PAPER

Expansions in the delay of quasi-periodic solutions for state dependent delay equations[*]

View the article online for updates and enhancements.

Expansions in the delay of quasi-periodic solutions for state dependent delay equations[*]

Alfonso Casal[1], Livia Corsi[2] and Rafael de la Llave[3,4]

[1] Department Matemática Aplicada, E.T.S.A. Universidad Politécnica de Madrid, Av. Juan de Herrera 4, Madrid 28040, Spain
[2] Dipartimento di Matematica e Fisica, Università di Roma Tre, L.go S. Leonardo Murialdo 1, 00146 Roma, Italy
[3] Department of Mathematics, Georgia Institute of Technology, 686 Cherry St, Atlanta, GA 30332-0160, United States of America

E-mail: alfonso.casal@upm.es, lcorsi@mat.uniroma3.it and rll6@math.gatech.edu

Abstract

We consider several models of state dependent delay differential equations (SDDEs), in which the delay is affected by a small parameter. This is a very singular perturbation since the nature of the equation changes. Under some conditions, we construct formal power series, which solve the SDDEs order by order. These series are quasi-periodic functions of time. This is very similar to the Lindstedt procedure in celestial mechanics. Truncations of these power series can be taken as input for *a posteriori* theorems, that show that near the approximate solutions there are true solutions. In this way, we hope that one can construct a catalogue of solutions for SDDEs, bypassing the need of a systematic theory of existence and uniqueness for all initial conditions.

Keywords: retarded potentials, series expansions, quasi-periodic solutions

1. Introduction

In special relativity, when we can ignore the effect of emitted radiations, [1–4], the motion of N charged particles can be described as solutions of the equation

$$M_i(\dot{x}_i)\ddot{x}_i = G\sum_{j\neq i} \frac{q_i q_j(x_i(t) - x_j(\tau_{ij}))}{|x_i(t) - x_j(\tau_{ij})|^3}, \tag{1.1}$$

where M is the relativistic mass, which in the one dimensional case is defined as

$$M_i(v) = m_i\left(1 - \frac{|v|^2}{c^2}\right)^{-3/2} + m_i\left(1 - \frac{|v|^2}{c^2}\right)^{-1/2},$$

m_i, q_i are the rest mass and the charge of the ith particle respectively, c is the speed of light and G is a physical constant. (In the higher dimensional case, the relativistic mass is a matrix). More importantly, τ_{ij} is the time that a signal emitted from particle j takes to reach particle i, and it is given by the implicit equation

$$\tau_{ij} = t - \frac{1}{c}|x_i(t) - x_j(\tau_{ij})|. \tag{1.2}$$

It is not difficult to show that if all the particles move with speed less that the speed of light, the solution of (1.2) is a unique function of t.

In Physics, it is common to consider $\varepsilon \equiv \frac{1}{c}$ as a small parameter and to try to predict the motion as a formal power series; see for instance [5]. It is important to observe that, if we consider x_i as given, we can find an asymptotic expansion of the solutions of (1.2). Indeed we can write

$$\tau_{ij} = t - \frac{1}{c}|x_i(t) - x_j(t)| + O(1/c^2) \tag{1.3}$$

The functional equation (1.1) is not a differential equation because τ_{ij} is in general not equal to t, hence the positions x_i in the rhs are evaluated at different times. If we take the approximation (1.3) for the delay, we obtain a state dependent delay equation (SDDE) because the delay is an explicit function of the state. The problem in (1.1), without making the simplification (1.3) is an of a more complicated nature since τ_{ij} depends implicitly on the whole trajectory of x_i, x_j. After we develop enough theory for SDDE, we will see in section 6 that the same ideas apply to the full model.

Other scientific problems, such as the dynamics in some population models with density dependent fertility age, are also naturally modeled with SDDEs; see for instance [6–8] and references therein for more examples.

In this paper we will study several models of delay equations (mainly state dependent delay equations.

Note that when the delay is a given constant T, there is a rather developed mathematical theory [9–11]. Precisely, if one prescribes as initial data a function defined on $[0, T]$, under the standard regularity assumptions for classical ODEs, one can obtain a rather satisfactory theory of existence, uniqueness, dependence on parameters and initial data, which constitutes the first step to developing a qualitative theory. However, when the delay depends on the state of the system (*a fortiori* on the whole trajectory) the situation is much more delicate and the theory of existence uniqueness and dependence on parameters and initial conditions is much more restricted [6]. There are indeed many examples of surprising behaviors which indicate that a systematic existence, uniqueness and regularity theory for SDDE will be significantly more complicated than the one for constant delay equations.

The goal of this paper is mathematically modest. We do not try to develop a general theory of existence and uniqueness. We only try to study special solutions for some type of SDDEs. Furthermore, we only try to study these solutions as formal power series.

Once we specify the class of solution we are looking for (mainly quasi-periodic[5]) we express the SDDEs as functional equations on the space of quasi-periodic functions, and we call such functional equation the *invariance equation*. In particular, we will consider SDDEs involving a small parameter ε, and obtain approximate solution of the invariance equation as a formal power series in ε.

One motivation for our study is that quasi-periodic solutions of (1.1) play an important rôle in chemistry since they are the basis for the 'old quantum theory'. For $\varepsilon = 0$, the quasi-periodic solutions of (1.1) satisfying the Bohr–Sommerfeld conditions have quantum analogues.

This is the background we mainly have in mind, and thus it motivates us to look for expansions in $\varepsilon = 1/c$ of quasi-periodic solutions for (1.1).

Thus, we will show that, under some mild non-degeneracy condition, it is possible to write systematically a formal power series expansions for the quasi-periodic solutions. Furthermore we will show that, if we trucate such expansions to a finite order, we obtain functions that, when substituted in the invariance equation, satisfy it up to a very small error.

Note that this is a very different procedure from the one followed sometimes in the physics literature (predictive mechanics, [12, 13] post-newtonian formalism [5] etc) in which one tries to find an ODE (often derived through a Lagrangian) which describes all the solutions. Our aim is to find expansions only for solutions of a certain type. It is quite possible, with the formalism developed here, that the perturbation expansions for solutions of different types are very different.

Note that obtaining a Lagrangian description of the motion of all particles, is forbidden by the 'no-interaction' theorems [14], which state that the only Lagrangian invariant under the Lorentz transformations are the free particles. Indeed, in general not even formal power series can be found [15]. The above results are not incompatible with our results, since we obtain the expansion only for solutions of very specific type. As observed in remark 2.3, our expansions depend very much in subtle properties of the unperturbed system, so it is quite possible that the approximate solutions we produce cannot be combined into a globally defined Lagrangian sytem, which is the only thing forbidden by [14].

The systematic construction of approximate solutions obtained in this paper matches very well with the recent development in *a posteriori* theorems, which show that near approximate solutions of a certain kind there will be true solutions. There are already such *a posteriori* results in quasi-periodic perturbations of some simple systems [16, 17] and in [18, 19]. Putting together these results, we obtain that some of the expansions we construct are asymptotic expansions of families of true solutions. One can hope that in the near future, the (rapidly growing) applicability of *a posteriori* theorems will be extended and more general theorems of this form will be proved, to cover at least the models considered in this paper. We call attention to [20] which implemented a very similar program of finding expansions in the delay and validating them.

Hence, the conjectural picture that emerges is that there are many solutions of the classical system that survive the inclusion of the delay. The set of solution that persists has a very complicated structure (number theory properties play a role) and the solutions depend in a very non-uniform way on the frequencies. Indeed we have to impose a condition on the frequency (see (1.22) or (1.23) below) in order to achieve our results. There is no way to make the solutions that persist fit in a common Lagrangian description, but nevertheless, the set of solutions that persist is large enough that they can be useful in practical problems.

---

[5] In section 5 we will consider also solutions converging exponentially to quasi-periodic.

1.1. Formulation of the problem.

We consider equations of the form

$$\dot{y}(t) = f_\varepsilon(y(t), y(t - \varepsilon r_1), \dots, y(t - \varepsilon r_\ell)), \tag{1.4}$$

or

$$\dot{y}(t) = f_\varepsilon(y(t), \varepsilon y(t - r_1), \dots, \varepsilon y(t - r_\ell)), \tag{1.5}$$

where $r_j = r_j(y(t)), j = 1, \dots, \ell$ are given functions, and the unknown is $y(t)$. In (1.1) the small parameter is $\varepsilon = 1/c$.

Remark 1.1.   For $\varepsilon = 0$ the resulting equation is an ODE in both cases (1.4) and (1.5).

Remark 1.2.   We can think of (1.1) as an equation of the form

$$\dot{y}(t) = f_\varepsilon(y(t), y(\tau)), \tag{1.6}$$

with $y \in \mathbb{R}^{6N}$ (positions and velocities), $\tau = \tau(y(t)) = \{\tau_{i,j}\}_{i,j=1}^N$ is implicitly defined by (1.2), and $\ell = N(N-1)/2$ is the number of pairs. In particular, for $\varepsilon = 0$ the equation (1.1) has an Hamiltonian structure.

For the sake of typographical simplicity, in this paper we will present mostly cases in which $\ell = 1$, and at the end we will make explicit the (typographical) changes needed to deal with the case $\ell \geqslant 2$ or the more complicated model of (1.1).

The search of quasi-periodic solutions of (1.4) with some frequency ω is equivalent to looking for a so-called *invariant torus*, i.e. a torus embedding

$$K : \mathbb{T}^d \to \mathbb{R}^n \tag{1.7}$$

satisfying

$$(\omega \cdot \partial_\theta K)(\theta) = f_\varepsilon(K(\theta), K(\theta - \varepsilon \omega r(K(\theta)))), \tag{1.8}$$

in such a way that the dynamics on the model torus $\mathbb{T}^d$ is given by

$$\dot{\theta} = \omega. \tag{1.9}$$

Of course if dealing with (1.5), we look for K satisfying

$$(\omega \cdot \partial_\theta K)(\theta) = f_\varepsilon(K(\theta), \varepsilon K(\theta - \omega r(K(\theta)))). \tag{1.10}$$

Observe that the case $d = 1$ corresponds to periodic solutions. Note also that (1.8) reduces to

$$(\omega \cdot \partial_\theta K)(\theta) = f_0(K(\theta), K(\theta)), \tag{1.11}$$

for $\varepsilon = 0$, while (1.10) reduces to

$$(\omega \cdot \partial_\theta K)(\theta) = f_0(K(\theta), 0). \tag{1.12}$$

We emphasize that in (1.8) and (1.10) both K and ω are unknown. In [16, 17] only the simpler case of quasi-periodically forced systems was considered, so that ω was externally fixed.

Remark 1.3. Note that the solutions of (1.8) or (1.10) are never unique. Indeed if K_ε is a solution, for any $\theta \in \mathbb{T}^d$ we have that $\widetilde{K}_\varepsilon(\theta) := K_\varepsilon(\theta + \omega)$ is also a solution. Hence the solutions appear as 1-parameter family. This lack of uniqueness has a physical interpretation, namely that the solution admits phase traslations, i.e. it traslates the origin in $\mathbb{T}^d$. A way to obtain uniqueness is by requesting, besides the invariance, a normalization: the most natural one seems to be

$$\frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \mathrm{d}\theta DK_0(\theta) \cdot K_\varepsilon(\theta) = 0, \tag{1.13}$$

first of all because it is very easy to impose, amd moreover because

Having the uniqueness of the solution is a useful property since it allows to compare results obtained by different methods, and discuss smooth dependence on perturbations.

The theory for the solutions of (1.8) is far from being a general theory for the solutions of (1.4) for all initial data. The goal of this paper is to show that if we have K_0 and ω_0 solving (1.8) for $\varepsilon = 0$ and we assume some mild non-degeneracy conditions, then we can systematically compute formal power series

$$K_\varepsilon = \sum_{j \geqslant 0} \varepsilon^j K_j \quad \omega_\varepsilon = \sum_{j \geqslant 0} \varepsilon^j \omega_j \tag{1.14}$$

solving (1.8) in the sense of formal power series. In other words, if we denote

$$K_\varepsilon^{[\leqslant N]} = \sum_{j=0}^{N} \varepsilon^j K_j \quad \omega_\varepsilon^{[\leqslant N]} = \sum_{j=0}^{N} \varepsilon^j \omega_j \tag{1.15}$$

we have that the function

$$y_\varepsilon^{[\leqslant N]}(t) := K_\varepsilon^{[\leqslant N]} \quad (\omega_\varepsilon^{[\leqslant N]} t) \tag{1.16}$$

satisfies

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} y_\varepsilon^{[\leqslant N]} - f_\varepsilon \left(y_\varepsilon^{[\leqslant N]}(t), y_\varepsilon^{[\leqslant N]}(t - \varepsilon r(y_\varepsilon^{[\leqslant N]}(t))) \right) \right| \leqslant C_N \varepsilon^{N+1}. \tag{1.17}$$

For (1.10) the analogue of (1.17) is

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} y_\varepsilon^{[\leqslant N]} - f_\varepsilon \left(y_\varepsilon^{[\leqslant N]}(t), \varepsilon y_\varepsilon^{[\leqslant N]}(t - r(y_\varepsilon^{[\leqslant N]}(t))) \right) \right| \leqslant C_N \varepsilon^{N+1}. \tag{1.18}$$

Expansion of the form (1.14) are called 'Lindstedt series' and the search of solutions for an ODE in the form of a Lindstedt series has been widely used in astronomy since the 19th century [21] and even before. Such expansions have been used also in delay equations; see for instance [22, 23] or [24] for further developement. The paper [20] includes also validation.

An important role will be played by the linearized equation around $\varepsilon = 0$. Postponing many details which we will make explicit later, a key result of this paper is the following meta-result.

Meta-lemma 1.4. Denote by D_1, D_2 the derivative w.r.t. the first and second argument of f_0 respectively. If given R it is possible to find δ 'small enough' and u such that

$$\omega_0 \cdot \partial_\theta u - (D_1 f_0(K_0(\theta), K_0(\theta)) + D_2 f_0(K_0(\theta), K_0(\theta))) u = R + \delta \partial_\theta K_0(\theta), \tag{1.19}$$

then we can determine the coefficients of the series (1.14) solving (1.8) to all orders.

Similarly, if it is possible to find δ 'small enough' and u such that

$$\omega_0 \cdot \partial_\theta u - (D_1 f_0(K_0(\theta), 0) + D_2 f_0(K_0(\theta), 0)) u = R + \delta \partial_\theta K_0(\theta), \tag{1.20}$$

then we can determine the coefficients of the series (1.14) solving (1.10) to all orders.

It is important to note that the the equations (1.19) and (1.20) only involve the unperturbed undelayed equation and that the same equations appear to all orders. Hence, some geometric properties of the solutions of the ODE, guarantee that we can get expansions to all orders in ε.

Proof.   The proof is quite straightforward and we shall show the details only for the case (1.8). We start by simply observing that

$$f_\varepsilon \left(K_\varepsilon(\theta), K_\varepsilon(\theta - \varepsilon \omega r(K_\varepsilon(\theta))) \right) = f_0(K_0, K_0) + \varepsilon D_1 f_0(K_0(\theta), K_0(\theta)) \cdot K_1(\theta)$$
$$+ \varepsilon D_2 f_0(K_0(\theta), K_0(\theta)) \cdot K_1(\theta) + \cdots$$

Thus, by a formal expansion in ε, we see that the terms $O(\varepsilon^n)$ have the form

$$(D_1 f_0(K_0, K_0) + D_2 f_0(K_0, K_0)) K_n$$
$$+ R_n \left(K_0, \ldots, K_{n-1}, DK_0, \ldots, DK_{n-1}, \ldots, D^{n-1} K_0, \ldots, D^{n-1} K_{n-1} \right) \tag{1.21}$$

where R_n is a polynomial in its variables, i.e. matching the coefficients at order ε^n both in K and in ω we obtain an equation of the form (1.19).

Of course the statement of meta-lemma 1.4 above is only formal since it does not specify the precise meaning of 'solve'. Such precise meaning entails the specification of the spaces in which the solution and the reminder lie; moreover we will need conditions on the frequency ω_0.

In the following we will present various cases in which the equation (1.19) (or (1.20)) is solvable. For each of the cases we formulate precisely the meaning of 'solvability' and the result on the existence of the Lindstedt series. The cases we will consider are well known to dynamicysts since they are also cases where one can prove persistence of the structure under the change of the differential equation, and they are the following.

Case 1:   The manifold $K_0(\mathbb{T}^d)$ is a normally hyperbolic invariant manifold (NHIM) and ω_0 satisfies a Diophantine condition.

Case 2:   The linearized evolution is reducible to constant coefficients and the eigenvalues of this constant coefficients matrix satisfy some Diophantine condition w.r.t. ω_0.

Case 3:   The unperturbed system is Hamiltonian, $K_0(\mathbb{T}^d)$ is a Lagrangian torus (i.e. the phase space has dimension $2d$), it satisfies a twist condition and ω_0 satisfies some Diophantine condition.

Case 4:   The unperturbed system is a two dimensional ODE which has a limit cycle.

Case 5:   The 'electrodynamics case' of (1.1) (The delay depends not only on the state, but also on the whole trajectory).

Here and henceforth we impose the standard Diophantine condition

$$|\omega_0 \cdot k| \geqslant \frac{\gamma}{|k|^\tau}, \quad \text{for all } k \in \mathbb{Z}^d \backslash \{0\}, \tag{1.22}$$

where, with abuse of notation, we denoted by $|\cdot|$ both the absolute value of a number and the ℓ_1-norm of a d-dimensional vector.

Some result can be obtained also in the case of *subexponential Diophanitne* ω_0, i.e.

$$\lim_{|k|\to\infty} \frac{1}{|k|} \log \frac{1}{|\omega_0 \cdot k|} = 0 \tag{1.23}$$

or equivalently

$$\forall\, \varepsilon > 0 \,\exists\, c = c(\varepsilon) \quad \text{such that } |\omega_0 \cdot k| \geqslant c(\varepsilon)\varepsilon^{-|k|\varepsilon} \quad \forall\, k \in \mathbb{Z}^d\backslash\{0\}.$$

It is remarkable that the Diophantine condition (1.22) is precisely one of the two main hypotheses of the celebrated KAM theorem.

Throughout the paper we shall use the following standard notations.

- Given $\xi > 0$ we denote by $\mathbb{T}^d_\xi$ the set

$$\mathbb{T}^d_\xi = \{\theta \in (\mathbb{C}/\mathbb{Z})^d : \operatorname{Re}(\theta_j) \in \mathbb{T},\ |\operatorname{Im}(\theta_j)| < \xi,\quad j = 1, \dots, d\}.$$

- We denote by $\mathcal{A}_\xi$ the space of functions $u : \mathbb{T}^d_\xi \to \mathbb{R}^n$ such that

$$\|u\|_\xi := \sum_{k\in\mathbb{Z}^d} e^{\xi 2\pi|k|}\|\hat{u}_k\| < \infty \tag{1.24}$$

where we denoted by $\hat{u}_k$ the kth Fourier coefficient of u and by $\|\cdot\|$ the standard Euclidean norm of an n-dimensional vector.

- For a function f of class C^r we denote its C^r-norm as $\|f\|_{C^r}$. $\qquad\qquad\square$

2. The case of quasi-periodic solutions which are also normally hyperbolic invariant manifolds

2.1. Basic definitions.

We recall that $M \subset \mathbb{R}^n$ a C^1 manifold is an NHIM for a C^r vector field f with $r \geqslant 2$ if

(a) $f(x) \in TM$ for all $x \in M$

(b) For every $x \in M$ there is a splitting

$$\mathbb{R}^n = T_x M \oplus E^s_x \oplus E^u_x, \tag{2.1}$$

such that there are positive constants C, ρ_+, ρ_- so that, denoting by F_t the time-t flow, one has

$$\|DF_{t|E^u}\|_{C^r} \leqslant C\varepsilon^{\rho_- t}, \quad t \leqslant 0$$

$$\|DF_{t|E^s}\|_{C^r} \leqslant C\varepsilon^{-\rho_+ t}, \quad t \geqslant 0$$

Note that if M is not compact, one needs to assume that the C^r properties of the manifold are uniform, and in this case the theory of [25–27] carries through; see also [28].

Here M is $K_0(\mathbb{T}^d)$ which is a compact manifold, so that we do not have to deal with the subtleties appearing in the case of non-compact manifolds.

We assume that the dynamics restricted to M is conjugated to a rotation. In such a case, the theory of [25] shows that if $f_0 \in C^r$, then M is a C^r submanifold and the splitting is C^{r-1}. The analytic case is more delicate, but it was proved in [29] that if ω_0 is Diophantine and f_0 is analytic, then M and the splitting are also analytic.

2.2. Solvability of the linearized equation.

Denote by $\Pi_x^s, \Pi_x^u, \Pi_x^c$ the projectors onto $E_x^s, E_x^u, E_x^c = T_x M$ corresponding to the splitting (2.1).
  Denoting

$$v_\theta^\alpha := \Pi_\theta^\alpha v,$$

$$F^\alpha(K_0(\theta)) := (D_1 f_0(K_0(\theta), K_0(\theta) + D_2 f_0(K_0(\theta), K_0(\theta)))|_{E_\theta^\alpha}, \quad \alpha = s, u, c,$$
(2.2)

the linearized equation (1.19) takes the form

$$\omega_0 \cdot \partial_\theta v^\alpha - F^\alpha(K_0(\theta))v^\alpha = R^\alpha, \quad \alpha = s, u$$
$$\omega_0 \cdot \partial_\theta v^c - F^c(K_0(\theta))v^c = R^c - \omega_n.$$
(2.3)

We need to show how to solve (2.3).

First of all we note that for $\alpha = s, u$ it sufficies to use the Duhamel formula. Indeed if $A_\theta^\alpha(t)$ satisfies

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} A_\theta^\alpha(t) = F^\alpha(K_0(\theta + \omega_0 t)) A_\theta^\alpha(t) \\ A_\theta^\alpha(0) = \mathbb{1} \end{cases}$$
(2.4)

then we can set

$$v^s(\theta) = \int_0^\infty \mathrm{d}t A_{\theta-\omega_0 t}^s(t) R^s(\theta - \omega_0 t), \quad v^u(\theta) = \int_{-\infty}^0 \mathrm{d}t A_{\theta-\omega_0 t}^u(t) R^u(\theta - \omega_0 t).$$
(2.5)

Since we have

$$|A_\theta^s(t)| \leqslant C\varepsilon^{-\rho+t}, \quad \forall\, t > 0 \qquad |A_\theta^u(t)| \leqslant C\varepsilon^{\rho-t}, \quad \forall\, t < 0$$

then the integral appearing in (2.5) is convergent, so $v^\alpha(\theta)$ is well-defined.

Lemma 2.1. *If $f_0 \in C^r$ and $R^\alpha \in C^{r-1}$ for $\alpha = s, u$, then $v^\alpha \in C^{r-1}$ and one has*

$$\|v^\alpha\|_{C^{r-1}} \leqslant C\|R^\alpha\|_{C^{r-1}}.$$
(2.6)

Proof. First of all note that since $f_0 \in C^r$ then the bundles E^s, E^u, TM are C^{r-1}. Moreover, since f_0 is of class C^r, then F is of class C^{r-1}, and hence $A_\theta(t)$ solving (2.4) depends on a C^{r-1} way on θ and the derivatives do not grow with t. This implies that we can take derivatives w.r.t. θ under integral sign in (2.5) and thus the bound (2.6) follows. $\qquad\square$

To deal with the center direction, since the manifold is normally hyperbolic we can write

$$v^c(\theta) = DK_0(\theta)w(\theta),$$

so that the equation for w is

$$\omega_0 \cdot \partial_\theta w = (DK_0(\theta))^{-1} R^c(\theta) + \omega_n.$$
(2.7)

Note that (2.7) is the standard cohomology equation appearing in KAM theory.

In order to solve (2.7), we can expand $w(\theta)$ in Fourier series

$$w(\theta) = \sum_{k \in \mathbb{Z}^d} \widehat{w}_k e^{2\pi i k \cdot \theta}, \tag{2.8}$$

so that, expanding also

$$g(\theta) := (DK_0(\theta))^{-1} R^c(\theta) = \sum_{k \in \mathbb{Z}^d} \hat{g}_k e^{2\pi i k \cdot \theta} \tag{2.9}$$

and integrating both sides w.r.t. θ, we see that (2.7) reads

$$\begin{aligned}
\omega_n &= -\langle g(\theta) \rangle \\
\widehat{w}_k &= \frac{1}{i2\pi\omega_0 \cdot k} \hat{g}_k, \quad k \neq 0.
\end{aligned} \tag{2.10}$$

In particular in the analytic case, under the hypothesis (1.23) we obtain $w(\theta)$ an analytic function defined in a domain $\mathbb{T}^d_{\xi'}$ for any $\xi' < \xi$.

Overall we thus obtained the following result.

Theorem 2.2. *Suppose that f_0 is analytic (resp. C^∞), K_0 is analytic on $\mathbb{T}^d_\xi$ (resp. C^∞) and ω_0 is subexponential Diophantine (resp. Diophantine). Then there exists Lindstedt series solving (1.8) to all orders; in particular the coefficients K_n are analytic on $\mathbb{T}^d_{\xi'}$ for any $\xi' < \xi$ (resp. C^∞). Moreover the Lindstedt series can be made so that (1.13) holds; with such normalization the series is unique.*

Note that the C^r case is much trickier to work out. Indeed the solution of the linearized equation loses derivatives and the composition to justify the derivatives requires justification. Our guess is that one gets $K_n \in C^{r-n\tau}$ and of course one obtains only finitely many terms in the expansion.

Remark 2.3. We say that ω_0 is Liouville if (1.23) fails, i.e. there is a sequence k_n with $|k_n| \to \infty$ such that $|\omega_0 \cdot k_n| < c(\varepsilon)e^{-\varepsilon|k_n|}$. In this case it is not possible to solve the cohomology equation (2.7), i.e. we cannot find a Lindstedt series for the solution of the invariance equation. This gives some insight on why the Postnewtonian formalism (which is global) fails. Indeed a global theory should work also for Liouville vectors.

We finally mention that, under some further technical assumptions (that could possibly be removed), one can apply the results of [16] and get validation for theorem 2.2, namely the existence of a true solution nearby. The discussion in [16] is very technical and goes why beyond the scope of the present paper.

3. The reducible case

For linear equations with quasi-periodic coefficients, it is natural to consider linear quasi-periodic changes of variables. If one is dealing with an equation of the form

$$\frac{\mathrm{d}}{\mathrm{d}t} v = A(\theta + \omega_0 t) v \tag{3.1}$$

a change of variables of the form

$$v(t) = M(\theta + \omega_0 t) w(t) \tag{3.2}$$

for some M, transforms the equation (3.1) into

$$\dot{w} = M^{-1}[-\omega_0 \cdot \partial_\theta M + AM](\theta + \omega_0 t)w(\theta). \tag{3.3}$$

We say that the equation (3.1) is *reducible* if it is possible to find

$$M : \mathbb{T}_\xi^d \to GL(n, \mathbb{C}), \quad \Lambda \in GL(n, \mathbb{C})$$

such that

$$M^{-1}[-\omega_0 \cdot \partial_\theta M + AM](\theta_0 + \omega t) = \Lambda$$

so that (3.3) has constant coefficients. Of course, without loss of generality we can look for Λ in Jordan normal form.

The question of reducibility has been considered extensively in many papers, both perturbatively [30, 31] and nonperturbatively [32, 33], in the sense that the smallness condition does not depend on the frequency; a good survey on the subject can be found in [34].

What is relevant to us is that, after a change of variables as in (3.2) we get that (1.19) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}w(t) = \Lambda w + M^{-1}(\theta + \omega_0 t)R_\theta + M^{-1}(\theta + \omega_0 t)DK_0(\theta + \omega_0 t)\omega_n \tag{3.4}$$

It is clear that, since

$$\frac{\mathrm{d}}{\mathrm{d}t}K_0(\theta + \omega_0 t) = f_0(K_0(\theta + \omega_0 t), K_0(\theta + \omega_0 t)) \tag{3.5}$$

deriving (3.5) w.r.t. θ on both sides we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}DK_0(\theta + \omega_0 t) = (D_1 f_0(K_0(\theta), K_0(\theta)) + D_2 f_0(K_0(\theta), K_0(\theta)))DK_0(\theta + \omega_0 t). \tag{3.6}$$

We can interpret (3.6) by saying that the vectors $\partial_{\theta_i}K_0(\theta)$ are eigenvectors of the linearized equations.

Since K is a torus embedding satisfying (1.9), there must be d zero-eigenvalues of Λ. An important assumption that needs to be made is that there are exactly d zero-eigenvalues of Λ, while the others are Diophantine w.r.t. ω_0; see definition 1 below. It is also important to note that the term R_θ is in the range of $DK_0(\theta)$. In other words we can split any vector u as

$$u = \Pi u + \Pi^\perp u$$

where Π is the projection onto the range of DK_0 and $\Pi^\perp$ is the projection onto the complementary space. Then the invariance equation restricted to the range of DK_0 takes the form

$$\omega_0 \cdot \partial_\theta \Pi w = \Pi(M^{-1}R + \omega_n), \tag{3.7}$$

where, with abuse of notation we are denoting by w, R the corresponding torus embedding. This is again a standard cohomology equation of the same type of (2.7), so we can solve it by imposing

$$\omega_n = -\langle (DK_0)^{-1}R \rangle$$

and assuming that ω_0 is Diophantine.

On the other hand on the Kernel of DK_0 we see that (3.7) is equivalent to the system

$$\omega_0 \cdot \partial_\theta \Pi^\perp w_i(\theta) = \mu_i \Pi^\perp w_i + \Pi^\perp (M^{-1}R)_i, \tag{3.8}$$

where $\mu_{n-d}, \dots \mu_n$ are the non-zero eigenvalues of Λ.

Note that, because of the previous calculation and the assumption of having no zero-eigenvalues except for the range of DK_0, ω_n does not appear in (3.8).

Similarly to the case of an NHIM, we can now pass to Fourier series as in (2.8) and we see that (3.8) is equivalent to

$$2\pi i(\omega_0 \cdot k)\widehat{w}_k = \mu_i \widehat{w}_k + \widehat{M^{-1}R}_{i,k}. \tag{3.9}$$

This motivates the following definition.

Definition 3.1.   We say that μ_i is (γ, τ)-Diophantine w.r.t. ω_0 if

$$|\mu_i - 2\pi i(\omega_0 \cdot k)| \geqslant \frac{\gamma}{|k|^\tau}, \quad \forall\, k \in \mathbb{Z}^d \backslash \{0\}. \tag{3.10}$$

We say that μ_i is subexponentially Diophantine w.r.t. ω_0 if

$$\lim_{|k|\to\infty} \frac{1}{|k|} \log|\mu_i - 2\pi i(\omega_0 \cdot k)| = 0. \tag{3.11}$$

Clearly if μ_i is (γ, τ)-Diophantine w.r.t. ω_0 (or subexponentially Diophanitne w.r.t ω_0) we can set

$$\widehat{w}_k = \frac{1}{\mu_i - 2\pi i(\omega_0 \cdot k)} \widehat{M^{-1}R}_{i,k}$$

It is straightforward to see that if $\mu_i \neq 0$ and it is (γ, τ)-Diophantine w.r.t. ω_0 we have

$$\|w\|_{\mathcal{A}_{\xi-\delta}} \leqslant \gamma \delta^{-(\tau+d)} \|M^{-1}R\|_{\mathcal{A}_\xi}$$

In the case that Λ has a non-trivial Jordan block with eigenvalue λ and multiplicity m we obtain a system of equations

$$\omega_0 \cdot \partial_\theta w^i - \lambda w^i - w^{i+1} - \cdots - w^m = R^m, \quad i = 1, \dots, m$$

which can be solved recursively starting from order m and going in decreasing order.

Therefore we have the following result.

Theorem 3.2.   *Assume that*

- *f_0 is analytic (resp. $\mathcal{C}^\infty$).*
- *The equation is reducible.*
- *The matrix Λ has exactly d eigenvalues and the rest of the eigenvalues of Λ is subexponentially Diophantine w.r.t. ω_0.*

Then there exists a Lindstedt series solving (1.8) to all orders. The coefficients K_n are analytic in $\mathbb{T}_{\xi'}^d$ for all $\xi' < \xi$ (resp. $\mathcal{C}^\infty$)

4. Lagrangian tori in the Hamiltonian case

If we assume that f_0 is Hamiltonian[6] then in the neighborhood of an invariant torus there is a very rigid structure that can be used to compute Lindstedt series. This structure (called automatic reducibility) was used in [35] to give a computationally efficent proof of the KAM theorem; we shall use the automatic reducibility to compute Lindstedt series.

Automatic reducibility has also been found in other systems which preserve geometric structures, such as conformally symplectic systems [36] and volume preserving systems [37]. The results in this section could also be adapted easily to the other automatically reducible systems.

The key observation is the following result; see [35, 38].

Lemma 4.3. *Assume that* $f_0 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *is Hamiltonian and* $K_0 : \mathbb{T}^d \to \mathbb{R}^n$ *satisfies* (3.5). *Assume* $n = 2d$. *Then the* $2d \times 2d$ *matrix-valued function*

$$M(\theta) = [DK_0(\theta), J^{-1}DK_0(\theta)N(\theta)]$$

satisfies

$$\omega_0 \cdot \partial_\theta M(\theta) = M(\theta) \begin{pmatrix} 0_d & L(\theta) \\ 0_d & 0_d \end{pmatrix} \tag{4.12}$$

where 0_d *denotes the* $d \times d$ *zero-matrix and* $L(\theta)$ *is an explicit matrix. By* $[\cdot, \cdot]$ *we denote the juxtaposition of two* $2d \times d$ *matrices to obtain a* $2d \times 2d$ *matrix.*

We refer to [35, 38] for the proof.

Lemma 4.3 has a very clear geometric meaning. The first columns of M have the same interpretation as in section 3. The last d columns are forced by the preservation of the symplectic structure.

Using (4.12) we see that, under the change of variables

$$U = MW$$

by the automatic reducibility the linearized equation becomes

$$\omega_0 \cdot \partial_\theta W = \begin{pmatrix} 0_d & A(\theta) \\ 0_d & 0_d \end{pmatrix} W + \begin{pmatrix} 0 \\ \omega_n \end{pmatrix} + \widetilde{R}_n$$

The reason why ω_n appears only in the second term is that it appears only in $DK_0(\theta)\omega_n$.

Again we obtain ω_n so that the second term has a solution using the theory of constant differential equations. Then the second component is determined up to an additive constant (the constant is uniquely determined if we impose for instance (1.13)).

To summarize, we obtained the following result.

Theorem 4.4. *Assume that* $n = 2d$ *and* f_0 *is an analytic (resp.* $\mathcal{C}^\infty$*) Hamiltonian vector field. Then there exists a Lindstedt series solving* (1.8) *to all orders. The coefficients* K_n *are analytic in* $\mathbb{T}^d_{\xi'}$ *for all* $\xi' < \xi$ *(resp.* $\mathcal{C}^\infty$*).*

---

[6] i.e. $f_0 = J\nabla H$ for some function $H : \mathbb{R}^n \to \mathbb{R}$ with n even, and J is the matrix of a 2-form $\Omega(\alpha, \beta) = \langle \alpha, J\beta \rangle$, which is symplectic (i.e. $d\Omega = 0$ and non-degenerate).

5. Limit cycles and isochrones

We now consider the case in which, for $\varepsilon = 0$, (1.4) or (1.5)) is a two dimensional ODE, which admits a limit cycle. This model is very common in applications in electronics, where the classical models of oscillators are limit cycles, but for fast electronics it is useful to include the delay.

Besides the limit cycle, it is useful to consider the solutions that converge exponentially to it. They were called isochrones in [39] which explained their physical and biological relevance. Their relation to stable manifolds was pointed out in [40].

In this section, we will show that there are Lindstedt series both for the limit cycles and for the isochrones. Furthemore we will mention that this fits very well with the recent developments in *a posteriori* theorems [18, 19] and that, in this case, we can prove that the series are asymptotic in a very strong sense. See theorem 5.6.

We also note that in this case we will develop a method to compute the Linstedt series in a much faster way. Each step of the algorithm will double the number of computed terms. This is in contrast with the methods discussed before, in which one step of the algorithm produced only one more term in the expansion. We may informally describe this method as overloading the Newton method to power series.

A convenient starting point for our analysis is the result in [41] that in a neighbourhood of the limit cycle, there is an embedding

$$W : \mathbb{T}^1 \times \mathbb{R} \to \mathbb{R}^2 \tag{5.13}$$

so that for every $\theta_0 \in \mathbb{T}$ and $s_0 \in \mathbb{R}$ with $|s_0| \ll 1$,

$$y(t) = W(\theta_0 + \omega_0 t, s_0 e^{\lambda t}) \tag{5.14}$$

is a solution of the equation of $\varepsilon = 0$. What we will do is to seek to modify the W, ω, λ so that (5.14) is a solution of the delay equation.

Note that in this case, we are not looking for a torus embedding as in (1.8) or (1.10), but we are also including the exponentially converging orbits. Hence, there are two parameters to be found, ω, the frequency of the torus and λ, the exponential factor of convergence.

Finding solutions of the delay equation (1.4) or (1.5) of the form (5.14) is equivalent to finding W, ω, λ satisfying

$$(\omega \cdot \partial_\theta + s\lambda\partial_s)W(\theta, s) = f_\varepsilon(W(\theta, s), W(\theta - \varepsilon\omega r(W(\theta, s)), se^{-\varepsilon\lambda r(W(\theta, s))})). \tag{5.15}$$

or

$$(\omega \cdot \partial_\theta + s\lambda\partial_s)W(\theta, s) = f_\varepsilon(W(\theta, s), \varepsilon W(\theta - \omega r(W(\theta, s)), se^{-\lambda r(W(\theta, s))})). \tag{5.16}$$

respectively.

Again we look for a solution $(W(\theta, s), \omega, \lambda)$ of (5.15) as a formal power series, i.e.

$$\lambda = \sum_{j \geqslant 0} \varepsilon^j \lambda_j, \quad \omega = \sum_{j \geqslant 0} \varepsilon^j \omega_j$$

$$W(\theta, s) = \sum_{j \geqslant 0} \varepsilon^j W_j(\theta, s), \tag{5.17}$$

The case is a particular case of the reducibility case, so we could get the series using the methods in section 3. In this section, however, we want to describe a different algorithm that is based on a Newton method and is quadratically convergent. When applied to the problem of Lindstedt series, we see that the method will double the number of coefficients that we have computed at every step (the step will be more complicated than in the order by order method). In this paper, we will not perform a comparision of the computational cost of the Newton method and the order by order method. In [41] such comparisons are performed in the ODE case.

Of course the meta-lemma 4 applies in a slightly different form also in this case, so we need to show that we can solve the linearized equation. Indeed, set

$$L_{\omega,\lambda} = \begin{pmatrix} \omega \\ \lambda s \end{pmatrix}, \tag{5.18}$$

denote

$$DWL_{\omega,\lambda} = (\omega \cdot \partial_\theta + s\lambda\partial_s)W(\theta, s) \tag{5.19}$$

and

$$F \circ W = f_\varepsilon(W(\theta, s), W(\theta - \varepsilon\omega r(W(\theta, s)), se^{-e\lambda r(W(\theta,s))})), \quad \text{or}$$
$$F \circ W = f_\varepsilon(W(\theta, s), \varepsilon W(\theta - \omega r(W(\theta, s)), se^{-\lambda r(W(\theta,s))})) \tag{5.20}$$

for (5.15) and (5.16) respectively, and assume that we have an approximate solution (W, ω, λ) of (5.15), i.e. such that

$$DWL_{\omega,\lambda} = F \circ W + E \tag{5.21}$$

for some small E. Thus for the Newton scheme we need to find an better approximation $(W + \Delta, \omega + \alpha, \lambda + \beta)$ such that the correction (Δ, α, β) elimitates the error E at the linear approximation. This means that indeed we need to solve the linearized equation

$$D\Delta L_{\omega,\lambda} + DWL_{\alpha,\beta} = (DF \circ W)\Delta + E. \tag{5.22}$$

Note that differentiating (5.21) we get

$$D^2 WL_{\omega,\lambda} + DWDL_{\omega,\lambda} = (DF \circ W)DW + DE \tag{5.23}$$

so, since the operator DW is invertible, the idea is to look for Δ of the form

$$\Delta = DWA. \tag{5.24}$$

Substituting (5.24) into (5.22) and rearranging we get

$$D^2 WAL_{\omega,\lambda} + DWDAL_{\omega,\lambda} - (DF \circ W)DWA = -DWL_{\alpha,\beta} + E. \tag{5.25}$$

Now, since $D^2 WAL_{\omega,\lambda} = D^2 WL_{\omega,\lambda}A$, and using (5.25) we obtain

$$(DE - DWDL_{\omega,\lambda})A + DWDAL_{\omega,\lambda} = -DWL_{\alpha,\beta} + E.$$

Due to the fact that the term DEA is 'quadratically small', we may drop it so that the equation for A becomes

$$-DWDL_{\omega,\lambda}A + DWDAL_{\omega,\lambda} = -DWL_{\alpha,\beta} + E. \tag{5.26}$$

This last step is called *quasi-Newton step* in the literature; see for instance [41] and references therein. If we now multiply by DW^{-1} we see that (5.26) reduces to

$$-DL_{\omega,\lambda}A + DAL_{\omega,\lambda} = -L_{\alpha,\beta} + \widetilde{E}, \quad \widetilde{E} := DW^{-1}E,$$

which in components $A = (A_1, A_2), \widetilde{E} = (\widetilde{E}_1, \widetilde{E}_2)$, takes the form

$$(\omega \cdot \partial_\theta + s\lambda\partial_s)A_1 + \alpha = \widetilde{E}_1$$

$$(\omega \cdot \partial_\theta + s\lambda\partial_s - \lambda)A_2 + \beta s = \widetilde{E}_2, \tag{5.27}$$

i.e. it is a linear equation with constant coefficients. Equations like (5.27) were studied in [41] with two methods. Here we follow the analysis based on power series. Indeed, expanding

$$A_h(\theta, s) = \sum_{k\in\mathbb{Z}^d} \sum_{p\geqslant 0} \hat{A}_{h,k}^{(p)} s^p e^{2\pi ik\theta}, \quad h = 1, 2$$

$$\widetilde{E}_h(\theta, s) = \sum_{k\in\mathbb{Z}^d} \sum_{p\geqslant 0} \hat{\widetilde{E}}_{h,k}^{(p)} s^p e^{2\pi ik\theta}, \quad h = 1, 2 \tag{5.28}$$

we see that (5.27) takes the form

$$(i2\pi\omega \cdot k + \lambda p)\hat{A}_{1,k}^{(p)} = \hat{\widetilde{E}}_{1,k}^{(p)} \quad p \geqslant 1$$

$$(i2\pi\omega \cdot k + \lambda p - \lambda)\hat{A}_{2,k}^{(p)} = \hat{\widetilde{E}}_{2,k}^{(p)} \quad p = 0, \; p \geqslant 2, \tag{5.29}$$

and

$$(i2\pi\omega \cdot k)\hat{A}_{1,k}^{(0)} + \alpha = \hat{\widetilde{E}}_{1,k}^{(0)}$$

$$(i2\pi\omega \cdot k)\hat{A}_{2,k}^{(1)} + \beta = \hat{\widetilde{E}}_{2,k}^{(1)} \tag{5.30}$$

Thus we can fix

$$\alpha = \hat{\widetilde{E}}_{1,0}^{(0)}, \quad \beta = \hat{\widetilde{E}}_{2,0}^{(1)},$$

$$\hat{A}_{1,k}^{(0)} = \frac{\hat{\widetilde{E}}_{1,k}^{(0)}}{(i2\pi\omega \cdot k)} \quad \hat{A}_{2,k}^{(1)} = \frac{\hat{\widetilde{E}}_{2,k}^{(1)}}{(i2\pi\omega \cdot k)}, \quad k \neq 0 \tag{5.31}$$

and

$$\hat{A}_{1,k}^{(p)} = \frac{\hat{\widetilde{E}}_{1,k}^{(p)}}{i2\pi\omega \cdot k + \lambda p} \quad \hat{A}_{2,k}^{(p)} = \frac{\hat{\widetilde{E}}_{2,0}^{(p)}}{i2\pi\omega \cdot k + \lambda(p-1)}. \tag{5.32}$$

Since λ and p are both real, then a subexponential Diophantine ω_0 ensures $A_1^{(p)}, A_2^{(p)}$ to be analytic functions of θ. Precisely we proved the following result.

Theorem 5.5. *Assume that f_0 is analytic (resp. $\mathcal{C}^\infty$). Then there exists Lindstedt series*

$$W = \sum_n \varepsilon^n \sum_{p\geqslant 0} s^p W_j^{(p)}, \quad \omega = \sum_n \varepsilon^n \omega_n \quad \lambda = \sum_n \varepsilon^n \lambda_n$$

solving (5.15) (resp. (5.16)) to all orders. The coefficients $W_j^{(p)}$ are analytic in $\mathbb{T}_{\xi'}^d$ for $\xi' < \xi$ (resp. $\mathcal{C}^\infty$)

We also mention that in this case there is an a-posteriori theory developed in [19], which takes as principal input the fact that there are approximate solutions that solve very approximately the equation (5.15) and conclude that there are true solutions.

Since the main conclusions of theorem 5.5 are precisely that we can construct series that satisfy (5.15) very accuratey, we can put together theorem 5.5 and the results of [19] and we obtain the following.

Theorem 5.6. *In the assumptipions of theorem 5.5, we can find solutions $W_\varepsilon, \omega_\varepsilon \lambda_\varepsilon$ of the equation (5.15). These W_ε are finitely differentiable functions for any $\varepsilon > 0$.*

Furthermore, there exists a function $r(\varepsilon)$, with $\lim_{\varepsilon \to 0} r(\varepsilon) = \infty$ in such a way that for all N, there exist numbers C_N such that

$$\|W_\varepsilon - W_\varepsilon^{[\leqslant N]}\|_{C^{r(\varepsilon)}} \leqslant C_N \varepsilon^{N+1}$$

$$|\omega_\varepsilon - \omega_\varepsilon^{[\leqslant N]}| \leqslant C_N \varepsilon^{N+1}$$

$$|\lambda_\varepsilon - \lambda_\varepsilon^{[\leqslant N]}| \leqslant C_N \varepsilon^{N+1}$$

Note that the conclusions are slightly stronger than the usual definition of asymptotic expansions since we conclude that the approximation is happening in stronger norms as ε goes to zero.

Remark 5.7. The discussion of the present section is restricted to a 2 dimensional case; see (5.13). In higher dimension unfortunately we do not know how to treat the isochrones. However the quasi-periodic solution are dealth with in section 2.

6. Systems with more delays and the electrodynamics case

We proved that it is possible to find solutions (in the sense of formal power series) to SDDE equations of the form (1.4) or (1.5) in various setting when $\ell = 1$. It is however clear that with a slight modification of the discussions above we could cover the case $\ell \geqslant 2$. Indeed the only difference is that the vector field f_ε depends on $\ell + 1$ arguments instead of only two, so it suffices to replace the operator

$$D_1 f_0(K_0(\theta), \ldots, K_0(\theta)) + D_2 f_0(K_0(\theta), \ldots, K_0(\theta))$$

with

$$\sum_{p=1}^{\ell+1} D_p f_0(K_0(\theta), \ldots, K_0(\theta))$$

where D_p denotes the derivative w.r.t. the pth argument.

We now discuss the physical case (1.1). We start by rewriting (1.1) as a dynamical system, i.e.

$$\dot{x}_i = v_i$$

$$\dot{v}_i = GM_i(v_i)^{-1} \sum_{j \neq i} \frac{q_i q_j (x_i(t) - x_j(t - \varepsilon|x_i(t) - x_j(t)| + O(\varepsilon^2)))}{|x_i(t) - x_j(t - \varepsilon|x_i(t) - x_j(t)| + O(\varepsilon^2))|^3} \tag{6.33}$$

where we denoted $\varepsilon = 1/c$ and we also exploited the expansion (1.3). We then look for a torus embedding

$$K : \mathbb{T}^d \to \mathbb{R}^{6N} \tag{6.34}$$

satisfying an equation of the form

$$\omega \cdot \partial_\theta K(\theta) = F(K(\theta), K(\theta - \varepsilon\omega r(K(\theta)) + O(\varepsilon^2))). \tag{6.35}$$

By remark 1.2 we see that if $d = 3N$ we are essentially in the same situation as in case 3, i.e. the case of Lagrangian tori, so we can apply the results of section 4. Unfortunately, for $d \neq 3N$ a similar result cannot be achieved.

Acknowledgments

Appendix A. Solutions of cohomology equations with frequency given by formal power series

In all the cases studied in the present paper, the frequencies are given by power series in ε. However we required (1.22) or (1.23) only for the first summand of the series defining ω. Indeed the following is true.

Lemma A.1. *Let*

$$\omega = \omega_\varepsilon = \sum_{j \geqslant 0} \varepsilon^j \omega_j \tag{A.1}$$

be an $\mathbb{R}^d$-valued formal power series. Let

$$\eta = \sum_{j \geqslant 0} \varepsilon^j \eta_j \tag{A.2}$$

be an $\mathcal{A}_\xi$-valued formal series (recall (1.24)), i.e. $\eta_j \in \mathcal{A}_\xi$ for all $j \geqslant 0$. Assume that ω_0 is subexponential Diophantine (recall (1.23)). Then for every $\delta > 0$ there is a unique $\mathcal{A}_{\xi-\delta}$-valued formal power series

$$\varphi = \sum_{j \geqslant 0} \varepsilon^j \varphi_j \tag{A.3}$$

solving

$$\omega \cdot \partial_\theta \varphi = \eta \tag{A.4}$$

in the sense of power series. Moreover the solution φ is unique if we impose

$$\frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \varphi_j(\theta) d\theta = 0, \quad j \geqslant 0. \tag{A.5}$$

The key observation to prove lemma A.1 is the following (very well known) result.

Proposition A.2.   *If ω is of the form* (A.1) *and ω_0 satisfies* (1.23), *then given any $\alpha \in \mathcal{A}_\xi$, for any $\xi' < \xi$ there is a solution to*

$$\omega \cdot \partial_\theta \beta = \alpha. \tag{A.6}$$

Proof.   If we Fourier-expand

$$\alpha(\theta) = \sum_{k \in \mathbb{Z}^d} \hat{\alpha}_k e^{2\pi i k \cdot \theta}, \quad \beta(\theta) = \sum_{k \in \mathbb{Z}^d} \hat{\beta}_k e^{2\pi i k \cdot \theta},$$

then we see that (A.6) is equivalent to

$$(2\pi i \omega \cdot k)\hat{\beta}_k = \hat{\alpha}_k.$$

If ω_0 satisfies (1.23) then

$$|\omega \cdot k|^{-1} \leqslant C e^{(\xi - \xi')|k|/2}.$$

By Cauchy estimates we have

$$|\hat{\alpha}_k| \leqslant e^{-2\pi \xi |k|} \|\alpha\|_\xi,$$

and hence

$$\|\beta\|_{\xi'} \leqslant \sum_{k \in \mathbb{Z}^d} e^{-2\pi \xi |k|} \|\alpha\|_\xi C e^{(\xi - \xi')|k|/2} \leqslant \widetilde{C} \|\alpha\|_\xi \sum_{k \in \mathbb{Z}^d} e^{-|k|(\xi - \xi')/2},$$

for some constant $\widetilde{C}$, so the assertion follows.

We are now ready to prove lemma A.1.   □

Proof.   (Lemma A.1) We can rewrite (A.4) as

$$\omega_0 \cdot \partial_\theta \varphi_n = \eta_n - \sum_{j=1}^{n} \omega_j \cdot \partial_\theta \varphi_{n-j}, \tag{A.7}$$

hence we can use recursively proposition A.2 to find

$$\varphi_n \in \mathcal{A}_{\xi - (1 - 2^{-n})\delta}$$

so the assertion follows.   □

ORCID iDs

Rafael de la Llave ⓘ https://orcid.org/0000-0002-0286-6233

References

[1] Landau L D and Lifshitz E M 1962 *The Classical Theory of Fields* (*Course of Theoretical Physics* vol 2) 2nd edn (London: Addison-Wesley) Translated from Russian by Morton Hamermesh
[2] Rohrlich F 2007 *Classical Charged Particles* 3rd edn (Hackensack, NJ: World Scientific Publishing)
[3] Spohn H 2004 *Dynamics of Charged Particles and Their Radiation Field* (Cambridge: Cambridge University Press)

[4] Wheeler J A and Feynman R P 1949 Classical electrodynamics in terms of direct inter-particle action *Rev. Mod. Phys.* **21** 425–33

[5] Misner C W, Thorne K S and Wheeler J A 1973 *Gravitation* (San Francisco: W H Freeman and Co.)

[6] Hartung F, Krisztin T, Walther H-O and Wu J 2006 Functional differential equations with state-dependent delays: theory and applications *Handbook of Differential Equations: Ordinary Differential Equations* vol 3 (Amsterdam: Elsevier) pp 435–545

[7] Iannelli M and Milner F 2017 The basic approach to age-structured population dynamics *Models, Methods and Numerics* (*Lecture Notes on Mathematical Modelling in the Life Sciences*) (Dordrecht: Springer)

[8] Kolmanovskiĭ V B and Nosov V R 1986 *Stability of Functional-Differential Equations* (*Mathematics in Science and Engineering* vol 180) (London: Academic)

[9] Diekmann O, van Gils S A, Sjoerd M, Lunel V and Walther H-O 1995 Functional, complex, and nonlinear analysis *Delay Equations* (*Applied Mathematical Sciences* vol 110) (New York: Springer)

[10] Hale J K 1977 *Theory of Functional Differential Equations* (*Applied Mathematical Sciences* vol 3) 2nd edn (New York-Heidelberg: Springer)

[11] Hale J K and Verduyn Lunel S M 1993 *Introduction to Functional-Differential Equations* (*Applied Mathematical Sciences* vol 99) (New York: Springer)

[12] Bel L 1970 Dynamique des systèmes de N particules ponctuelles en relativité restreinte *Ann. Inst. Henri Poincare, Sect.* A **12** 307–21 (http://www.numdam.org/item/AIHPA_1970__12_3_307_0/)

[13] Martin J and Sanz J L 1978 Slow motion approximation in predictive relativistic mechanics. I. Approximated dynamics up to order c^{-4} *J. Math. Phys.* **19** 1887–91

[14] Currie D G, Jordan T F and Sudarshan E C G 1963 Relativistic invariance and Hamiltonian theories of interacting particles *Rev. Mod Phys.* **35** 350–75

[15] Martin J and Sanz J L 1978 Nointeraction theorem of Currie, Jordan, and Sudarshan expansions in c^{-1} *J. Math. Phys.* **19** 780–8

[16] He X and de la Llave R 2016 Construction of quasi-periodic solutions of state-dependent delay differential equations by the parameterization method II: analytic case *J. Differ. Equ.* **261** 2068–108

[17] He X and de la Llave R 2017 Construction of quasi-periodic solutions of state-dependent delay differential equations by the parameterization method I: finitely differentiable, hyperbolic case *J. Dynam. Differ. Equ.* **29** 1503–17

[18] Gimeno J, Yang J and de la Llave R 2019 Numerical computation approach for the parametrization method for state-dependent delay perturbation of an ode in the plane *Preprint*

[19] Yang J, Gimeno J and de la Llave R 2019 Parameterization method for state dependent delay perturbation of an ordinary differential equation *Preprint*

[20] Chicone C 2003 Inertial and slow manifolds for delay equations with small delays *J. Differ. Equ.* **190** 364–406

[21] Poincaré H 1987 *Les méthodes nouvelles de la mécanique céleste* Tome II. *Les Grands Classiques Gauthier-Villars. [Gauthier-Villars Great Classics]. Librairie Scientique et Technique Albert Blanchard, Paris,. Méthodes de MM. Newcomb, Gyldén, Lindstedt et Bohlin.* [The methods of Newcomb, Gyldén, Lindstedt and Bohlin], Reprint of the 1893 original, Bibliothèque Scientifique Albert Blanchard. [Albert Blanchard Scientific Library]

[22] Casal A and Freedman M 1980 A Poincaré–Lindstedt approach to bifurcation problems for differential-delay equations *IEEE Trans. Automat. Control* **25** 967–73

[23] Casal A and Freedman M 1980 The Poincaré–Lindstedt method for differential equations with delay *Proc. 2nd Conf. on Differential Equations and Their Applications, I* (*Valldoreix 1979*) pp 73–80

[24] Rand R 2011 *Differential-delay Equations* (*Complex Systems, Nonlinear Phys. Sci.*) (Beijing: Higher Ed Press) pp 83–117

[25] Fenichel N 1973/74 Asymptotic stability with rate conditions *Indiana Univ. Math. J.* **23** 1109–37

[26] Hirsch M W, Pugh C C and Shub M 1977 *Invariant Manifolds* (*Lecture Notes in Mathematics* vol 583) (Berlin: Springer)

[27] Pesin Y B 2004 *Lectures on Partial Hyperbolicity and Stable Ergodicity* (*Zurich Lectures in Advanced Mathematics*) (Zürich: European Mathematical Society (EMS))

[28] Bates P W, Lu K and Zeng C 2001 Persistence of C^k normally hyperbolic invariant manifolds for infinite dimensional dynamical systems *First Int Congress of Chinese Mathematicians* (*Beijing 1998*) (AMS/IP Stud. Adv. Math. vol 20) (Providence, RI: Amer. Math. Soc.) pp 403–10

[29] Canadell M and Haro A 2017 Computation of quasiperiodic normally hyperbolic invariant tori: rigorous results *J. Nonlinear Sci.* **27** 1869–904

[30] Dinaburg E I and Sinaĭ J G 1975 The one-dimensional Schrödinger equation with quasiperiodic potential *Funkcional. Anal. i Priložen.* **9** 8–21

[31] Eliasson L H 1992 Floquet solutions for the 1-dimensional quasi-periodic Schrödinger equation *Commun. Math. Phys.* **146** 447–82

[32] Hou X and You J 2012 Almost reducibility and non-perturbative reducibility of quasi-periodic linear systems *Invent. Math.* **190** 209–60

[33] Puig J 2006 A nonperturbative Eliasson's reducibility theorem *Nonlinearity* **19** 355–76

[34] Puig J 2002 Reducibility of linear equations with quasi-periodic coefficients. A survey (http://www.maia.ub.edu/dsg/2002/0201puig.pdf)

[35] de la Llave R, González A, Jorba À and Villanueva J 2005 KAM theory without action-angle variables *Nonlinearity* **18** 855–95

[36] Calleja R C, Celletti A and de la Llave R 2013 A KAM theory for conformally symplectic systems: efficient algorithms and their validation *J. Differ. Equ.* **255** 978–1049

[37] de la Llave R and Mireles-James J D 2012 Parameterization of invariant manifolds by reducibility for volume preserving and symplectic maps *Discrete Contin. Dyn. Syst.* **32** 4321–60

[38] de la Llave R 2001 A tutorial on KAM theory *Smooth Ergodic Theory and its Applications* (*Seattle, WA 1999*) (*Proc. Sympos. Pure Math.* vol 69) (Providence, RI: Amer. Math. Soc.) pp 175–292

[39] Winfree A T 1974 Patterns of phase compromise in biological cycles *J. Math. Biol.* **1** 73–95

[40] Guckenheimer J 1975 Isochrons and phaseless sets *J. Math. Biol.* **1** 259–73

[41] Huguet G and de la Llave R 2013 Computation of limit cycles and their isochrons: fast algorithms and their convergence *SIAM J. Appl. Dyn. Syst.* **12** 1763–802