1)^{1/2}}\int_0^\infty e^{-2y^2}\,dy.$$

By the α-Hölder continuity of R, we then have

$$V(x_i, c_i) - J_i^N$$

$$= E\big[(R_\mu(\tau_i) - R_{\bar{\mu}^N}(\tau_i))|x_i, c_i\big] \leq L_\alpha E\big[|F_{\bar{\mu}^N}(\tau_i) - F_\mu(\tau_i)|^\alpha|x_i, c_i\big]$$

$$= L_\alpha E\bigg[\bigg|\frac{1}{N}(1 - F_\mu(\tau_i)) + \frac{N-1}{N}(F_{\bar{\mu}_{-i}^N}(\tau_i) - F_\mu(\tau_i))\bigg|^\alpha\bigg|x_i, c_i\bigg]$$

$$\leq L_\alpha E\bigg[\bigg(\frac{1}{N} + \frac{N-1}{N}\|F_{\bar{\mu}_{-i}^N} - F_\mu\|_\infty\bigg)^\alpha\bigg]$$

$$\leq L_\alpha\bigg(\frac{1}{N} + \frac{N-1}{N}E\|F_{\bar{\mu}_{-i}^N} - F_\mu\|_\infty\bigg)^\alpha = O(N^{-\alpha/2}) \quad \text{as } N \to \infty.$$

Note that we can drop the conditioning because $\bar{\mu}_{-i}^N$ is independent of (x_i, c_i).

Next, consider the system where player i makes a unilateral deviation from the candidate approximate Nash equilibrium $\bar{\mathbf{a}}$; say, she chooses an admissible control β. Denote her controlled state process by X_i^β, and the state processes of all other players by X_j as before for $j \neq i$. Let

$$\bar{\nu}^N := \frac{1}{N}\bigg(\delta_{\tau_i^\beta} + \sum_{j\neq i}\delta_{\tau_j}\bigg) = \frac{1}{N}\delta_{\tau_i^\beta} + \frac{N-1}{N}\bar{\mu}_{-i}^N$$

be the corresponding empirical measure of the completion times, and

$$J_i^{N,\beta} := E\bigg[R_{\bar{\nu}^N}(\tau_i^\beta) - \int_0^{\tau_i^\beta \wedge T} c_i\beta_t^2\,dt\bigg|x_i, c_i\bigg]$$

be the corresponding net gain for player i. We have

$$
J_i^{N,\beta} - V(x_i, c_i)
$$
$$
= E\big[R_{\bar{\nu}^N}(\tau_i^\beta) - R_\mu(\tau_i^\beta)|x_i, c_i\big]
$$
$$
+ E\bigg[R_\mu(\tau_i^\beta) - \int_0^{\tau_i^\beta \wedge T} c_i \beta_t^2 \, dt \bigg| x_i, c_i\bigg] - E\bigg[R_\mu(\tau_i) - \int_0^{\tau_i \wedge T} c_i \bar{a}_{i,t}^2 \, dt \bigg| x_i, c_i\bigg]
$$
$$
\leq E\big[R_{\bar{\nu}^N}(\tau_i^\beta) - R_\mu(\tau_i^\beta)|x_i, c_i\big],
$$

where we have used the optimality of $\bar{a}_i$ for the ith player's problem (in response to μ) in the last step. ($\bar{a}_i$ is optimal in the filtration of $B_1, \ldots, B_N$ by Remark 2.1.) Similar to how we estimated $V(x_i, c_i) - J_i^N$, we can show

$$
J_i^{N,\beta} - V(x_i, c_i)
$$
$$
\leq L_\alpha E\bigg[\bigg|\frac{1}{N}\big(1 - F_\mu(\tau_i^\beta)\big) + \frac{N-1}{N}\big(F_{\bar{\mu}_{-i}^N}(\tau_i^\beta) - F_\mu(\tau_i^\beta)\big)\bigg|^\alpha \bigg| x_i, c_i\bigg]
$$
$$
\leq L_\alpha E\bigg[\bigg(\frac{1}{N} + \frac{N-1}{N}\|F_{\bar{\mu}_{-i}^N} - F_\mu\|_\infty\bigg)^\alpha\bigg] = O\big(N^{-\alpha/2}\big) \quad \text{as } N \to \infty.
$$

Combining the two estimates, we obtain

$$
J_i^{N,\beta} - J_i^N = J_i^{N,\beta} - V(x_i, c_i) + \big(V(x_i, c_i) - J_i^N\big) \leq O\big(N^{-\alpha/2}\big) \quad \text{as } N \to \infty.
$$

This shows $\bar{\mathbf{a}}$ is an $O(N^{-\alpha/2})$-Nash equilibrium action. $\square$

REMARK 4.1. For reward functions that are merely continuous in the rank variable, we still have convergence of the approximation, but we do not know the convergence rate.

5. A case with heterogeneous players. We end our discussion with a numerically computed example with heterogeneous agents, assuming $R(t, r) = 1_{\{t \leq T\}} 15(1 - r)^2$ under which the (Nash) equilibrium is unique. When the players are heterogeneous, there is generally no explicit characterization of the equilibrium. We numerically solve the fixed-point equation

$$
F_\mu(t) = E\bigg[\int_0^t \frac{u(z, 0; \mu, c)}{u(0, \xi; \mu, c)} f_{\tau_{\xi/\sigma}^\circ}(z) \, dz\bigg], \quad t \in [0, T]
$$

by discretizing the rank space, which is equivalent to approximating $R \in \mathcal{R}$ by elements in $\mathcal{R}_S$.[4] From the stability result (Theorem 4.3), we know that the sequence

---

[4]There is an advantage of discretizing the rank variable instead of time variable: when T is large, a fine discretization of $[0, T]$ may result in a high computational burden. In contrast, the rank space $[0, 1]$ remains fixed.

of equilibria of the discretized games, if converging, will converge weakly to the unique equilibrium of the original game. We also know from Section 4.1 that the equilibrium distribution μ of the discretized game can be represented by a vector $(T_1^\mu, \ldots, T_d^\mu)$, where T_k^μ is the (r_k)-quantile of μ and $0 = r_0 < r_1 \cdots < r_d < 1$ is a partition of the rank space $[0, 1]$. This allows us to compute μ by solving a nonlinear system of algebraic equations for $(T_1^\mu, \ldots, T_d^\mu)$.

We consider players that differ in the initial location and cost, with $x_0 \in \{1, 2\}$ and $c \in \{1, 4\}$. Other model parameters are fixed to be $T = 1$ and $\sigma = 0.25$. Denote by β the population terminal completion rate and by $\beta_{\mathbf{x}}$ the terminal completion rate of the players of type $\mathbf{x}$ relative to its initial weight. Similarly, $V_{\mathbf{x}}$ refers to the game value for a player of type $\mathbf{x}$. The total welfare in the heterogeneous setting is given by

$$\text{Total welfare} = \int V(x_0, c) \, dm(x_0, c).$$

Let us call "disadvantaged" (DA) the players who start from a larger distance, or who have a higher cost of effort. Analogously for "advantaged" (AD) players.

Table 2 shows that for those parameters, the following happens:

- As the fraction of DA players in the population increases, the population completion rate decreases, but the completion rates of the DA group and of the AD group both increase. That is, the aggregate percentage of completions is lower because more of the population is disadvantaged, and not because of working less hard. In fact, both the value of DA players and the value of AD players go up with the population fraction of DA players. This is because DA players find

TABLE 2
Equilibrium completion rates and game values under different population composition

Case	m	β	β_{AD}	β_{DA}	V_{AD}	V_{DA}	Total welfare
0	$\delta_{(1,1)}$	75.9%	75.9%	–	0.178	–	0.178
1	$\frac{4}{5}\delta_{(1,1)} + \frac{1}{5}\delta_{(2,1)}$	73.8%	92.2%	0%	0.319	0	0.255
2	$\frac{3}{5}\delta_{(1,1)} + \frac{2}{5}\delta_{(2,1)}$	60%	100%	0%	1.701	0	1.020
3	$\frac{2}{5}\delta_{(1,1)} + \frac{3}{5}\delta_{(2,1)}$	49.8%	100%	16.4%	4.276	0.022	1.724
4	$\frac{1}{5}\delta_{(1,1)} + \frac{4}{5}\delta_{(2,1)}$	49.8%	100%	37.3%	7.338	0.058	1.514
5	$\delta_{(2,1)}$	49.8%	–	49.8%	–	0.086	0.086
6	$\frac{4}{5}\delta_{(1,1)} + \frac{1}{5}\delta_{(1,4)}$	73.8%	92.2%	0.1%	0.319	0	0.255
7	$\frac{3}{5}\delta_{(1,1)} + \frac{2}{5}\delta_{(1,4)}$	60.4%	100%	0.9%	1.675	0.005	1.007
8	$\frac{2}{5}\delta_{(1,1)} + \frac{3}{5}\delta_{(1,4)}$	51.9%	100%	19.8%	4.030	0.110	1.678
9	$\frac{1}{5}\delta_{(1,1)} + \frac{4}{5}\delta_{(1,4)}$	51.8%	100%	39.8%	7.091	0.253	1.621
10	$\delta_{(1,4)}$	51.8%	–	51.8%	–	0.365	0.365

it more advantageous to put in higher effort when the fraction of AD players is lower, due to weaker competition, so that a higher percentage of DA players completes the task and enjoys a higher value.

- The aggregate welfare is not monotone in the percentage of DA players: it goes up with the fraction of DA players only in the lower and mid range of the DA fraction, unlike the groups' welfares that always go up. When the fraction of DA players becomes very high, the total population welfare starts decreasing. This is the result of two conflicting forces: quality (higher individual value) versus quantity (fewer AD individuals).

These results support two (not very surprising) empirical predictions: (i) in low-growth industries, or in nonprofit institutions we should not see tournament-based compensation, especially when they have a high percentage of high-skilled employees; (ii) in undeveloped countries with large differences in access to good education, those with less access would give up early.

To recap, a more efficient society (more AD workers) has a higher productivity (larger β) and may have a higher total welfare, but it makes, in this example, the individuals worse off, because the efficient workers tend to work too hard and the inefficient workers tend to work too little.

Figure 7 shows the equilibrium effort corresponding to Cases 6 and 9 in Table 2, varying the fraction of low cost versus high cost players. (Other cases lead to similar observations.) Comparing the effort level between the left two panels and the right two panels, we see that the low-cost players exert higher effort than the high-cost players. Their effort region is also larger. On the other hand, players also tend to play to their opponents' level. A comparison of the top left panel with the bottom left panel demonstrates that the effort peak of AD players is significantly lower when they face weaker competition. Similarly, DA players tend to raise their effort when they face stronger competition until the competition gets too strong for them to keep up.

APPENDIX

A.1. Proofs of Theorems 3.4, 3.5 and 3.6. We first solve an auxiliary problem. Define

$$
\text{(A.1)} \qquad \mathcal{J}(h) := \int_0^\alpha h(r)\, dr, \qquad \mathcal{K}(h) := \int_0^\alpha -\ln h(r)\, dr
$$

and

$$
\text{(A.2)} \qquad
\begin{aligned}
\mathfrak{h} := \Big\{ h : [0, \alpha] \to \mathbb{R} : h \text{ is increasing, } 0 < h \le \exp\Big(-\frac{R_\infty}{2c\sigma^2}\Big), \\
\mathcal{K}(h) \le \frac{K - R_\infty(1 - \alpha)}{2c\sigma^2} \Big\}.
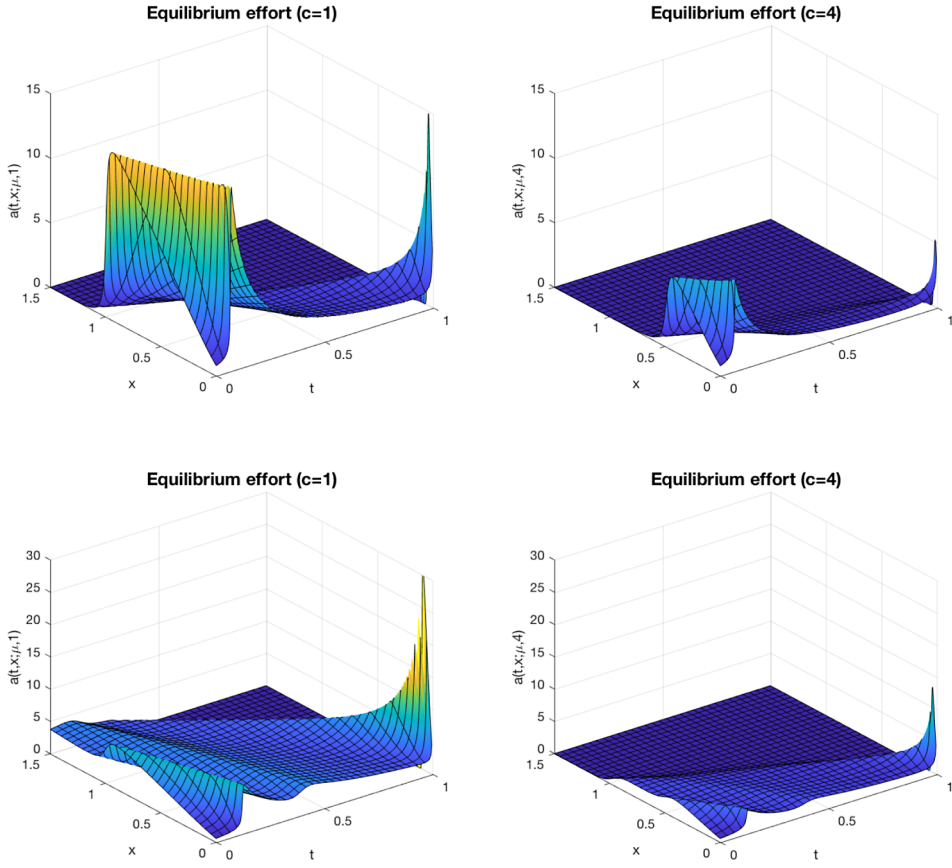\end{aligned}
$$

FIG. 7. *Effort function in an equilibrium formed by* 80% *low-cost players and* 20% *high-cost play-ers* (*top two panels*) *and* 20% *low-cost players and* 80% *high-cost players* (*bottom two panels*). *Everyone starts at $x_0 = 1$.*

LEMMA A.1. *Let $\alpha \in (0, 1]$ and $K \geq R_\infty$. Then $\inf_{h \in \mathfrak{h}} \mathcal{J}(h)$ is uniquely attained (up to a.e. equivalence) at*

$$h^* \equiv \exp\left(-\frac{K - R_\infty(1 - \alpha)}{2\alpha c \sigma^2}\right).$$

PROOF. We first show uniqueness. Suppose $h_1, h_2 \in \mathfrak{h}$ both minimize $\mathcal{J}$. Then, since $\mathcal{J}$ is linear and $\mathfrak{h}$ is convex, $h_\varepsilon := (1 - \varepsilon)h_1 + \varepsilon h_2$ is optimal for any $\varepsilon \in [0, 1]$. Note that any optimizer h necessarily satisfies $\mathcal{K}(h) = (K - R_\infty(1 - \alpha))/(2c\sigma^2)$, otherwise we can find $\lambda \in (0, 1)$ such that $\lambda h \in \mathfrak{h}$ and $\mathcal{J}(\lambda h) < \mathcal{J}(h)$. This implies $\mathcal{K}(h_\varepsilon) = (1 - \varepsilon)\mathcal{K}(h_1) + \varepsilon\mathcal{K}(h_2)$ for any $\varepsilon \in [0, 1]$. However, by the strict convexity of $z \mapsto -\ln z$ (and thus of $\mathcal{K}$), this can only happen when $h_1 = h_2$ a.e. on $[0, \alpha]$.

Next, we show h^* is an optimizer. Let $h \in \mathfrak{h}$ be arbitrary. We have by Jensen's inequality that

$$
\begin{aligned}
\mathcal{J}(h^*) &= \alpha \exp\left(-\frac{K - R_\infty(1-\alpha)}{2\alpha c\sigma^2}\right) \leq \alpha \exp\left(-\frac{1}{\alpha}\mathcal{K}(h)\right) \\
&= \alpha \exp\left(\frac{1}{\alpha}\int_0^\alpha \ln h(r)\, dr\right) \leq \alpha \exp\left(\ln\left(\frac{1}{\alpha}\int_0^\alpha h(r)\, dr\right)\right) = \mathcal{J}(h). \quad \square
\end{aligned}
$$

PROOF OF THEOREM 3.4. By Theorem 3.1(ii), we have

$$
T_\alpha(H) = F_{\tau_{x_0/\sigma}^\circ}^{-1}\left(\frac{\int_0^\alpha \exp(-\frac{H(r)}{2c\sigma^2})\, dr}{\int_0^1 \exp(-\frac{H(r)}{2c\sigma^2})\, dr}\right).
$$

First of all, it is clear from the above expression that one should only pay R_∞ beyond rank α, because $H 1_{[0,\alpha]} + R_\infty 1_{(\alpha,1]}$ performs no worse than H. Assuming $H = R_\infty$ on $(\alpha, 1]$, we further let $h(r) := \exp(-\frac{H(r)}{2c\sigma^2})$ and write

$$
T_\alpha(H) = F_{\tau_{x_0/\sigma}^\circ}^{-1}\left(\frac{\int_0^\alpha h(r)\, dr}{\int_0^\alpha h(r)\, dr + (1-\alpha)\exp(-\frac{R_\infty}{2c\sigma^2})}\right).
$$

Then, to minimize $T_\alpha(H)$, it suffices to minimize $\mathcal{J}(h)$. Moreover, the feasibility constraint $H \in \mathcal{H}$ precisely translates to $h \in \mathfrak{h}$, under the assumption that $H = R_\infty$ on $(\alpha, 1]$. Thus, the solution to our minimum quantile problem is in one-to-one correspondence to the solution to our auxiliary problem, given by Lemma A.1. All the statements of the theorem can now be derived in a straightforward manner. $\square$

PROOF OF THEOREM 3.5. By Theorem 3.1(i), we have

$$
T_\alpha(H; T) = \begin{cases} F_{\tau_{x_0/\sigma}^\circ}^{-1}\left(\frac{1 - F_{\tau_{x_0/\sigma}^\circ}(T)}{1 - \beta(H)}\int_0^\alpha \exp\left(\frac{R_\infty - H(r)}{2c\sigma^2}\right) dr\right) \leq T \\ \qquad\qquad \alpha \leq \beta(H), \\ \Delta \quad \alpha > \beta(H), \end{cases}
$$

where $\beta(H) = F_{\mathcal{E}_T(H)}(T) \in (0, 1)$ is the unique solution of

$$
\frac{F_{\tau_{x_0/\sigma}^\circ}(T)}{1 - F_{\tau_{x_0/\sigma}^\circ}(T)} = \frac{1}{1-z}\int_0^z \exp\left(\frac{R_\infty - H(r)}{2c\sigma^2}\right) dr = \phi(z; H).
$$

CLAIM. *The optimal value does not change if we restrict ourselves to reward function H which satisfies $H(r) = R_\infty$ for all $r > \alpha$.*

To prove the claim, let $H \in \mathcal{H}_T^\alpha$ and define $\hat{H} := H 1_{[0,\alpha]} + R_\infty 1_{(\alpha,1]}$. We first check the feasibility of $\hat{H}$. Since $\phi(\beta(\hat{H}); \hat{H}) = \phi(\beta(H); H) \leq \phi(\beta(H); \hat{H})$ and

$\phi(z; \hat{H})$ is increasing, we must have $\beta(\hat{H}) \leq \beta(H)$. Using this, together with $H \geq \hat{H} \geq R_\infty$, we get

$$\int_0^{\beta(\hat{H})} \hat{H}(r)\, dr + (1 - \beta(\hat{H}))R_\infty \leq \int_0^{\beta(H)} H(r)\, dr + (1 - \beta(H))R_\infty \leq K.$$

Thus, $\hat{H}$ also satisfies the budget constraint. Since $\beta(H) \geq \alpha$, we have by the monotonicity of ϕ that $\phi(\beta(\hat{H}); \hat{H}) = \phi(\beta(H); H) \geq \phi(\alpha; H) = \phi(\alpha; \hat{H})$, and thus, $\beta(\hat{H}) \geq \alpha$. So we conclude that $\hat{H} \in \mathcal{H}_T^\alpha$. Next, we have

$$\begin{aligned}
F_{\tau_{x_0/\sigma}^\circ}(T_\alpha(\hat{H}; T)) &= \frac{1 - F_{\tau_{x_0/\sigma}^\circ}(T)}{1 - \beta(\hat{H})} \int_0^\alpha \exp\left(\frac{R_\infty - \hat{H}(r)}{2c\sigma^2}\right) dr \\
&\leq \frac{1 - F_{\tau_{x_0/\sigma}^\circ}(T)}{1 - \beta(H)} \int_0^\alpha \exp\left(\frac{R_\infty - H(r)}{2c\sigma^2}\right) dr \\
&= F_{\tau_{x_0/\sigma}^\circ}(T_\alpha(H; T)),
\end{aligned}$$

which is equivalent to $T_\alpha(\hat{H}; T) \leq T_\alpha(H; T)$, and the claim has been verified.

Its immediate consequence is that if $\beta(H) \geq \alpha$ and $H(r) = R_\infty$ for all $r > \alpha$, then

$$\int_0^{\beta(H)} H(r)\, dr + (1 - \beta(H))R_\infty = \int_0^1 H(r)\, dr,$$

which implies $\mathcal{H}_T^\alpha \cap \{H|_{(\alpha,1]} \equiv R_\infty\} = \mathcal{H} \cap \{\beta(H) \geq \alpha\} \cap \{H|_{(\alpha,1]} \equiv R_\infty\}$. Thus, we can work with the simpler feasible set $\mathcal{H}$ instead of the equilibrium-dependent $\mathcal{H}_T^\alpha$.

As in the $T = \infty$ case, we let $h(r) := \exp(-\frac{H(r)}{2c\sigma^2})$. In view of the claim above, we can set $h(r) \equiv \exp(-\frac{R_\infty}{2c\sigma^2})$ for $r \in (\alpha, 1]$ and only search for the optimal h on $[0, \alpha]$. With a slight abuse of notation, we also use $\beta(h)$ to denote the unique solution z in $(0, 1)$ of

$$(\text{A.3}) \quad C_T := \frac{F_{\tau_{x_0/\sigma}^\circ}(T)}{1 - F_{\tau_{x_0/\sigma}^\circ}(T)} = \frac{1}{1 - z}\left(\int_0^{z \wedge \alpha} \exp\left(\frac{R_\infty}{2c\sigma^2}\right) h(r)\, dr + (z - z \wedge \alpha)\right).$$

The feasibility condition translates to $h \in \mathfrak{h}_T := \mathfrak{h} \cap \{\beta(h) \geq \alpha\}$ where $\mathfrak{h}$ is defined in (A.2). Equation (A.3) implies that $\beta(h) \geq \alpha$ if and only if

$$\mathcal{J}(h) = \int_0^\alpha h(r)\, dr \leq C_T(1 - \alpha)\exp\left(-\frac{R_\infty}{2c\sigma^2}\right).$$

By Lemma A.1,

$$\inf_{h \in \mathfrak{h}} \mathcal{J}(h) = \mathcal{J}(h^*) = \alpha \exp\left(-\frac{K - (1 - \alpha)R_\infty}{2\alpha c\sigma^2}\right).$$

Therefore, we arrived at the following feasibility criteria:

$$\mathfrak{h}_T \neq \varnothing \iff \beta(h^*) \geq \alpha \iff \mathcal{J}(h^*) \leq C_T(1-\alpha)\exp\left(-\frac{R_\infty}{2c\sigma^2}\right)$$

$$\iff T \geq F_{\tau_{x_0/\sigma}^\circ}^{-1}\left(\frac{\alpha}{\alpha + (1-\alpha)\exp(\frac{K-R_\infty}{2\alpha c\sigma^2})}\right) = T_\alpha^*.$$

If $\mathfrak{h}_T = \varnothing$, then there is no feasible reward scheme which attains, in equilibrium, the desired completion rate of α by time T. Suppose $h \in \mathfrak{h}_T \neq \varnothing$. Then

$$F_{\tau_{x_0/\sigma}^\circ}\big(T_\alpha(H;T)\big) = \big(1 - F_{\tau_{x_0/\sigma}^\circ}(T)\big)\exp\left(\frac{R_\infty}{2c\sigma^2}\right)\frac{\int_0^\alpha h(r)\,dr}{1-\beta(h)}.$$

Thus, to minimize $T_\alpha(H;T)$, we only need to minimize

$$\mathcal{L}(h) := \frac{\mathcal{J}(h)}{1-\beta(h)}\exp\left(\frac{R_\infty}{2c\sigma^2}\right), \quad h \in \mathfrak{h}_T.$$

From (A.3), we also obtain

$$\beta(h) = \frac{C_T + \alpha - \exp(\frac{R_\infty}{2c\sigma^2})\mathcal{J}(h)}{1+C_T}, \quad h \in \mathfrak{h}_T.$$

It follows that

$$\mathcal{L}(h) = \frac{\mathcal{J}(h)}{1-\alpha + \exp(\frac{R_\infty}{2c\sigma^2})\mathcal{J}(h)}, \quad h \in \mathfrak{h}_T.$$

Hence, to minimize $\mathcal{L}(h)$, it suffices to minimize $\mathcal{J}(h)$ over $h \in \mathfrak{h}_T$. Note that when $\mathfrak{h}_T \neq \varnothing$, we have $\beta(h^*) \geq \alpha$ and thus, $h^* \in \mathfrak{h}_T$. This implies $h^* = \arg\min_{h \in \mathfrak{h}} \mathcal{J}(h)$ is also the (unique) optimizer of $\inf_{h \in \mathfrak{h}_T} \mathcal{J}(h)$. From h^*, we can recover the optimal H^* and the associated $T_\alpha^*(T)$.

As in the case $T = \infty$, all the statements of the theorem but the last two can now be derived in a straightforward manner. For the last two, setting $T_\alpha^* = T$ yields the minimum budget and the maximum terminal completion rate. $\square$

PROOF OF THEOREM 3.6. (i) By (3.7), maximizing V_∞ over $\mathcal{H}$ is equivalent to minimizing $\mathcal{J}(h) = \int_0^1 h(z)\,dz$ over $\mathfrak{h}$ (defined in (A.2)) with $\alpha = 1$. By Lemma A.1, $\arg\min_{h \in \mathfrak{h}} \mathcal{J}(h) = h^* \equiv \exp(-K/(2c\sigma^2))$. It follows that $H^* = \arg\max_{H \in \mathcal{H}} V_\infty(H) = -2c\sigma^2 \ln h^* \equiv K$ and $V_\infty(H^*) = K$.

(ii) By (3.5) and Proposition 3.2, maximizing the welfare or the expected total effort is equivalent to maximizing the terminal completion rate, which we have solved in Theorem 3.5. $\square$

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