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Local Well-Posedness of Vlasov–Poisson–Boltzmann Equation with Generalized Diffuse Boundary Condition

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Abstract
The Vlasov–Poisson–Boltzmann equation is a classical equation governing the dynamics of charged particles with the electric force being self-imposed. We consider the system in a convex domain with the Cercignani–Lampis boundary condition. We construct a uniqueness local-in-time solution based on an L^∞-estimate and $W^{1,p}$-estimate. In particular, we develop a new iteration scheme along the characteristic with the Cercignani–Lampis boundary for the L^∞-estimate, and an intrinsic decomposition of boundary integral for $W^{1,p}$-estimate.

Keywords Cercignani-Lampis boundary · Vlasov–Poisson–Boltzmann system · Boundary value problem · Local well-posedness

1 Introduction

In this paper we study the existence and uniqueness of Vlasov–Poisson–Boltzmann (VPB) system with generalized diffuse boundary condition. VPB is a classical model that describes the dynamics of dilute charged particles (such as plasma) with a self-imposed electric field (see [2,13] and reference therein). We denote $F(t, x, v)$ the phase-space-distribution function of charged particles at time t, location $x \in \Omega$, a bounded domain in $\mathbb{R}^3$, moving with velocity $v \in \mathbb{R}^3$. The equation writes:

$$\partial_t F + v \cdot \nabla_x F - E \cdot \nabla_v F = Q(F, F), \quad F|_{t=0} = F_0(x, v). \tag{1.1}$$

The characteristics solves the following Hamilton ODEs

$$\dot{x} = v, \quad \dot{v} = -E. \tag{1.2}$$

The collision operator Q on the right, as a functional of F, describes the binary collisions between particles and takes the form of

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$$
\begin{aligned}
Q(F_1, F_2)(v) &= Q_{\text{gain}}(F_1, F_2)(v) - Q_{\text{loss}}(F_1, F_2)(v) = Q_{\text{gain}}(F_1, F_2) - \nu(F_1)F_2 \\
&:= \iint_{\mathbb{R}^3\times\mathbb{S}^2} B(v-u,\omega)F_1(u')F_2(v')d\omega du \\
&\quad - \left(\iint_{\mathbb{R}^3\times\mathbb{S}^2} B(v-u,\omega)F_1(u)d\omega du\right)F_2(v).
\end{aligned}
\tag{1.3}
$$

In the collision process, momentum and energy are conserved, namely,

$$u' + v' = u + v, \quad |u'|^2 + |v'|^2 = |u|^2 + |v|^2,$$

where the post-velocities are denoted as

$$u' = u - [(u-v)\cdot\omega]\omega, \qquad v' = v + [(u-v)\cdot\omega]\omega. \tag{1.4}$$

In (1.3), B is called a collision kernel, and we use the hard potential model in this paper:

$$
\begin{aligned}
&B(v-u,\omega) = |v-u|^{\mathcal{K}} q_0\Big(\frac{v-u}{|v-u|}\cdot\omega\Big), \quad \text{with} \quad 0 < \mathcal{K} \le 1, \\
&0 \le q_0\Big(\frac{v-u}{|v-u|}\cdot\omega\Big) \le C\Big|\frac{v-u}{|v-u|}\cdot\omega\Big|.
\end{aligned}
$$

In (1.1), E denotes the electrostatic field, and we consider a self-imposed electric field in this paper: namely, the charged particles themselves form a potential that in turn drives their own dynamics. This is in particular a relevant model for plasma particles without extra magnetic field. More specifically,

$$E(t,x) = -\nabla_x\phi(t,x), \tag{1.5}$$

with the electrostatic potential ϕ determined by the Poisson equation

$$-\Delta_x\phi(t,x) = \int_{\mathbb{R}^3} F(t,x,v)dv - \rho_0 \text{ in } \Omega, \quad \frac{\partial\phi}{\partial n} = 0 \text{ on } \partial\Omega, \tag{1.6}$$

where ρ_0 is a background constant charge density. We set ρ_0 as an average of the initial total mass:

$$\rho_0 = \frac{1}{|\Omega|}\int_{\Omega\times\mathbb{R}^3} F_0(x,v)dvdx. \tag{1.7}$$

The boundary condition of F is determined by the interaction between the charged particles and the physical boundary. We denote the boundary of the phase space as $\gamma := \{(x,v) \in \partial\Omega\times\mathbb{R}^3\}$. Let $n = n(x)$ be the outward normal direction at $x \in \partial\Omega$. We split the phase boundary into an incoming (γ_-) and outgoing (γ_+) set as:

$$\gamma_\mp := \{(x,v) \in \partial\Omega\times\mathbb{R}^3 : n(x)\cdot v \lessgtr 0\} \quad \text{or} \quad \gamma_\mp(x) := \{v \in \partial\Omega\times\mathbb{R}^3 : n(x)\cdot v \lessgtr 0\}. \tag{1.8}$$

The boundary condition determines the distribution on γ_-, and describes how particles, once hit the boundary, bounce back into the domain. It is characterized through a scattering kernel $R(u \to v; x, t)$ that satisfies a general balance law

$$F(t,x,v)|n(x)\cdot v| = \int_{\gamma_+(x)} R(u\to v; x,t)F(t,x,u)\{n(x)\cdot u\}du, \quad \text{on } \gamma_-. \tag{1.9}$$

Physically, $R(u \to v; x, t)$ represents the probability of a molecule striking in the boundary at $x \in \partial\Omega$ with velocity u to be bounced back to the domain with velocity v at the same location x and time t. In this paper we use a model proposed by Cercignani and Lampis in [4,5]. With two accommodation coefficients

$$0 < r_\perp \le 1, \quad 0 < r_\parallel < 2, \tag{1.10}$$

the Cercignani–Lampis boundary condition (C-L boundary condition) can be written as

$$\begin{aligned} &R(u \to v; x, t) \\ &:= \frac{1}{r_\perp r_\parallel (2 - r_\parallel)\pi/2} \frac{|n(x)\cdot v|}{(2T_w(x))^2} \\ &\exp\left(-\frac{1}{2T_w(x)}\left[\frac{|v_\perp|^2 + (1-r_\perp)|u_\perp|^2}{r_\perp} + \frac{|v_\parallel - (1-r_\parallel)u_\parallel|^2}{r_\parallel(2-r_\parallel)}\right]\right) \\ &\times I_0\left(\frac{1}{2T_w(x)} \frac{2(1-r_\perp)^{1/2} v_\perp u_\perp}{r_\perp}\right). \end{aligned} \tag{1.11}$$

Here $T_w(x)$ is a wall temperature on the boundary and

$$I_0(y) := \pi^{-1}\int_0^\pi e^{y\cos\phi}\, d\phi.$$

In this formula, $v_\perp$ and $v_\parallel$ denote the normal and tangential components of the velocity respectively:

$$v_\perp = v \cdot n(x), \quad v_\parallel = v - v_\perp n(x). \tag{1.12}$$

Similarly $u_\perp = u \cdot n(x)$ and $u_\parallel = u - u_\perp n(x)$.

This model can be considered as a generalization of fundamental boundary conditions. For instance if we set $r_\perp = 1$ and $r_\parallel = 1$, the scattering kernel equals

$$R(u \to v; x, t) = \frac{2}{\pi (2T_w(x))^2} e^{-\frac{|v|^2}{2T_w(x)}} |n(x)\cdot v|.$$

This corresponds the so-called diffuse boundary condition:

$$F(t,x,v) = \frac{2}{\pi(2T_w(x))^2} e^{-\frac{|v|^2}{2T_w(x)}} \int_{n(x)\cdot u>0} F(t,x,u)\{n(x)\cdot u\} du \text{ on } (x,v)\in\gamma_-. \tag{1.13}$$

With $r_\perp = 0$, $r_\parallel = 0$, the scattering kernel is given by

$$R(u \to v; x, t) = \delta(u - \mathfrak{R}_x v),$$

with $\mathfrak{R}_x v = v - 2n(x)(n(x)\cdot v)$. This corresponds the specular reflection boundary condition $F(t,x,v) = F(t,x,\mathfrak{R}_x v)$.

Finally with $r_\perp = 0$, $r_\parallel = 2$, the scattering kernel is given by

$$R(u \to v; x, t) = \delta(u + v),$$

which corresponds the bounce-back reflection reflection boundary condition $F(t,x,v) = F(t,x,-v)$. The C–L model is related to the Maxwell boundary condition since both models can describe the intermediate reflection law between diffuse and specular reflection boundary conditions. The comparison of the two is found in [6].

It is important to note that the C–L boundary condition satisfies the reciprocity property

$$R(u \to v; x, t) = R(-v \to -u; x, t)\frac{e^{-|v|^2/(2T_w(x))}}{e^{-|u|^2/(2T_w(x))}} \frac{|n(x)\cdot v|}{|n(x)\cdot u|}, \tag{1.14}$$

and the normalization property (see the proof in appendix)

$$\int_{\gamma_-(x)} R(u \to v; x, t) dv = 1. \tag{1.15}$$

We note that the normalization (1.15) property immediately leads to the null flux condition for F:

$$\int_{\mathbb{R}^3} F(t, x, v)\{n(x) \cdot v\} dv = 0, \quad \text{for } x \in \partial\Omega. \tag{1.16}$$

This guarantees the conservation of total mass:

$$\int_{\Omega\times\mathbb{R}^3} F(t, x, v) dv dx = \int_{\Omega\times\mathbb{R}^3} F(0, x, v) dv dx \text{ for all } t \geq 0. \tag{1.17}$$

We note that from the conservation of mass (1.17) and our choice (1.7), the Neumann boundary condition of (1.6) is automatically compatible.

The generality of the C-L model allows it to be applicable to many problems, including the rarefied gas flow studied in [18,21,22]; gas surface interaction model presented in [19,23]; and rigid-sphere interaction model investigated in [10,11], to name a few. There also emerged many other derivations of C-L model besides the original one, and we refer interested readers to [3,4,7].

1.1 Main Result

We now discuss the main result of this paper. Throughout this paper we assume the domain is C^3, which means for any $p \in \partial\Omega$, there exists sufficiently small $\delta_1 > 0$, $\delta_2 > 0$, and an one-to-one and onto C^3-map η_p so that

$$\begin{aligned} \eta_p : \{x_\parallel \in \mathbb{R}^2 : |x_\parallel| < \delta_1\} &\to \partial\Omega \cap B(p, \delta_2), \\ x_\parallel = (x_{\parallel,1}, x_{\parallel,2}) &\mapsto \eta_p(x_{\parallel,1}, x_{\parallel,2}). \end{aligned} \tag{1.18}$$

We further assume the domain is *convex*: there exists $C_\eta > 0$ and $C_\Omega > 0$ such that at all $p \in \partial\Omega$, the Hessian of the corresponding η_p, defined in (1.18) are upper and lower bounded for all $x_\parallel$ in (1.18) as

$$-C_\eta|\zeta|^2 \leq \sum_{i,j=1}^{2} \zeta_i\zeta_j\partial_i\partial_j\eta_p(x_\parallel) \cdot n(x_\parallel) \leq -C_\Omega|\zeta|^2, \quad \forall \zeta \in \mathbb{R}^2. \tag{1.19}$$

We define the global Maxwellian using the maximum wall temperature as

$$\mu := e^{-\frac{|v|^2}{2T_M}}, \quad \text{with } T_M := \max_{x\in\partial\Omega}\{T_w(x)\}. \tag{1.20}$$

By setting

$$F = \sqrt{\mu} f, \tag{1.21}$$

we have:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x\phi \cdot \nabla_v f + \frac{1}{2T_M} f v \cdot \nabla_x \phi = \Gamma(f, f) \\ f(t = 0, x, v) = f_0(x, v) := \mu^{-1/2} F_0 \\ f(t, x, v)|n(x) \cdot v||_{\gamma_-} = \frac{1}{\sqrt{\mu}} \int_{n(x)\cdot u>0} R(u \to v; x, t) f(t, x, u)\sqrt{\mu(u)}\{n(x) \cdot u\} du \end{cases} \tag{1.22}$$

where the collision operator becomes

$$\Gamma(f_1, f_2) = \Gamma_{\text{gain}}(f_1, f_2) - \nu(F_1)F_2/\mu = \frac{1}{\sqrt{\mu}} Q_{\text{gain}}(\sqrt{\mu} f_1, \sqrt{\mu} f_2) - \nu(F_1) f_2, \tag{1.23}$$

and ϕ solves

$$-\Delta_x \phi(t, x) = \int_{\mathbb{R}^3} f(t, x, v)\sqrt{\mu(v)}dv - \rho_0 \text{ in } \Omega, \quad \frac{\partial \phi}{\partial n} = 0 \text{ on } \partial\Omega. \tag{1.24}$$

The reciprocity property (1.14) is also translated to: for $(x, v) \in \gamma_-$,

$$f(t, x, v)|n(x) \cdot v| = \frac{1}{\sqrt{\mu}} \int_{n(x)\cdot u>0} R(-v \to -u; x, t) \frac{e^{-|v|^2/(2T_w(x))}}{e^{-|u|^2/(2T_w(x))}} f(t, x, u)\sqrt{\mu(u)} \frac{|n(x) \cdot v|}{|n(x) \cdot u|} \{n(x) \cdot u\} du.$$

Denote

$$d\sigma(u, v) := R(-v \to -u; x, t)du, \tag{1.25}$$

then according to the normalization property (1.15), $d\sigma$ is a probability measure in space $\gamma_+(x)$, reducing the boundary condition for f to:

$$f(t, x, v)|_{\gamma_-} = e^{[\frac{1}{4T_M} - \frac{1}{2T_w(x)}]|v|^2} \int_{n(x)\cdot u>0} f(t, x, u) e^{-[\frac{1}{4T_M} - \frac{1}{2T_w(x)}]|u|^2} d\sigma(u, v). \tag{1.26}$$

For easier notation, we furthermore denote, for all θ:

$$w_\theta := e^{\theta|v|^2}, \quad \langle v \rangle := \sqrt{|v|^2 + 1}. \tag{1.27}$$

Now we state our main theorem of the paper:

Theorem 1 *Assume $\Omega \subset \mathbb{R}^3$ is an open bounded, and convex C^3 domain, the wall temperature $T_w(x) > 0$ is smooth, and that the two accommodation coefficients of (1.10) satisfy*

$$\frac{\min_{x\in\partial\Omega}\{T_w(x)\}}{\max_{x\in\partial\Omega}\{T_w(x)\}} > \max\Big(\frac{1 - r_\parallel}{2 - r_\parallel}, \frac{\sqrt{1 - r_\perp} - (1 - r_\perp)}{r_\perp}\Big). \tag{1.28}$$

Assume further that

$$\|w_\theta f_0\|_\infty < \infty, \tag{1.29}$$

$$\|w_{\tilde{\theta}} \nabla_v f_0\|_{L^3_{x,v}} < \infty, \tag{1.30}$$

$$\|w_{\tilde{\theta}} \alpha^\beta_{f_0,\epsilon} \nabla_{x,v} f_0\|_{L^p(\Omega\times\mathbb{R}^3)} < \infty \quad \text{for} \quad 3 < p < 6,\ 1 - \frac{2}{p} < \beta < \frac{2}{3}, \tag{1.31}$$

with

$$0 < \tilde{\theta} < \theta < \frac{1}{4\max_{x\in\partial\Omega}\{T_w(x)\}}, \tag{1.32}$$

and a weight function $\alpha_{f,\epsilon}$, to be defined in (1.41), then there is a unique solution $f(t, x, v)$ to (1.22) in a time interval of $t \in [0, \bar{t}]$ with

$$\bar{t} = \bar{t}(\|w_\theta f_0\|_\infty, \|w_{\tilde{\theta}} \alpha^\beta_{f_0,\epsilon} \nabla_{x,v} f_0\|_{L^p(\Omega\times\mathbb{R}^3)}, \|w_{\tilde{\theta}} \nabla_v f_0\|_{L^3_{x,v}}, r_\parallel, r_\perp, \Omega, T_M, \min(T_w(x))). \tag{1.33}$$

Moreover, there are $\mathfrak{C} > 0$ *and* $\lambda > 0$*, so that* f *satisfies*

$$\sup_{0\le t\le \bar{t}} \|w_\theta e^{-\mathfrak{C}\langle v\rangle^2 t} f(t)\|_\infty \lesssim \|w_\theta f_0\|_\infty, \tag{1.34}$$

$$\sup_{0\le t\le \bar{t}} \|\nabla_v f(t)\|_{L^3_x L^{1+\delta}_v} < \infty, \tag{1.35}$$

$$\sup_{0\le t\le \bar{t}} \left\{ \|w_{\tilde{\theta}} e^{-\lambda t\langle v\rangle} \alpha^\beta_{f,\epsilon} \nabla_{x,v} f(t)\|^p_p + \int_0^t |w_{\tilde{\theta}} e^{-\lambda s\langle v\rangle} \alpha^\beta_{f,\epsilon} \nabla_{x,v} f(t)|^p_{p,+} \right\} < \infty. \tag{1.36}$$

Remark 1 We do not assume the smallness of our initial data, but we need the small scale of the time $\bar{t}$. Setting $r_\perp = 1$ and $r_\parallel = 1$, this theorem also provides the first large date well-posedness of VPB system with the standard diffuse boundary condition (1.13). A small data result had been established in [2]. We use the condition (1.28) in the proof of the L^∞ bound, which itself serves as an important a-priori estimate for the existence and the $W^{1,p}$ estimate (1.35) and (1.36).

Remark 2 To the best of our knowledge, Theorem 1 provides the *first* local in time solution to the Vlasov–Poisson–Boltzmann system in bounded domains with the Cercignani–Lampis boundary condition. The local in time result for the Boltzmann equation without field can be found in [6].

1.2 Strategy of the Proof

In this section we discuss the major difficulties and describe the main strategy utilized in the proof.

The main difficulty comes from the singularity at the boundary. Consider the simple Vlasov–Poisson (VP) equation without the collision:

$$\partial_t f + v\cdot\nabla_x f - \nabla_x\phi_f \cdot \nabla_v f = 0. \tag{1.37}$$

Suppose one has two solutions f and g, then taking the difference we have:

$$\partial_t(f-g) + v\cdot\nabla_x(f-g) - \nabla_x\phi_f\cdot\nabla_v(f-g) = (\nabla_x\phi_f - \nabla_x\phi_g)\cdot\nabla_v g.$$

To show the uniqueness using the stability argument essentially comes down to controling $\nabla_v g$. This is hard to achieve in general: it is a rather well-known result that transport equation in a bounded domain could potentially form singularities [1,16].

This could be better understood by following the trajectory of the Hamiltonian system (1.2). Denote $(X(s;t,x,v), V(s;t,x,v))$ the solution to it that starts with $(X(t;t,x,v), V(t;t,x,v)) = (x,v)$, then follow the ODE, we have

$$\frac{d}{ds}\begin{bmatrix} X(s;t,x,v) \\ V(s;t,x,v) \end{bmatrix} = \begin{bmatrix} V(s;t,x,v) \\ -\nabla_x\phi_f(s, X(s;t,x,v)) \end{bmatrix} \text{ for } -\infty < s,t < \infty. \tag{1.38}$$

For $(t,x,v)\in\mathbb{R}\times\Omega\times\mathbb{R}^3$, we define *the backward exit time* $t_{\mathbf{b}}(t,x,v)$:

$$t_{\mathbf{b}}(t,x,v) := \sup\{s\ge 0 : X(\tau;t,x,v)\in\Omega \text{ for all } \tau\in(t-s,t)\}, \tag{1.39}$$

and the corresponding existing location and velocity:

$$x_{\mathbf{b}}(t,x,v) := X(t-t_{\mathbf{b}}(t,x,v);t,x,v) \quad\text{and}\quad v_{\mathbf{b}}(t,x,v) := V(t-t_{\mathbf{b}}(t,x,v);t,x,v).$$

Call the boundary condition $f|_{\gamma_-} = h$, then (1.37) has an explicit solution

$$f(t,x,v) = h(t - t_{\mathbf{b}}(t,x,v), x_{\mathbf{b}}(t,x,v), v_{\mathbf{b}}(t,x,v)).$$

This leads to a fact that the derivatives of f may contain singularities from a direct computation of $\nabla_x x_{\mathbf{b}}(t,x,v)$ as

$$\nabla_x f(t,x,v) \sim \nabla_x x_{\mathbf{b}}(t,x,v) \sim \frac{1}{n(x_{\mathbf{b}}(t,x,v)) \cdot v_{\mathbf{b}}(t,x,v)}. \tag{1.40}$$

The term blows up as $v_{\mathbf{b}}$ becomes tangential to the surface at the backward exit time. This difficulty sits at the core of many boundary problems of Boltzmann-type equations.

To account for this difficulty, we follow the strategy of incorporating a kinetic weight [2, 15]:

Definition 1 (Kinetic Weight) For $\epsilon > 0$, let f solve (1.37), define

$$\begin{aligned} \alpha_{f,\epsilon}(t,x,v) := \chi\Big(\frac{t - t_{\mathbf{b}}(t,x,v) + \epsilon}{\epsilon}\Big)|n(x_{\mathbf{b}}(t,x,v)) \cdot v_{\mathbf{b}}(t,x,v)| \\ + \Big[1 - \chi\Big(\frac{t - t_{\mathbf{b}}(t,x,v) + \epsilon}{\epsilon}\Big)\Big]. \end{aligned} \tag{1.41}$$

Here we use a smooth function $\chi : \mathbb{R} \to [0,1]$ satisfying

$$\begin{aligned} \chi(\tau) = 0,\ \tau \le 0,\ \text{and}\ \chi(\tau) = 1,\ \tau \ge 1, \\ \frac{d}{d\tau}\chi(\tau) \in [0,4]\ \text{ for all } \tau \in \mathbb{R}. \end{aligned} \tag{1.42}$$

Note that $\alpha_{f,\epsilon}(0,x,v) \equiv \alpha_{f_0,\epsilon}(0,x,v)$ is determined by the initial data f_0. There are two important features of this weight. First it is invariant under the transport operator, namely:

$$[\partial_t + v \cdot \nabla_x - \nabla_x \phi_f \cdot \nabla_v]\alpha_{f,\epsilon}(t,x,v) = 0. \tag{1.43}$$

Second, it takes the value of $|n(x_{\mathbf{b}}^f(t,x,v)) \cdot v_{\mathbf{b}}^f(t,x,v)|$ for $t > t_{\mathbf{b}}^f(t,x,v)$, which is exactly the singularity in (1.40). With the weight term applied, the singularity term can be canceled.

The proof of the main theorem consists two parts: an L^∞-estimate and a weighted $W^{1,p}$ estimate. These estimates are based on the uniform estimates of the following iterative sequence:

$$\begin{aligned} \partial_t f^{m+1} + v \cdot \nabla_x f^{m+1} - \nabla_x \phi^m \cdot \nabla_v f^{m+1} + \frac{1}{2T_M} f^{m+1} v \\ \cdot \nabla_x \phi^m = \Gamma_{\text{gain}}(f^m, f^m) - \nu(F^m) f^{m+1}, \end{aligned} \tag{1.44}$$

with boundary condition:

$$f^{m+1}(t,x,v)\Big|_{\gamma_-} = e^{[\frac{1}{4T_M} - \frac{1}{2T_w(x)}]|v|^2} \int_{n(x)\cdot u > 0} f^m(t,x,u) e^{-[\frac{1}{4T_M} - \frac{1}{2T_w(x)}]|u|^2} d\sigma(u,v), \tag{1.45}$$

and initial condition

$$f^{m+1}(0,x,v) = f(0,x,v).$$

Here we denote

$$\phi^m := \phi_{f^m}.$$

Now we discuss the roadmap for getting these estimates respectively.

L^∞ *estimate:* For obtaining the L^∞ estimate, we derive the trajectory formula and trace each (x, v, t) back along the characteristic till it either hits the boundary or the initial datum for f^m.

It may so happen that some particles bounce back and forth in the domain multiple times before tracing back to $t = 0$ (say k times), and then a k-layered integral will appear. This multiple integral includes v_i, the parameter we use to represent the integral variable at the i-th iteration with the boundary (see more precise definition in Definition 2), and the integral formula will be derived in Lemma 2. There are two main problems one need to handle here: 1. how to integrate the k-fold integral, and 2. what is the chance for a particle to interact with the wall finite times?

To deal with the first difficulty amounts to carefully trace and compute the integration. In case of the diffuse boundary condition with constant temperature where $R = \frac{1}{2\pi} e^{-\frac{|v|^2}{2}} |n(x) \cdot v| = c_\mu e^{\frac{-|v|^2}{2}} |n(x) \cdot v|$, the computation can be simplified. According to (1.45), the boundary condition here is:

$$f = c_\mu \sqrt{\mu(v_{i-1})} \int_{n \cdot v_i > 0} f(v_i) \sqrt{\mu(v_i)} |n \cdot v_i| \mathrm{d}v_i.$$

Trace back further for the next interaction of $i+1$, one arrives at the final integral with respect to v_i to be simply

$$\int_{n \cdot v_i > 0} c_\mu \mu(v_i) |n \cdot v_i| dv_i.$$

Since the form of this v_i-integral is uniform for all $1 \leq i \leq k$, the multiple integral can be treated by Fubini's theorem. Such lucky coincidence no longer holds true for the C-L boundary condition. From (1.25) the integrand is a function of both v and u. As a result the v_i-integral is not uniform over i, making the Fubini's theorem not applicable. The multiple integral thus needs to be computed with the fixed order $v_k, v_{k-1}, \cdots, v_1$, bringing extra computational difficulty. We now perform this integral order by order. To do so we start with v_k, the most outer layer. The integral contains

$$\int_{n \cdot u_k > 0} e^{-[\frac{1}{4T_M} - \frac{1}{2T_w(x)}]|u_k|^2} d\sigma(v_k, v_{k-1}), \tag{1.46}$$

with appropriate $d\sigma(v_k, v_{k-1})$ definition. This integral then becomes a function of v_{k-1}, which is then computed in the second most outer layer. Using Lemma 16 one can show that (1.46) can be approximately explicitly computed – $e^{c|v_{k-1}|^2}$. We perform this iteratively over i by counting back from k to 1, and inductively compute this k-fold integral. This result is presented in Lemma 3.

To deal with the second difficulty, one needs to quantize the probability of a particle that interacts with the wall more than k times, or equivalently, we need to give an estimate of the measure $\mathbf{1}_{\{t_k > 0\}}$. In [2,14] the authors studied the diffuse boundary condition in which they decompose the boundary as

$$\gamma_+^\delta = \{u \in \gamma_+ : |n \cdot u| > \delta, |u| \leq \delta^{-1}\}, \quad \text{and} \quad \gamma_+ \backslash \gamma_+^\delta,$$

and show that there can be only finite number of v_j that belongs to γ_+^δ. Meanwhile, the integration over $\gamma_+ \backslash \gamma_+^\delta$ can be controlled by the small δ. As k increases, one obtains a larger power of δ, leading to a decay factor for the measure of $\mathbf{1}_{\{t_k > 0\}}$. When C-L condition is given, the strategy needs to be revised. In particular, the integrand in equation (1.11) and (1.25) contains $e^{-|u_\parallel-(1-r_\parallel)v_\parallel|^2}$, and even if $|u_\parallel| \gg 1$, $|u_\parallel-(1-r_\parallel)v_\parallel|$ can still be small, meaning the integration over the $\gamma_+\backslash\gamma_+^\delta$ does not provide the smallness. One key observation here is separate the discussion based on the distance between $u_\parallel$ and $(1-r_\parallel)v_\parallel$. Let $|u_\parallel|$ large enough, with $1-r_\parallel < 1$. The bad case is when $|u_\parallel-(1-r_\parallel)v_\parallel| < \delta^{-1}$, then $|v_\parallel| \geq |u_\parallel|+\delta^{-1}$. For example let $1-r_\parallel = 1/2$, then if $|u_\parallel - \frac{1}{2}v_\parallel| < \delta^{-1}$, we take $|u_\parallel| \geq 3\delta^{-1}$ to have:

$$\frac{1}{2}|v_\parallel| > |u_\parallel| - \delta^{-1} > \frac{1}{2}|u_\parallel| + \frac{1}{2}\delta^{-1}, \quad |v_\parallel| > |u_\parallel| + \delta^{-1},$$

which brings up the value of $v_\parallel$. Consequently, if these 'bad' cases $|u_\parallel-(1-r_\parallel)v_\parallel| < \delta^{-1}$ take place many times in the k-fold integral, a very big v_i will be generated. Then the application of the boundary condition that provides a fast decay for big $|v_i|$ can be used to balance out all the growing factors, leading to a small measure of $\mathbf{1}_{t_k} > 0$ in the end.

Consider this, we further decompose γ_+ into

$$\gamma_+^\eta = \{u \in \gamma_+ : |n\cdot u| > \eta\delta, |u| \leq \eta\delta^{-1}\}, \quad \text{and} \quad \gamma_+\backslash\gamma_+^\eta,$$

where η is selected to be a small number (depending on $r_\parallel$) so that

$$|u_\parallel - (1-r_\parallel)v_\parallel| < \delta^{-1} \quad \Rightarrow \quad |v_\parallel| \geq |u_\parallel| + \delta^{-1}.$$

We comment here that such property only works when the coefficient $1-r_\parallel < 1$. In the real computation the wall temperature is involved in the boundary condition, thus the actual coefficient contains $T_w(x)$ and is more complicated than $1-r_\parallel$. In order to ensure such constant to be less than 1, we impose the condition (1.28). See Lemma 4 for detail.

$W^{1,p}$ *estimate:* For getting the $W^{1,p}$ estimate (1.36), we rely on the energy-type estimate for $\nabla_{x,v}f$ with weight $\alpha_{f,\epsilon}^\beta$, for which $\int_0^t\int_{\partial\Omega}\int_{n\cdot v<0}|\alpha_{f,\epsilon}^\beta\nabla_{x,v}f|^p|n\cdot v|dvdS_xds$ needs to be controlled. Using the fact that $\alpha_{f,\epsilon}(t,x,v) = |n(x)\cdot v|$ on γ_-, the singularity of (1.40) can be controlled by first setting:

$$\beta > \frac{p-2}{p}, \quad |n\cdot v|^{p\beta-p+1} \in L^1_{\text{loc}}(\mathbb{R}^3).$$

Then with some further calculation, shown in (3.23) (3.24), we roughly need to estimate:

$$\int_{\gamma_-}|\alpha^\beta\partial f|^p \lesssim \int_{\gamma_-} e^{[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]p|v|^2}\left(\int_{n\cdot u>0}|\partial f(u)|e^{-[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]p|u|^2}d\sigma(u,v)\right)^p. \tag{1.47}$$

To handle the integration of u, in [2,15], the authors studied the diffusion boundary condition and proposed to split the term into the integration over the grazing set

$$\gamma_+^\epsilon = \{(x,u) \in \gamma_+ : u\cdot n(x) < \epsilon \text{ or } |u| > 1/\epsilon\} \quad \text{and} \quad \gamma_+\backslash\gamma_+^\epsilon.$$

However, this is not enough since we do not have direct smallness even for big u (in the grazing set), and thus are not able to bound $\int_{\{(x,u)\in\gamma_+^\epsilon\}}$ by ε. To handle C-L boundary condition, we propose in this paper to add another layer of splitting. Besides the standard grazing/non-grazing sets, we also split the γ_+ integral into the grazing sets defined by v, approximately:

$$\gamma_+^{v,x,\epsilon} = \{(x,u) \in \gamma_+ : u\cdot n(x) < \epsilon \text{ or } |u-v| > 1/\epsilon\} \quad \text{and} \quad \gamma_+\backslash\gamma_+^{v,x,\epsilon}.$$

With this decomposition we have

$$
\begin{aligned}
(1.47) &= \int_{\gamma_-} e^{[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]p|v|^2}\left(\int_{\{u:(x,u)\in\gamma_+^{v,x,\epsilon}\}} |\partial f(u)| e^{-[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]|u|^2} d\sigma(u,v)\right)^p \\
&+ \int_{\gamma_-} e^{[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]p|v|^2}\left(\int_{\{u:(x,u)\in\gamma_+\backslash\gamma_+^{v,x,\epsilon}\}} |\partial f(u)| e^{-[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]|u|^2} d\sigma(u,v)\right)^p \\
&\lesssim \int_{\gamma_-} e^{[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]p|v|^2} \\
&\times\Bigg[\left(\int_{\{u:(x,u)\in\gamma_+^{v,x,\epsilon}\}} e^{-[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]q|u|^2}(\text{all terms in } d\sigma\ (1.26))^q du\right)^{p/q} \\
&\quad \int_{\gamma_+^{v,x,\epsilon}} |\alpha^\beta \partial f|^p \qquad (1.48)\\
&+\left(\int_{\{u:(x,u)\in\gamma_+\backslash\gamma_+^{v,x,\epsilon}\}} e^{-[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]q|u|^2}(\text{all terms in } d\sigma\ (1.26))^q du\right)^{p/q} \\
&\quad \int_{\gamma_+\backslash\gamma_+^{v,x,\epsilon}} |\alpha^\beta \partial f|^p\Bigg]. \qquad (1.49)
\end{aligned}
$$

Now with the application of C-L boundary condition, one has the smallness in terms of ϵ for the integral over $\gamma_+^{v,x,\epsilon}$. And after direct computation one has the L^1_v for dv and thus bounds

$$(1.48) \lesssim O(\varepsilon)\int_{\gamma_+^{v,x,\varepsilon}} |\alpha^\beta\partial f|^p \le O(\varepsilon)\int_{\gamma_+} |\alpha^\beta\partial f|^p.$$

On the set of $\gamma_+\backslash\gamma_+^{v,x,\epsilon}$, one still has L^1_v integrand for dv but the smallness is lost. We now recycle the standard grazing/non-grazing set definition, by further splitting the v-integration into $\mathbf{1}_{|v|\le\epsilon^{-1}}$ and $\mathbf{1}_{|v|\ge\epsilon^{-1}}$. While the integration is naturally bounded by $O(\epsilon)$ when integrated on $|v|\ge\epsilon^{-1}$, the $|v|\le\epsilon^{-1}$ case leads to $|u|\le 2\epsilon^{-1}$, making u falling in the non-grazing set $\gamma_+\backslash\gamma_+^{\epsilon/2}$. We now stand on the same footing as the situation discussed in [2,15]. Apply Lemma 10 we obtain an upper bound for the integration in the bulk (the terms not involving boundaries) and initial data, meaning:

$$(1.49) \lesssim O(\varepsilon)\int_{\gamma_+} |\alpha^\beta\partial f|^p + \text{ initial condition } + \text{ bulk }.$$

The bulk part is treated similarly as in the proof of [2]. The entire proof for the weighted $W^{1,p}$ estimate is presented in Sect. 3.

The non-weighted $L^3_x L^{1+\delta}$ bound for the velocity derivative $\nabla_v f$ is discussed in Sect. 4. Characteristics and the energy-type estimate are the main tools used. The boundary terms are treated similarly as is done for the $W^{1,p}$ estimate, and the bulk terms are similar to those estimated in [2]. This estimate in the end leads to the $L^{1+\delta}$ stability $\|f-g\|_{L^{1+\delta}} \lesssim \|f_0-g_0\|_{L^{1+\delta}}$.

1.3 Outline

In Sect. 2 we prove the L^∞ bound for the sequence solution f^m. In Sect. 3, we prove the weight $W^{1,p}$ estimate for the sequence solution f^m. Then we derive the $L^3_x L^{1+\delta}_v$ estimate and the

$L^{1+\delta}$ stability for f^m in Sect. 4. The $L^{1+\delta}$ stability is the key to the well-poseness. In Sect. 5 we combine all the estimates for the sequence solution f^m and conclude the existence and uniqueness. More specifically, in Theorem 1, the existence is given by Proposition 9 and the uniqueness is given by Proposition 10. In the appendix we prove some necessary estimates.

2 L^∞ Estimate

For any given constants $\mathfrak{C}, \theta \in \mathbb{R}$, define a Gaussian-weighted solution:

$$h^{m+1}(t,x,v) = e^{\theta|v|^2} e^{-\mathfrak{C}t\langle v\rangle^2} f^{m+1}(t,x,v), \tag{2.1}$$

then according to (1.44), we have:

$$\begin{aligned} &\partial_t h^{m+1} + v\cdot\nabla_x h^{m+1} - \nabla_x\phi^m\cdot\nabla_v h^{m+1} + \nu^m h^{m+1} \\ &= e^{\theta|v|^2} e^{-\mathfrak{C}t\langle v\rangle^2}\Gamma_{\text{gain}}\left(\frac{h^m}{e^{-\mathfrak{C}\langle v\rangle^2 t}e^{\theta|v|^2}}, \frac{h^m}{e^{-\mathfrak{C}\langle v\rangle^2 t}e^{\theta|v|^2}}\right), \end{aligned} \tag{2.2}$$

equipped with boundary condition

$$\begin{aligned} h^{m+1}|_{\gamma_-(x)} &= e^{\theta|v|^2} e^{-\mathfrak{C}t\langle v\rangle^2} e^{[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]|v|^2} \\ &\int_{\gamma_+(x)} h^m(t,x,u) e^{-[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]|u|^2} e^{-\theta|u|^2} e^{\mathfrak{C}t\langle u\rangle^2} d\sigma(u,v), \end{aligned} \tag{2.3}$$

where $\gamma_\pm$ is defined in (1.8) and

$$\nu^m(t) = \mathfrak{C}\langle v\rangle^2 + \nabla_x\phi^m\cdot\nabla_v\big(-\mathfrak{C}t\langle v\rangle^2 + \theta|v|^2\big) + \frac{1}{2T_M} v\cdot\nabla_x\phi^m + \nu(F^m). \tag{2.4}$$

This equation is linear for h^{m+1} with h^m serving as a source term, ν^m serving as a damping coefficient and ϕ^m serving as the electric field. The main purpose of this section is to show that h^m, and thus f^m form a bounded sequence in L^∞. More precisely:

Proposition 2 *Let h^{m+1} satisfy (2.3) with the Cercignani–Lampis boundary condition (2.3). Assume the constraints for θ and T_w hold true ((1.32) and (1.28)) and $\|h_0(x,v)\|_{L^\infty} < \infty$. Then if*

$$\sup_{i\le m}\|h^i(t,x,v)\|_{L^\infty} \le C_\infty\|h_0(x,v)\|_\infty, \quad t\le t_\infty, \tag{2.5}$$

we have

$$\sup_{0\le t\le t_\infty}\|h^{m+1}(t,x,v)\|_{L^\infty} \le C_\infty\|h_0(x,v)\|_{L^\infty}. \tag{2.6}$$

Here $C_\infty = C_\infty(T_M, \min\{T_w(x)\}, \theta, r_\perp, r_\parallel, \Omega)$ is a constant and

$$t\le t_\infty = t_\infty(\|h_0(x,v)\|_{L^\infty}, T_M, \min\{T_w(x)\}, \theta, r_\perp, r_\parallel, \Omega) \ll 1. \tag{2.7}$$

Remark 3 Two remarks are in line:

- The smallness only depends on the initial data, wall temperature, domain, the accommodation coefficients $r_{\parallel,\perp}$.
- We will also trace the dependence of the constants C_∞ and t_∞ in the proof. C_∞ will be explicitly defined in (2.139).

This proposition implies the uniform-in-m L^∞ estimate for $h^m(t,x,v)$, and this allows us to further bound

$$\sup_m \|w_{\theta'} f^m\|_\infty < \infty, \tag{2.8}$$

which lays the foundation for later sections.

To show the proposition, we start with Lemma 1 in which we control the acceleration term $\nabla_x\phi$. We then explicitly derive the formula using the information of the trajectory for h^m, as will be presented in Lemma 2. This will bring a k-fold integration for particles that collide with the boundary k-times before the final time. We will further show that all the terms in this integration (more precisely, all terms in (2.11) (2.12)), can be bounded in Lemmas 3 and 4. We then summarize the estimates and give the proof of the proposition.

We now present Lemmas 1 and 2. Then we split this section into three subsections, the first subsection concludes the proof for Lemma 2. We present both Lemmas 3 and 4 in the second subsection. In last subsection we combine the estimates in Lemmas 3 and 4 with the formula in Lemma 2 to conclude Proposition 2.

We first give an estimate of the bound of $\|\nabla_x\phi^m\|_\infty$.

Lemma 1 *For any* $0<\delta<1, \theta<\frac{1}{4T_M}$, $0\le t\le 1$, *if* (f,ϕ_f) *satisfy the condition* (1.24) *then*

$$\|\phi_f(t)\|_{C^{1,1-\delta}(\bar{\Omega})} \le C\|h(t)\|_{L^\infty} + C\rho_0. \tag{2.9}$$

Proof For any $p>1$,

$$\begin{aligned}\Big\|\int_{\mathbb{R}^3} f(t,x,v)\sqrt{\mu(v)}dv-\rho_0\Big\|_{L^p(\Omega)} &\le \Big\|\int_{\mathbb{R}^3} f(t,x,v)\sqrt{\mu(v)}dv\Big\|_{L^p(\Omega)} + \|\rho_0\|_{L^p(\Omega)}\\ &\le |\Omega|^{1/p}\left(\int_{\mathbb{R}^3} e^{-\theta|v|^2}e^{\mathfrak{C}\langle v\rangle^2 t}\sqrt{\mu(v)}dv\right)\|h(t)\|_{L^\infty}\\ &\quad+\rho_0.\end{aligned}$$

By the elliptic estimate with condition (1.17)

$$\|\phi_f(t)\|_{W^{2,p}(\Omega)} \le C\|h(t)\|_{L^\infty}+\rho_0,$$

which further leads to, according to the Morrey inequality for $p>3$, $\Omega\subset\mathbb{R}^3$, and $\partial\Omega$ being C^1:

$$\|\phi_f(t)\|_{C^{1,1-3/p}(\Omega)} \le C\|\phi_f(t)\|_{W^{2,p}(\Omega)} \le C\|h(t)\|_{L^\infty}+C\rho_0.$$

□

We represent h^{m+1} with the stochastic cycles defined as follows.

Definition 2 Define an Hölder continuous characteristics which solves (since $\nabla\phi^m$ is quasi-Lipschitz continuous from Lemma 1, this is possible, see also chapter 8 of [20] for example)

$$\frac{d}{ds}\begin{pmatrix} X^m(s;t,x,v)\\ V^m(s;t,x,v)\end{pmatrix} = \begin{pmatrix} V^m(s;t,x,v)\\ -\nabla_x\phi^m(s,X^m(s;t,x,v))\end{pmatrix}, \tag{2.10}$$

and we trace back in time and determine the boundary-colliding time and location, namely:

$$t_1(t,x,v)=\sup\{s<t: X^m(s;t,x,v)\in\partial\Omega\},\quad x_1(t,x,v)=X^m(t_1(t,x,v);t,x,v).$$

We then build the probability measure at $x = x_1$ as $d\sigma(v_1, V^m(t_1; t, x, v))$, supported on $\mathcal{V}_1 = \gamma_+(x_1)$:

$$\int_{\mathcal{V}_1} d\sigma(v_1, V^m(t_1; t, x, v)) = 1.$$

Inductively, define t_k and x_k the time and position of a particle striking the boundary for the k-th time:

$$t_k(t, x, v, v_1, \cdots, v_{k-1}) = \sup\{s < t_{k-1} : X^{m-k+1}(s; t_{k-1}, x_{k-1}, v_{k-1}) \in \partial\Omega\},$$
$$x_k(t, x, v, v_1, \cdots, v_{k-1}) = X^{m-k+1}\left(t_k(t, x, v, v_{k-1}); t_{k-1}(t, x, v), x_{k-1}(t, x, v), v_{k-1}\right),$$

and correspondingly build probability measure $d\sigma(v_k, V^{m-k+1}(t_k; t_{k-1}, x_{k-1}, v_{k-1}))$ at x_k over $\mathcal{V}_k = \gamma_+(x_k)$ for:

$$\int_{\mathcal{V}_k} d\sigma(v_k, V^{m-k+1}(t_k; t_{k-1}, x_{k-1}, v_{k-1})) = 1.$$

For simplicity, we denote for all $l \leq m$:

$$V^{m-l}(s) := V^{m-l}(s; t_l, x_l, v_l), \quad X^{m-l}(s) := X^l(s; t_l, x_l, v_l).$$

Lemma 2 *Let h^{m+1} satisfies (2.2) with the Cercignani–Lampis boundary condition (2.3), and assume (2.5) holds true, then with properly chosen $\mathfrak{C}$ and θ, point-wise in (t, x, v), one has: if $t_1 \leq 0$:*

$$\begin{aligned} |h^{m+1}(t, x, v)| \leq & |h_0\left(X^m(0), V^m(0)\right)| \\ & + \int_0^t e^{-\int_s^t \frac{\mathfrak{C}}{2}\langle V^m(\tau)\rangle^2 d\tau} e^{\theta|V^m(s)|^2} e^{-\mathfrak{C}s\langle V^m(s)\rangle^2} \Gamma^m_{gain}(s) ds. \end{aligned} \tag{2.11}$$

If $t_1 > 0$, for arbitrary $k \geq 2$, one has:

$$\begin{aligned} |h^{m+1}(t, x, v)| \leq & \int_{t_1}^t e^{-\int_s^t \frac{\mathfrak{C}}{2}\langle V^m(\tau)\rangle d\tau} e^{\theta|V^m(s)|^2} e^{-\mathfrak{C}\langle V^m(s)\rangle s} \Gamma^m_{gain}(s) ds \\ & + e^{\theta|V^m(t_1)|^2} e^{-\mathfrak{C}t_1\langle V^m(t_1)\rangle^2} e^{[\frac{1}{4T_M} - \frac{1}{2T_w(x_1)}]|V^m(t_1)|^2} \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} H, \end{aligned} \tag{2.12}$$

with H given by

$$\begin{aligned} & \sum_{l=1}^{k-1} \mathbf{1}_{\{t_l > 0, t_{l+1} \leq 0\}} |h_0\left(X^{m-l}(0), V^{m-l}(0)\right)| d\Sigma^k_{l,m}(0) \\ & + \sum_{l=1}^{k-1} \int_{\max\{0, t_{l+1}\}}^{t_l} e^{\theta|V^{m-l}(s)|^2} e^{-\mathfrak{C}s\langle V^{m-l}(s)\rangle^2} |\Gamma^{m-l}_{gain}(s)| d\Sigma^k_{l,m}(s) ds \\ & + \mathbf{1}_{\{t_k > 0\}} |h^{m-k+2}\left(t_k, x_k, V^{m-k+1}(t_k)\right)| d\Sigma^k_{k-1,m}(t_k), \end{aligned} \tag{2.13}$$

where

$$
\begin{aligned}
d\Sigma^k_{l,m}(s) =& \Big\{ \prod_{j=l+1}^{k-1} d\sigma\left(v_j, V^{m-j+1}(t_j)\right) \Big\} \\
& \Big\{ e^{-\int_s^{t_l} \frac{\mathfrak{C}}{2} \langle V^{m-l}(\tau)\rangle^2 d\tau} e^{-\theta|v_l|^2} e^{\mathfrak{C} t_l \langle v_l\rangle^2} e^{-[\frac{1}{4T_M} - \frac{1}{2T_w(x_l)}]|v_l|^2} d\sigma(v_l, V^{m-l+1}(t_l)) \Big\} \\
& \Big\{ \prod_{j=1}^{l-1} 2 e^{\mathfrak{C}(t_j - t_{j+1})\langle v_j\rangle^2} e^{[\frac{1}{2T_w(x_j)} - \frac{1}{2T_w(x_{j+1})}]|v_j|^2} d\sigma\left(v_j, V^{m-j+1}(t_j)\right) \Big\}.
\end{aligned}
\tag{2.14}
$$

Here we use a notation

$$
\Gamma^m_{gain}(s) := \Gamma_{gain}\left(\frac{h^m(s, X^m(s), V^m(s))}{e^{\theta|V^m(s)|^2} e^{-\mathfrak{C}s\langle V^m(s)\rangle^2}}, \frac{h^m(s, X^m(s), V^m(s))}{e^{\theta|V^m(s)|^2} e^{-\mathfrak{C}s\langle V^m(s)\rangle^2}} \right). \tag{2.15}
$$

2.1 Proof of Lemma 2

We present the proof of Lemma 2. Most of the proof is tedious but straightforward derivation.

Proof of Lemma 2 For given ϕ^m, we choose small enough t and big enough $\mathfrak{C}$:

$$
\nabla_x \phi^m \cdot \nabla_v\big(- \mathfrak{C} t \langle v\rangle^2 + \theta|v|^2\big) + \frac{1}{2T_M} v \cdot \nabla_x \phi^m \le \frac{\mathfrak{C}}{2}\langle v\rangle^2, \tag{2.16}
$$

and thus, noting $\nu(F^m) \ge 0$, from (2.4), we have:

$$
\nu^m(t) \ge \frac{\mathfrak{C}}{2}\langle v\rangle^2. \tag{2.17}
$$

From (1) we first denote

$$
C_{\phi^m} := \sup_{0\le i\le m} \|\nabla_x \phi^i\|_\infty \lesssim \sup_{0\le i\le m} \|h^i\|_{L^\infty} < \infty, \tag{2.18}
$$

and

$$
\tilde{\mu}(t, x, v) := e^{-\theta|v|^2} e^{\mathfrak{C} t\langle v\rangle^2} e^{-[\frac{1}{4T_M} - \frac{1}{2T_w(x)}]|v|^2}. \tag{2.19}
$$

If $t_1(t, x, v) \le 0$, the particle has been following a fixed trajectory without scattering, then according to (2.2), for $0 \le s \le t$,

$$
\frac{d}{ds}\left[e^{-\int_s^t \nu^m(\tau) d\tau} h^{m+1}(s, X^m(s), V^m(s)) \right] = e^{-\int_s^t \nu^m(\tau) d\tau} e^{\theta|V^m(s)|^2} e^{-\mathfrak{C}s\langle V^m(s)\rangle^2} \Gamma^m_{\text{gain}}(s), \tag{2.20}
$$

then (2.11) follows by applying (2.17).

If $t_1(t, x, v) > 0$, the trajectory of the particle can be split into a few discontinuous sections. In particular:

$$
\begin{aligned}
|h^{m+1}(t, x, v)\mathbf{1}_{\{t_1>0\}}| \le & |h^{m+1}\left(t_1, x_1, V^m(t_1)\right)| e^{-\int_{t_1}^t \frac{\mathfrak{C}}{2}\langle V^m(\tau)\rangle^2 d\tau} \\
& + \int_{t_1}^t e^{-\int_s^t \frac{\mathfrak{C}}{2}\langle V^m(\tau)\rangle^2 d\tau} e^{\theta|V^m(s)|^2} e^{-\mathfrak{C}s\langle V^m(s)\rangle^2} |\Gamma^m_{\text{gain}}(s)| ds.
\end{aligned}
\tag{2.21}
$$

Note the first term of the RHS of (2.21) can be expressed by the boundary condition. In particular, for $1 \le k \le m$, the boundary condition (2.3) can be written as, using (2.19):

$$h^{m-k+2}(t_k, x_k, V^{m-k+1}(t_k)) = \frac{1}{\tilde{\mu}(t_k, x_k, V^{m-k+1}(t_k))} \int_{\mathcal{V}_k} h^{m-k+1}(t_k, x_k, v_k)\tilde{\mu}(t_k, x_k, v_k) d\sigma(v_k, V^{m-k+1}(t_k)). \tag{2.22}$$

We now use induction on k to show (2.12). Directly applying (2.22) with $k = 1$, the first term of the RHS of (2.21) is bounded by

$$\frac{1}{\tilde{\mu}\,(t_1, x_1, V^m(t_1))} \int_{\mathcal{V}_1} h^m(t_1, x_1, v_1)\tilde{\mu}(t_1, x_1, v_1) d\sigma(v_1, V^m(t_1)). \tag{2.23}$$

Noting (2.11) and (2.21), this term is to be controlled by:

$$\begin{aligned}
(2.23) \le \frac{1}{\tilde{\mu}(t_1, x_1, V^m(t_1))}\Big[&\int_{\mathcal{V}_1} \mathbf{1}_{\{t_2 \le 0 < t_1\}} e^{-\int_0^{t_1} \frac{\mathfrak{C}}{2}\langle V^{m-1}(\tau)\rangle^2 d\tau} h^m(0, X^{m-1}(0), \\
&V^{m-1}(0))\tilde{\mu}(t_1, x_1, v_1) d\sigma\left(v_1, V^m(t_1)\right) \\
&+ \int_0^{t_1} \int_{\mathcal{V}_1} \mathbf{1}_{\{t_2 \le 0 < t_1\}} e^{-\int_s^{t_1} \frac{\mathfrak{C}}{2}\langle V^{m-1}(\tau)\rangle^2 d\tau} e^{\theta |V^{m-1}(s)|^2} e^{-\mathfrak{C}s\langle V^{m-1}(s)\rangle^2} \\
&|\Gamma_{\text{gain}}^{m-1}(s)|\tilde{\mu}(t_1, x_1, v_1) d\sigma\left(v_1, V^m(t_1)\right) ds \\
&+ \int_{\mathcal{V}_1} \mathbf{1}_{\{t_2 > 0\}} e^{-\int_{t_2}^{t_1} \frac{\mathfrak{C}}{2}\langle V^{m-1}(\tau)\rangle^2 d\tau} |h^m(t_2, x_2, V^{m-1}(t_2))\tilde{\mu}(t_1, x_1, v_1) d\sigma\left(v_1, V^m(t_1)\right) \\
&+ \int_{t_2}^{t_1} \int_{\mathcal{V}_1} \mathbf{1}_{\{t_2 > 0\}} e^{-\int_s^{t_1} \frac{\mathfrak{C}}{2}\langle V^{m-1}(\tau)\rangle^2 d\tau} e^{\theta |V^{m-1}(s)|^2} e^{-\mathfrak{C}s\langle V^{m-1}(s)\rangle^2} \\
&|\Gamma_{\text{gain}}^{m-1}(s)|\tilde{\mu}(t_1, x_1, v_1) d\sigma\left(v_1, V^m(t_1)\right) ds\Big],
\end{aligned}$$

showing the validity of (2.12) $k = 2$. For higher k, we use induction. Assume (2.12) is valid for $k \ge 2$(induction hypothesis) we prove so for $k + 1$. We express the last term in (2.13) using the boundary condition. In (2.22), since $\frac{1}{\tilde{\mu}(t_k, x_k, V^{m-k+1}(t_k))}$ depends on v_{k-1}, we move this term to the integration over $\mathcal{V}_{k-1}$ in (2.12). Using the second line of (2.14) with $l = k - 1$, $s = t_k$, the integration over $\mathcal{V}_{k-1}$ is

$$\int_{\mathcal{V}_{k-1}} \frac{e^{-\int_{t_k}^{t_{k-1}} \frac{\mathfrak{C}}{2}\langle V^{m-k+1}(\tau)\rangle^2 d\tau}}{\tilde{\mu}\left(t_k, x_k, V^{m-k+1}(t_k)\right)} \tilde{\mu}(t_{k-1}, x_{k-1}, v_{k-1}) d\sigma(v_{k-1}, V^{m-k}(t_{k-1})). \tag{2.24}$$

By Definition 2 we have $|V^{m-k+1}(t_{k-1})| = |v_{k-1}|$. By (2.10) for $t_k \le \tau \le t_{k-1}$ we have

$$\langle v_{k-1}\rangle - C_{\phi^m}(t_{k-1} - \tau) \le V^{m-k+1}(\tau)\rangle \le \langle v_{k-1}\rangle + C_{\phi^m}(t_{k-1} - \tau), \tag{2.25}$$

with C_{ϕ^m} is defined in (2.18). This leads to

$$|V^{m-k+1}(t_k)|^2 \le |v_{k-1}|^2 + 2C_{\phi^m}(t_{k-1} - t_k)|v_{k-1}| + (C_{\phi^m})^2(t_{k-1} - t_k)^2, \tag{2.26}$$

and

$$\langle V^{m-k+1}(t_k)\rangle^2 \ge \langle v_{k-1}\rangle^2 - 2C_{\phi^m}(t_{k-1} - t_k)\langle v_{k-1}\rangle, \tag{2.27}$$

which further suggests:

$$e^{-\int_{t_k}^{t_{k-1}}\frac{\mathfrak{C}}{2}\langle V^{m-k+1}(\tau)\rangle^2 d\tau} \leq e^{(t_{k-1}-t_k)\langle v_{k-1}\rangle\left(-\frac{\mathfrak{C}}{2}\langle v_{k-1}\rangle+\mathfrak{C}C_{\phi^m}(t_{k-1}-t_k)\right)}. \tag{2.28}$$

Considering the definition of $\tilde{\mu}$ in (2.19), and utilizing the inequalities above, we finally arrive at, taking $C_T := \frac{1}{2\min\{T_w(x)\}} - \frac{1}{4T_M}$:

$$\begin{aligned}
&e^{-\int_{t_k}^{t_{k-1}}\frac{\mathfrak{C}}{2}\langle V^{m-k+1}(\tau)\rangle^2 d\tau}/\tilde{\mu}\left(t_k, x_k, V^{m-k+1}(t_k)\right)\\
&\leq e^{-\mathfrak{C}\langle v_{k-1}\rangle^2 t_k+\theta|v_{k-1}|^2} e^{[\frac{1}{4T_M}-\frac{1}{2T_w(x_k)}]|v_{k-1}|^2}\\
&\quad\times \exp\left(\Big[-\frac{\mathfrak{C}}{2}\langle v_{k-1}\rangle + \mathfrak{C}C_{\phi^m}(t_{k-1}-t_k) + 2\mathfrak{C}C_{\phi^m}t_k\right.\\
&\quad\left. +2\theta C_{\phi^m} + 2C_T C_{\phi^m}\Big](t_{k-1}-t_k)\langle v_{k-1}\rangle\right)\\
&\quad\times \exp\left(\big[\theta C_{\phi^m}^2 + C_T C_{\phi^m}^2\big](t_{k-1}-t_k)^2\right).
\end{aligned} \tag{2.29}$$

Set $t = t(T, \theta, \phi^m)$ small enough such that

$$\exp\left(\big[\theta C_{\phi^m}^2 + C_T C_{\phi^m}^2\big](t_{k-1}-t_k)^2\right) \leq \exp\left(\big[\theta C_{\phi^m}^2 + C_T C_{\phi^m}^2\big]t\right) \leq 2, \quad \text{and} \quad C_{\phi^m}t \leq \frac{1}{8}.$$

Furthermore, we take $\mathfrak{C}$ to be big enough so that

$$\begin{aligned}
&-\frac{\mathfrak{C}}{2}\langle v_{k-1}\rangle + \mathfrak{C}C_{\phi^m}(t_{k-1}-t_k) + 2\mathfrak{C}C_{\phi^m}t_k + 2\theta C_{\phi^m} + 2C_T C_{\phi^m}\\
&\leq -\frac{\mathfrak{C}}{2} + \mathfrak{C}/8 + \mathfrak{C}/4 + 2\theta C_{\phi^m} + 2C_T C_{\phi^m} \leq -\frac{\mathfrak{C}}{8} + 2\theta C_{\phi^m} + 2C_T C_{\phi^m} \leq 0.
\end{aligned} \tag{2.30}$$

We simplify (2.29):

$$(2.29) \leq 2e^{-\mathfrak{C}\langle v_{k-1}\rangle^2 t_k+\theta|v_{k-1}|^2} e^{[\frac{1}{4T_M}-\frac{1}{2T_w(x_k)}]|v_{k-1}|^2}. \tag{2.31}$$

This leads to the boundedness of the integrand in (2.24) by:

$$(2.31) \times \tilde{\mu}(t_{k-1}, x_{k-1}, v_{k-1}) = 2e^{[\frac{1}{2T_w(x_{k-1})}-\frac{1}{2T_w(x_k)}]|v_{k-1}|^2} e^{\mathfrak{C}(t_{k-1}-t_k)\langle v_{k-1}\rangle^2}, \tag{2.32}$$

and in turn gives the estimate shown in (2.14) with $l = k-1$.

For the remaining term in (2.22), we split the integration over $\mathcal{V}_k$ into two terms as

$$\begin{aligned}
&\int_{\mathcal{V}_k} h^{m-k+1}(t_k, x_k, v_k)\tilde{\mu}(t_k, x_k, V^{m-k+1}(t_k))d\sigma(v_k, v_{k-1})\\
&= \underbrace{\int_{\mathcal{V}_k} \mathbf{1}_{\{t_{k+1}\leq 0<t_k\}}}_{(2.33)_1} + \underbrace{\int_{\mathcal{V}_k} \mathbf{1}_{\{t_{k+1}>0\}}}_{(2.33)_2}.
\end{aligned} \tag{2.33}$$

We use the similar bound of (2.11) and derive that

$$
\begin{aligned}
(2.33)_1 \leq & \int_{\mathcal{V}_k} \mathbf{1}_{\{t_{k+1}\leq 0<t_k\}} e^{\int_0^{t_k} -\frac{\mathfrak{C}}{2}\langle V^{m-k}(\tau)\rangle^2 d\tau} h^{m-k+1}(0, X^{m-k}(0), V^{m-k}(0)) \\
& \tilde{\mu}(t_k, x_k, v_k) d\sigma(v_k, V^{m-k+1}(t_k)) \\
& + \int_0^{t_k} \int_{\mathcal{V}_k} \mathbf{1}_{\{t_{k+1}\leq 0<t_k\}} e^{\int_s^{t_k} -\frac{\mathfrak{C}}{2}\langle V^{m-k}(\tau)\rangle^2 d\tau} e^{-\mathfrak{C}\langle V^{m-k}(s)\rangle^2 s} e^{\theta |V^{m-k}(s)|^2} \Gamma_{\text{gain}}^{m-k}(s) \\
& \tilde{\mu}(t_k, x_k, v_k) d\sigma(v_k, V^{m-k+1}(t_k)) ds.
\end{aligned}
\tag{2.34}
$$

In the first line of (2.34), is consistent with the second bracket of the first line of (2.14) with $l = k$, $s = t_k$. In the second line of (2.34),

$$
e^{\int_s^{t_k} -\frac{\mathfrak{C}}{2}\langle V^{m-k}(\tau)\rangle^2 d\tau} \tilde{\mu}(t_k, x_k, v_k) d\sigma(v_k, V^{m-k+1}(t_k))
$$

is consistent with the second line of (2.14) with $l = k$.

From the induction hypothesis((2.12) is valid for k), we derive the integration over $\mathcal{V}_j$ for $j \leq k-1$ is consistent with the third line of (2.14). After taking integration $\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}$ we change $d\Sigma_{k-1,m}^k$ in (2.14) to $d\Sigma_{k,m}^{k+1}$. Thus (2.34) becomes

$$
\begin{aligned}
& \int_{\prod_{j=1}^k \mathcal{V}_j} \mathbf{1}_{\{t_{k+1}\leq 0<t_k\}} |h_0\left(X^{k+1}(0), v_k\right)| d\Sigma_{k,m}^{k+1}(0) \\
& \quad + \int_{\prod_{j=1}^k \mathcal{V}_j} \int_0^{t_k} e^{-\mathfrak{C}\langle V^{m-k}(s)\rangle^2 s} e^{\theta|V^{m-k}(s)|^2} \Gamma_{\text{gain}}^{m-k}(s) d\Sigma_{k,m}^{k+1}(s) ds.
\end{aligned}
\tag{2.35}
$$

Then we use the same estimate as (2.21) and derive

$$
\begin{aligned}
(2.33)_2 \leq & \int_{\mathcal{V}_k} \mathbf{1}_{\{t_{k+1}>0\}} e^{\int_{t_{k+1}}^{t_k} -\frac{\mathfrak{C}}{2}\langle V^{m-k}(\tau)\rangle^2 d\tau} h^{m+1}\left(t_{k+1}, x_{k+1}, V^{m-k}(t_{k+1})\right) \\
& \tilde{\mu}(t_k, x_k, v_k) d\sigma\left(v_k, V^{m-k+1}(t_k)\right) \\
& + \int_{t_{k+1}}^{t_k} \int_{\mathcal{V}_k} \mathbf{1}_{\{t_{k+1}>0\}} e^{\int_s^{t_k} -\frac{\mathfrak{C}}{2}\langle V^{m-k}(\tau)\rangle^2 d\tau} e^{-\mathfrak{C}\langle V^{m-k}(s)\rangle s} e^{\theta|V^{m-k}(s)|^2} \Gamma_{\text{gain}}^{m-k}(s) \\
& \tilde{\mu}(t_k, x_k, v_k) d\sigma(v_k, V^{m-k+1}(t_k)) ds.
\end{aligned}
\tag{2.36}
$$

Similarly as (2.35), after taking integration over $\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}$ (2.36) is

$$
\begin{aligned}
& \int_{\prod_{j=1}^k \mathcal{V}_j} \mathbf{1}_{\{t_{k+1}>0\}} |h^{m-k+1}\left(t_{k+1}, x_{k+1}, V^{m-k}(t_k)\right)| d\Sigma_{k,m}^{k+1}(t_{k+1}) \\
& \quad + \int_{\prod_{j=1}^k \mathcal{V}_j} \int_{t_{k+1}}^{t_k} e^{-\mathfrak{C}\langle V^{m-k}(s)\rangle^2 s} e^{\theta|V^{m-k}(s)|^2} \Gamma_{\text{gain}}^{m-k}(s) d\Sigma_{k,m}^{k+1}(s) ds.
\end{aligned}
\tag{2.37}
$$

From (2.37) (2.35), the summation in the first and second lines of (2.13) extends to k. And the index of the third line of (2.13) changes from k to $k+1$. For the rest terms with index $l \leq k-1$, we haven not done any change to them in the previous step. Thus their integration are over $\prod_{l=1}^{k-1}\mathcal{V}_j$. We add $\int_{\mathcal{V}_k} d\sigma(v_k, V^{m-k+1}(t_k)) = 1$ to all of them, so that all the integrations are over $\prod_{l=1}^k \mathcal{V}_j$ and we change $d\Sigma_{l,m}^{k-1}$ to $d\Sigma_{l,m}^k$ by

$$
d\Sigma_{l,m}^k = d\sigma\left(v_k, V^{m-k+1}(t_k)\right) d\Sigma_{l,m}^{k-1}.
$$

Therefore, the formula (2.13) is valid for $k+1$ and we derive the lemma. $\square$

As the lemma implies, to have L_∞ bound of h^{m+1}, it is crucial to obtain an estimate of H that is controlled in (2.13). It is rather clear that the first two terms in (2.13) include all finite collisions $< k$ in finite time, while the last term collects all trajectories whose corresponding particles collide with boundaries more than k times within t. These two types of estimates will be obtained in Lemmas 3 and 4 respectively in the next subsection. Namely we need only boundedness for the first two terms, but need decaying in k for the third, in which we essentially need to show the chance for a particle to collide with boundaries more than k times within a small time window t is very small.

2.2 *k*-Fold Integral

As a preparation, we first define:

$$r_{max} := \max(r_\|(2-r_\|), r_\perp), \quad r_{min} := \min(r_\|(2-r_\|), r_\perp). \tag{2.38}$$

Then immediately, one has $1 \geq r_{max} \geq r_{min} > 0$. We then define

$$\xi := \frac{1}{4T_M\theta} > 1, \tag{2.39}$$

from (2.1) considering $\theta < \frac{1}{4T_M}$. In the calculation of the k-fold integration over $\prod_{j=1}^k \mathcal{V}_j$, we inductively use the following notations:

$$\begin{aligned} T_{l,l} &= \frac{2\xi}{\xi+1}T_M, \quad T_{l,l-1} = r_{min}T_M + (1-r_{min})T_{l,l}, \cdots, \\ T_{l,1} &= r_{min}T_M + (1-r_{min})T_{l,2}, \end{aligned} \tag{2.40}$$

and thus naturally for $1 \leq i \leq l$, we have

$$T_{l,i} = \frac{2\xi}{\xi+1}T_M + (T_M - \frac{2\xi}{\xi+1}T_M)[1-(1-r_{min})^{l-i}]. \tag{2.41}$$

Moreover, we will use

$$\begin{aligned} d\Phi^{k,l}_{p,m}(s) :=& \Big\{ \prod_{j=l+1}^{k-1} d\sigma(v_j, v_{j-1}) \Big\} \\ & \times \Big\{ e^{-\int_s^{t_l} \frac{\mathfrak{C}}{2}\langle V^{m-l}(\tau)\rangle^2 d\tau} e^{-\theta|v_l|^2} e^{\mathfrak{C}t_l\langle v_l\rangle^2} e^{-[\frac{1}{4T_M} - \frac{1}{2T_w(x_l)}]|v_l|^2} d\sigma(v_l, V^{m-l+1}(t_l)) \Big\} \\ & \times \Big\{ \prod_{j=p}^{l-1} 2e^{\mathfrak{C}(t_j - t_{j+1})\langle v_j\rangle^2} e^{[\frac{1}{2T_w(x_j)} - \frac{1}{2T_w(x_{j+1})}]|v_j|^2} d\sigma\left(v_j, V^{m-j+1}(t_j)\right) \Big\}, \end{aligned} \tag{2.42}$$

and

$$d\Upsilon_p^{p'} := \{ \prod_{j=p}^{p'} 2e^{\mathfrak{C}(t_j - t_{j+1})\langle v_j\rangle^2} e^{[\frac{1}{2T_w(x_j)} - \frac{1}{2T_w(x_{j+1})}]|v_j|^2} d\sigma(v_j, v_{j-1}) \}, \tag{2.43}$$

to simplify the notation. Note that if $p = 1$, $d\Phi^{k,l}_{1,m}(s) = d\Sigma^k_{l,m}(s)$, defined in (2.14), and according to the definition in (2.42) and (2.14), we have

$$d\Phi^{k,l}_{p,m}(s) = d\Phi^{k,l}_{p',m}(s)d\Upsilon^{p'-1}_p, \quad \text{and} \quad d\Sigma^k_{l,m}(s) = d\Phi^{k,l}_{p,m}(s)d\Upsilon^{p-1}_1. \tag{2.44}$$

Now we state the lemma.

Lemma 3 *There exists*

$$t^* = t^*(T_M, \xi, \mathcal{C}, \mathfrak{C}, k), \tag{2.45}$$

such that for $t \leq t^*$, and $0 \leq s \leq t_l$, we have

$$\int_{\prod_{j=p}^{k-1}\mathcal{V}_j} \mathbf{1}_{\{t_l>0\}} d\Phi^{k,l}_{p,m}(s) \leq (2C_{T_M,\xi})^{2(l-p+1)}\mathcal{A}_{l,p}, \tag{2.46}$$

with $C_{T_M,\xi}$ and $\mathcal{C}$ being constants defined in (2.56) and (2.59) respectively, and

$$\mathcal{A}_{l,p} = \exp\left(\Big[\frac{[T_{l,p} - T_w(x_p)][1 - r_{min}]}{2T_w(x_p)[T_{l,p}(1 - r_{min}) + r_{min}T_w(x_p)]} + (2\mathcal{C})^{l-p+1}(\mathfrak{C}t)\Big]|V^{m-p+1}(t_p)|^2\right). \tag{2.47}$$

Moreover, for any $p < p' \leq l$, we have

$$\begin{aligned} &\int_{\prod_{j=p}^{k-1}\mathcal{V}_j} \mathbf{1}_{\{t_l>0\}} d\Phi^{k,l}_{p,m}(s) \leq (2C_{T_M,\xi})^{2(l-p'+1)}\mathcal{A}_{l,p'} \\ &\int_{\prod_{j=p}^{p'-1}\mathcal{V}_j} \mathbf{1}_{\{t_l>0\}} d\Upsilon^{p'-1}_p \leq (2C_{T_M,\xi})^{2(l-p+1)}\mathcal{A}_{l,p}. \end{aligned} \tag{2.48}$$

Remark 4 We comment here that this lemma indeed include the information for the k-fold integral in Lemma 2 by setting $p = 1$. To derive the decaying factor in Lemma 4, we need to extract smallness from the integral over v_p for $p \leq k$, for example, in Lemmas 5 and 7. This is the reason that we introduce the notation (2.42), (2.44) and incorporate them in this lemma.

Proof From (1.15) and (1.25), consider the first bracket of the first line in (2.14), for $l + 1 \leq j \leq k - 1$ we have

$$\int_{\prod_{j=l+1}^{k-1}\mathcal{V}_j} \prod_{j=l+1}^{k-1} d\sigma(v_j, V^{m-j+1}(t_j)) = 1.$$

Without loss of generality we can assume $k = l + 1$. Thus $d\Phi^{k,l}_{p,m} = d\Phi^{l+1,l}_{p,m}$. We use an induction of p with $1 \leq p \leq l$ to prove (2.46).

When $p = l$, by the second line of (2.42), the integration over $\mathcal{V}_l$ is written as

$$\int_{\mathcal{V}_l} e^{-\theta|v_l|^2} e^{\mathfrak{C}\langle v_l\rangle^2 t_l} e^{\int_s^{t_l} -\frac{\mathfrak{C}}{2}\langle V^{m-l}(\tau)\rangle^2 d\tau} e^{-[\frac{1}{4T_M} - \frac{1}{2T_w(x_l)}]|v_l|^2} d\sigma(v_l, V^{m-l+1}(t_l)). \tag{2.49}$$

In order to compute (2.49), we bound

$$e^{-\theta|v_l|^2} e^{\mathfrak{C}\langle v_l\rangle^2 t_l} \leq 2e^{(-\theta+\mathfrak{C}t_l)|v_l|^2}, \tag{2.50}$$

where we take $t \leq t^* = t^*(\mathfrak{C})$ small enough such that $e^{\mathfrak{C}t} \leq 2$ and thus $e^{\mathfrak{C}\langle v_l\rangle^2 t_l} \leq e^{\mathfrak{C}|v_l|^2 t_l} e^{\mathfrak{C}t_l} \leq 2e^{\mathfrak{C}|v_l|^2 t_l}$.

By (2.50) with $\theta = \frac{1}{4T_M\xi}$ in (2.39) we have

$$(2.49) \le 2\int_{\mathcal{V}_l} e^{-[\frac{1}{2T_M}\frac{\xi+1}{2\xi}-\frac{1}{2T_w(x_l)}-\mathfrak{C}t_l]|v_l|^2} d\sigma(v_l, V^{m-l+1}(t_l)). \tag{2.51}$$

Expanding the $d\sigma(v_l, V^{m-l+1}(t_l))$ using (1.11) and (1.25), we rewrite (2.51) as

$$\begin{aligned} &2\int_{\mathcal{V}_{l,\perp}} \frac{|v_{l,\perp}|}{r_\perp T_w(x_l)} e^{-[\frac{1}{2T_M}\frac{\xi+1}{2\xi}-\frac{1}{2T_w(x_l)}-\mathfrak{C}t_l]|v_{l,\perp}|^2} I_0\left(\frac{(1-r_\perp)^{1/2}v_{l,\perp}V_\perp^{m-l+1}(t_l)}{T_w(x_l)r_\perp}\right) \\ &\quad e^{-\frac{|v_{l,\perp}|^2+(1-r_\perp)|V_\perp^{m-l+1}(t_l)|^2}{2T_w(x_l)r_\perp}} dv_{l,\perp} \\ &\times \int_{\mathcal{V}_{l,\parallel}} \frac{1}{\pi r_\parallel(2-r_\parallel)(2T_w(x_l))} e^{-[\frac{1}{2T_M}\frac{\xi+1}{2\xi}-\frac{1}{2T_w(x_l)}-\mathfrak{C}t_l]|v_{l,\parallel}|^2} e^{-\frac{1}{2T_w(x_l)}\frac{|v_{l,\parallel}-(1-r_\parallel)V_\parallel^{m-l+1}(t_l)|^2}{r_\parallel(2-r_\parallel)}} dv_{l,\parallel}, \end{aligned} \tag{2.52}$$

where $v_{l,\parallel}$, $v_{l,\perp}$, $\mathcal{V}_{l,\perp}$ and $\mathcal{V}_{l,\parallel}$ are defined as

$$v_{l,\perp} = v_l \cdot n(x_l),\ \ v_{l,\parallel} = v_l - v_{l,\perp}n(x_l),\ \ \mathcal{V}_{l,\perp} = \{v_{l,\perp} : v_l \in \mathcal{V}_l\},\ \ \mathcal{V}_{l,\parallel} = \{v_{l,\parallel} : v_l \in \mathcal{V}_l\}. \tag{2.53}$$

$V_\perp^{m-l+1}(t_l)$ and $V_\parallel^{m-l+1}(t_l)$ are defined similarly.

First we compute the integration over $\mathcal{V}_{l,\parallel}$, the second line of (2.52). To apply (6.6) in Lemma 16, we set

$$\varepsilon = \mathfrak{C}t_l,\ \ w = (1-r_\parallel)V_\parallel^{m-l+1}(t_l)\ ,\ v = v_{l,\parallel},$$

$$a = -[\frac{1}{2T_M\frac{2\xi}{\xi+1}} - \frac{1}{2T_w(x_l)}],\ \ b = \frac{1}{2T_w(x_l)r_\parallel(2-r_\parallel)}. \tag{2.54}$$

By $\xi > 1$ in (2.39), we take $t^* = t^*(\mathfrak{C}, \xi, T_M) \ll 1$ such that when $t_l < t \le t^*$, we have

$$b - a - \varepsilon = \frac{1}{2T_w(x_l)r_\parallel(2-r_\parallel)} - \frac{1}{2T_w(x_l)} + \frac{1}{2T_M\frac{2\xi}{\xi+1}} - \mathfrak{C}t_l \ge \frac{1}{2T_M\frac{2\xi}{\xi+1}} - \mathfrak{C}t \ge \frac{1}{4T_M}. \tag{2.55}$$

Also we take $t^* = t^*(\mathfrak{C}, \xi, T_M)$ small enough to obtain $1+4T_M\mathfrak{C}t_l \le 1+4T_M\mathfrak{C}t \le 2$ when $t \le t^*$. Hence

$$\begin{aligned} \frac{b}{b-a-\varepsilon} &= \frac{b}{b-a}[1+\frac{\varepsilon}{b-a-\varepsilon}] \\ &\le \frac{\frac{2\xi}{\xi+1}T_M}{\frac{2\xi}{\xi+1}T_M + [T_w(x_l) - \frac{2\xi}{\xi+1}T_M]r_\parallel(2-r_\parallel)}[1+4T_M\mathfrak{C}t_l] \\ &\le \frac{\frac{4\xi}{\xi+1}T_M}{\frac{2\xi}{\xi+1}T_M + [\min\{T_w(x)\} - \frac{2\xi}{\xi+1}T_M]r_{max}} := C_{T_M,\xi}, \end{aligned} \tag{2.56}$$

where we have used (2.38).

In regard to (6.6), we have

$$\frac{(a+\varepsilon)b}{b-a-\varepsilon} = \frac{ab}{b-a}\left[1+\frac{\varepsilon}{b-a-\varepsilon}\right] + \frac{b}{b-a-\varepsilon}\varepsilon. \tag{2.57}$$

By (2.56) and $t_l < t$, we obtain

$$\frac{b}{b-a-\varepsilon}\varepsilon \le \frac{\frac{4\xi}{\xi+1}T_M}{\frac{2\xi}{\xi+1}T_M + [\min\{T_w(x)\} - \frac{2\xi}{\xi+1}T_M]r_{\max}}\mathfrak{C}t.$$

By (2.54), we have

$$\frac{ab}{b-a} = \frac{\frac{2\xi}{\xi+1}T_M - T_w(x_l)}{2T_w(x_l)[\frac{2\xi}{\xi+1}T_M + [T_w(x_l) - \frac{2\xi}{\xi+1}T_M]r_\parallel(2-r_\parallel)]}.$$

Therefore, by (2.55) and (2.57) we obtain

$$\frac{(a+\varepsilon)b}{b-a-\varepsilon} \le \frac{\frac{2\xi}{\xi+1}T_M - T_w(x_l)}{2T_w(x_l)[\frac{2\xi}{\xi+1}T_M + [T_w(x_l) - \frac{2\xi}{\xi+1}T_M]r_\parallel(2-r_\parallel)]} + \mathcal{C}(\mathfrak{C}t), \tag{2.58}$$

where we define

$$\mathcal{C} := \frac{4T_M\left(\frac{2\xi}{\xi+1}T_M - \min\{T_w(x)\}\right)}{2\min\{T_w(x)\}[\frac{2\xi}{\xi+1}T_M + [\min\{T_w(x)\} - \frac{2\xi}{\xi+1}T_M]r_{max}]} + \frac{\frac{4\xi}{\xi+1}T_M}{\frac{2\xi}{\xi+1}T_M + [\min\{T_w(x)\} - \frac{2\xi}{\xi+1}T_M]r_{max}}. \tag{2.59}$$

By (2.56), (2.58) and Lemma 16, using $w = (1-r_\parallel)V_\parallel^{m-l+1}(t_l)$ we bound the second line of (2.52) by

$$C_{T_M,\xi}\exp\left(\left[\frac{[\frac{2\xi}{\xi+1}T_M - T_w(x_l)]}{2T_w(x_l)[\frac{2\xi}{\xi+1}T_M(1-r_\parallel)^2 + r_\parallel(2-r_\parallel)T_w(x_l)]} + \mathcal{C}(\mathfrak{C}t)\right]|(1-r_\parallel)V_\parallel^{m-l+1}(t_l)|^2\right) \tag{2.60}$$

$$\le C_{T_M,\xi}\exp\left(\left[\frac{[\frac{2\xi}{\xi+1}T_M - T_w(x_l)][1-r_{min}]}{2T_w(x_l)\left[\frac{2\xi}{\xi+1}T_M(1-r_{min}) + r_{min}T_w(x_l)\right]} + \mathcal{C}(\mathfrak{C}t)\right]|V_\parallel^{m-l+1}(t_l)|^2\right), \tag{2.61}$$

where we have used (2.38).

Next we compute the first line of (2.52). To apply (6.9) in Lemma 17, we set

$$\varepsilon = \mathfrak{C}t_l,\ w = \sqrt{1-r_\perp}V_\perp^{m-l+1}(t_l)\ ,\ v = v_{l,\perp},$$
$$a = -[\frac{1}{2T_M\frac{2\xi}{\xi+1}} - \frac{1}{2T_w(x_l)}],\ b = \frac{1}{2T_w(x_l)r_\perp}.$$

Thus we can compute $\frac{b}{b-a-\varepsilon}$ and $\frac{(a+\varepsilon)b}{b-a-\varepsilon}$ using (2.56) and (2.58). Hence replacing $r_\parallel(2-r_\parallel)$ by $r_\perp$ and replacing $V_\parallel^{m-l+1}(t_l)$ by $V_\perp^{m-l+1}(t_l)$ in (2.60), we bound the first line of (2.52) by

$$2C_{T_M,\xi}\exp\left(\Big[\frac{[\frac{2\xi}{\xi+1}T_M-T_w(x_l)]}{2T_w(x_l)[\frac{2\xi}{\xi+1}T_M(1-r_\perp)+r_\perp T_w(x_l)]}+\mathcal{C}(\mathfrak{C}t)\Big]|\sqrt{1-r_\perp}V_\perp^{m-l+1}(t_l)|^2\right)$$
$$\le 2C_{T_M,\xi}\exp\left(\Big[\frac{[\frac{2\xi}{\xi+1}T_M-T_w(x_l)][1-r_{min}]}{2T_w(x_l)\big[\frac{2\xi}{\xi+1}T_M(1-r_{min})+r_{min}T_w(x_l)\big]}+\mathcal{C}(\mathfrak{C}t)\Big]|V_\perp^{m-l+1}(t_l)|^2\right). \tag{2.62}$$

where we use (2.38).

Collecting (2.61) (2.62), we derive

$$(2.52)\le 2(C_{T_M,\xi})^2\exp\left(\left[\frac{[\frac{2\xi}{\xi+1}T_M-T_w(x_l)][1-r_{min}]}{2T_w(x_l)\big[\frac{2\xi}{\xi+1}T_M(1-r_{min})+r_{min}T_w(x_l)\big]}+\mathcal{C}(\mathfrak{C}t)\right]|V^{m-l+1}(t_l)|^2\right)$$
$$\le (2C_{T_M,\xi})^2\mathcal{A}_{l,l},$$

where $\mathcal{A}_{l,l}$ is defined in (2.47) and $T_{l,l}=\frac{2\xi}{\xi+1}T_M$.

Therefore, (2.46) is valid for $p=l$.

Suppose (2.46) is valid for the $p=q+1$(induction hypothesis) with $q+1\le l$, then

$$\int_{\prod_{j=q+1}^{l}\mathcal{V}_j}\mathbf{1}_{\{t_l>0\}}d\Phi_{q+1,m}^{l+1,l}(s)\le(2C_{T_M,\xi})^{2(l-q)}\mathcal{A}_{l,q+1}.$$

We want to show (2.46) holds for $p=q$. By the hypothesis and the third line of (2.42),

$$\int_{\prod_{j=q}^{l}\mathcal{V}_j}\mathbf{1}_{\{t_l>0\}}d\Phi_{q,m}^{l+1,l}(s)\le(2C_{T_M,\xi})^{2(l-q)}\mathcal{A}_{l,q+1}$$
$$\times\int_{\mathcal{V}_q}2e^{\mathfrak{C}(t_q-t_{q+1})\langle v_q\rangle^2}e^{[\frac{1}{2T_w(x_q)}-\frac{1}{2T_w(x_{q+1})}]|v_q|^2}d\sigma(v_q,V^{m-q}(t_{q+1})). \tag{2.63}$$

Using the definition of $\mathcal{A}_{l,q+1}$ in (2.47), we obtain

$$(2.63)\le(2C_{T_M,\xi})^{2(l-q)}\int_{\mathcal{V}_q}2\exp\left(\frac{(T_{l,q+1}-T_w(x_{q+1}))(1-r_{min})}{2T_w(x_{q+1})[T_{l,q+1}(1-r_{min})+r_{min}T_w(x_{q+1})]}|V^{m-q}(t_{q+1})|^2\right.$$
$$\left.+(2\mathcal{C})^{l-q}(\mathfrak{C}t)|V^{m-q}(t_{q+1})|^2\right)e^{\mathfrak{C}(t_q-t_{q+1})\langle v_q\rangle^2}e^{[\frac{1}{2T_w(x_q)}-\frac{1}{2T_w(x_{q+1})}]|v_q|^2}d\sigma(v_q,V^{m-q+1}(t_q)). \tag{2.64}$$

Let $\mathfrak{C}$ in (2.17) satisfy

$$2C_{\phi^m}\frac{(T_{l,q+1}-T_w(x_{q+1}))(1-r_{min})}{2T_w(x_{q+1})[T_{l,q+1}(1-r_{min})+r_{min}T_w(x_{q+1})]}$$
$$\le\frac{C_{\phi^m}}{\min(T_w(x))(1-r_{min})}\le\mathfrak{C},\quad(C_{\phi^m})^2\le\mathfrak{C}. \tag{2.65}$$

Similarly to (2.25) and (2.26),

$$|V^{m-q}(t_{q+1})|\le|v_q|+C_{\phi^m}(t_q-t_{q+1}),$$
$$|V^{m-q}(t_{q+1})|^2\le C_{\phi^m}^2(t_q-t_{q+1})^2+2C_{\phi^m}(t_q-t_{q+1})|v_q|+|v_q|^2.$$

Then we apply (2.65) to get

$$\exp\left[\left(\frac{(T_{l,q+1}-T_w(x_{q+1}))(1-r)}{2T_w(x_{q+1})[T_{l,q+1}(1-r)+rT_w(x_{q+1})]}+(2\mathcal{C})^{l-q}(\mathfrak{C}t)\right)|V^{m-q+1}(t_{q+1})|^2\right]$$
$$\times e^{\mathfrak{C}(t_q-t_{q+1})\langle v_q\rangle^2}$$
$$\leq \exp\left[\left(\frac{(T_{l,q+1}-T_w(x_{q+1}))(1-r)}{2T_w(x_{q+1})[T_{l,q+1}(1-r)+rT_w(x_{q+1})]}+(2\mathcal{C})^{l-q}(\mathfrak{C}t)\right)|v_q|^2\right]$$
$$\times e^{\mathfrak{C}(t_q-t_{q+1})|v_q|}e^{\mathfrak{C}(t_q-t_{q+1})^2}e^{\mathfrak{C}(t_q-t_{q+1})|v_q|^2}e^{\mathfrak{C}(t_q-t_{q+1})}$$
$$\leq 2\exp\left[\left(\frac{(T_{l,q+1}-T_w(x_{q+1}))(1-r)}{2T_w(x_{q+1})[T_{l,q+1}(1-r)+rT_w(x_{q+1})]}+2(2\mathcal{C})^{l-q}(\mathfrak{C}t)\right)|v_q|^2\right],$$

where we haved use $e^{\mathfrak{C}(t_q-t_{q+1})|v_q|}\leq e^{\mathfrak{C}(t_q-t_{q+1})}e^{\mathfrak{C}(t_q-t_{q+1})|v_q|^2}$ and thus

$$\begin{aligned}e^{\mathfrak{C}(t_q-t_{q+1})|v_q|}e^{\mathfrak{C}(t_q-t_{q+1})^2}\times e^{\mathfrak{C}(t_q-t_{q+1})|v_q|^2}e^{\mathfrak{C}(t_q-t_{q+1})}&\leq e^{2\mathfrak{C}t|v_q|^2}e^{3\mathfrak{C}(t_q-t_{q+1})}\\&\leq 2e^{(2\mathcal{C})^{l-q}(\mathfrak{C}t)|v_q|^2}.\end{aligned}$$

Here we take $t^*=t^*(\mathfrak{C})$ small enough such that when $t\leq t^*$,

$$e^{3\mathfrak{C}(t_q-t_{q+1})}\leq e^{3\mathfrak{C}t}\leq 2.$$

Thus we obtain

$$\begin{aligned}(2.64)\leq 4(2C_{T_M,\xi})^{2(l-q)}\int_{\mathcal{V}_q}\exp\Bigg[\Bigg(&\frac{(T_{l,q+1}-T_w(x_{q+1}))(1-r_{min})}{2T_w(x_{q+1})[T_{l,q+1}(1-r_{min})+r_{min}T_w(x_{q+1})]}|v_q|^2\\&+2(2\mathcal{C})^{l-q}(\mathfrak{C}t)|v_q|^2\Bigg)\Bigg]e^{[\frac{1}{2T_w(x_q)}-\frac{1}{2T_w(x_{q+1})}]|v_q|^2}d\sigma(v_q,V^{m-q+1}(t_q)).\end{aligned}$$

We focus on the coefficient of $|v_q|^2$ in (2.64), we derive

$$\begin{aligned}&\frac{(T_{l,q+1}-T_w(x_{q+1}))(1-r_{min})}{2T_w(x_{q+1})[T_{l,q+1}(1-r_{min})+r_{min}T_w(x_{q+1})]}|v_q|^2+[\frac{1}{2T_w(x_q)}-\frac{1}{2T_w(x_{q+1})}]|v_q|^2\\&=\frac{(T_{l,q+1}-T_w(x_{q+1}))(1-r_{min})-[T_{l,q+1}(1-r_{min})+r_{min}T_w(x_{q+1})]}{2T_w(x_{q+1})[T_{l,q+1}(1-r_{min})+r_{min}T_w(x_{q+1})]}|v_q|^2+\frac{|v_q|^2}{2T_w(x_q)}\\&=\frac{-T_w(x_{q+1})(1-r_{min})-r_{min}T_w(x_{q+1})}{2T_w(x_{q+1})[T_{l,q+1}(1-r_{min})+r_{min}T_w(x_{q+1})]}|v_q|^2+\frac{|v_q|^2}{2T_w(x_q)}\\&=\frac{-|v_q|^2}{2[T_{l,q+1}(1-r_{min})+r_{min}T_w(x_{q+1})]}+\frac{|v_q|^2}{2T_w(x_q)}.\end{aligned}$$

By the Definition 2, $x_{q+1}=x_{q+1}(t,x,v,v_1,\cdots,v_q)$, thus $T_w(x_{q+1})$ depends on v_q. In order to explicitly compute the integration over $\mathcal{V}_q$, we need to get rid of the dependence of the $T_w(x_{q+1})$ on v_q, thus we bound

$$\begin{aligned}\exp\left(\frac{-|v_q|^2}{2[T_{l,q+1}(1-r_{min})+r_{min}T_w(x_{q+1})]}\right)&\leq\exp\left(\frac{-|v_q|^2}{2[T_{l,q+1}(1-r_{min})+r_{min}T_M]}\right)\\&=\exp\left(\frac{-|v_q|^2}{2T_{l,q}}\right),\end{aligned}\tag{2.66}$$

where we use (2.40).

Expanding $d\sigma(v_q, V^{m-q+1}(t_q))$ by (1.25) and using (2.66), we derive

$$\begin{aligned}
(2.64) &\le 4(2C_{T_M,\xi})^{2(l-q)} \\
&\int_{\mathcal{V}_{q,\perp}} \frac{2}{r_\perp} \frac{|v_{q,\perp}|}{2T_w(x_q)} e^{-[\frac{1}{2T_{l,q}} - \frac{1}{2T_w(x_q)} - 2(2\mathcal{C})^{l-q}(\mathfrak{C}t)]|v_{q,\perp}|^2} I_0\left(\frac{(1-r_\perp)^{1/2} v_{q,\perp} V_\perp^{m-q+1}(t_q)}{T_w(x_q) r_\perp}\right) \\
&e^{-\frac{|v_{q,\perp}|^2 + (1-r_\perp)|V_\perp^{m-q+1}(t_q)|^2}{2T_w(x_q) r_\perp}} dv_{q,\perp} \\
&\times \int_{\mathcal{V}_{q,\parallel}} \frac{1}{\pi r_\parallel (2-r_\parallel)(2T_w(x_q))} \\
&e^{-[\frac{1}{2T_{l,q}} - \frac{1}{2T_w(x_q)} - 2(2\mathcal{C})^{l-q}(\mathfrak{C}t)]|v_{q,\parallel}|^2} e^{-\frac{1}{2T_w(x_q)} \frac{|v_{q,\parallel} - (1-r_\parallel) V_\parallel^{m-q+1}(t_q)|^2}{r_\parallel (2-r_\parallel)}} dv_{q,\parallel}.
\end{aligned} \tag{2.67}$$

In the third line of (2.67), to apply (6.6) in Lemma 16, we set

$$a = -\left[\frac{1}{2T_{l,q}} - \frac{1}{2T_w(x_q)}\right], \ b = \frac{1}{2T_w(x_q) r_\parallel (2-r_\parallel)},$$
$$\varepsilon = 2(2\mathcal{C})^{l-q}(\mathfrak{C}t), \ w = (1-r_\parallel) v_{q-1,\parallel}.$$

Comparing with (2.54), we can replace $\frac{2\xi}{\xi+1} T_M$ by $T_{l,q}$ and replace $\mathfrak{C}t$ by $2(2\mathcal{C})^{l-q}(\mathfrak{C}t)$. Then we apply the replacement to (2.55) and obtain

$$b - a - \varepsilon \ge \frac{1}{2T_{l,q}} - 2(2\mathcal{C})^{l-q}(\mathfrak{C}t) \ge \frac{1}{2T_M \frac{2\xi}{\xi+1}} - 2(2\mathcal{C})^k(\mathfrak{C}t) \ge \frac{1}{4T_M},$$

where we take $t^* = t^*(T_M, \xi, \mathcal{C}, \mathfrak{C}, k)$ small enough and $t \le t^*$. Also we require the t satisfy

$$\frac{\varepsilon}{b-a-\varepsilon} \le 4T_M \times 2(2\mathcal{C})^k(\mathfrak{C}t) \le 2.$$

We conclude the t^* here only depends on the parameters $T_M, \xi, \mathcal{C}, \mathfrak{C}, k$. Thus by the same computation as (2.56) we obtain

$$\frac{b}{b-a-\varepsilon} \le \frac{2T_{l,q}}{T_{l,q} + [\min\{T_w(x)\} - T_{l,q}] r_\parallel (2-r_\parallel)} \le C_{T_M,\xi},$$

where we use $T_{l,q} \le \frac{2\xi}{\xi+1} T_M$ from (2.40) and (2.38). $C_{T_M,\xi}$ is defined in (2.56)

By the same computation as (2.58), we obtain

$$\begin{aligned}
\frac{(a+\varepsilon)b}{b-a-\varepsilon} &= \frac{ab}{b-a} + \frac{ab}{b-a} \frac{\varepsilon}{b-a-\varepsilon} + \frac{b}{b-a-\varepsilon} \varepsilon \\
&\le \frac{T_{l,q} - T_w(x_q)}{2T_w(x_q)[T_{l,q} + [T_w(x_q) - T_{l,q}] r_\parallel (2-r_\parallel)]} + (2\mathcal{C})^{l-q+1}(\mathfrak{C}t).
\end{aligned}$$

Here we have used $T_{l,q} \le \frac{2\xi}{\xi+1} T_M$ and (2.38) to obtain

$$\begin{aligned}
&\frac{ab}{b-a} \frac{\varepsilon}{b-a-\varepsilon} + \frac{b\varepsilon}{b-a-\varepsilon} \\
&\le \frac{4T_M\big(T_{l,q} - \min\{T_w(x)\}\big)}{2\min\{T_w(x)\}[T_{l,q} + [\min\{T_w(x)\} - T_{l,q}] r_\parallel (2-r_\parallel)]} 2(2\mathcal{C})^{l-q}(\mathfrak{C}t) \\
&+ \frac{2T_{l,q}}{\frac{2\xi}{\xi+1} T_M + [\min\{T_w(x)\} - T_{l,q}] r_\parallel (2-r_\parallel)} 2(2\mathcal{C})^{l-q}(\mathfrak{C}t) \le (2\mathcal{C})^{l-q+1}(\mathfrak{C}t),
\end{aligned}$$

with $\mathcal{C}$ defined in (2.59).

Thus by Lemma 16 with $w = (1-r_\parallel)V_\parallel^{m-q+1}(t_q)$, the third line of (2.67) is bounded by

$$\begin{aligned}
&C_{T_M,\xi}\exp\bigg(\Big[\frac{[T_{l,q}-T_w(x_q)]}{2T_w(x_q)[T_{l,q}(1-r_\parallel)^2+r(2-r_\parallel)T_w(x_q)]}\\
&\quad+(2\mathcal{C})^{l-q+1}(\mathfrak{C}t)\Big]|(1-r_\parallel)V_\parallel^{m-q+1}(t_q)|^2\bigg)\\
&\leq C_{T_M,\xi}\exp\bigg(\Big[\frac{[T_{l,q}-T_w(x_q)][1-r_{min}]}{2T_w(x_q)[T_{l,q}(1-r_{min})+r_{min}T_w(x_q)]}\\
&\quad+(2\mathcal{C})^{l-q+1}(\mathfrak{C}t)\Big]|V_\parallel^{m-q+1}(t_q)|^2\bigg).
\end{aligned} \tag{2.68}$$

By the same computation the second line of (2.67) is bounded by

$$C_{T_M,\xi}\exp\bigg(\Big[\frac{[T_{l,q}-T_w(x_q)][1-r_{min}]}{2T_w(x_q)[T_{l,q}(1-r_{min})+r_{min}T_w(x_q)]}+(2\mathcal{C})^{l-q+1}(\mathfrak{C}t)\Big]|V_\perp^{m-q+1}(t_q)|^2\bigg). \tag{2.69}$$

By (2.68) and (2.69), we derive that

$$\begin{aligned}
(2.67) &\leq (2C_{T_M,\xi})^{2(l-q+1)}\exp\bigg(\Big[\frac{[T_{l,q}-T_w(x_q)][1-r_{min}]}{2T_w(x_q)[T_{l,q}(1-r_{min})+r_{min}T_w(x_q)]}\\
&\quad+(2\mathcal{C})^{l-q+1}(\mathfrak{C}t)\Big]|V^{m-q+1}(t_q)|^2\bigg)\\
&=(2C_{T_M,\xi})^{2(l-q+1)}\mathcal{A}_{l,q},
\end{aligned}$$

which is consistent with (2.46) with $p=q$. The induction is valid we derive (2.46).

Now we focus on (2.48). The first inequality in (2.48) follows directly from (2.46) and (2.44). For the second inequality, by (2.43) we have

$$\begin{aligned}
&(2C_{T_M,\xi})^{2(l-p'+1)}\mathcal{A}_{l,p'}\int_{\prod_{j=p}^{p'-1}\mathcal{V}_j}\mathbf{1}_{\{t_l>0\}}d\Upsilon_p^{p'-1}\\
&=(2C_{T_M,\xi})^{2(l-p'+1)}\mathcal{A}_{l,p'}\int_{\prod_{j=p}^{p'-2}\mathcal{V}_j}d\Upsilon_p^{p'-2}\\
&\int_{\mathcal{V}_{p'-1}}\mathbf{1}_{\{t_l>0\}}2e^{\mathfrak{C}(t_{p'-1}-t_{p'})\langle v_{p'-1}\rangle^2}e^{[\frac{1}{2T_w(x_{p'-1})}-\frac{1}{2T_w(x_{p'})}]|v_{p'-1}|^2}d\sigma(v_{p'-1},V^{m-p'+2}(t_{p'-1})).
\end{aligned} \tag{2.70}$$

In the proof for (2.46) we have

$$(2.63)\leq(2.64)\leq(2.67)\leq(C_{T_M,\xi})^{2(l-q+1)}\mathcal{A}_{l,q}.$$

Then by replacing q by $p'-1$ in the estimate (2.63)$\leq(2C_{T_M,\xi})^{2(l-q+1)}\mathcal{A}_{l,q}$ we have

$$\begin{aligned}
&(2C_{T_M,\xi})^{2(l-p'+1)}\mathcal{A}_{l,p'}\int_{\prod_{j=p}^{p'-1}\mathcal{V}_j}\mathbf{1}_{\{t_l>0\}}d\Upsilon_p^{p'-1}\\
&=(2.70)\leq(2C_{T_M,\xi})^{2(l-p'+2)}\mathcal{A}_{l,p'-1}\int_{\prod_{j=p}^{p'-2}\mathcal{V}_j}\mathbf{1}_{\{t_l>0\}}d\Upsilon_p^{p'-2}.
\end{aligned}$$

Keep doing this computation until integrating over $\mathcal{V}_p$ we obtain the second inequality in (2.48). □

In the following lemma, we prepare for showing the smallness of the last term in (2.13).

Lemma 4 *Assume the constraint for T_w holds true ((1.28)). There exists*

$$k_0 = k_0(\Omega, C_{T_M,\xi}, \mathcal{C}, T_M, r_\perp, r_\parallel, \min\{T_w(x)\}, \xi, \mathfrak{C}) \gg 1, \tag{2.71}$$

$$t' = t'(k_0, \xi, T_M, \min\{T_w(x)\}, \mathcal{C}, r_\perp, r_\parallel, \mathfrak{C}, C_{\phi^m}) \le t^* \ll 1 \tag{2.72}$$

such that for all $t \in [0, t']$, we have

$$\int_{\prod_{j=1}^{k_0-1} \mathcal{V}_j} \mathbf{1}_{\{t_{k_0}>0\}} d\Sigma_{k_0-1,m}^{k_0}(t_{k_0}) \le (\frac{1}{2})^{k_0} \mathcal{A}_{k_0-1,1}, \tag{2.73}$$

where $\mathcal{A}_{k_0-1,1}$ is defined in (2.47) *and t^* is defined in* (2.45).

Remark 5 We comment that the main difference between this lemma and Lemma 3 is that here a decaying factor $(\frac{1}{2})^{k_0}$ is needed. This lemma implies for $k = k_0$ large enough, the last term of (2.13) is negligible.

The main idea to prove Lemma 4 is to use the decomposition (2.97) for the integral domain. In Lemmas 5–8 we use Lemma 3 to show that such decomposition indeed make contribution in obtaining the smallness. Among them Lemma 8 is the most important one as it summarizes all estimates in Lemmas 5–7 and directly provides the decaying factor for the k-fold integral. Echoing the difficulties for obtaining L_∞ bound as discussed in Sect. 1.2, where we proposed splitting γ_+ into γ_+^δ and the remainders, in Lemma 8, we detail such splitting and the trajectories' behavior in these sets.

After proving Lemmas 5–8 as preparation, we will conclude the proof for Lemma 4.

Lemma 5 *Recall* (2.46) *in Lemma 3.*

For $1 \le i \le k-1$, if

$$|v_i \cdot n(x_i)| < \delta, \tag{2.74}$$

then

$$\int_{\prod_{j=i}^{k-1} \mathcal{V}_j} \mathbf{1}_{\{t_k>0\}} \mathbf{1}_{\{v_i \in \mathcal{V}_i : |v_i \cdot n(x_i)| < \delta\}} d\Phi_{i,m}^{k,k-1}(t_k) \le \delta (2C_{T_M,\xi})^{2(k-i)} \mathcal{A}_{k-1,i}. \tag{2.75}$$

If

$$|v_{i,\parallel} - \eta_{i,\parallel} V_\parallel^{m-i+1}(t_i)| > \delta^{-1}, \tag{2.76}$$

then

$$\int_{\prod_{j=i}^{k-1} s\mathcal{V}_j} \mathbf{1}_{\{t_k>0\}} \mathbf{1}_{\{|v_{i,\parallel} - \eta_{i,\parallel} V_\parallel^{m-i+1}(t_i)| > \delta^{-1}\}} d\Phi_{i,m}^{k,k-1}(t_k) \le \delta (2C_{T_M,\xi})^{2(k-i)} \mathcal{A}_{k-1,i}. \tag{2.77}$$

Here $\eta_{i,\parallel}$ is a constant defined in (2.85).

If

$$|v_{i,\perp} - \eta_{i,\perp} V_\perp^{m-i+1}(t_i)| > \delta^{-1}, \tag{2.78}$$

then

$$\int_{\prod_{j=i}^{k-1} \mathcal{V}_j} \mathbf{1}_{\{t_k>0\}} \mathbf{1}_{\{|v_{i,\perp} - \eta_{i,\perp} V_\perp^{m-i+1}(t_i)| > \delta^{-1}\}} d\Phi_{i,m}^{k,k-1}(t_k) \le \delta (2C_{T_M,\xi})^{2(k-i)} \mathcal{A}_{k-1,i}. \tag{2.79}$$

Here $\eta_{i,\perp}$ is a constant defined in (2.88).

Proof First we focus on (2.75). By (2.67) in Lemma 3, we can replace l by $k-1$ and replace q by i to obtain

$$\begin{aligned}&\int_{\prod_{j=i}^{k-1}\mathcal{V}_j}\mathbf{1}_{\{t_k>0\}}d\Phi_{i,m}^{k,k-1}(t_k)\le 4(2C_{T_M,\xi})^{2(k-1-i)}\\&\int_{\mathcal{V}_{i,\perp}}\frac{2}{r_\perp}\frac{|v_{i,\perp}|}{2T_w(x_i)}e^{-[\frac{1}{2T_{k-1,i}}-\frac{1}{2T_w(x_i)}-2(2\mathcal{C})^{k-1-i}(\mathfrak{C}t)]|v_{i,\perp}|^2}I_0\left(\frac{(1-r_\perp)^{1/2}v_{i,\perp}V_\perp^{m-i+1}(t_i)}{T_w(x_i)r_\perp}\right)\\&e^{\frac{|v_{i,\perp}|^2+(1-r_\perp)|V_\perp^{m-i+1}(t_i)|^2}{2T_w(x)r_\perp}}dv_{i,\perp}\\&\times\int_{\mathcal{V}_{i,\parallel}}\frac{1}{\pi r_\parallel(2-r_\parallel)(2T_w(x_i))}\\&e^{-[\frac{1}{2T_{k-1,i}}-\frac{1}{2T_w(x_i)}-2(2\mathcal{C})^{k-1-i}(\mathfrak{C}t)]|v_{i,\parallel}|^2}e^{-\frac{1}{2T_w(x_i)}\frac{|v_{i,\parallel}-(1-r_\parallel)V_\parallel^{m-i+1}(t_i)|^2}{r_\parallel(2-r_\parallel)}}dv_{i,\parallel}.\end{aligned}\tag{2.80}$$

Under the condition (2.74), we consider the second line of (2.80) with integrating over $\{v_{i,\perp}\in\mathcal{V}_{i,\perp}:|v_i\cdot n(x_i)|<\frac{1-\eta}{2(1+\eta)}\delta\}$. To apply (6.10) in Lemma 17, we set

$$\begin{aligned}&a=-\left[\frac{1}{2T_{k-1,i}}-\frac{1}{2T_w(x_i)}\right],\ b=\frac{1}{2T_w(x_i)r_\perp},\ \varepsilon=(2\mathcal{C})^{k-1-i}(\mathfrak{C}t),\\&w=\sqrt{1-r_\perp}V_\perp^{m-i+1}(t_i).\end{aligned}$$

Under the condition $|v_i\cdot n(x_i)|<\frac{1-\eta}{2(1+\eta)}\delta$, applying (6.10) in Lemma 17 and using (2.69) with $q=i,l=k-1$, we bound the second line of (2.80) by

$$\delta C_{T_M,\xi}\exp\left(\Big[\frac{[T_{k-1,i}-T_w(x_i)][1-r_{min}]}{2T_w(x_i)[T_{k-1,i}(1-r_{min})+r_{min}T_w(x_i)]}+(2\mathcal{C})^{k-i}(\mathfrak{C}t)\Big]|V_\perp^{m-i+1}(t_i)|^2\right).\tag{2.81}$$

Comparing with (2.69), we conclude the second line of (2.80) provides one more constant term δ. The third line of (2.80) is bounded by (2.68) with $q=i,l=k-1$. Therefore, we derive (2.75).

Then we focus on (2.77). We consider the third line of (2.80). To apply (6.8) in Lemma 16, we set

$$\begin{aligned}&a=-\frac{1}{2T_{k-1,i}}+\frac{1}{2T_w(x_i)},\quad b=\frac{1}{2T_w(x_i)r_\parallel(2-r_\parallel)},\\&\varepsilon=2(2\mathcal{C})^{k-1-i}(\mathfrak{C}t),\ w=(1-r_\parallel)V_\parallel^{m-i+1}(t_i).\end{aligned}\tag{2.82}$$

We define

$$B_{i,\parallel}:=b-a-\varepsilon.\tag{2.83}$$

In regard to (6.8),

$$\frac{b}{b-a-\varepsilon}w=\frac{b}{b-a}[1+\frac{\varepsilon}{b-a-\varepsilon}]w.$$

By (2.82),

$$\frac{b}{b-a}=\frac{T_{k-1,i}}{T_{k-1,i}(1-r_\parallel)^2+T_w(x_i)r_\parallel(2-r_\parallel)},\quad\frac{\varepsilon}{b-a-\varepsilon}=\frac{2(2\mathcal{C})^{k-1-i}(\mathfrak{C}t)}{B_{i,\parallel}}.$$

Thus we obtain

$$\frac{b}{b-a-\varepsilon} w = \eta_{i,\parallel} V_{\parallel}^{m-i+1}(t_i), \tag{2.84}$$

where we define

$$\eta_{i,\parallel} := \frac{T_{k-1,i}[1 + 2(2\mathcal{C})^{k-1-i}(\mathfrak{C}t)/B_{i,\parallel}]}{T_{k-1,i}(1-r_{\parallel})^2 + T_w(x_i)r_{\parallel}(2-r_{\parallel})}(1-r_{\parallel}). \tag{2.85}$$

Thus under the condition (2.76), applying (6.8) in Lemma 6.6 with $\frac{b}{b-a-\varepsilon} w = \eta_{i,\parallel} V_{\parallel}^{m-i+1}(t_i)$ and using (2.68) with $q = i, l = k-1$, we bound the third line of (2.80) by

$$\delta C_{T_M,\xi} \exp\left(\left[\frac{[T_{k-1,i} - T_w(x_i)][1-r_{min}]}{2T_w(x_i)[T_{k-1,i}(1-r_{min}) + r_{min}T_w(x_i)]} + 2(2\mathcal{C})^{k-1-i}(\mathfrak{C}t)\right]|V_{\parallel}^{m-i+1}(t_i)|^2\right).$$

By the same computation in Lemma 5, we derive (2.77) because of the extra constant δ.

Last we focus on (2.79). We consider the second line of (2.80). To apply (6.10) in Lemma 18, we set

$$a = -\frac{1}{2T_{k-1,i}} + \frac{1}{2T_w(x_i)}, \quad b = \frac{1}{2T_w(x_i)r_{\perp}},$$
$$\varepsilon = 2(2\mathcal{C})^{k-1-i}(\mathfrak{C}t), \ w = \sqrt{1-r_{\perp}} V_{\perp}^{m-i+1}(t_i). \tag{2.86}$$

Define

$$B_{i,\perp} := b - a - \varepsilon. \tag{2.87}$$

By the same computation as (2.84),

$$\frac{b}{b-a-\varepsilon} w = \eta_{i,\perp} V_{\perp}^{m-i+1}(t_i),$$

where we define

$$\eta_{i,\perp} := \frac{T_{k-1,i}[1 + \frac{2(2\mathcal{C})^{k-1-i}(\mathfrak{C}t)}{B_{i,\perp}}]}{T_{k-1,i}(1-r_{\perp}) + T_w(x_i)r_{\perp}}\sqrt{1-r_{\perp}}. \tag{2.88}$$

Thus under the condition (2.78), applying (6.13) in Lemma 18 with $\frac{b}{b-a-\varepsilon} w = \eta_{i,\perp} V_{\perp}^{m-i+1}(t_i)$ and using (2.69) with $q = i, l = k-1$, we bound the second line of (2.80) by

$$\delta C_{T_M,\xi} \exp\left(\left[\frac{[T_{k-1,i} - T_w(x_i)][1-r_{min}]}{2T_w(x_i)[T_{k-1,i}(1-r_{min}) + r_{min}T_w(x_i)]} + 2(2\mathcal{C})^{k-i}(\mathfrak{C}t)\right]|V^{m-i+1})_{\perp}(t_i)|^2\right).$$

Then we derive (2.77) because of the extra constant δ. □

Lemma 6 *For $\eta_{i,\parallel}$ and $\eta_{i,\perp}$ defined in Lemma 5, suppose there exists $\eta < 1$ such that*

$$\max\{\eta_{i,\parallel}, \eta_{i,\perp}\} < \eta < 1. \tag{2.89}$$

If

$$|v_{i,\parallel}| > \frac{1+\eta}{1-\eta}\delta^{-1} \text{ and } |v_{i,\parallel} - \eta_{i,\parallel} V_{\parallel}^{m-i+1}(t_i)| < \delta^{-1}, \tag{2.90}$$

then we have

$$|V_{\parallel}^{m-i+1}(t_i)| > |v_{i,\parallel}| + \delta^{-1}. \tag{2.91}$$

Also if

$$|v_{i,\perp}| > \frac{1+\eta}{1-\eta}\delta^{-1} \text{ and } |v_{i,\perp} - \eta_{i,\perp} V_{\perp}^{m-i+1}(t_i)| < \delta^{-1}, \tag{2.92}$$

then we have

$$|V_{\perp}^{m-i+1}(t_i)| > |v_{i,\perp}| + \delta^{-1}. \tag{2.93}$$

Remark 6 Lemma 5 includes all the cases that are controllable since they provides the small number δ, which direct contributes in obtaining the exponential decay in (2.73). This lemma discuss the rest cases that does not directly provide the smallness, which are the main difficulty.

Proof Under the condition (2.90) we have

$$\eta_{i,\parallel}|V_{\parallel}^{m-i+1}(t_i)| > |v_{i,\parallel}| - \delta^{-1}.$$

Thus we derive

$$\begin{aligned}
|V_{\parallel}^{m-i+1}(t_i)| &> |v_{i,\parallel}| + \frac{1-\eta_{i,\parallel}}{\eta_{i,\parallel}}|v_{i,\parallel}| - \frac{1}{\eta_{i,\parallel}}\delta^{-1} \\
&> |v_{i,\parallel}| + \frac{1-\eta_{i,\parallel}}{\eta_{i,\parallel}}\frac{1+\eta}{1-\eta}\delta^{-1} - \frac{1}{\eta_{i,\parallel}}\delta^{-1} \\
&> |v_{i,\parallel}| + \frac{1-\eta_{i,\parallel}}{\eta_{i,\parallel}}\frac{1+\eta_{i,\parallel}}{1-\eta_{i,\parallel}}\delta^{-1} - \frac{1}{\eta_{i,\parallel}}\delta^{-1} \\
&> |v_{i,\parallel}| + \frac{1+\eta_{i,\parallel}}{\eta_{i,\parallel}}\delta^{-1} - \frac{1}{\eta_{i,\parallel}}\delta^{-1} > |v_{i,\parallel}| + \delta^{-1},
\end{aligned}$$

where we use $|v_{i,\parallel}| > \frac{1+\eta}{1-\eta}\delta^{-1}$ in the second line and $1 > \eta \geq \eta_{i,\parallel}$ in the third line. Then we obtain (2.91).

Under the condition (2.92), we apply the same computation above to obtain (2.93). □

Lemma 7 *Suppose there are n number of v_j such that*

$$|v_{j,\parallel} - \eta_{j,\parallel} V_{\perp}^{m-j+1}(t_j)| \geq \delta^{-1}, \tag{2.94}$$

and also suppose the index j in these v_j are $i_1 < i_2 < \cdots < i_n$, then

$$\int_{\prod_{j=i_1}^{k-1}\mathcal{V}_j} \mathbf{1}_{\{t_k>0\}}\mathbf{1}_{\{(2.94) \text{ holds for } j=i_1,i_2,\cdots,i_n\}} d\Phi_{i_1,m}^{k,k-1}(t_k) \leq (\delta)^n (2C_{T_M,\xi})^{2(k-i_1)}\mathcal{A}_{k-1,i_1}. \tag{2.95}$$

Proof By (2.48) in Lemma 2 with $l = k-1$, $p = i_1$, $p' = i_n$ and using (2.77) with $i = i_n$, we have

$$\begin{aligned}
&\int_{\prod_{j=i_1}^{k-1}\mathcal{V}_j} \mathbf{1}_{\{t_k>0\}}\mathbf{1}_{\{(2.94) \text{ holds for } j=i_1,\cdots,i_n\}} d\Phi_{i_1,m}^{k,k-1}(t_k) \\
&\leq \delta(2C_{T_M,\xi})^{2(k-i_n)}\mathcal{A}_{k-1,i_n}\int_{\prod_{j=i_1}^{i_n-1}\mathcal{V}_j} \mathbf{1}_{\{t_k>0\}}\mathbf{1}_{\{(2.94) \text{ holds for } j=i_1,\cdots,i_{n-1}\}} d\Upsilon_{i_1}^{i_n-1} \\
&= \delta(2C_{T_M,\xi})^{2(k-i_n)}\mathcal{A}_{k-1,i_n}\int_{\prod_{j=i_1}^{i_{n-1}-1}\mathcal{V}_j} \\
&\quad \int_{\prod_{j=i_{n-1}}^{(i_n)-1}\mathcal{V}_j} \mathbf{1}_{\{t_k>0\}}\mathbf{1}_{\{(2.94) \text{ holds for } j=i_1,\cdots,i_{n-1}\}} d\Upsilon_{i_{n-1}}^{(i_n)-1} d\Upsilon_{i_1}^{i_{n-1}-1}.
\end{aligned} \tag{2.96}$$

Again by (2.48) and (2.77) with $i = i_{n-1}$ we have

$$(2.962.96) \leq \delta^2 (C_{T_M,\xi})^{2(k-i_{n-1})} \mathcal{A}_{k-1,i_{n-1}}$$
$$\int_{\prod_{j=i_1}^{i_{n-1}-1} \mathcal{V}_j} \mathbf{1}_{\{t_k>0\}} \mathbf{1}_{\{\text{ (2.94) holds for } j=i_1,\cdots,i_{n-2}\}} d\Upsilon_{i_1}^{i_{n-1}-1}.$$

Keep doing this computation until integrating over $\mathcal{V}_{i_1}$ we derive (2.95). □

Lemma 8 *Assume $t \leq t_*$(so that we can apply Lemma 3) satisfies (2.108) and (2.110). For $0 < \delta \ll 1$, we define*

$$\mathcal{V}_j^\delta := \{v_j \in \mathcal{V}_j : |v_j \cdot n(x_j)| > \delta, |v_j| \leq \delta^{-1}\}. \tag{2.97}$$

For the sequence $\{v_1, v_2, \cdots, v_{k-1}\}$, consider a subsequence $\{v_{l+1}, v_{l+2}, \cdots, v_{l+L}\}$ with $l + 1 < l + L \leq k - 1$ as following:

$$\underbrace{v_l}_{\in \mathcal{V}_l^{\frac{1-\eta}{2(1+\eta)}\delta}}, \underbrace{v_{l+1}, v_{l+2} \cdots v_{l+L},}_{all \in \mathcal{V}_{l+j} \backslash \mathcal{V}_{l+j}^{\frac{1-\eta}{2(1+\eta)}\delta}} \quad \underbrace{v_{l+L+1}}_{\in \mathcal{V}_{l+L+1}^{\frac{1-\eta}{2(1+\eta)}\delta}}. \tag{2.98}$$

In (2.98), if $L \geq 100\frac{1+\eta}{1-\eta}$, then we have

$$\int_{\prod_{j=l}^{k-1} \mathcal{V}_j} \mathbf{1}_{\{t_k>0\}} \mathbf{1}_{\{v_{l+j} \in \mathcal{V}_{l+j} \backslash \mathcal{V}_{l+j}^{\frac{1-\eta}{2(1+\eta)}\delta} \text{ for } 1 \leq j \leq L\}} d\Phi_{l,m}^{k,k-1}(t_k) \leq (3\delta)^{L/2} (2C_{T_M,\xi})^{2(k-l)} \mathcal{A}_{k-1,l}. \tag{2.99}$$

Here the η satisfies the condition (2.89).

Remark 7 In order to apply Lemma 6 we need to create the condition (2.90) and (2.92). This is the main reason that we consider the space $\mathcal{V}_j^{\frac{1-\eta}{2(1+\eta)}\delta}$.

This lemma asserts that implies that when L is large enough, such subsequence (2.98), without further considering the constraint for $|v_{i,\parallel} - \eta_{i,\parallel} V_\parallel^{m-i+1}(t_i)|$ for $l + 1 \leq i \leq l + L$ as (2.90),(2.76), provides a decay factor $(3\delta)^{L/2}$. Such decay factor is the key the obtain the decay factor $(\frac{1}{2})^{k_0}$ in Lemma 4. In fact in the proof we consider all possible cases for each $v_{i,\parallel}$ in the subsequence (2.98) and apply the estimates in Lemmas 5–7 to obtain the decay factor $(3\delta)^{L/2}$ for all cases. We will heavily rely on this lemma to prove Lemma 4.

Proof By the definition (2.97) we have

$$\mathcal{V}_i \backslash \mathcal{V}_i^{\frac{1-\eta}{2(1+\eta)}\delta} = \{v_i \in \mathcal{V}_i : |v_i \cdot n(x_i)| < \frac{1-\eta}{2(1+\eta)}\delta \text{ or } |v_i| \geq \frac{2(1+\eta)}{1-\eta}\delta^{-1}\}.$$

Here we summarize the result of Lemmas 5 and 6. With $\frac{1-\eta}{1+\eta}\delta < \delta$, when $v_i \in \mathcal{V}_i \backslash \mathcal{V}_i^{\frac{1-\eta}{2(1+\eta)}\delta}$

(1) When $|v_i \cdot n(x_i)| < \frac{1-\eta}{2(1+\eta)}\delta$, we have (2.75).
(2) When $|v_i| > \frac{2(1+\eta)}{1-\eta}\delta^{-1}$,
   (a) when $|v_{i,\parallel}| > \frac{1+\eta}{1-\eta}\delta^{-1}$, if $|v_{i,\parallel} - \eta_{i,\parallel} V_\parallel^{m-i+1}(t_i)| < \delta^{-1}$, then $|V_\parallel^{m-i+1}(t_i)| > |v_{i,\parallel}| + \delta^{-1}$.
   (b) when $|v_{i,\parallel}| > \frac{1+\eta}{1-\eta}\delta^{-1}$, if $|v_{i,\parallel} - \eta_{i,\parallel} V_\parallel^{m-i+1}(t_i)| \geq \delta^{-1}$, then we have (2.77).

(c) when $|v_{i,\perp}| > \frac{1+\eta}{1-\eta}\delta^{-1}$, if $|v_{i,\perp} - \eta_{i,\perp} V_{\perp}^{m-i+1}(t_i)| < \delta^{-1}$, then $|V_{\perp}^{m-i+1}(t_i)| > |v_{i,\perp}| + \delta^{-1}$.

(d) when $|v_{i,\perp}| > \frac{1+\eta}{1-\eta}\delta^{-1}$, if $|v_{i,\perp} - \eta_{i,\perp} V_{\perp}^{m-i+1}(t_i)| \geq \delta^{-1}$, then we have (2.79).

We define $\mathcal{W}_{i,\delta}$ as the space that provides the smallness:

$$\begin{aligned}\mathcal{W}_{i,\delta} := \{v_i \in \mathcal{V}_i : |v_{i,\perp}| < \frac{1-\eta}{2(1+\eta)}\delta\} \bigcup \{v_i \in \mathcal{V}_i : |v_{i,\perp}| > \frac{1+\eta}{1-\eta}\delta^{-1} \\ \text{and } |v_{i,\perp} - \eta_{i,\perp} V_{\perp}^{m-i+1}(t_i)| > \delta^{-1}\} \\ \bigcup \{v_i \in \mathcal{V}_i : |v_{i,\parallel}| > \frac{1+\eta}{1-\eta}\delta^{-1} \text{ and } |v_{i,\parallel} - \eta_{i,\parallel} V_{\parallel}^{m-i+1}(t_i)| > \delta^{-1}\}.\end{aligned}$$

Then we have

$$\begin{aligned}\mathcal{V}_i \backslash \mathcal{V}_i^{\frac{1-\eta}{2(1+\eta)}\delta} \subset & \mathcal{W}_{i,\delta} \bigcup \{v_{i,\perp} \in \mathcal{V}_{i,\perp} : |v_{i,\perp}| > \frac{1+\eta}{1-\eta}\delta^{-1} \text{ and } |v_{i,\perp} - \eta_{i,\perp} V_{\perp}^{m-i+1}(t_i)| < \delta^{-1}\} \\ & \bigcup \{v_{i,\parallel} \in \mathcal{V}_{i,\parallel} : |v_{i,\parallel}| > \frac{1+\eta}{1-\eta}\delta^{-1} \text{ and } |v_{i,\parallel} - \eta_{i,\parallel} V_{\parallel}^{m-i+1}(t_i)| < \delta^{-1}\}.\end{aligned} \tag{2.100}$$

By (2.75), (2.77) and (2.79) with $\frac{1-\eta}{1+\eta}\delta < \delta$, we obtain

$$\int_{\prod_{j=i}^{k-1} \mathcal{V}_j} \mathbf{1}_{\{v_i \in \mathcal{W}_{i,\delta}\}} \mathbf{1}_{\{t_k > 0\}} d\Phi_{i,m}^{k,k-1}(t_k) \leq 3\delta (2C_{T_M,\xi})^{2(k-i)} \mathcal{A}_{k-1,i}. \tag{2.101}$$

For the subsequence $\{v_{l+1}, \cdots, v_{l+L}\}$ in (2.98), when the number of $v_j \in \mathcal{W}_{j,\delta}$ is larger than $L/2$, by (2.95) in Lemma 7 with $n = L/2$ and replacing the condition (2.94) by $v_j \in \mathcal{W}_{j,\delta}$, we obtain

$$\begin{aligned}&\int_{\prod_{j=l}^{k-1} \mathcal{V}_j} \mathbf{1}_{\{\text{Number of } v_j \in \mathcal{W}_{j,\delta} \text{ is large than } L/2\}} \mathbf{1}_{\{t_k > 0\}} d\Phi_{l,m}^{k,k-1}(t_k) \\ &\leq (3\delta)^{L/2} (2C_{T_M,\xi})^{2(k-l_i)} \mathcal{A}_{k-1,l}.\end{aligned} \tag{2.102}$$

This finish the discussion with the cases (1),(2b),(2d). Then we focus on the cases (2a),(2c).

When the number of $v_j \notin \mathcal{W}_{j,\delta}$ is larger than $L/2$, by (2.100) we further consider two cases. The first case is that the number of $v_j \in \{v_j : |v_{j,\parallel}| > \frac{1+\eta}{1-\eta}\delta^{-1}$ and $|v_{j,\parallel} - \eta_{j,\parallel} V_{\parallel}^{m-j+1}(t_j)| < \delta^{-1}\}$ is larger than $L/4$. According to the relation of $v_{j,\parallel}$ and $V_{\parallel}^{m-j+1}(t_j)$, we categorize them into

Set1: $\{v_j \notin \mathcal{W}_{j,\delta} : |v_{j,\parallel}| > \frac{1+\eta}{1-\eta}\delta^{-1}$ and $|v_{j,\parallel} - \eta_{j,\parallel} V_{\parallel}^{m-j+1}(t_j)| < \delta^{-1}\}$.

Denote $M = |\text{Set1}|$ and the corresponding index in Set1 as $j = p_1, p_2, \cdots, p_M$. Then we have

$$L/4 \leq M \leq L. \tag{2.103}$$

By (2.91) in Lemma 6, for those v_{p_j}, we have

$$|v_{p_j,\parallel}| - |V_{\parallel}^{m-p_j+1}(t_{p_j})| < -\delta^{-1}. \tag{2.104}$$

Set2: $\{v_j \in \mathcal{V}_j \backslash \mathcal{V}_j^{\frac{1-\eta}{2(1+\eta)\delta}} : |v_{j,\parallel}| \geq |V_{\parallel}^{m-j+1}(t_j)|\}$.

Denote $\mathcal{M} = |\text{Set2}|$ and the corresponding index in Set2 as $j = q_1, q_2, \cdots, q_{\mathcal{M}}$. By (2.103) we have

$$1 \le \mathcal{M} \le L - M \le \frac{3}{4}L. \tag{2.105}$$

Then for those v_{q_j} we define

$$a_j := |v_{q_j,\|}| - |V_\|^{m-q_j+1}(t_{q_j})| > 0. \tag{2.106}$$

Set3: $\{v_j \in \mathcal{V}_j \backslash \mathcal{V}_j^{\frac{1-\eta}{2(1+\eta)\delta}} : |v_{j,\|}| \le |V_\|^{m-j+1}(t_j)| \le |v_{j,\|}| + \delta^{-1}\}$.

Denote $N = |\text{Set3}|$ and the corresponding index in Set3 as $j = o_1, o_2, \cdots, o_N$. Then for those o_j, we have

$$|v_{o_j,\|}| \le |V_\|^{m-o_j+1}(t_{o_j})| \le |v_{o_j,\|}| + \delta^{-1}. \tag{2.107}$$

From (2.98), we have $v_l \in \mathcal{V}_l^{\frac{1-\eta}{2(1+\eta)}\delta}$, thus we obtain

$$\begin{aligned} -\frac{2(1+\eta)}{1-\eta}\delta^{-1} < |v_{l+L,\|}| - |v_{l,\|}| &= \sum_{j=1}^{L}\left(|v_{l+j,\|}| - |v_{l+j-1,\|}|\right) \\ &= \sum_{j=1}^{L}\left(|v_{l+j,\|}| - |V_\|^{m-(l+j)+1}(t_{l+j})|\right) \\ &\quad + \sum_{j=1}^{L}\left(|V_\|^{m-(l+j)+1}(t_{l+j})| - |v_{l+j-1,\|}|\right) \\ &\le \sum_{j=1}^{L}\left(|v_{l+j,\|}| - |V_\|^{m-(l+j)+1}(t_{l+j})|\right) + \sum_{j=1}^{L} C_{\phi^m}(t_{l+j-1} - t_{l+j}), \end{aligned}$$

where C_{ϕ^m} is defined in (2.18). Take $t = t(\phi^m)$ small enough such that

$$\sum_{j=1}^{L+1} C_{\phi^m}(t_{l+j-1} - t_{l+j}) \le C_{\phi^m} t \le 1. \tag{2.108}$$

By (2.104), (2.106) and (2.107), we derive that

$$\begin{aligned} \frac{-2(1+\eta)}{1-\eta}\delta^{-1} - 1 < &\sum_{j=1}^{M}\left(|v_{p_j,\|}| - |V_\|^{m-p_j+1}(t_{p_j})|\right) + \sum_{j=1}^{\mathcal{M}}\left(|v_{q_j,\|}| - |V_\|^{m-q_j+1}(t_{q_j})|\right) \\ &+ \sum_{j=1}^{N}\left(|v_{o_j,\|}| - |V_\|^{m-o_j+1}(t_{o_j})|\right) \le -M\delta^{-1} + \sum_{j=1}^{\mathcal{M}} a_j. \end{aligned}$$

Therefore, by $L \ge 100\frac{1+\eta}{1-\eta}$ and (2.103), we obtain

$$\frac{2(1+\eta)}{1-\eta}\delta^{-1} + 1 \le \frac{L}{10}\delta^{-1} \le \frac{M}{2}\delta^{-1}$$

and thus

$$\sum_{j=1}^{\mathcal{M}} a_j \geq M\delta^{-1} - \frac{2(1+\eta)}{1-\eta}\delta^{-1} - 1 > \frac{M\delta^{-1}}{2}. \tag{2.109}$$

We focus on integrating over $\mathcal{V}_{q_i}$ with $1 \leq i \leq \mathcal{M}$, those index satisfy (2.106). We consider the third line of (2.80) with $i = q_i$ and with integrating over $\{v_{q_i,\parallel} \in \mathcal{V}_{q_i,\parallel} : |v_{q_i,\parallel}| - |V_\parallel^{m-q_j+1}(t_{q_j})| = a_i\}$. To apply (6.7) in Lemma 16, we set

$$a = -\frac{1}{2T_{k-1,q_i}} + \frac{1}{2T_w(x_{q_i})}, \quad b = \frac{1}{2T_w(x_{q_i})r_\parallel(2-r_\parallel)}, \quad \varepsilon = 2(2\mathcal{C})^{k-1-q_i}(\mathfrak{C}t).$$

We take $t = t(\xi, k, T_M, \mathcal{C}, \mathfrak{C})$ small enough such that

$$a + \varepsilon - b = -\frac{1}{2T_{k-1,q_i}} + \frac{1}{2T_w(x_{q_i})} - \frac{1}{2T_w(x_{q_i})r_\parallel(2-r_\parallel)} + 2(2\mathcal{C})^{k-1-q_i}(\mathfrak{C}t) < -\frac{1}{4T_M}. \tag{2.110}$$

Then we use $\eta_{q_i,\parallel} < 1$ to obtain

$$\mathbf{1}_{\{|v_{q_i,\parallel}|-|V_\parallel^{m-q_i+1}(t_{q_i})|=a_i\}} \leq \mathbf{1}_{\{|v_{q_i,\parallel}|-\eta_{q_i,\parallel}|V_\parallel^{m-q_i+1}(t_{q_i})|>a_i\}} \leq \mathbf{1}_{\{|v_{q_i,\parallel}-\eta_{q_i,\parallel}V_\parallel^{m-q_i+1}(t_{q_i})|>a_i\}}. \tag{2.111}$$

By (6.7) in Lemma 16 and (2.111), we apply (2.68) with $q = q_i$ to bound the third line of (2.80)(the integration over $\mathcal{V}_{q_i,\parallel}$) by

$$e^{-\frac{a_i^2}{4T_M}} C_{T_M,\xi} \exp\bigg(\Big[\frac{[T_{k-1,q_i} - T_w(x_{q_i})][1-r_{min}]}{2T_w(x_{q_i})[T_{k-1,q_i}(1-r_{min}) + r_{min}T_w(x_{q_i})]} + 2(2\mathcal{C})^{k-q_i}(\mathfrak{C}t)\Big]|V_\parallel^{m-q_j+1}(t_{q_i})|^2\bigg). \tag{2.112}$$

Hence by the constant in (2.112) we draw a similar conclusion as (2.101):

$$\int_{\prod_{j=q_i}^{k-1}\mathcal{V}_j} \mathbf{1}_{\{t_k>0\}}\mathbf{1}_{\{|v_{q_i,\parallel}|-|V_\parallel^{m-q_i+1}(t_{q_i})|=a_i\}} d\Phi_{q_i,m}^{k,k-1}(t_k) \leq e^{-\frac{a_i^2}{4T_M}}(2C_{T_M,\xi})^{2(k-q_i)}\mathcal{A}_{k-1,q_i}. \tag{2.113}$$

Therefore, by Lemma 7, after integrating over $\mathcal{V}_{q_1,\parallel}, \mathcal{V}_{q_2,\parallel}, \cdots, \mathcal{V}_{q_\mathcal{M},\parallel}$ we obtain an extra constant

$$\begin{aligned} e^{-[a_i^2+a_2^2+\cdots+a_\mathcal{M}^2]/4T_M} &\leq e^{-[a_i+a_2+\cdots+a_\mathcal{M}]^2/(4T_M\mathcal{M})} \leq e^{-[M\delta^{-1}/2]^2/(4T_M\mathcal{M})} \\ &\leq e^{-[\frac{L}{8}\delta^{-1}]^2/(4T_M\frac{3}{4}L)} \leq e^{-\frac{1}{96T_M}L(\delta^{-1})^2} \leq e^{-L\delta^{-1}}, \end{aligned}$$

where we have used (2.109) in the last step of first line, (2.103), (2.105) in the first step of second line and take $\delta \ll 1$ in the last step of second line. Then $e^{-L\delta^{-1}}$ is smaller than $(3\delta)^{L/2}$ in (2.102) and we conclude

$$\int_{\prod_{j=l}^{k-1}\mathcal{V}_j} \mathbf{1}_{\{M=|\text{Set1}|\geq L/4\}}\mathbf{1}_{\{t_k>0\}} d\Phi_{l,m}^{k,k-1}(t_k) \leq (3\delta)^{L/2}(2C_{T_M,\xi})^{2(k-l_i)}\mathcal{A}_{k-1,l}. \tag{2.114}$$

The second case is that the number of $v_j \in \{v_j \notin \mathcal{W}_{j,\delta} : |v_{j,\perp}| > \frac{1+\eta}{1-\eta}\delta^{-1}\}$ is larger than $L/4$. We categorize $v_{j,\perp}$ into

Set4: $\{v_j \notin \mathcal{W}_{j,\delta} : |v_{j,\perp}| > \frac{1+\eta}{1-\eta}\delta^{-1}$ and $|v_{j,\perp} - \eta_{j,\perp} V_\perp^{m-j+1}(t_j)| < \delta^{-1}\}$.

Set5: $\{v_j \in \mathcal{V}_j \backslash \mathcal{V}_j^{\frac{1-\eta}{2(1+\eta)\delta}} : |v_{j,\perp}| > |V_\perp^{m-j+1}(t_j)|\}$.

Set6: $\{v_j \in \mathcal{V}_j \backslash \mathcal{V}_j^{\frac{1-\eta}{2(1+\eta)\delta}} : |v_{j,\perp}| \le |V_\perp^{m-j+1}(t_j)| \le |v_{j,\perp}| + \delta^{-1}\}$.

Denote $|\text{Set4}| = M_1$ with $L/4 \le M_1 \le L$ and the corresponding index as $p'_1, p'_2, \cdots, p'_{M_1}$, $|\text{Set5}| = \mathcal{M}_1$ and the corresponding index as $q'_1, q'_2, \cdots, q'_{\mathcal{M}_1}$, $|\text{Set6}| = N_1$ and the corresponding index as $o'_1, o'_2, \cdots, o'_{N_1}$. Also define $b_j := |v_{q'_j,\perp}| - |V_\perp^{m-q'_j+1}(t_{q'_j})|$. By the same computation as (2.109), we have

$$\sum_{j=1}^{\mathcal{M}_1} b_j \ge M_1\delta^{-1} - \frac{2(1+\eta)}{1-\eta}\delta^{-1} > \frac{M_1\delta^{-1}}{2}.$$

We focus on the integration over $v_{q'_j}$. Let $1 \le i \le \mathcal{M}_1$, we consider the second line of (2.80) with $i = q'_i$ and with integrating over $\{v_{q'_i,\perp} \in \mathcal{V}_{q'_i,\perp} : |v_{q'_i,\perp}| - |V_\perp^{m-q'_i+1}(t_{q'_i})| = b_i\}$. To apply (6.12) in Lemma 16, we set

$$a = -\frac{1}{2T_{k-1,q'_i}} + \frac{1}{2T_w(x_{q'_i})}, \quad b = \frac{1}{2T_w(x_{q'_i})r_\perp}, \quad \varepsilon = 2(2\mathcal{C})^{k-q'_i-1}(\mathfrak{C}t).$$

By the same computation as (2.110), we have $a + \varepsilon - b < -\frac{1}{4T_M}$. Similarly to (2.111), we have

$$\mathbf{1}_{\{|v_{q'_i,\perp}| - |V_\perp^{m-q'_i+1}(t_{q'_i})| = b_i\}} \le \mathbf{1}_{\{|v_{q'_i,\perp} - \eta_{q'_i,\perp} V_\perp^{m-q'_i+1}(t_{q'_i})| > b_i\}}.$$

Hence by (6.12) in Lemma 18 and applying (2.69), we bound the integration over $\mathcal{V}_{q'_i,\perp}$ by

$$e^{-\frac{b_i^2}{16T_M}} C_{T_M,\xi}$$
$$\exp\left(\left[\frac{[T_{k-1,q'_i} - T_w(x_{q'_i})][1-r_{min}]}{2T_w(x_{q'_i})[T_{k-1,q'_i}(1-r_{min}) + r_{min}T_w(x_{q'_i})]} + (2\mathcal{C})^{k-q'_i}(\mathfrak{C}t)\right] |V_\perp^{m-q'_i+1}(t_{q'_i})|^2\right).$$

Therefore,

$$\int_{\prod_{j=q'_i}^{k-1} \mathcal{V}_j} \mathbf{1}_{\{t_k>0\}} \mathbf{1}_{\{|v_{q'_i,\perp}| - |V_\perp^{m-q'_i+1}(t_{q'_i})| = b_i\}} d\Phi_{q'_i,m}^{k,k-1}(t_k) \le e^{-\frac{b_i^2}{16T_M}} (C_{T_M,\xi})^{2(k-q'_i)} \mathcal{A}_{k-1,q'_i}.$$

The integration over $\mathcal{V}_{q'_1,\perp}, \mathcal{V}_{q'_2,\perp}, \cdots, \mathcal{V}_{q'_{\mathcal{M}_1},\perp}$ provides an extra constant

$$e^{-[b_1^2+b_2^2+\cdots+b_{\mathcal{M}_1}^2]/16T_M} \le e^{-\frac{1}{400T_M}L(\delta^{-1})^2} \le e^{-L\delta^{-1}},$$

where we set $\delta \ll 1$ in the last step. Then $e^{-L\delta^{-1}}$ is smaller than $(3\delta)^{L/2}$ in (2.102) and we conclude

$$\int_{\prod_{j=l}^{k-1} \mathcal{V}_j} \mathbf{1}_{\{M_1 = |\text{Set4}| \ge L/4\}} \mathbf{1}_{\{t_k>0\}} d\Phi_{l,m}^{k,k-1}(t_k) \le (3\delta)^{L/2} (2C_{T_M,\xi})^{2(k-l)} \mathcal{A}_{k-1,l}. \tag{2.115}$$

Finally collecting (2.102), (2.114) and (2.115) we derive the lemma. □

Now we prove the Lemma 4.

Proof of Lemma 4 We mainly apply Lemma 8 during the proof. In order to apply Lemma 8, here we consider the space $\mathcal{V}_i^{\frac{1-\eta}{2(1+\eta)}\delta}$ and ensure η satisfy the condition (2.89). Also we let $t' = t'(\xi, k, T_M, \mathcal{C}, \mathfrak{C}, C_{\phi^m})$ (consistent with (2.72)) satisfy condition (2.108) and (2.110).

In the proof we first construct the η that satisfies the condition (2.89) in Step 1. Then we prove there can be at most finite number of $v_j \in \mathcal{V}\backslash\mathcal{V}_j^{\frac{1-\eta}{2(1+\eta)}\delta}$ in Step 2. With such conclusion in Step 2, we apply Lemma 8 and consider the contribution of all possible subsequence (2.98) in Step 3. In Step 4 we conclude the lemma.

Step 1

In this step we mainly focus on constructing the η, which is defined in (2.126).

First we consider $\eta_{i,\parallel}$, which is defined in (2.85). In regard to (2.82) and (2.83), by (2.110) with $t \le t'$,

$$B_{i,\parallel} \ge \frac{1}{2T_{k-1,i}} - 2(2\mathcal{C})^{k-1-i}t \ge \frac{1}{2^{\frac{2\xi}{\xi+1}}T_M} - (2\mathcal{C})^k(\mathfrak{C}t) \ge \frac{1}{4T_M}. \tag{2.116}$$

By (2.41), $T_{k-1,i} \to T_M$ as $k - i \to \infty$. For any $\varepsilon_1 > 0$, there exists k_1 s.t when

$$k \ge k_1, \quad i \le k/2, \text{ we have } T_{k-1,i} \le (1+\varepsilon_1)T_M. \tag{2.117}$$

Moreover, by (1.28), there exists ε_2 s.t

$$\frac{\min\{T_w(x)\}}{T_M} > \frac{1-r_\parallel}{2-r_\parallel}(1+\varepsilon_2). \tag{2.118}$$

Then we have

$$\varepsilon_2 = \varepsilon_2(\min\{T_w(x)\}, T_M, r_\parallel, r_\perp). \tag{2.119}$$

We use (2.117) and (2.118) to bound $T_w(x_i)$ in the $\eta_{i,\parallel}$(defined in (2.85)) below as

$$T_w(x_i) = T_{k-1,i}\frac{T_w(x_i)}{T_{k-1,i}} \ge T_{k-1,i}\frac{T_w(x_i)}{T_M}\frac{1}{1+\varepsilon_1} > \frac{1-r_\parallel}{2-r_\parallel}T_{k-1,i}\frac{1+\varepsilon_2}{1+\varepsilon_1}. \tag{2.120}$$

Thus we obtain

$$\eta_{i,\parallel} < \frac{1+2\frac{(2\mathcal{C})^k(\mathfrak{C}t)}{B_{i,\parallel}}}{(1-r_\parallel)^2 + \frac{1-r_\parallel}{2-r_\parallel}\frac{1+\varepsilon_2}{1+\varepsilon_1}r_\parallel(2-r_\parallel)}(1-r_\parallel) = \frac{1+\frac{(2\mathcal{C})^k(\mathfrak{C}t)}{B_{i,\parallel}}}{1-r_\parallel+r_\parallel\frac{1+\varepsilon_2}{1+\varepsilon_1}}. \tag{2.121}$$

By (2.117), we take

$$k = k_1 = k_1(\varepsilon_2, T_M, r_{\min}) \tag{2.122}$$

large enough such that $\varepsilon_1 < \varepsilon_2/4$. By (2.116) and (2.121), we derive that when $k = k_1$,

$$\sup_{i\le k/2} \eta_{i,\parallel} \le \frac{1+4T_M(2\mathcal{C})^k(\mathfrak{C}t)}{1-r_\parallel+r_\parallel\frac{1+\varepsilon_2}{1+\varepsilon_2/4}} < \eta_\parallel < 1. \tag{2.123}$$

Here we define

$$\eta_\parallel := \frac{1}{1-r_\parallel+r_\parallel\frac{1+\varepsilon_2}{1+\varepsilon_2/2}} < 1. \tag{2.124}$$

where we take $t' = t'(k, T_M, \varepsilon_2, \mathcal{C}, \mathfrak{C}, r_{\|})$ small enough and $t \leq t'$ such that $4T_M(2\mathcal{C})^k(\mathfrak{C}t) \ll 1$ to ensure the second inequality in (2.123). Combining (2.119) and (2.122), we conclude the t' we choose here only depends on the parameter in (2.72).

Then we consider $\eta_{i,\perp}$ which is defined in (2.88). In regard to (2.86) and (2.87), by (2.116) we have $B_{i,\perp} \geq \frac{1}{4T_M}$. By $\frac{\min\{T_w(x)\}}{T_M} > \frac{\sqrt{1-r_\perp}-(1-r_\perp)}{r_\perp}$ in (1.28) we can use the same computation as (2.120) to obtain

$$T_w(x_i) > \frac{\sqrt{1-r_\perp}-(1-r_\perp)}{r_\perp} T_{k-1,i} \frac{1+\varepsilon_2}{1+\varepsilon_1},$$

with $\varepsilon_1 < \varepsilon_2/4$. Thus we obtain

$$\eta_{i,\perp} < \eta_\perp < 1.$$

where we define

$$\eta_\perp := \frac{1}{\sqrt{1-r_\perp} + (1-\sqrt{1-r_\perp})\frac{1+\varepsilon_2}{1+\varepsilon_2/2}} < 1, \tag{2.125}$$

with $t' = t'(k, T_M, \varepsilon_2, \mathcal{C}, \mathfrak{C}, r_{\|})$(consistent with (2.72)) small enough and $t \leq t'$.

Finally we define

$$\eta := \max\{\eta_\perp, \eta_{\|}\} < 1. \tag{2.126}$$

Step 2

We claim that for $t \ll 1$,

$$|t_j - t_{j+1}| \gtrsim \left(\frac{1-\eta}{2(1+\eta)}\delta\right)^3, \text{ for } v_j \in \mathcal{V}_j^{\frac{1-\eta}{2(1+\eta)}\delta}. \tag{2.127}$$

For $t_j \leq 1$,

$$\begin{aligned}
&\left|\int_{t_j}^{t_{j+1}} V^{m-j}(s; t_j, x_j, v_j)ds\right|^2 \\
&= |x_{j+1} - x_j|^2 \gtrsim |(x_{j+1} - x_j) \cdot n(x_j)| \\
&= \left|\int_{t_j}^{t_{j+1}} V^{m-j}(s; t_j, x_j, v_j) \cdot n(x_j)ds\right| \\
&= \left|\int_{t_j}^{t_{j+1}} \left(v_j - \int_{t_j}^{s} \nabla\phi^{m-j}(\tau, X(\tau; t_j, x_j, v_j))d\tau\right) \cdot n(x_j)ds\right| \\
&\geq |v_j \cdot n(x_j)||t_j - t_{j+1}| - \left|\int_{t_j}^{t_{j+1}} \int_{t_j}^{s} \nabla\phi^{m-j}(\tau, X(\tau; t_j, x_j, v_j))d\tau) \cdot n(x_j)ds\right|.
\end{aligned}$$

Here we have used the fact that if $x, y \in \partial\Omega$ and $\partial\Omega$ is C^2 and Ω is bounded then $|x-y|^2 \gtrsim_\Omega |(x-y)\cdot n(x)|$(see the proof in [8] and [9]). Thus

$$\begin{aligned}|v_j \cdot n(x_j)| &\lesssim \frac{1}{|t_j - t_{j+1}|}\Big|\int_{t_j}^{t_{j+1}} V(s;t_j,x_j,v_j)ds\Big|^2 \\ &\quad + \frac{1}{|t_j - t_{j+1}|}\Big|\int_{t_j}^{t_{j+1}}\int_{t_j}^{s} \nabla\phi^{m-j}(\tau, X(\tau;t_j,x_j,v_j))d\tau)\cdot n(x_j)ds\Big| \\ &\lesssim |t_j - t_{j+1}|\{|v_j|^2 + |t_j - t_{j+1}|^3\|\nabla\phi^{m-j}\|_\infty^2 \\ &\quad + \frac{1}{2}\sup_{t_{j+1}\le\tau\le t_j}|\nabla\phi^{m-j}(\tau, X(\tau;t_j,x_j,v_j))\cdot n(x_j)|\}.\end{aligned}$$

Since $v_j \in \mathcal{V}_j^{\frac{1-\eta}{2(1+\eta)}\delta}$,

$$|v_j\cdot n(x_j)| \lesssim |t_j - t_{j+1}|\{\delta^{-2} + t^3\|\nabla\phi^{m-j}\|_\infty^2 + \|\nabla\phi^{m-j}\|_\infty\}. \tag{2.128}$$

By $t \ll 1$ and $\|\nabla\phi^j\|_\infty$ is bounded due to Lemma 1, we can prove (2.127).

In consequence, when $t_k > 0$, by (2.127) and $t \ll 1$, there can be at most $\{[C_\Omega(\frac{2(1+\eta)}{(1-\eta)\delta})^3]+1\}$ numbers of $v_j \in \mathcal{V}_j^{\frac{1-\eta}{2(1+\eta)}\delta}$. Equivalently there are at least $k - [C_\Omega(\frac{2(1+\eta)}{(1-\eta)\delta})^3]$ numbers of $v_j \in \mathcal{V}_j \backslash \mathcal{V}_j^{\frac{1-\eta}{2(1+\eta)}\delta}$.

Step 3

In this step we combine Step 1 and Step 2 and focus on the integration over $\prod_{j=1}^{k-1}\mathcal{V}_j$.

By (2.127) in Step 2, we define

$$N := \Big[C_\Omega\big(\frac{2(1+\eta)}{\delta(1-\eta)}\big)^3\Big] + 1. \tag{2.129}$$

For the sequence $\{v_1, v_2, \cdots, v_{k-1}\}$, suppose there are p number of $v_j \in \mathcal{V}_j^{\frac{1-\eta}{2(1+\eta)}\delta}$ with $p \le N$, we conclude there are at $\binom{k-1}{p}$ number of these sequences. Below we only consider a single sequence of them.

In order to get (2.124),(2.125)< 1, we need to ensure the condition (2.117). Thus we take $k = k_1(T_M, \xi, r_\perp, r_\parallel)$ and only use the decomposition $\mathcal{V}_j = \Big(\mathcal{V}_j\backslash\mathcal{V}_j^{\frac{1-\eta}{2(1+\eta)}\delta}\Big) \cup \mathcal{V}_j^{\frac{1-\eta}{2(1+\eta)}\delta}$ for $1 \le j \le k/2$. Thus we only consider the half sequence $\{v_1, v_2, \cdots, v_{k/2}\}$. We derive that when $t_k > 0$, there are at most N number of $v_j \in \mathcal{V}_j^{\frac{1-\eta}{2(1+\eta)}\delta}$ and at least $k/2 - N$ number of $v_j \in \mathcal{V}_j\backslash\mathcal{V}_j^{\frac{1-\eta}{2(1+\eta)}\delta}$ in $\prod_{j=1}^{k/2}\mathcal{V}_j$.

In this single half sequence $\{v_1, \cdots, v_{k/2}\}$, in order to apply Lemma 8, we only want to consider the subsequence (2.98) with $l+1 < l+L \le k/2$ and $L \ge 100\frac{1+\eta}{1-\eta}$. Thus we need to ignore those subsequence with $L < 100\frac{1+\eta}{1-\eta}$. By (2.98) one can see at the end of this subsequence, it is adjacent to a $v_l \in \mathcal{V}_l^{\frac{1-\eta}{2(1+\eta)}\delta}$. By (2.129), we conclude

$$\textit{There are at most } N \textit{ number of subsequences (2.98) with} L \le 100\frac{1+\eta}{1-\eta}. \tag{2.130}$$

We ignore these subsequences. Then we define the parameters for the remaining subsequence(with $L \geq 100\frac{1+\eta}{1-\eta}$) as:

$$M_1 := \text{the number of } v_j \in \mathcal{V}_j \backslash \mathcal{V}_j^{\frac{1-\eta}{2(1+\eta)}\delta} \text{ in the first subsequence starting from } v_1.$$
$$n := \text{the number of these subsequences.}$$

Similarly we can define $M_2, M_3, \cdots, M_n$ as the number in the second, third, $\cdots$, n-th subsequence. Recall that we only consider $\prod_{j=1}^{k/2} \mathcal{V}_j$, thus we have

$$100\frac{1+\eta}{1-\eta} \leq M_i \leq k/2, \text{ for } 1 \leq i \leq n. \tag{2.131}$$

By (2.130), we obtain

$$k/2 \geq M_1 + \cdots M_n \geq k/2 - 100\frac{1+\eta}{1-\eta}N. \tag{2.132}$$

Take M_i with $1 \leq i \leq n$ as an example. Suppose this subsequence starts from v_{l_i+1} to $v_{l_i+M_i}$, by (2.99) in Lemma 8 with replacing l by l_i and L by M_i, we obtain

$$\begin{aligned}\int_{\prod_{j=l_i}^{k-1} \mathcal{V}_j} &\mathbf{1}_{\{t_k>0\}} \mathbf{1}_{\{v_{l_i+j} \in \mathcal{V}_{l_i+j} \backslash \mathcal{V}_{l_i+j}^{\frac{1-\eta}{2(1+\eta)}\delta} \text{ for } 1 \leq j \leq M_i\}} d\Phi_{l_i,m}^{k,k-1}(t_k) \\ &\leq (3\delta)^{M_i/2} (2C_{T_M,\xi})^{2(k-l)} \mathcal{A}_{k-1,l_i}.\end{aligned} \tag{2.133}$$

Since (2.133) holds for all $1 \leq i \leq n$, by Lemma 7 we can draw the conclusion for the Step 3 as following. For a single sequence $\{v_1, v_2, \cdots, v_{k-1}\}$, when there are p number $v_j \in \mathcal{V}_j^{\frac{1-\eta}{2(1+\eta)}\delta}$, we have

$$\begin{aligned}\int_{\prod_{j=1}^{k-1} \mathcal{V}_j} &\mathbf{1}_{\{p \text{ number } v_j \in \mathcal{V}_j^{\frac{1-\eta}{2(1+\eta)}\delta} \text{ for a single sequence}\}} \mathbf{1}_{\{t_k>0\}} d\Sigma_{k-1,m}^k(t_k) \\ &\leq (3\delta)^{(M_1+\cdots+M_n)/2} (2C_{T_M,\xi})^{2k} \mathcal{A}_{k-1,1}.\end{aligned} \tag{2.134}$$

Step 4
Now we are ready to prove the lemma. By (2.129), we have

$$\int_{\prod_{j=1}^{k-1} \mathcal{V}_j} \mathbf{1}_{\{t_k>0\}} d\Sigma_{k-1,m}^k(t_k) \leq \sum_{p=1}^{N} \int_{\{\text{Exactly } p \text{ number of } v_j \in \mathcal{V}_j^{\frac{1-\eta}{2(1+\eta)}\delta}\}} \mathbf{1}_{\{t_k>0\}} d\Sigma_{k-1,m}^k(t_k). \tag{2.135}$$

Since (2.134) holds for a single sequence, we derive

$$\begin{aligned}(2.135) &\leq (2C_{T_M,\xi})^{2k} \sum_{p=1}^{N} \binom{k-1}{p} (3\delta)^{(M_1+M_2+\cdots M_n)/2} \mathcal{A}_{k-1,1} \\ &\leq (2C_{T_M,\xi})^{2k} N(k-1)^N (3\delta)^{k/4-101\frac{1+\eta}{1-\eta}N} \mathcal{A}_{k-1,1},\end{aligned} \tag{2.136}$$

where we use (2.132) in the second line.

Take $k = N^3$, the coefficient in (2.136) is bounded by

$$(2C_{T_M,\xi})^{2N^3} N^{3N+1} (3\delta)^{N^3/4-101\frac{1+\eta}{1-\eta}N} \leq (2C_{T_M,\xi})^{2N^3} N^{4N} (3\delta)^{N^3/5}, \tag{2.137}$$

where we choose $N = N(\eta)$ large such that $N^3/4 - 101\frac{1+\eta}{1-\eta}N \geq N^3/5$.

Using (2.129), we derive

$$3\delta = C(\Omega,\eta)N^{-1/3}.$$

Finally we bound (2.137) by

$$\begin{aligned}&(2C_{T_M,\xi})^{2N^3}N^{4N}(C(\Omega,\eta)N^{-1/3})^{N^3/5}\\ &\leq e^{2N^3\log(2C_{T_M,\xi})}e^{4N\log N}e^{(N^3/5)\log(C(\Omega,\eta)N^{-1/3})}\\ &= e^{4N\log N}e^{(N^3/5)(log(C(\Omega,\eta))-\frac{1}{3}\log N)}e^{2N^3\log(2C_{T_M,\xi})}\\ &= e^{4N\log N-\frac{N^3}{15}(\log N-3\log C_{\Omega,\eta}-30\log(2C_{T_M,\xi}))}\\ &\leq e^{4N\log N-\frac{N^3}{30}\log N}\leq e^{-\frac{N^3}{50}\log N}=e^{-\frac{k}{150}\log k}\leq(\frac{1}{2})^k,\end{aligned}$$

where we choose δ small enough in the second line such that $N = N(\Omega,\eta,C_{T_M,\xi})$ is large enough to satisfy

$$\begin{aligned}&\log N - 3\log C(\Omega,\eta) - 30\log(2C_{T_M,\xi}) \geq \frac{\log N}{2},\\ &4N\log N - \frac{N^3}{30}\log N \leq -\frac{N^3}{50}\log N.\end{aligned}$$

And thus we choose $k = N^3 = k_2 = k_2(\Omega,\eta,C_{T_M,\xi})$ and we also require $\log k > 150$ in the last step. Then we get (2.73).

Therefore, by the condition (2.117), eventually we choose $k = k_0 = \max\{k_1,k_2\}$. By the definition of η (2.126) with (2.124) and (2.125), we obtain $\eta = \eta(T_M,\mathcal{C},r_\perp,r_\parallel,\varepsilon_2)$. Thus by (2.119) and (2.122), we conclude the k_0 we choose here does not depend on t and only depends on the parameter in (2.71). We derive the lemma. □

Now we are ready to prove the Proposition 2, we will combine Lemmas 2– 4 to close the estimate.

2.3 Proof of Proposition 2

Proof of Proposition 2 First we take

$$t_\infty \leq t'. \tag{2.138}$$

with t' defined in (2.72). Then we let $k = k_0$ with k_0 defined in (2.71) so that we can apply Lemmas 4 and 3. Define the constant in (2.5) as

$$C_\infty = 3(2C_{T_M,\xi})^{k_0}, \tag{2.139}$$

where $C_{T_M,\xi}$ is defined in (2.56), k_0 in defined in (2.71).

We mainly use the formula given in Lemma 2 and we use Lemmas 3, 4 to control every term in (2.13). We consider two cases.

Case1: $t_1 \leq 0$,

We consider (2.11) in Lemma 2. Since

$$e^{-\int_s^t\frac{\mathfrak{C}}{2}\langle V^m(\tau)\rangle^2d\tau}\leq e^{\frac{\mathfrak{C}}{2}(s-t)\langle v\rangle^2}e^{\mathfrak{C}C_{\phi^m}(t-s)^2\langle v\rangle},$$

by (2.11) and using the definition of $\Gamma^m_{\text{gain}}(s)$ in (2.15) we have

$$|h^{m+1}(t,x,v)| \le |h_0(X^m(0;t,x,v),V^m(0;t,x,v))| \tag{2.140}$$

$$+\int_0^t e^{\frac{\mathfrak{C}}{2}(s-t)\big(\langle v\rangle^2-2C_{\phi^m}(t-s)\langle v\rangle\big)}e^{-\mathfrak{C}\langle V^m(s)\rangle^2 s}e^{\theta|V^m(s)|^2}$$

$$\int_{\mathbb{R}^3\times\mathbb{S}^2} B(V^m(s)-u,w)\sqrt{\mu(u)} \tag{2.141}$$

$$+\Big|\frac{h^m(t,x,v)}{e^{-\mathfrak{C}\langle v\rangle^2 t+\theta|v|^2}}\Big(s,X^m(s),u'\big(u,V^m(s)\big)\Big)\Big|\Big|\frac{h^m(t,x,v)}{e^{-\mathfrak{C}\langle v\rangle^2 t+\theta|v|^2}}$$

$$\Big(s,X^m(s),v'\big(u,V^m(s)\big)\Big)\Big|d\omega du ds, \tag{2.142}$$

where $u'\big(u,V^m(s)\big)$ and $v'\big(u,V^m(s)\big)$ are defined by (1.4). Then we have

$$\begin{aligned}
(2.142) &\le (\sup_{0\le s\le t}\|h^m(s)\|_{L^\infty})^2\times\int_0^t\int_{\mathbb{R}^3\times\mathbb{S}^2} e^{\frac{\mathfrak{C}}{2}(s-t)\big(\langle v\rangle^2-2C_{\phi^m}(t-s)\langle v\rangle\big)}B\big(V^m(s)-u,w\big)\\
&\quad\times\sqrt{\mu(u)}e^{-\mathfrak{C}\langle V^m(s)\rangle^2 s}e^{\theta|V^m(s)|^2}e^{-\theta(|u|^2+|V^m(s)|^2)}e^{\mathfrak{C}(\langle u\rangle^2+\langle V^m(s)\rangle^2)s}d\omega du ds\\
&\lesssim (\sup_{0\le s\le t}\|h^m(s)\|_{L^\infty})^2\int_0^t\int_{\mathbb{R}^3} e^{\frac{\mathfrak{C}}{2}(s-t)\big(\langle v\rangle^2-2C_{\phi^m}(t-s)\langle v\rangle\big)}|V^m(s)\\
&\quad -u|^{\mathcal{K}}\sqrt{\mu}e^{-\theta|u|^2}e^{\mathfrak{C}\langle u\rangle^2 s}du ds\\
&\lesssim_{C_\infty}\|h_0\|^2_{L^\infty}\int_0^t e^{\frac{\mathfrak{C}}{2}(s-t)\big(\langle v\rangle^2-2C_{\phi^m}(t-s)\langle v\rangle\big)}\langle V^m(s)\rangle^{\mathcal{K}}ds\\
&\lesssim \|h_0\|^2_{L^\infty}\int_0^t e^{\frac{\mathfrak{C}}{2}(s-t)\big(\langle v\rangle^2-2C_{\phi^m}(t-s)\langle v\rangle\big)}\big(\langle v\rangle^{\mathcal{K}}+(t-s)^{\mathcal{K}}\big)ds\\
&\le \|h_0\|^2_{L^\infty}\int_0^t e^{\frac{\mathfrak{C}}{2}(s-t)\big(\langle v\rangle^2-2C_{\phi^m}(t-s)\langle v\rangle\big)}\big(\langle v\rangle^{\mathcal{K}}+1\big)\{\mathbf{1}_{|v|>N}+\mathbf{1}_{|v|\le N}\}ds\\
&\lesssim \|h_0\|^2_{L^\infty}\Big[\int_0^t e^{\frac{\mathfrak{C}}{4}(s-t)\langle v\rangle^2}\langle v\rangle^{\mathcal{K}}\mathbf{1}_{|v|>N}ds+\int_0^t\langle v\rangle^{\mathcal{K}}\mathbf{1}_{|v|>N}\Big]\\
&\lesssim_{\|h_0\|_\infty}\big(\frac{1}{N^2}+Nt\big),
\end{aligned}$$

where $0\le\mathcal{K}\le 1$. Therefore, we obtain

$$(2.142)\le C(C_\infty,\|h_0\|_\infty)(\frac{1}{N^2}+Nt)\le\frac{1}{k_0}\|h_0\|_\infty, \tag{2.143}$$

where we choose

$$N=N(C_\infty,\|h_0\|_\infty,k_0)\gg 1,\quad t_\infty=t_\infty(N,C_\infty,\|h_0\|_\infty,k_0)\ll 1, \tag{2.144}$$

with $t\le t_\infty$ to obtain the last inequality in (2.143).

Finally collecting (2.140) and (2.142) we obtain

$$\|h^{m+1}(t,x,v)\mathbf{1}_{\{t_1\le 0\}}\|_\infty\le 2\|h_0\|_\infty\le C_\infty\|h_0\|_\infty, \tag{2.145}$$

where C_∞ is defined in (2.139).

Case2: $t_1\ge 0$,

We consider (2.12) in Lemma 2. First we focus on the first line. By (2.143) we obtain

$$\int_{t_1}^t e^{-\int_s^t\frac{\mathfrak{C}}{2}\langle V^m(\tau)\rangle^2 d\tau}e^{-\mathfrak{C}\langle V^m(s)\rangle^2 s}e^{\theta|V^m(s)|^2}\Gamma^m_{\text{gain}}(s)ds\le\frac{1}{k_0}\|h_0\|_\infty. \tag{2.146}$$

Then we focus on the second line of (2.12). Using $\theta = \frac{1}{4T_M\xi}$ we bound the second line of (2.12) by

$$\exp\left(\Big[\frac{1}{2T_M\frac{2\xi}{\xi+1}} - \frac{1}{2T_w(x_1)}\Big]|V^m(t_1)|^2\right)\int_{\prod_{j=1}^{k_0-1}\mathcal{V}_j} H. \tag{2.147}$$

Now we focus on $\int_{\prod_{j=1}^{k_0-1}\mathcal{V}_j} H$. We compute H term by term with the formula given in (2.13). First we compute the first line of (2.13). By Lemma 3 with $p=1$, for every $1\le l\le k_0-1$, we have

$$\begin{aligned}
&\int_{\prod_{j=1}^{k_0-1}\mathcal{V}_j} \mathbf{1}_{\{t_{l+1}\le 0<t_l\}}|h_0\big(X^{m-l}(0),V^{m-l}(0)\big)|d\Sigma_{l,m}^{k_0}(0)\\
&\le \|h_0\|_\infty \int_{\prod_{j=1}^{k_0-1}\mathcal{V}_j} \mathbf{1}_{\{t_{l+1}\le 0<t_l\}}d\Sigma_{l,m}^{k_0}(0)\\
&\le (2C_{T_M,\xi})^l\|h_0\|_\infty \exp\left(\frac{(T_{l,1}-T_w(x_1))(1-r_{min})}{2T_w(x_1)[T_{l,1}(1-r_{min})+r_{min}T_w(x_1)]}|V^m(t_1)|^2\right.\\
&\quad\left. + (2\mathcal{C})^l(\mathfrak{C}t)|V^m(t_1)|^2\right).
\end{aligned} \tag{2.148}$$

In regard to (2.147) we have

$$\begin{aligned}
&\exp\left(\Big[\frac{1}{2T_M\frac{2\xi}{\xi+1}} - \frac{1}{2T_w(x_1)}\Big]|V^m(t_1)|^2\right)\times (2.148)\\
&= (2C_{T_M,\xi})^l\|h_0\|_\infty \exp\left(\Big[\frac{-1}{2\big(T_w(x_1)r_{min}+T_{l,1}(1-r_{min})\big)} + \frac{1}{2T_M\frac{2\xi}{\xi+1}}\Big]|V^m(t_1)|^2\right.\\
&\quad\left. + (2\mathcal{C})^l(\mathfrak{C}t)|V^m(t_1)|^2\right).
\end{aligned}$$

Using the definition (2.40) we have $T_w(x_1) < \frac{2\xi}{\xi+1}T_M$ and $T_{l,1} < \frac{2\xi}{\xi+1}T_M$, then we take

$$t_\infty = t_\infty(T_M, k_0, \xi, \mathcal{C}, \mathfrak{C}) \tag{2.149}$$

small enough and $t\le t_\infty$ so that the coefficient for $|V^m(t_1)|^2$ is

$$\begin{aligned}
&\frac{-1}{2\big(T_w(x_1)r_{min}+T_{l,1}(1-r_{min})\big)} + \frac{1}{2T_M\frac{2\xi}{\xi+1}} + (2\mathcal{C})^l(\mathfrak{C}t)\\
&\le \frac{-1}{2\big(T_M r_{min}+T_{l,1}(1-r_{min})\big)} + \frac{1}{2T_M\frac{2\xi}{\xi+1}} + (2\mathcal{C})^{k_0}(\mathfrak{C}t) \le 0.
\end{aligned} \tag{2.150}$$

Since (2.148) holds for all $1\le l\le k_0-1$, by (2.150) the contribution of the first line of (2.13) in (2.147) is bounded by

$$(2C_{T_M,\xi})^{k_0}\|h_0\|_\infty. \tag{2.151}$$

Then we compute the second line of (2.13). For each $1\le l\le k_0-1$ such that $\max\{0,t_{l+1}\}\le s\le t_l$, by (2.14), we have

$$d\Sigma_{l,m}^{k_0}(s) = e^{-\int_s^{t_l}\frac{\mathfrak{C}}{2}\langle V^{m-l}(\tau)\rangle^2 d\tau}d\Sigma_{l,m}^{k_0}(t_l).$$

Therefore, we derive

$$\begin{aligned}
&\int_{\max\{0,t_l\}}^{t_l}\int_{\prod_{j=1}^{k_0-1}\mathcal{V}_j} e^{-\mathfrak{C}\langle V^{m-l}(s)\rangle^2 s}e^{\theta|V^{m-l}(s)|^2}|\Gamma_{\text{gain}}^{m-l}(s)|d\Sigma_{l,m}^{k_0}(s)ds\\
&\quad\le \int_{\prod_{j=1}^{k_0-1}\mathcal{V}_j}\int_{\max\{0,t_l\}}^{t_l} e^{-\int_s^{t_l}\frac{\mathfrak{C}}{2}\langle V^{m-l}(\tau)\rangle^2 d\tau}e^{-\mathfrak{C}\langle V^{m-l}(s)\rangle^2 s}e^{\theta|V^{m-l}(s)|^2}|\Gamma_{\text{gain}}^{m-l}(s)|ds\,d\Sigma_{l,m}^{k_0}(t_l)\\
&\quad\le \frac{1}{k_0}\|h_0\|_\infty\int_{\prod_{j=1}^{k_0-1}\mathcal{V}_j}\Sigma_{l,m}^{k_0}(t_l) \qquad (2.152)\\
&\quad\le \frac{1}{k_0}\|h_0\|_\infty(2C_{T_M,\xi})^l\exp\bigg(\frac{(T_{l,1}-T_w(x_1))(1-r_{min})}{2T_w(x_1)[T_{l,1}(1-r_{min})+r_{min}T_w(x_1)]}|V^m(t_1)|^2\\
&\qquad+(2\mathcal{C})^l(\mathfrak{C}t)|V^m(t_1)|^2\bigg),
\end{aligned}$$

where we apply (2.143) in the third line and apply Lemma 3 in the last line.

In regard to (2.147), by (2.150) we obtain

$$\exp\bigg(\Big[\frac{1}{2T_M\frac{2\xi}{\xi+1}}-\frac{1}{2T_w(x_1)}\Big]|V^m(t_1)|^2\bigg)\times (2.152)\le\frac{1}{k_0}(2C_{T_M,\xi})^l\|h_0\|_\infty.$$

Since (2.152) holds for all $1\le l\le k_0-1$, the contribution of the second line of (2.13) in (2.147) is bounded by

$$\frac{k_0-1}{k_0}(2C_{T_M,\xi})^{k_0}\|h_0\|_\infty. \qquad (2.153)$$

Last we compute the third term of (2.13). By Lemma 4 and the Assumption (2.5) we obtain

$$\begin{aligned}
&\int_{\prod_{j=1}^{k_0-1}\mathcal{V}_j}\mathbf{1}_{\{0<t_{k_0}\}}|h^{m-k_0+2}\big(t_{k_0},x_{k_0},V^{m-k_0+1}(t_{k_0})\big)|d\Sigma_{k_0-1,m}^{k_0}(t_{k_0})\\
&\quad\le \|h^{m-k_0+2}\|_\infty\int_{\prod_{j=1}^{k_0-1}\mathcal{V}_j}\mathbf{1}_{\{0<t_{k_0}\}}d\Sigma_{k_0-1,m}^{k_0}(t_{k_0}) \qquad (2.154)\\
&\quad\le 3(2C_{T_M,\xi})^{k_0}(\frac{1}{2})^{k_0}\|h_0\|_\infty\exp\bigg(\frac{(T_{l,1}-T_w(x_1))(1-r_{min})}{2T_w(x_1)[T_{l,1}(1-r_{min})+r_{min}T_w(x_1)]}|V^m(t_1)|^2\\
&\qquad+(2\mathcal{C})^l(\mathfrak{C}t)|V^m(t_1)|^2\bigg).
\end{aligned}$$

In regard to (2.147), by (2.150) we have

$$\exp\bigg(\Big[\frac{1}{2T_M\frac{2\xi}{\xi+1}}-\frac{1}{2T_w(x_1)}\Big]|V^m(t_1)|^2\bigg)\times (2.154)\le(2C_{T_M,\xi})^{k_0}\|h_0\|_\infty.$$

Thus the contribution of the third line of (2.13) in (2.147) is bounded by

$$(2C_{T_M,\xi})^{k_0}\|h_0(x,v)\|_\infty. \qquad (2.155)$$

Collecting (2.151) (2.153) (2.155) we conclude that the second line of (2.12) is bounded by

$$(2C_{T_M,\xi})^{k_0}\times(2+\frac{k_0-1}{k_0})\|h_0\|_\infty. \qquad (2.156)$$

Adding (2.156) to (2.146) we use (2.12) to derive

$$\|h^{m+1}(t,x,v)\mathbf{1}_{\{t_1\geq 0\}}\|_\infty \leq 3(2C_{T_M,\xi})^{k_0}\|h_0\|_\infty. \tag{2.157}$$

Combining (2.145) and (2.157) we derive (2.6).

Last we focus the parameters for t_∞ in (2.7). In the proof the constraints for t_∞ are (2.138), (2.144) and (2.149). We obtain

$$\begin{aligned} t_\infty &= t_\infty(t', N, C_\infty, \|h_0\|_\infty, T_M, k_0, \xi, \mathcal{C}, \mathfrak{C}) \\ &= t_\infty(k_0, \xi, T_M, \min\{T_w(x)\}, \mathcal{C}, r_\perp, r_\parallel, \mathfrak{C}, C_{T_M,\xi}, \|h_0\|_\infty, C_{\phi^m}). \end{aligned}$$

By the definition of k_0 in (2.71), definition of $C_{T_M,\xi}$ in (2.56), definition of $\mathcal{C}$ in (2.59), definition of C_{ϕ^m} in (2.18) and the condition for $\mathfrak{C}$ in (2.30), (2.16), (2.65), we derive (2.7). □

3 Weighted $W^{1,p}$ Estimate for f^{m+1}

For proving the uniqueness of the solution as mentioned in the introduction, we rely on the estimate for $\nabla_x f$. In this section we prove the weighted $W^{1,p}$ estimates for $f^{m+1} = F^{m+1}/\sqrt{\mu}$ that satisfies (1.44) with boundary condition (1.45). We will be proving the following proposition.

Proposition 3 *Assume all the assumption in Proposition 2 holds true (so that we have* (2.8) *). Let f^{m+1} solving* (1.44) *with boundary condition* (1.45)*. Define*

$$\begin{aligned} \mathcal{E}^m(t) := \sup_{l\leq m}\Big[&\lambda\|w_{\tilde{\theta}}e^{-\lambda t\langle v\rangle}f^l(t)\|_p^p + \lambda\int_0^t |w_{\tilde{\theta}}e^{-\lambda s\langle v\rangle}f^l(s)|_{p,+}^p \\ &+ \frac{1}{2}\|e^{-\lambda t\langle v\rangle}w_{\tilde{\theta}}\alpha_{f^{l-1},\epsilon}^\beta \nabla_{x,v}f^l(t)\|_p^p \\ &+ \int_0^t |e^{-\lambda s\langle v\rangle}w_{\tilde{\theta}}\alpha_{f^{l-1},\epsilon}^\beta \nabla_{x,v}f^l(s)|_{p,+}^p \\ &+ \frac{\lambda}{4}\int_0^t \|e^{-\lambda s\langle v\rangle}\langle v\rangle w_{\tilde{\theta}}\alpha_{f^{l-1},\epsilon}^\beta \nabla_{x,v}f^l(s)\|_p^p\Big]. \end{aligned} \tag{3.1}$$

Then for small enough $\tilde{\theta}$ so that $0<\tilde{\theta}<\theta\ll 1$ and $\lambda\gg 1$, there exists $t_W\ll 1$ ($t_W\leq t_\infty$) and $C_W\gg 1$ such that for

$$\frac{p-2}{p} < \beta < \frac{2}{3} \quad \textit{for} \quad 3<p<6, \tag{3.2}$$

if

$$\sup_{0\leq t\leq t_W} \mathcal{E}^m(t) \leq 2C_W\{\|w_{\tilde{\theta}}f_0\|_p^p + \|w_{\tilde{\theta}}\alpha_{f_0,\epsilon}^\beta \nabla_{x,v}f_0\|_p^p\} < \infty, \tag{3.3}$$

then

$$\sup_{0\leq t\leq t_W} \mathcal{E}^{m+1}(t) \leq 2C_W\{\|w_{\tilde{\theta}}f_0\|_p^p + \|w_{\tilde{\theta}}\alpha_{f_0,\epsilon}^\beta \nabla_{x,v}f_0\|_p^p\}. \tag{3.4}$$

Here C_W is a constant defined in (3.65)*, and t_W satisfies the condition* (3.66).

This proposition implies the uniform in m bound of the weighted $W^{1,p}$ norm of f^m. This gives us an a-priori estimate for the later proof for the uniqueness. The "energy" term defined in (3.1) has two components that depend on p-norm of f, and three components on p-norm of ∂f. Therefore in the proof for the proposition, we need to provide the estimates for these components. Some lemmas from [2,15] will be repeatedly used and we cite them here first.

We note that in bulk this part of proof is rather similar to that in [2]. The main difficulty comes in through the boundary treatment as mentioned in the introduction, and this complexity is reflected Step 1 for f and Step 5 for ∂f in the proof.

For the initial problems of the transport equation with time-independent field $E(t,x)$, source $H(t,x,v)$, and damping term $\psi(t,x,v) \geq 0$, let h solves:

$$\partial_t h + v\cdot\nabla_x h + E\cdot\nabla_v h + \psi h = H, \tag{3.5}$$

then we have the following estimates for h:

Lemma 9 (Lemma 5 in [2]) *For $p \in [1,\infty)$ assume that $h, \partial_t h + v\cdot\nabla_x h - \nabla\phi\cdot\nabla_v h \in L^p([0,T]; L^p(\Omega\times\mathbb{R}^3))$ and $h_{\gamma_-} \in L^p([0,T]; L^p(\gamma))$. Then $h \in C^0([0,T]; L^p(\Omega\times\mathbb{R}^3))$ and $h_{\gamma_+} \in L^p([0,T]; L^p(\gamma))$ and for almost every $t \in [0,T]$:*

$$\begin{aligned}\|h(t)\|_p^p + \int_0^t |h|_{\gamma_+,p}^p &= \|h(0)\|_p^p + \int_0^t |h|_{\gamma_-,p}^p \\ &\quad + \int_0^t \iint_{\Omega\times\mathbb{R}^3} \{\partial_t h + v\cdot\nabla_x h + E\cdot\nabla_v h + \psi h\}|h|^{p-1}.\end{aligned} \tag{3.6}$$

Lemma 10 (Lemma 6 in [2]) *Assume $E \in L^\infty$, then for $t \ll 1$, $\varepsilon > 0$,*

$$\begin{aligned}&\int_0^t \int_{\gamma_+\setminus\gamma_+^\varepsilon} |h| \mathrm{d}\gamma \mathrm{d}s \\ &\leq C(\varepsilon)\left\{ \|h_0\|_1 + \int_0^t \|h(s)\|_1 + \|[\partial_t + v\cdot\nabla_x + E\cdot\nabla_v + \psi]h(s)\|_1 \mathrm{d}s \right\},\end{aligned} \tag{3.7}$$

where

$$\gamma_+^\epsilon = \{(x,v) \in \gamma_+ : n(x)\cdot v < \epsilon \text{ or } |v| > \epsilon^{-1}\}. \tag{3.8}$$

The next result is about the integrability of α,

Proposition 4 (Proposition 3 in [2]) *Assume $E(t,x) \in C_x^1$ is given. Then for $0 \leq s \leq t \ll 1$, $0 < \sigma < 1$ and $N > 1$ and $x \in \bar{\Omega}$,*

$$\int_{|u|\leq N} \frac{\mathrm{d}u}{\alpha_{f,\epsilon}(s,x,u)^\sigma} \lesssim_{\sigma,\Omega,N} 1, \tag{3.9}$$

and, for any $0 < \kappa \leq 2$,

$$\int_{|u|\geq N} \frac{e^{-C|v-u|^2}}{|v-u|^{2-\kappa}} \frac{1}{\alpha_{f,\epsilon}(s,x,u)^\sigma} \mathrm{d}u \lesssim_{\sigma,\Omega,N,\kappa} 1. \tag{3.10}$$

We will also need the C^2 estimate for ϕ:

Lemma 11 *Assume (3.2). If ϕ solves (1.24) then*

$$\|\phi(t)\|_{C^{2,1-\frac{3}{p}}} \leq (C_1)^{1/p}\{\|f(t)\|_p + \|e^{-\lambda t\langle v\rangle}\alpha_{f,\epsilon}^\beta \nabla_x f(t)\|_p\} \quad \textit{for} \quad p > 3. \tag{3.11}$$

Proof Applying the Schauder estimate to (1.24) we deduce

$$\|\phi\|_{C^{2,1-\frac{3}{p}}} \lesssim_{p,\Omega} \Big\| \int_{\mathbb{R}^3} f(t)\sqrt{\mu} dv \Big\|_{C^{0,1-\frac{3}{p}}(\bar{\Omega})} \quad \text{for} \quad p > 3. \tag{3.12}$$

By the Morrey inequality, $W^{1,p} \subset C^{0,1-\frac{3}{p}}$ for $p > 3$, we derive

$$\begin{aligned}\Big\| \int_{\mathbb{R}^3} f(t)\sqrt{\mu} dv \Big\|_{C^{0,1-\frac{n}{p}}(\bar{\Omega})} &\lesssim \Big\| \int_{\mathbb{R}^3} f(t)\sqrt{\mu} dv \Big\|_{W^{1,p}(\Omega)} \\ &\lesssim \Big(\int_{\mathbb{R}^3} \mu^{q/2} dv \Big)^{1/q} \|f(t)\|_{L^p(\Omega\times\mathbb{R}^3)} \\ &\quad + \Big\| \int_{\mathbb{R}^3} \nabla_x f(t)\sqrt{\mu} dv \Big\|_{L^p(\Omega)}.\end{aligned}$$

By the Hölder inequality,we have

$$\begin{aligned}&\Big| \int_{\mathbb{R}^3} \nabla_x f(t,x,v)\sqrt{\mu(v)} dv \Big| \\ &\le \Big\| \frac{\sqrt{e^{\lambda t\langle\cdot\rangle}\mu(\cdot)}}{\alpha_{f,\epsilon}(t,x,\cdot)^\beta} \Big\|_{L^{\frac{p}{p-1}}(\mathbb{R}^3)} \big\| e^{-\lambda t\langle\cdot\rangle}\alpha_{f,\epsilon}(t,x,\cdot)^\beta \nabla_x f(t,x,\cdot)\big\|_{L^p(\mathbb{R}^3)} \\ &= \Big(\int_{\mathbb{R}^3} \frac{\mu(v)^{\frac{p}{2(p-1)}}}{\alpha(t,x,v)^{\frac{\beta p}{p-1}}} dv \Big)^{\frac{p-1}{p}} \| e^{-\lambda t\langle\cdot\rangle}\alpha^\beta_{f,\epsilon} \nabla_x f(t,x,\cdot)\|_{L^p(\mathbb{R}^3)}.\end{aligned}$$

From the assumption $\frac{p-2}{p-1} < \frac{\beta p}{p-1} < \frac{2}{3}\frac{p}{p-1} < 1$. We draw the conclusion. □

We need some estimates about the collision operator Γ for Proposition 3. Define a notation

$$\mathbf{k}_\rho(v,u) = \frac{1}{|v-u|} \exp\{-\rho|v-u|^2 - \rho\frac{||v|-|u||^2}{|v-u|^2}\}. \tag{3.13}$$

The velocity derivative for the nonlinear Boltzmann operator reads

$$\begin{aligned}&\nabla_v \big(\Gamma_{\text{gain}}(f^m, f^m) - \Gamma_{\text{loss}}(f^m, f^{m+1})\big) \\ &\ = \Gamma_{\text{gain}}(\nabla_v f^m, f^m) + \Gamma_{\text{gain}}(f^m, \nabla_v f^m) - \Gamma_{\text{loss}}(\nabla_v f^m, f^{m+1}) \\ &\quad - \Gamma_{\text{loss}}(f^m, \nabla_v f^{m+1}) \\ &\quad + \Gamma_{v,\text{gain}}(f^m, f^m) - \Gamma_{v,\text{loss}}(f^m, f^{m+1}).\end{aligned} \tag{3.14}$$

Here we have defined

$$\begin{aligned}&\Gamma_{v,\text{gain}}(f^m, f^m) - \Gamma_{v,\text{loss}}(f^m, f^{m+1}) \\ &\ := \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |u\cdot\omega| f^m(v+u_\perp) f^m(v+u_\parallel)\nabla_v\sqrt{\mu(v+u)} \mathrm{d}\omega\mathrm{d}u \\ &\quad - \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |u\cdot\omega| f^m(v+u) f^{m+1}(v)\nabla_v\sqrt{\mu(v+u)} \mathrm{d}\omega\mathrm{d}u.\end{aligned} \tag{3.15}$$

Lemma 12 *For* $0 < \frac{\theta}{4} < \rho$, *if* $0 < \tilde{\rho} < \rho - \frac{\theta}{4}$, $0 \le s \le t \ll \tilde{\rho}$, *then*

$$\mathbf{k}_\varrho(v,u)\frac{e^{\theta|v|^2}}{e^{\subseteq|u|^2}}\frac{e^{\lambda s\langle u\rangle}}{e^{\lambda s\langle v\rangle}} \lesssim \mathbf{k}_{\tilde{\varrho}}(v,u). \tag{3.16}$$

Moreover,

$$\int_{\mathbb{R}^3} \mathbf{k}_{\tilde{\rho}}(v,u)du \lesssim \langle v\rangle^{-1}. \tag{3.17}$$

For the nonlinear Boltzmann operator we have

$$\begin{aligned}&|\Gamma_{gain}(f^m,f^m)-\Gamma_{loss}(f^m,f^{m+1})|\\&\lesssim \left(\|w_{\theta'}f^m\|_\infty+\|w_{\theta'}f^{m+1}\|\right)\int_{\mathbb{R}^3}\mathbf{k}_{\varrho}(v,u)|f^m(u)|\mathrm{d}u.\end{aligned} \tag{3.18}$$

For (3.14) *we have*

$$\begin{aligned}&|w_{\tilde{\theta}}\Gamma_{gain}(\nabla_v f^m,f^m)|+|w_{\tilde{\theta}}\Gamma_{gain}(f^m,\nabla_v f^m)|\\&\quad\lesssim \|w_{\theta'}f^m\|_\infty\int_{\mathbb{R}^3}\mathbf{k}_{\tilde{\varrho}}(v,u)|w_{\tilde{\theta}}\nabla_v f^m(u)|\mathrm{d}u.\end{aligned} \tag{3.19}$$

$$\begin{aligned}&|w_{\tilde{\theta}}\Gamma_{loss}(\nabla_v f^m,f^{m+1})|\\&\quad\lesssim \|w_{\theta'}f^{m+1}\|_\infty\int_{\mathbb{R}^3}\mathbf{k}_{\tilde{\varrho}}(v,u)|w_{\tilde{\theta}}\nabla_v f^m(u)|\mathrm{d}u,\\&|w_{\tilde{\theta}}\Gamma_{loss}(f^m,\nabla_v f^{m+1})|\\&\quad\lesssim \langle v\rangle\|w_{\theta'}f^m\|_\infty|w_{\tilde{\theta}}\nabla_v f^{m+1}(v)|.\end{aligned} \tag{3.20}$$

$$\begin{aligned}&|w_{\tilde{\theta}}\Gamma_{v,loss}(f^m,f^{m+1})|\\&\quad\lesssim \langle v\rangle\frac{w_{\tilde{\theta}}(v)}{w_{\theta'}(v)}\|w_{\theta'}f^{m+1}\|_\infty\|e^{-\lambda\langle u\rangle s}w_{\tilde{\theta}}(u)f^m\|_p.\end{aligned} \tag{3.21}$$

$$\begin{aligned}&|w_{\tilde{\theta}}\Gamma_{v,gain}(f^m,f^m)|\\&\quad\lesssim \|w_{\theta'}f^m\|_\infty\int_{\mathbb{R}^3}\mathbf{k}_{\tilde{\varrho}}(v,u)|w_{\theta'}f^m(u)|\mathrm{d}u.\end{aligned} \tag{3.22}$$

For $(x,v)\in\gamma_-$*, we have the following bound for* $\nabla_{x,v}f^{m+1}$ *on the boundary:*

$$|\nabla_{x,v}f^{m+1}(t,x,v)| \lesssim \langle v\rangle^2 e^{[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]|v|^2}\left(1+\frac{1}{|n(x)\cdot v|}\right)\times \text{(3.24)} \tag{3.23}$$

with

$$\begin{aligned}&\int_{n(x)\cdot u>0}\Big[\langle u\rangle|\nabla_{x,v}f^m(t,x,u)|\\&\quad+\langle u\rangle^2|f^m|+\|w_{\theta'}f^m\|_\infty\int_{\mathbb{R}^3}\mathbf{k}_\rho(u,u')|f^{m-1}(u')|du'+\langle u\rangle|f^m|\|\nabla_x\phi^{m-1}\|_\infty\Big]\\&\quad e^{-[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]|u|^2}d\sigma(u,v).\end{aligned} \tag{3.24}$$

Proof The proof of (3.16) is given in appendix.

The nonlinear Boltzmann operator (1.23) equals

$$\begin{aligned}&\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|u\cdot\omega|g_1(v+u_1)g_2(v+u_2)\sqrt{\mu(v+u)}\mathrm{d}\omega\mathrm{d}u\\&\quad-\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|u\cdot\omega|g_1(v+u)g_2(v)\sqrt{\mu(v+u)}\mathrm{d}\omega\mathrm{d}u,\end{aligned} \tag{3.25}$$

where $u_1 = (u \cdot \omega)\omega$ and $u_2 = u - u_1$. By exchanging the role of $\sqrt{\mu}$ and w^{-1}, we conclude (3.18).

The estimates (3.19)– (3.22) follows from the standard way using (3.25). The readers can also find them in chapter 4 of [2].

Then we focus on the derivative on the boundary. By (1.22) we have

$$\begin{aligned}\partial_n f^{m+1}(t,x,v) = \frac{-1}{n(x)\cdot v}\Bigg\{&\partial_t f^{m+1} + \sum_{i=1}^{2}(v\cdot\tau_i)\partial_{\tau_i} f^{m+1} - \nabla_x\phi^m\cdot\nabla_v f^{m+1} \\ &+\left(\frac{v}{2T_M}\cdot\nabla_x\phi^m\right)f^{m+1} - \Gamma_{\text{gain}}(f^m, f^m) + \Gamma_{\text{loss}}(f^m, f^{m+1})\Bigg\}.\end{aligned} \tag{3.26}$$

Let $\tau_1(x)$ and $\tau_2(x)$ be unit tangential vectors to $\partial\Omega$ satisfying $\tau_1(x)\cdot n(x) = 0 = \tau_2(x)\cdot n(x)$ and $\tau_1(x)\times\tau_2(x) = n(x)$. Define the orthonormal transformation from $\{n, \tau_1, \tau_2\}$ to the standard bases $\{e_1, e_2, e_3\}$,

$$\mathcal{T}(x)n(x) = e_1,\ \ \mathcal{T}(x)\tau_1(x) = e_2,\ \ \mathcal{T}(x)\tau_2(x) = e_3,\ \ \mathcal{T}^{-1} = \mathcal{T}^T.$$

By a change of variable $u' = \mathcal{T}(x)u$, $v' = \mathcal{T}(x)v$ we have

$$\begin{aligned}u_\perp &= n(x)\cdot u = n(x)\cdot\mathcal{T}^T(x)u' = n(x)^T\mathcal{T}^T(x)u' = [\mathcal{T}(x)n(x)]^T u' = e_1\cdot u' = u_1',\\ u_\parallel &= [\tau_1(x)\cdot u]\tau_1(x) + [\tau_2(x)\cdot u]\tau_2(x) = [\tau_1(x)\cdot\mathcal{T}^T(x)u']\tau_1(x) + [\tau_2(x)\cdot\mathcal{T}^T(x)u']\tau_2(x)\\ &= \{[\mathcal{T}\tau_1(x)]^T u'\}\tau_1(x) + \{[\mathcal{T}\tau_2(x)]^T u'\}\tau_2(x) = u_2'\tau_1(x) + u_3'\tau_2(x)\\ &= u_2'\mathcal{T}^T(x)e_2 + u_3'\mathcal{T}^T(x)e_3,\\ v_\perp &= v_1',\quad v_\parallel = v_2'\mathcal{T}^T(x)e_2 + v_3'\mathcal{T}^T(x)e_3.\end{aligned}$$

Then the boundary condition becomes

$$f^{m+1}(t,x,v) = e^{[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]|v|^2}\int_{u_1'>0} f^m(t,x,\mathcal{T}^T(x)u')e^{-[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]|u'|^2}d\sigma'(u,v),$$

where we define

$$\begin{aligned}d\sigma'(u,v) = &\frac{1}{r_\perp r_\parallel(2-r_\parallel)\pi/2}\frac{|u_1'|}{(2T_w(x))^2}I_0\left(\frac{1}{2T_w(x)}\frac{2(1-r_\perp)^{1/2}v_1'u_1'}{r_\perp}\right)\\ &\exp\left(-\frac{1}{2T_w(x)}\left[\frac{|u_1'|^2 + (1-r_\perp)|v_1'|^2}{r_\perp}\right.\right.\\ &\left.\left.+\frac{|u_2'\mathcal{T}^T(x)e_2 + u_3'\mathcal{T}^T(x)e_3 - (1-r_\parallel)[v_2'\mathcal{T}^T(x)e_2 + v_3'\mathcal{T}^T(x)e_3]|^2}{r_\parallel(2-r_\parallel)}\right]\right)du'.\end{aligned}$$

We can further take the tangential derivatives ∂_{τ_i}, for $(x, v) \in \gamma_-$,

$$
\begin{aligned}
&|\partial_{\tau_i} f^{m+1}(t,x,v)| \\
&\quad \lesssim e^{[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]|v|^2}\Big(\Big|\frac{|v|^2\partial_{\tau_i}T_w(x)}{2[T_w(x)]^2}\int_{n(x)\cdot u>0} f^m(t,x,u)e^{-[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]|u|^2}d\sigma(u,v)\Big| \\
&\qquad +\Big|\int_{n(x)\cdot u>0}\partial_{\tau_i} f^m(t,x,u)e^{-[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]|u|^2}d\sigma(u,v)\Big| \\
&\qquad +\Big|\int_{n(x)\cdot u>0}\nabla_v f^m(t,x,u)\partial_{\tau_i}\mathcal{T}^T(x)\mathcal{T}(x)ue^{-[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]|u|^2}d\sigma(u,v)\Big| \\
&\qquad +\Big|\int_{n(x)\cdot u>0} f^m(t,x,u)e^{-[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]|u|^2}\frac{-|u|^2\partial_{\tau_i}T_w(x)}{2[T_w(x)]^2}d\sigma(u,v)\Big| \\
&\qquad +\Big|\int_{n(x)\cdot u>0} f^m(t,x,u)e^{-[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]|u|^2}\frac{-2\partial_{\tau_i}T_w(x)}{T_w(x)}d\sigma(u,v)\Big| \\
&\qquad +\Big|\int_{n(x)\cdot u>0} f^m(t,x,u)e^{-[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]|u|^2}[-\frac{\partial_{\tau_i}T_w(x)}{T_w^2(x)}\frac{(1-r_\perp)^{1/2}v_1'u_1'}{r_\perp}]d\sigma(u,v)\Big| \\
&\qquad +\Big|\int_{n(x)\cdot u>0} f^m(t,x,u)e^{-[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]|u|^2}\frac{\partial_{\tau_i}T_w(x)}{2(T_w(x))^2}(|v|^2+|u|^2)d\sigma(u,v)\Big| \\
&\qquad +\Big|\int_{n(x)\cdot u>0} f^m(t,x,u)e^{-[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]|u|^2}\frac{1}{2T_w(x)}\big[|u|^2+\frac{|v|^2\partial_{\tau_i}\mathcal{T}^T(x)}{r_\parallel(2-r_\parallel)}\big]d\sigma(u,v)\Big|\Big).
\end{aligned}
\tag{3.27}
$$

Then we take velocity derivatives and obtain for $(x, v) \in \gamma_-$,

$$
\begin{aligned}
&\nabla_v f^{m+1}(t,x,v) \\
&\quad = v[\frac{1}{2T_M}-\frac{1}{T_w(x)}]e^{[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]|v|^2}\int_{n(x)\cdot u>0} f^m(t,x,u)e^{-[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]|u|^2}d\sigma(u,v) \\
&\qquad + e^{[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]|v|^2}\int_{n(x)\cdot u>0} f^m(t,x,u)e^{-[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]|u|^2}\frac{(1-r_\perp)^{1/2}u_\perp}{T_w(x)r_\perp}n(x)d\sigma(u,v) \\
&\qquad + e^{[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]|v|^2}\int_{n(x)\cdot u>0} f^m(t,x,u)e^{-[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]|u|^2} \\
&\qquad \Big(-\frac{1}{T_w}\Big[\frac{(1-r_\perp)v_\perp}{r_\perp}n(x)-\frac{u_\parallel-(1-r_\parallel)v_\parallel}{r_\parallel(2-r_\parallel)}(1-r_\parallel)(\mathbf{I}_{3\times 3}-n(x)\otimes n(x))\Big]\Big)d\sigma(u,v),
\end{aligned}
\tag{3.28}
$$

where we use

$$
\nabla_v v_\perp = n(x),\ \nabla_v v_\parallel = \nabla_v(v - v_\perp\cdot n(x)) = \mathbf{I}_{3\times3} - n(x)\otimes n(x).
$$

From (1.22), the temporal derivative is

$$
\begin{aligned}
\partial_t f^{m+1}(t,x,v) &= e^{[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]|v|^2}\int_{n(x)\cdot u>0}\partial_t f^m(t,x,u)e^{-[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]|u|^2}d\sigma(u,v) \\
&= e^{[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]|v|^2}\int_{n(x)\cdot u>0}\Big[-u\cdot\nabla_x f^m+\nabla_x\phi^{m-1}\cdot\nabla_v f^m \\
&\quad -(\frac{u}{2T_M}\cdot\nabla_x\phi^{m-1})f^m+\Gamma_{\text{gain}}(f^{m-1},f^{m-1})-\Gamma_{\text{loss}}(f^{m-1},f^m)\Big] \\
&\quad e^{-[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]|u|^2}d\sigma(u,v).
\end{aligned}
\tag{3.29}
$$

Combine (3.26)-(3.29) we conclude (3.23), where we use $T_w \in C^1_x$. □

We are now ready to show Proposition 3.

Proof of Proposition 3 Setting $t \le t_W \le t_\infty$ so that Proposition 2 holds valid. In the following, we first examine the terms related to p-norm of f in Step 1, and it will be followed by Step 2, in which we examine the boundedness of ∂f terms. In Step 3 we collect these estimates to form the conclusion. The Green's identity used in Step 2 leads to two terms (bulk and boundary), to bound which, heavy computation is involved and we present the details in Step 4 and 5 respectively.

Step 1: estimate of p-norm of f:
Since f^{m+1} solves (1.44), its weighted version then satisfies:

$$\begin{aligned}&\partial_t\big[e^{-\lambda t\langle v\rangle}w_{\tilde\theta}f^{m+1}\big]+v\cdot\nabla_x\big[e^{-\lambda t\langle v\rangle}w_{\tilde\theta}f^{m+1}\big]-\nabla_x\phi^m\cdot\nabla_v\big[e^{-\lambda t\langle v\rangle}w_{\tilde\theta}f^{m+1}\big]+\big[\nu(F^m)\\&\quad+\lambda\langle v\rangle+\frac{v}{2T_M}\cdot\nabla_x\phi^m-\lambda t\partial_v\langle v\rangle+2\tilde\theta v\cdot\nabla_x\phi^m\big]\big[e^{-\lambda t\langle v\rangle}w_{\tilde\theta}f^{m+1}\big]\\&=e^{-\lambda t\langle v\rangle}w_{\tilde\theta}\Gamma_{\text{gain}}(f^m,f^m).\end{aligned}\tag{3.30}$$

For $t_W = t_W(\lambda) \ll 1$, one can take $\lambda = \lambda(t, C_{\phi^m})$ large enough so that

$$\nu(F^m)+\lambda\langle v\rangle+\frac{v}{2T_M}\cdot\nabla_x\phi^m-\lambda t\partial_v\langle v\rangle+2\tilde\theta v\cdot\nabla_x\phi^m\ge\nu(F^m)+\frac{\lambda}{2}\langle v\rangle>\frac{\lambda}{2}\langle v\rangle,\tag{3.31}$$

and thus we apply Lemma 9, and combine with (3.18) to have:

$$\begin{aligned}&\|e^{-\lambda t\langle v\rangle}w_{\tilde\theta}f^{m+1}(t)\|_p^p+\int_0^t|e^{-\lambda s\langle v\rangle}w_{\tilde\theta}f^{m+1}|_{p,+}^p+\frac{\lambda}{2}\int_0^t\|\langle v\rangle^{1/p}e^{-\lambda s\langle v\rangle}w_{\tilde\theta}f^{m+1}\|_p^p\\&\lesssim_p\|w_{\tilde\theta}f(0)\|_p^p+\int_0^t|e^{-\lambda s\langle v\rangle}w_{\tilde\theta}f^{m+1}|_{p,-}^p\\&\quad+\|w_{\theta'}f^m\|_\infty\int_0^t\int_{\Omega\times\mathbb{R}^3}|e^{-\lambda s\langle v\rangle}w_{\tilde\theta}f^{m+1}(v)|^{p-1}e^{-\lambda s\langle v\rangle}w_{\tilde\theta}\int_{\mathbb{R}^3}\mathbf{k}_\varrho(v,u)|f^m(u)|du.\end{aligned}\tag{3.32}$$

To deal with the last term in (3.32), we note that by using Hölder inequality and Young's inequality, with (3.16), we have:

$$\begin{aligned}&\int_{\mathbb{R}^3}|e^{-\lambda s\langle v\rangle}w_{\tilde\theta}f^{m+1}(v)|^{p-1}\int_{\mathbb{R}^3}\mathbf{k}_\varrho(v,u)\frac{w_{\tilde\theta}(v)}{w_{\tilde\theta}(u)}\frac{e^{\lambda s\langle u\rangle}}{e^{\lambda s\langle v\rangle}}|e^{-\lambda s\langle u\rangle}w_{\tilde\theta}(u)f^m(u)|\mathrm{d}u\mathrm{d}v\\&\lesssim_p\|e^{-\lambda s\langle v\rangle}w_{\tilde\theta}f^{m+1}\|_{L^p_v}^{p-1}\left\|\int_{\mathbb{R}^3}\mathbf{k}_{\tilde\varrho}(v,u)^{1/q}\mathbf{k}_{\tilde\varrho}(v,u)^{1/p}|e^{-\lambda s\langle u\rangle}w_{\tilde\theta}f^m(u)|\mathrm{d}u\right\|_{L^p_v}\\&\lesssim\|e^{-\lambda s\langle v\rangle}w_{\tilde\theta}f^{m+1}\|_{L^p_v}^{p-1}\left(\int_{\mathbb{R}^3}\mathbf{k}_{\tilde\varrho}(v,u)\mathrm{d}u\right)^{1/q}\left\|\left(\int_{\mathbb{R}^3}\mathbf{k}_{\tilde\varrho}(v,u)|e^{-\lambda s\langle u\rangle}w_{\tilde\theta}f^m(u)|^p\mathrm{d}u\right)^{1/p}\right\|_{L^p_v}\\&\lesssim\|e^{-\lambda s\langle v\rangle}w_{\tilde\theta}f^{m+1}\|_{L^p_v}^{p-1}\|e^{-\lambda s\langle v\rangle}w_{\tilde\theta}f^m\|_{L^p_v}\left(\int_{\mathbb{R}^3}\mathbf{k}_{\tilde\varrho}(v,u)\mathrm{d}u\right)^{1/q}\left(\int_{\mathbb{R}^3}\mathbf{k}_{\tilde\varrho}(v,u)\mathrm{d}v\right)^{1/p}\\&\lesssim\|e^{-\lambda s\langle v\rangle}w_{\tilde\theta}f^{m+1}\|_{L^p_v}^{p-1}\|e^{-\lambda s\langle v\rangle}w_{\tilde\theta}f^m\|_{L^p_v}\lesssim_p\|e^{-\lambda s\langle v\rangle}w_{\tilde\theta}f^{m+1}\|_{L^p_v}^p+\|e^{-\lambda s\langle v\rangle}w_{\tilde\theta}f^m\|_{L^p_v}^p,\end{aligned}\tag{3.33}$$

which gives a bound for the last term in (3.32) as:

$$C(p)\sup_m\|w_{\theta'}f^m\|_\infty\Big(\int_0^t\|e^{-\lambda s\langle v\rangle}w_{\tilde\theta}f^{m+1}\|_p^p+\int_0^t\|e^{-\lambda s\langle v\rangle}w_{\tilde\theta}f^m\|_{L^p_v}^p\Big).\tag{3.34}$$

One can further absorb the first term above to the left hand side of (3.32) by choosing large enough λ:

$$\frac{\lambda}{4} \geq C(p) \sup_m \|w_{\theta'} f^m\|_\infty. \tag{3.35}$$

To deal with $\int_0^t |e^{-\lambda s\langle v\rangle} w_{\tilde{\theta}} f^{m+1}|_{p,-}^p$ in (3.32), we first decompose

$$\gamma_+ = \gamma_{+,1}^{v,x,\epsilon} \cup \Big(\gamma_+(x)\backslash\gamma_{+,1}^{v,x,\epsilon}\Big),$$

where

$$\begin{aligned}\gamma_{+,1}^{v,x,\epsilon} &= \{(x,u) \in \gamma_+ : |n(x)\cdot u| \leq \epsilon \text{ or } |u_\| - \frac{2T_M(1-r_\|)}{2T_M + (T_w(x) - 2T_M)r_\|(2-r_\|)} v_\|| \geq \epsilon^{-1} \\ &\quad \text{or } |u_\perp - \frac{2T_M\sqrt{1-r_\perp}}{2T_M + \big(T_w(x) - 2T_M\big)r_\perp} v_\perp| \geq \epsilon^{-1}\}.\end{aligned} \tag{3.36}$$

This leads to

$$\begin{aligned}\int_0^t |e^{-\lambda s\langle v\rangle} w_{\tilde{\theta}} f^{m+1}|_{p,-}^p &= \int_0^t \int_{\partial\Omega} \int_{n(x)\cdot v<0} |n(x)\cdot v| e^{-p\lambda s\langle v\rangle} w_{\tilde{\theta}}^p |f^{m+1}|^p \\ &\lesssim_{\tilde{\theta}} \int_0^t |n(x)\cdot v| \int_{\partial\Omega}\int_{n(x)\cdot v<0} e^{-p\lambda s\langle v\rangle} w_{\tilde{\theta}}^p(v) e^{p[\frac{1}{4T_M} - \frac{1}{2T_w(x)}]|v|^2} \\ &\quad \times \bigg(\Big[\int_{\gamma_{+,1}^{v,x,\epsilon}} + \int_{\gamma_+\backslash\gamma_{+,1}^{v,x,\epsilon}}\Big] |e^{-\lambda s\langle u\rangle} w_{\tilde{\theta}}(u) f^m| e^{-[\frac{1}{4T_M} - \frac{1}{2T_w(x)}]|u|^2} d\sigma(u,v)\bigg)^p,\end{aligned} \tag{3.37}$$

where we used

$$|f^m| = |e^{-\lambda s\langle u\rangle} w_{\tilde{\theta}} f^m||e^{\lambda s\langle u\rangle} w_{\tilde{\theta}}^{-1}(u)| \lesssim_{\tilde{\theta}} |e^{-\lambda s\langle u\rangle} w_{\tilde{\theta}} f^m|.$$

We further expand $d\sigma(v,u)$ by (1.25) and apply Hölder inequality using $1 = \frac{1}{p} + \frac{1}{p^*}$ for:

$$\begin{aligned}(3.37) &\lesssim_p \int_0^t \int_{\partial\Omega} \bigg(\int_{\gamma_{+,1}^{v,x,\epsilon}(x)} |e^{-\lambda s\langle u\rangle} w_{\tilde{\theta}}(u) f^m|^p \{n(x)\cdot u\} du\bigg) \\ &\quad \int_{n(x)\cdot v<0} |n(x)\cdot v| e^{-p\lambda s\langle v\rangle} w_{\tilde{\theta}}^p(v) e^{p[\frac{1}{4T_M} - \frac{1}{2T_w(x)}]|v|^2} \\ &\quad \bigg(\int_{\gamma_{+,1}^{v,x,\epsilon}(x)} |n(x)\cdot u| e^{-p^*[\frac{1}{4T_M} - \frac{1}{2T_w(x)}]|u|^2} I_0\Big(p^* \frac{(1-r_\perp)^{1/2} v_\perp u_\perp}{T_w(x) r_\perp}\Big) \\ &\quad e^{-\frac{p^*}{2T_w(x)}\Big[\frac{|u_\perp|^2 + (1-r_\perp)|v_\perp|^2}{r_\perp} + \frac{|u_\| - (1-r_\|)v_\||^2}{r_\|(2-r_\|)}\Big]} du\bigg)^{p/p^*} \\ &+ \int_0^t \int_{\partial\Omega} \bigg(\int_{\gamma_+\backslash\gamma_{+,1}^{v,x,\epsilon}} |e^{-\lambda s\langle u\rangle} w_{\tilde{\theta}}(u) f^m|^p \{n(x)\cdot u\}\bigg) \\ &\quad \int_{n(x)\cdot v<0} |n(x)\cdot v| e^{-p\lambda s\langle v\rangle} w_{\tilde{\theta}}^p(v) e^{p[\frac{1}{4T_M} - \frac{1}{2T_w(x)}]|v|^2} \\ &\quad \bigg(\int_{\gamma_+\backslash\gamma_{+,1}^{v,x,\epsilon}(x)} |n(x)\cdot u| e^{-p^*[\frac{1}{4T_M} - \frac{1}{2T_w(x)}]|u|^2} I_0\Big(p^* \frac{(1-r_\perp)^{1/2} v_\perp u_\perp}{T_w(x) r_\perp}\Big)\end{aligned} \tag{3.38}$$

$$e^{-\frac{p^*}{2T_w(x)}\left[\frac{|u_\perp|^2+(1-r_\perp)|v_\perp|^2}{r_\perp}+\frac{|u_\|-(1-r_\|)v_\||^2}{r_\|(2-r_\|)}\right]}du\Big)^{p/p^*}, \tag{3.39}$$

where we apply Hölder inequality for I_0 to have

$$I_0^{p^*}(y) = (\frac{1}{\pi}\int_0^\pi e^{y\cos\theta}d\theta)^{p^*} \le \frac{1}{\pi^{p^*}}\int_0^\pi e^{p^* y\cos\theta}d\theta\pi^{1/p} = \pi^{1/p-1/p^*}I_0(p^*y).$$

We now separate the discussion of (3.38) and (3.39).

- Estimate of (3.38): to control this term, we will first control the integrand, which itself is an integration in u, shown on the second line, and with this term bounded, we move forward to control the next layer integration in v.
  - Based on the decomposition (3.36), the u-integration in the second line of (3.38) is further split into

$$\underbrace{\int_{|n(x)\cdot u|\le\varepsilon}}_{\text{term I}} + \underbrace{\int_{|u_\| - \frac{2T_M(1-r_\|)}{2T_M+(T_w(x)-2T_M)r_\|(2-r_\|)}v_\||\ge\varepsilon^{-1}}}_{\text{term II}} + \underbrace{\int_{|u_\perp - \frac{2T_M\sqrt{1-r_\perp}}{2T_M+(T_w(x)-2T_M)r_\perp}v_\perp|\ge\varepsilon^{-1}}}_{\text{term III}} \tag{3.40}$$

To control term I, we draw the similarity to (2.75) in Lemma 5. To be more specific, we apply (6.10) with

$$a = -\left[\frac{p^*}{4T_M} - \frac{p^*}{2T_w(x)}\right], b = \frac{p^*}{2T_w(x)r_\perp}, \varepsilon = 0, w = \sqrt{1-r_\perp}v_\perp.$$

Thus by (2.81) with $T_{k-1,i}$ replaced by $2T_M$, term I is bounded by

$$\varepsilon \exp\left(\frac{p^*[2T_M - T_w(x)][1-r_{min}]}{2T_w(x)[2T_M(1-r_{min}) + r_{min}T_w(x)]}|v|^2\right). \tag{3.41}$$

Similar techniques can be applied to analyze term II and term III. With

$$a = -\left[\frac{p^*}{4T_M} - \frac{p^*}{2T_w(x)}\right], b = \frac{p^*}{2T_w(x)r_\|(2-r_\|)}, \varepsilon = 0, w = (1-r_\|)v_\|$$

and

$$a = -\left[\frac{p^*}{4T_M} - \frac{p^*}{2T_w(x)}\right], b = \frac{p^*}{2T_w(x)r_\perp}, \varepsilon = 0, w = \sqrt{1-r_\perp}v_\perp$$

respectively, we have either

$$\frac{b}{b-a-\varepsilon}w = \frac{2T_M(1-r_\|)}{2T_M + (T_w(x)-2T_M)r_\|(2-r_\|)}v_\|,$$

or

$$\frac{b}{b-a-\varepsilon}w = \frac{2T_M\sqrt{1-r_\|}}{2T_M + (T_w(x)-2T_M)r_\perp}v_\perp,$$

which further bound the two terms by (3.41). Putting them back into (3.40) we have:

$$(3.40) \lesssim \varepsilon \exp\left(\frac{p[2T_M - T_w(x)][1-r_{min}]}{2T_w(x)[2T_M(1-r_{min}) + r_{min}T_w(x)]}|v|^2\right). \tag{3.42}$$

– With the integrand controlled, we move to the v-integration in (3.38). Plugging (3.42) into (3.38), we have the boundedness of the integrand:

$$\lesssim_p \varepsilon |n(x)\cdot v| e^{-p\lambda s\langle v\rangle} w^p_{\tilde\theta}(v) \exp\left(p\left[\frac{1}{4T_M} - \frac{1}{2[2T_M(1-r_{min})+r_{min}T_w(x)]}\right]|v|^2\right). \tag{3.43}$$

Taking $\tilde\theta = \tilde\theta(T_M, r_{min}) \ll 1$ such that

$$p\tilde\theta + p\Big[\frac{1}{4T_M} - \frac{1}{2[2T_M(1-r_{min})+r_{min}T_w(x)]}\Big]$$

$$\le p\tilde\theta + p\Big[\frac{1}{4T_M} - \frac{1}{2[2T_M(1-r_{min})+r_{min}T_M]}\Big] < 0, \tag{3.44}$$

one has (3.43)$\in L^1_v(\mathbb{R}^3)$.

Pull out the constant we finally conclude with

$$(3.38) \lesssim_{p,T_M,r_{min}} \epsilon \int_0^t |e^{-\lambda s\langle u\rangle} w_{\tilde\theta}(u) f^m|^p_{p,+}. \tag{3.45}$$

– Estimate of (3.39): note that comparing with the integrand in (3.38), here the integration in u is taken on $\gamma_+\backslash\gamma_{+,1}^{v,x,\epsilon}$, which does not provide a small ϵ. With brute-force calculation we only get:

$$\lesssim_p |n(x)\cdot v| e^{-p\lambda s\langle v\rangle} w^p_{\tilde\theta}(v)$$
$$\exp\left(p\Big[\frac{1}{4T_M} - \frac{1}{2[2T_M(1-r_{min})+r_{min}T_w(x)]}\Big]|v|^2\right) \in L^1_v(\mathbb{R}^3).$$

Now we decompose the v-integration into

$$\int_{n(x)\cdot v<0} = \int_{n(x)\cdot v<0} \mathbf{1}_{|v|>\epsilon^{-1}} + \mathbf{1}_{|v|\le\epsilon^{-1}}.$$

– When $|v| > \varepsilon^{-1}$, using the exponential decaying function (3.44) we obtain,

$$(3.39)\mathbf{1}_{|v|>\varepsilon^{-1}} \lesssim_{p,T_M,r_{min}} \epsilon \int_0^t |e^{-\lambda s\langle u\rangle} w_{\tilde\theta}(u) f^m|^p_{p,+}. \tag{3.46}$$

– When $|v| \le \epsilon^{-1}$, since $u \in \gamma_+\backslash\gamma_{+,1}^{v,x,\epsilon}$, for any $x\in\Omega$,

$$\begin{aligned}|u| \le |u_\perp| + |u_\parallel| \le\ & |u_\perp - \frac{2T_M\sqrt{1-r_\perp}}{2T_M + \big(T_w(x)-2T_M\big)r_\perp} v_\perp| \\ & + |\frac{2T_M\sqrt{1-r_\perp}}{2T_M + \big(T_w(x)-2T_M\big)r_\perp} v_\perp| \\ & + |u_\parallel - \frac{2T_M(1-r_\parallel)}{2T_M + (T_w(x)-2T_M)r_\parallel(2-r_\parallel)} v_\parallel| \\ & + |\frac{2T_M(1-r_\parallel)}{2T_M + (T_w(x)-2T_M)r_\parallel(2-r_\parallel)} v_\parallel| \le 6\epsilon^{-1}.\end{aligned} \tag{3.47}$$

In the derivation we used (1.28), $r_\parallel \le 1$ in the assumption (1.10) for

$$\frac{2T_M(1-r_\parallel)}{2T_M+(T_w(x)-2T_M)r_\parallel(2-r_\parallel)} < \frac{1}{(1-r_\parallel)+\frac{T_w(x)r_\parallel(2-r_\parallel)}{2T_M(1-r_\parallel)}} \le \frac{1}{1-\frac{1}{2}r_\parallel} \le 2, \tag{3.48}$$

and similarly to have

$$\frac{2T_M\sqrt{1-r_\perp}}{2T_M+[T_w(x)-2T_M]r_\perp} \le 2.$$

Then $u \in \gamma_+(x)\backslash\gamma_+^{\epsilon/6}(x)$, where $\gamma_+^{\varepsilon/6}$ is defined in (3.8). By Lemma 10 we obtain

$$\begin{aligned}(3.39)\mathbf{1}_{|v|\le\varepsilon^{-1}} &\lesssim_\varepsilon \int_0^t \int_{\partial\Omega}\int_{\gamma_+(x)/\gamma_+^{\epsilon/6}(x)} |e^{-\lambda s\langle u\rangle} w_{\tilde\theta}(u) f^m|^p \{n(x)\cdot u\} du dS_x ds \\ &\lesssim_\varepsilon \left[\|w_{\tilde\theta}(v) f(0)\|_p^p + \int_0^t \|e^{-\lambda s\langle v\rangle} w_{\tilde\theta}(v) f^m\|_p^p + (3.50)\right]\end{aligned} \tag{3.49}$$

with

$$\begin{aligned}&\int_0^t \iint_{\Omega\times\mathbb{R}^3} [\partial_t + v\cdot\nabla_x - \nabla_x\phi^{m-1}\cdot\nabla_v + \text{ LHS of (3.31)}]|e^{-\lambda s\langle u\rangle} w_{\tilde\theta}(u) f^m|^p \\ &\lesssim_p \sup_m \|w_{\theta'} f^m\|_\infty \Big(\int_0^t \|e^{-\lambda s\langle v\rangle} w_{\tilde\theta}(v) f^m\|_p^p + \int_0^t \|e^{-\lambda s\langle v\rangle} w_{\tilde\theta}(v) f^{m-1}\|_p^p\Big),\end{aligned} \tag{3.50}$$

where we apply (3.34) and replace $m+1$, m by m, $m-1$ respectively.

Adding (3.45), (3.46) and (3.49) back into (3.37), one has:

$$\begin{aligned}&\int_0^t |e^{-\lambda s\langle v\rangle} w_{\tilde\theta}(v) f^{m+1}|_{p,-}^p \\ &\le C(p,T_M,r_{min}) \times \epsilon \int_0^t |e^{-\lambda s\langle v\rangle} w_{\tilde\theta}(v) f^m|_{p,+}^p \\ &\quad + C(p,T_M,r_{min}) C(\varepsilon) \sup_m \|w_{\theta'} f^m\|_\infty \\ &\quad \times \Big(\|w_{\tilde\theta}(v) f(0)\|_p^p + \int_0^t \|e^{-\lambda s\langle v\rangle} w_{\tilde\theta}(v) f^m\|_p^p + \int_0^t \|e^{-\lambda s\langle v\rangle} w_{\tilde\theta}(v) f^{m-1}\|_p^p\Big),\end{aligned} \tag{3.51}$$

where $C(\varepsilon)$ comes from (3.49), $C(p, T_M, r_{min})$ comes from $\lesssim_{p,T_M,r_{min}}$ and $C(p)$ comes in (3.34).

We finally plug (3.34) and (3.51) back in (3.32), with condition for λ in (3.35) satisfied, we conclude with

$$\begin{aligned}
&\|e^{-\lambda t\langle v\rangle} w_{\tilde{\theta}}(v) f^{m+1}(t)\|_p^p + \int_0^t |e^{-\lambda s\langle v\rangle} w_{\tilde{\theta}}(v) f^{m+1}|_{p,+}^p \\
&\quad + \frac{\lambda}{4}\int_0^t \|\langle v\rangle^{1/p} e^{-\lambda s\langle v\rangle} w_{\tilde{\theta}}(v) f^{m+1}(t)\|_p^p \\
&\leq C(p, T_M, r_{min}) \times \epsilon \int_0^t |e^{-\lambda s\langle v\rangle} w_{\tilde{\theta}}(v) f^m|_{p,+}^p \\
&\quad + C(p, T_M, r_{min}) C(\varepsilon) \sup_m \|w_{\theta'} f^m\|_\infty \Big(\|w_{\tilde{\theta}}(v) f(0)\|_p^p \\
&\quad + t \sup_{0\leq s\leq t} \|e^{-\lambda s\langle v\rangle} w_{\tilde{\theta}} f^{m-1}(s)\|_p^p + t \sup_{0\leq s\leq t} \|e^{-\lambda s\langle v\rangle} w_{\tilde{\theta}} f^m(s)\|_p^p \Big).
\end{aligned} \tag{3.52}$$

Step 2: estimate of p-norm of ∂f:

We first write down the equation for $e^{-\lambda t\langle v\rangle} w_{\tilde{\theta}} \partial f^{m+1}$ with $\partial \in \{\nabla_{x_i}, \nabla_{v_i}\}$. According to (1.44) one has

$$[\partial_t + v\cdot\nabla_x - \nabla_x\phi^m\cdot\nabla_v + \nu_{\lambda,\phi^m,w}](e^{-\lambda t\langle v\rangle} w_{\tilde{\theta}} \partial f^{m+1}) = e^{-\lambda t\langle v\rangle} w_{\tilde{\theta}} \mathcal{G}^m, \tag{3.53}$$

with

$$\begin{aligned}
\mathcal{G}^m &= -\partial v\cdot\nabla_x f^{m+1} + \partial\nabla\phi^m\cdot\nabla_v f^{m+1} + \partial\Gamma_{\text{gain}}(f^m, f^m) - \partial\Gamma_{\text{loss}}(f^m, f^{m+1}) \\
&\quad -\partial\left(\frac{v}{2T_M}\cdot\nabla\phi^m(t,x)\right) f^{m+1}.
\end{aligned} \tag{3.54}$$

Considering (3.15) we have:

$$\begin{aligned}
|\mathcal{G}^m| &\lesssim |\nabla_x f^{m+1}| + |\nabla^2\phi^m||\nabla_v f^{m+1}| + |\Gamma_{\text{gain}}(\partial f^m, f^m)| \\
&\quad + |\Gamma_{\text{gain}}(f^m, \partial f^m)| + |\Gamma_{v,\text{gain}}(f^m, f^m)| \\
&\quad + |\Gamma_{\text{loss}}(\partial f^m, f^{m+1})| + |\Gamma_{\text{loss}}(f^m, \partial f^{m+1})| + |\Gamma_{v,\text{loss}}(f^m, f^{m+1})| \\
&\quad + w_\theta^{-1/2}(|\nabla\phi^m| + |\nabla^2\phi^m|)\|w_{\theta'} f^{m+1}\|_\infty.
\end{aligned} \tag{3.55}$$

By (3.31), we have

$$\nu_{\lambda,\phi^m,w} := \lambda\langle v\rangle + \frac{v}{2T_M}\cdot\nabla_x\phi^m(t,x) + \frac{\nabla_x\phi^m\cdot\nabla_v[e^{-\lambda t\langle v\rangle} w_{\tilde{\theta}}]}{e^{-\lambda t\langle v\rangle} w_{\tilde{\theta}}} \geq \frac{\lambda}{2}\langle v\rangle. \tag{3.56}$$

Since α is invariant to the transport equation, according to (1.43), we have

$$\begin{aligned}
&p|e^{-\lambda t\langle v\rangle} w_{\tilde{\theta}} \alpha^\beta_{f^m,\epsilon} \partial f^{m+1}|^{p-1} [\partial_t + v\cdot\nabla_x - \nabla_x\phi\cdot\nabla_v + \nu_{\lambda,\phi^m,w}] |e^{-\lambda t\langle v\rangle} w_{\tilde{\theta}} \alpha^\beta_{f^m,\epsilon} \partial f^{m+1}| \\
&= p e^{-\lambda p t\langle v\rangle} w^p_{\tilde{\theta}} \alpha^{\beta p}_{f^m,\epsilon} |\partial f^{m+1}|^{p-1} \mathcal{G}^m.
\end{aligned} \tag{3.57}$$

These allow us to apply Lemma 9 to (3.57) for

$$
\begin{aligned}
&\|e^{-\lambda t\langle v\rangle} w_{\tilde\theta}\alpha^{\beta}_{f^m,\epsilon}\partial f^{m+1}(t)\|_p^p + \frac{\lambda}{2}\int_0^t \|e^{-\lambda s\langle v\rangle}\langle v\rangle^{1/p} w_{\tilde\theta}\alpha^{\beta}_{f^m,\epsilon}\partial f^{m+1}\|_p^p \\
&\quad + \int_0^t |e^{-\lambda s\langle v\rangle} w_{\tilde\theta}\alpha^{\beta}_{f^m,\epsilon}\partial f^{m+1}|_{p,+}^p \\
&\quad \le \|w_{\tilde\theta}\alpha^{\beta}_{f,\epsilon}\partial f(0)\|_p^p + \underbrace{\int_0^t |e^{-\lambda s\langle v\rangle} w_{\tilde\theta}\alpha^{\beta}_{f^m,\epsilon}\partial f^{m+1}|_{p,-}^p}_{(3.58)_{\gamma_-}} \\
&\quad + \underbrace{\int_0^t \iint_{\Omega\times\mathbb{R}^3} p e^{-\lambda p s\langle v\rangle}\alpha^{\beta p}_{f^m,\epsilon} w^p_{\tilde\theta}|\partial f^{m+1}|^{p-1}|\mathcal{G}^m|}_{(3.58)_{\mathcal{G}^m}}.
\end{aligned}
\tag{3.58}
$$

The two terms will be separately considered in the later steps (Step 4 and 5 respectively). In the end we will obtain:

$$
\begin{aligned}
(3.58)_{\gamma_-} &\le C(p, T_M, r_{min}) \times \epsilon \int_0^t |e^{-\lambda s\langle v\rangle} w_{\tilde\theta}\alpha^{\beta}_{f^{m-1},\epsilon}\partial f^m|_{p,+}^p \\
&\quad + C(p, T_M, r_{min})C(\varepsilon)\|w_{\tilde\theta}\alpha^{\beta}_{f^{m-1},\epsilon}\nabla_{x,v} f(0)\|_p^p + \\
&\quad + C(p, T_M, r_{min})C(\varepsilon)\sup_m \|w_{\theta'} f^m\|_\infty \Big(\int_0^t \|e^{-\lambda s\langle v\rangle}\langle v\rangle^{1/p} w_{\tilde\theta}\alpha^{\beta}_{f^{m-1},\epsilon}\nabla_{x,v} f^m\|_p^p \\
&\quad + \int_0^t |e^{-\lambda s\langle v\rangle} w_{\tilde\theta} f^m|_{p,+}^p\Big) \\
&\quad + C(p, T_M, r_{min})C(\varepsilon) \times (\sup_m \|w_{\theta'} f^m\|_\infty + \sup_{l\le m}\|\nabla^2\phi^l\|_\infty) \\
&\quad \times \Big(\int_0^t \|e^{-\lambda s\langle v\rangle} w_{\tilde\theta} f^{m-1}\|_p^p + \int_0^t \|e^{-\lambda s\langle v\rangle} w_{\tilde\theta}\alpha^{\beta}_{f^{m-2},\epsilon}\nabla_{x,v} f^{m-1}\|_p^p \\
&\quad + \int_0^t \|e^{-\lambda s\langle v\rangle} w_{\tilde\theta}\alpha^{\beta}_{f^{m-1},\epsilon}\nabla_{x,v} f^m\|_p^p\Big),
\end{aligned}
\tag{3.59}
$$

and that

$$
\begin{aligned}
(3.58)_{\mathcal{G}^m} \le & C(p)\Big[\Big(1 + \sup_m \|w_{\theta'} f^m\|_\infty\Big)\int_0^t \|e^{-\lambda s\langle v\rangle}\langle v\rangle^{1/p} w_{\tilde\theta}\alpha^{\beta}_{f^m,\epsilon}\partial f^{m+1}\|_p^p \\
& + \Big(1 + \sup_m \|w_{\theta'} f^m\|_\infty + \|\nabla^2\phi^m\|_\infty\Big) \\
& \int_0^t \Big(\|e^{-\lambda s\langle v\rangle} w_{\tilde\theta}\alpha^{\beta}_{f^{m-1},\epsilon}\partial f^m\|_p^p + \|e^{-\lambda s\langle v\rangle} w_{\tilde\theta}\alpha^{\beta}_{f^m,\epsilon}\partial f^{m+1}\|_p^p\Big) \\
& + \Big(1 + \sup_m \|w_{\theta'} f^m\|_\infty\Big)\int_0^t \|e^{-\lambda s\langle v\rangle} w_{\tilde\theta} f^m\|_p^p\Big].
\end{aligned}
\tag{3.60}
$$

Inserting these back in (3.58) and using $\sup_{l\le m}\sup_{0\le s\le t}\|\nabla^2\phi^l(s)\|_\infty \lesssim \mathcal{E}^m < \infty$ and $\sup_m \|w_{\theta'} f^m\|_\infty < \infty$ according to Proposition 2, we have:

$$\begin{aligned}
&\|e^{-\lambda t\langle v\rangle} w_{\tilde\theta}\alpha^\beta_{f^m,\epsilon}\partial f^{m+1}(t)\|_p^p + \frac{\lambda}{4}\int_0^t \|e^{-\lambda s\langle v\rangle}\langle v\rangle^{1/p} w_{\tilde\theta}\alpha^\beta_{f^m,\epsilon}\partial f^{m+1}\|_p^p \\
&\quad + \int_0^t |e^{-\lambda s\langle v\rangle} w_{\tilde\theta}\alpha^\beta_{f^m,\epsilon}\partial f^{m+1}|_{p,+}^p \\
&\quad \le C(p,T_M,r_{min}) \times \epsilon \int_0^t |e^{-\lambda s\langle v\rangle} w_{\tilde\theta}\alpha^\beta_{f^{m-1},\epsilon}\partial f^m|_{p,+}^p \\
&\quad + C(p,T_M,r_{min})C(\varepsilon)\|w_{\tilde\theta}\alpha^\beta_{f^{m-1},\epsilon}\partial f(0)\|_p^p \\
&\quad + C(p,T_M,r_{min})C(\varepsilon)\sup_m \|w_{\theta'} f^m\|_\infty \Big(\int_0^t |e^{-\lambda s\langle v\rangle} w_{\tilde\theta} f^m|_{p,+}^p \\
&\quad + \int_0^t \|e^{-\lambda s\langle v\rangle}\langle v\rangle^{1/p} w_{\tilde\theta}\alpha^\beta_{f^{m-1},\epsilon}\partial f^m\|_p^p\Big) \\
&\quad + tC(p,T_M,r_{min},\varepsilon) \times (\sup_m \|w_{\theta'} f^m\|_\infty + \mathcal{E}^m) \\
&\quad \times \sup_{0\le s\le t}\Big(\|e^{-\lambda s\langle v\rangle} w_{\tilde\theta} f^{m-1}\|_p^p + \|e^{-\lambda s\langle v\rangle} w_{\tilde\theta} f^m\|_p^p \\
&\quad + \|e^{-\lambda s\langle v\rangle} w_{\tilde\theta}\alpha^\beta_{f^m,\epsilon}\partial f^{m+1}\|_p^p + \|e^{-\lambda s\langle v\rangle} w_{\tilde\theta}\alpha^\beta_{f^{m-1},\epsilon}\partial f^m\|_p^p \\
&\quad + \|e^{-\lambda s\langle v\rangle} w_{\tilde\theta}\alpha^\beta_{f^{m-2},\epsilon}\partial f^{m-1}\|_p^p\Big).
\end{aligned} \tag{3.61}$$

Step 3: summarize (collecting (3.52) **and** (3.61) **for the conclusion**):
Multiplying $\lambda \gg 1$ to (3.52) and adding to (3.61) we derive that

$$\begin{aligned}
&\lambda\|e^{-\lambda t\langle v\rangle} w_{\tilde\theta} f^{m+1}(t)\|_p^p + \|e^{-\lambda t\langle v\rangle} w_{\tilde\theta}\alpha^\beta_{f^m,\epsilon}\partial f^{m+1}(t)\|_p^p \\
&\quad + \frac{\lambda}{4}\int_0^t \|e^{-\lambda s\langle v\rangle}\langle v\rangle^{1/p} w_{\tilde\theta}\alpha^\beta_{f^m,\epsilon}\partial f^{m+1}(s)\|_p^p \\
&\quad + \int_0^t |e^{-\lambda s\langle v\rangle} w_{\tilde\theta}\alpha^\beta_{f^m,\epsilon}\partial f^{m+1}(s)|_{p,+}^p + \lambda\int_0^t |e^{-\lambda s\langle v\rangle} w_{\tilde\theta} f^{m+1}|_{p,+}^p \\
&\le C(p,T_M,r_{min})C(\varepsilon)\lambda\big(\|w_{\tilde\theta} f(0)\|_p^p + \|w_{\tilde\theta}\alpha^\beta_{f,\epsilon}\partial f(0)\|_p^p\big) + C(p,T_M,r_{min}) \\
&\quad \times \varepsilon\int_0^t |e^{-\lambda s\langle v\rangle} w_{\tilde\theta}\alpha^\beta_{f^{m-1},\epsilon}\partial f^m|_{p,+}^p \\
&\quad + C(p,T_M,r_{min})\Big(\frac{C(\varepsilon)\sup_m \|w_{\theta'} f^m\|_\infty}{\lambda} + \varepsilon\Big)\lambda\int_0^t |e^{-\lambda s\langle v\rangle} w_{\tilde\theta} f^m|_{p,+}^p \\
&\quad + C(p,T_M,r_{min})\frac{4C(\varepsilon)}{\lambda}\frac{\lambda}{4}\int_0^t \|e^{-\lambda s\langle v\rangle}\langle v\rangle^{1/p} w_{\tilde\theta}\alpha^\beta_{f^{m-1},\epsilon}\partial f^m\|_p^p) \\
&\quad + tC(p,T_M,r_{min})C(\varepsilon) \times (\sup_m \|w_{\theta'} f^m\|_\infty + \mathcal{E}^m)\lambda \\
&\quad \times \Bigg[\sup_{l=m,m-1}\sup_{0\le s\le t}\Big(\|e^{-\lambda s\langle v\rangle} w_{\tilde\theta} f^l\|_p^p \\
&\quad + \|e^{-\lambda s\langle v\rangle} w_{\tilde\theta}\alpha^\beta_{f^{l-1},\varepsilon}\partial f^l\|_p^p\Big) + \|e^{-\lambda s\langle v\rangle} w_{\tilde\theta}\alpha^\beta_{f^m,\varepsilon}\partial f^{m+1}\|_p^p\Bigg].
\end{aligned} \tag{3.62}$$

Recall the definition of $\mathcal{E}^m$ in (3.4), we have

$$\begin{aligned}(3.62) \leq\ & C(p, T_M, r_{min})C(\varepsilon)\lambda\big(\|w_{\tilde{\theta}} f(0)\|_p^p + \|w_{\tilde{\theta}}\alpha_{f,\epsilon}^{\beta}\partial f(0)\|_p^p\big) + \mathcal{E}^m \times C(p, T_M, r_{min}) \\ & \Big[\varepsilon + \big(\frac{C(\varepsilon)\sup_m \|w_{\theta'} f^m\|_\infty}{\lambda} + \varepsilon\big) + \frac{4C(\varepsilon)}{\lambda} + t \times C(\varepsilon)(\sup_m \|w_{\theta'} f^m\|_\infty + \mathcal{E}^m)\lambda\Big] \\ & + C(p, T_M, r_{min})tC(\varepsilon)(\sup_m \|w_{\theta'} f^m\|_\infty + \mathcal{E}^m)\lambda\|e^{-\lambda s\langle v\rangle} w_{\tilde{\theta}}\alpha_{f^m,\varepsilon}^{\beta}\partial f^{m+1}\|_p^p.\end{aligned} \tag{3.63}$$

First we take $\varepsilon = \varepsilon(p, T_M, r_{min}) \ll 1$ such that $2\varepsilon C(p, T_M, r_{min}) \leq \frac{1}{10}$. Then with ε fixed we let $\lambda = \lambda(p, T_M, r_{min}, \varepsilon) \gg 1$ satisfy

$$C(p, T_M, r_{min}) \times \Big(\frac{C(\varepsilon)\sup_m \|w_{\theta'} f^m\|_\infty}{\lambda} + \frac{4C(\varepsilon)}{\lambda}\Big) \leq \frac{1}{10}. \tag{3.64}$$

Then with ε, λ fixed we can define the constant C_W in (3.4) as

$$C_W := C(p, T_M, r_{min})C(\varepsilon)\lambda \gg 1, \tag{3.65}$$

where $C(p, T_M, r_{min})C(\varepsilon)\lambda$ is the coefficient for the first term in the RHS of (3.63).

Last we take $t_W = t_W(p, T_M, r_{min}, \varepsilon, \lambda, C_W, f_0)$ small with $t \leq t_W$ and apply the assumption in (3.4) such that

$$t_W \times C(p, T_M, r_{min})C(\varepsilon) \times (\sup_m \|w_{\theta'} f^m\|_\infty + \mathcal{E}^m)\lambda \leq \frac{1}{10}. \tag{3.66}$$

Finally collecting (3.62), (3.64), (3.65) and (3.66), since (3.62) holds for all $0 < t \leq t_W$, we obtain

$$\begin{aligned}& \sup_{t\leq t_W}\Big(\lambda\|e^{-\lambda t\langle v\rangle} w_{\tilde{\theta}} f^{m+1}(t)\|_p^p + \frac{9}{10}\|e^{-\lambda t\langle v\rangle} w_{\tilde{\theta}}\alpha_{f^m,\epsilon}^{\beta}\partial f^{m+1}(t)\|_p^p \\ & \quad + \frac{\lambda}{4}\int_0^t \|e^{-\lambda s\langle v\rangle}\langle v\rangle^{1/p} w_{\tilde{\theta}}\alpha_{f^m,\epsilon}^{\beta}\partial f^{m+1}(s)\|_p^p \\ & \quad + \int_0^t |e^{-\lambda s\langle v\rangle} w_{\tilde{\theta}}\alpha_{f^m,\epsilon}^{\beta}\partial f^{m+1}(s)|_{p,+}^p + \lambda\int_0^t |e^{-\lambda s\langle v\rangle} w_{\tilde{\theta}} f^{m+1}|_{p,+}^p\Big) \\ & \leq C(p, T_M, r_{min})C(\varepsilon)\lambda\big(\|w_{\tilde{\theta}} f(0)\|_p^p + \|w_{\tilde{\theta}}\alpha_{f,\epsilon}^{\beta}\partial f(0)\|_p^p\big) + \frac{3}{10}\sup_{t\leq t_W}\mathcal{E}^m(t) \\ & \leq (C_W + \frac{3\times 2}{10}C_W)\big(\|w_{\tilde{\theta}} f(0)\|_p^p + \|w_{\tilde{\theta}}\alpha_{f,\epsilon}^{\beta}\partial f(0)\|_p^p\big) \\ & \leq 2C_W\big(\|w_{\tilde{\theta}} f(0)\|_p^p + \|w_{\tilde{\theta}}\alpha_{f,\epsilon}^{\beta}\partial f(0)\|_p^p\big).\end{aligned}$$

Thus we prove (3.4) and conclude Proposition 3.

Step 4: estimate of (3.58)$\mathcal{G}^m$:

First we consider (3.58)$\mathcal{G}^m$. Directly the first two terms $|\nabla_x f^{m+1}| + |\nabla^2\phi^m||\nabla_v f^{m+1}|$ of (3.54) in (3.58) is bounded by

$$(1 + \|\nabla^2\phi^m\|_\infty)\int_0^t \|e^{-\lambda s\langle v\rangle} w_{\tilde{\theta}}\alpha_{f^m,\epsilon}^{\beta}\partial f^{m+1}(s)\|_p^p. \tag{3.67}$$

From (3.19) (3.20), the contribution of

$$|\Gamma_{\text{gain}}(\partial f^m, f^m)| + |\Gamma_{\text{loss}}(\partial f^m, f^{m+1})| + |\Gamma_{\text{gain}}(f^m, \partial f^m)|$$

of (3.54) in (3.58)$\mathcal{G}^m$ is bounded by

$$\begin{aligned}&\Big(1+\|w_{\theta'}f^m\|_\infty+\|w_{\theta'}f^{m+1}\|_\infty\Big)\\&\quad\times\int_0^t\iint_{\Omega\times\mathbb{R}^3}|e^{-\lambda s\langle v\rangle}\alpha^\beta_{f^m,\epsilon}w_{\tilde\theta}\partial f^{m+1}(v)|^{p-1}\\&\quad\int_{\mathbb{R}^3}\alpha^\beta_{f^m,\epsilon}(v)\mathbf{k}_\rho(v,u)w_{\tilde\theta}(v)|\partial f^m(u)|dudvdxds.\end{aligned}\tag{3.68}$$

The estimate of (3.68) will be carried out in Step3.

From (3.20), the contribution of $|\Gamma_{\text{loss}}(f^m,\partial f^{m+1})|$ of (3.54) in (3.58)$\mathcal{G}^m$ is bounded by

$$\|w_{\theta'}f^m\|_\infty\int_0^t\|e^{-\lambda s\langle v\rangle}\langle v\rangle^{1/p}w_{\tilde\theta}\alpha^\beta_{f^m,\epsilon}\partial f^{m+1}\|_p^p.\tag{3.69}$$

From (3.21), the contribution of $|\Gamma_{v,\text{loss}}(f^m,f^{m+1})|$ of (3.54) in (3.58)$\mathcal{G}^m$ is bounded by

$$\begin{aligned}&\|w_{\theta'}f^{m+1}\|_\infty\int_0^t\iint_{\Omega\times\mathbb{R}^3}pe^{-\lambda(p-1)s\langle v\rangle}|\alpha^\beta_{f^m,\epsilon}w_{\tilde\theta}\partial f^{m+1}|^{p-1}\\&\quad e^{-\lambda s\langle v\rangle}\alpha^\beta_{f^m,\epsilon}\langle v\rangle\frac{w_{\tilde\theta}}{w_{\theta'}}\|e^{-\lambda s\langle u\rangle}w_{\tilde\theta}(u)f^m(s,x,u)\|_{L^p(\mathbb{R}^3)}\\&\lesssim\|w_{\theta'}f^{m+1}\|_\infty\Big(\int_0^t\iint_{\Omega\times\mathbb{R}^3}|e^{-\lambda s\langle v\rangle}\alpha^\beta_{f^m,\epsilon}\partial f^{m+1}|^p\\&\quad+\int_0^t\iint_{\Omega\times\mathbb{R}^3}|e^{-\lambda s\langle u\rangle}w_{\tilde\theta}(u)f^m(u)|^p\Big),\end{aligned}\tag{3.70}$$

where we use

$$e^{-\lambda s\langle v\rangle}\alpha^\beta_{f^m,\epsilon}\langle v\rangle\frac{w_{\tilde\theta}}{w_{\theta'}}\lesssim w_{\theta'}^{-1/2}.$$

From (3.22), the contribution of $|\Gamma_{v,\text{gain}}(f^m,f^m)|$ in (3.58)$\mathcal{G}$ is bounded by

$$\begin{aligned}&\|w_{\theta'}f^m\|_\infty\int_0^t\iint_{\Omega\times\mathbb{R}^3}e^{-\lambda(p-1)s\langle v\rangle}\alpha^{\beta p}_{f^m,\epsilon}|w_{\tilde\theta}\partial f^{m+1}(v)|^{p-1}\\&\quad\int_{\mathbb{R}^3}\mathbf{k}_\rho(v,u)\frac{e^{\lambda s\langle u\rangle}w_{\tilde\theta}(v)}{e^{\lambda s\langle v\rangle}w_{\tilde\theta}(u)}e^{-\lambda s\langle u\rangle}w_{\tilde\theta}(u)|f^m(u)|\\&\lesssim_p\|w_{\theta'}f^m\|_\infty\Big(\int_0^t\|\langle v\rangle^{1/p}e^{-\lambda s\langle v\rangle}\alpha^\beta_{f^m,\epsilon}w_{\tilde\theta}\partial f^{m+1}\|_p^p\\&\quad+\int_0^t\|e^{-\lambda s\langle v\rangle}w_{\tilde\theta}f^m\|_p^p\Big),\end{aligned}\tag{3.71}$$

where we have used, for $1/p + 1/p^* = 1$ and $0 < \tilde{\rho} \ll \rho$, from (3.16) and (3.17)

$$\begin{aligned}
&\int_0^t \iint_{\Omega\times\mathbb{R}^3} e^{-\lambda(p-1)s\langle v\rangle}\alpha_{f^m,\epsilon}^{\beta p}|w_{\tilde{\theta}}\partial f^{m+1}(v)|^{p-1} \\
&\quad \int_{\mathbb{R}^3}\mathbf{k}_\rho(v,u)\frac{e^{\lambda s\langle u\rangle}w_{\tilde{\theta}}(v)}{e^{\lambda s\langle v\rangle}w_{\tilde{\theta}}(u)}e^{-\lambda s\langle u\rangle}w_{\tilde{\theta}}(u)|f^m(u)| \\
&\leq \int_0^t \iint_{\Omega\times\mathbb{R}^3}\frac{\alpha_{f^m,\epsilon}^{\beta}(v)}{\langle v\rangle^{(p-1)/p}}|\langle v\rangle^{1/p}e^{-\lambda s\langle v\rangle}\alpha_{f^m,\epsilon}^{\beta}w_{\tilde{\theta}}\partial f^{m+1}|^{p-1} \\
&\quad \times\Big(\int_{\mathbb{R}^3}\mathbf{k}_{\tilde{\rho}}(v,u)du\Big)^{1/p^*}\Big(\mathbf{k}_{\tilde{\rho}}(v,u)|e^{-\lambda s\langle u\rangle}w_{\tilde{\theta}}f^m(u)|^p du\Big)^{1/p}dv \\
&\leq \int_0^t\int_\Omega\Big(\int_{\mathbb{R}^3}|\langle v\rangle^{1/p}e^{-\lambda s\langle v\rangle}\alpha_{f^m,\epsilon}^{\beta}w_{\tilde{\theta}}\partial f^{m+1}|^p\Big)^{\frac{p-1}{p}}\Big(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\mathbf{k}_{\tilde{\rho}}(v,u)|e^{-\lambda s\langle u\rangle}w_{\tilde{\theta}}f^m(u)|^p\Big)^{1/p} \\
&\lesssim_p \int_0^t\iint_{\Omega\times\mathbb{R}^3}|e^{-\lambda s\langle v\rangle}\langle v\rangle^{1/p}\alpha_{f^m,\epsilon}^{\beta}w_{\tilde{\theta}}\partial f^{m+1}|^p + |e^{-\lambda s\langle u\rangle}w_{\tilde{\theta}}f^m|^p.
\end{aligned}$$

In the last step we have applied the Young's inequality.

We focus on (3.68). We split the u-integration of (3.68) into the integration over $\{|u| \leq N\}$ and $\{|u| > N\}$.

The contribution of $\{|u| \geq N\}$ in (3.68) is bounded by

$$\begin{aligned}
&\int_0^t\int_{\Omega\times\mathbb{R}^3}|e^{-\lambda s\langle v\rangle}\langle v\rangle^{1/p}w_{\tilde{\theta}}\alpha_{f^m,\epsilon}^{\beta}\partial f^{m+1}(v)|^{p-1}\frac{\alpha_{f^m,\epsilon}^{\beta}}{\langle v\rangle^{p/(p-1)}} \\
&\quad\times\int_{|u|\geq N}\mathbf{k}_\rho(v,u)\frac{w_{\tilde{\theta}}(v)}{w_{\tilde{\theta}}(u)}\frac{e^{-\lambda s\langle v\rangle}}{e^{-\lambda s\langle u\rangle}}|e^{-\lambda s\langle u\rangle}w_{\tilde{\theta}}\partial f^m(u)|dudvdxds \\
&\leq\int_0^t\int_\Omega\Big(\int_{\mathbb{R}^3}|e^{-\lambda s\langle v\rangle}\langle v\rangle^{1/p}w_{\tilde{\theta}}\alpha_{f^m,\epsilon}^{\beta}\partial f^{m+1}(v)|^p\Big)^{1/p^*} \qquad (3.72)\\
&\quad\Big(\int_{|u|\geq N}|e^{-\lambda s\langle u\rangle}w_{\tilde{\theta}}\alpha_{f^{m-1},\epsilon}^{\beta}\partial f^m(u)|^p\int_v\mathbf{k}_{\tilde{\rho}}(v,u)\Big)^{1/p} \\
&\leq\int_0^t\|e^{-\lambda s\langle v\rangle}\langle v\rangle^{1/p}w_{\tilde{\theta}}\alpha_{f^m,\epsilon}^{\beta}\partial f^{m+1}(s)\|_p^p ds+\int_0^t\|e^{-\lambda s\langle v\rangle}w_{\tilde{\theta}}\alpha_{f^{m-1},\epsilon}^{\beta}\partial f^m(s)\|_p^p ds,
\end{aligned}$$

where we use Hölder inequality, Proposition 4 with $\beta\frac{p}{p-1} < 1$, $\frac{\alpha_{f^m,\epsilon}^{\beta}}{\langle v\rangle^{p/(p-1)}} \leq 1$. And we apply (3.16) to get

$$\begin{aligned}
&\int_{|u|\geq N}\mathbf{k}_\rho(v,u)\frac{w_{\tilde{\theta}}(v)}{w_{\tilde{\theta}}(u)}\frac{e^{-\lambda s\langle v\rangle}}{e^{-\lambda s\langle u\rangle}}|e^{-\lambda s\langle u\rangle}w_{\tilde{\theta}}\partial f^m(u)|du \\
&\leq\int_{|u|\geq N}\mathbf{k}_{\tilde{\rho}}(v,u)\frac{1}{\alpha_{f^{m-1},\epsilon}^{\beta}}|e^{-\lambda s\langle u\rangle}\alpha_{f^{m-1},\epsilon}^{\beta}w_{\tilde{\theta}}(u)\partial f^m(u)|du \\
&\lesssim\Big(\int_{|u|\geq N}\mathbf{k}_{\tilde{\rho}}(v,u)\frac{1}{\alpha_{f^{m-1},\epsilon}^{\beta p^*}(u)}\Big)^{1/p^*}\Big(\int_{|u|\geq N}\mathbf{k}_{\tilde{\rho}}(v,u)|e^{-\lambda s\langle u\rangle}\alpha_{f^{m-1},\epsilon}^{\beta}w_{\tilde{\theta}}(u)\partial f^m(u)|^p\Big)^{1/p} \\
&\lesssim\Big(\int_{|u|\geq N}\mathbf{k}_{\tilde{\rho}}(v,u)|e^{-\lambda s\langle u\rangle}\alpha_{f^{m-1},\epsilon}^{\beta}w_{\tilde{\theta}}(u)\partial f^m(u)|^p\Big)^{1/p}.
\end{aligned}$$

The contribution of $\{|u| \leq N\}$ in (3.68) is bounded by, from Hölder inequality,

$$\begin{aligned}
&\int_0^t \iint_{\Omega\times\mathbb{R}^3} |e^{-\lambda s\langle v\rangle}\langle v\rangle^{1/p} w_{\tilde{\theta}}\alpha_{f^m,\epsilon}(v)^\beta \partial f^{m+1}(v)|^{p-1} \\
&\quad\times \int_{|u|\leq N} \mathbf{k}_\rho(v,u)\frac{w_{\tilde{\theta}}(v)}{w_{\tilde{\theta}}(u)}\frac{e^{-\lambda s\langle v\rangle}}{e^{-\lambda s\langle u\rangle}}\frac{\alpha_{f^m,\epsilon}(v)^\beta |e^{-\lambda s\langle u\rangle} w_{\tilde{\theta}}\alpha^\beta_{f^{m-1},\epsilon}\partial f^m(u)|}{\langle v\rangle^{(p-1)/p}\alpha^\beta_{f^{m-1},\epsilon}(u)} dudvdxds \\
&\leq \int_0^t \|e^{-\lambda s\langle v\rangle}\langle v\rangle^{1/p} w_{\tilde{\theta}}\alpha^\beta_{f^m,\epsilon}\partial f^{m+1}(v)\|_p^{p-1} \\
&\quad\times \left[\iint_{\Omega\times\mathbb{R}^3}\left(\int_{|u|\leq N}\mathbf{k}_{\tilde{\rho}}(v,u)\frac{|e^{-\lambda s\langle u\rangle} w_{\tilde{\theta}}\alpha^\beta_{f^{m-1},\epsilon}\partial f^m(u)|}{\alpha^\beta_{f^{m-1},\epsilon}(u)}du\right)^p dvdx\right]^{1/p} ds.
\end{aligned} \tag{3.73}$$

By the Hölder inequality, the u-integration part of (3.73) as

$$\|e^{-\lambda s\langle v\rangle} w_{\tilde{\theta}}\alpha^\beta_{f^{m-1},\epsilon}\partial f^m(\cdot)\|_{L^p(\mathbb{R}^3)} \times \left(\int_{\mathbb{R}^3}\frac{e^{-p^*\tilde{\rho}|v-u|^2}}{|v-u|^{p^*}}\frac{\mathbf{1}_{|u|\leq N}}{\alpha^{\beta p^*}_{f^{m-1},\epsilon}(u)}\right)^{1/p^*}. \tag{3.74}$$

Note that

$$\left(\int_{\mathbb{R}^3}\frac{e^{-p^*\tilde{\rho}|v-u|^2}}{|v-u|^{p^*}}\frac{\mathbf{1}_{|u|\leq N}}{\alpha^{\beta p^*}_{f^{m-1},\epsilon}(u)}\right)^{1/p^*} \leq \left|\frac{1}{|\cdot|^{p^*}} * \frac{\mathbf{1}_{|\cdot|\leq N}}{\alpha^{\beta p^*}_{f^{m-1},\epsilon}}\right|^{1/p^*}.$$

By the Hardy-Littlewood-Sobolev inequality with

$$1 + \frac{1}{p/p^*} = \frac{1}{3/p^*} + \frac{1}{\frac{3}{2}\frac{p-1}{p}},$$

we have

$$\begin{aligned}
&\left\|\left|\frac{1}{|\cdot|^{p^*}} * \frac{\mathbf{1}_{|\cdot|\leq N}}{\alpha^{\beta p^*}_{f^{m-1},\epsilon}(u)}\right|\right\|_{L^p(\mathbb{R}^3)} = \left\|\frac{1}{|\cdot|^{p^*}} * \frac{\mathbf{1}_{|\cdot|\leq N}}{\alpha^{\beta p^*}_{f^{m-1},\epsilon}(u)}\right\|_{L^{p/p^*}(\mathbb{R}^3)} \\
&\lesssim \left\|\frac{\mathbf{1}_{|\cdot|\leq N}}{\alpha^{\beta p^*}_{f^{m-1},\epsilon}(\cdot)}\right\|_{L^{\frac{3(p-1)}{2p}}(\mathbb{R}^3)} \lesssim \left(\int_{\mathbb{R}^3}\frac{\mathbf{1}_{|v|\leq N}}{\alpha_{f^{m-1},\epsilon}(v)^{\frac{p}{p-1}\beta\frac{3(p-1)}{2p}}}dv\right)^{\frac{2p}{3(p-1)}\frac{p-1}{p}} \\
&\lesssim \left(\int_{\mathbb{R}^3}\frac{\mathbf{1}_{|v|\leq N}}{\alpha^{3\beta/2}_{f^{m-1},\epsilon}(v)}\right)^{2/3} \lesssim 1,
\end{aligned} \tag{3.75}$$

where we use $3\beta/2 < 1$ and Proposition 4. Using (3.75) (3.74) (3.72) we have

$$\begin{aligned}
(3.68) \lesssim (1+\sup_m \|w_{\theta'}f^m\|_\infty)\Big[&\int_0^t \|e^{-\lambda s\langle v\rangle}\langle v\rangle^{1/p} w_{\tilde{\theta}}\alpha^\beta_{f^m,\epsilon}\partial f^{m+1}\|_p^p \\
&+ \int_0^t \|e^{-\lambda s\langle v\rangle} w_{\tilde{\theta}}\alpha^\beta_{f^{m-1},\epsilon}\partial f^m\|_p^p\Big].
\end{aligned} \tag{3.76}$$

Finally from (3.67) (3.69) (3.72) (3.71) (3.70) (3.76), (3.58)$\mathcal{G}^m$ has a bound as

$$\begin{aligned} &C(p)\Big[\Big(1+\sup_m \|w_{\theta'} f^m\|_\infty\Big)\int_0^t \|e^{-\lambda s\langle v\rangle}\langle v\rangle^{1/p} w_{\tilde\theta}\alpha^\beta_{f^m,\epsilon}\partial f^{m+1}\|_p^p \\ &+\Big(1+\sup_m \|w_{\theta'} f^m\|_\infty + \|\nabla^2\phi^m\|_\infty\Big)\int_0^t \Big(\|e^{-\lambda s\langle v\rangle} w_{\tilde\theta}\alpha^\beta_{f^{m-1},\epsilon}\partial f^m\|_p^p \\ &+\|e^{-\lambda s\langle v\rangle} w_{\tilde\theta}\alpha^\beta_{f^m,\epsilon}\partial f^{m+1}\|_p^p\Big) \\ &+\Big(1+\sup_m \|w_{\theta'} f^m\|_\infty\Big)\int_0^t \|e^{-\lambda s\langle v\rangle} w_{\tilde\theta} f^m\|_p^p\Big]. \end{aligned} \tag{3.77}$$

In order to control the first line in (3.77) by $\frac{\lambda}{2}\int_0^t \|e^{-\lambda s\langle v\rangle}\langle v\rangle^{1/p} w_{\tilde\theta}\alpha^\beta_{f^m,\epsilon}\partial f^{m+1}\|_p^p$ in (3.58), we require the λ satisfy

$$\frac{\lambda}{4} \geq C(p)(1+\sup_m \|w_{\theta'} f^m\|_\infty). \tag{3.78}$$

Step 5: estimate of (3.58)γ_-:
We focus on (3.58)γ_-. The overall strategy is similar to (3.51). From (3.23) (3.24)

$$\begin{aligned} &\int_0^t |e^{-\lambda s\langle v\rangle} w_{\tilde\theta}\alpha^\beta_{f^m,\epsilon}\partial f^{m+1}|_{p,-}^p \\ &= \int_0^t\int_{\partial\Omega}\int_{n(x)\cdot v<0} |n(x)\cdot v|^{\beta p}|e^{-\lambda s\langle v\rangle} w_{\tilde\theta}\nabla_{x,v} f^{m+1}(t,x,v)|^p |n(x)\cdot v| dv \\ &\lesssim \int_0^t\int_{\partial\Omega}\int_{n(x)\cdot v<0}\langle v\rangle^{2p} \\ &\quad e^{-\lambda ps\langle v\rangle} e^{p[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]|v|^2} w_{\tilde\theta}^p\Big(|n(x)\cdot v|^{\beta p+1}+|n(x)\cdot v|^{(\beta-1)p+1}\Big)\times|(3.24)|^p dv. \end{aligned} \tag{3.79}$$

Now we bound $|(3.24)|^p$.

– First line of (3.24), we split the u-integration into $\gamma_{+,2}^{v,x,\epsilon}\cup\Big(\gamma_+(x)\backslash\gamma_{+,2}^{v,x,\epsilon}\Big)$, where

$$\begin{aligned} \gamma_{+,2}^{v,x,\epsilon} &= \{(x,u)\in\gamma_+ : |n(x)\cdot u|\leq\epsilon \text{ or } |u_\parallel - \frac{2T_\zeta(1-r_\parallel)}{2T_\zeta+(T_w(x)-2T_\zeta)r_\parallel(2-r_\parallel)}v_\parallel|\geq\epsilon^{-1} \\ &\text{or } |u_\perp - \frac{2T_\zeta\sqrt{1-r_\perp}}{2T_\zeta+\big(T_w(x)-2T_\zeta\big)r_\perp}v_\perp|\geq\epsilon^{-1}\}, \end{aligned} \tag{3.80}$$

and T_ζ will be defined later in (3.88).

By the Hölder inequality

$$\Big(\int_{n(x)\cdot u>0}|e^{-\lambda s\langle u\rangle}w_{\tilde\theta}\alpha^{\beta}_{f^{m-1},\epsilon}\nabla_{x,v}f^m(s,x,u)|\{e^{\lambda s\langle v\rangle}w_{\tilde\theta}^{-1}\alpha^{-\beta}_{f^{m-1},\epsilon}(u)\}\langle u\rangle$$

$$e^{-[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]|u|^2}d\sigma(u,v)\Big)^p$$

$$\lesssim\Big(\int_{\gamma_{+,2}^{v,x,\epsilon}(x)}|e^{-\lambda s\langle u\rangle}w_{\tilde\theta}\alpha^{\beta}_{f^{m-1},\epsilon}\nabla_{x,v}f^m(s,x,u)|^p\{n(x)\cdot u\}du\Big)$$

$$\times\Big(\int_{\gamma_{+,2}^{v,x,\epsilon}(x)}\underbrace{\{e^{-\lambda s\langle u\rangle}w_{\tilde\theta}\alpha^{\beta}_{f^{m-1},\epsilon}(u)\}^{-p^*}}_{(I)}|n(x)\cdot u|\langle u\rangle^{p^*}e^{-p^*[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]|u|^2}$$

$$\times I_0\Big(\frac{(1-r_\perp)^{1/2}u_\perp v_\perp}{T_w(x)r_\perp}\Big)^{p^*}e^{-\frac{p^*}{2T_w(x)}\Big[\frac{|u_\perp|^2+(1-r_\perp)|v_\perp|^2}{r_\perp}+\frac{|u_\|-(1-r_\|)v_\||^2}{r_\|(2-r_\|)}\Big]}du\Big)^{p/p^*} \tag{3.81}$$

$$+\Big(\int_{\gamma_+(x)\backslash\gamma_{+,2}^{v,x,\epsilon}(x)}|e^{-\lambda s\langle u\rangle}w_{\tilde\theta}\alpha^{\beta}_{f^{m-1},\epsilon}\nabla_{x,v}f^m(s,x,u)|^p\{n(x)\cdot u\}du\Big)$$

$$\times\Big(\int_{\gamma_+(x)\backslash\gamma_{+,2}^{v,x,\epsilon}(x)}\{e^{-\lambda s\langle u\rangle}w_{\tilde\theta}\alpha^{\beta}_{f^{m},\epsilon}(s,x,u)\}^{-p^*}|n(x)\cdot u|\langle u\rangle^{p^*}e^{-p^*[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]|u|^2}$$

$$\times I_0\Big(\frac{(1-r_\perp)^{1/2}u_\perp v_\perp}{T_w(x)r_\perp}\Big)^{p^*}e^{-\frac{p^*}{2T_w(x)}\Big[\frac{|u_\perp|^2+(1-r_\perp)|v_\perp|^2}{r_\perp}+\frac{|u_\|-(1-r_\|)v_\||^2}{r_\|(2-r_\|)}\Big]}du\Big)^{p/p^*}. \tag{3.82}$$

Similar to Step 1, we separate the discussion of (3.81) and (3.82).

- estimate of (3.81).
  - To compute the u-integration, for any $c>0$ we bound

$$e^{\lambda s p^*\langle u\rangle}|n(x)\cdot u|^c\langle u\rangle^{p^*}\lesssim e^{p^*\zeta|u|^2}, \tag{3.83}$$

where ζ will be defined later in (3.89). Then we introduce $c_1>1$ with $1=\frac{1}{c_1}+\frac{1}{c_1^*}$ to deal with the $\alpha_{f^{m-1},\varepsilon}$ in (I). Then the u-integration is

$$\begin{aligned}&\lesssim\int_{\gamma_{+,2}^{v,x,\epsilon}(x)}\{e^{-\lambda s\langle u\rangle}w_{\tilde\theta}\alpha^{\beta}_{f^{m-1},\epsilon}(u)\}^{-p^*}|n(x)\cdot u|^{1/c_1}|n(x)\cdot u|^{1/c_1^*}\langle u\rangle^{p^*}e^{-p^*[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]|u|^2}\\&\quad\times I_0\Big(\frac{(1-r_\perp)^{1/2}u_\perp v_\perp}{T_w(x)r_\perp}\Big)^{p^*}e^{-\frac{p^*}{2T_w(x)}\Big[\frac{|u_\perp|^2+(1-r_\perp)|v_\perp|^2}{r_\perp}+\frac{|u_\|-(1-r_\|)v_\||^2}{r_\|(2-r_\|)}\Big]}du\\&\lesssim\int_{\gamma_{+,2}^{v,x,\epsilon}(x)}\Big[w_{\tilde\theta}\alpha^{\beta}_{f^{m-1},\epsilon}(u)\Big]^{-p^*}|n(x)\cdot u|^{1/c_1^*}e^{p^*\zeta|u|^2}e^{-p^*\Big[\frac{1}{4T_M}-\frac{1}{2T_w(x)}\Big]|u|^2}\\&\quad\times I_0\Big(p^*\frac{(1-r_\perp)^{1/2}u_\perp v_\perp}{T_w(x)r_\perp}\Big)e^{-\frac{p^*}{2T_w(x)}\Big[\frac{|u_\perp|^2+(1-r_\perp)|v_\perp|^2}{r_\perp}+\frac{|u_\|-(1-r_\|)v_\||^2}{r_\|(2-r_\|)}\Big]}du,\end{aligned} \tag{3.84}$$

where we have applied (3.83). Applying the Hölder inequality once more with $1 = \frac{1}{c_1} + \frac{1}{c_1^*}$, we obtain

$$(3.84) \lesssim \Big(\int_{\gamma_{+,2}^{v,x,\epsilon}(x)} \big[w_{\tilde{\theta}}\alpha^{\beta}_{f^{m-1},\epsilon}(u)\big]^{-p^*c_1} du\Big)^{\frac{1}{c_1}} \tag{3.85}$$

$$\times \Big(\int_{\gamma_{+,2}^{v,x,\epsilon}(x)} |n(x)\cdot u| e^{-c_1^* p^*[\frac{1}{4T_M} - \frac{1}{2T_w(x)} - \zeta]|u|^2} I_0\Big(c_1^* p^* \frac{(1-r_\perp)^{1/2}u_\perp v_\perp}{T_w(x) r_\perp}\Big) \tag{3.86}$$

$$e^{-\frac{c_1^* p^*}{2T_w(x)}\Big[\frac{|u_\perp|^2 + (1-r_\perp)|v_\perp|^2}{r_\perp} + \frac{|u_\parallel - (1-r_\parallel)v_\parallel|^2}{r_\parallel(2-r_\parallel)}\Big]} du\Big)^{\frac{1}{c_1^*}}. \tag{3.87}$$

We choose c_1 to be close to 1^+ to guarantee $\beta p^* c_1 < 1$. Using Proposition 4 with $v = 0$ and $w_{\tilde{\theta}}^{-p^*c_1} = e^{-\tilde{\theta}p^*c_1|u|^2}$, we have (3.85)$\lesssim_p 1$.
For (3.87) we let $\zeta < \frac{1}{4T_M}$ and denote

$$\frac{1}{4T_\zeta} = \frac{1}{4T_M} - \zeta, \quad T_\zeta > T_M. \tag{3.88}$$

By $0 < r_{min} \le 1$, we choose $\zeta = \zeta(T_M, r_{min})$ to be small such that

$$2T_\zeta(1 - r_{min}) + T_M r_{min} < 2T_M, \quad \frac{T_M}{T_\zeta} > 1/2. \tag{3.89}$$

- To control (3.87), recall the definition of (3.80). Here we simply replace the T_M in (3.36) by T_ζ. Thus we can apply the same decomposition as in (3.40) and obtain the result as (3.42) in Step 1 with replacing T_M by T_ζ. We get

$$(3.87) \lesssim \varepsilon \exp\Big(\frac{2T_\zeta - T_w(x)}{2T_w(x)[2T_\zeta(1-r_{min}) + r_{min}T_w(x)]}(1-r_{min})p^*|v|^2\Big). \tag{3.90}$$

Thus we obtain

$$(3.81) \lesssim_p \exp\Big(\frac{2T_\zeta - T_w(x)}{2T_w(x)[2T_\zeta(1-r_{min}) + rT_w(x)]}(1-r_{min})p|v|^2\Big)$$
$$\times \epsilon \int_{\gamma_{+,2}^{v,x,\epsilon}(x)} |e^{-\lambda s\langle u\rangle} w_{\tilde{\theta}}\alpha^{\beta}_{f^{m-1},\epsilon}\nabla_{x,v} f^m(s,x,u)|^p \{n(x)\cdot u\} du. \tag{3.91}$$

- With the integrand controlled, we move to the v-integration (3.79). Plugging (3.91) into (3.79) we have the boundedness of the integrand:

$$\langle v\rangle^{2p} e^{-\lambda ps\langle v\rangle}\Big(|n(x)\cdot v|^{\beta p+1} + |n(x)\cdot v|^{(\beta-1)p+1}\Big) w_{\tilde{\theta}}^p$$
$$\times \exp\Big(p\Big[\frac{1}{4T_M} - \frac{1}{2[2T_\zeta(1-r_{min}) + r_{min}T_w(x)]}\Big]|v|^2\Big), \tag{3.92}$$

where we apply the same computation as (3.43).
By (3.2), $(\beta-1)p+1 > -1$, thus $|n(x)\cdot v|^{(\beta-1)p+1} \in L^1_{\text{loc}}$. Using (3.89) we derive

$$2[2T_\zeta(1-r_{min}) + T_w(x)r_{min}] < 4T_M,$$

We take $\tilde{\theta} = \tilde{\theta}(\zeta, r_{min}, T_M) \ll 1$ to obtain

$$p\tilde{\theta} + p[\frac{1}{4T_M} - \frac{1}{2[2T_\zeta(1-r_{min}) + r_{min}T_w(x)]}] < 0. \tag{3.93}$$

Thus we derive (3.92)$\in L^1_v$.
Therefore, the contribution of (3.81) in (3.58)γ_- is

$$\lesssim_{p,T_M,r_{min}} \varepsilon \int_0^t |e^{-\lambda s\langle u\rangle} w_{\tilde{\theta}} \alpha^\beta_{f^{m-1},\epsilon} \nabla_{x,v} f^m|^p_{p,+}. \tag{3.94}$$

– estimate of (3.82). Similar to the Step 1, the integration in u does not provide a small ε. Thus we have

$$\begin{aligned}(3.82) \lesssim \exp\Big(&\frac{2T_\zeta - T_w(x)}{2T_w(x)[2T_\zeta(1-r_{min}) + r_{min}T_w(x)]}(1-r_{min})p|v|^2\Big)\\ &\times\Big(\int_{\gamma_+(x)\backslash\gamma_{+,2}^{v,x,\epsilon}(x)} |e^{-\lambda s\langle u\rangle} w_{\tilde{\theta}} \alpha^\beta_{f^{m-1},\epsilon} \nabla_{x,v} f^m(s,x,u)|^p\{n(x)\cdot u\}du\Big).\end{aligned} \tag{3.95}$$

Plugging (3.95) into (3.79) we conclude the integrand is given by (3.92)$\in L^1_v$. Again we decompose v into $\mathbf{1}_{|v|\le\epsilon^{-1}}$ and $\mathbf{1}_{|v|>\epsilon^{-1}}$.

- When $|v| > \varepsilon^{-1}$, by the exponential decaying function in (3.92) the contribution of (3.82)$\mathbf{1}_{|v|>\epsilon^{-1}}$ in (3.58)γ_-

$$\lesssim_{p,T_M,r_{min}} \epsilon \int_0^t \int_{\partial\Omega} (3.95) \le \epsilon \int_0^t |e^{-\lambda s\langle u\rangle} w_{\tilde{\theta}} \alpha^\beta_{f^m,\epsilon} \nabla_{x,v} f^m|^p_{p,+}. \tag{3.96}$$

- When $|v| \le \epsilon^{-1}$, since $u \in \gamma_+\backslash\gamma_{+,2}^{v,x,\varepsilon}$, for any $x \in \partial\Omega$ we have

$$\begin{aligned}|u| \le 2\varepsilon^{-1} &+ |\frac{2T_\zeta\sqrt{1-r_\perp}}{2T_\zeta + (T_w(x) - 2T_\zeta)r_\perp} v_\parallel|\\ &+ |\frac{2T_\zeta(1-r_\parallel)}{2T_\zeta + (T_w(x) - 2T_\zeta)r_\parallel(2-r_\parallel)} v_\parallel| \le 10\varepsilon^{-1}.\end{aligned}$$

In the derivation we used (3.88) and (3.48) to conclude

$$\frac{2T_\zeta\sqrt{1-r_\perp}}{2T_\zeta + (T_w(x) - 2T_\zeta)r_\perp} \le \frac{1}{(1-r_\parallel) + \frac{T_w(x)r_\parallel(2-r_\parallel)}{4T_M(1-r_\parallel)}} \le 4,$$

and similarly to have

$$\frac{2T_\zeta\sqrt{1-r_\perp}}{2T_\zeta + [T_w(x) - 2T_\zeta]r_\perp} \le 4.$$

Thus $u \in \gamma_+(x)/\gamma_+^{\epsilon/10}$, where $\gamma_+^{\varepsilon/6}$ is defined in (3.8). From Lemma 10 the contribution of (3.82)$\mathbf{1}_{|v|\le\epsilon^{-1}}$ in (3.58)γ_- is

$$\begin{aligned}&\lesssim_\varepsilon \int_0^t \int_{\partial\Omega} \int_{\gamma_+(x)/\gamma_+^{\epsilon/10}(x)} |e^{-\lambda s\langle u\rangle} w_{\tilde{\theta}} \alpha^\beta_{f^{m-1},\epsilon} \nabla_{x,v} f^m(s,x,u)|^p\{n(x)\cdot u\}dudS_x ds\\ &\lesssim_\varepsilon \|w_{\tilde{\theta}} \alpha^\beta_{f^{m-1},\epsilon} \nabla_{x,v} f(0)\|^p_p + \int_0^t \|e^{-\lambda s\langle v\rangle} w_{\tilde{\theta}} \alpha^\beta_{f^{m-1},\epsilon} \nabla_{x,v} f^m\|^p_p + (3.98)\end{aligned} \tag{3.97}$$

with

$$\int_0^t \iint_{\Omega\times\mathbb{R}^3} [\partial_t + v\cdot\nabla_x - \nabla_x\phi^{m-1}\cdot\nabla_v + \nu_{\phi^{m-1},\lambda,w_{\tilde\theta}}]|e^{-\lambda s\langle v\rangle} w_{\tilde\theta}\alpha^\beta_{f^{m-1},\epsilon}\nabla_{x,v} f^m|^p \tag{3.98}$$

$$\leq \int_0^t \iint_{\Omega\times\mathbb{R}^3} pe^{-\lambda ps\langle v\rangle} w^p_{\tilde\theta}\alpha^{\beta p}_{f^{m-1},\epsilon}|\nabla_{x,v} f^m|^{p-1}|\mathcal{G}^{m-1}|. \tag{3.99}$$

Clearly (3.99)$\lesssim$ (3.77) with replacing all $m+1$ by m and m by $m-1$.

Collecting (3.94) (3.96) (3.97) (3.98), the contribution of the first line (3.24) in (3.58)γ_- is

$$\begin{aligned}\lesssim_{p,T_M,r_{min}} \epsilon \int_0^t |e^{-\lambda s\langle v\rangle} w_{\tilde\theta}\alpha^\beta_{f^{m-1},\epsilon}\partial f^m|^p_{p,+} + C(\varepsilon)\Big[\|w_{\tilde\theta}\alpha^\beta_{f^m,\epsilon}\nabla_{x,v} f(0)\|^p_p \\ + \int_0^t \|e^{-\lambda s\langle v\rangle} w_{\tilde\theta}\alpha^\beta_{f^{m-1},\epsilon}\nabla_{x,v} f^m\|^p_p + (3.58)\mathcal{G}^{m-1}\Big],\end{aligned} \tag{3.100}$$

where $C(\varepsilon)$ comes from (3.97).

– Second line of (3.24). By the Hölder inequality, we have

$$\begin{aligned}&\Bigg(\int_{n(x)\cdot u>0}\Big(\langle u\rangle^2\|\nabla_x\phi^{m-1}\|_{L^\infty}|f^m|\Big) \\ &\quad + \|w_{\theta'} f^m\|_{L^\infty}\int_{\mathbb{R}^3}\mathbf{k}_\rho(u,u')|f^{m-1}(u')|du' e^{-[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]|u|^2}d\sigma(u,v)\Bigg)^p \\ &\lesssim \Bigg(\int_{n(x)\cdot u>0}\Big(\langle u\rangle^2\|\nabla_x\phi^{m-1}\|_{L^\infty}|e^{-\lambda s\langle u\rangle}w_{\tilde\theta}(u) f^m|\Big)e^{\lambda s\langle u\rangle}w^{-1}_{\tilde\theta}(u)d\sigma(u,v) \\ &\quad + \|w_{\theta'} f^m\|_{L^\infty}\int_{\mathbb{R}^3}\mathbf{k}_\rho(u,u')|e^{-\lambda s\langle u'\rangle}w_{\tilde\theta}(u') f^{m-1}(u')|du' e^{-[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]|u|^2} \\ &\quad e^{\lambda s\langle u\rangle}w^{-1}_{\tilde\theta}(u)d\sigma(u,v)\Bigg)^p.\end{aligned} \tag{3.101}$$

Similarly to (3.83), we bound $\langle u\rangle^2$ as $\langle u\rangle^2 \lesssim e^{\zeta|u|^2}$ with the same ζ satisfying (3.89). Using $e^{\lambda s\langle u\rangle}w^{-1}_{\tilde\theta}(u) \lesssim 1$ we obtain

$$\begin{aligned}(3.101) &\lesssim \int_{n(x)\cdot u>0}|e^{-\lambda s\langle u\rangle}w_{\tilde\theta}(u) f^m|^p\{n\cdot u\}du \times \Bigg(\int_{n(x)\cdot u>0} e^{p^*\zeta|u|^2}e^{-p^*[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]|u|^2} \\ &\quad \times I_0\Big(\frac{p^*(1-r_\perp)^{1/2}u_\perp v_\perp}{T_w(x)r_\perp}\Big)e^{-\frac{p^*}{2T_w(x)}\Big[\frac{|u_\perp|^2+(1-r_\perp)|v_\perp|^2}{r_\perp}+\frac{|u_\parallel-(1-r_\parallel)v_\parallel|^2}{r_\parallel(2-r_\parallel)}\Big]}du\Bigg)^{p/p^*} \\ &\quad + \|w_{\theta'} f^m\|_{L^\infty}\Bigg(\int_{n(x)\cdot u>0}\Big(\int_{\mathbb{R}^3}\mathbf{k}_\rho(u,u')du'\Big)^{p/q} \\ &\quad \Big(\int_{\mathbb{R}^3}\mathbf{k}_\rho(u,u')|e^{-\lambda s\langle u'\rangle}w_{\tilde\theta}(u') f^{m-1}(u')|^p du'\Big)^{1/p} \\ &\quad \times e^{-[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]|u|^2}\{n(x)\cdot u\}I_0 e^{-\frac{1}{2T_w(x)}\Big[\frac{|u_\perp|^2+(1-r_\perp)|v_\perp|^2}{r_\perp}+\frac{|u_\parallel-(1-r_\parallel)v_\parallel|^2}{r_\parallel(2-r_\parallel)}\Big]}du\Bigg)^p.\end{aligned}$$

Then we can apply the same computation as in (3.81), (3.82) for the u-integration. Thus by (3.91) we derive

$$
\begin{aligned}
(3.101) \lesssim & \int_{n(x)\cdot u>0} |e^{-\lambda s\langle u\rangle} w_{\tilde{\theta}}(u) f^m|^p \{n\cdot u\} du \times (3.91) \\
& + \|w_{\theta'} f^m\|_{L^\infty} \Big(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathbf{k}_\rho(u,u') |e^{-\lambda s\langle u'\rangle} w_{\tilde{\theta}}(u') f^{m-1}(u')|^p du' du \Big) \times (3.91) \\
\lesssim & \Big(\int_{n(x)\cdot u>0} |e^{-\lambda s\langle u\rangle} w_{\tilde{\theta}}(u) f^m|^p \{n\cdot u\} du + \|e^{-\lambda s\langle u\rangle} w_{\tilde{\theta}}(u) f^{m-1}\|_p^p \Big) \times (3.91).
\end{aligned}
$$

By exactly the same computation as (3.92), the integrand for the v-integration in (3.79) is $\in L^1_v$. Thus the contribution of the second line of (3.24) in (3.58)γ_- is

$$
\lesssim_{p,T_M,r_{min}} \|w_{\theta'} f^m\|_\infty \Big(\int_0^t |e^{-\lambda s\langle u\rangle} w_{\tilde{\theta}}(u) f^m|_{p,+}^p + \int_0^t \|e^{-\lambda s\langle u\rangle} w_{\tilde{\theta}}(u) f^{m-1}\|_p^p \Big). \tag{3.102}
$$

Collecting (3.100) (3.102) we conclude that

$$
\begin{aligned}
& \int_0^t |e^{-\lambda s\langle v\rangle} w_{\tilde{\theta}} \alpha^\beta_{f^m,\epsilon} \partial f^{m+1}|_{p,-}^p \\
& \le C(p,T_M,r_{min}) \times \epsilon \int_0^t |e^{-\lambda s\langle v\rangle} w_{\tilde{\theta}} \alpha^\beta_{f^{m-1},\epsilon} \partial f^m|_{p,+}^p \\
& \quad + C(p,T_M,r_{min}) C(\varepsilon) \|w_{\tilde{\theta}} \alpha^\beta_{f^{m-1},\epsilon} \nabla_{x,v} f(0)\|_p^p + \\
& \quad + C(p,T_M,r_{min}) C(\varepsilon) \sup_m \|w_{\theta'} f^m\|_\infty \Big(\int_0^t \|e^{-\lambda s\langle v\rangle} \langle v\rangle^{1/p} w_{\tilde{\theta}} \alpha^\beta_{f^{m-1},\epsilon} \nabla_{x,v} f^m\|_p^p \\
& \quad + \int_0^t |e^{-\lambda s\langle v\rangle} w_{\tilde{\theta}} f^m|_{p,+}^p \Big) \\
& \quad + C(p,T_M,r_{min}) C(\varepsilon) \times (\sup_m \|w_{\theta'} f^m\|_\infty + \sup_{l\le m} \|\nabla^2 \phi^l\|_\infty) \\
& \quad \times \Big(\int_0^t \|e^{-\lambda s\langle v\rangle} w_{\tilde{\theta}} f^{m-1}\|_p^p + \int_0^t \|e^{-\lambda s\langle v\rangle} w_{\tilde{\theta}} \alpha^\beta_{f^{m-2},\epsilon} \nabla_{x,v} f^{m-1}\|_p^p \\
& \quad + \int_0^t \|e^{-\lambda s\langle v\rangle} w_{\tilde{\theta}} \alpha^\beta_{f^{m-1},\epsilon} \nabla_{x,v} f^m\|_p^p \Big),
\end{aligned} \tag{3.103}
$$

where $C(p,T_M,r_{min})$ comes from $\lesssim_{p,T_M,r_{min}}$ and $C(p)$ in (3.77). Similar to Step 1, here it's important to note that in the second line, the first term has $\varepsilon \ll 1$, while the second term has $C(\varepsilon)$, which can be large number depends on ε.

Remark 8 We comment the largeness of λ comes from (3.31), the boundedness of f (3.35), the boundedness of ∂f (3.78) and in (3.64). The smallness of $\tilde{\theta}$ comes from (3.44) and (3.93). The largeness of the constant C_W in (3.4) comes from (3.65). The smallness the time t_W in (3.4) comes from (3.66).

4 $L^3_x L^{1+}_v$-Estimate of $\nabla_v f$ and L^{1+}-Stability

As we mention in the introduction, to conclude the uniqueness we need to control $\nabla_v f$ with certain norm. With $W^{1,p}$ estimate for $\nabla_x f$ in Sect. 3, we will establish the $L^3_x L^{1+}_v$-estimate for the sequence solution $\nabla_v f^{m+1}$ in Proposition 5 in the section. With such estimate for $\nabla_v f$, we then show the sequence f^{m+1} is L^{1+} Cauchy in Proposition 6. The L^{1+} Cauchy is crucial to show the existence of the VPB equation. These two propositions lead to the $L^3_x L^{1+}_v$-estimate for $\nabla_v f$ and the L^{1+}-stability for f that satisfies (1.1) under good initial condition. These two propositions are given in Proposition 7, 8 respectively. The L^{1+} stability directly leads to the uniqueness of VPB system.

Proposition 5 *Assume f^{m+1} solves (2.2) and satisfy all assumptions in Proposition 3. We also assume extra initial condition*

$$\|w_{\tilde{\theta}} \nabla_v f_0\|_{L^3_{x,v}} < \infty. \tag{4.1}$$

There exists $t_\delta \ll 1$ ($t_\delta < t_W$) and C_δ such that when $0 \leq t < t_\delta$, if

$$\sup_{0\leq s\leq t} \|e^{-\lambda s\langle v\rangle} \nabla_v f^m(s)\|_{L^3_x L^{1+\delta}_v} \leq \underbrace{2C_\delta\Big[\|w_{\tilde{\theta}} \nabla_v f(0)\|_{L^3_{x,v}} + \sup_n \sup_{0\leq s\leq t} \|w_{\theta'} f^n(s)\|_\infty + \sup_n \sup_{0\leq s\leq t} \|e^{-\lambda s\langle u\rangle} w_{\tilde{\theta}} \alpha^\beta_{f^{n-1},\epsilon} \nabla_{x,v} f^n(s)\|_p\Big]}_{(4.2)_*}, \tag{4.2}$$

then we have

$$\|e^{-\lambda t\langle v\rangle} \nabla_v f^{m+1}(t)\|_{L^3_x L^{1+\delta}_v} \leq (4.2)_*. \tag{4.3}$$

Here C_δ is defined in (4.31) and t_δ satisfies (4.32).

Proof of Proposition 5 First we take $t_\delta \leq t_W$ with t_W defined in Proposition 3 so that we can apply Proposition 3 and Proposition 2. We have

$$\begin{aligned}
&[\partial_t + v\cdot\nabla_x - \nabla_x\phi^m\cdot\nabla_v](e^{-\lambda t\langle v\rangle}\partial_v f^{m+1}) \\
&\quad + [\lambda\langle v\rangle - \lambda t + \frac{v}{2T_M}\cdot\nabla_x\phi^m + \nu(F^m)](e^{-\lambda t\langle v\rangle}\partial_v f^{m+1}) \\
&= e^{-\lambda t\langle v\rangle} \times \Big[-\nabla_x f^{m+1} - \frac{1}{2T_M}\nabla_x\phi^m f^{m+1} + \partial_v\Big(\Gamma_{\text{gain}}(f^m, f^m)\Big)\Big].
\end{aligned} \tag{4.4}$$

By (3.28), we have boundary bound for $(x, v) \in \gamma_-$

$$\big|\partial_v f^{m+1}(t,x,v)\big| \lesssim |v|^2 e^{[\frac{1}{4T_M} - \frac{1}{2T_w(x)}]|v|^2} \int_{n\cdot u>0} |f^m(t,x,u)||u| e^{-[\frac{1}{4T_M} - \frac{1}{2T_w(x)}]|u|^2} \mathrm{d}\sigma(u,v) \quad \text{on } \gamma_-. \tag{4.5}$$

By (3.31) we have

$$\lambda\langle v\rangle - \lambda t\partial_v\langle v\rangle + \frac{v}{2T_M}\cdot\nabla_x\phi^m + \nu(F^m) \geq \frac{\lambda}{2}\langle v\rangle.$$

Then we bound $|e^{-\lambda t\langle v\rangle}\nabla_v f^{m+1}|$ along the characteristics

$$
\begin{aligned}
&|e^{-\lambda t\langle v\rangle}\partial_v f^{m+1}(t,x,v)|\\
&\quad\le \mathbf{1}_{t_{\mathbf{b}}(t,x,v)>t}|\partial_v f(0,X(0;t,x,v),V(0;t,x,v))| &(4.6)\\
&\qquad+\mathbf{1}_{t_{\mathbf{b}}(t,x,v)<t}|v_b|^2 e^{[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]|v_b|^2} &(4.7)\\
&\int_{n(x_{\mathbf{b}})\cdot u>0}|f^m(t-t_{\mathbf{b}},x_{\mathbf{b}},u)||u|e^{-[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]|u|^2}\mathrm{d}\sigma(u,v_b) &(4.8)\\
&\qquad+\int_{\max\{t-t_{\mathbf{b}},0\}}^t\|\nabla_x\phi^m\|_\infty|v||w_{\theta'}(V(s;t,x,v))|^{-1}\|w_{\theta'}f^{m+1}\|_\infty\mathrm{d}s &(4.9)\\
&\qquad+\int_{\max\{t-t_{\mathbf{b}},0\}}^t|\nabla_x f^{m+1}(s,X(s;t,x,v),V(s;t,x,v))|\mathrm{d}s &(4.10)\\
&\qquad+\int_{\max\{t-t_{\mathbf{b}},0\}}^t(1+\|w_{\theta'}f^m\|_\infty)\int_{\mathbb{R}^3}\mathbf{k}_\varrho(V(s;t,x,v),u)|\partial_v f^m(s,X(s),u)|\mathrm{d}u\mathrm{d}s. &(4.11)
\end{aligned}
$$

We will discuss every term in (4.6)-(4.11) separately. In Step 1 we analyze (4.6)-(4.9). In Step 2,3 we analyze (4.10),(4.11) respectively. In Step 4 we conclude this lemma by summarizing all the estimates in previous steps.

Step 1.

Note that if $|v|>2C_{\phi^m}t$, for $0\le s\le t$,

$$
|V(s;t,x,v)|\ge|v|-\int_0^t|\nabla_x\phi^m(\tau;t,x,v)|\mathrm{d}\tau\ge|v|-C_{\phi^m}t\ge\frac{|v|}{2}. \tag{4.12}
$$

Therefore

$$
\sup_{s,t,x}\left\|\frac{1}{w_{\tilde\theta}(V(s;t,x,v))}\right\|_{L^r_v}\lesssim_{\tilde\theta}1\ \text{ for any } 1\le r\le\infty. \tag{4.13}
$$

- Estimate of (4.6). We derive

$$
\begin{aligned}
&\|(4.6)\|_{L^3_xL^{1+\delta}_v}\\
&\lesssim\left(\int_\Omega\left(\int_{\mathbb{R}^3}|w_{\tilde\theta}\partial_v f(0,X(0),V(0))|^3\right)\left(\int_{\mathbb{R}^3}\frac{1}{|w_{\tilde\theta}(V(0))|^{(1+\delta)\frac{3}{2-\delta}}}dv\right)^{\frac{2-\delta}{1+\delta}}\right)^{1/3} &(4.14)\\
&\lesssim_{\tilde\theta}\left(\iint_{\Omega\times\mathbb{R}^3}|w_{\tilde\theta}(V(0;t,x,v))\partial_v f(0,X(0;t,x,v),V(0;t,x,v))|^3\mathrm{d}v\mathrm{d}x\right)^{1/3}\\
&=\|w_{\tilde\theta}\partial_v f(0)\|_{L^3_{x,v}},
\end{aligned}
$$

  where we have used a change of variables $(x,v)\mapsto(X(0;t,x,v),V(0;t,x,v))$ and (4.13).
- Estimate of (4.9). Clearly with $\theta'>0$,

$$
\|(4.9)\|_{L^3_xL^{1+\delta}_v}\lesssim_{\theta'}\sup_{0\le s\le t}\|w_{\theta'}f^{m+1}(s)\|_\infty. \tag{4.15}
$$

– Estimate of (4.7),(4.8). We bound (4.7) (4.8) by

$$\begin{aligned}
&\|w_{\theta'} f^m\|_\infty |v_b|^2 e^{[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]|v_b|^2} \int_{n(x_b)\cdot u>0} e^{-\theta' u^2}|u| e^{-[\frac{1}{4T_M}-\frac{1}{2T_w(x_b)}]|u|^2} d\sigma(u,v_b)\\
&\lesssim \|w_{\theta'} f^m\|_\infty |v_b|^2 e^{[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]|v_b|^2} \int_{n(x_b)\cdot u>0} e^{-[\frac{1}{4T_M}-\frac{1}{2T_w(x_b)}]|u|^2} d\sigma(u,v_b)\\
&\lesssim \|w_{\theta'} f^m\|_\infty |v_b|^2 e^{[\frac{1}{4T_M}-\frac{1}{2T_w(x_b)}]|v_b|^2} \exp\Big(\Big[\frac{[2T_M-T_w(x_b)][1-r_{min}]}{2T_w(x_b)\big[2T_M(1-r_{min})+r_{min}T_w(x_b)\big]}\Big]|v_b|^2\Big)\\
&= \|w_{\theta'} f^m\|_\infty |v_b|^2 \exp\Big(\Big[\frac{1}{4T_M}-\frac{1}{2[2T_M(1-r_{min})+r_{min}T_w(x_b)]}\Big]|v_b|^2\Big),
\end{aligned}$$

where we use (2.52) and directly apply (2.60) with replacing $\frac{2\xi}{\xi+1}T_M$ by $2T_M$, t by 0 in the third line for the u-integration. Using

$$\frac{1}{4T_M}-\frac{1}{2[2T_M(1-r_{min})+r_{min}T_w(x_b)]}<0,$$

we obtain

$$\|(4.7)\ (4.8)\|_{L^3_x L^{1+\delta}_v} \lesssim_{T_M,r_{min}} \sup_{0\le s\le t}\|w_{\theta'} f^m(s)\|_\infty. \tag{4.16}$$

Step 2.

– Estimate of (4.10). We claim

$$\|(4.10)\|_{L^3_x L^{1+\delta}_v} \lesssim_{\tilde\theta,\beta,p} \int_0^t \|e^{-\lambda s\langle v\rangle} w_{\tilde\theta}\alpha^\beta_{f^m,\epsilon}\nabla_x f^{m+1}(s)\|_{L^p_{x,v}}. \tag{4.17}$$

For $3<p<6$, by the Hölder inequality $\frac{1}{1+\delta}=\frac{1}{\frac{p+p\delta}{p-1-\delta}}+\frac{1}{p}$,

$$\begin{aligned}
&\Big\|\Big\|\int_{\max\{t-t_{\mathbf{b}},0\}}^t \partial_x f^{m+1}(s,X(s;t,x,v),V(s;t,x,v))\mathrm{d}s\Big\|_{L^{1+\delta}_v(\mathbb{R}^3)}\Big\|_{L^3_x}\\
&\lesssim \Big\|\Big\|\int_{\max\{t-t_{\mathbf{b}},0\}}^t \frac{e^{-\lambda s\langle V(s;t,x,v)\rangle} w_{\tilde\theta}\alpha^\beta_{f^m,\epsilon}\partial_x f^{m+1}(s,X(s;t,x,v),V(s;t,x,v))}{e^{-\lambda s\langle V(s;t,x,v)\rangle} w_{\tilde\theta}\alpha^\beta_{f^m,\epsilon}(s,X(s;t,x,v),V(s;t,x,v))}\mathrm{d}s\Big\|_{L^{1+\delta}_v(\mathbb{R}^3)}\Big\|_{L^3_x}\\
&\lesssim \sup_{s,x}\Big\|\frac{e^{\lambda s\langle v\rangle}w_{\tilde\theta}(v)^{-1}}{\alpha_{f^m,\epsilon}(s,x,v)^\beta}\Big\|_{L_v^{\frac{p+p\delta}{p-1-\delta}}(\mathbb{R}^3)}\\
&\quad\times\Big\|\Big\|\int_0^t e^{-\lambda s\langle V(s;t,x,v)\rangle} w_{\tilde\theta}\alpha^\beta_{f^m,\epsilon}\partial_x f^{m+1}(s,X(s;t,x,v),V(s;t,x,v))\mathrm{d}s\Big\|_{L^p_v(\mathbb{R}^3)}\Big\|_{L^3_x}\\
&\lesssim \sup_{s,x}\Big\|\frac{e^{\lambda s\langle v\rangle}w_{\tilde\theta}(v)^{-1}}{\alpha_{f^m,\epsilon}(s,x,v)^\beta}\Big\|_{L_v^{\frac{p+p\delta}{p-1-\delta}}(\mathbb{R}^3)} \times \int_0^t \|e^{-\lambda s\langle v\rangle} w_{\tilde\theta}\alpha^\beta_{f^m,\epsilon}\partial_x f^{m+1}(s)\|_{L^p_{x,v}} ds,
\end{aligned} \tag{4.18}$$

where we have used $\alpha_{f^m,\epsilon}(t,x,v)=\alpha_{f^m,\epsilon}(s,X(s;t,x,v),V(s;t,x,v))$ for $t-t_{\mathbf{b}}(t,x,v)\le s\le t$ and the change of variables $(x,v)\mapsto(X(s;t,x,v),V(s;t,x,v))$ and the Minkowski inequality.
For β in (3.2), we have $\beta\frac{p}{p-1}<1$ since $\frac{2}{3}<\frac{p-1}{p}$ for $3<p$. Therefore, we can choose $0<\delta=\delta(\beta,p)\ll 1$ so that β satisfies

$$\beta\times\frac{p+p\delta}{p-1-\delta}<1. \tag{4.19}$$

We apply Proposition 4 to conclude that

$$\sup_{s,x}\left\|\frac{e^{\lambda s\langle v\rangle}w_{\tilde\theta}(v)^{-1}}{\alpha_{f^m,\epsilon}(s,x,v)^{\beta}}\right\|^{\frac{p+p\delta}{p-1-\delta}}_{L_v^{\frac{p+p\delta}{p-1-\delta}}(\mathbb{R}^3)} \lesssim_{\tilde\theta} \sup_{s,x}\int_{\mathbb{R}^3}\frac{1}{\alpha_{f^m,\epsilon}(s,x,v)^{\beta\frac{p+p\delta}{p-1-\delta}}}\mathrm{d}v \lesssim_{\beta,p} 1. \tag{4.20}$$

Finally, from (4.18), (4.20), we conclude the claim (4.17).

Step 3.

– Estimate of (4.11). We consider (4.11). We split the u-integration of (4.11) into two parts with $N \gg 1$ as

$$\int_{|u|\le N}\mathbf{k}_{\varrho}(V(s;x,t,v),u)|\nabla_v f^m(s,X(s),u)|\mathrm{d}u \tag{4.21}$$

$$+\int_{|u|\ge N}\mathbf{k}_{\varrho}(V(s;t,x,v),u)|\nabla_v f^m(s,X(s),u)|\mathrm{d}u. \tag{4.22}$$

First we bound (4.21). From the change of variables $(x,v)\mapsto(X(s;t,x,v),V(s;t,x,v))$ for $t-t_{\mathbf{b}}(t,x,v)\le s\le t$

$$\begin{aligned}&\left\|\int_{|u|\le N}\mathbf{k}_{\varrho}(V(s;t,x,v),u)|\nabla_v f^m(s,X(s;t,x,v),u)|\mathrm{d}u\right\|_{L^3_xL^{1+\delta}_v}\\ &\lesssim \left\|\int_{|u|\le N}\mathbf{k}_{\varrho}(v,u)|e^{-\lambda s\langle u\rangle}\nabla_v f^m(s,x,u)|\mathrm{d}u\right\|_{L^3_xL^{1+\delta}_v},\end{aligned} \tag{4.23}$$

where we use $e^{\lambda s\langle u\rangle}\lesssim 1$ when $|u|\le N$. If $|v|\ge 2N$ then $|v-u|^2\gtrsim|v|^2$ and $|v-u|\ge N$, thus for $|v|\ge 2N$ and $|u|\le N$,

$$\mathbf{k}_{\varrho}(v,u)\lesssim\frac{e^{-C_1|v|^2}}{|v-u|}=O(1/N).$$

If $|v|\le 2N$, for $0<\delta\ll 1$ with $\frac{3(1+\delta)}{1-2\delta}>3$,

$$\begin{aligned}(4.23)\lesssim&\left\|\left\|\int_{|u|\le N}\mathbf{k}_{\varrho}(v,u)|e^{-\lambda s\langle u\rangle}\nabla_v f^m(s,x,u)|\mathrm{d}u\right\|_{L_v^{1+\delta}(\{|v|\ge 2N\})}\right\|_{L^3_x}\\ &+\left\|\left\|e^{-C|v|^2}\right\|_{L_v^{3/2}}\left\|\int_{|u|\le N}\frac{1}{|v-u|}|e^{-\lambda s\langle u\rangle}\nabla_v f^m(s,x,u)|\mathrm{d}u\right\|_{L_v^{\frac{3(1+\delta)}{1-2\delta}}(\{|v|\le 2N\})}\right\|_{L^3_x}\\ \lesssim&\|e^{-\lambda s\langle v\rangle}\nabla_v f^m(s)\|_{L^3_xL^{1+\delta}_v}+\left\|\left\|\frac{\mathbf{1}_{|v|\le 2N}}{|v-\cdot|}*|e^{-\lambda s\langle\cdot\rangle}\nabla_v f^m(s,x,\cdot)|\right\|_{L_v^{\frac{3(1+\delta)}{1-2\delta}}}\right\|_{L^3_x}.\end{aligned} \tag{4.24}$$

Then by the Hardy-Littlewood-Sobolev inequality with $1+\frac{1}{\frac{3(1+\delta)}{1-2\delta}}=\frac{1}{3}+\frac{1}{1+\delta}$, we derive that

$$(4.24)\lesssim_{\delta}\left\|\|e^{-\lambda s\langle v\rangle}\nabla_v f^m(s,x,v)\|_{L_v^{1+\delta}}\right\|_{L^3_x}=\|e^{-\lambda s\langle v\rangle}\nabla_v f^m(s)\|_{L^3_xL^{1+\delta}_v}.$$

Combining the last estimate with (4.23), (4.24), we prove that

$$\|(4.21)\|_{L^3_x L^{1+\delta}_v} \lesssim_\delta \|e^{-\lambda s\langle v\rangle}\nabla_v f^m(s)\|_{L^3_x L^{1+\delta}_v}. \tag{4.25}$$

Now we consider (4.22). We have

$$\begin{aligned}(4.22) \lesssim \int_{|u|\geq N} &\frac{1}{w_{\tilde{\theta}-t}(V(s;t,x,v))^{1/2}} \frac{w_{\tilde{\theta}-t}(V(s;t,x,v))}{w_{\tilde{\theta}}(u)} \frac{\mathbf{k}_\varrho(V(s;t,x,v),u)e^{\lambda s\langle u\rangle}}{\alpha_{f^{m-1},\epsilon}(s,X(s;t,x,v),u)^\beta}\\ &\times \frac{1}{w_{\tilde{\theta}-t}(V(s;t,x,v))^{1/2}}\\ &e^{-\lambda s\langle u\rangle} w_{\tilde{\theta}}(u)\alpha_{f^{m-1},\epsilon}(s,X(s;t,x,v),u)^\beta\\ &|\nabla_v f^m(s,X(s;t,x,v),u)|\mathrm{d}u.\end{aligned}$$

By the Hölder inequality with $\frac{1}{p}+\frac{1}{p^*}=1$ with $3<p<6$,

$$\begin{aligned}|(4.22)| \lesssim &\frac{1}{w_{\tilde{\theta}-t}(V(s;t,x,v))^{1/2}}\\ &\left\| \frac{w_{\tilde{\theta}-t}(V(s;t,x,v))}{w_{\tilde{\theta}}(u)} \frac{\mathbf{k}_\varrho(V(s;t,x,v),u)e^{\lambda s\langle u\rangle}}{\alpha_{f^{m-1},\epsilon}(s,X(s;t,x,v),u)^\beta}\right\|_{L^{p^*}(\{|u|\geq N\})}\\ &\times \left\| \frac{e^{-\lambda s\langle u\rangle}w_{\tilde{\theta}}(u)}{w_{\tilde{\theta}-t}(V(s;t,x,v))^{1/2}}\alpha_{f^{m-1},\epsilon}(s,X(s;t,x,v),u)^\beta\right.\\ &\left.|\nabla_v f^m(s,X(s;t,x,v),u)|\right\|_{L^p_u(\mathbb{R}^3)}.\end{aligned} \tag{4.26}$$

Then by the Hölder inequality with $\frac{1}{1+\delta}=\frac{1}{p}+\frac{1}{\frac{(1+\delta)p}{p-(1+\delta)}}$,

$$\begin{aligned}\|(4.22)\|_{L^{1+\delta}_v} \lesssim &\left\|\frac{1}{w_{\tilde{\theta}-t}(V(s;t,x,v))^{1/2}}\right\|_{L_v^{\frac{(1+\delta)p}{p-(1+\delta)}}}\\ &\times \sup_v \left\| \frac{w_{\tilde{\theta}-t}(V(s;t,x,v))}{w_{\tilde{\theta}}(u)} \frac{\mathbf{k}_\varrho(V(s;t,x,v),u)e^{\lambda s\langle u\rangle}}{\alpha_{f^{m-1},\epsilon}(s,X(s;t,x,v),u)^\beta}\right\|_{L^{p^*}(\{|u|\geq N\})}\\ &\times \left\|\left\| \frac{e^{-\lambda s\langle u\rangle}w_{\tilde{\theta}}(u)}{w_{\tilde{\theta}-t}(V(s;t,x,v))^{1/2}}\alpha_{f^{m-1},\epsilon}(s,X(s;t,x,v),u)^\beta\right.\right.\\ &\left.\left.|\nabla_v f^m(s,X(s;t,x,v),u)|\right\|_{L^p_u}\right\|_{L^p_v}.\end{aligned}$$

From (3.16), for some $0<\tilde{\varrho}<\varrho$ we have

$$\mathbf{k}_\varrho(v,u)\frac{e^{(\tilde{\theta}-t)|v|^2}e^{\lambda s\langle u\rangle}}{e^{\tilde{\theta}|u|^2}} \lesssim \mathbf{k}_\varrho(v,u)\frac{e^{(\tilde{\theta}-t)|v|^2}}{e^{(\tilde{\theta}-t)|u|^2}}\frac{e^{\lambda t\langle u\rangle}}{e^{t|u|^2}} \lesssim \mathbf{k}_{\tilde{\varrho}}(v,u).$$

Hence using (4.13) we derive,

$$
\begin{aligned}
&\Big\| \|(4.22)\|_{L^{1+\delta}_v} \Big\|_{L^3_x} \\
&\lesssim_{\Omega,\beta,p} \sup_{X,V} \left\| \frac{e^{-\frac{\tilde{\varrho}}{10}|V-u|^2}}{|V-u|} \frac{1}{\alpha_{f^{m-1},\epsilon}(s,X,u)^\beta} \right\|_{L^{p^*}(\{|u|\geq N\})} \\
&\quad \times \left\| \frac{e^{-\lambda s\langle u\rangle} w_{\tilde{\theta}}(u)}{w_{\tilde{\theta}-t}(V(s;t,x,v))^{1/2}} \alpha_{f^{m-1},\epsilon}(s,X(s;t,x,v),u)^\beta |\nabla_v f^m(s,X(s;t,x,v),u)| \right\|_{L^p_{u,v,x}}.
\end{aligned}
$$

By (3.10) in Proposition 4 with $\frac{p-2}{p-1} < \beta p^* < 1$ from (3.2) and applying the change of variables $(x,v) \mapsto (X(s;t,x,v), V(s;t,x,v))$, we derive that

$$
\begin{aligned}
\Big\| \|(4.22)\|_{L^{1+\delta}_v} \Big\|_{L^3_x} &\lesssim_{\Omega,p,\beta} \left\| \left\| \frac{e^{-\lambda s\langle u\rangle}}{w_{\tilde{\theta}-t}(v)^{1/2}} w_{\tilde{\theta}}(u) \alpha_{f^{m-1},\epsilon}(s,x,u)^\beta |\nabla_v f^m(s,x,u)| \right\|_{L^p_v} \right\|_{L^p_{u,x}} \\
&\lesssim \left\| \frac{1}{w_{\tilde{\theta}-t}(v)^{1/2}} \right\|_{L^p_v} \left\| e^{-\lambda s\langle u\rangle} w_{\tilde{\theta}}(u) \alpha_{f^{m-1},\epsilon}(s,x,u)^\beta |\nabla_v f^m(s,x,u)| \right\|_{L^p_{u,x}} \\
&\lesssim_{\tilde{\theta}} \left\| e^{-\lambda s\langle u\rangle} w_{\tilde{\theta}} \alpha^\beta_{f^{m-1},\epsilon} |\nabla_v f^m(s)| \right\|_{L^p}.
\end{aligned} \tag{4.27}
$$

Combining (4.26) and (4.27) we conclude that

$$
\|(4.22)\|_{L^3_x L^{1+\delta}_v} \lesssim_{\Omega,\beta,p,\tilde{\theta}} \|e^{-\lambda s\langle u\rangle} w_{\tilde{\theta}} \alpha^\beta_{f^{m-1},\epsilon} \nabla_v f^m(s)\|_{L^p_{x,v}}. \tag{4.28}
$$

Finally from (4.25) and (4.28), and using the Minkowski inequality, we conclude that

$$
\begin{aligned}
\|(4.11)\|_{L^3_v L^{1+\delta}_x} \lesssim_{\Omega,\beta,p,\tilde{\theta}} (1+\|w_{\theta'} f^m\|_\infty) \int_0^t \big[&\|e^{-\lambda s\langle v\rangle} \nabla_v f^m(s)\|_{L^3_x L^{1+\delta}_v} \\
&+ \|e^{-\lambda s\langle v\rangle} w_{\tilde{\theta}} \alpha^\beta_{f^{m-1},\epsilon} \nabla_v f^m(s)\|_{L^p_{x,v}} \big] ds.
\end{aligned} \tag{4.29}
$$

Step 4. Since all assumption in Proposition 2,3 are satisfied, we have the uniform in n bound

$$
\sup_n \|w_{\theta'} f^n\|_\infty < \infty, \quad \sup_n \|e^{-\lambda s\langle v\rangle} w_{\tilde{\theta}} \alpha^\beta_{f^{m-1},\varepsilon} \nabla_{x,v} f^n\|_p < \infty.
$$

Collecting terms from (4.14), (4.15),(4.16), (4.17), and (4.29), we derive

$$
\begin{aligned}
&\|e^{-\lambda t\langle v\rangle} \nabla_v f^{m+1}(t)\|_{L^3_x L^{1+\delta}_v} \\
&\quad \leq C(\Omega,p,\beta,\tilde{\theta}) \times (1+\|w_{\theta'} f^m\|_\infty) \Big[\|w_{\tilde{\theta}} \nabla_v f(0)\|_{L^3_{x,v}} + \sup_n \sup_{0\leq s\leq t} \|w_{\theta'} f^n(s)\|_\infty \\
&\qquad + \sup_n \sup_{0\leq s\leq t} \|e^{-\lambda s\langle u\rangle} w_{\tilde{\theta}} \alpha^\beta_{f^{n-1},\epsilon} \nabla_{x,v} f^n(s)\|_p + t \sup_{0\leq s\leq t} \|e^{-\lambda s\langle v\rangle} \nabla_v f^m(s)\|_{L^3_x L^{1+\delta}_v} \Big].
\end{aligned} \tag{4.30}
$$

Now we define the constant in (4.2) as

$$
C_\delta = C(\Omega,p,\beta,\tilde{\theta})(1+\sup_n \|w_{\theta'} f^n\|_\infty). \tag{4.31}
$$

For the last term in (4.30) by the assumption (4.2), we take $t < t_\delta = t_\delta(C_\delta) \ll 1$ small enough such that

$$
\begin{aligned}
&C(\Omega, p, \beta, \tilde{\theta})(1 + \sup_n \|w_{\theta'} f^n\|_\infty) t \sup_{0\le s\le t} \|e^{-\lambda s\langle v\rangle} \nabla_v f^m(s)\|_{L^3_x L^{1+\delta}_v} \\
&\le t_\delta C_\delta^2 \Big[\|w_{\tilde{\theta}} \nabla_v f(0)\|_{L^3_{x,v}} + \sup_n \sup_{0\le s\le t} \|w_{\theta'} f^n(s)\|_\infty \\
&\quad + \sup_n \sup_{0\le s\le t} \|e^{-\lambda s\langle u\rangle} w_{\tilde{\theta}} \alpha^\beta_{f^{n-1},\epsilon} \nabla_{x,v} f^n(s)\|_p \Big] \\
&\le C_\delta \Big[\|w_{\tilde{\theta}} \nabla_v f(0)\|_{L^3_{x,v}} + \sup_n \sup_{0\le s\le t} \|w_{\theta'} f^n(s)\|_\infty \\
&\quad + \sup_n \sup_{0\le s\le t} \|e^{-\lambda s\langle u\rangle} w_{\tilde{\theta}} \alpha^\beta_{f^{n-1},\epsilon} \nabla_{x,v} f^n(s)\|_p \Big].
\end{aligned}
\tag{4.32}
$$

Finally we get

$$
\begin{aligned}
&\|e^{-\lambda t\langle v\rangle} \nabla_v f^{m+1}(t)\|_{L^3_x L^{1+\delta}_v} \\
&\le 2C_\delta \Big[\|w_{\tilde{\theta}} \nabla_v f(0)\|_{L^3_{x,v}} + \sup_n \sup_{0\le s\le t} \|w_{\theta'} f^n(s)\|_\infty \\
&\quad + \sup_n \sup_{0\le s\le t} \|e^{-\lambda s\langle u\rangle} w_{\tilde{\theta}} \alpha^\beta_{f^{n-1},\epsilon} \nabla_{x,v} f^n(s)\|_p \Big].
\end{aligned}
$$

We prove (4.3) and derive the proposition. □

The next proposition follows from the Proposition 5.

Proposition 6 *Suppose f^{m+1} and f^m solve (1.44) with boundary condition (1.45), and satisfy all assumption in Proposition 2 3 5. Then there exists $\bar{t} \ll 1$ ($\bar{t} \le t_\delta$) with $t \le \bar{t}$ such that*

$$
\begin{aligned}
&\sup_{0\le s\le t} \|e^{-\lambda s\langle v\rangle}[f^{m+1}(s) - f^m(s)]\|_{L^{1+\delta}(\Omega\times\mathbb{R}^3)} + \int_0^t |e^{-\lambda s\langle v\rangle}(f^{m+1} - f^m)(s)|^{1+\delta}_{1+\delta,+} \\
&\le \frac{1}{2} \sup_{0\le s\le t} \|e^{-\lambda s\langle v\rangle}[f^m(s) - f^{m-1}(s)]\|_{L^{1+\delta}(\Omega\times\mathbb{R}^3)} \\
&\quad + \frac{1}{2}\int_0^t |e^{-\lambda s\langle v\rangle}(f^m - f^{m-1})(s)|^{1+\delta}_{1+\delta,+}.
\end{aligned}
\tag{4.33}
$$

Here $\bar{t}$ satisfies (4.43).

Remark 9 This proposition is crucial to show the existence of the solution. In Proposition 9 we will use the L^{1+} Cauchy with (2.8) to conclude the existence of the solution f.

Proof First we take $\bar{t} \le t_\delta$ with t_δ defined in Proposition 5 so that we can apply all the previous Propositions.

Assume f^{m+1} and f^m solve (1.44), then

$$
\begin{aligned}
&\partial_t\big[e^{-\lambda t\langle v\rangle}(f^{m+1} - f^m)\big] + v\cdot\nabla_x\big[e^{-\lambda t\langle v\rangle}(f^{m+1} - f^m)\big] - \nabla_x\phi^m\cdot\nabla_v\big[e^{-\lambda t\langle v\rangle}(f^{m+1} - f^m)\big] \\
&\quad + \Big(\lambda\langle v\rangle + \frac{v}{2T_M}\cdot\nabla_x\phi^m - \lambda t\partial_v\langle v\rangle + \nu(F^m)\Big)\big[e^{-\lambda t\langle v\rangle}(f^{m+1} - f^m)\big] \\
&= (\nabla_x\phi^m - \nabla_x\phi^{m-1})\nabla_v(e^{-\lambda t\langle v\rangle} f^m) - \frac{v}{2T_M}\cdot(\nabla_x\phi^m - \nabla_x\phi^{m-1})(e^{-\lambda t\langle v\rangle} f^m) \\
&\quad + e^{-\lambda t\langle v\rangle}\Big[\Gamma_{\text{gain}}(f^m, f^m) - \Gamma_{\text{gain}}(f^{m-1}, f^{m-1}) + f^m\Big(\nu(F^{m-1}) - \nu(F^m)\Big)\Big].
\end{aligned}
\tag{4.34}
$$

By (3.31) we have

$$\lambda\langle v\rangle + \frac{v}{2T_M}\cdot\nabla_x\phi^m - \lambda t\partial_v\langle v\rangle + \nu(F^m) \geq \frac{\lambda}{2}\langle v\rangle.$$

Then using Lemma 9 for $L^{1+\delta}$-space with $0<\delta\ll 1$, we obtain

$$\begin{aligned}&\|e^{-\lambda t\langle v\rangle}(f^{m+1}-f^m)(t)\|_{1+\delta}^{1+\delta} + \int_0^t |e^{-\lambda s\langle v\rangle}(f^{m+1}-f^m)(s)|_{1+\delta,+}^{1+\delta}\\ &\quad + \frac{\lambda}{2}\int_0^t \|\langle v\rangle e^{-\lambda s\langle v\rangle}(f^{m+1}-f^m)(s)\|_{1+\delta}^{1+\delta}\\ &\quad \leq \|[f^{m+1}-f^m](0)\|_{1+\delta}^{1+\delta} + \int_0^t\iint_{\Omega\times\mathbb{R}^3}|\text{RHS of (4.34)}||e^{-\lambda s\langle v\rangle}(f^{m+1}-f^m)|^\delta\\ &\quad + \int_0^t |e^{-\lambda s\langle v\rangle}(f^{m+1}-f^m)|_{1+\delta,-}^{1+\delta}.\end{aligned} \tag{4.35}$$

We now analyze the three terms in RHS of (4.34).

- Estimate of the first term. For $0<\delta\ll 1$, by the Hölder inequality with $1=\frac{1}{\frac{3(1+\delta)}{2-\delta}}+\frac{1}{3}+\frac{1}{\frac{1+\delta}{\delta}}$ and the Sobolev embedding $W^{1,1+\delta}(\Omega)\subset L^{\frac{3(1+\delta)}{2-\delta}}(\Omega)$ when $\Omega\subset\mathbb{R}^3$, the contribution of the first term of the RHS of (4.34) is bounded by

$$\begin{aligned}&\int_0^t\int_{\Omega\times\mathbb{R}^3}|(\nabla_x\phi^m-\nabla_x\phi^{m-1})\cdot\nabla_v\big(e^{-\lambda s\langle v\rangle}f^m\big)||e^{-\lambda s\langle v\rangle}(f^{m+1}-f^m)|^\delta\\ &\quad\lesssim \int_0^t \|\nabla_x\phi^m-\nabla_x\phi^{m-1}\|_{L_x^{\frac{3(1+\delta)}{2-\delta}}}\|e^{-\lambda s\langle v\rangle}\nabla_v f^m\|_{L^3_xL^{1+\delta}_v}\|e^{-\lambda s\langle v\rangle}[f^{m+1}-f^m]^\delta\|_{L_{x,v}^{\frac{1+\delta}{\delta}}}\\ &\quad\lesssim \sup_{0\leq s\leq t}\|e^{-\lambda s\langle v\rangle}\nabla_v f^m(s)\|_{L^3_xL^{1+\delta}_v}\times\int_0^t\|e^{-\lambda s\langle v\rangle}(f^{m+1}-f^m)(s)\|_{1+\delta}^{1+\delta}ds.\end{aligned} \tag{4.36}$$

- Estimate of the second term. By the Hölder inequality with $1=\frac{\delta}{1+\delta}+\frac{1}{1+\delta}$, the contribution of the second term of the RHS of (4.34) is bounded by

$$\begin{aligned}&\int_0^t\int_{\Omega\times\mathbb{R}^3}\frac{v}{2T_M}\cdot(\nabla_x\phi^m-\nabla_x\phi^{m-1})(e^{-\lambda s\langle v\rangle}f^m)|e^{-\lambda s\langle v\rangle}(f^{m+1}-f^m)|^\delta\\ &\quad\lesssim\int_0^t\sup_x\|\langle v\rangle f^m\|_{L_v^{1+\delta}}\|\nabla_x\phi^m-\nabla_x\phi^{m-1}\|_{L_x^{1+\delta}}\|e^{-\lambda s\langle v\rangle}(f^{m+1}-f^m)^\delta\|_{L_{x,v}^{\frac{1+\delta}{\delta}}}\\ &\quad\lesssim_\Omega\big[\|w_{\theta'}f^m\|_\infty+\|w_{\theta'}f^{m+1}\|_\infty\big]\int_0^t\|e^{-\lambda s\langle v\rangle}(f^{m+1}-f^m)(s)\|_{1+\delta}^{1+\delta}ds.\end{aligned} \tag{4.37}$$

- Estimate of the third term. By (3.18), using the Hölder inequality with $1 = \frac{\delta}{1+\delta} + \frac{1}{1+\delta}$, the contribution of the last term of the RHS of (4.34) is bounded by

$$\begin{aligned}
&\int_0^t \int_{\Omega\times\mathbb{R}^3} e^{-\lambda s\langle v\rangle}\big[\Gamma_{\text{gain}}(f^m,f^m) - \Gamma_{\text{gain}}(f^{m-1},f^{m-1}) + \Gamma_{\text{loss}}(f^{m-1},f^m) \\
&\quad - \Gamma_{\text{loss}}(f^m,f^m)\big]|e^{-\lambda s\langle v\rangle}(f^{m+1}-f^m)|^\delta \\
&\lesssim \int_0^t \int_{\Omega\times\mathbb{R}^3} \big[\Gamma_{\text{gain}}(f^m,f^m) - \Gamma_{\text{gain}}(f^m,f^{m-1}) \\
&\quad + \Gamma_{\text{gain}}(f^m,f^{m-1}) - \Gamma_{\text{gain}}(f^{m-1},f^{m-1}) \\
&\quad + \Gamma_{\text{loss}}(f^{m-1},f^m) - \Gamma_{\text{loss}}(f^m,f^m)\big]|e^{-\lambda s\langle v\rangle}(f^{m+1}-f^m)|^\delta \\
&\lesssim \big[\|w_{\theta'}f^m\|_\infty + \|w_{\theta'}f^{m-1}\|_\infty\big]\int_0^t\int_x\int_v\int_u e^{-\lambda s\langle v\rangle}\mathbf{k}_\rho(v,u)|f^m(u)-f^{m-1}(u)| \qquad (4.38)\\
&\quad |e^{-\lambda s\langle v\rangle}\big[f^{m+1}(v)-f^m(v)\big]|^\delta \\
&\lesssim \int_0^t\int_x\int_v\int_u \big[\|w_{\theta'}f^m\|_\infty + \|w_{\theta'}f^{m-1}\|_\infty\big]\mathbf{k}_\rho(v,u)|e^{-\lambda s\langle v\rangle}\big[f^{m+1}(v)-f^m(v)\big]|^\delta \\
&\lesssim \int_0^t\int_x\int_v |e^{-\lambda s\langle v\rangle}\big[f^{m+1}(v)-f^m(v)\big]|^{1+\delta}\int_u \big(\mathbf{k}_\rho(v,u)\big)^{1+\delta} \\
&\lesssim \int_0^t \|e^{-\lambda s\langle v\rangle}\big[f^{m+1}(v)-f^m(v)\big]\|_{1+\delta}^{1+\delta}.
\end{aligned}$$

Since all assumptions in Proposition 2 and Proposition 5 are satisfied, we have

$$\sup_{0\le s\le t}\sup_n \big\{\|e^{-\lambda s\langle v\rangle}\nabla_v f^n(s)\|_{L^3_xL^{1+\delta}_v} + \|w_{\theta'}f^n(s)\|_\infty\big\} < \infty. \tag{4.39}$$

Collecting (4.36) (4.37) and (4.38), in (4.35) we have

$$\begin{aligned}
&\int_0^t \iint_{\Omega\times\mathbb{R}^3} |\text{RHS of (4.34)}||e^{-\lambda s\langle v\rangle}(f^{m+1}-f^m)|^\delta \\
&\lesssim_\Omega (4.39)\times\int_0^t \|e^{-\lambda s\langle v\rangle}(f^{m+1}-f^m)\|_{1+\delta}^{1+\delta}.
\end{aligned} \tag{4.40}$$

Following the proof of the Step 1 in Proposition 3, we apply the same decomposition (3.38) to $\gamma_+(x)$. By (4.40), we can obtain

$$\begin{aligned}
&\int_0^t |e^{-\lambda s\langle v\rangle}[f^{m+1}-f^m]|_{1+\delta,-}^{1+\delta} \\
&\lesssim_{\delta,T_M,r_{min},\Omega} \varepsilon\int_0^t |e^{-\lambda s\langle v\rangle}[f^m-f^{m-1}]|_{1+\delta,+}^{1+\delta} + C(\varepsilon)\|[f^m-f^{m-1}](0)\|_{1+\delta}^{1+\delta} \qquad (4.41)\\
&\quad + C(\varepsilon)(4.39)\times\int_0^t \|e^{-\lambda s\langle v\rangle}(f^m-f^{m-1})\|_{1+\delta}^{1+\delta}.
\end{aligned}$$

By (4.35) (4.40) and (4.41), using $f^m(0) = f^{m+1}(0) = f_0$ and $\frac{\lambda}{2}\int_0^t \|\langle v\rangle e^{-\lambda s\langle v\rangle}(f^{m+1} - f^m)(s)\|_{1+\delta}^{1+\delta} \ge 0$,

$$\begin{aligned}
&\|e^{-\lambda t\langle v\rangle}(f^{m+1}-f^m)(t)\|_{1+\delta}^{1+\delta} + \int_0^t |e^{-\lambda s\langle v\rangle}(f^{m+1}-f^m)(s)|_{1+\delta,+}^{1+\delta} \\
&\le C(\delta,T_M,r_{min},\Omega)(4.39)\times\Big(t\sup_{0\le s\le t}\|e^{-\lambda s\langle v\rangle}(f^{m+1}-f^m)(s)\|_{1+\delta}^{1+\delta} \qquad (4.42)\\
&\quad + C(\varepsilon)t\sup_{0\le s\le t}\|e^{-\lambda s\langle v\rangle}(f^m-f^{m-1})(s)\|_{1+\delta}^{1+\delta} + \varepsilon\int_0^t |e^{-\lambda s\langle v\rangle}[f^m-f^{m-1}]|_{1+\delta,+}^{1+\delta}\Big).
\end{aligned}$$

Now we take ε and $t < \bar{t}(\delta, T_M, r_{min}, \Omega, \varepsilon)$ small enough such that

$$\begin{aligned} C(\delta, T_M, r_{min}, \Omega)\varepsilon \times (4.39) &< \frac{1}{10}, \\ C(\delta, T_M, r_{min}, \Omega)C(\varepsilon)\bar{t} \times (4.39) &\leq \frac{1}{10}, \end{aligned} \tag{4.43}$$

we derive (4.33) and prove the Proposition. □

The Proposition 5 suggests, according to (4.3), that the $L^3_x L^{1+\delta}_v$ estimate of $\nabla_v f$ is obtained upon a good initial condition, the boundedness in L^∞ and the weighted $W^{1,p}$ estimate. In particular, we have the following proposition.

Proposition 7 *Assume f and ϕ solve (1.22) (1.24) (1.9), and satisfy estimates*

$$\|w_{\theta'} f\| < \infty, \tag{4.44}$$

$$\|w_{\tilde{\theta}} e^{-\lambda t\langle v\rangle} \alpha^{\beta}_{f,\epsilon} \nabla_{x,v} f\|_p < \infty. \tag{4.45}$$

We also assume extra initial condition

$$\|w_{\tilde{\theta}} \nabla_v f_0\|_{L^3_{x,v}} < \infty. \tag{4.46}$$

Then

$$\|e^{-\lambda t\langle v\rangle} \nabla_v f\|_{L^3_x L^{1+\delta}_v} < \infty. \tag{4.47}$$

Proof By replacing f^{m+1} and f^m by f in (4.4), we obtain bound for $\partial_v f$ using (4.6)-(4.9) with replacing f^m and f^{m+1} into f. Following exactly the same proof in Proposition 5, by (4.30), we obtain

$$\begin{aligned} &\|e^{-\lambda t\langle v\rangle} \nabla_v f\|_{L^3_x L^{1+\delta}_v} \\ &\leq C(\Omega, p, \beta, \tilde{\theta}) \times (1 + \|w_{\theta'} f\|_\infty)\Big[\|w_{\tilde{\theta}} \nabla_v f(0)\|_{L^3_{x,v}} + \|w_{\theta'} f\|_\infty \\ &\quad + \|e^{-\lambda s\langle v\rangle} w_{\tilde{\theta}} \alpha^{\beta}_{f,\epsilon} \nabla_{x,v} f\|_p \\ &\quad + \int_0^t \|e^{-\lambda s\langle v\rangle} \nabla_v f(s)\|_{L^3_x L^{1+\delta}_v} ds\Big]. \end{aligned} \tag{4.48}$$

By assumption (4.44) and (4.45), the first line of the RHS of (4.48) is bounded. We derive the proposition by the Gronwall's inequality. □

The Proposition 6 suggests that the L^{1+} stability of f can be also obtained upon a good initial condition.

Proposition 8 *Suppose f and g solve (1.22) (1.24) (1.9), and satisfy all assumption in Proposition 7. Then*

$$\|e^{-\lambda t\langle v\rangle}\big[f(t) - g(t)\big]\|_{L^{1+\delta}(\Omega\times\mathbb{R}^3)} \lesssim \|f_0 - g_0\|_{L^{1+\delta}(\Omega\times\mathbb{R}^3)}. \tag{4.49}$$

Remark 10 Clearly, this proposition serves as a criteria for showing the uniqueness of the solution. The main assumption is that the solution needs to satisfy the initial condition (4.46) and the estimates (4.44) and (4.45). In Sect. 5 where we show the uniqueness in Proposition 10, the effort is devoted to bounding (4.45) for the solution f.

Proof of Proposition 8 By replacing $f^{m+1} = f$ and $f^m = g$ in Proposition 6, using (4.35) (4.40) and (4.41), we obtain

$$\|e^{-\lambda t\langle v\rangle}(f-g)(t)\|_{1+\delta}^{1+\delta} + \int_0^t |e^{-\lambda s\langle v\rangle}(f-g)(s)|_{1+\delta,+}^{1+\delta}$$
$$\leq C(\delta, T_M, r_{min}, \Omega)C(\varepsilon)\|f_0 - g_0\|_{1+\delta}^{1+\delta} + C(\delta, T_M, r_{min}, \Omega)\int_0^t \|e^{-\lambda s\langle v\rangle}(f-g)(s)\|_{1+\delta}^{1+\delta}$$
$$+ C(\delta, T_M, r_{min})\varepsilon \int_0^t |e^{-\lambda s\langle v\rangle}(f-g)|_{1+\delta,+}^{1+\delta}. \tag{4.50}$$

We pick $\varepsilon \ll 1$ such that $C(\delta, T_M, r_{min}, \Omega)\varepsilon < \frac{1}{10}$. With ε fixed we applying the Gronwall inequality and derive the $L^{1+\delta}$-stability (4.49). □

5 Existence and Uniqueness

In this section we finalize the existence and uniqueness of the VPB system. The existence is stated in Proposition 9 and the uniqueness is given in Proposition 10. The combination of these two leads to the final Theorem 1.

To show the existence, we first realize that due to the linearity of the boundary condition, the boundary contribution in the integration of the equation tested with a test function converges to that of its weak limit. The strategy used in the proof for diffuse boundary condition is carried over, and we adapt the proof of Theorem 6 in [2] to fit our setting.

For the valid application of the propositions in the previous sections, we let $t \leq \bar{t}$ with $\bar{t}$ given in Proposition 6. Then from the assumption in Proposition 2, 3, 5, 6, we have

$$\bar{t} \leq t_\delta \leq t_W \leq t_\infty.$$

The condition for these four terms are (4.43),(4.32),(3.66) and (2.7) respectively. Thus we conclude the condition for $\bar{t}$ as stated in (1.33) in Theorem 1.

Proposition 9 *Given the assumption in Proposition 2 and Proposition 6, for $t \leq \bar{t}$ there exists at least one solution f that satisfies*

$$\partial_t f + v\cdot\nabla_x f - \nabla_x\phi_f\cdot\nabla_v f + \frac{v}{2T_M}\cdot\nabla_x\phi f = \Gamma(f, f).$$

Moreover,

$$\|w_{\theta'} f\|_\infty < \infty. \tag{5.1}$$

To prove this proposition we first cite a lemma. This lemma will be used to apply the average lemma in (5.7).

Lemma 13 (Lemma 14 of [2]) *Assume $f(s, x, v) = e^s f_0(x, v)$ for $s < 0$. Assume Ω is convex and* $\sup_{0\leq t\leq T}\|E\|_{L^\infty(\Omega)} < \infty$. *Let $\bar{E}(t, x) = 1_\Omega(x)E(t, x)$ for $x \in \mathbb{R}^3$. There exists $\bar{f}(t, x, v) \in L^2(\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3)$, an extension of f_δ, such that*

$$f|_{\Omega\times\mathbb{R}^3} = f_\delta \text{ and } \bar{f}|_\gamma = f_\delta|_\gamma, \text{ and } \bar{f}|_{t=0} = f_\delta|_{t=0}.$$

Proof of Proposition 9 Since assumptions on Proposition 6 are all satisfied, we apply the result for:

$$\sup_{0\le s\le t} \|e^{-\lambda s\langle v\rangle}(f^l - f^m)(s)\|_{L^{1+\delta}} \le (\frac{1}{2})^{\min\{l,m\}}. \tag{5.2}$$

Thus $e^{-\lambda s\langle v\rangle} f^m$ is a Cauchy sequence in $L^{1+\delta}$ and there exists f such that

$$e^{-\lambda t\langle v\rangle} f^m \to e^{-\lambda t\langle v\rangle} f \text{ strongly in } L^{1+\delta}(\Omega\times\mathbb{R}^3). \tag{5.3}$$

By (2.6) and (5.3), there is a unique weak-* limit(up to subsequence) $(w_{\theta'} f^m, w_{\theta'} f^{m+1}) \rightharpoonup^* (w_{\theta'} f, w_{\theta'} f)$ weakly-* in $L^\infty(\mathbb{R}\times\Omega\times\mathbb{R}^3)\cap L^\infty(\mathbb{R}\times\gamma)$ with $\|w_{\theta'} f\|_\infty < \infty$. This means, if let $\varphi \in C_c^\infty([0,\bar t]\times\bar\Omega\times\mathbb{R}^3)$,

$$\begin{aligned}
&\int_0^{\bar t}\int_{\Omega\times\mathbb{R}^3} f^{m+1}[-\partial_t - v\cdot\nabla_x]\varphi + f^{m+1}\{\nabla_x\phi^m\cdot\nabla_v\varphi + \frac{v}{2T_M}\cdot\nabla_x\phi^m\varphi\}\\
&\quad + \int_{\Omega\times\mathbb{R}^3} f^{m+1}(\bar t,x,v)\varphi(\bar t,x,v) - \int_{\Omega\times\mathbb{R}^3} f_0(x,v)\phi(0,x,v)\\
&\quad = \int_0^{\bar t}\int_{\Omega\times\mathbb{R}^3}\Gamma_{\text{gain}}(f^m,f^m)\varphi - \Gamma_{\text{loss}}(f^m,f^{m+1})\varphi\\
&\quad + \int_0^{\bar t}\int_{\gamma_+} f^{m+1}\varphi - \int_0^{\bar t}\int_{\gamma_-} e^{[\frac{1}{4T_M}-\frac{1}{2T_W(x)}]|v|^2}\int_{n\cdot u>0} f^m(u) e^{[\frac{1}{2T_W(x)}-\frac{1}{4T_M}]|u|^2} d\sigma(u,v)\varphi.
\end{aligned}$$

then all the terms converge to the limit with f replacing f^{m+1} and f^m, except $f^{m+1}\{\nabla_x\phi^m\cdot\nabla_v\varphi + \frac{v}{2T_M}\cdot\nabla_x\phi^m\varphi\}$, $\Gamma_{\text{gain}}(f^m,f^m)\varphi$, $\Gamma_{\text{loss}}(f^m,f^{m+1})\varphi$. We now discuss the three terms respectively.

We define, for $(x,v)\in\bar\Omega\times\mathbb{R}^3$ and for $0<\delta\ll 1$,

$$\mathrm{k}_\delta(x,v) = \chi(\frac{|n(x)\cdot v|}{\delta})[1-\chi(\delta|v|)]\chi(\frac{|v|}{\delta}-1) \tag{5.4}$$

with smooth function

$$\chi(x) = \begin{cases} 0, & x<0;\\ 1, & x\ge 1.\end{cases} \tag{5.5}$$

Then $\mathrm{k}_\delta(t,x,v)=0$ if either $|n(x)\cdot v|\le\delta$, $|v|\ge\frac{1}{\delta}$, or $|v|<\delta$.

- For the term $\Gamma_{\text{loss}}(f^m,f^{m+1})\varphi$, we note:

$$\begin{aligned}
&|\int_0^{\bar t}\int_{\Omega\times\mathbb{R}^3}\Gamma_{\text{loss}}(f^m,f^{m+1})\varphi - \Gamma_{\text{loss}}(f,f)\varphi|\\
&\quad\le |\int_0^{\bar t}\int_{\Omega\times\mathbb{R}^3}\int_{\mathbb{R}^3}|v-u|[f^m(u)-f(u)]\sqrt{\mu(u)}du f^{m+1}(v)\varphi(t,x,v)dvdxdt|\\
&\quad\quad + |\int_0^{\bar t}\int_{\Omega\times\mathbb{R}^3}\int_{\mathbb{R}^3}|v-u|f(u)\sqrt{\mu(u)}du[f^{m+1}(v)-f(v)]\varphi(t,x,v)dvdxdt|.
\end{aligned}$$

The second term vanishes due to (2.6), and to handle the first term, we apply the Hölder inequality:

$$
\begin{aligned}
&\int_0^{\bar t}\int_{\Omega\times\mathbb{R}^3}\int_{\mathbb{R}^3}(|v|+|u|)e^{-\theta'|v|^2}\big(\mathrm{k}_\delta(x,u)+1-\mathrm{k}_\delta(x,u)\big)[f^m(u)-f(u)]\sqrt{\mu(u)}du\\
&\quad\times\varphi(t,x,v)dvdxdt\sup_{0\le t\le\bar t}\|e^{\theta'|v|^2}f^{m+1}(t)\|_\infty\\
&\lesssim\Big[\int_0^{\bar t}\int_{\Omega\times\mathbb{R}^3}|v|e^{-\theta'|v|^2}\Big(\int_{\mathbb{R}^3}\langle u\rangle\sqrt{\mu(u)}\mathrm{k}_\delta(x,u)[f^m(u)-f(u)]du\Big)^2dvdxdt\Big]^{1/2} \qquad (5.6)\\
&\quad\times\Big[\int_0^{\bar t}\int_{\Omega\times\mathbb{R}^3}\varphi^2(t,x,v)dvdxdt\Big]^{1/2}+O(\delta)\\
&\lesssim\Big[\int_{\mathbb{R}^3}|v|e^{-\theta'|v|^2}\Big\|\int_{\mathbb{R}^3}\mathrm{k}_\delta(x,u)[f^m(t,x,u)\\
&\quad-f(t,x,u)]\langle u\rangle\sqrt{\mu}(u)du\Big\|_{L^2(\Omega\times[0,T])}dv\Big]^{1/2}+O(\delta).
\end{aligned}
$$

The $O(\delta)$ comes from the integration with $1-\mathrm{k}_\delta(x,u)$, a nonzero term only when $|u|\le 2\delta$ or $|u|\le\frac{1}{\delta}$, or $|n(x)\cdot u|\le\delta$, and thus

$$
\int_0^{\bar t}\int_{\Omega\times\mathbb{R}^3}\int_{\mathbb{R}^3}|u|\sqrt{\mu}\mathbf{1}_{|u|\le 2\delta\text{ or }|u|\ge\delta^{-1}}[\cdots]=O(\delta).
$$

We now extend the results in Lemma 13 that treats $\bar f^m(t,x,v)$ to deal with $\mathrm{k}_\delta(x,u)f^m(t,x,v)$. Apply the average lemma in [12] to $f^m(t,x,v)$, we have:

$$
\sup_m\Big\|\int_{\mathbb{R}^3}\bar f^m(t,x,u)\langle u\rangle\sqrt{\mu(u)}du\Big\|_{H^{1/4}_{t,x}(\mathbb{R}\times\mathbb{R}^3)}<\infty. \qquad (5.7)
$$

Since $H^{1/4}\subset\subset L^2$, we conclude that up to subsequence:

$$
\int_{\mathbb{R}^3}\mathrm{k}_\delta(x,u)f^m(t,x,u)\langle u\rangle\sqrt{\mu(u)}du\to\int_{\mathbb{R}^3}\mathrm{k}_\delta(x,u)f(t,x,u)\langle u\rangle\sqrt{\mu(u)}du\text{ strongly in }L^2_{t,x},
$$

meaning (5.6) goes to 0 as $m\to\infty$.

- For the term $\Gamma_{\text{gain}}(f^m,f^m)\varphi$, we use a test function $\varphi_1(v)\varphi_2(t,x)$. By the standard change of variables $(v,u)\to(v',u')$ and $(v,u)\to(u',u)$, we get

$$
\begin{aligned}
&\int_0^{\bar t}\int_{\Omega\times\mathbb{R}^3}\Gamma_{\text{gain}}(f^m,f^m)\varphi-\Gamma_{\text{gain}}(f,f)\varphi\\
&=\int_0^{\bar t}\int_{\Omega\times\mathbb{R}^3}\Gamma_{\text{gain}}(f^m-f,f^m)\varphi+\int_0^{\bar t}\iint_{\Omega\times\mathbb{R}^3}\Gamma_{\text{gain}}(f,f^m-f)\varphi\\
&=\int_0^{\bar t}\int_{\Omega\times\mathbb{R}^3}\Big(\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}\big(f^m(t,x,u)-f(t,x,u)\big)\sqrt{\mu(u')}|(v-u)\cdot\omega|\varphi_1(u)d\omega du\Big)\\
&\quad\times f^m(t,x,v)\varphi_2(t,x)dvdxdt \qquad (5.8)\\
&\quad+\int_0^{\bar t}\int_{\Omega\times\mathbb{R}^3}\Big(\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}\big(f^m(t,x,u)-f(t,x,u)\big)\sqrt{\mu(u)}|(v-u)\cdot\omega|\varphi_1(u')d\omega du\Big)\\
&\quad\times f(t,x,v)\varphi_2(t,x)dvdxdt. \qquad (5.9)
\end{aligned}
$$

Let $N \gg 1$ we decompose the integration of (5.9) and (5.8) using

$$1 = \{1 - \chi(|u| - N)\}\{1 - \chi(|v| - N)\} + \chi(|u| - N) \\ + \chi(|v| - N) - \chi(|u| - N)\chi(|v| - N). \tag{5.10}$$

The $\chi(|u|-N)+\chi(|v|-N)-\chi(|u|-N)\chi(|v|-N)$ component can be easily controlled. For example, (5.8) becomes:

$$\int_0^{\bar{t}} \int_{\Omega\times\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} [\cdots] \times \{\chi(|u| - N) + \chi(|v| - N) - \chi(|u| - N)\chi(|v| - N)\} \\ \lesssim \sup_m \|e^{\theta'|v|^2} f^m\|_\infty \|e^{\theta'|v|^2} f\|_\infty \Big(\int_0^{\bar{t}} \int_{\Omega\times\mathbb{R}^3} \int_{\mathbb{R}} e^{-\frac{\theta'}{2}|v|^2} e^{-\frac{\theta'}{2}|u|^2} [\mathbf{1}_{|v|\geq N} \\ + \mathbf{1}_{|u|\geq N}] du dv dx dt \Big) \leq O(\frac{1}{N}). \tag{5.11}$$

To consider the component involving $\{1 - \chi(|u| - N)\}\{1 - \chi(|v| - N)\}$, we note that this term is nontrivial if $|v| \leq N + 1$ and $|u| \leq N + 1$. Consider its effect in (5.8), we have:

$$\int_0^{\bar{t}} \int_{\Omega\times\mathbb{R}^3} \int_{\mathbb{R}^3} \Big(f^m(t, x, v) - f(t, x, v) \Big) \\ \times \{1 - \chi(|u| - N)\} \Big(\int_{\mathbb{S}^2} \sqrt{\mu(u')} |(v - u) \cdot \omega| \varphi_1(u) d\omega \Big) du \\ \times \{1 - \chi(|v| - N)\} f^m(t, x, v) \varphi_2(t, x) dv dx dt. \tag{5.12}$$

Define

$$\Phi_v(u) = \{1 - \chi(|u| - N)\} \int_{\mathbb{S}^2} \sqrt{\mu(u')} |(v - u) \cdot \omega| \varphi_1(u) d\omega \text{ for } |v| \leq N + 1, \tag{5.13}$$

then (5.12) is further written as

$$\int_0^{\bar{t}} \int_{\Omega} \int_{\mathbb{R}^3} (1 - \mathrm{k}_\delta) \Big(f^m(t, x, v) - f(t, x, v) \Big) \Phi_v(u) \{1 - \chi(|v| - N)\} \\ f^m(t, x, v) \varphi_2(t, x) dv dx dt. \\ + \mathrm{k}_\delta \Big(f^m(t, x, v) - f(t, x, v) \Big) \Phi_v(u) \{1 - \chi(|v| - N)\} f^m(t, x, v) \varphi_2(t, x) dv dx dt. \tag{5.14}$$

The first term in (5.14) is bounded by $O(\delta) \sup_m \|e^{\theta'|v|^2} f^m\|_\infty$, introducing $O(\delta)$ error, and to handle the second term in (5.14), we form an open cover of $\{v \in \mathbb{R}^3 : |v| \leq N + 1\} \subset \bigcup_{i=1}^{O(N^3/\delta^3)} B(v_i, \delta)$. δ is small enough so that

$$|\Phi_v(u) - \Phi_{v_i}(u)| < \epsilon, \text{ if } v \in B(v_i, \delta). \tag{5.15}$$

This leads to

$$\int_0^{\bar{t}} \int_{\Omega} \int_{\mathbb{R}^3} \sum_i 1_{v\in B(v_i,\delta)} \int_{\mathbb{R}^3} \big(f^m(t, x, u) - f(t, x, u) \big) \big(\Phi_v(u) - \Phi_{v_i}(u) \big) du \\ \times \{1 - \chi(|v| - N)\} f^m(t, x, v) \varphi_2(t, x) dv dx dt = O(\varepsilon).$$

By rewriting $\Phi_v(u)$ in the second term of (5.14) as $\Phi_v(u) - \Phi_{v_i}(u) + \Phi_{v_i} u$ we finally obtain

$$(5.14) \le O(\epsilon) + O(\delta) + \int_0^{\bar{t}} \int_\Omega \sum_i \int_{\mathbb{R}^3} 1_{v \in B(v_i,\delta)} \int_{\mathbb{R}^3} \mathrm{k}_\delta(x,u)(f^m(t,x,u) \\ - f(t,x,u))\Phi_{v_i}(u)du \\ \times \{1 - \chi(|v| - N)\} f^m(t,x,v)\varphi_2(t,x) dv dx dt. \tag{5.16}$$

By the average lemma we conclude

$$\max_{1 \le i \le O(\frac{N^3}{\delta^3})} \sup_m \| \int_{\mathbb{R}^3} \mathrm{k}_\delta(x,u) f^m(t,x,u)\Phi_{v_i}(u)du \|_{H^{1/4}_{t,x}(\mathbb{R}\times\mathbb{R}^3)} < \infty. \tag{5.17}$$

For $i = 1$ we extract a subsequence $m_1 \subset \mathcal{M}_1$ such that

$$\int_{\mathbb{R}^3} \mathrm{k}_\delta(x,u) f^m(t,x,u)\Phi_{v_i}(u)du \to \int_{\mathbb{R}^3} \mathrm{k}_\delta(x,u) f(t,x,u)\Phi_{v_i}(u)du \text{ strongly in } L^2_{t,x}. \tag{5.18}$$

Then we follow the Cantor diagonal argument to extract convergent subsequences $\mathcal{M}_{O(\frac{N^3}{\delta^3})} \subset \cdots \subset \mathcal{M}_2 \subset \mathcal{M}_1$. Denote f^m the subsequence extracted from $m \in \mathcal{M}_{O(\frac{N^3}{\delta^3})}$, then we have (5.18) for all i and conclude

$$(5.12) \le C_{\varphi_2,N} \sup_m \|e^{\theta'|v|^2} f^m\|_\infty \\ \max_i \int_0^{\bar{t}} \left\| \int_{\mathbb{R}^3} \mathrm{k}_\delta(x,u)(f^m(t,x,u) - f(t,x,u))\Phi_{v_i}(u)du \right\|_{L^2_{t,x}} \to 0. \tag{5.19}$$

This, combined with (5.11) provides (5.8) goes to 0. Similar argument is applied to show the convergence of (5.9) and will not be shown here. These together lead to the convergence of $\Gamma_{\text{gain}}(f^m, f^m)$.

- For the term $f^{m+1}\{\nabla_x\phi^m \cdot \nabla_v\varphi + \frac{v}{2T_M} \cdot \nabla_x\phi^m\varphi\}$, we note:

$$-(\Delta\phi^m - \Delta\phi) = \int \mathrm{k}_\delta(f^m - f)\sqrt{\mu} + \int (1 - \mathrm{k}_\delta)(f^m - f)\sqrt{\mu}.$$

Using the standard elliptic estimate:

$$\|\nabla_x\phi^m - \nabla_x\phi\|_{L^2_{t,x}} \le \|\mathrm{k}_\delta(f^m - f)\sqrt{\mu}\|_{L^2_{t,x}} + O(\delta)\sup_m \|e^{\theta|v|^2} f^m\|_\infty \to 0, \tag{5.20}$$

where we run the same argument by applying the average lemma for the strong convergence in $L^2_{t,x}$. Finally

$$\int_0^{\bar{t}} \int_{\Omega\times\mathbb{R}^3} f^{m+1}\{\nabla_x\phi^m \cdot \nabla_v\varphi + \frac{v}{2} \cdot \nabla_x\phi^m\varphi\} - f\{\nabla_x\phi_f \cdot \nabla_v\varphi + \frac{v}{2} \cdot \nabla_x\phi_f\varphi\} dv dx dt \\ \le \int_0^{\bar{t}} \int_{\Omega\times\mathbb{R}^3} (f^{m+1} - f)\{\nabla_x\phi^m \cdot \nabla_v\varphi + \frac{v}{2} \cdot \nabla_x\phi^m\varphi\} dv dx dt \\ + \int_0^{\bar{t}} \int_{\Omega\times\mathbb{R}^3} f\{\nabla_x(\phi^m - \phi_f) \cdot \nabla_v\varphi + \frac{v}{2} \cdot \nabla_x(\phi^m - \phi_f)\varphi\} dv dx dt.$$

Here the first term goes to 0 by the weak$*$ convergence of $e^{\theta|v|^2} f$ in L^∞, the control of the second term comes from (5.20).

This proves the existence of a weak solution $f \in L^\infty$. □

Proposition 10 states the uniqueness of the VPB system.

Proposition 10 *Assume* $\|w_\theta f_0\|_\infty < \infty$, T_w *satisfies* (1.28), $\|w_{\tilde\theta}\alpha^{\beta}_{f_0,\epsilon}\nabla_{x,v} f_0\|_p < \infty$ *for* $0 < \epsilon \ll 1$, *and that* (p, β) *satisfy* (3.2), *then if* $\|w_{\tilde\theta}\nabla_v f_0\|_{L^3_{x,v}} < +\infty$, *there is a unique solution to* (1.22) *satisfying* (1.35) *and* (1.36) *for* $t \le \bar t$.

The proof of the proposition is built upon the following lemma.

Lemma 14 *Assume that* Ω *is convex* (1.19). *Suppose that* $\sup_t \|E(t)\|_{C^1_x} < \infty$ *and*

$$n(x)\cdot E(t,x) = 0 \ \text{ for } x \in \partial\Omega \text{ and for all } t. \tag{5.21}$$

Assume $(t,x,v) \in \mathbb{R}_+ \times \bar\Omega \times \mathbb{R}^3$ *and* $t+1 \ge t_{\mathbf{b}}(t,x,v)$. *If* $x \in \partial\Omega$ *then we further assume that* $n(x)\cdot v > 0$. *Then we have*

$$n(x_{\mathbf{b}}(t,x,v)) \cdot v_{\mathbf{b}}(t,x,v) < 0. \tag{5.22}$$

Proof *Step 1*. Note that locally we can parameterize the trajectory (see Lemma 15 in [15] for details). We consider local parametrization (1.18). We drop the subscript p for the sake of simplicity. If $X(s;t,x,v)$ is near the boundary then we can define $(X_n, X_\parallel)$ to satisfy

$$X(s;t,x,v) = \eta(X_\parallel(s;t,x,v)) + X_n(s;t,x,v)[-n(X_\parallel(s;t,x,v))]. \tag{5.23}$$

For the normal velocity we define

$$V_n(s;t,x,v) := V(s;t,x,v)\cdot[-n(X_\parallel(s;t,x,v))]. \tag{5.24}$$

We define $V_\parallel$ tangential to the level set $\big(\eta(X_\parallel) + X_n(-n(X_\parallel))\big)$ for fixed X_n. Note that

$$\frac{\partial\big(\eta(x_\parallel) + x_n(-n(x_\parallel))\big)}{\partial x_{\parallel,i}} \perp n(x_\parallel) \ \text{ for } i = 1,2.$$

We define $(V_{\parallel,1}, V_{\parallel,2})$ as

$$V_{\parallel,i} := \Big(V - V_n[-n(X_\parallel)]\Big)\cdot\Big(\frac{\partial\eta(X_\parallel)}{\partial x_{\parallel,i}} + X_n\Big[-\frac{\partial n(X_\parallel)}{\partial x_{\parallel,i}}\Big]\Big). \tag{5.25}$$

Therefore we obtain

$$V(s;t,x,u) = V_n[-n(X_\parallel)] + V_\parallel\cdot\nabla_{x_\parallel}\eta(X_\parallel) - X_n V_\parallel\cdot\nabla_{x_\parallel} n(X_\parallel). \tag{5.26}$$

Directly we have

$$\dot X(s;t,x,u) = \dot X_\parallel\cdot\nabla_{x_\parallel}\eta(X_\parallel) + \dot X_n[-n(X_\parallel)] - X_n\dot X_\parallel\cdot\nabla_{x_\parallel} n(X_\parallel).$$

Comparing coefficients of normal and tangential components, we obtain that

$$\dot X_n(s;t,x,v) = V_n(s;t,x,v), \quad \dot X_\parallel(s;t,x,v) = V_\parallel(s;t,x,v). \tag{5.27}$$

On the other hand, from (5.26),

$$\begin{aligned}\dot V(s) = {}& \dot V_n[-n(X_\parallel)] - V_n\nabla_{x_\parallel} n(X_\parallel)\dot X_\parallel + V_\parallel\cdot\nabla^2_{x_\parallel}\eta(X_\parallel)\dot X_\parallel + \dot V_\parallel\cdot\nabla_{x_\parallel}\eta(X_\parallel)\\ & - \dot X_n\nabla_{x_\parallel} n(X_\parallel)V_\parallel - X_n\nabla_{x_\parallel} n(X_\parallel)\dot V_\parallel - X_n V_\parallel\cdot\nabla^2_{x_\parallel} n(X_\parallel)\dot X_\parallel.\end{aligned} \tag{5.28}$$

From (5.28)$\cdot[-n(X_\parallel)]$, (5.27), and $\dot{V} = E$, we obtain that

$$\begin{aligned}\dot{V}_n(s) &= [V_\parallel(s)\cdot\nabla^2\eta(X_\parallel(s))\cdot V_\parallel(s)]\cdot n(X_\parallel(s)) + E(s,X(s))\cdot[-n(X_\parallel(s))]\\ &\quad - X_n(s)[V_\parallel(s)\cdot\nabla^2 n(X_\parallel(s))\cdot V_\parallel(s)]\cdot n(X_\parallel(s)).\end{aligned} \tag{5.29}$$

Step 2. We prove (5.22) by the contradiction argument. Assume we choose (t,x,v) satisfying the assumptions of Lemma 14. Let us assume

$$X_n(t-t_{\mathbf{b}};t,x,v) + V_n(t-t_{\mathbf{b}};t,x,v) = 0. \tag{5.30}$$

First we choose $0<\epsilon\ll 1$ such that $X_n(s;t,x,v)\ll 1$ and

$$V_n(s;t,x,v)\geq 0 \ \text{ for } t-t_{\mathbf{b}}(t,x,v) < s < t-t_{\mathbf{b}}(t,x,v)+\epsilon. \tag{5.31}$$

The sole case that we cannot choose such $\epsilon>0$ is when there exists $0<\delta\ll 1$ such that $V_n(s;t,x,v)<0$ for all $s\in(t-t_{\mathbf{b}}(t,x,v), t-t_{\mathbf{b}}(t,x,v)+\delta)$. But from (5.27) for $s\in(t-t_{\mathbf{b}}(t,x,v), t-t_{\mathbf{b}}(t,x,v)+\delta)$

$$0\leq X_n(s;t,x,v) = X_n(t-t_{\mathbf{b}}(t,x,v);t,x,v) + \int_{t-t_{\mathbf{b}}(t,x,v)}^{s} V_n(\tau;t,x,v)\mathrm{d}\tau < 0.$$

Now with $\epsilon>0$ in (5.31), temporarily we define that $t_* := t-t_{\mathbf{b}}(t,x,v)+\epsilon$, $x_* = X(t-t_{\mathbf{b}}(t,x,v)+\epsilon;t,x,v)$, and $v_* = V(t-t_{\mathbf{b}}(t,x,v)+\epsilon;t,x,v)$. Then $(X_n(s;t,x,v), X_\parallel(s;t,x,v)) = (X_n(s;t_*,x_*,v_*), X_\parallel(s;t_*,x_*,v_*))$ and $(V_n(s;t,x,v), V_\parallel(s;t,x,v)) = (V_n(s;t_*,x_*,v_*), V_\parallel(s;t_*,x_*,v_*))$.

Now we consider the RHS of (5.29). From (1.19), the first term $[V_\parallel(s)\cdot\nabla^2\eta(X_\parallel(s))\cdot V_\parallel(s)]\cdot n(X_\parallel(s))\leq 0$. By an expansion and (5.21) we can bound the second term

$$\begin{aligned}&E(s,X(s))\cdot n(X_\parallel(s))\\ &\quad= E(s,X_n(s),X_\parallel(s))\cdot n(X_\parallel(s))\\ &\quad= E(s,0,X_\parallel(s))\cdot n(X_\parallel(s)) + \|E(s)\|_{C^1_x}O(|X_n(s)|)\\ &\quad= \|E(s)\|_{C^1_x}O(|X_n(s)|).\end{aligned} \tag{5.32}$$

From (1.38) and assumptions of Lemma 14,

$$|V_\parallel(s;t,x,v)|\leq |v| + t_{\mathbf{b}}(t,x,v)\|E\|_\infty \leq |v| + (1+t)\|E\|_\infty.$$

Combining the above results with (5.29), we conclude that

$$\dot{V}_n(s;t_*,x_*,v_*)\lesssim (|v|+(1+t)\|E\|_\infty)^2 X_n(s;t_*,x_*,v_*), \tag{5.33}$$

and hence from (5.27) for $t-t_{\mathbf{b}}(t,x,v)\leq s\leq t_*$

$$\begin{aligned}&\frac{d}{ds}[X_n(s;t_*,x_*,v_*) + V_n(s;t_*,x_*,v_*)]\\ &\quad\lesssim (|v|+(1+t)\|E\|_\infty)^2[X_n(s;t_*,x_*,v_*) + V_n(s;t_*,x_*,v_*)].\end{aligned} \tag{5.34}$$

By the Gronwall inequality and (5.30), for $t-t_{\mathbf{b}}(t,x,v)\leq s\leq t_*$

$$\begin{aligned}&[X_n(s;t_*,x_*,v_*) + V_n(s;t_*,x_*,v_*)]\\ &\quad\lesssim [X_n(t-t_{\mathbf{b}}(t,x,u)) + V_n(t-t_{\mathbf{b}}(t,x,u))]e^{C\epsilon(|v|+(1+t)\|E\|_\infty)^2)}\\ &\quad= 0.\end{aligned}$$

From (5.31) we conclude that $X_n(s;t,x,v)\equiv 0$ and $V_n(s;t,x,v)\equiv 0$ for all $s\in[t-t_{\mathbf{b}}(t,x,u), t-t_{\mathbf{b}}(t,x,u)+\epsilon]$. We can continue this argument successively to deduce that $X_n(s;t,x,v) \equiv 0$ and $V_n(s;t,x,v) \equiv 0$ for all $s \in [t - t_{\mathbf{b}}(t,x,v), t]$. Therefore $x_n = 0 = v_n$ which implies $x \in \partial\Omega$ and $n(x)\cdot v = 0$. This is a contradiction since we chose $n(x)\cdot v > 0$ if $x \in \partial\Omega$. □

Now we finish the proof for the uniqueness.

Proof of Proposition 10 Since all the assumption in Proposition 2 3 6 are valid, from Proposition 9 we have the existence of the solution f to (1.22). To conclude the uniqueness, we need to apply Proposition 8 and need to verify the condition (4.44) and (4.45). The first condition is already given in (5.1) in Proposition 9. We here focus on establishing the second condition.

For f satisfying (1.22), we claim

$$\begin{aligned} &\sup_{0\le t\le \bar t}\|w_{\tilde\theta} f(t)\|_p^p + \sup_{0\le t\le \bar t}\|e^{-\lambda t\langle v\rangle} w_{\tilde\theta}\alpha^\beta_{f,\epsilon}\nabla_{x,v} f(t)\|_p^p \\ &\quad + \int_0^{\bar t} |e^{-\lambda t\langle v\rangle} w_{\tilde\theta}\alpha^\beta_{f,\epsilon}\nabla_{x,v} f(t)|^p_{p,+} \\ &\lesssim \|w_{\tilde\theta} f_0\|_p^p + \|w_{\tilde\theta}\alpha^\beta_{f_0,\epsilon}\nabla_{x,v} f_0\|_p^p. \end{aligned} \tag{5.35}$$

By the weak lower-semicontinuity of L^p we know that

$$w_{\tilde\vartheta}\alpha^\beta_{f^\ell,\varepsilon}\nabla_{x,v} f^{\ell+1} \rightharpoonup \mathcal{F}, \quad \sup_{0\le t\le\bar t}\|\mathcal{F}(t)\|_p^p \le \liminf \sup_{0\le t\le \bar t}\|w_{\tilde\vartheta}\alpha^\beta_{f^\ell,\varepsilon}\nabla_{x,v} f^{\ell+1}(t)\|_p^p,$$

and

$$\int_0^{\bar t} |\mathcal{F}|^p_{p,+} \le \liminf \int_0^{\bar t} |w_{\tilde\vartheta}\alpha^\beta_{f^\ell,\varepsilon}\nabla_{x,v} f^{\ell+1}(t)|^p_{p,+}.$$

We need to prove that

$$\mathcal{F} = w_{\tilde\vartheta}\alpha^\beta_{f,\varepsilon}\nabla_{x,v} f \text{ almost everywhere except } \gamma_0. \tag{5.36}$$

We claim that, up to some subsequence, for any given smooth test function $\psi \in C_c^\infty(\bar\Omega\times\mathbb{R}^3\backslash\gamma_0)$

$$\lim_{\ell\to\infty}\int_0^t\iint_{\Omega\times\mathbb{R}^3} w_{\tilde\vartheta}\alpha^\beta_{f^\ell,\varepsilon}\nabla_{x,v} f^{\ell+1}\psi \mathrm{d}x\mathrm{d}v = \int_0^t\iint_{\Omega\times\mathbb{R}^3} w_{\tilde\vartheta}\alpha^\beta_{f,\varepsilon}\nabla_{x,v} f\psi\mathrm{d}x\mathrm{d}v. \tag{5.37}$$

We note that we need to extract a single subsequence, let say $\{\ell_*\}\subset\{\ell\}$, satisfying (5.37) for all test functions in $C_c^\infty(\bar\Omega\times\mathbb{R}^3\backslash\gamma_0)$.

We will exam (5.37) by the identity obtained from the integration by parts

$$\int_0^t\iint_{\Omega\times\mathbb{R}^3} w_{\tilde\vartheta}\alpha^\beta_{f^\ell,\varepsilon}\nabla_{x,v} f^{\ell+1}\psi\mathrm{d}x\mathrm{d}v$$

$$= -\int_0^t\iint_{\Omega\times\mathbb{R}^3}\alpha^\beta_{f^\ell,\varepsilon} f^{\ell+1}\nabla_{x,v}(w_{\tilde\vartheta}\psi)\mathrm{d}x\mathrm{d}v \tag{5.38}$$

$$+\int_0^t\iint_\gamma n\alpha^\beta_{f^\ell,\varepsilon} f^{\ell+1}(w_{\tilde\vartheta}\psi) \tag{5.39}$$

$$-\int_0^t\iint_{\Omega\times\mathbb{R}^3}\nabla_{x,v}\alpha^\beta_{f^\ell,\varepsilon} f^{\ell+1}(w_{\tilde\vartheta}\psi)\mathrm{d}x\mathrm{d}v. \tag{5.40}$$

For each $N \in \mathbb{N}$ we define a set

$$\mathcal{S}_N := \left\{(x,v) \in \bar{\Omega} \times \mathbb{R}^3 : \mathrm{dist}(x, \partial\Omega) \leq \frac{1}{N} \text{ and } |n(x) \cdot v| \leq \frac{1}{N}\right\} \cup \{|v| > N\}. \tag{5.41}$$

For a given test function we can always find $N \gg 1$ such that

$$supp(\psi) \subset (\mathcal{S}_N)^c := \bar{\Omega} \times \mathbb{R}^3 \backslash \mathcal{S}_N. \tag{5.42}$$

We focus on proving the convergence of (5.38) and (5.39). From (1.41), Lemma 1 and the uniform in ℓ estimate (2.6) , if $(x,v) \in (\mathcal{S}_N)^c$ then

$$\begin{aligned}\sup_{\ell \geq 0} |\alpha^{\beta}_{f^\ell,\varepsilon}(t,x,v)| \lesssim & \quad |v|^\beta + (t+\varepsilon)^\beta \sup_{\ell \geq 0} \|\nabla \phi^\ell\|_\infty^\beta \\ \lesssim & \quad N^\beta + (\bar{t}+\varepsilon)^\beta \sup_{\ell \geq 0} \|w_\vartheta f^\ell\|_\infty^\beta \leq C_N < +\infty.\end{aligned}$$

Hence we extract a subsequence (let say $\{\ell_N\}$) out of subsequence in Proposition 9 such that $\alpha^\beta_{f^{\ell_N},\varepsilon} \overset{*}{\rightharpoonup} A \in L^\infty$ *weakly* $- *$ *in* $L^\infty((0,\bar{t}) \times (\mathcal{S}_N)^c) \cap L^\infty((0,\bar{t}) \times (\gamma \cap (\mathcal{S}_N)^c))$. Note that $\alpha^\beta_{f^{\ell_N},\varepsilon}$ satisfies $[\partial_t + v \cdot \nabla_x - \nabla_x \phi^{\ell_N} \cdot \nabla_v]\alpha^\beta_{f^{\ell_N},\varepsilon} = 0$ and $\alpha^\beta_{f^{\ell_N},\varepsilon}|_{\gamma_-} = |n \cdot v|^\beta$. By passing a limit in the weak formulation we conclude that $[\partial_t + v \cdot \nabla_x - \nabla_x \phi_f \cdot \nabla_v]A = 0$ and $A|_{\gamma_-} = |n \cdot v|^\beta$. By the uniqueness of the Vlasov equation ($\nabla \phi_f \in W^{1,p}$ for any $p < \infty$) we derive $A = \alpha^\beta_{f,\varepsilon}$ almost everywhere and hence conclude that

$$\alpha^\beta_{f^{\ell_N},\varepsilon} \overset{*}{\rightharpoonup} \alpha^\beta_{f,\varepsilon} \textit{ weakly} - * \textit{ in } L^\infty((0,\bar{t}) \times (\mathcal{S}_N)^c) \cap L^\infty((0,\bar{t}) \times (\gamma \cap (\mathcal{S}_N)^c)). \tag{5.43}$$

Now the convergence of (5.38) and (5.39) is a direct consequence of strong convergence of (5.3) and the weak$-*$ convergence of (5.43):

$$\lim_{\ell \to \infty} (5.38) + (5.39) = -\int_0^t \iint_{\Omega \times \mathbb{R}^3} \alpha^\beta_{f,\varepsilon} f \nabla_{x,v}(w_{\tilde{\vartheta}} \psi) \mathrm{d}x\mathrm{d}v + \int_0^t \iint_\gamma n\alpha^\beta_{f,\varepsilon} f(w_{\tilde{\vartheta}}\psi). \tag{5.44}$$

We now show the convergence of (5.40).

Step 1. Let us choose $(x,v) \in (\mathcal{S}_N)^c$. From (1.41),

$$\text{If } t^{f^\ell}_{\mathbf{b}} \geq t + \varepsilon \textit{ then } \alpha_{f^\ell,\varepsilon}(t,x,v) = 1. \tag{5.45}$$

From now we only consider that case

$$t^{f^\ell}_{\mathbf{b}}(t,x,v) \leq \varepsilon + t. \tag{5.46}$$

If $|v| \geq 2(\varepsilon + \bar{t}) \sup_\ell \|\nabla \phi^\ell\|_\infty$ then

$$\begin{aligned}|V^{f^\ell}(s;t,x,v)| &\geq |v| - \int_s^t \|\nabla \phi^\ell(\tau)\|_\infty \mathrm{d}\tau \\ &\geq (\varepsilon + \bar{t}) \sup_\ell \|\nabla \phi^\ell\|_\infty \quad \textit{for all } \ell \textit{ and } s \in [-\varepsilon, \bar{t}].\end{aligned}$$

Then we apply a velocity lemma derived in (3.32) of [2]. We define

$$\tilde{\alpha}(t,x,v) := \sqrt{\xi(x)^2 + |\nabla \xi(x) \cdot u|^2 - 2(u \cdot \nabla_x^2 \xi(x) \cdot u)\xi(x)}. \tag{5.47}$$

For $|u| \geq N$ and $t - t_{\mathbf{b}}(t,x,u) \geq -\varepsilon/2$,

$$\alpha_{f,\varepsilon}(t,x,u)^2 \lesssim \tilde{\alpha}(t,x,u)^2 \lesssim \alpha_{f,\varepsilon}(t,x,u)^2. \tag{5.48}$$

At $s = t - t_{\mathbf{b}}^{f^\ell}(t,x,v)$, we obtain

$$|n(x_{\mathbf{b}}^{f^\ell}) \cdot v_{\mathbf{b}}^{f^\ell}| \geq \frac{e^{-\frac{C_\Omega}{\sup_\ell \|\nabla \phi^\ell\|_\infty}}}{C_\Omega} \times \frac{1}{N} \quad \textit{for all } \ell. \tag{5.49}$$

Step 2. From now on we assume (5.46) and

$$\begin{aligned} &|v| \leq 2(\varepsilon + \bar{t}) \sup_\ell \|\nabla \phi^\ell\|_\infty, \\ &\textit{or, from (1.38)},\ |V^{f^\ell}(s;t,x,v)| \leq 3(\varepsilon + \bar{t}) \sup_\ell \|\nabla \phi^\ell\|_\infty \ \ \textit{for}\ \ s \in [-\varepsilon, \bar{t}]. \end{aligned} \tag{5.50}$$

Let $(X_n^{f^\ell}, X_\parallel^{f^\ell}, V_n^{f^\ell}, V_\parallel^{f^\ell})$ satisfy (5.27), (5.25), and (5.29) with $E = -\nabla \phi^\ell$.

Let us define

$$\tau_1 := \sup\left\{\tau \geq 0 : V_n^{f^\ell}(s;t,x,v) \geq 0 \textit{ for all } s \in [t - t_{\mathbf{b}}^{f^\ell}(t,x,v), \tau]\right\}. \tag{5.51}$$

Since $(X^{f^\ell}(s;t,x,v), V^{f^\ell}(s;t,x,v))$ is C^1 (note that $\nabla \phi^\ell \in C^1_{t,x}$) in s we have $V_n^{f^\ell}(\tau_1;t,x,v) = 0$.

We claim that, there exists some constant $\delta_{**} = O_{\varepsilon,\bar{t},\sup_\ell \|\nabla \phi^\ell\|_{C^1}}(\frac{1}{N})$ in (5.57) which does not depend on ℓ such that

$$\begin{aligned} &\text{If } 0 \leq V_n^{f^\ell}(t - t_{\mathbf{b}}^{f^\ell}(t,x,v);t,x,v) < \delta_{**} \text{ and } (5.50), \\ &\quad \textit{then}\ \ V_n^{f^\ell}(s;t,x,v) \leq e^{C|s-(t-t_{\mathbf{b}}^{f^\ell}(t,x,v))|^2} V_n^{f^\ell}(t - t_{\mathbf{b}}^{f^\ell}(t,x,v);t,x,v) \\ &\quad \textit{for}\ s \in [t - t_{\mathbf{b}}^{f^\ell}, \tau_1]. \end{aligned} \tag{5.52}$$

For the proof we regard the equations (5.27), (5.25), and (5.29) as the forward-in-time problem with an initial datum at $s = t - t_{\mathbf{b}}^{f^\ell}(t,x,v)$. Clearly we have $X_n^{f^\ell}(t - t_{\mathbf{b}}^{f^\ell}(t,x,v);t,x,v) = 0$ and $V_n^{f^\ell}(t - t_{\mathbf{b}}^{f^\ell}(t,x,v);t,x,v) \geq 0$ from Lemma 14. Again from Lemma 14, if $V_n^{f^\ell}(t - t_{\mathbf{b}}^{f^\ell}(t,x,v);t,x,v) = 0$ then $X_n^{f^\ell}(s;t,x,v) = 0$ for all $s \geq t - t_{\mathbf{b}}^{f^\ell}(t,x,v)$. From now on we assume $V_n^{f^\ell}(t - t_{\mathbf{b}}^{f^\ell}(t,x,v);t,x,v)] > 0$. From (5.29), as long as $t - t_{\mathbf{b}}^{f^\ell}(t,x,v) \leq s \leq \bar{t}$ and

$$V_n^{f^\ell}(s;t,x,v) \geq 0 \ \ \textit{and}\ \ X_n^{f^\ell}(s;t,x,v) \leq \frac{1}{N} \ll 1, \tag{5.53}$$

then we have

$$
\begin{aligned}
\dot{V}_n^{f^\ell}(s) = & \underbrace{[V_\parallel^{f^\ell}(s)\cdot\nabla^2\eta(X_\parallel^{f^\ell}(s))\cdot V_\parallel^{f^\ell}(s)]\cdot n(X_\parallel^{f^\ell}(s))}_{\leq 0 \ from \ (1.19)} \\
& - \underbrace{\nabla\phi^\ell(s, X^{f^\ell}(s))\cdot[-n(X_\parallel^{f^\ell}(s))]}_{=O(1)\sup_\ell\|\nabla\phi^\ell\|_{C^1}\times X_n^{f^\ell}(s) \ from \ (5.32)} \\
& - \underbrace{X_n^{f^\ell}(s)[V_\parallel^{f^\ell}(s)\cdot\nabla^2 n(X_\parallel^{f^\ell}(s))\cdot V_\parallel^{f^\ell}(s)]\cdot n(X_\parallel^{f^\ell}(s))}_{=O(1)\{3(\varepsilon+\bar{t})\sup_\ell\|\nabla\phi^\ell\|_\infty\}^2\times X_n^{f^\ell}(s) \ from \ (5.50)} \\
& \leq C(1+\varepsilon+\bar{t})^2(\sup_\ell\|\nabla\phi^\ell\|_{C^1}\sup_\ell\|\nabla\phi^\ell\|_\infty)\times X_n^{f^\ell}(s).
\end{aligned}
\tag{5.54}
$$

Let us consider (5.54) together with $\dot{X}_n^{f^\ell}(s;t,x,v) = V_n^{f^\ell}(s;t,x,v)$. Then, as long as s satisfies (5.53),

$$
\begin{aligned}
V_n^{f^\ell}(s) &= V_n^{f^\ell}(t-t_{\mathbf{b}}^{f^\ell}) + \int_{t-t_{\mathbf{b}}^{f^\ell}}^s \dot{V}_n^{f^\ell}(\tau)\mathrm{d}\tau \\
&\leq V_n^{f^\ell}(t-t_{\mathbf{b}}^{f^\ell}) + \int_{t-t_{\mathbf{b}}^{f^\ell}}^s C(1+\varepsilon+\bar{t})^2(\sup_\ell\|\nabla\phi^\ell\|_{C^1}\sup_\ell\|\nabla\phi^\ell\|_\infty)\times X_n^{f^\ell}(\tau)\mathrm{d}\tau \\
&= V_n^{f^\ell}(t-t_{\mathbf{b}}^{f^\ell}) + \int_{t-t_{\mathbf{b}}^{f^\ell}}^s C(1+\varepsilon+\bar{t})^2(\sup_\ell\|\nabla\phi^\ell\|_{C^1}\sup_\ell\|\nabla\phi^\ell\|_\infty) \\
&\quad \int_{t-t_{\mathbf{b}}^{f^\ell}}^\tau V_n^{f^\ell}(\tau')\mathrm{d}\tau'\mathrm{d}\tau \\
&\leq C(1+\varepsilon+\bar{t})^2(\sup_\ell\|\nabla\phi^\ell\|_{C^1}\sup_\ell\|\nabla\phi^\ell\|_\infty)\int_{t-t_{\mathbf{b}}^{f^\ell}}^s |s-(t-t_{\mathbf{b}}^{f^\ell})|V_n^{f^\ell}(\tau')\mathrm{d}\tau'.
\end{aligned}
$$

From the Gronwall's inequality, we derive that, as long as (5.53) holds,

$$
V_n^{f^\ell}(s;t,x,v) \leq V_n^{f^\ell}(t-t_{\mathbf{b}}^{f^\ell}(t,x,v))e^{C(1+\varepsilon+\bar{t})^2(\sup_\ell\|\nabla\phi^\ell\|_{C^1}\sup_\ell\|\nabla\phi^\ell\|_\infty)\times|s-(t-t_{\mathbf{b}}^{f^\ell}(t,x,v))|^2}.
\tag{5.55}
$$

Now we verify the conditions of (5.53) for all $-\varepsilon \leq t-t_{\mathbf{b}}^{f^\ell}(t,x,v) \leq s \leq \bar{t}$. Note that we are only interested in the case of $V_n^{f^\ell}(t-t_{\mathbf{b}}^{f^\ell}(t,x,v);t,x,v) < \delta_{**}$. From the argument of (5.54), ignoring negative curvature term,

$$
\begin{aligned}
&|X_n^{f^\ell}(s;t,x,v)| \\
&\quad \leq (\varepsilon+\bar{t})|V_n^{f^\ell}(t-t_{\mathbf{b}}^{f^\ell};t,x,v)| \\
&\qquad + C[1+(\varepsilon+\bar{t})^2\sup_\ell\|\nabla\phi^\ell\|_\infty]\sup_\ell\|\nabla\phi^\ell\|_{C^1}\int_{t-t_{\mathbf{b}}^{f^\ell}}^s\int_{t-t_{\mathbf{b}}^{f^\ell}}^\tau |X_n^{f^\ell}(\tau;t,x,v)|\mathrm{d}\tau\mathrm{d}s \\
&\quad \leq (\varepsilon+\bar{t})|V_n^{f^\ell}(t-t_{\mathbf{b}}^{f^\ell};t,x,v)| + C\int_{t-t_{\mathbf{b}}^{f^\ell}}^s |\tau-(t-t_{\mathbf{b}}^{f^\ell})||X_n^{f^\ell}(\tau;t,x,v)|\mathrm{d}\tau.
\end{aligned}
$$

Then by the Gronwall's inequality we derive that, in case of (5.46),

$$|X_n^{f^\ell}(s;t,x,v)| \le C_{\varepsilon+\bar{t}}|V_n^{f^\ell}(t-t_{\mathbf{b}}^{f^\ell};t,x,v)| \;\; for\ all\ -\varepsilon \le t-t_{\mathbf{b}}^{f^\ell} \le s \le t \le \bar{t}. \tag{5.56}$$

If we choose

$$\delta_{**} = \frac{o(1)}{|\bar{t}+\varepsilon|} \times \frac{1}{N}, \tag{5.57}$$

then (5.55) holds for $-\varepsilon \le t - t_{\mathbf{b}}^{f^\ell}(t,x,v) \le s \le \bar{t}$. Hence we complete the proof of (5.52).

Step 3. Suppose that (5.50) holds and $0 \le V_n^{f^\ell}(t-t_{\mathbf{b}}^{f^\ell}(t,x,v);t,x,v) < \delta_{**}$ with δ_{**} of (5.57). Recall the definition of τ_1 in (5.51). Inductively we define $\tau_2 := \sup\big\{\tau \ge 0 : V_n^{f^\ell}(s;t,x,v) \le 0\ for\ all\ s \in [\tau_1,\tau]\big\}$ and $\tau_3, \tau_4, \cdots$. Clearly such points can be countably many at most in an interval of $[t-t_{\mathbf{b}}^{f^\ell}, t]$. Suppose $\lim_{k\to\infty}\tau_k = t$. Then choose $k_0 \gg 1$ such that $|\tau_{k_0} - t| \ll |V_n^{f^\ell}(t-t_{\mathbf{b}}^{f^\ell};t,x,v)|$. Then, for $s \in [\tau_{k_0}, t]$, from (5.54) and (5.50),

$$|V_n^{f^\ell}(t;t,x,v)| \lesssim |V_n^{f^\ell}(t-t_{\mathbf{b}}^{f^\ell};t,x,v)|. \tag{5.58}$$

Now we assume that $\tau_{k_0} < t \le \tau_{k_0+1}$. From the definition of τ_i in (5.51) we split the case in two.

<u>Case 1:</u> *Suppose* $V_n^{f^\ell}(s;t,x,v) > 0\ for\ s \in (\tau_{k_0}, t)$.
From (5.54) and (5.56)

$$V_n^{f^\ell}(t;t,x,v) \lesssim \int_{\tau_{k_0}}^{\bar{t}} X_n^{f^\ell}(s) \lesssim |V_n^{f^\ell}(t-t_{\mathbf{b}}^{f^\ell};t,x,v)|. \tag{5.59}$$

<u>Case 2:</u> *Suppose* $V_n^{f^\ell}(s;t,x,v) < 0\ for\ s \in (\tau_{k_0}, t)$.
Suppose

$$-V_n^{f^\ell}(t;t,x,v) = |V_n^{f^\ell}(t;t,x,v)| \ge \frac{1}{\varepsilon}|V_n^{f^\ell}(t-t_{\mathbf{b}}^{f^\ell};t,x,v)|^{1/2}. \tag{5.60}$$

From (5.54), now taking account of the curvature term this time, we derive that

$$\begin{aligned}-V_n^{f^\ell}(t;t,x,v) \le &\int_{\tau_{k_0}}^{t} (-1)[V_\parallel^{f^\ell}(s)\cdot\nabla^2\eta(X_\parallel^{f^\ell}(s))\cdot V_\parallel^{f^\ell}(s)]\cdot n(X_\parallel^{f^\ell}(s))\mathrm{d}s \\ &+ C|V_n^{f^\ell}(t-t_{\mathbf{b}}^{f^\ell}(t,x,v);t,x,v)|,\end{aligned}$$

where we have used (5.50) and (5.56). From (5.60) the above inequality implies that, for $|V_n^{f^\ell}(t-t_{\mathbf{b}}^{f^\ell}(t,x,v);t,x,v)| \ll 1$,

$$\frac{1}{2\varepsilon}|V_n^{f^\ell}(t-t_{\mathbf{b}}^{f^\ell};t,x,v)|^{1/2} \le \int_{\tau_{k_0}}^{t} (-1)[V_\parallel^{f^\ell}(s)\cdot\nabla^2\eta(X_\parallel^{f^\ell}(s))\cdot V_\parallel^{f^\ell}(s)]\cdot n(X_\parallel^{f^\ell}(s))\mathrm{d}s.$$

Note that $|\frac{d}{ds}V_\parallel^{f^\ell}(s)|$ and $|\frac{d}{ds}X_\parallel^{f^\ell}(s)|$ are all bound from $\nabla\phi^\ell \in C^1$, (5.50), and (5.56). By (5.50) and (1.19) we can take ε to be sufficiently small such that

$$\begin{aligned}&\int_{t-|V_n^{f^\ell}(t-t_{\mathbf{b}}^{f^\ell};t,x,v)|^{1/2}}^{t} (-1)[V_\parallel^{f^\ell}(s)\cdot\nabla^2\eta(X_\parallel^{f^\ell}(s))\cdot V_\parallel^{f^\ell}(s)]\cdot n(X_\parallel^{f^\ell}(s))\mathrm{d}s \\ &\quad\le |V_n^{f^\ell}(t-t_{\mathbf{b}}^{f^\ell};t,x,v)|^{1/2}C_\eta C_{\varepsilon,\bar{t},\sup_\ell\|\nabla\phi^\ell\|_\infty} \le \frac{1}{4\varepsilon}|V_n^{f^\ell}(t-t_{\mathbf{b}}^{f^\ell};t,x,v)|^{1/2}.\end{aligned}$$

Hence we obtain

$$\begin{aligned}&\frac{1}{4\varepsilon}|V_n^{f^\ell}(t-t_{\mathbf{b}}^{f^\ell};t,x,v)|^{1/2}\\&\leq \int_{\tau_{k_0}}^{t-|V_n^{f^\ell}(t-t_{\mathbf{b}}^{f^\ell};t,x,v)|^{1/2}}(-1)[V_\parallel^{f^\ell}(s)\cdot\nabla^2\eta(X_\parallel^{f^\ell}(s))\cdot V_\parallel^{f^\ell}(s)]\cdot n(X_\parallel^{f^\ell}(s))\mathrm{d}s.\end{aligned}\tag{5.61}$$

On the other hand, if $t-|V_n^{f^\ell}(t-t_{\mathbf{b}}^{f^\ell};t,x,v)|^{1/2}\leq\tau_{k_0}$ then $|t-\tau_{k_0}|\leq|V_n^{f^\ell}(t-t_{\mathbf{b}}^{f^\ell};t,x,v)|^{1/2}$, which implies that, from (5.54), (5.50), and (5.56),

$$|V_n^{f^\ell}(t;t,x,v)|\lesssim|V_n^{f^\ell}(t-t_{\mathbf{b}}^{f^\ell};t,x,v)|^{1/2}.\tag{5.62}$$

Now we consider $X_n^{f^\ell}(t;t,x,v)$. From (5.54) and $\dot{X}_n^{f^\ell}(s;t,x,v)=V_n^{f^\ell}(s;t,x,v)$ together with (5.56) and (5.50)

$$\begin{aligned}&X_n^{f^\ell}(t;t,x,v)\\&\leq(\bar{t}+\varepsilon)|V_n^{f^\ell}(t-t_{\mathbf{b}}^{f^\ell};t,x,v)|\\&\quad+\int_{\tau_{k_0}}^{t}\int_{\tau_{k_0}}^{\tau}\underbrace{[V_\parallel^{f^\ell}(s)\cdot\nabla^2\eta(X_\parallel^{f^\ell}(s))\cdot V_\parallel^{f^\ell}(s)]\cdot n(X_\parallel^{f^\ell}(s))}_{\leq 0}\mathrm{d}s\mathrm{d}\tau\\&\leq(\bar{t}+\varepsilon)|V_n^{f^\ell}(t-t_{\mathbf{b}}^{f^\ell};t,x,v)|\\&\quad+|V_n^{f^\ell}(t-t_{\mathbf{b}}^{f^\ell};t,x,v)|^{1/2}\\&\quad\int_{\tau_{k_0}}^{t-|V_n^{f^\ell}(t-t_{\mathbf{b}}^{f^\ell};t,x,v)|^{1/2}}[V_\parallel^{f^\ell}(s)\cdot\nabla^2\eta(X_\parallel^{f^\ell}(s))\cdot V_\parallel^{f^\ell}(s)]\cdot n(X_\parallel^{f^\ell}(s))\mathrm{d}s\\&\leq(\bar{t}+\varepsilon)|V_n^{f^\ell}(t-t_{\mathbf{b}}^{f^\ell};t,x,v)|-\frac{1}{4\varepsilon}|V_n^{f^\ell}(t-t_{\mathbf{b}}^{f^\ell};t,x,v)|^1\quad \textit{from } (5.61)\\&<0.\end{aligned}\tag{5.63}$$

Clearly this cannot happen since $x\in\bar{\Omega}$ and $x_n\geq 0$. Therefore our assumption (5.60) was wrong and we conclude (5.62).

Step 4 From (5.52), (5.58), (5.59), and (5.62) in *Step 1* and *Step 2*, we conclude that the same estimate (5.62) for $|V_n^{f^\ell}(t-t_{\mathbf{b}}^{f^\ell};t,x,v)|\ll 1$ in the case of (5.46) and (5.50). Finally from (5.45), (5.49), (5.52), and (5.62) Therefore we conclude that

$$|V_n^{f^\ell}(t-t_{\mathbf{b}}^{f^\ell}(t,x,v);t,x,v)|\gtrsim\left(\frac{1}{N^2}\right)\quad(t,x,v)\in[0,\bar{t}]\times(\mathcal{S}_N)^c.\tag{5.64}$$

From (2.36), (2.37), (2.40), and (2.41) in Lemma 2.4 in [17],

$$\sup_{\substack{\ell\in\mathbb{N},\ (x,v)\in(\mathcal{S}_N)^c,\\ -\varepsilon\leq t-t_{\mathbf{b}}^{f^\ell}(t,x,v)\leq t\leq\bar{t}}}|\nabla_{x,v}\alpha^\beta_{f^\ell,\varepsilon}(t,x,v)|\lesssim\frac{1}{|V_n^{f^\ell}(t-t_{\mathbf{b}}^{f^\ell};t,x,v)|^{2-\beta}}\lesssim_{\varepsilon,N,\bar{t}}1.$$

Hence we extract another subsequence out of all previous steps (and redefine this as $\{\ell_N\}$) such that

$$\nabla_{x,v}\alpha^\beta_{f^{\ell_N},\varepsilon}\overset{*}{\rightharpoonup}\nabla_{x,v}\alpha^\beta_{f,\varepsilon}\ \textit{weakly}-*\ \textit{in}\ L^\infty((-\varepsilon,\bar{t})\times(\mathcal{S}_N)^c).\tag{5.65}$$

Note that the limiting function is identified from (5.43). Finally the trong convergence of (5.3) and the weak−∗ convergence of (5.65) justifies the convergence of (5.40):

$$\lim_{\ell\to\infty}(5.40) = -\int_0^t \iint_{\Omega\times\mathbb{R}^3} \nabla_{x,v}\alpha^{\beta}_{f,\varepsilon} f(w_{\tilde{\vartheta}}\psi)\mathrm{d}x\mathrm{d}v. \tag{5.66}$$

Now we extract the final subsequence $\{\ell_*\}$ from the previous subsequence: By the Cantor's diagonal argument we define

$$\ell_* = \ell_\ell. \tag{5.67}$$

Combining (5.44) and (5.66) we have (5.37) with this subsequence for any test function ψ. For any $\psi \in C_c^\infty(\bar{\Omega}\times\mathbb{R}^3\backslash\gamma_0)$ there exists $N_\psi \in \mathbb{N}$ such that $supp(\psi) \subset (\mathcal{S}_{N_\psi})^c$. Hence (5.36) follows from (5.37).

Finally we obtain (5.35). Assumptions in Proposition 7 thus hold. Applying Proposition 7 8, assuming f_1 and f_2 are both solutions, then

$$\|e^{-\lambda t\langle v\rangle}[f_1(t)-f_2(t)]\|_{L^{1+\delta}(\Omega\times\mathbb{R}^3)} \lesssim \|f_1(0)-f_2(0)\|_{L^{1+\delta}(\Omega\times\mathbb{R}^3)},$$

so the solution is unique. □

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6 Appendix

Lemma 15 *For $R(u\to v;x,t)$ given by (1.11), given any u such that $u\cdot n(x)>0$,*

$$\int_{n(x)\cdot v<0} R(u\to v;x,t)dv = 1. \tag{6.1}$$

Proof We can transform the basis from $\{n,\tau_1,\tau_2\}$ to the standard bases $\{e_1,e_2,e_3\}$. For the sake of simplicity, we assume $T_w(x)=1$. The integration over $\mathcal{V}_\|$, after the orthonormal transformation, becomes integration over $\mathbb{R}^2$. We have

$$\int_{\mathbb{R}^2}\frac{1}{r_\|(2-r_\|)}\exp\Big(\frac{|v_\| - (1-r_\|)u_\||^2}{r_\|(2-r_\|)}\Big)dv_\|,$$

which is obviously normalized.

Then we consider the integration over $\mathcal{V}_\perp$, which is $e_3<0$ after the transformation. We want to show

$$\frac{2}{r_\perp}\int_{-\infty}^0 -v_\perp e^{-\frac{|v_\perp|^2}{r_\perp}} e^{\frac{-(1-r_\perp)|u_\perp|^2}{r_\perp}} I_0\Big(\frac{2(1-r_\perp)^{1/2}v_\perp u_\perp}{r_\perp}\Big)dv_\perp = 1. \tag{6.2}$$

The Bessel function reads

$$J_0(y) = \frac{1}{\pi}\int_0^\pi e^{iy\cos\theta}d\theta = \sum_{k=0}^\infty \frac{1}{\pi}\int_0^\pi \frac{(iy\cos\theta)^k}{k!}d\theta = \sum_{k=0}^\infty \int_0^\pi \frac{(iy\cos\theta)^{2k}}{(2k)!}d\theta$$
$$\sum_{k=0}^\infty \int_0^\pi \frac{(-1)^k(y)^{2k}(\cos\theta)^{2k}}{(2k)!}d\theta = \sum_{k=0}^\infty (-1)^k\frac{(\frac14 y^2)^k}{(k!)^2},$$

where we use the Fubini's theorem and the fact that

$$\int_0^\pi \cos^{2k}\theta = \frac{\pi}{2^{2k}}\binom{2k}{k}.$$

Hence

$$I_0(y) = \frac{1}{\pi}\int_0^\pi e^{i(-iy)\cos\theta}d\theta = J_0(-iy) = \sum_{k=0}^{\infty}\frac{(\frac{1}{4}y^2)^k}{(k!)^2}, \quad I_0(y) = I_0(-y). \tag{6.3}$$

By taking the change of variable $v_\perp \to -v_\perp$, the LHS of (6.2) can be written as

$$\frac{2}{r_\perp}\int_0^\infty v_\perp e^{-\frac{|v_\perp|^2}{r_\perp}} e^{\frac{-(1-r_\perp)|u_\perp|^2}{r_\perp}} I_0(\frac{2(1-r_\perp)^{1/2}v_\perp u_\perp}{r_\perp})dv_\perp.$$

Using (6.3) we rewrite the above term as

$$\sum_{k=0}^{\infty}\frac{2}{r_\perp}\int_0^\infty v_\perp e^{\frac{-|v_\perp|^2}{r_\perp}} e^{\frac{-(1-r_\perp)|u_\perp|^2}{r_\perp}} \frac{(1-r_\perp)^k v_\perp^{2k} u_\perp^{2k}}{(k!)^2 r_\perp^{2k}}dv, \tag{6.4}$$

where we use the Tonelli theorem. Rescale $v_\perp = \sqrt{r_\perp}v_\perp$ we have

$$\begin{aligned}
&\frac{2}{r_\perp}\int_0^\infty v_\perp e^{\frac{-|v_\perp|^2}{r_\perp}} e^{\frac{-(1-r_\perp)|u_\perp|^2}{r_\perp}} \frac{(1-r_\perp)^k v_\perp^{2k} u_\perp^{2k}}{(k!)^2 r_\perp^{2k}}dv \\
&= 2\int_0^\infty v_\perp e^{-|v_\perp|^2} e^{\frac{-(1-r_\perp)|u_\perp|^2}{r_\perp}} \frac{(1-r_\perp)^k v_\perp^{2k} u_\perp^{2k}}{(k!)^2 r_\perp^{k}}dv \\
&= 2\int_0^\infty v_\perp^{2k+1} e^{-|v_\perp|^2} dv e^{\frac{-(1-r_\perp)|u_\perp|^2}{r_\perp}} \frac{(1-r_\perp)^k u_\perp^{2k}}{(k!)^2 r_\perp^{k}} \\
&= 2\frac{k!}{2} e^{\frac{-(1-r_\perp)|u_\perp|^2}{r_\perp}} \frac{(1-r_\perp)^k u_\perp^{2k}}{(k!)^2 r_\perp^{k}} = e^{\frac{-(1-r_\perp)|u_\perp|^2}{r_\perp}} \frac{(1-r_\perp)^k u_\perp^{2k}}{k! r_\perp^{k}}.
\end{aligned} \tag{6.5}$$

Therefore, the LHS of (6.2) can be written as

$$e^{\frac{-(1-r_\perp)|u_\perp|^2}{r_\perp}} \sum_{k=0}^{\infty} \frac{(1-r_\perp)^k u_\perp^{2k}}{k! r_\perp^{k}} = e^{\frac{-(1-r_\perp)|u_\perp|^2}{r_\perp}} e^{\frac{(1-r_\perp)|u_\perp|^2}{r_\perp}} = 1.$$

□

Lemma 16 *For any $a > 0, b > 0, \varepsilon > 0$ with $a + \varepsilon < b$,*

$$\frac{b}{\pi}\int_{\mathbb{R}^2} e^{\varepsilon|v|^2} e^{a|v|^2} e^{-b|v-w|^2} dv = \frac{b}{b-a-\varepsilon} e^{\frac{(a+\varepsilon)b}{b-a-\varepsilon}|w|^2}. \tag{6.6}$$

And when $\delta \ll 1$,

$$\begin{aligned}
\frac{b}{\pi}\int_{|v-\frac{b}{b-a-\varepsilon}w|>\delta^{-1}} e^{\varepsilon|v|^2} e^{a|v|^2} e^{-b|v-w|^2} dv &\le e^{-(b-a-\varepsilon)\delta^{-2}} \frac{b}{b-a-\varepsilon} e^{\frac{(a+\varepsilon)b}{b-a-\varepsilon}|w|^2} &(6.7)\\
&\le \delta \frac{b}{b-a-\varepsilon} e^{\frac{(a+\varepsilon)b}{b-a-\varepsilon}|w|^2}. &(6.8)
\end{aligned}$$

Proof

$$\begin{aligned}
\frac{b}{\pi}\int_{\mathbb{R}^2} e^{\varepsilon|v|^2}e^{a|v|^2}e^{-b|v-w|^2}dv &= \frac{b}{\pi}\int_{\mathbb{R}^2} e^{(a+\varepsilon-b)|v|^2}e^{2bv\cdot w}e^{-b|w|^2}dv\\
&= \frac{b}{\pi}\int_{\mathbb{R}^2} e^{(a+\varepsilon-b)|v+\frac{b}{a+\varepsilon-b}w|^2}e^{\frac{-b^2}{a+\varepsilon-b}|w|^2}e^{-b|w|^2}dv\\
&= \frac{b}{\pi}\int_{\mathbb{R}^2} e^{(a+\varepsilon-b)|v|^2}dv\, e^{\frac{(a+\varepsilon)b}{b-a-\varepsilon}|w|^2} = \frac{b}{b-a-\varepsilon}e^{\frac{(a+\varepsilon)b}{b-a-\varepsilon}|w|^2},
\end{aligned}$$

where we apply change of variable $v+\frac{b}{a+\varepsilon-b}w \to v$ in the first step of the last line, then we obtain (6.6).

Following the same derivation

$$\begin{aligned}
&\frac{b}{\pi}\int_{|v-\frac{b}{b-a-\varepsilon}w|>\delta^{-1}} e^{\varepsilon|v|^2}e^{a|v|^2}e^{-b|v-w|^2}dv\\
&\quad= \frac{b}{\pi}\int_{|v-\frac{b}{b-a-\varepsilon}w|>\delta^{-1}} e^{(a+\varepsilon-b)|v-\frac{b}{b-a-\varepsilon}w|^2}dv\, e^{\frac{(a+\varepsilon)b}{b-a-\varepsilon}|w|^2}\\
&\quad\le e^{-(b-a-\varepsilon)\delta^{-2}}\frac{b}{b-a-\varepsilon}e^{\frac{(a+\varepsilon)b}{b-a-\varepsilon}|w|^2} \le \delta\frac{b}{b-a-\varepsilon}e^{\frac{(a+\varepsilon)b}{b-a-\varepsilon}|w|^2},
\end{aligned}$$

thus we obtain (6.8). □

Lemma 17 *For any $a>0, b>0, \varepsilon>0$ with $a+\varepsilon<b$,*

$$2b\int_{\mathbb{R}^+} ve^{\varepsilon v^2}e^{av^2}e^{-bv^2}e^{-bw^2}I_0(2bvw)dv = \frac{b}{b-a-\varepsilon}e^{\frac{(a+\varepsilon)b}{b-a-\varepsilon}w^2}. \tag{6.9}$$

And when $\delta \ll 1$,

$$2b\int_{0<v<\delta} ve^{\varepsilon v^2}e^{av^2}e^{-bv^2}e^{-bw^2}I_0(2bvw)dv \le \delta\frac{b}{b-a-\varepsilon}e^{\frac{(a+\varepsilon)b}{b-a-\varepsilon}w^2}. \tag{6.10}$$

Proof

$$\begin{aligned}
&2b\int_{\mathbb{R}^+} ve^{\varepsilon v^2}e^{av^2}e^{-bv^2}e^{-bw^2}I_0(2bvw)dv\\
&\quad= 2b\int_{\mathbb{R}^+} ve^{(a+\varepsilon-b)v^2}I_0(2bvw)e^{\frac{b^2}{a+\varepsilon-b}w^2}e^{\frac{b^2}{b-a-\varepsilon}w^2}dv e^{-bw^2}\\
&\quad= 2(b-a-\varepsilon)\int_{\mathbb{R}^+} ve^{(a+\varepsilon-b)v^2}I_0(2bvw)e^{\frac{(bw)^2}{a+\varepsilon-b}}dv\frac{b}{b-a-\varepsilon}e^{\frac{(a+\varepsilon)b}{b-a-\varepsilon}w^2}\\
&\quad= \frac{b}{b-a-\varepsilon}e^{\frac{(a+\varepsilon)b}{b-a-\varepsilon}w^2},
\end{aligned}$$

where we use (6.2) in Lemma 15 in the last line, then we obtain (6.9).

Following the same derivation we have

$$\begin{aligned}
&2b\int_{0<v<\delta} ve^{\varepsilon v^2}e^{av^2}e^{-bv^2}e^{-bw^2}I_0(2bvw)dv\\
&\quad= 2(b-a-\varepsilon)\int_{0<v<\delta} ve^{(a+\varepsilon-b)v^2}I_0(2bvw)e^{\frac{(bw)^2}{a+\varepsilon-b}}dv\frac{b}{b-a-\varepsilon}e^{\frac{(a+\varepsilon)b}{b-a-\varepsilon}w^2}.
\end{aligned}$$

Using the definition of I_0 we have

$$I_0(y) = \frac{1}{\pi}\int_0^{\pi} e^{y\cos\phi}d\phi \le e^y.$$

Thus when $a - b + \varepsilon < 0$,

$$\begin{aligned} &2(b-a-\varepsilon)\int_{0<v<\delta} ve^{(a+\varepsilon-b)v^2} I_0(2bvw) e^{\frac{(bw)^2}{a+\varepsilon-b}} dv \\ &\leq 2(b-a-\varepsilon)\int_{0<v<\delta} ve^{(a-b+\varepsilon)v^2} e^{2vbw} e^{\frac{(bw)^2}{a-b+\varepsilon}} \\ &= 2(b-a-\varepsilon)\int_{0<v<\delta} ve^{(a-b+\varepsilon)(v+\frac{bw}{a-b+\varepsilon})^2} dv \\ &\leq 2(b-a-\varepsilon)\int_{0<v<\delta} vdv < \delta, \end{aligned}$$

where we use $\delta \ll 1$ in the last step, then we obtain (6.10). Then we derive (6.13). □

Lemma 18 *For any $m, n > 0$, when $\delta \ll 1$, we have*

$$2m^2 \int_{\frac{n}{m}u_\perp+\delta^{-1}}^{\infty} v_\perp e^{-m^2 v_\perp^2} I_0(2mnv_\perp u_\perp) e^{-n^2 u_\perp^2} dv_\perp \lesssim e^{-\frac{m^2}{4\delta^2}}. \tag{6.11}$$

In consequence, for any $a > 0, b > 0, \varepsilon > 0$ with $a + \varepsilon < b$,

$$2b \int_{\frac{b}{b-a-\varepsilon}w+\delta^{-1}}^{\infty} ve^{\varepsilon v^2} e^{av^2} e^{-bv^2} e^{-bw^2} I_0(2bvw) dv \leq e^{\frac{-(b-a-\varepsilon)}{4\delta^2}} \frac{b}{b-a-\varepsilon} e^{\frac{(a+\varepsilon)b}{b-a-\varepsilon}w^2} \tag{6.12}$$

$$\leq \delta \frac{b}{b-a-\varepsilon} e^{\frac{(a+\varepsilon)b}{b-a-\varepsilon}w^2}. \tag{6.13}$$

Proof We discuss two cases. The first case is $v_\perp > 2\frac{n}{m}u_\perp$. We bound I_0 as

$$I_0(2mnv_\perp u_\perp) \leq \frac{1}{\pi}\int_0^\pi \exp\Big(2mnv_\perp u_\perp\Big) d\theta = \exp\Big(2mnv_\perp u_\perp\Big).$$

The LHS of (6.11) is bounded by

$$2m^2 \int_{\max\{2\frac{n}{m}u_\perp, \frac{n}{m}u_\perp+\delta^{-1}\}}^{\infty} ve^{-m^2(v_\perp - \frac{n}{m}u_\perp)^2} dv.$$

Using $v_\perp > 2\frac{n}{m}u_\perp$ we have

$$(v_\perp - \frac{n}{m}u_\perp)^2 \geq (\frac{v_\perp}{2} + \frac{v_\perp}{2} - \frac{n}{m}u_\perp)^2 \geq \frac{v_\perp^2}{4}.$$

Thus we can further bound LHS of (6.11) by

$$2m^2 \int_{\max\{2\frac{n}{m}u_\perp, \frac{n}{m}u_\perp+\delta^{-1}\}}^{\infty} v_\perp e^{-\frac{m^2 v_\perp^2}{4}} dv_\perp \lesssim e^{-\frac{m^2}{4\delta^2}}.$$

The second case is $0 \leq v_\perp \leq 2\frac{n}{m}u_\perp$. Since $\frac{n}{m}u_\perp + \delta^{-1} < v_\perp$, without loss of generality, we can assume $u_\perp > \delta^{-1}$. We compare the Taylor series of $v_\perp I_0(2mnv_\perp u_\perp)$ and $\exp\Big(2mnv_\perp u_\perp\Big)$. We have

$$v_\perp I_0(2mnv_\perp u_\perp) = \sum_{k=0}^{\infty} \frac{m^{2k} n^{2k} v_\perp^{2k+1} u_\perp^{2k}}{(k!)^2}, \tag{6.14}$$

and

$$\exp\Big(2mnv_\perp u_\perp\Big) = \sum_{k=0}^{\infty} \frac{2^k m^k n^k v_\perp^k u_\perp^k}{k!}. \tag{6.15}$$

We choose k_1 such that when $k > k_1$, we can apply the Sterling formula such that

$$\frac{1}{2} \le |\frac{k!}{k^k e^{-k}\sqrt{2\pi k}}| \le 2.$$

Then we observe the quotient of the k-th term of (6.14) and the $2k+1$-th term of (6.15),

$$\begin{aligned}
&\frac{m^{2k}n^{2k}v_\perp^{2k+1}u_\perp^{2k}}{(k!)^2}\Big/\Big(\frac{2^{2k+1}m^{2k+1}n^{2k+1}v_\perp^{2k+1}u_\perp^{2k+1}}{(2k+1)!}\Big)\\
&\le \frac{4}{k^{2k}e^{-2k}2\pi k}\Big/\Big(\frac{2^{2k+1}mnu_\perp}{(2k+1)^{2k+1}e^{-(2k+1)}\sqrt{2\pi(2k+1)}}\Big)\\
&= \frac{4e}{2\pi mn}\Big(\frac{k+1/2}{k}\Big)^{2k+1}\frac{\sqrt{2\pi(2k+1)}}{u_\perp}\\
&= \frac{4e}{2\pi mn}\Big(\frac{2k+1}{2k}\Big)^{2k+1}\frac{\sqrt{2\pi(2k+1)}}{u_\perp} \le \frac{4e^2}{\sqrt{\pi}mn}\frac{\sqrt{k}}{u_\perp}.
\end{aligned}$$

Thus we can take $k_u = u_\perp^2$ such that when $k \le k_u$,

$$\sum_{k=k_1}^{k_u} \frac{m^{2k}n^{2k}v_\perp^{2k+1}u_\perp^{2k}}{(k!)^2} \le \frac{4e^2}{\sqrt{\pi}mn}\sum_{k=k_1}^{k_u}\frac{2^{2k+1}m^{2k+1}n^{2k+1}v_\perp^{2k+1}u_\perp^{2k+1}}{(2k+1)!}. \tag{6.16}$$

Similarly we observe the quotient of the k-th term of (6.14) and the $2k$-th term of (6.15),

$$\begin{aligned}
&\frac{m^{2k}n^{2k}v_\perp^{2k+1}u_\perp^{2k}}{(k!)^2}\Big/\Big(\frac{2^{2k}m^{2k}n^{2k}v_\perp^{2k}u_\perp^{2k}}{(2k)!}\Big)\\
&\le \frac{4v_\perp}{k^{2k}e^{-2k}2\pi k}\Big/\Big(\frac{2^{2k}}{(2k)^{2k}e^{-2k}\sqrt{4\pi k}}\Big) = \frac{4v_\perp}{\sqrt{\pi}\sqrt{k}}.
\end{aligned}$$

When $k > k_u = u_\perp^2$, by $u_\perp > \delta^{-1}$ and $v_\perp < 2\frac{n}{m}u_\perp$ we have

$$\frac{4v_\perp}{\sqrt{\pi}\sqrt{k}} \le \frac{4v_\perp}{\sqrt{\pi}u_\perp} \le \frac{8n}{m\sqrt{\pi}}.$$

Thus we have

$$\sum_{k=k_u}^{\infty} \frac{m^{2k}n^{2k}v_\perp^{2k+1}u_\perp^{2k}}{(k!)^2} \le \frac{8n}{m\sqrt{\pi}}\sum_{k=k_u}^{\infty}\frac{2^{2k}m^{2k}n^{2k}v_\perp^{2k}u_\perp^{2k}}{(2k)!}. \tag{6.17}$$

Collecting (6.17) (6.16), when $v_\perp < 2\frac{n}{m}u_\perp$, we obtain

$$v_\perp I_0(2mnv_\perp u_\perp) \lesssim \exp\Big(\frac{2(1-r_\perp)^{1/2}v_\perp u_\perp}{r_\perp}\Big). \tag{6.18}$$

By (6.18), we have

$$\int_{\frac{n}{m}u_\perp+\delta^{-1}}^{2\frac{n}{m}u_\perp} v_\perp I_0(2mnv_\perp u_\perp))e^{-m^2v_\perp^2}e^{n^2v_\perp^2}dv \lesssim \int_{\frac{n}{m}u_\perp+\delta^{-1}}^{2\frac{n}{m}u_\perp} e^{-m^2(v_\perp-\frac{n}{m}u_\perp)^2}dv \le e^{-m^2\delta^{-2}}. \tag{6.19}$$

Collecting (6.15) and (6.19) we prove (6.11).

Then following the same derivation as (6.9),

$$\begin{aligned} &2b\int_{\frac{b}{b-a-\varepsilon}w+\delta^{-1}}^{\infty} ve^{\varepsilon v^2}e^{av^2}e^{-bv^2}e^{-bw^2}I_0(2bvw)dv \\ &= 2(b-a-\varepsilon)\int_{\frac{b}{b-a-\varepsilon}w+\delta^{-1}}^{\infty} ve^{(a+\varepsilon-b)v^2}I_0(2bvw)e^{\frac{(bw)^2}{a+\varepsilon-b}}dv\frac{b}{b-a-\varepsilon}e^{\frac{(a+\varepsilon)b}{b-a-\varepsilon}w^2} \\ &\le e^{\frac{-(b-a-\varepsilon)}{4\delta^2}}\frac{b}{b-a-\varepsilon}e^{\frac{(a+\varepsilon)b}{b-a-\varepsilon}w^2} \le \delta\frac{b}{b-a-\varepsilon}e^{\frac{(a+\varepsilon)b}{b-a-\varepsilon}w^2}, \end{aligned}$$

where we apply (6.11) in the first step in the third line and take $\delta \ll 1$ in the last step of the third line. □

Lemma 19 *If* $0 < \frac{\theta}{4} < \rho$, *if* $0 < \tilde{\rho} < \rho - \frac{\theta}{4}$, $0 \le \lambda t < \theta$,

$$\mathbf{k}_\varrho(v,u)\frac{e^{\theta|v|^2}}{e^{\subseteq|u|^2}}\frac{e^{\lambda t\langle u\rangle}}{e^{\lambda t\langle v\rangle}} \lesssim \mathbf{k}_{\tilde{\varrho}}(v,u). \tag{6.20}$$

Proof When $\langle u\rangle - \langle v\rangle \le 1$,

$$\frac{e^{\lambda s\langle u\rangle}}{e^{\lambda s\langle v\rangle}} \le e^{\lambda s}.$$

When $\langle u\rangle - \langle v\rangle \ge 1$,

$$\langle u\rangle^2 - \langle v\rangle^2 = (\langle u\rangle - \langle v\rangle)(\langle u\rangle + \langle v\rangle) \ge \langle u\rangle - \langle v\rangle.$$

Thus by $\langle u\rangle^2 = |u|^2 + 1$,

$$\frac{e^{\lambda s\langle u\rangle}}{e^{\lambda s\langle v\rangle}} \lesssim 1 + \frac{e^{\lambda s|u|^2}}{e^{\lambda s|v|^2}}.$$

Note

$$\mathbf{k}_\varrho(v,u)\frac{e^{\vartheta|v|^2}}{e^{\vartheta|u|^2}} = \frac{1}{|v-u|}\exp\left\{-\varrho|v-u|^2 - \varrho\frac{||v|^2-|u|^2|^2}{|v-u|^2} + \vartheta|v|^2 - \vartheta|u|^2\right\}.$$

Let $v - u = \eta$ and $u = v - \eta$. Then the exponent equals

$$\begin{aligned} &-\varrho|\eta|^2 - \varrho\frac{||\eta|^2 - 2v\cdot\eta|^2}{|\eta|^2} - \vartheta\{|v-\eta|^2 - |v|^2\} - \lambda t\{|v| - |v-\eta|\} \\ &= -2\varrho|\eta|^2 + 4\varrho v\cdot\eta - 4\varrho\frac{|v\cdot\eta|^2}{|\eta|^2} - \vartheta\{|\eta|^2 - 2v\cdot\eta\} \\ &= (-2\varrho - \vartheta)|\eta|^2 + (4\varrho + 2\vartheta)v\cdot\eta - 4\varrho\frac{\{v\cdot\eta\}^2}{|\eta|^2}. \end{aligned}$$

If $0 < \vartheta < 4\varrho$ then the discriminant of the above quadratic form of $|\eta|$ and $\frac{v\cdot\eta}{|\eta|}$ is

$$(4\varrho + 2\vartheta)^2 - 4(-2\varrho - \vartheta)(-4\varrho) = 4\vartheta^2 - 16\varrho\vartheta < 0.$$

Hence, the quadratic form is negative definite. We thus have, for $0 < \tilde{\varrho} < \varrho - \frac{\vartheta}{4}$, the following perturbed quadratic form is still negative definite

$$-(\varrho - \tilde{\varrho})|\eta|^2 - (\varrho - \tilde{\varrho})\frac{||\eta|^2 - 2v\cdot\eta|^2}{|\eta|^2} - \vartheta\{|\eta|^2 - 2v\cdot\eta\} \le 0.$$

For

$$\begin{aligned}&\mathbf{k}_{\varrho}(v,u)\frac{e^{\vartheta|v|^2}}{e^{\vartheta|u|^2}}\frac{e^{\lambda t\langle u\rangle^2}}{e^{\lambda t|v|^2}}\\ &\quad= \frac{1}{|v-u|}\exp\left\{-\varrho|v-u|^2 - \varrho\frac{||v|^2-|u|^2|^2}{|v-u|^2} + (\theta - \lambda t)|v|^2 - (\theta-\lambda t)|u|^2\right\}.\end{aligned}$$

We just need to replace θ by $\theta - \lambda t$ in the previous computation. By $\lambda t \ll \theta$,

$$-(\varrho - \tilde{\varrho})|\eta|^2 - (\varrho - \tilde{\varrho})\frac{||\eta|^2 - 2v\cdot\eta|^2}{|\eta|^2} - (\theta - \lambda t)\{|\eta|^2 - 2v\cdot\eta\} \le 0.$$

Therefore, we conclude the lemma. □

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