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Averaging Gaussian functionals*

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Abstract

This paper consists of two parts. In the first part, we focus on the average of a functional over shifted Gaussian homogeneous noise and as the averaging domain covers the whole space, we establish a Breuer-Major type Gaussian fluctuation based on various assumptions on the covariance kernel and/or the spectral measure. Our methodology for the first part begins with the application of Malliavin calculus around Nualart-Peccati's Fourth Moment Theorem, and in addition we apply the Fourier techniques as well as a soft approximation argument based on Bessel functions of first kind.

The same methodology leads us to investigate a closely related problem in the second part. We study the spatial average of a linear stochastic heat equation driven by space-time Gaussian colored noise. The temporal covariance kernel γ_0 is assumed to be *locally integrable* in this paper. If the spatial covariance kernel is *nonnegative and integrable on the whole space*, then the spatial average admits Gaussian fluctuation; with some extra mild integrability condition on γ_0, we are able to provide a functional central limit theorem. These results complement recent studies on the spatial average for SPDEs. Our analysis also allows us to consider the case where the spatial covariance kernel is not integrable: For example, in the case of the Riesz kernel, the first chaotic component of the spatial average is dominant so that the Gaussian fluctuation also holds true.

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Contents

1 Introduction

Motivated by the Breuer-Major central limit theorem (CLT) [2] and recent studies on the spatial averages of SPDEs [14, 15, 7], we devote this paper to seeking general conditions that lead to the Gaussian fluctuations of averages of Gaussian functionals.

Let us briefly introduce our framework. Let W be a d-dimensional homogenous Gaussian noise with covariance kernel γ, that is, $W = \big\{W(\phi), \phi \in C_c^\infty(\mathbb{R}^d)\big\}$ is a centered Gaussian family of real random variables, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with covariance structure given by

$$\mathbb{E}\big[W(\phi)W(\varphi)\big] = \int_{\mathbb{R}^{2d}} \phi(x)\varphi(y)\gamma(x-y)\,dx\,dy\,,\ \forall \phi, \varphi \in C_c^\infty(\mathbb{R}^d), \tag{1.1}$$

where $\gamma : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is symmetric with $\gamma^{-1}(\{\infty\}) \subset \{0\}$ and $\gamma(x) = (\mathscr{F}\mu)(x) = \int_{\mathbb{R}^d} e^{-\mathbf{i}x\cdot\xi}\mu(d\xi)$ for some nonnegative tempered measure μ on $\mathbb{R}^d$. These assumptions on γ ensure that (1.1) defines a nonnegative definite covariance functional and μ is known as the spectral measure. Notice that $\gamma(0) \in \mathbb{R}$ is equivalent to the finiteness of $\mu(\mathbb{R}^d)$.

It is clear that (1.1) defines an inner product, under which the space $C_c^\infty(\mathbb{R}^d)$ can be extended into a real Hilbert space $\mathfrak{H}$. Furthermore, the mapping $\phi \in C_c^\infty(\mathbb{R}^d) \mapsto W(\phi)$ extends to a linear isometry between $\mathfrak{H}$ and the Gaussian Hilbert space spanned by W. We write $W(\phi) = \int_{\mathbb{R}^d} \phi(x)\,W(dx)$ and $\mathbb{E}[W(\phi)W(\varphi)] = \langle \phi, \varphi \rangle_{\mathfrak{H}}$, for any $\phi, \varphi \in \mathfrak{H}$. This gives us an *isonormal Gaussian process* over $\mathfrak{H}$.

Now consider a real random variable $F \in L^2(\Omega)$ that is measurable with respect to W and has the following Wiener chaos expansion:

$$F(W) = \mathbb{E}[F] + \sum_{p\geq 1} I_p^W(f_p)\,, \tag{1.2}$$

where $I_p^W(\cdot)$ denotes the pth multiple stochastic integral with respect to W and f_p belongs to the symmetric subspace $\mathfrak{H}^{\odot p}$ of the pth tensor product $\mathfrak{H}^{\otimes p}$, $\forall p \in \mathbb{N}$; see [21] for more details. Along the paper we will denote by $\Pi_p F$ the orthogonal projection of F onto the pth Wiener chaos.

In order to formulate our results, we need to introduce the *spatial shifts* $\{U_x, x \in \mathbb{R}^d\}$. For each $x \in \mathbb{R}^d$ and F given as in (1.2), $U_x F$ is defined by

$$U_x F := \mathbb{E}[F] + \sum_{p\geq 1} I_p^W(f_p^x), \tag{1.3}$$

with[1] $f_p^x(y_1, \dots, y_p) = f_p(y_1 - x, \dots, y_p - x)$ for any $x, y_1, \dots, y_p \in \mathbb{R}^d$ and $p \in \mathbb{N}$. Here is another look at the above definition. For any $x \in \mathbb{R}^d$ and any $\varphi \in C_c^\infty(\mathbb{R}^d)$, we write $\varphi^x(y) = \varphi(y - x)$ and we introduce W_x, the shifted Gaussian field, defined by $W_x(\phi) = W(\phi^x)$, for any $\phi \in C_c^\infty(\mathbb{R}^d)$, and by extension for any $\phi \in \mathfrak{H}$. The family W_x has the same covariance structure as W and the associated multiple stochastic integrals satisfy $I_p^{W_x}(f) = I_p^W(f^x)$ for any $f \in \mathfrak{H}^{\odot p}$, so that $U_x F(W) = F(W_x)$ shall give us (1.3).

Let F be given as in (1.2). We are interested in the spatial averages of $U_x F$ over $B_R = \{x \in \mathbb{R}^d : \|x\| \leq R\}$, with the particular aim at general conditions on the kernels $\{f_p, p \in \mathbb{N}\}$ and the covariance kernel γ (and/or the associated spectral measure μ) that imply

$$\frac{1}{\sigma(R)} \int_{B_R} U_x F\, dx \xrightarrow[R\to+\infty]{\text{law}} N(0,1)\,, \tag{1.4}$$

where $\sigma(R)$ is a normalization constant and $N(m, v^2)$ stands for a real normal distribution with mean m and variance v^2.

To illustrate how this spatial averaging is related to the aforementioned Breuer-Major theorem and to give a flavor of our results, we provide below a particular case (see Example 1.2) and refer to Section 2 for more general results. Let us first recall the continuous-time Breuer-Major theorem (in a slightly different form).

Theorem 1.1. *Suppose $g \in L^2(\mathbb{R}, e^{-x^2/2}dx)$ has the following orthogonal expansion in Hermite polynomials $\{H_p = (-1)^p e^{x^2/2} \frac{d^p}{dx^p} e^{-x^2/2}, p \in \mathbb{N}\}$:*

$$g = \sum_{p\geq m} c_p H_p \text{ with } c_m \neq 0,\ m \geq 1 \text{ known as the Hermite rank of } g.$$

[1] For a generalized function $f \in \mathfrak{H}$, we can define f^x as follows. Let $\{f_n, n \in \mathbb{N}\} \subset C_c^\infty(\mathbb{R}^d)$ be an approximating sequence of f in $\mathfrak{H}$, we can define f_n^x for each $n \in \mathbb{N}$ and f^x to be the limit of the Cauchy sequence $\{f_n^x, n \in \mathbb{N}\}$ in $\mathfrak{H}$. It is routine to verify that the definition of f^x does not depend on the particular choice of the approximating sequence.

Let $Y = \{Y_x, x \in \mathbb{R}^d\}$ be a centered Gaussian stationary process with covariance function $\mathbb{E}[Y_a Y_b] = \rho(a-b)$ such that $\rho(0) = 1$. Under the condition $\rho \in L^m(\mathbb{R}^d, dx)$, we have

$$R^{-d/2}\int_{B_R} g(Y_x)\,dx \xrightarrow[R\to+\infty]{\text{law}} N(0,\sigma^2)\,,$$

with $\sigma^2 := \omega_d \sum_{q\geq m} c_q^2 q! \int_{\mathbb{R}^d} \rho(x)^m\, dx \in [0,\infty)$, ω_d being the volume of B_1; see also [3, 25].

Example 1.2. Now fix a unit vector $e \in \mathfrak{H}$ and put $F = g\big(W(e)\big)$, then $U_x F = g\big(W_x(e)\big) = g\big(Y_x\big)$, with $Y_x = W(e^x)$. If $g \in L^2(\mathbb{R}, e^{-x^2/2}dx)$ has Hermite rank $m \geq 1$ and

$$\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^{2d}} e(a)e(b)\gamma(a-b-x)dadb\right|^m dx < +\infty\,,$$

then Theorem 1.1 produces an example of (1.4). Note that in this example, the Gaussian functional $F = g\big(W(e)\big)$ depends only on one coordinate while our principal concern is for Gaussian functionals that may depend on infinitely many coordinates.

Recall the chaos expansions (1.2) and (1.3), and from now on, we consider the case where F has *Hermite rank* $m \geq 1$, meaning that:

$$\mathbb{E}[F] = 0,\ \{f_j, j = 1,\dots,m-1\} \text{ are zero vectors and } f_m \in \mathfrak{H}^{\odot m} \text{ is nonzero.}$$

In this case, we write

$$\int_{B_R} U_x F\, dx = \sum_{p\geq m} I_p^W\big(g_{p,R}\big) \text{ with } g_{p,R} = \int_{B_R} f_p^x\, dx \text{ for each } p \geq m.$$

In view of Hu and Nualart's chaotic central limit theorem [11], based on the Fourth Moment Theorems of Nualart, Peccati and Tudor [23, 26], it is enough to look for conditions that guarantee the central limit theorem on each fixed chaos, provided one has some uniform control of the variance of each chaotic component. More precisely, we have the following general result.

Theorem 1.3. *Consider a sequence of centered square integrable random variables $(F_n, n \in \mathbb{N})$ with Wiener chaos expansions $F_n = \sum_{q\geq 1} I_q^W(f_{q,n})$, where $f_{q,n} \in \mathfrak{H}^{\odot q}$ for each $q, n \in \mathbb{N}$. Suppose that:*

(i) $\forall q \geq 1$, $q!\|f_{q,n}\|^2_{\mathfrak{H}^{\otimes q}} \to \sigma_q^2$, *as* $n \to +\infty$;

(ii) $\forall q \geq 2$ *and* $\forall r \in \{1,\dots,q-1\}$, $\|f_{q,n} \otimes_r f_{q,n}\|_{\mathfrak{H}^{\otimes(2q-2r)}} \to 0$, *as* $n \to +\infty$;

(iii) $\lim_{N\to+\infty}\limsup_{n\to+\infty}\sum_{q\geq N} q!\|f_{q,n}\|^2_{\mathfrak{H}^{\otimes q}} = 0$.

Then, as $n \to \infty$, F_n converges in law to $N(0,\sigma^2)$, with $\sigma^2 = \sum_{q\geq 1}\sigma_q^2$.

We refer to [20, 22] for more details on this result and to Section 2 for the definition of the r-contraction $\otimes_r$.

Now let us look at the central limit theorem on each chaos. We fix an integer $p \geq 2$ and put

$$G_{p,R} = I_p^W\big(g_{p,R}\big)$$

with $\sigma_{p,R}^2 := \mathrm{Var}\big(G_{p,R}\big)$. Assume $\sigma_{p,R} > 0$ for large R, then according to the Fourth Moment Theorem of Nualart and Peccati [23], we know that

$$\frac{G_{p,R}}{\sigma_{p,R}} \xrightarrow[R\to+\infty]{\text{law}} N(0,1)$$

if and only if

$$\lim_{R\to+\infty} \frac{1}{\sigma_{p,R}^2} \sum_{r=1}^{p-1} \|g_{p,R} \otimes_r g_{p,R}\|_{\mathfrak{H}^{\otimes(2p-2r)}} = 0\,. \tag{1.5}$$

Moreover, we have the following rate of convergence in the total variation distance, as a consequence of the Nourdin-Peccati bound (see [20, Chapter 5]):

$$d_{\mathrm{TV}}\left(\frac{G_{p,R}}{\sigma_{p,R}}, N(0,1)\right) \leq \frac{C}{\sigma_{p,R}^2} \sum_{r=1}^{p-1} \|g_{p,R} \otimes_r g_{p,R}\|_{\mathfrak{H}^{\otimes(2p-2r)}}\,. \tag{1.6}$$

Throughout this paper, we write C for immaterial constants that may vary from line to line.

In the first part of this paper (Section 2), we will exploit the above ideas to derive sufficient conditions for (1.4) to hold, *with $\sigma(R)$ growing like $CR^{d/2}$*. Note that the order of $\sigma(R)$ matches the result in Theorem 1.1. Without introducing further notation, we provide another example of (1.4), which is a corollary of our main result (Theorem 2.15); see Remark 2.16.

Theorem 1.4. *Let the above notation prevail. Assume $\gamma(0) \in (0,\infty)$ and $\gamma \in L^m(\mathbb{R}^d, dx)$, where $m \geq 1$ is the Hermite rank of F. If we assume in addition that the kernels $f_p \in L^1(\mathbb{R}^{pd}) \cap \mathfrak{H}^{\odot p}$, $p \geq m$, satisfy*

$$\sum_{p\geq m} p!\gamma(0)^p \|f_p\|_{L^1(\mathbb{R}^{pd})}^2 < +\infty\,, \tag{1.7}$$

then, $R^{-d/2}\int_{B_R} U_x F\,dx \xrightarrow[R\to+\infty]{\text{law}} N(0,\sigma^2)$, with

$$\sigma^2 = \omega_d \sum_{p\geq m} p! \int_{\mathbb{R}^{2dp}} f_p(\boldsymbol{s_p}) f_p(\boldsymbol{t_p}) \left(\int_{\mathbb{R}^d} \prod_{j=1}^{p} \gamma(t_j - s_j + z)\, dz\right) d\boldsymbol{s_p} d\boldsymbol{t_p} \in [0,\infty)$$

with $\boldsymbol{s_p} = (s_1,\dots,s_p)$, $d\boldsymbol{t_p} = dt_1\cdots dt_p$ and ω_d being the volume of $B_1 = \{\|x\| \leq 1\}$.

One may want to compare our Theorem 1.4 with Theorem 1.1 and Example 1.2. We refer the readers to Section 2 for more results with this flavor and here we briefly give a literature overview:

1. To the best of our knowledge, problem (1.4) first received attention in the 1976 paper [18] by Maruyama, using the method of moments. Proofs and extensions of Maruyama's CLT were published in his 1985 paper [19].

2. In 1983, Breuer and Major provided a CLT [2], motivated by the non-central limit theorems of Dobrushin, Major, Rosenblatt and Taqqu during 1977-1981 (see [8, 17, 27, 28, 29]). Unlike these works, Breuer and Major were interested at the asymptotic normality of nonlinear functionals over stationary Gaussian fields when the corresponding correlation function decay fast enough. Although Breuer-Major's theorem (see Theorem 1.1) takes a simpler form compared to Maruyama's CLT, it has found a tremendous amount of applications in theory and practice.
3. Chambers and Slud established further extensions to Maruyama's CLT in [4] and obtained the Breuer-Major theorem as a corollary (when assuming the existence of spectral density). In both [4] and Maruyama's work [18, 19] , the story always begins with a real stationary Gaussian process with time-shifts $\{U_s, s\in\mathbb{R}\}$ and they formulated the chaos expansion based on the spectral (probability) measure.
4. In the present work, we provide sufficient conditions for (1.4) in terms of the spectral measure. Comparing our assumptions based on the spectral measure with those in [4], both sets of assumptions essentially cover our Theorem 1.4 as a particular case, while they are different in their full generality. Moreover, we also provide sufficient conditions for (1.4) in terms of the covariance kernel.

Our methodology from the first part can be applied to the study of spatial averages of the stochastic heat equation driven by Gaussian colored noise and this constitutes the second part of our paper. More precisely, we consider the following stochastic heat equation with a multiplicative Gaussian colored noise on $\mathbb{R}_+\times\mathbb{R}^d$:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u\dot{W} \tag{1.8}$$

where the Laplacian $\Delta=\sum_{i=1}^d \partial_{x_i}^2$ concerns only spatial variables and the initial condition is fixed to be $u_{0,x}\equiv 1$.

The notation $\dot{W}$ stands for $\frac{\partial^{d+1}W}{\partial t\partial x_1\cdots\partial x_d}$ and the noise W is formally defined as a centered Gaussian family $\big\{W(\phi), \phi\in C_c^\infty(\mathbb{R}_+\times\mathbb{R}^d)\big\}$, with covariance structure

$$\begin{aligned}\mathbb{E}[W(\phi)W(\psi)] &= \int_{\mathbb{R}_+^2} dsdt\gamma_0(t-s)\langle\phi(s,\bullet),\gamma_1*\psi(t,\bullet)\rangle_{L^2(\mathbb{R}^d)}\\ &= \int_{\mathbb{R}_+^2} dsdt\gamma_0(t-s)\int_{\mathbb{R}^d}\mu_1(d\xi)\mathscr{F}\phi(s,\xi)\mathscr{F}\psi(t,-\xi)\,,\end{aligned} \tag{1.9}$$

for any $\phi,\psi\in C_c^\infty(\mathbb{R}_+\times\mathbb{R}^d)$, where $\mathscr{F}$ denotes the Fourier transform with respect to the spatial variables and the following two conditions are satisfied:

1. $\gamma_0:\mathbb{R}\to[0,\infty]$ is *locally integrable* and nonnegative-definite,
2. γ_1 is a measure, such that $\gamma_1=\mathscr{F}\mu_1$ for some nonnegative tempered measure μ_1, called the spectral measure, satisfying Dalang's condition (see *e.g.* [6])

$$\int_{\mathbb{R}^d}\frac{\mu_1(d\xi)}{1+\|\xi\|^2}<+\infty. \tag{1.10}$$

If γ_1 is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$, we still denote by γ_1 its density and then

$$\langle \phi(s,\bullet), \gamma_1 * \psi(t,\bullet)\rangle_{L^2(\mathbb{R}^d)} = \int_{R^{2d}} \phi(s,x)\gamma_1(x-y)\psi(t,y)dxdy.$$

We will use this notation even if γ_1 is a measure. The basic example is $d=1$ and $\gamma_1 = \delta_0$ and in this case μ_1 is $(2\pi)^{-1}$ times Lebesgue measure.

We point out that (1.9) defines an inner product, under which $C_c^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$ can be extended into a Hilbert space $\mathscr{H}$. As we did before, we can build an isonormal process $\{W(h), h \in \mathscr{H}\}$ from $\{W(h), h \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R}^d)\}$. We denote by $I_p^W(f)$ the pth multiple integral of a symmetric element $f \in \mathscr{H}^{\odot p}$. For general $f \in \mathscr{H}^{\otimes p}$, we denote by $\widetilde{f}$ the canonical symmetrization of f, that is,

$$\widetilde{f}(s_1,y_1,s_2,y_2,\dots,s_p,y_p) = \frac{1}{p!}\sum_{\sigma\in\mathfrak{S}_p} f\big(s_{\sigma(1)},y_{\sigma(1)},\dots,s_{\sigma(p)},y_{\sigma(p)}\big),$$

where the sum runs over the permutation group $\mathfrak{S}_p$ over $\{1,\dots,p\}$. Quite often in this paper, we write $f(\boldsymbol{s_p},\boldsymbol{y_p})$ for $f(s_1,y_1,\dots,s_p,y_p)$, whenever it is convenient.

For each $t \ge 0$, let $\mathcal{F}_t$ be the σ-algebra generated by $\big\{W(\phi) : \phi$ is continuous with support contained in $[0,t]\times\mathbb{R}^d\big\}$. We say that a random field $u = \{u_{t,x}, (t,x) \in \mathbb{R}_+\times\mathbb{R}^d\}$ is adapted if for each (t,x), the random variable $u_{t,x}$ is $\mathcal{F}_t$-measurable.

We interpret equation (1.8) in the Skorokhod sense and recall the definition of mild solution from [9, Definition 3.1].

Definition 1.5. *An adapted random field* $u = \big\{u_{t,x}, t\ge 0, x \in \mathbb{R}^d\big\}$ *such that* $\mathbb{E}\big[u_{t,x}^2\big] < +\infty$ *for all* (t,x) *is said to be a mild solution to equation* (1.8) *with initial conditoin* $u_{0,\cdot} = 1$*, if for any* $t \in \mathbb{R}_+$, $x\in\mathbb{R}^d$*, the process* $\{G(t-s,x-y)u_{s,y}\mathbf{1}_{[0,t]}(s) : s\ge 0, y \in \mathbb{R}^d\}$ *is Skorokhod integrable and*

$$u_{t,x} = 1 + \int_0^t\int_{\mathbb{R}^d} G(t-s,x-y)u_{s,y}W(ds,dy)\,,$$

where $G(t,x) = (2\pi t)^{-d/2}\exp\big(-\|x\|^2/(2t)\big)$ *for* $t>0$ *and* $x \in \mathbb{R}^d$*.*

The above stochastic heat equation has a unique mild solution u with explicit Wiener chaos expansion given by (see [9, Theorem 3.2])

$$u_{t,x} = 1 + \sum_{n\ge 1} I_n^W(f_{t,x,n}),$$

where

$$f_{t,x,n}(\boldsymbol{s_n},\boldsymbol{y_n}) = \frac{1}{n!}\prod_{i=0}^{n-1} G(s_{\sigma(i)} - s_{\sigma(i+1)}, y_{\sigma(i)} - y_{\sigma(i+1)}), \tag{1.11}$$

with $\sigma \in \mathfrak{S}_n$ being such that $t > s_{\sigma(1)} > \cdots > s_{\sigma(n)} > 0$. In the above expression we have used the convention $s_{\sigma(0)} = t$ and $y_{\sigma(0)} = x$. We also refer interested readers to [10, 13] for more general noises.

Notice that $u_{t,x} - \mathbb{E}[u_{t,x}]$ has Hermite rank 1 and it is known that for any fixed $t \in \mathbb{R}_+$, $\{u_{t,x} : x \in \mathbb{R}^d\}$ is strictly stationary meaning that the finite-dimensional distributions of the process $\{u_{t,x+y}, x \in \mathbb{R}^d\}$ do not depend on y. So the following integral

$$\int_{B_R} \big(u_{t,x} - 1\big)\, dx \tag{1.12}$$

resembles the object in (1.4) and we are able to establish its Gaussian fluctuation under some mild assumptions. The spatial averages (1.12) have been studied in recent articles [14, 15, 7]:

(i) Huang, Nualart and Viitasaari [14] initiated their study by looking at the one-dimensional (nonlinear) stochastic heat equation driven by a space-time white noise.

(ii) Huang, Nualart, Viitasaari and Zheng [15] continued to study the d-dimensional stochastic heat equation driven by Gaussian noise that is white in time and colored in space, with the spatial covariance described by the Riesz kernel.

(iii) Delgado-Vences, Nualart and Zheng [7] carried out similar investigation for the one-dimensional stochastic wave equation.

In the above references, the Gaussian noise is assumed to be white in time, which gives rise to a martingale structure. This is important for applying Itô calculus (*e.g.* Burkholder-Davis-Gundy inequality and Clark-Ocone formula) to obtain quantitative central limit theorems for (1.12).

In the present paper, we consider a linear stochastic heat equation driven by space-time colored noise, so Itô calculus can not be applied anymore; while due to the linearity, an explicit chaos expansion of the solution is available for us to apply the chaotic central limit theorem (Theorem 1.3).

We define

$$A_t(R) := \int_{B_R} \big(u_{t,x} - 1\big)\, dx$$

and let $\Pi_p A_t(R)$ be the projection of $A_t(R)$ on the pth Wiener chaos, that is,

$$\Pi_p A_t(R) := I_p^W \left(\int_{B_R} f_{t,x,p} dx \right).$$

Throughout this paper, we assume that γ_0, γ_1 are nontrivial, meaning that

$$\gamma_1(\mathbb{R}^d) > 0 \quad \text{and} \quad \int_0^t \int_0^t \gamma_0(r-v) dr dv > 0$$

for any $t > 0$. The following is our main result.

Theorem 1.6. *Suppose $\gamma_0 : \mathbb{R} \to \mathbb{R}_+ \cup \{+\infty\}$ is locally integrable, γ_1 satisfies Dalang's condition* (1.10) *and $\gamma_1(\mathbb{R}^d) < \infty$. Then as $R \to +\infty$, $\{R^{-d/2} A_t(R),\, t \geq 0\}$ converges to a centered continuous Gaussian process $\{\mathcal{G}_t, t \geq 0\}$ in finite-dimensional distributions. The covariance structure of $\mathcal{G}$ is given by*

$$\mathbb{E}[\mathcal{G}_s \mathcal{G}_t] =: \Sigma_{s,t} = \omega_d \int_{\mathbb{R}^d} \left(\mathbb{E}\left[e^{\beta_{s,t}(z)} \right] - 1 \right) dz \in (0, \infty), \tag{1.13}$$

where

$$\beta_{s,t}(z) := \int_0^s \int_0^t \gamma_0(r-v)\gamma_1(X_r^1 - X_v^2 + z)drdv$$

with X^1, X^2 *two independent standard Brownian motions on* $\mathbb{R}^d$.

If in addition, there exist some $t_0 > 0$ *and some* $\alpha \in (0, 1/2)$ *such that*

$$\int_0^{t_0} \int_0^{t_0} \gamma_0(r-v) r^{-\alpha} v^{-\alpha} drdv < +\infty, \tag{1.14}$$

then as $R \to +\infty$, $\{R^{-d/2} A_t(R),\, t \geq 0\}$ *converges weakly to* $\{\mathcal{G}_t, t \geq 0\}$ *in the space of continuous functions* $C(\mathbb{R}_+)$.

Notice that (1.14) is satisfied when $\gamma_0 = \delta_0$. In this case γ_0 is not a function but the result can be properly formulated.

One may ask what happens if $\gamma_1(\mathbb{R}^d)$ is not finite, and this includes an important example, the Riesz kernel $\gamma_1(z) = \|z\|^{-\beta}$ with $\beta \in (0, 2 \wedge d)$.

Theorem 1.7. *Suppose* $\gamma_0 : \mathbb{R} \to \mathbb{R}_+ \cup \{\infty\}$ *is locally integrable and* $\gamma_1(\mathbb{R}^d) = +\infty$.

(1) *Assume that* μ_1 *admits a density* φ_1 *that satisfies*

$$\int_{\mathbb{R}^d} \frac{\varphi_1(\xi) + \varphi_1(\xi)^2}{1 + \|\xi\|^2} d\xi < +\infty. \tag{1.15}$$

Then, $R^{-d}\mathrm{Var}\big(\Pi_1 A_t(R)\big)$ *diverges to infinity as* $R \to +\infty$ *and*

$$\lim_{R\to+\infty} R^{-d} \sum_{p\geq 2} \mathrm{Var}\big(\Pi_p A_t(R)\big) = \omega_d \int_{\mathbb{R}^d} \mathbb{E}\big(e^{\beta_{t,t}(z)} - \beta_{t,t}(z) - 1\big)dz \in (0, \infty).$$

As a consequence, we have

$$\frac{A_t(R)}{\sqrt{\mathrm{Var}\big(A_t(R)\big)}} \xrightarrow[R\to+\infty]{\text{law}} N(0,1).$$

(2) *When* $\gamma_1(z) = \|z\|^{-\beta}$ *for some* $\beta \in (0, 2 \wedge d)$, *we have*

$$\frac{A_t(R)}{R^{d-\frac{\beta}{2}}} \xrightarrow[R\to+\infty]{\text{law}} N(0, \kappa_\beta), \tag{1.16}$$

with

$$\kappa_\beta := \left(\int_0^t \int_0^t drdv\gamma_0(r-v)\right) \int_{B_1^2} dxdy\|x-y\|^{-\beta}.$$

Note that the Riesz kernel in part (2) satisfies the modified version of Dalang's condition (1.15) if and only if $d/2 < \beta < 2 \wedge d$, which is equivalent to

$$\begin{cases} \beta \in (1/2, 1) & \text{for } d = 1 \\ \beta \in (1, 2) & \text{for } d = 2 \\ \beta \in (3/2, 2) & \text{for } d = 3. \end{cases} \tag{1.17}$$

In particular, in dimension one, $\beta \in (1/2, 1)$ is equivalent to the fractional noise with Hurst parameter $H \in (1/2, 3/4)$.

Remark 1.8. Unlike previous studies, we consider a noise that is colored in time, and our results complement, in particular, those in [14, 15]. In [14] where the noise is white in space and time, the authors were able to obtain the chaotic central limit theorem for the linear equation (parabolic Anderson model), proving also a rate of convergence in the total variation distance. The quantitative CLT in the case $\gamma_0 = \delta_0$ and $\gamma_1(z) = \|z\|^{-\beta}$, was obtained in [15] for the nonlinear equation, and the authors of [15] also proved that for the linear equation, the first chaos is dominant so the central limit theorem is not chaotic.

We point out that in both parts of Theorem 1.7 the first chaos dominates, that is, the central limit theorem is not chaotic. Moreover, we are able to provide the following functional version of Theorem 1.7.

Theorem 1.9. *Suppose $\gamma_0 : \mathbb{R} \to \mathbb{R}_+ \cup \{\infty\}$ is locally integrable and $\gamma_1(\mathbb{R}^d) = +\infty$.*

(1) *Let the assumptions in part* (1) *of Theorem 1.7 hold and we assume that the condition* (1.14) *is satisfied. We put*

$$\widehat{A}_t(R) := \sum_{p\geq 2} \Pi_p\big(A_t(R)\big),$$

then as $R \to \infty$, the process $\big(R^{-d/2}\widehat{A}_t(R) : t \in \mathbb{R}_+\big)$ converges in law to a centered continuous Gaussian process $\widehat{\mathcal{G}}$ with covariance given by

$$\mathbb{E}\big[\widehat{\mathcal{G}}_s\widehat{\mathcal{G}}_t\big] := \omega_d \int_{\mathbb{R}^d} \mathbb{E}\left[e^{\beta_{s,t}(z)} - \beta_{s,t}(z) - 1\right] dz.$$

(2) *If condition* (1.14) *is satisfied for some $\alpha \in (0, 1/2)$ and $\gamma_1(z) = \|z\|^{-\beta}$ for some $\beta \in (0, 2 \wedge d)$, then the process $\big(R^{-d+\frac{\beta}{2}} A_t(R) : t \in \mathbb{R}_+\big)$ converges in law to a centered continuous Gaussian process $\widetilde{\mathcal{G}}$, as $R \to \infty$. Here the covariance structure of $\widetilde{\mathcal{G}}$ is given by*

$$\mathbb{E}\big[\widetilde{\mathcal{G}}_s\widetilde{\mathcal{G}}_t\big] = \left(\int_0^t \int_0^s drdv\gamma_0(r-v)\right) \int_{B_1^2} dxdy\|x-y\|^{-\beta}.$$

We will organize the rest of our article into three sections. Section 2 begins with a subsection on some preliminary knowledge, where we provide some important lemmas for our later analysis. We devote Section 2.2 to the investigation of the central limit theorems on a fixed chaos by looking at assumptions on the covariance kernel and on the spectral measure separately. We derive the corresponding chaotic central limit theorems in Section 2.3. Section 3 is devoted to the proof of Theorems 1.6, 1.7 and 1.9. For Theorem 1.6. we show the convergence of the finite-dimensional distributions and the tightness. Theorem 1.7 and Theorem 1.9 are proved as a by-product of the estimations in the proof of Theorem 1.6. Finally, Section 4 provides the proofs of some technical results stated in previous sections.

2 Infinite version of the Breuer-Major theorem

2.1 Preliminaries

In this section, we introduce some notation for later reference and we provide several lemmas needed for our proofs.

Recall from our introduction that $\{W(h), h \in \mathfrak{H}\}$ is an isonormal Gaussian process such that for any $\phi, \psi \in \mathfrak{H}$,

$$\mathbb{E}\big[W(\phi)W(\psi)\big] = \langle \phi, \psi\rangle_{\mathfrak{H}} = \int_{\mathbb{R}^{2d}} \phi(x)\psi(y)\gamma(x-y)dxdy = \int_{\mathbb{R}^d} \mathscr{F}\phi(\xi)\mathscr{F}\psi(-\xi)\mu(d\xi),$$

where γ is the covariance kernel and μ is the spectral measure whose Fourier transform is γ, understood in the generalized sense. Let $\mathfrak{H}_\mu$ be the Hilbert space of functions $g : \mathbb{R}^d \to \mathbb{C}$ such that $g(-x) = \overline{g}(x)$ for μ-almost every $x \in \mathbb{R}^d$ and

$$\int_{\mathbb{R}^d} |g(\xi)|^2 \, \mu(d\xi) < +\infty\,.$$

Here $\overline{z}$ is the complex conjugate of $z \in \mathbb{C}$. It is clear that the Fourier transform stands as a linear isometry from $\mathfrak{H}$ to $\mathfrak{H}_\mu$.

For any integer $p \geq 2$, let $\mathfrak{H}^{\otimes p}$ (resp. $\mathfrak{H}^{\odot p}$) the pth tensor product (resp. symmetric tensor product) of $\mathfrak{H}$. Note that for any integer $p \geq 2$, the pth multiple stochastic integral I_p^W is a linear and continuous operator from $\mathfrak{H}^{\otimes p}$ into $L^2(\Omega)$. We can define spaces like $\mathfrak{H}_\mu^{\otimes p}$ and $\mathfrak{H}_\mu^{\odot p}$ in the obvious manner.

To simplify the display, we introduce some compact notation below.

Notation A: For any $R > 0$, $B_R(x)$ stands for the d-dimensional Euclidean (closed) ball centered at x with radius R and we have used B_R for $B_R(0)$. We write vol(A) for the volume of $A \subset \mathbb{R}^d$ and $\omega_d =$ vol(B_1). We use $\|\cdot\|$ to denote the Euclidean norm in any dimension.

For $r \in \mathbb{N}$ and $\boldsymbol{x_r} = (x_1, \dots, x_r)$, we write $-\boldsymbol{x_r}$ for $(-x_1, \dots, -x_r)$, $d\boldsymbol{x_r} = dx_1 \cdots dx_r$ and $\mu(d\boldsymbol{x_r}) = \mu(dx_1)\cdots\mu(dx_r)$; we also write $\tau(\boldsymbol{x_r}) = x_1 + \cdots + x_r$. For integers $1 \leq r < p$, we write $(\xi_1, \dots, \xi_p) = \boldsymbol{\xi_p} = (\boldsymbol{\xi_r}, \boldsymbol{\eta_{p-r}})$ with $\boldsymbol{\xi_r} = (\xi_1, \dots, \xi_r)$ and $\boldsymbol{\eta_{p-r}} = (\xi_{r+1}, \dots, \xi_p)$. With the above compact notation, we define the contraction operators $\otimes_r$ as follows. For $f \in \mathfrak{H}^{\otimes p}$ and $g \in \mathfrak{H}^{\otimes q}$ $(p, q \in \mathbb{N})$, their r-contraction, with $0 \leq r \leq p \wedge q$, belongs to $\mathfrak{H}^{\otimes p+q-2r}$ and is defined by

$$(f \otimes_r g)\big(\boldsymbol{\xi_{p-r}}, \boldsymbol{\eta_{q-r}}\big) := \int_{\mathbb{R}^{2rd}} f\big(\boldsymbol{\xi_{p-r}}, \boldsymbol{a_r}\big) g\big(\boldsymbol{\eta_{q-r}}, \widetilde{\boldsymbol{a}}_{\boldsymbol{r}}\big) \prod_{j=1}^{r} \gamma(a_j - \widetilde{a}_j) d\boldsymbol{a_r} d\widetilde{\boldsymbol{a}}_{\boldsymbol{r}}$$

for $\boldsymbol{\xi_{p-r}} \in \mathbb{R}^{pd-rd}$ and $\boldsymbol{\eta_{q-r}} \in \mathbb{R}^{qd-rd}$. In particular, $f \otimes_0 g = f \otimes g$ is the usual tensor product and if $p = q$, $f \otimes_p g = \langle f, g\rangle_{\mathfrak{H}^{\otimes p}}$; see also [20, Appendix B]. Let us introduce some useful lemmas now.

For p positive, we denote by J_p the Bessel function of first kind with order p:

$$J_p(x) = \frac{(x/2)^p}{\sqrt{\pi}\Gamma(p+\frac{1}{2})} \int_0^\pi (\sin\theta)^{2p} \cos\big(x\cos\theta\big)\, d\theta, \quad x \in \mathbb{R}\,; \tag{2.1}$$

see [16, (5.10.4)]. Let us also record here

$$\omega_d = \mathrm{vol}(B_1) = \frac{\pi^{d/2}}{\Gamma\big(1+\frac{d}{2}\big)}, \tag{2.2}$$

with Γ the Euler's Gamma function.

Lemma 2.1. (1) *Given $\xi \in \mathbb{R}^d$ and $R > 0$, we have*

$$\int_{B_R} e^{-\mathbf{i}\xi\cdot u}\, du = (2\pi R)^{d/2}\|\xi\|^{-d/2} J_{d/2}\big(R\|\xi\|\big)\,,$$

where $J_{d/2}$ is the Bessel function of the first kind with order $d/2$.

(2) *Given a positive real number p, we have*

$$J_p(x) \sim \sqrt{2/(\pi x)}\cos\Big(x - \frac{(2p+1)\pi}{4}\Big) \quad \text{as } x \to +\infty, \tag{2.3}$$

$$J_p(x) \sim \frac{x^p}{2^p\Gamma(p+1)} \quad \text{as } x \to 0. \tag{2.4}$$

As a consequence, we have $\sup\{|J_p(x)| : x \in \mathbb{R}_+\} < +\infty$ and $|J_p(x)| \le C|x|^{-1/2}$ for any $x \in \mathbb{R}$, here C is some absolute constant.

(3) *Put $\ell_R(x) = \omega_d^{-1}\|x\|^{-d} J_{d/2}(R\|x\|)^2$, then $\{\ell_R : R > 0\}$ is an approximation of the identity.*

Proof. (1) Let us suppose first that $R = 1$. In this case, one sees that the Fourier transform of $\mathbf{1}_{\{\|u\|\le 1\}}$ is rotationally symmetric, so without losing any generality, we assume $\xi = (0,\dots,0,\rho)$ with $\rho = \|\xi\| > 0$. Then for $d \ge 2$,

$$\begin{aligned}
&\int_{\mathbb{R}^d} e^{-\mathbf{i}\xi\cdot u}\mathbf{1}_{\{\|u\|\le 1\}}\, du \\
&= \int_{-1}^{1} e^{-\mathbf{i}\rho x_d}\int_{\mathbb{R}^{d-1}} \mathbf{1}_{\{\|\boldsymbol{x_{d-1}}\|^2 \le 1 - x_d^2\}} d\boldsymbol{x_{d-1}}\, dx_d = \int_{-1}^{1} e^{-\mathbf{i}\rho x_d}\omega_{d-1}\big(1-x_d^2\big)^{\frac{d-1}{2}}\, dx_d \\
&= \omega_{d-1}\int_{-1}^{1}\cos(\rho y)\big(1-y^2\big)^{\frac{d-1}{2}}\, dy = \omega_{d-1}\int_0^{\pi}\cos(\rho\cos(\theta))\sin(\theta)^d\, d\theta \\
&= (2\pi)^{d/2}\rho^{-d/2}J_{d/2}(\rho),
\end{aligned}$$

where the last equality follows from the expressions (2.2) and (2.1). That is, for $d \ge 2$,

$$\int_{\mathbb{R}^d} e^{-\mathbf{i}\xi\cdot u}\mathbf{1}_{\{\|u\|\le 1\}}\, du = (2\pi)^{d/2}\|\xi\|^{-d/2}J_{d/2}(\|\xi\|).$$

The above equality also holds true for $d = 1$, as one can verify by a direct computation for both sides. So the result in part (1) is established for $R = 1$. The general case follows from a change of variable.

(2) The asymptotic behavior of Bessel functions can be found in *e.g.* page 134 of the book [16]. The uniform boundedness of J_p on $\mathbb{R}_+$ follows immediately from this

asymptotic behavior. By (2.3), we can find some $L > 0$ such that $|J_p(x)| \le 1/\sqrt{x}$ for any $x \ge L$, while it follows from (2.1) that $|J_p(x)| \le C_1 x^p$ for any $x \ge 0$. It suffices to pick $C = 1 + C_1 L^{p+\frac{1}{2}}$ such that $C_1 \le C L^{-p-\frac{1}{2}}$ to conclude that $|J_p(x)| \le C|x|^{-1/2}$ for any $x \in \mathbb{R}$.

(3) It suffices to show $1 = \|\ell_1\|_{L^1(\mathbb{R}^d)}$. It follows from point (1) that

$$\begin{aligned}
&\int_{\mathbb{R}^d} \|x\|^{-d} J_{d/2}(\|x\|)^2 dx = \int_{\mathbb{R}^d} \left(\lim_{a\downarrow 0} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp\left(-\mathbf{i}\xi \cdot x - \frac{a}{4}\|x\|^2 \right) \mathbf{1}_{\{\|\xi\|\le 1\}} d\xi \right)^2 dx \\
&= \lim_{a\downarrow 0} \int_{\mathbb{R}^{2d}} d\xi d\xi' \mathbf{1}_{\{\xi,\xi'\in B_1\}} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \exp\left(-\mathbf{i}(\xi+\xi')\cdot x - \frac{a}{2}\|x\|^2 \right) dx \\
&= \lim_{a\downarrow 0} \int_{\mathbb{R}^{2d}} d\xi d\xi' \mathbf{1}_{\{\xi,\xi'\in B_1\}} \frac{\exp\big(-\|\xi+\xi'\|^2/(2a) \big)}{(2\pi a)^{d/2}} \\
&= \lim_{a\downarrow 0} \int_{\mathbb{R}^d} \mathrm{vol}\big(B_1 \cap B_1(\xi)\big) \frac{e^{-\|\xi\|^2/(2a)}}{(2\pi a)^{d/2}} = \omega_d \,,
\end{aligned}$$

where interchanges of integrals and limits are valid due to the dominated convergence theorem. Our proof of this lemma is finished. ☐

The following lemma has its discrete analogue in [20, (7.2.7)] and for the sake of completeness, we provide a short proof; see also [25, (3.3)].

Lemma 2.2. *If $\phi : \mathbb{R}^d \to \mathbb{R}$ belongs to $L^p(\mathbb{R}^d, dx)$ for some positive number p. Then for any $r \in (0,p)$, one has*

$$\frac{1}{R^{d(1-rp^{-1})}} \int_{B_R} |\phi(x)|^r dx \xrightarrow{R\to+\infty} 0\,.$$

Proof. Fix $\delta \in (0,1)$. We deduce from Hölder's inequality that

$$\begin{aligned}
&\frac{1}{R^{d(1-rp^{-1})}} \int_{B_R} |\phi(x)|^r dx = \frac{1}{R^{d(1-rp^{-1})}} \int_{B_{\delta R}} |\phi(x)|^r dx + \frac{1}{R^{d(1-rp^{-1})}} \int_{B_R\setminus B_{\delta R}} |\phi(x)|^r dx \\
&\le C\delta^{d(1-rp^{-1})} \left(\int_{\mathbb{R}^d} |\phi(x)|^p dx \right)^{r/p} + C\Big(1-\delta^{d(1-rp^{-1})}\Big) \left(\int_{B_R\setminus B_{\delta R}} |\phi(x)|^p dx \right)^{r/p} .
\end{aligned}$$

Note that for any fixed $\delta \in (0,1)$, the second term goes to zero, as $R \to +\infty$, while the first term can be made arbitrarily small by choosing sufficiently small δ. ☐

At the end of this section, we record a consequence of Young's inequality.

Lemma 2.3. *Suppose $\varphi : \mathbb{R}^d \to \mathbb{R}$ belongs to $L^q(\mathbb{R}^d, dx)$ with $q = p/(p-1)$ for some integer $p \ge 2$. Then,*

$$\big\|\varphi^{*p}\big\|_\infty \le \|\varphi\|^p_{L^q(\mathbb{R}^d)}\,, \tag{2.5}$$

where the p-convolution can be defined iteratively: $\varphi^{*2} = \varphi * \varphi$, ..., $\varphi^{*p} = \varphi * \varphi^{*p-1}$.

Proof. Young's convolution inequality states that

$$\|h_1 * h_2\|_{L^r(\mathbb{R}^d)} \le \|h_1\|_{L^p(\mathbb{R}^d)}\|h_2\|_{L^q(\mathbb{R}^d)}$$

for any $h_1 \in L^p(\mathbb{R}^d)$ and $h_2 \in L^q(\mathbb{R}^d)$ with $p^{-1}+q^{-1} = 1 + r^{-1}$ and $1 \le p,q,r \le \infty$. As a consequence, we obtain the following inequalities:

$$\begin{cases} \|\varphi^{*p}\|_\infty = \|\varphi * \varphi^{*p-1}\|_\infty \le \|\varphi\|_{L^q(\mathbb{R}^d)}\|\varphi^{*p-1}\|_{L^{q_1}(\mathbb{R}^d)} & \text{with } q_1 = p, \\ \|\varphi^{*p-1}\|_{L^{q_1}(\mathbb{R}^d)} = \|\varphi * \varphi^{*p-2}\|_{L^{q_1}(\mathbb{R}^d)} \le \|\varphi\|_{L^q(\mathbb{R}^d)}\|\varphi^{*p-2}\|_{L^{q_2}(\mathbb{R}^d)} & \text{with } q_2 = p/2, \\ \|\varphi^{*p-2}\|_{L^{q_2}(\mathbb{R}^d)} = \|\varphi * \varphi^{*p-3}\|_{L^{q_2}(\mathbb{R}^d)} \le \|\varphi\|_{L^q(\mathbb{R}^d)}\|\varphi^{*p-3}\|_{L^{q_3}(\mathbb{R}^d)} & \text{with } q_3 = p/3, \\ \quad \dots & \\ \|\varphi^{*2}\|_{L^{q_{p-2}}(\mathbb{R}^d)} = \|\varphi * \varphi\|_{L^{q_{p-2}}(\mathbb{R}^d)} \le \|\varphi\|_{L^q(\mathbb{R}^d)}\|\varphi\|_{L^{q_{p-1}}(\mathbb{R}^d)} & \text{with } q_{p-1} = \frac{p}{p-1}. \end{cases}$$

This completes the proof of (2.5). □

Recall from our introduction that we consider the case where $F = \sum_{k\ge m} I_k^W(f_k)$ has Hermite rank $m \ge 1$ with $f_k \in \mathfrak{H}^{\odot k}$ for each $k \ge m$. We write

$$G_R := \int_{B_R} U_x F\, dx = \sum_{k\ge m} I_k^W(g_{k,R}) =: \sum_{k\ge m} G_{k,R} \quad \text{with} \quad g_{k,R} = \int_{B_R} f_k^x\, dx\,.$$

In what follows, we first investigate the central limit theorem on each chaos based on two sets of assumptions. One involves the covariance kernel γ and the other is based on the spectral measure μ. This is the content of Section 2.2, and in Section 2.3, we consider the case where F has a general chaos expansion. In each situation, the random variable may depend on infinitely many coordinates, which shall be distinguished from the classical Breuer-Major theorem.

2.2 Central limit theorems on a fixed chaos

Fix an integer $p \ge 2$ and note that the random field $\{I_p^W(f_p^x), x \in \mathbb{R}^d\}$ is centered, strictly stationary. We put

$$\mathbb{E}[I_p^W(f_p^x)I_p^W(f_p^y)] =: \Phi_p(x-y).$$

Then, if

$$\int_{\mathbb{R}^d} |\Phi_p(x)|dx < \infty, \tag{2.6}$$

we have, with the notation $G_{p,R} = I_p^W(g_{p,R})$,

$$\lim_{R\to+\infty} \frac{\mathrm{Var}(G_{p,R})}{R^d} = \omega_d \int_{\mathbb{R}^d} \Phi_p(x)dx. \tag{2.7}$$

Indeed,

$$\mathrm{Var}(G_{p,R}) = \int_{B_R^2} \Phi_p(x-y)dxdy = \int_{B_R} \mathrm{vol}\big(B_R \cap B_R(-z)\big)\Phi_p(z)dz.$$

Because $\mathrm{vol}\big(B_R \cap B_R(-z)\big)/\mathrm{vol}(B_R)$ is bounded by one and convergent to one, as $R \to +\infty$, (2.7) follows from (2.6) and the dominated convergence theorem. This fact leads us to stick on the situation that the normalization $\sigma(R)$ in (1.4) is of order $R^{d/2}$, as $R \to +\infty$. Such an order is also consistent with the Breuer-Major theorem (see Theorem 1.1).

2.2.1 CLT under assumptions on the covariance kernel

We write

$$\Phi_p(x) = p!\langle f_p^x, f_p\rangle_{\mathfrak{H}^{\otimes p}} = p! \int_{\mathbb{R}^{2pd}} f_p(\boldsymbol{\xi_p}) f_p(\boldsymbol{\eta_p}) \prod_{i=1}^p \gamma\big(\xi_i - \eta_i + x\big) d\boldsymbol{\xi_p}\, d\boldsymbol{\eta_p}\,.$$

Therefore, a sufficient condition for (2.6) to hold is the following hypothesis:

(H1) $f_p \in \mathfrak{H}^{\odot p}$ satisfies $\displaystyle\int_{\mathbb{R}^d}\int_{\mathbb{R}^{2pd}} |f_p(\boldsymbol{\xi_p}) f_p(\boldsymbol{\eta_p})| \prod_{i=1}^p |\gamma|\big(\xi_i - \eta_i + x\big) d\boldsymbol{\eta_p}\, d\boldsymbol{\xi_p} dx < \infty.$

Define

$$\kappa_p(\boldsymbol{\xi_p} - \boldsymbol{\eta_p}) = \int_{\mathbb{R}^d} \prod_{i=1}^p \gamma(\xi_i - \eta_i + z) dz. \tag{2.8}$$

Then, under **(H1)**,

$$\int_{\mathbb{R}^d} \Phi_p(x) dx = p! \int_{\mathbb{R}^{2pd}} f_p(\boldsymbol{\xi_p}) f_p(\boldsymbol{\eta_p}) \kappa_p(\boldsymbol{\xi_p} - \boldsymbol{\eta_p})\, d\boldsymbol{\xi_p}\, d\boldsymbol{\eta_p}.$$

Suppose that $\gamma \in L^p(\mathbb{R}^d)$ and $f_p \in L^1(\mathbb{R}^{pd})$. Then, hypothesis **(H1)** is satisfied. In fact, using Hölder's inequality, we obtain

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^{2pd}} |f_p(\boldsymbol{\xi_p}) f_p(\boldsymbol{\eta_p})| \prod_{i=1}^p |\gamma|(\xi_i - \eta_i + x) d\boldsymbol{\xi_p}\, d\boldsymbol{\eta_p} dx \le \|\gamma\|^p_{L^p(\mathbb{R}^d)} \|f_p\|^2_{L^1(\mathbb{R}^{pd})} < \infty\,.$$

Remark 2.4. (i) In the particular case where $p = 1$, the conditions $f_1 \in L^1(\mathbb{R}^d) \cap \mathfrak{H}$ and $\gamma \in L^1(\mathbb{R}^d)$ are necessary, since hypothesis **(H1)** becomes

$$\int_{\mathbb{R}^{2d}} \big|f_1(t) f_1(s)\big| \int_{\mathbb{R}^d} |\gamma|(t - s + z) dz dt ds = \|f_1\|^2_{L^1(\mathbb{R}^d)} \|\gamma\|_{L^1(\mathbb{R}^d)} < \infty\,.$$

Under these necessary conditions, it is clear that

$$\int_{B_R} I_1^W(f_1^x)\, dx$$

is a centered Gaussian random variable with

$$\mathrm{Var}\left(\int_{B_R} I_1^W(f_1^x)\, dx\right) \sim \omega_d R^d \|f_1\|^2_{L^1(\mathbb{R}^d)} \int_{\mathbb{R}^d} \gamma(z) dz, \text{ as } R \to +\infty.$$

(ii) Here is an example of non-integrable covariance kernel: $\gamma(x) = \|x\|^{-\beta}$, with $\beta \in (0, d)$. Now let us search for sufficient condition for κ_p to be well defined. Notice that

$$\int_{\mathbb{R}^d} \prod_{i=1}^{p} \gamma(a_i + z)\, dz = \int_{\mathbb{R}^d} \prod_{i=1}^{p} \|a_i + z\|^{-\beta}\, dz$$

and for $a_1, \ldots, a_p$ mutually distinct, the product $\prod_{i=1}^{p} \|a_i + z\|^{-\beta}$ is integrable near the singularities. Indeed, choosing $\varepsilon = \frac{1}{2}\min\{|a_i - a_k| \, : 1 \le i < k \le p\}$, we can write for each $j = 1, \ldots, p$,

$$\int_{B_\varepsilon(a_j)} \prod_{i=1}^{p} \|a_i + z\|^{-\beta}\, dz \le C \int_{B_\varepsilon(a_j)} \|a_j + z\|^{-\beta}\, dz = C \int_{B_\varepsilon} \|z\|^{-\beta}\, dz = C \int_0^{\varepsilon} r^{-\beta} r^{d-1}\, dr,$$

which is finite. Thus, we only need to control the integral at infinity. Notice that for $L > 0$ large (that may depend on the a_i's), there exist two constants C_1, C_2 such that

$$C_1 \int_{\|z\| \ge L} \|z\|^{-\beta p}\, dz \le \int_{\|z\| \ge L} \prod_{i=1}^{p} \|a_i + z\|^{-\beta}\, dz \le C_2 \int_{\|z\| \ge L} \|z\|^{-\beta p}\, dz.$$

Then the finiteness of the integral at infinity is equivalent to $p > d/\beta$. In other words, the function κ_p, given in (2.8), makes sense only for $p > d/\beta$. This forces us to consider chaoses of order at least $\lfloor d/\beta \rfloor + 1 =: m_0$. Now for $p \ge m_0$, the kernel $f_p \in \mathfrak{H}^{\odot p}$ satisfies **(H1)** if

$$\int_{\mathbb{R}^{2pd}} \big|f_p(\boldsymbol{x_p}) f_p(\boldsymbol{y_p})\big| \int_{\mathbb{R}^d} \prod_{i=1}^{p} \|x_i - y_i + z\|^{-\beta}\, dz d\boldsymbol{x_p} d\boldsymbol{y_p} < \infty.$$

The following result is a central limit theorem under some restrictions on γ.

Theorem 2.5. *Fix an integer $p \ge 2$, $f_p \in \mathfrak{H}^{\odot p}$ and assume that the hypothesis* **(H1)** *holds. Moreover, suppose that one of the following two conditions hold true:*

(i) The kernel f_p has the form[2] $f_p = \mathsf{sym}\big(h_1 \otimes \cdots \otimes h_p\big)$, where the $h_j \in \mathfrak{H}$ satisfy

$$\sum_{i,j=1}^{p} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^{2d}} h_i(s) h_j(t) \gamma(s - t + z) ds dt \right|^p dz < \infty\,. \tag{2.9}$$

(ii) $\gamma \in L^p(\mathbb{R}^d)$ and $f_p \in L^1(\mathbb{R}^{pd})$. (Note that (ii) *implies* **(H1)***.)*

Then

$$\frac{G_{p,R}}{R^{d/2}} \xrightarrow[R \to +\infty]{\text{law}} N(0, \sigma_p^2),$$

where

$$\sigma_p^2 = p! \omega_d \int_{\mathbb{R}^{2pd}} f_p(\boldsymbol{s_p}) f_p(\boldsymbol{t_p}) \kappa_p(\boldsymbol{t_p} - \boldsymbol{s_p})\, d\boldsymbol{t_p}\, d\boldsymbol{s_p}.$$

[2] If $h_1, \ldots, h_p \in \mathfrak{H}$, we denote by $\mathsf{sym}\big(h_1 \otimes \cdots \otimes h_p\big)$ the symmetrization of the tensor product $h_1 \otimes \cdots \otimes h_p$:

$$\mathsf{sym}\big(h_1 \otimes \cdots \otimes h_p\big) := \frac{1}{p!} \sum_{\pi \in \mathfrak{S}_p} h_{\pi(1)} \otimes \cdots \otimes h_{\pi(p)}\,,$$

where $\mathfrak{S}_p$ is the permutation group on the first p positive integers.

Proof. In view of the Fourth Moment Theorem of Nualart and Peccati [23], to prove this central convergence it suffices to establish

$$\lim_{R\to+\infty}\frac{1}{R^{2d}}\big\|g_{p,R}\otimes_r g_{p,R}\big\|^2_{\mathfrak{H}^{\otimes(2p-2r)}}=0$$

for $r=1,\dots,p-1$. By definition, we can write

$$\big(g_{p,R}\otimes_r g_{p,R}\big)(\boldsymbol{s_{p-r}},\boldsymbol{t_{p-r}})=\int_{\mathbb{R}^{2rd}}g_{p,R}(\boldsymbol{s_{p-r}},\boldsymbol{a_r})g_{p,R}(\boldsymbol{t_{p-r}},\boldsymbol{b_r})\prod_{i=1}^{r}\gamma(a_i-b_i)\,d\boldsymbol{a_r}\,d\boldsymbol{b_r}\,.$$

As a consequence,

$$\begin{aligned}
&\big\|g_{p,R}\otimes_r g_{p,R}\big\|^2_{\mathfrak{H}^{\otimes(2p-2r)}}\\
&=\int_{\mathbb{R}^{4pd}}d\boldsymbol{a_r}\,d\boldsymbol{b_r}d\widetilde{\boldsymbol{a}}_{\boldsymbol{r}}\,d\widetilde{\boldsymbol{b}}_{\boldsymbol{r}}\,d\boldsymbol{t_{p-r}}\,d\boldsymbol{s_{p-r}}\,d\widetilde{\boldsymbol{t}}_{\boldsymbol{p-r}}\,d\widetilde{\boldsymbol{s}}_{\boldsymbol{p-r}}g_{p,R}(\boldsymbol{s_{p-r}},\boldsymbol{a_r})g_{p,R}(\boldsymbol{t_{p-r}},\boldsymbol{b_r})\\
&\times g_{p,R}(\widetilde{\boldsymbol{s}}_{\boldsymbol{p-r}},\widetilde{\boldsymbol{a}}_{\boldsymbol{r}})g_{p,R}(\widetilde{\boldsymbol{t}}_{\boldsymbol{p-r}},\widetilde{\boldsymbol{b}}_{\boldsymbol{r}})\left(\prod_{i=1}^{r}\gamma(a_i-b_i)\gamma(\widetilde{a}_i-\widetilde{b}_i)\right)\left(\prod_{j=1}^{p-r}\gamma(t_j-\widetilde{t}_j)\gamma(\widetilde{s}_j-s_j)\right)\\
&=\int_{B_R^4}d\boldsymbol{x_4}\int_{\mathbb{R}^{4dp}}d\boldsymbol{a_r}\,d\boldsymbol{b_r}d\widetilde{\boldsymbol{a}}_{\boldsymbol{r}}\,d\widetilde{\boldsymbol{b}}_{\boldsymbol{r}}\,d\boldsymbol{t_{p-r}}\,d\boldsymbol{s_{p-r}}\,d\widetilde{\boldsymbol{t}}_{\boldsymbol{p-r}}\,d\widetilde{\boldsymbol{s}}_{\boldsymbol{p-r}}f_p^{x_1}(\boldsymbol{s_{p-r}},\boldsymbol{a_r})f_p^{x_2}(\boldsymbol{t_{p-r}},\boldsymbol{b_r})\\
&\times f_p^{x_3}(\widetilde{\boldsymbol{s}}_{\boldsymbol{p-r}},\widetilde{\boldsymbol{a}}_{\boldsymbol{r}})f_p^{x_4}(\widetilde{\boldsymbol{t}}_{\boldsymbol{p-r}},\widetilde{\boldsymbol{b}}_{\boldsymbol{r}})\left(\prod_{i=1}^{r}\gamma(a_i-b_i)\gamma(\widetilde{a}_i-\widetilde{b}_i)\right)\prod_{j=1}^{p-r}\gamma(t_j-\widetilde{t}_j)\gamma(\widetilde{s}_j-s_j)\,. \qquad (2.10)
\end{aligned}$$

Shifting the variables from the kernels to the covariance, we write

$$\begin{aligned}
&\big\|g_{p,R}\otimes_r g_{p,R}\big\|^2_{\mathfrak{H}^{\otimes(2p-2r)}}\\
&=\int_{B_R^4}d\boldsymbol{x_4}\int_{\mathbb{R}^{4dp}}d\boldsymbol{a_r}\,d\boldsymbol{b_r}d\widetilde{\boldsymbol{a}}_{\boldsymbol{r}}\,d\widetilde{\boldsymbol{b}}_{\boldsymbol{r}}\,d\boldsymbol{t_{p-r}}\,d\boldsymbol{s_{p-r}}\,d\widetilde{\boldsymbol{t}}_{\boldsymbol{p-r}}\,d\widetilde{\boldsymbol{s}}_{\boldsymbol{p-r}}f_p(\boldsymbol{s_{p-r}},\boldsymbol{a_r})f_p(\boldsymbol{t_{p-r}},\boldsymbol{b_r})\\
&\qquad\times f_p(\widetilde{\boldsymbol{s}}_{\boldsymbol{p-r}},\widetilde{\boldsymbol{a}}_{\boldsymbol{r}})f_p(\widetilde{\boldsymbol{t}}_{\boldsymbol{p-r}},\widetilde{\boldsymbol{b}}_{\boldsymbol{r}})\left(\prod_{i=1}^{r}\gamma(a_i-b_i+x_1-x_2)\gamma(\widetilde{a}_i-\widetilde{b}_i+x_3-x_4)\right)\\
&\qquad\quad\times\left(\prod_{j=1}^{p-r}\gamma(t_j-\widetilde{t}_j+x_2-x_4)\gamma(\widetilde{s}_j-s_j+x_3-x_1)\right)\,.
\end{aligned}$$

Making the change of variables $x_1-x_2=z_1$, $x_3-x_4=z_2$ and $x_2-x_4=z_3$ (so $x_3-x_1=z_2-z_3-z_1$), we obtain

$$\begin{aligned}
&R^{-2d}\big\|g_{p,R}\otimes_r g_{p,R}\big\|^2_{\mathfrak{H}^{\otimes(2p-2r)}}\\
&\le CR^{-d}\int_{B_{2R}^3}d\boldsymbol{z_3}\left|\int_{\mathbb{R}^{4dp}}d\boldsymbol{a_r}\,d\boldsymbol{b_r}d\widetilde{\boldsymbol{a}}_{\boldsymbol{r}}\,d\widetilde{\boldsymbol{b}}_{\boldsymbol{r}}\,d\boldsymbol{t_{p-r}}\,d\boldsymbol{s_{p-r}}\,d\widetilde{\boldsymbol{t}}_{\boldsymbol{p-r}}\,d\widetilde{\boldsymbol{s}}_{\boldsymbol{p-r}}f_p(\boldsymbol{s_{p-r}},\boldsymbol{a_r})\right.\\
&\qquad\times f_p(\boldsymbol{t_{p-r}},\boldsymbol{b_r})f_p(\widetilde{\boldsymbol{s}}_{\boldsymbol{p-r}},\widetilde{\boldsymbol{a}}_{\boldsymbol{r}})f_p(\widetilde{\boldsymbol{t}}_{\boldsymbol{p-r}},\widetilde{\boldsymbol{b}}_{\boldsymbol{r}})\left(\prod_{i=1}^{r}\gamma(a_i-b_i+z_1)\gamma(\widetilde{a}_i-\widetilde{b}_i+z_2)\right)
\end{aligned}$$

$$\times \left(\prod_{j=1}^{p-r} \gamma(t_j - \widetilde{t}_j + z_3)\gamma(\widetilde{s}_j - s_j + z_2 - z_1 - z_3) \right) \Bigg| . \tag{2.11}$$

The rest of our proof will be split into two cases.

Proof under (i). Using the tensor-product structure of the kernels, we can further bound (2.11) by

$$CR^{-d} \int_{B_{2R}^3} d\boldsymbol{z_3} \phi(z_1)^r \phi(z_2)^r \phi(z_3)^{p-r} \phi(z_2 - z_1 - z_3)^{p-r} ,$$

with

$$\phi(z) := \sum_{i,j=1}^{p} \left| \int_{\mathbb{R}^{2d}} h_i(a) h_j(b) \gamma(a - b + z) dadb \right| .$$

In view of (2.9), the function ϕ belong to $L^p(\mathbb{R}^d)$. It follows immediately from Hölder's inequality that

$$\begin{aligned} R^{-2d} \| g_{p,R} \otimes_r g_{p,R} \|^2_{\mathfrak{H}^{\otimes(2p-2r)}} &\leq C \left(\int_{\mathbb{R}^d} \phi(z_1)^p dz_1 \right) R^{-d} \int_{B_{2R}^2} dz_2 dz_3 \phi(z_2)^r \phi(z_3)^{p-r} \\ &= C \left(\int_{\mathbb{R}^d} \phi(z_1)^p dz_1 \right) R^{-d} \left(\int_{B_{2R}} \phi(z_2)^r dz_2 \right) \left(\int_{B_{2R}} \phi(z_3)^{p-r} dz_3 \right) . \end{aligned}$$

Then, we can conclude our proof under the condition (i) by using Lemma 2.2. □

Proof under (ii). Note first that due to Hölder's inequality,

$$\int_{B_{2R}} \left(\prod_{i=1}^{r} |\gamma|(a_i - b_i + z_1) \right) \left(\prod_{j=1}^{p-r} |\gamma|(\widetilde{s}_j - s_j + z_2 - z_1 - z_3) \right) dz_1 \leq \int_{\mathbb{R}^d} |\gamma(z)|^p \, dz ,$$

which implies that (2.11) can be further bounded by

$$\begin{aligned} & C \|\gamma\|^p_{L^p(\mathbb{R}^d)} \|f_p\|_{L^1(\mathbb{R}^{pd})} R^{-d} \int_{B_{2R}^2 \times \mathbb{R}^{3dp}} dz_2 dz_3 d\boldsymbol{b_r} d\widetilde{\boldsymbol{a}}_{\boldsymbol{r}} \, d\widetilde{\boldsymbol{b}}_{\boldsymbol{r}} \, d\boldsymbol{t_{p-r}} \, d\widetilde{\boldsymbol{t}}_{\boldsymbol{p-r}} \, d\widetilde{\boldsymbol{s}}_{\boldsymbol{p-r}} \\ & \times \left| f_p(\boldsymbol{t_{p-r}}, \boldsymbol{b_r}) f_p(\widetilde{\boldsymbol{s}}_{\boldsymbol{p-r}}, \widetilde{\boldsymbol{a}}_{\boldsymbol{r}}) f_p(\widetilde{\boldsymbol{t}}_{\boldsymbol{p-r}}, \widetilde{\boldsymbol{b}}_{\boldsymbol{r}}) \right| \left(\prod_{i=1}^{r} |\gamma|(\widetilde{a}_i - \widetilde{b}_i + z_2) \right) \left(\prod_{j=1}^{p-r} |\gamma|(t_j - \widetilde{t}_j + z_3) \right) \\ & \leq C \int_{\mathbb{R}^{3dp}} d\boldsymbol{b_r} d\widetilde{\boldsymbol{a}}_{\boldsymbol{r}} \, d\widetilde{\boldsymbol{b}}_{\boldsymbol{r}} \, d\boldsymbol{t_{p-r}} \, d\widetilde{\boldsymbol{t}}_{\boldsymbol{p-r}} \, d\widetilde{\boldsymbol{s}}_{\boldsymbol{p-r}} \left| f_p(\boldsymbol{t_{p-r}}, \boldsymbol{b_r}) f_p(\widetilde{\boldsymbol{s}}_{\boldsymbol{p-r}}, \widetilde{\boldsymbol{a}}_{\boldsymbol{r}}) f_p(\widetilde{\boldsymbol{t}}_{\boldsymbol{p-r}}, \widetilde{\boldsymbol{b}}_{\boldsymbol{r}}) \right| \times \mathbf{L}_R , \end{aligned}$$

where $\mathbf{L}_R = \mathbf{L}_R(\widetilde{\boldsymbol{a}}_{\boldsymbol{r}}, \widetilde{\boldsymbol{b}}_{\boldsymbol{r}}, \widetilde{\boldsymbol{t}}_{\boldsymbol{p-r}}, \boldsymbol{t_{p-r}})$ is given by

$$\mathbf{L}_R = R^{-d} \left(\int_{B_{2R}} \prod_{i=1}^{r} |\gamma|(\widetilde{a}_i - \widetilde{b}_i + z_2) dz_2 \right) \left(\int_{B_{2R}} \prod_{j=1}^{p-r} |\gamma|(t_j - \widetilde{t}_j + z_3) dz_3 \right) .$$

Note that by Hölder's inequality and Lemma 2.2,

$$\mathbf{L}_R \le \left(\prod_{i=1}^{r} \frac{1}{R^{d(1-rp^{-1})}} \int_{B_{2R}} |\gamma|^r(\widetilde{a}_i - \widetilde{b}_i + z_2)dz_2\right)^{1/r}$$
$$\times \left(\prod_{j=1}^{p-r} \frac{1}{R^{d(1-(p-r)p^{-1})}} \int_{B_{2R}} |\gamma|^{p-r}(t_j - \widetilde{t}_j + z_3)dz_3\right)^{1/(p-r)} \xrightarrow{R\to+\infty} 0\,,$$

and that

$$\mathbf{L}_R \le CR^{-d}\|\gamma\|^r_{L^p(\mathbb{R}^d)} R^{d\frac{p-r}{p}} \|\gamma\|^{p-r}_{L^p(\mathbb{R}^d)} R^{d\frac{r}{p}} = C\|\gamma\|^p_{L^p(\mathbb{R}^d)} < +\infty\,.$$

Thus, it follows from the dominated convergence theorem that, as $R \to \infty$,

$$R^{-2d}\|g_{p,R} \otimes_r g_{p,R}\|^2_{\mathfrak{H}^{\otimes(2p-2r)}} \to 0$$

for all $r \in \{1,\dots,p-1\}$. This completes the proof. □

2.2.2 CLT under assumptions on the spectral measure

Let us first study the asymptotic variance using the Fourier transform. Throughout this section, we are going to assume that $\mu(d\xi) = \varphi(\xi)d\xi$, that is, the spectral measure is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$. Note that $\varphi(\xi) = \varphi(-\xi)$.

We first write,

$$\Phi_p(x-y) = p!\langle f_x, f_y\rangle_{\mathfrak{H}^{\otimes p}} = p!\int_{\mathbb{R}^{pd}} (\mathscr{F} f^x_p)(\boldsymbol{\xi_p})(\mathscr{F} f^y_p)(-\boldsymbol{\xi_p})\ \mu(d\boldsymbol{\xi_p})$$
$$= p!\int_{\mathbb{R}^{pd}} \exp\Big(-\mathbf{i}(x-y)\cdot\tau(\boldsymbol{\xi_p})\Big)|\mathscr{F} f_p|^2(\boldsymbol{\xi_p})\ \mu(d\boldsymbol{\xi_p})\,,$$

where $\tau(\boldsymbol{\xi_p}) := \xi_1 + \cdots + \xi_p$. As a consequence of Lemma 2.1, we obtain

$$\mathrm{Var}(G_{p,R}) = p!\int_{B_R^2}\int_{\mathbb{R}^{pd}} \exp\Big(-\mathbf{i}(x-y)\cdot\tau(\boldsymbol{\xi_p})\Big)|\mathscr{F} f_p|^2(\boldsymbol{\xi_p})\ \mu(d\boldsymbol{\xi_p})\ dxdy$$
$$= p!(2\pi R)^d \int_{\mathbb{R}^{pd}} \|\tau(\boldsymbol{\xi_p})\|^{-d} J_{d/2}\big(R\|\tau(\boldsymbol{\xi_p})\|\big)^2 |\mathscr{F} f_p|^2(\boldsymbol{\xi_p})\ \mu(d\boldsymbol{\xi_p})\,. \tag{2.12}$$

Now making the change of variables $\tau(\boldsymbol{\xi_p}) = x$ yields

$$\mathrm{Var}(G_{p,R})R^{-d} = p!(2\pi)^d \int_{\mathbb{R}^d} \|x\|^{-d} J_{d/2}\big(R\|x\|\big)^2 \Psi_p(x)dx,$$

where

$$\Psi_p(x) := \int_{\mathbb{R}^{pd-d}} |\mathscr{F} f_p|^2\big(\boldsymbol{\xi_{p-1}}, x-\tau(\boldsymbol{\xi_{p-1}})\big)\varphi\big(x-\tau(\boldsymbol{\xi_{p-1}})\big)\prod_{i=1}^{p-1}\varphi(\xi_i)d\boldsymbol{\xi_{p-1}}. \tag{2.13}$$

We remark that Ψ_p is defined almost everywhere on $\mathbb{R}^d$ and recall that

$$\left\{\ell_R(x) := \omega_d^{-1}\|x\|^{-d} J_{d/2}(R\|x\|)^2\right\}_{R>0}$$

is an *approximation of the identity*. Therefore, it is natural to introduce the following hypothesis:

(H2) Ψ_p, defined in (2.13), is uniformly bounded on $\mathbb{R}^d$ and continuous at zero.

Under **(H2)**, we have

$$\lim_{R\to+\infty} \frac{\mathrm{Var}(G_{p,R})}{R^d} = p!(2\pi)^d \omega_d \Psi_p(0)\,,$$

where

$$\Psi_p(0) = \int_{\mathbb{R}^{(p-1)d}} |\mathscr{F} f_p|^2\big(\boldsymbol{\xi_{p-1}}, -\tau(\boldsymbol{\xi_{p-1}})\big)\varphi\big(\tau(\boldsymbol{\xi_{p-1}})\big) \prod_{i=1}^{p-1} \varphi(\xi_i) d\boldsymbol{\xi_{p-1}}. \tag{2.14}$$

Note that for the particular case $p = 1$, $\Psi_1(x) = |\mathscr{F} f_1|^2(x)\varphi(x)$; if $f_1 \in L^1(\mathbb{R}^d)$ and φ is uniformly bounded with continuity at zero, then the function Ψ_1 is uniformly bounded and continuous at zero.

Remark 2.6. (1) Heuristically, we can rewrite $\Psi_p(0)$ as follows:

$$\Psi_p(0) = \int_{\{\tau(\boldsymbol{\xi_p})=0\}} |\mathscr{F} f_p|^2(\boldsymbol{\xi_p}) \prod_{i=1}^{p} \varphi(\xi_i)\nu(d\boldsymbol{\xi_p}),$$

where ν is the surface measure on the hyperplane $\{\tau(\boldsymbol{\xi_p}) = 0\}$. This is an informal expression, because the trace of $\mathscr{F} f_p$ on the hyperplane $\{\tau(\boldsymbol{\xi_p}) = 0\}$ is not properly defined for an arbitrary kernel f_p.

(2) Notice that the quantity $\frac{\mathrm{Var}(G_{p,R})}{(2\pi R)^d p!\omega_d}$ is equal to

$$\int_{\mathbb{R}^{pd-d}} \left(\int_{\mathbb{R}^d} dx \ell_R(x)\varphi\big(x-\tau(\boldsymbol{\xi_{p-1}})\big)|\mathscr{F} f_p|^2\big(\boldsymbol{\xi_{p-1}}, x-\tau(\boldsymbol{\xi_{p-1}})\big)\right) \prod_{i=1}^{p-1} \varphi(\xi_i) d\boldsymbol{\xi_{p-1}}.$$

It is clear that $|\mathscr{F} f_p|^2(\boldsymbol{\xi_{p-1}}, x-\tau(\boldsymbol{\xi_{p-1}}))$ is well-defined almost everywhere with respect to $\varphi\big(x-\tau(\boldsymbol{\xi_{p-1}})\big)dx$, and $\varphi\big(x-\tau(\boldsymbol{\xi_{p-1}})\big)|\mathscr{F} f_p|^2(\boldsymbol{\xi_{p-1}}, x-\tau(\boldsymbol{\xi_{p-1}}))$ is integrable with respect to the probability measure $\ell_R(x)dx$. We can also read from (2.14) that the function $\boldsymbol{\xi_{p-1}} \mapsto |\mathscr{F} f_p|^2\big(\boldsymbol{\xi_{p-1}}, -\tau(\boldsymbol{\xi_{p-1}})\big)$ is integrable with respect to the measure $\varphi\big(\tau(\boldsymbol{\xi_{p-1}})\big)\prod_{i=1}^{p-1}\varphi(\xi_i)d\boldsymbol{\xi_{p-1}}$.

To obtain the Gaussian fluctuation of $G_{p,R}$, one shall first establish the order of the variance and then compute the contractions. Our hypothesis **(H2)** gives the exact asymptotic behavior of $\mathrm{Var}(G_{p,R})$. In fact, it is enough to impose a weaker condition, known as the Maruyama's condition concerning the variance; see [18].

Proposition 2.7 (Maruyama's condition)**.** *Put*

$$\widehat{\Psi}_p(h) := \int_{\{\|\tau(\boldsymbol{\xi_p})\|\le h\}} |\mathscr{F} f_p|^2(\boldsymbol{\xi_p})\mu(d\boldsymbol{\xi_p})\,.$$

If

$$0 < \liminf_{h\downarrow 0} h^{-d}\, \widehat{\Psi}_p(h) \le \limsup_{h\downarrow 0} h^{-d}\, \widehat{\Psi}_p(h) < \infty, \tag{2.15}$$

then we have, with $\sigma^2_{p,R} = \mathrm{Var}(G_{p,R})$

$$0 < \liminf_{R\to+\infty} \sigma^2_{p,R} R^{-d} \le \limsup_{R\to+\infty} \sigma^2_{p,R} R^{-d} < \infty.$$

We will provide a proof of Proposition 2.7 in Section 4, see also [4, Corollary 2.2].

The following lemma provides sufficient conditions for **(H2)** to hold. One of the conditions is $\varphi \in L^q(\mathbb{R}^d)$, which is the condition imposed on the spectral density in the version of the classical Breuer-Major theorem proved in [1, Theorem 2.10].

Lemma 2.8. *Suppose that* $f_p \in L^1(\mathbb{R}^{pd}) \cap \mathfrak{H}^{\odot p}$ *and* $\varphi \in L^q(\mathbb{R}^d)$*, with* $q = p/(p-1)$*. Then* Ψ_p *is bounded and continuous on* $\mathbb{R}^d$*, in particular hypothesis* **(H2)** *is true.*

The proof of Lemma 2.8 is given in Section 4.

Remark 2.9. It is worth comparing the sufficient conditions for the hypotheses **(H1)** and **(H2)** here:

$$\left\{\gamma \in L^p(\mathbb{R}^d) \text{ and } f_p \in L^1(\mathbb{R}^{pd})\right\} \Rightarrow (\mathbf{H1})$$
$$\left\{\varphi \in L^q(\mathbb{R}^d) \text{ and } f_p \in L^1(\mathbb{R}^{pd})\right\} \Rightarrow (\mathbf{H2}).$$

This is natural in view of the Hausdorff-Young's inequality. Indeed, $q = p/(p-1) \in (1, 2]$, so $\gamma = \mathscr{F}\varphi$ belongs to $L^p(\mathbb{R}^d)$, provided $\varphi \in L^q(\mathbb{R}^d)$. Note that both hypotheses imply that the fluctuation of $G_{p,R}$ is of order $R^{d/2}$; moreover, as we will see shortly, both hypotheses ($\gamma \in L^p(\mathbb{R}^d)$ and $\varphi \in L^q(\mathbb{R}^d)$) imply that the fluctuation of $G_{p,R}$ is Gaussian, as R tends to infinity.

Let us introduce the following hypothesis, which can be seen as the contraction-analogue of **(H2)**.

(H3) For $1 \le r \le p-1$ and any $\delta > 0$, $\Psi_p^{(r,\delta)}$ is uniformly bounded on $\mathbb{R}^d$ and continuous at zero, where

$$\begin{aligned}
&\Psi_p^{(r,\delta)}(x,y) \\
&= \int_{\mathbb{R}^{2pd-2d}} d\boldsymbol{\xi_r} d\boldsymbol{\eta_{p-r}} d\widetilde{\boldsymbol{\xi}}_{\boldsymbol{r-1}} d\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r-1}} |\mathscr{F} f_p|^2\Big(\boldsymbol{\eta_{p-r}}, \widetilde{\boldsymbol{\xi}}_{\boldsymbol{r-1}}, x - \tau(\boldsymbol{\eta_{p-r}}) - \tau(\widetilde{\boldsymbol{\xi}}_{\boldsymbol{r-1}})\Big)\varphi(\xi_r) \\
&\times |\mathscr{F} f_p|^2\Big(\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r-1}}, y - \tau(\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r-1}}) - \tau(\boldsymbol{\xi_r}), \boldsymbol{\xi_r}\Big) \left(\prod_{i=1}^{r-1} \varphi(\xi_i)\varphi(\widetilde{\xi}_i)\right) \mathbf{1}_{\{\|\tau(\boldsymbol{\xi_r}) + \tau(\boldsymbol{\eta_{p-r}})\| < \delta\}}
\end{aligned} \tag{2.16}$$

$$\times\,\varphi(\eta_{p-r})\varphi\big(\tau(\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r-1}})+\tau(\boldsymbol{\xi_r})-y\big)\left(\prod_{j=1}^{p-r-1}\varphi(\eta_j)\varphi(\widetilde{\eta}_j)\right)\varphi\big(\tau(\boldsymbol{\eta_{p-r}})+\tau(\widetilde{\boldsymbol{\xi}}_{\boldsymbol{r-1}})-x\big).$$

We remark that the function $\Psi_p^{(r,\delta)}$ is defined almost everywhere on $\mathbb{R}^{2d}$ and with the same proof as in Lemma 2.8, we can show that $f_p\in L^1(\mathbb{R}^{pd})$ and $\varphi\in L^q(\mathbb{R}^d)$ for $q=p/(p-1)$ guarantee **(H3)**.

Lemma 2.10. *Suppose that $f_p\in L^1(\mathbb{R}^{pd})\cap\mathfrak{H}^{\odot p}$ and $\varphi\in L^q(\mathbb{R}^d)$, with $q=p/(p-1)$. Then for every $r\in\{1,\dots,p-1\}$ and $\delta>0$, $\Psi_p^{(r,\delta)}$ is bounded continuous on $\mathbb{R}^{2d}$. In particular hypothesis* **(H3)** *is true.*

For the sake of completeness, we provide a proof in Section 4.

Theorem 2.11. *Fix an integer $p\geq 2$ and $f_p\in\mathfrak{H}^{\odot p}$ satisfying hypotheses* **(H2)** *and* **(H3)**. *Then,*

$$\frac{G_{p,R}}{R^{d/2}}\xrightarrow[R\to+\infty]{\text{law}}N(0,\sigma_p^2),$$

where $\sigma_p^2=p!(2\pi)^d\omega_d\Psi_p(0)$, with $\Psi_p(0)$ given by (2.14).

If **(H2)** is replaced by the Maruyama's condition (2.15), we have the following corollary.

Corollary 2.12. *Fix an integer $p\geq 2$ and $f_p\in\mathfrak{H}^{\odot p}$ satisfying hypotheses* **(H3)**. *Assume that Maruyama's condition* (2.15) *holds true. Then,*

$$\frac{G_{p,R}}{\sigma_{p,R}}\xrightarrow[R\to+\infty]{\text{law}}N(0,1),$$

with $\sigma_{p,R}$ being the standard deviation of $G_{p,R}$.

We will omit the proof of this corollary, as it follows simply from Proposition 2.7 and the following proof of Theorem 2.11.

Proof of Theorem 2.11. It suffices to show the contraction condition (1.5). We spilt the proof into several steps. We will use Fourier transform to rewrite (2.10) in Steps 1-3 and we will carry out the asymptotic analysis in Step 4.

Step 1: Plancherel's formula implies

$$\begin{aligned}&\int_{\mathbb{R}^{2rd}}f_p^{x_1}(\boldsymbol{s_{p-r}},\boldsymbol{a_r})f_p^{x_2}(\boldsymbol{t_{p-r}},\boldsymbol{b_r})\prod_{i=1}^{r}\gamma(a_i-b_i)d\boldsymbol{a_r}\,d\boldsymbol{b_r}\\&=\int_{\mathbb{R}^{rd}}(\mathscr{F}_rf_p^{x_1})(\boldsymbol{s_{p-r}},\boldsymbol{\xi_r})(\mathscr{F}_rf_p^{x_2})(\boldsymbol{t_{p-r}},-\boldsymbol{\xi_r})\,\mu(d\boldsymbol{\xi_r}).\end{aligned}$$

and

$$\int_{\mathbb{R}^{2rd}}f_p^{x_3}(\widetilde{\boldsymbol{s}}_{\boldsymbol{p-r}},\widetilde{\boldsymbol{a}}_{\boldsymbol{r}})f_p^{x_4}(\widetilde{\boldsymbol{t}}_{\boldsymbol{p-r}},\widetilde{\boldsymbol{b}}_{\boldsymbol{r}})\prod_{i=1}^{r}\gamma(\widetilde{a}_i-\widetilde{b}_i)d\widetilde{\boldsymbol{a}}_{\boldsymbol{r}}\,d\widetilde{\boldsymbol{b}}_{\boldsymbol{r}}$$

$$= \int_{\mathbb{R}^{rd}} (\mathscr{F}_r f_p^{x_3})(\widetilde{\boldsymbol{s}}_{\boldsymbol{p-r}}, \widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}})(\mathscr{F}_r f_p^{x_4})(\widetilde{\boldsymbol{t}}_{\boldsymbol{p-r}}, -\widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}})\, \mu(d\widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}})\,,$$

where $\mathscr{F}_r$ denotes the Fourier transform with respect to the right-most r variables.

Step 2: Similarly, we have

$$\begin{aligned}
&\int_{\mathbb{R}^{4(p-r)d}} (\mathscr{F}_r f_p^{x_1})(\boldsymbol{s}_{\boldsymbol{p-r}}, \boldsymbol{\xi}_{\boldsymbol{r}})(\mathscr{F}_r f_p^{x_2})(\boldsymbol{t}_{\boldsymbol{p-r}}, -\boldsymbol{\xi}_{\boldsymbol{r}})(\mathscr{F}_r f_p^{x_3})(\widetilde{\boldsymbol{s}}_{\boldsymbol{p-r}}, \widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}})(\mathscr{F}_r f_p^{x_4})(\widetilde{\boldsymbol{t}}_{\boldsymbol{p-r}}, -\widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}})\\
&\qquad\qquad \times \left(\prod_{j=1}^{p-r} \gamma(t_i - \widetilde{t}_i)\gamma(\widetilde{s}_i - s_i)\right) d\boldsymbol{t}_{\boldsymbol{p-r}}\, d\boldsymbol{s}_{\boldsymbol{p-r}}\, d\widetilde{\boldsymbol{t}}_{\boldsymbol{p-r}}\, d\widetilde{\boldsymbol{s}}_{\boldsymbol{p-r}}\\
&= \left(\int_{\mathbb{R}^{2(p-r)d}} (\mathscr{F}_r f_p^{x_1})(\boldsymbol{s}_{\boldsymbol{p-r}}, \boldsymbol{\xi}_{\boldsymbol{r}})(\mathscr{F}_r f_p^{x_3})(\widetilde{\boldsymbol{s}}_{\boldsymbol{p-r}}, \widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}}) \prod_{j=1}^{p-r} \gamma(\widetilde{s}_i - s_i) d\boldsymbol{s}_{\boldsymbol{p-r}}\, d\widetilde{\boldsymbol{s}}_{\boldsymbol{p-r}}\right)\\
&\quad \times \left(\int_{\mathbb{R}^{2(p-r)d}} (\mathscr{F}_r f_p^{x_2})(\boldsymbol{t}_{\boldsymbol{p-r}}, -\boldsymbol{\xi}_{\boldsymbol{r}})(\mathscr{F}_r f_p^{x_4})(\widetilde{\boldsymbol{t}}_{\boldsymbol{p-r}}, -\widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}}) \prod_{j=1}^{p-r} \gamma(t_i - \widetilde{t}_i) d\boldsymbol{t}_{\boldsymbol{p-r}}\, d\widetilde{\boldsymbol{t}}_{\boldsymbol{p-r}}\right)\\
&= \left(\int_{\mathbb{R}^{pd-rd}} (\mathcal{F}_{p-r}\mathscr{F}_r f_p^{x_1})(\boldsymbol{\eta}_{\boldsymbol{p-r}}, \boldsymbol{\xi}_{\boldsymbol{r}})(\mathcal{F}_{p-r}\mathscr{F}_r f_p^{x_3})(-\boldsymbol{\eta}_{\boldsymbol{p-r}}, -\widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}})\, \mu(d\boldsymbol{\eta}_{\boldsymbol{p-r}})\right)\\
&\qquad \times \left(\int_{\mathbb{R}^{pd-rd}} (\mathcal{F}_{p-r}\mathscr{F}_r f_p^{x_2})(\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}}, -\boldsymbol{\xi}_{\boldsymbol{r}})(\mathcal{F}_{p-r}\mathscr{F}_r f_p^{x_4})(-\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}}, \widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}})\, \mu(d\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}})\right),
\end{aligned}$$

where $\mathcal{F}_{p-r}$ denotes the Fourier transform with respect to the left-most $p-r$ variables. It is clear that the composition of $\mathcal{F}_{p-r}$ and $\mathscr{F}_r$ is the usual Fourier transform.

Step 3: Using basic properties of the Fourier transform, we have $(\mathcal{F}_{p-r}\mathscr{F}_r f_p^{x})(\boldsymbol{\xi}_{\boldsymbol{p}}) = e^{-\mathbf{i}x\cdot\tau(\boldsymbol{\xi}_{\boldsymbol{p}})}(\mathscr{F} f_p)(\boldsymbol{\xi}_{\boldsymbol{p}})$. So combining facts from the above steps yields that the second integral in (2.10) is equal to

$$\begin{aligned}
&\int_{\mathbb{R}^{2pd}} \mu(d\boldsymbol{\xi}_{\boldsymbol{r}})\, \mu(d\widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}})\mu(d\boldsymbol{\eta}_{\boldsymbol{p-r}})\mu(d\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}})(\mathscr{F} f_p)(\boldsymbol{\eta}_{\boldsymbol{p-r}}, \boldsymbol{\xi}_{\boldsymbol{r}})(\mathscr{F} f_p)(-\boldsymbol{\eta}_{\boldsymbol{p-r}}, -\widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}})\\
&\quad \times (\mathscr{F} f_p)(\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}}, -\boldsymbol{\xi}_{\boldsymbol{r}})(\mathscr{F} f_p)(-\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}}, \widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}})\;\; e^{-\mathbf{i}x_1\cdot(a+b)}e^{-\mathbf{i}x_2\cdot(\widetilde{b}-a)}e^{-\mathbf{i}x_3\cdot(-\widetilde{a}-b)}e^{-\mathbf{i}x_4\cdot(\widetilde{a}-\widetilde{b})}\,,
\end{aligned}$$

with the notation $a = \tau(\boldsymbol{\xi}_{\boldsymbol{r}}), b = \tau(\boldsymbol{\eta}_{\boldsymbol{p-r}}), \widetilde{a} = \tau(\widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}})$ and $\widetilde{b} = \tau(\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}})$ *throughout this proof.*

It follows from Lemma 2.1 that

$$\begin{aligned}
&\int_{B_R^4} e^{-\mathbf{i}x_1\cdot(a+b)}e^{-\mathbf{i}x_2\cdot(\widetilde{b}-a)}e^{-\mathbf{i}x_3\cdot(-\widetilde{a}-b)}e^{-\mathbf{i}x_4\cdot(\widetilde{a}-\widetilde{b})}\, d\boldsymbol{x}_{\boldsymbol{4}}\\
&= (2\pi R)^{2d}\|a+b\|^{-d/2}\|\widetilde{b}-a\|^{-d/2}\|\widetilde{a}+b\|^{-d/2}\|\widetilde{a}-\widetilde{b}\|^{-d/2}\\
&\qquad \times J_{d/2}\big(R\|a+b\|\big)J_{d/2}\big(R\|\widetilde{b}-a\|\big)J_{d/2}\big(R\|\widetilde{a}+b\|\big)J_{d/2}\big(R\|\widetilde{a}-\widetilde{b}\|\big).
\end{aligned}$$

Thus, we have for $r \in \{1, \dots, p-1\}$,

$$\mathcal{I}_R := (2\pi R)^{-2d}\big\|g_{p,R} \otimes_r g_{p,R}\big\|^2_{\mathfrak{H}^{\otimes(2p-2r)}} \tag{2.17}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^{2pd}} \mu(d\boldsymbol{\xi_r})\,\mu(d\widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}})\mu(d\boldsymbol{\eta_{p-r}})\mu(d\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}})(\mathscr{F}f_p)(\boldsymbol{\eta_{p-r}},\boldsymbol{\xi_r})(\mathscr{F}f_p)(-\boldsymbol{\eta_{p-r}},-\widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}})\\
&\quad\times(\mathscr{F}f_p)(\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}},-\boldsymbol{\xi_r})(\mathscr{F}f_p)(-\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}},\widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}})\|a+b\|^{-d/2}\|\widetilde{b}-a\|^{-d/2}\|\widetilde{a}+b\|^{-d/2}\\
&\qquad\times\|\widetilde{a}-\widetilde{b}\|^{-d/2}J_{d/2}\big(R\|a+b\|\big)J_{d/2}\big(R\|\widetilde{b}-a\|\big)J_{d/2}\big(R\|\widetilde{a}+b\|\big)J_{d/2}\big(R\|\widetilde{a}-\widetilde{b}\|\big).
\end{aligned}
$$

Step 4: In what follows, we prove that $\lim_{R\to+\infty}\mathcal{I}_R=0$.

We decompose the above integral into two parts: $\mathcal{I}_R=\int_{\mathbb{R}^{pd}\times\mathcal{D}_\delta}+\int_{\mathbb{R}^{pd}\times\mathcal{D}_\delta^c}$, with

$$
\mathcal{D}_\delta=\{(\boldsymbol{\xi_r},\boldsymbol{\eta_{p-r}})\in\mathbb{R}^{pd}:\|a+b\|\geq\delta\}.
$$

To ease the presentation, we introduce for every $\delta\in[0,\infty)$,

$$
\mathbf{T}_\delta(R):=\int_{\{\|\tau(\boldsymbol{\xi_p})\|\geq\delta\}}\mu(d\boldsymbol{\xi_p})|\mathscr{F}f_p|^2(\boldsymbol{\xi_p})\|\tau(\boldsymbol{\xi_p})\|^{-d}J_{d/2}\big(R\|\tau(\boldsymbol{\xi_p})\|\big)^2.
$$

Note that, by (2.12) and the symmetry of μ, we have

$$
\mathbf{T}_0(R)=\frac{\mathrm{Var}(G_{p,R})}{p!(2\pi R)^d},
$$

which, under the hypothesis **(H2)**, converges to $\omega_d\Psi_p(0)$, as $R\to+\infty$.

Now on $\mathbb{R}^{pd}\times\mathcal{D}_\delta$, we can write, using Cauchy-Schwarz inequality,

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^{pd}\times\mathcal{D}_\delta}\right|\leq\int_{\mathbb{R}^{pd}}\mu(d\widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}})\mu(d\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}})|\mathscr{F}f_p|(-\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}},\widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}})\|\widetilde{a}-\widetilde{b}\|^{-d/2}\big|J_{d/2}\big(R\|\widetilde{a}-\widetilde{b}\|\big)\big|\\
&\times\int_{\mathcal{D}_\delta}\mu(d\boldsymbol{\xi_r})\mu(d\boldsymbol{\eta_{p-r}})|\mathscr{F}f_p|(\boldsymbol{\eta_{p-r}},\boldsymbol{\xi_r})\|a+b\|^{-d/2}\big|J_{d/2}\big(R\|a+b\|\big)\big||\mathscr{F}f_p|(-\boldsymbol{\eta_{p-r}},-\widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}})\\
&\times|\mathscr{F}f_p|(\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}},-\boldsymbol{\xi_r})\|\widetilde{b}-a\|^{-d/2}\|\widetilde{a}+b\|^{-d/2}\big|J_{d/2}\big(R\|\widetilde{b}-a\|\big)J_{d/2}\big(R\|\widetilde{a}+b\|\big)\big|\\
&\leq\sqrt{\mathbf{T}_\delta(R)}\int_{\mathbb{R}^{pd}}\mu(d\widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}})\mu(d\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}})|\mathscr{F}f_p|(-\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}},\widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}})\|\widetilde{a}-\widetilde{b}\|^{-d/2}\big|J_{d/2}\big(R\|\widetilde{a}-\widetilde{b}\|\big)\big|\\
&\quad\times\Bigg(\int_{\mathcal{D}_\delta}\mu(d\boldsymbol{\xi_r})\mu(d\boldsymbol{\eta_{p-r}})|\mathscr{F}f_p|^2(-\boldsymbol{\eta_{p-r}},-\widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}})|\mathscr{F}f_p|^2(\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}},-\boldsymbol{\xi_r})\\
&\qquad\qquad\times\|\widetilde{b}-a\|^{-d}\|\widetilde{a}+b\|^{-d}J_{d/2}\big(R\|\widetilde{b}-a\|\big)^2J_{d/2}\big(R\|\widetilde{a}+b\|\big)^2\Bigg)^{1/2}\\
&\leq\sqrt{\mathbf{T}_\delta(R)\mathbf{T}_0(R)}\Bigg(\int_{\mathbb{R}^{2pd}}\mu(d\widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}})\mu(d\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}})\mu(d\boldsymbol{\xi_r})\mu(d\boldsymbol{\eta_{p-r}})|\mathscr{F}f_p|^2(-\boldsymbol{\eta_{p-r}},-\widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}})\\
&\quad\times|\mathscr{F}f_p|^2(\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}},-\boldsymbol{\xi_r})\|\widetilde{b}-a\|^{-d}\|\widetilde{a}+b\|^{-d}J_{d/2}\big(R\|\widetilde{b}-a\|\big)^2J_{d/2}\big(R\|\widetilde{a}+b\|\big)^2\Bigg)^{1/2}\\
&=\mathbf{T}_0(R)^{3/2}\sqrt{\mathbf{T}_\delta(R)}\,.
\end{aligned}
$$

We claim that

$$\text{for any fixed } \delta > 0,\ \mathbf{T}_\delta(R) \to 0, \text{ as } R \to +\infty. \tag{2.18}$$

Indeed, on $\{\|\tau(\boldsymbol{\xi_p})\| \geq \delta > 0\}$, $J_{d/2}\big(R\|\tau(\boldsymbol{\xi_p})\|\big)^2$ converges to zero, as $R \to +\infty$; and clearly,

$$\mathbf{T}_\delta(R) \leq \delta^{-d}\left(\sup_{t\in\mathbb{R}_+} J_{d/2}(t)^2\right)\int_{\|\tau(\boldsymbol{\xi_p})\|\geq\delta} \mu(d\boldsymbol{\xi_p})|\mathscr{F}f_p|^2(\boldsymbol{\xi_p}) < \infty\,,$$

so claim (2.18) follows from the dominated convergence theorem. Therefore, the first part $\int_{\mathbb{R}^{pd}\times\mathcal{D}_\delta}$ goes to zero, as R tends to infinity.

Then, it remains to estimate the integral over $\mathbb{R}^{pd}\times\mathcal{D}_\delta^c$. Similarly, we obtain, by applying Cauchy-Schwarz inequality,

$$\begin{aligned}\left|\int_{\mathbb{R}^{pd}\times\mathcal{D}_\delta^c}\right| &\leq \int_{\mathcal{D}_\delta^c}\mu(d\boldsymbol{\xi_r})\mu(d\boldsymbol{\eta_{p-r}})\|a+b\|^{-d/2}\big|J_{d/2}\big(R\|a+b\|\big)\big|\,|\mathscr{F}f_p|(\boldsymbol{\eta_{p-r}},\boldsymbol{\xi_r})\\ &\quad\times\sqrt{\mathbf{T}_0(R)}\bigg(\int_{\mathbb{R}^{pd}}\|\widetilde{a}+b\|^{-d}\|\widetilde{b}-a\|^{-d}J_{d/2}\big(R\|\widetilde{a}+b\|\big)^2J_{d/2}\big(R\|\widetilde{b}-a\|\big)^2\\ &\quad\times|\mathscr{F}f_p|^2(-\boldsymbol{\eta_{p-r}},-\widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}})|\mathscr{F}f_p|^2(\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}},-\boldsymbol{\xi_r})\mu(d\widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}})\mu(d\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}})\bigg)^{1/2}.\end{aligned}$$

Recall that μ is symmetric. We can write, after the change of variable ($\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}} \to -\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}}$) and then applying Cauchy-Schwarz inequality,

$$\left|\int_{\mathbb{R}^{pd}\times\mathcal{D}_\delta^c}\right| \leq \mathbf{T}_0(R)\mathbf{K}_R,$$

where

$$\begin{aligned}\mathbf{K}_R := \int_{\mathbb{R}^{pd}\times\{\|a+b\|<\delta\}}&\mu(d\boldsymbol{\xi_r})\mu\big(d\boldsymbol{\eta_{p-r}}\big)\mu\big(d\widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}}\big)\mu\big(d\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}}\big)\|\widetilde{a}+b\|^{-d}\|a+\widetilde{b}\|^{-d}\\ &\times J_{d/2}\big(R\|\widetilde{a}+b\|\big)^2J_{d/2}\big(R\|a+\widetilde{b}\|\big)^2|\mathscr{F}f_p|^2(\boldsymbol{\eta_{p-r}},\widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}})|\mathscr{F}f_p|^2(\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}},\boldsymbol{\xi_r}).\end{aligned}$$

From previous discussion, it holds under hypothesis **(H2)** that

$$\sup\big\{\mathbf{T}_0(R) : R > 0\big\} < +\infty.$$

So it remains to show that $\mathbf{K}_R \to 0$, as $R \to +\infty$.

Making the following change of variables

$$\begin{aligned}\widetilde{a}+b \to x\,, &\quad (\boldsymbol{\eta_{p-r}},\widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}}) \to \Big(\boldsymbol{\eta_{p-r}},\widetilde{\boldsymbol{\xi}}_{\boldsymbol{r-1}}, x-\tau(\boldsymbol{\eta_{p-r}})-\tau(\widetilde{\boldsymbol{\xi}}_{\boldsymbol{r-1}})\Big)\\ \widetilde{b}+a \to y\,, &\quad (\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}},\boldsymbol{\xi_r}) \to \Big(\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r-1}}, y-\tau(\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r-1}})-\tau(\boldsymbol{\xi_r}),\boldsymbol{\xi_r}\Big)\end{aligned}$$

yields

$$\mathbf{K}_R = \omega_d^2 \int_{\mathbb{R}^{2d}} dxdy \ell_R(x)\ell_R(y)\Psi_p^{(r,\delta)}(x,y),$$

where $\Psi_p^{(r,\delta)}(x,y)$ is defined in (2.16). By our hypothesis **(H3)**, we have as $R\to+\infty$, that $\omega_d^{-2}\mathbf{K}_R$ is convergent to

$$\begin{aligned}
&\Psi_p^{(r,\delta)}(0,0) \\
&= \int_{\mathbb{R}^{2pd-2d}} d\boldsymbol{\xi_r} d\boldsymbol{\eta_{p-r}} d\widetilde{\boldsymbol{\xi}}_{\boldsymbol{r-1}} d\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r-1}} |\mathscr{F}f_p|^2\Big(\boldsymbol{\eta_{p-r}}, \widetilde{\boldsymbol{\xi}}_{\boldsymbol{r-1}}, -\tau(\boldsymbol{\eta_{p-r}}) - \tau(\widetilde{\boldsymbol{\xi}}_{\boldsymbol{r-1}})\Big) \\
&\times |\mathscr{F}f_p|^2\Big(\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r-1}}, -\tau(\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r-1}}) - \tau(\boldsymbol{\xi_r}), \boldsymbol{\xi_r}\Big) \left(\prod_{i=1}^{r-1}\varphi(\xi_i)\varphi(\widetilde{\xi}_i)\right) \varphi\big(\tau(\boldsymbol{\eta_{p-r}}) + \tau(\widetilde{\boldsymbol{\xi}}_{\boldsymbol{r-1}})\big) \\
&\times \varphi(\xi_r)\varphi(\eta_{p-r})\varphi\big(\tau(\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r-1}}) + \tau(\boldsymbol{\xi_r})\big) \left(\prod_{j=1}^{p-r-1}\varphi(\eta_j)\varphi(\widetilde{\eta}_j)\right) \mathbf{1}_{\{\|\tau(\boldsymbol{\xi_r}) + \tau(\boldsymbol{\eta_{p-r}})\| < \delta\}},
\end{aligned}$$

which converges to zero, as $\delta\downarrow 0$. This concludes our proof. □

Recall the Hilbert-space notation $\mathfrak{H}_\mu$ and $\mathfrak{H}_\mu^{\otimes p}$ from the beginning of Section 2. It is clear that

$$\boldsymbol{\xi_p} \in \mathbb{R}^{pd} \longmapsto F_R(\boldsymbol{\xi_p}) := (\mathscr{F}f_p)(\boldsymbol{\xi_p})\|\tau(\boldsymbol{\xi_p})\|^{-d/2} J_{d/2}(R\|\tau(\boldsymbol{\xi_p})\|)$$

belongs to $\mathfrak{H}_\mu^{\otimes p}$ for each $R>0$, since $\mathscr{F}f_p \in \mathfrak{H}_\mu^{\otimes p}$ and $\|\tau(\boldsymbol{\xi_p})\|^{-d/2}J_{d/2}(R\|\tau(\boldsymbol{\xi_p})\|)$ is uniformly bounded for any given $R>0$ (see Lemma 2.1). We can also define the corresponding contractions in this framework. For $h_1\in\mathfrak{H}_\mu^{\otimes p}$ and $h_2\in\mathfrak{H}_\mu^{\otimes q}$ ($p,q\in\mathbb{N}$), their r-contraction, with $0\le r\le p\wedge q$, belongs to $\mathfrak{H}_\mu^{\otimes p+q-2r}$ and is defined by

$$(h_1\otimes_{r,\mu} h_2)\big(\boldsymbol{\xi_{p-r}}, \boldsymbol{\eta_{p-r}}\big) = \int_{\mathbb{R}^{rd}} h_1(\boldsymbol{\xi_{p-r}}, \boldsymbol{a_r})\overline{h_2}\big(\boldsymbol{\eta_{p-r}}, \boldsymbol{a_r}\big)\,\mu(d\boldsymbol{a_r}).$$

One should not confuse this notion with the one introduced in **Notation A**.

With the notation F_R and $\otimes_{r,\mu}$, we can rewrite $\mathcal{I}_R$ in (2.17) as follows:

$$\begin{aligned}
\mathcal{I}_R &= \int_{\mathbb{R}^{2pd}} d\mu\; F_R(\boldsymbol{\eta_{p-r}}, \boldsymbol{\xi_r})\overline{F_R}(\boldsymbol{\eta_{p-r}}, \widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}})\overline{F_R}(\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}}, \boldsymbol{\xi_r})F_R(\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}}, \widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}}) \\
&= \int_{\mathbb{R}^{2pd}} \mu(d\boldsymbol{\eta_{p-r}})\mu(d\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}})\big(F_R\otimes_{r,\mu}F_R\big)\big(\boldsymbol{\eta_{p-r}}, \widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}}\big)\big(F_R\otimes_{r,\mu}F_R\big)\big(\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}}, \boldsymbol{\eta_{p-r}}\big) \\
&= \big\|F_R\otimes_{r,\mu}F_R\big\|^2_{\mathfrak{H}_\mu^{\otimes 2p-2r}},
\end{aligned}$$

where we used the fact that $\big(F_R\otimes_{r,\mu}F_R\big)\big(\boldsymbol{\eta_{p-r}}, \widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}}\big) = \overline{\big(F_R\otimes_{r,\mu}F_R\big)}\big(\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}}, \boldsymbol{\eta_{p-r}}\big)$, which follows simply from the definition of contraction. Hence, we can formulate the following Fourth Moment Theorem.

Theorem 2.13. *Fix an integer $p \geq 2$ and $f_p \in \mathfrak{H}^{\odot p}$. Assume* **(H2)***, which implies that, in view of* (2.12)*,*

$$\sigma_p^2 := p!(2\pi)^d \lim_{R\to+\infty} \big\|F_R\big\|^2_{\mathfrak{H}_\mu^{\otimes p}} \in [0,+\infty)\,. \tag{2.19}$$

Then, the following statements are equivalent:

(S1) $\dfrac{G_{p,R}}{R^{d/2}}$ *converges in law to* $N(0,\sigma_p^2)$*, as* $R\to+\infty$*;*

(S2) $\mathbb{E}\big[G_{p,R}^4\big]R^{-2d}$ *converges to* $3\sigma_p^4$*, as* $R\to+\infty$*;*

(S3) *For every* $r\in\{1,\dots,p-1\}$*,* $\|F_R\otimes_{r,\mu}F_R\|_{\mathfrak{H}_\mu^{\otimes 2p-2r}}\to 0$*, as* $R\to+\infty$*.*

Remark 2.14. (i) Recall from Lemma 2.1 that on $\mathbb{R}_+$, $J_{d/2}(x)\leq C\big(1\wedge\frac{1}{\sqrt{x}}\big)$. Therefore, we obtain the following estimates:

$$\|F_R\otimes_{r,\mu}F_R\|_{\mathfrak{H}_\mu^{\otimes 2p-2r}} \leq C\big\|F^{(1)}\otimes_{r,\mu}F^{(1)}\big\|_{\mathfrak{H}_\mu^{\otimes 2p-2r}}$$

and

$$\|F_R\otimes_{r,\mu}F_R\|_{\mathfrak{H}_\mu^{\otimes 2p-2r}} \leq \frac{C}{R^2}\big\|F^{(2)}\otimes_{r,\mu}F^{(2)}\big\|_{\mathfrak{H}_\mu^{\otimes 2p-2r}},$$

with $F^{(j)}(\boldsymbol{\xi_p}) := |\mathscr{F}f_p|(\boldsymbol{\xi_p})\|\tau(\boldsymbol{\xi_p})\|^{-\frac{d+j-1}{2}}$, $j=1,2$. As a consequence,

(1) if $\|F^{(1)}\otimes_{r,\mu}F^{(1)}\|_{\mathfrak{H}_\mu^{\otimes 2p-2r}}<\infty$ and μ admits a spectral density, then by the dominated convergence theorem, we have $\|F_R\otimes_{r,\mu}F_R\|_{\mathfrak{H}_\mu^{\otimes 2p-2r}}\to 0$, which implies the Gaussian fluctuation;

(2) if $\|F^{(2)}\otimes_{r,\mu}F^{(2)}\|_{\mathfrak{H}_\mu^{\otimes 2p-2r}}<\infty$, we deduce from (1.6) that

$$d_{\rm TV}\Big(G_{p,R}/\sigma_{p,R}, N(0,1)\Big)\leq C/R\,.$$

(ii) In view of the Cauchy-Schwarz inequality for contractions, one has

$$\|F^{(j)}\otimes_{r,\mu}F^{(j)}\|_{\mathfrak{H}_\mu^{\otimes 2p-2r}}\leq \|F^{(j)}\|^2_{\mathfrak{H}_\mu^{\otimes p}}\quad \text{for } j=1,2.$$

So one may intend to assume

$$\|F^{(1)}\|_{\mathfrak{H}_\mu^{\otimes p}}\wedge\|F^{(2)}\|_{\mathfrak{H}_\mu^{\otimes p}}<\infty\,, \tag{2.20}$$

which, however, is not reasonable in our framework. In fact, (2.19) and (2.4) tell us that $\|F_R\|^2_{\mathfrak{H}_\mu^{\otimes p}}$, which is equal to

$$\frac{R^d}{2^d\Gamma(\frac{d}{2}+1)^2}\int_{\{\tau(\boldsymbol{\xi_p})=0\}}|\mathscr{F}f_p|^2(\boldsymbol{\xi_p})\mu(d\boldsymbol{\xi_p}) + \int_{\{\|\tau(\boldsymbol{\xi_p})\|>0\}}|\mathscr{F}f_p|^2(\boldsymbol{\xi_p})\ell_R(\tau(\boldsymbol{\xi_p}))\mu(d\boldsymbol{\xi_p}),$$

converges to $\frac{\sigma_p^2}{p!(2\pi)^d}$; if we assume (2.20) or we assume the weaker condition

$$\int_{\mathbb{R}^{pd}} \left(\|\tau(\boldsymbol{\xi_p})\|^{-d-1} \wedge \|\tau(\boldsymbol{\xi_p})\|^{-d}\right) |\mathscr{F} f_p|^2(\boldsymbol{\xi_p})\, \mu(d\boldsymbol{\xi_p}) < \infty\,,$$

then the integral over $\{\|\tau(\boldsymbol{\xi_p})\| > 0\}$ vanishes asymptotically, so that we can write

$$\frac{R^d}{2^d\Gamma(\frac{d}{2}+1)^2} \int_{\{\tau(\boldsymbol{\xi_p})=0\}} |\mathscr{F} f_p|^2(\boldsymbol{\xi_p})\, \mu(d\boldsymbol{\xi_p}) \xrightarrow{R\to+\infty} \frac{\sigma_p^2}{p!(2\pi)^d}\,. \tag{2.21}$$

This forces the integral in (2.21) to be zero by dominated convergence, so that $\sigma_p^2 = 0$.

2.3 Chaotic central limit theorems

As a continuation of previous section, we consider the case of infinitely many chaoses and we derive a chaotic central limit theorem. Recall $F \in L^2(\Omega)$ admits the following chaos expansion (1.2) with Hermite rank $m \geq 1$:

$$F(W) = \sum_{p\geq m} I_p^W(f_p) \quad \text{with} \quad f_p \in \mathfrak{H}^{\odot p}\,.$$

Let us introduce the following natural hypothesis:

$$\textbf{(H4)} \qquad \sum_{p\geq m} p! \int_{\mathbb{R}^{2pd}} d\boldsymbol{t_p}\, d\boldsymbol{s_p}\, |f_p|(\boldsymbol{s_p})|f_p|(\boldsymbol{t_p}) \int_{\mathbb{R}^d} \prod_{i=1}^p |\gamma|\big(t_i - s_i + z\big) dz < \infty.$$

Recall the notation κ_p from (2.8) and we put

$$\|f_p\|_{\kappa_p}^2 := \int_{\mathbb{R}^{2dp}} f_p(\boldsymbol{s_p}) f_p(\boldsymbol{t_p}) \kappa_p(\boldsymbol{t_p} - \boldsymbol{s_p})\, d\boldsymbol{t_p}\, d\boldsymbol{s_p}\,.$$

So under **(H4)**,

$$\sigma^2 := \omega_d \sum_{p\geq m} p! \|f_p\|_{\kappa_p}^2 \in [0,\infty). \tag{2.22}$$

Note that an immediate consequence of our hypothesis **(H4)** is the following result

$$\lim_{N\to+\infty} \sup_{R>0} R^{-d} \sum_{q\geq N+1} \mathrm{Var}\left(\int_{B_R} I_p^W(f_p^x)\, dx\right) = 0\,. \tag{2.23}$$

In fact, one can write, similarly as before,

$$\begin{aligned} &\sup_{R>0} \frac{1}{\omega_d R^d} \sum_{q\geq N+1} \mathrm{Var}\left(\int_{B_R} I_p^W(f_p^x)\, dx\right) \\ &= \sum_{q\geq N+1} p! \int_{\mathbb{R}^{2pd}} d\boldsymbol{t_p}\, d\boldsymbol{s_p}\, f_p(\boldsymbol{s_p}) f_p(\boldsymbol{t_p}) \left(\int_{\mathbb{R}^d} \frac{\mathrm{vol}\big(B_R \cap B_R(-z)\big)}{\mathrm{vol}(B_R)} \prod_{i=1}^p \gamma\big(t_i - s_i + z\big) dz\right) \end{aligned}$$

$$\leq \sum_{q\geq N+1} p! \int_{\mathbb{R}^{2pd}} d\boldsymbol{t_p}\, d\boldsymbol{s_p} \left|f_p(\boldsymbol{s_p})f_p(\boldsymbol{t_p})\right| \left(\int_{\mathbb{R}^d} \prod_{i=1}^p |\gamma|\big(t_i - s_i + z\big)dz\right) \xrightarrow{N\to+\infty} 0\,.$$

Now we state our main result as a consequence of (2.23), Theorems 2.5 and 1.3.

Theorem 2.15. *Suppose $F \in L^2(\Omega)$ admits the chaos expansion* (1.2) *with Hermite rank $m \geq 2$ and assume that* **(H4)** *is satisfied. Suppose that for each $p \geq m$, the kernel $f_p \in \mathfrak{H}^{\odot p}$ satisfies* (i) *or* (ii) *in Theorem 2.5. Let σ^2 be given by* (2.22)*. Then, as $R \to +\infty$,*

$$R^{-d/2} \int_{B_R} U_x F(W)\, dx \quad \textit{converges in law to} \quad N(0, \sigma^2)\,.$$

Remark 2.16. (1) In Theorem 2.15, we exclude the first chaos for the following obvious reason. Under the assumption that $\{f_1, \gamma\} \subset L^1(\mathbb{R}^d)$, $R^{-d/2}\int_{B_R} I_1^W(f_1^x)dx$ is a centered Gaussian random variable with variance tending to $\omega_d\|f_1\|^2_{L^1(\mathbb{R}^d)}\int_{\mathbb{R}^d}\gamma(z)dz$, as $R \to +\infty$; see point (i) in Remark 2.4.

(2) Suppose $\gamma(0) < +\infty$ or equivalently $\mu(\mathbb{R}^d) < +\infty$, then $\gamma = \mathscr{F}\mu$ is a function bounded by $\gamma(0)$. If $\gamma \in L^m(\mathbb{R}^d)$ (for some integer $m \geq 1$), then $\gamma \in L^p(\mathbb{R}^d)$ for any $p \geq m$, so that $\|\gamma\|^p_{L^p(\mathbb{R}^d)} \leq \gamma(0)^{p-m}\|\gamma\|^m_{L^m(\mathbb{R}^d)}$. As a result,

$$\begin{aligned}&\sum_{p\geq m} p! \int_{\mathbb{R}^{2pd}} d\boldsymbol{t_p}\, d\boldsymbol{s_p}\, |f_p|(\boldsymbol{s_p})|f_p|(\boldsymbol{t_p}) \int_{\mathbb{R}^d} \prod_{i=1}^p |\gamma|\big(t_i - s_i + z\big)dz \\ &\leq \sum_{p\geq m} p!\|\gamma\|^p_{L^p(\mathbb{R}^d)}\|f_p\|^2_{L^1(\mathbb{R}^{pd})} \leq C\sum_{p\geq m} p!\gamma(0)^p\|f_p\|^2_{L^1(\mathbb{R}^{pd})}.\end{aligned}$$

This tells us that condition (1.7) implies **(H4)**, so Theorem 1.4 stands as an easy corollary of our Theorem 2.15 and previous point (1).

We can formulate another chaotic CLT based on the spectral measure.

Theorem 2.17. *Suppose that $F \in L^2(\Omega)$ admits the chaos expansion* (1.2) *with Hermite rank $m \geq 1$. Assume that the spectral measure has a density. Suppose that for each $p \geq m$, the function Ψ_p defined in* (2.13) *is continuous at zero and the following boundedness condition holds (which implies* **(H2)** *for each p):*

$$\textbf{(H4}'\textbf{)} \qquad \sum_{p\geq m} p!\|\Psi_p\|_\infty < \infty.$$

Assume additionally that hypothesis **(H3)** *holds for each $p \geq m$. Then,*

$$R^{-d/2} \int_{B_R} U_x F(W)\, dx \xrightarrow[\text{law}]{R\to+\infty} N\left(0, (2\pi)^d\omega_d \sum_{p\geq m} p!\Psi_p(0)\right)\,.$$

Proof. For $m = 1$, we should consider the first chaos and it is clear that $R^{-d/2}G_{1,R}$ is centered Gaussian with variance tending to $\omega_d(2\pi)^d\Psi_1(0)$.

Now let us consider higher-order chaoses. For each $p \geq m \vee 2$, hypotheses **(H2)** and **(H3)** hold true. This implies that $G_{p,R}R^{-d/2}$ converges in law to $N(0, \sigma_p^2)$, with σ_p

introduced in Theorem 2.5. In view of the chaotic central limit theorem (Theorem 1.3), it remains to check condition (2.23). We can write

$$\sum_{p\geq N+1}\frac{\mathrm{Var}\big(G_{p,R}\big)}{\omega_d R^d}=(2\pi)^d\sum_{p\geq N+1}p!\int_{\mathbb{R}^d}\ell_R(x)\Psi_p(x)\,dx\leq(2\pi)^d\sum_{p\geq N+1}p!\|\Psi_p\|_\infty\,,$$

where the last inequality follows from the fact that $\ell_R(x)dx$ is a probability measure on $\mathbb{R}^d$; so hypothesis **(H4')** implies (2.23). Hence, our proof is finished. □

Corollary 2.18. *Suppose that $F\in L^2(\Omega)$ admits the chaos expansion* (1.2) *with Hermite rank $m\geq 1$ and for each $p\geq m$, the kernel f_p belongs to $L^1(\mathbb{R}^{pd})\cap\mathfrak{H}^{\odot p}$. Assume that the spectral measure μ is finite with spectral density φ such that φ is uniformly bounded with continuity at zero and*

$$\sum_{p\geq m}p!\|\mathscr{F}f_p\|_\infty^2\|\varphi\|_{L^1(\mathbb{R}^d)}^p<\infty\,. \tag{2.24}$$

Then, $R^{-d/2}\int_{B_R}U_xF(W)\,dx\xrightarrow[\text{law}]{R\to+\infty}N\left(0,(2\pi)^d\omega_d\sum_{p\geq m}p!\Psi_p(0)\right)$.

Proof. Note that μ is finite, which is equivalent to $\varphi\in L^1(\mathbb{R}^d)$. This implies with boundedness of φ that $\varphi\in L^q(\mathbb{R}^d)$ for any $q>1$. It is clear that for any $p\geq 2\vee m$, $f_p\in L^1(\mathbb{R}^d)\cap\mathfrak{H}^{\odot p}$ and $\gamma\in L^{p/(p-1)}(\mathbb{R}^d)$, so Lemma 2.10 and Lemma 2.8 ensure that hypotheses **(H2)** and **(H3)** are valid on the pth chaos.

If F has the first chaos with $f_1\in L^1(\mathbb{R}^d)$, then Ψ_1 is uniformly bounded with continuity at zero (the continuity of φ at zero is only required at this point). Therefore, $G_{1,R}R^{-d/2}$ converges in law to a centered Gaussian with variance $(2\pi)^d\Psi_1(0)$.

It remains to notice that $\Psi_p(x)\leq\big\||\mathscr{F}f_p|^2\big\|_\infty\varphi^{*p}(x)\leq\big\||\mathscr{F}f_p|^2\big\|_\infty\|\varphi\|_{L^{p/(p-1)}(\mathbb{R}^d)}^p$ by (2.5). We know that $\|\varphi\|_{L^{p/(p-1)}(\mathbb{R}^d)}^p\leq\|\varphi\|_\infty\|\varphi\|_{L^1(\mathbb{R}^d)}^{p-1}$ so that **(H4')** holds in this setting. To see this, we write

$$\sum_{p\geq m}p!\|\Psi_p\|_\infty\leq C\sum_{p\geq m}p!\|\mathscr{F}f_p\|_\infty^2\|\varphi\|_{L^1(\mathbb{R}^d)}^p\,,$$

that is, **(H4')** is implied by (2.24). Hence, the proof is done by applying Theorem 2.17. □

3 Proof of Theorems 1.6, 1.7 and 1.9

Let $u_{t,x}$ be the mild solution to the linear stochastic heat equation (1.8) with initial condition $u_{0,x}=1$ for all $x\in\mathbb{R}^d$, driven by a Gaussian noise with temporal and spatial covariance kernels being γ_0 and γ_1, respectively. We assume $\gamma_0:\mathbb{R}\to[0,\infty]$ locally integrable and the Fourier transform of γ_1 is a nonnegative tempered measure μ_1 that satisfies the Dalang's condition (1.10).

Recall that

$$A_t(R)=\int_{B_R}\big(u_{t,x}-1\big)\,dx=\sum_{p=1}^\infty I_p^W\left(\int_{B_R}f_{t,x,p}dx\right),$$

where, for any integer $p \geq 1$, $f_{t,x,p}$ is the kernel appearing in the Wiener chaos expansion of $u_{t,x}$ (see (1.11)).

Let us introduce some notation for later convenience.

Notation B. For given $t > 0$ and $p \in \mathbb{N}$, $\Delta_p(t) = \{\boldsymbol{s_p} \in \mathbb{R}_+^p : t > s_1 > \ldots > s_p > 0\}$ and $\mathrm{SIM}_p(t) = \{\boldsymbol{s_p} \in \mathbb{R}_+^p : s_1 + \cdots + s_p \leq t\}$. For $\sigma \in \mathfrak{S}_p$, we write $\boldsymbol{x_p^\sigma} = (x_1^\sigma, \ldots, x_p^\sigma) = (x_{\sigma(1)}, \ldots, x_{\sigma(p)})$, so $\boldsymbol{s_p^\sigma} \in \Delta_p(t)$ means $t > s_{\sigma(1)} > \cdots > s_{\sigma(p)}$ and we write $\int_{\Delta_p(t)} d\boldsymbol{s_p^\sigma}$ for $\int_{[0,t]^p} d\boldsymbol{s_p} \mathbf{1}_{\Delta_p(t)}(\boldsymbol{s_p^\sigma})$. For fixed integers $1 \leq r \leq p-1$, the r-contraction $f \otimes_r g$ of $f, g \in \mathscr{H}^{\otimes p}$ is the element in $\mathscr{H}^{\otimes 2p-2r}$ given by

$$
\begin{aligned}
\big(f \otimes_r g\big)\big(\boldsymbol{s_{p-r}}, \widetilde{\boldsymbol{s}}_{\boldsymbol{p-r}}, \boldsymbol{\xi_{p-r}}, \widetilde{\boldsymbol{\xi}}_{\boldsymbol{p-r}}\big) &= \int_{\mathbb{R}_+^{2r}} d\boldsymbol{a_r} d\widetilde{\boldsymbol{a}}_{\boldsymbol{r}} \left(\prod_{i=1}^r \gamma_0(a_i - \widetilde{a}_i)\right) \int_{\mathbb{R}^{2dr}} d\boldsymbol{x_r} d\widetilde{\boldsymbol{x}}_{\boldsymbol{r}} \\
&\times \left(\prod_{i=1}^r \gamma_0(x_i - \widetilde{x}_i)\right) f\big(\boldsymbol{s_{p-r}}, \boldsymbol{a_r}, \boldsymbol{\xi_{p-r}}, \boldsymbol{x_r}\big) g\big(\widetilde{\boldsymbol{s}}_{\boldsymbol{p-r}}, \widetilde{\boldsymbol{a}}_{\boldsymbol{r}}, \widetilde{\boldsymbol{\xi}}_{\boldsymbol{p-r}}, \widetilde{\boldsymbol{x}}_{\boldsymbol{r}}\big),
\end{aligned}
$$

which may be a generalized function.

Here is the plan for the proof of Theorems 1.6 and 1.7. Section 3.1 deals with computing the limit of the covariance function of the process $A_t(R)$ as $R \to +\infty$, provided that $\gamma_1(\mathbb{R}^d)$ is finite. Section 3.2 is devoted to the proof of the *convergence of the finite-dimensional distributions*, and we prove the *tightness* of $\{R^{-d/2} A_t(R), t \geq 0\}$ in Section 3.3 under the extra assumption (1.14). As a by-product of the computations in Section 3.1, we provide a proof of Theorem 1.7 in Section 3.4.

3.1 Limiting covariance structure in Theorem 1.6

The main ingredient is the following *Feymann-Kac representation*.

Lemma 3.1 (Feynman-Kac formula)**.** *Let γ_0, γ_1 be given as in Theorem 1.6 and we fix $t, s > 0$. Then for any $x, y \in \mathbb{R}^d$, we have*

$$
\phi_{t,s}(x-y) := \mathbb{E}\big[u_{t,x} u_{s,y}\big] = \mathbb{E}\left[e^{\beta_{t,s}(x-y)}\right]
$$

with

$$
\beta_{t,s}(z) := \int_0^t \int_0^s \gamma_0(u-v)\gamma_1(X_u^1 - X_v^2 + z) du dv,
$$

where X^1, X^2 are two independent standard Brownian motions on $\mathbb{R}^d$ that start at zero.

We refer to [9, Theorem 3.6] for the proof of a more general statement. We point out that in this reference, the moment formula is stated for $x = y$ and $t = s$, see equation (3.21) therein; one can prove the case $x \neq y$ or $t \neq s$ verbatim.

It follows from Lemma 3.1 that

$$
\begin{aligned}
\Sigma_{s,t} : &= \lim_{R+\infty} R^{-d} \mathbb{E}\big[A_t(R) A_s(R)\big] = \lim_{R\to+\infty} R^{-d} \int_{B_R^2} \big(\phi_{t,s}(x-y) - 1\big) dx dy \\
&= \lim_{R\to+\infty} R^{-d} \int_{\mathbb{R}^d} \big(\phi_{t,s}(z) - 1\big) \mathrm{vol}\big(B_R \cap B_R(-z)\big) dz = \omega_d \int_{\mathbb{R}^d} \big(\phi_{t,s}(z) - 1\big) dz,
\end{aligned}
$$

provided the integral $\int_{\mathbb{R}^d}\big(\phi_{t,s}(z)-1\big)dz$ is finite. Note that in our setting, $\phi(z)\geq 1$ for every $z\in\mathbb{R}^d$; note also that, since γ_1 is integrable,

$$\begin{aligned}\int_{\mathbb{R}^d}\big(\phi_{t,s}(z)-1\big)dz &\geq \int_{\mathbb{R}^d}\mathbb{E}[\beta_{t,s}(z)]dz\\ &= \left(\int_0^t\int_0^s\gamma_0(u-v)dudv\right)\int_{\mathbb{R}^d}\gamma_1(z)dz\in(0,\infty),\end{aligned}\tag{3.1}$$

where the equality follows from Fubini's theorem.

Note that

$$\int_{\mathbb{R}^d}\big(\phi_{t,s}(z)-1\big)dz=\sum_{p\geq 1}\frac{1}{p!}\int_{\mathbb{R}^d}\mathbb{E}\big[\beta_{t,s}(z)^p\big]dz,$$

where the object $\beta_{t,s}(z)$ can be understood as the "weighted" intersection local time of two independent Brownian motions X^1 and X^2.

In order to show that $\int_{\mathbb{R}^d}\big(\phi_{t,s}(z)-1\big)dz<\infty$, we first estimate the pth moment of $\beta_{t,s}(z)$. Without losing any generality, we assume $s\leq t$. Using that γ_1 is the Fourier transform of the spectral density φ_1, which is continuous and bounded due to the finiteness of $\gamma_1(\mathbb{R}^d)$, we can write

$$\begin{aligned}\mathbb{E}\big[\beta_{s,t}(z)^p\big]&=\int_{[0,s]^p\times[0,t]^p}\left(\prod_{j=1}^p\gamma_0(s_j-r_j)\right)\mathbb{E}\left[\prod_{j=1}^p\gamma_1(X^1_{s_j}-X^2_{r_j}+z)\right]d\boldsymbol{s_p}d\boldsymbol{r_p}\\ &=\int_{[0,s]^p\times[0,t]^p}\int_{\mathbb{R}^{pd}}d\boldsymbol{\xi_p}d\boldsymbol{s_p}d\boldsymbol{r_p}\left(\prod_{j=1}^p\gamma_0(s_j-r_j)\right)\left(\prod_{j=1}^p\varphi_1(\xi_j)\right)\\ &\qquad\times\mathbb{E}\left(\prod_{j=1}^p e^{-\mathbf{i}\xi_j\cdot(X^1_{s_j}-X^2_{r_j}+z)}\right)\\ &=\int_{[0,s]^p\times[0,t]^p}\int_{\mathbb{R}^{pd}}d\boldsymbol{\xi_p}d\boldsymbol{s_p}d\boldsymbol{r_p}\left(\prod_{j=1}^p\gamma_0(s_j-r_j)\right)\left(\prod_{j=1}^p\varphi_1(\xi_j)\right)e^{-\mathbf{i}z\cdot\tau(\boldsymbol{\xi_p})}\\ &\qquad\times\exp\left(-\frac{1}{2}\sum_{1\leq i,j\leq p}(s_i\wedge s_j+r_i\wedge r_j)\xi_i\cdot\xi_j\right),\end{aligned}\tag{3.2}$$

which is a nonnegative, uniformly continuous and uniformly bounded function in z. Indeed, it is clear that $0\leq\mathbb{E}[\beta_{s,t}(z)^p]\leq\mathbb{E}[(\beta_{s,t}(0)^p]<+\infty$ and the uniform continuity follows from the dominated convergence theorem. Then by the monotone convergence theorem, we write

$$\int_{\mathbb{R}^d}\mathbb{E}\big[\beta_{s,t}(z)^p\big]dz=\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^d}\mathbb{E}\big[\beta_{s,t}(z)^p\big]\exp\left(-\frac{\varepsilon}{2}\|z\|^2\right)dz\in[0,\infty].$$

Recall from (3.2) that the finiteness of $\mathbb{E}\big[\beta_{s,t}(0)^p\big]$ allows us to apply Fubini's theorem to get for any $\varepsilon>0$,

$$T_{p,\varepsilon}:=\int_{\mathbb{R}^d}\mathbb{E}\big[\beta_{s,t}(z)^p\big]\exp\left(-\frac{\varepsilon}{2}\|z\|^2\right)dz$$

$$= (2\pi)^d \int_{[0,s]^p\times[0,t]^p} \int_{\mathbb{R}^{pd}} d\boldsymbol{\xi_p} d\boldsymbol{s_p} d\boldsymbol{r_p} \left(\prod_{j=1}^p \gamma_0(s_j - r_j)\right) \left(\prod_{j=1}^p \varphi_1(\xi_j)\right)$$
$$\times G\big(\varepsilon, \tau(\boldsymbol{\xi_p})\big) \exp\left(-\frac{1}{2}\sum_{1\le i,j\le p}(s_i\wedge s_j + r_i\wedge r_j)\xi_i\cdot\xi_j\right),$$

which is finite.

Consider first the case $p \ge 2$. Using that $s \le t$ and

$$\exp\left(-\frac{1}{2}\sum_{1\le i,j\le p}(r_i\wedge r_j)\xi_i\cdot\xi_j\right) \le 1,$$

we can bound $T_{p,\varepsilon}$ as follows

$$T_{p,\varepsilon} \le (2\pi)^d \Gamma_t^p \int_{\mathbb{R}^{pd}} d\boldsymbol{\xi_p} \int_{[0,t]^p} d\boldsymbol{s_p} \left(\prod_{j=1}^p \varphi_1(\xi_j)\right)$$
$$\times G\big(\varepsilon, \tau(\boldsymbol{\xi_p})\big) \exp\left(-\frac{1}{2}\sum_{1\le i,j\le p}(s_i\wedge s_j)\xi_i\cdot\xi_j\right),$$

where the constant $\Gamma_t := \int_{-t}^t \gamma_0(u)du$ is finite for each $t > 0$ in view of the local integrability of γ_0. Making the change of variables $\boldsymbol{\xi_p} = (\eta_1 - \eta_2, \dots, \eta_{p-1} - \eta_p, \eta_p)$, yields, with the convention $s_{p+1} = 0$ and $\eta_0 = 0$,

$$T_{p,\varepsilon} \le (2\pi)^d \Gamma_t^p p! \int_{\mathbb{R}^{pd}} d\boldsymbol{\xi_p} \int_{\Delta_p(t)} d\boldsymbol{s_p} e^{-\frac{1}{2}\sum_{j=1}^p (s_j - s_{j+1})\|\xi_1+\cdots+\xi_j\|^2} G(\varepsilon, \tau(\boldsymbol{\xi_p})) \prod_{j=1}^p \varphi_1(\xi_j)$$
$$= (2\pi)^d \Gamma_t^p p! \int_{\mathbb{R}^d} d\eta_p G(\varepsilon, \eta_p) \int_{\mathbb{R}^{pd-d}} d\boldsymbol{\eta_{p-1}} \int_{\mathrm{SIM}_p(t)} d\boldsymbol{w_p} \prod_{j=1}^p \varphi_1(\eta_j - \eta_{j-1}) e^{-\frac{1}{2}w_j\|\eta_j\|^2}.$$

Put

$$Q_p(\eta_p) = \int_{\mathbb{R}^{pd-d}} d\boldsymbol{\eta_{p-1}} \int_{\mathrm{SIM}_p(t)} d\boldsymbol{w_p} \left(\prod_{j=1}^p \varphi_1(\eta_j - \eta_{j-1}) e^{-\frac{1}{2}w_j\|\eta_j\|^2}\right),$$

then we just obtained

$$T_{p,\varepsilon} \le (2\pi)^d \Gamma_t^p p! \int_{\mathbb{R}^d} d\eta_p G(\varepsilon, \eta_p) Q_p(\eta_p).$$

In the following, we will prove that $Q_p(\eta_p)$ is uniformly bounded and provide an estimate. We rewrite $Q_p(\eta_p)$ as follows. With $h_j(\eta) = \exp\big(-\frac{1}{2}w_j\|\eta\|^2\big)$,

$$Q_p(\eta_p) = \int_{\mathrm{SIM}_p(t)} d\boldsymbol{w_p}\ h_p(\eta_p) \int_{\mathbb{R}^{pd-d}} d\boldsymbol{\eta_{k-1}}\ \varphi_1(\eta_1)h_1(\eta_1)\varphi_1(\eta_2 - \eta_1)h_2(\eta_2)$$

$$\times \varphi_1(\eta_3-\eta_2)h_3(\eta_3)\times\cdots\times\varphi_1(\eta_{p-1}-\eta_{p-2})h_{p-1}(\eta_{p-1})\varphi_1(\eta_p-\eta_{p-1})\,.$$

Using that φ_1 is bounded, we get

$$\int_{\mathbb{R}^d}\varphi_1(\eta_1)\varphi_1(\eta_2-\eta_1)h_1(\eta_1)d\eta_1\le\|\varphi_1\|_\infty\int_{\mathbb{R}^d}\varphi_1(\eta_1)h_1(\eta_1)d\eta_1. \tag{3.3}$$

On the other hand, using (4.3), we have

$$\int_{\mathbb{R}^d}d\eta_j h_j(\eta_j)\varphi_1(\eta_{j+1}-\eta_j)\le\int_{\mathbb{R}^d}d\eta_j\varphi_1(\eta_j)h_j(\eta_j)$$

for $j=2,\dots,p-1$. So,

$$\begin{aligned}
Q_p(\eta_p)&\le\|\varphi_1\|_\infty\int_{\mathrm{SIM}_p(t)}d\boldsymbol{w_p}e^{-\frac12 w_p\|\eta_p\|^2}\prod_{j=1}^{p-1}\int_{\mathbb{R}^d}e^{-w_j\|\eta_j\|^2}\varphi_1(\eta_j)d\eta_j\\
&\le t\|\varphi_1\|_\infty\int_{\mathbb{R}^{pd-d}}\int_{\mathrm{SIM}_{p-1}(t)}\prod_{i=1}^{p-1}\varphi_1(\xi_i)e^{-\frac12 w_i\|\xi_i\|^2}d\boldsymbol{w_{p-1}}d\boldsymbol{\xi_{p-1}}\\
&\le t\|\varphi_1\|_\infty\sum_{j=0}^{p-1}\binom{p-1}{j}\frac{t^j}{j!}D_N^j(2C_N)^{p-1-j},
\end{aligned}\tag{3.4}$$

where the last inequality follows from Lemma 3.3 in [9], with the notation

$$C_N=\int_{\{\|\xi\|\ge N\}}\frac{\varphi_1(\xi)}{\|\xi\|^2}d\xi \tag{3.5}$$

and

$$D_N=\int_{\{\|\xi\|\le N\}}\varphi_1(\xi)d\xi.$$

Notice that these quantities are finite for any $N>0$ by condition (1.10). We fix N such that $0<4\Gamma_tC_N<1$. This gives us the uniform boundedness of Q_p and moreover,

$$T_{p,\varepsilon}\le(2\pi)^d\Gamma_t^p p!\|Q_p\|_\infty\le\|\varphi_1\|_\infty(2\pi)^d\Gamma_t^p p!t(4C_N)^{p-1}\exp\left(\frac{tD_N}{2C_N}\right),$$

which immediately implies

$$\int_{\mathbb{R}^d}\mathbb{E}\big[\beta_{s,t}(z)^p\big]dz\le\|\varphi_1\|_\infty(2\pi)^d\Gamma_t^p p!t(4C_N)^{p-1}\exp\left(\frac{tD_N}{2C_N}\right)<\infty \tag{3.6}$$

and

$$\begin{aligned}
\sum_{p\ge2}\frac{1}{p!}\int_{\mathbb{R}^d}\mathbb{E}\big[\beta_{s,t}(z)^p\big]dz&\le\frac{(2\pi)^d\|\varphi_1\|_\infty t}{4C_N}\exp\left(\frac{tD_N}{2C_N}\right)\sum_{p\ge2}(4\Gamma_tC_N)^p\\
&=\frac{4\|\varphi_1\|_\infty(2\pi)^d tC_N\Gamma_t^2}{1-4\Gamma_tC_N}\exp\left(\frac{tD_N}{2C_N}\right)
\end{aligned}\tag{3.7}$$

is finite, since $0 < 4\Gamma_t C_N < 1$.

To show the integrability of $\phi_{s,t} - 1$, it remains to check that

$$\int_{\mathbb{R}^d} \mathbb{E}\big[\beta_{s,t}(z)\big] dz < \infty, \tag{3.8}$$

which follows from (3.1). Therefore,

$$\int_{\mathbb{R}^d} \big(\phi_{s,t}(z) - 1\big) dz \leq t\Gamma_t \|\gamma_1\|_{L^1(\mathbb{R}^d)} + \frac{4\|\varphi_1\|_\infty (2\pi)^d t C_N \Gamma_t^2}{1 - 4\Gamma_t C_N} \exp\Big(\frac{tD_N}{2C_N}\Big) < \infty.$$

As a consequence, we proved that, for any $s, t \in \mathbb{R}_+$,

$$\lim_{R\to+\infty} \frac{\mathbb{E}[A_t(R)A_s(R)]}{R^d} = \Sigma_{s,t} = \omega_d \int_{\mathbb{R}^d} \big(\phi_{s,t}(z) - 1\big) dz \in (0, \infty).$$

3.2 Convergence of the finite-dimensional distributions in Theorem 1.6

Fix $0 < t_1 < \cdots < t_n < \infty$ and put

$$g_{q,R}(t) = R^{-d/2} \int_{B_R} f_{t,x,q} dx\,.$$

Then $A_R := R^{-d/2}\big(A_{t_1}(R), \ldots, A_{t_n}(R)\big)$ falls into the framework of the following Proposition 3.2, the multivariate chaotic central limit theorem borrowed from [3, Theorem 2.1].

Proposition 3.2. *Fix an integer $n \geq 1$ and consider a family $\big\{A_R, R > 0\big\}$ of random vectors in $\mathbb{R}^n$ such that each component of $A_R = (A_{R,1}, \ldots, A_{R,n})$ belongs to $L^2(\Omega, \sigma\{W\}, \mathbb{P})$ and has the following chaos expansion*

$$A_{R,j} = \sum_{q\geq 1} I_q^W(g_{q,j,R}) \quad \text{with } g_{q,j,R} \text{ symmetric kernels.}$$

Suppose the following conditions (a)-(d) *hold:*

(a) $\forall i, j \in \{1, \ldots, n\}$ *and* $\forall q \geq 1$, $\mathbb{E}\big[I_q^W(g_{q,j,R}) I_q^W(g_{q,i,R})\big] \xrightarrow{R\to+\infty} \sigma_{i,j,q}$.

(b) $\forall i \in \{1, \ldots, n\}$, $\displaystyle\sum_{q\geq 1} \sigma_{i,i,q} < \infty$.

(c) For any $1 \leq r \leq q - 1$, $\big\|g_{q,i,R} \otimes_r g_{q,i,R}\big\|_{\mathscr{H}^{\otimes(2q-2r)}} \xrightarrow{R\to+\infty} 0$.

(d) $\forall i \in \{1, \ldots, n\}$, $\displaystyle\lim_{N\to+\infty} \sup_{R>0} \sum_{q\geq N+1} \mathbb{E}\big[I_q^W(g_{q,i,R})^2\big] = 0$.

Then A_R converges in law to $N(0, \Sigma)$ as $R \to +\infty$, where $\Sigma = \big(\sigma_{i,j}\big)_{i,j=1}^n$ is given by $\sigma_{i,j} = \sum_{q\geq 1} \sigma_{i,j,q}$.

Proof of conditions (a), (b) **and** (d): It suffices to prove that for any $t, s \in \mathbb{R}_+$ and for any $p \geq 1$, $p!\langle g_{p,R}(t), g_{p,R}(s)\rangle_{\mathscr{H}^{\otimes p}}$ is convergent to some limit, denoted by $\sigma_p(t,s)$ and for each $t \geq 0$,

$$\sum_{p\geq 1} \sigma_p(t,t) < +\infty \tag{3.9}$$

and

$$\lim_{N\to+\infty} \sup_{R>0} \sum_{q\geq N+1} p!\|g_{p,R}(t)\|^2_{\mathscr{H}^{\otimes p}} = 0. \tag{3.10}$$

It is well-known in the literature that the pth moment of $\beta_{t,t}(0)$ coincides with the variance of the pth chaotic component of the solution $u_{t,x}$; see for instance [12]. Then, it is natural to expect that our verification of condition (a) in Proposition 3.2 will resemble the computations we have done for $\mathbb{E}\big[\beta_{t,s}(z)^p\big]$. Moreover, we will see that condition (3.9) is a consequence of the finiteness of the integral $\int_{\mathbb{R}^d}\big(\phi_{t,s}(z)-1\big)dz$ proved in Section 3.1. The verification of condition (3.10) will be straightforward, as a by-product of the computations in Section 3.1.

Let us start with the case $p = 1$. By an easy computation,

$$\begin{aligned}\langle g_{1,R}(t), g_{1,R}(s)\rangle_{\mathscr{H}} &= R^{-d}\int_{B_R^2} \langle G(t-\bullet, x-\bullet), G(s-\bullet, y-\bullet)\rangle_{\mathscr{H}} dxdy \\ &= (2\pi)^d\omega_d \int_0^t\int_0^s dudv\gamma_0(u-v)\int_{\mathbb{R}^d} d\xi\ \ell_R(\xi)\varphi_1(\xi)e^{-\frac12(t-u+s-v)\|\xi\|^2}, \end{aligned} \tag{3.11}$$

where $\ell_R(\xi)$ is the approximation of the identity introduced in Point (3) of Lemma 2.1. Since γ_1 is integrable on $\mathbb{R}^d$, φ_1 is uniformly continuous and uniformly bounded. Then, taking the limit as $R\to+\infty$ in (3.11), yields

$$\langle g_{1,R}(t), g_{1,R}(s)\rangle_{\mathscr{H}} \xrightarrow{R\to+\infty} (2\pi)^d\omega_d\varphi_1(0)\int_0^t\int_0^s dudv\gamma_0(u-v) = \sigma_1(t,s).$$

Notice that $\sigma_1(t,s) = \omega_d\int_{\mathbb{R}^d}\mathbb{E}\big[\beta_{s,t}(z)\big]dz$, in view of (3.1) and $(2\pi)^d\varphi_1(0) = \gamma_1(\mathbb{R}^d)$.

Now let us consider higher-order chaos. For a fixed $p \geq 2$, we write

$$\mathbb{E}\Big[I_p^W\big(g_{p,R}(t)\big)I_p^W\big(g_{p,R}(s)\big)\Big] = \frac{p!}{R^d}\int_{B_R^2} dxdy\ \big\langle f_{t,x,p}, f_{s,y,p}\big\rangle_{\mathscr{H}^{\otimes p}}.$$

The kernel $f_{t,x,p}$ is a nonnegative function on $\mathbb{R}_+^p\times\mathbb{R}^{pd}$, so $\langle f_{t,x,p}, f_{s,y,p}\rangle_{\mathscr{H}^{\otimes p}} \geq 0$. We first write, by using the Fourier transform in space,

$$\begin{aligned}&\big\langle f_{t,x,p}, f_{s,y,p}\big\rangle_{\mathscr{H}^{\otimes p}} \\ &= \int_{\mathbb{R}_+^{2p}} d\boldsymbol{s_p}d\widetilde{\boldsymbol{s}}_{\boldsymbol{p}}\prod_{j=1}^p\gamma_0(s_j-\widetilde{s}_j)\int_{\mathbb{R}^{pd}}\mu_1(\boldsymbol{\xi_p})\mathscr{F}f_{t,x,p}(\boldsymbol{s_p},\boldsymbol{\xi_p})\mathscr{F}f_{s,y,p}(\widetilde{\boldsymbol{s}}_{\boldsymbol{p}},-\boldsymbol{\xi_p}).\end{aligned} \tag{3.12}$$

Note that for $\boldsymbol{s}_{\boldsymbol{p}}^{\boldsymbol{\sigma}} \in \Delta_p(t)$, by the change of variables $y_1 = x_1^\sigma - x$, $y_j = x_j^\sigma - x_{j-1}^\sigma$ for $j \geq 2$, we can write, with X^1 standard Brownian motion on $\mathbb{R}^d$ as before,

$$\begin{aligned}
&\mathbf{1}_{\Delta_p(t)}(\boldsymbol{s}_{\boldsymbol{p}}^{\boldsymbol{\sigma}}) \int_{\mathbb{R}^{dp}} d\boldsymbol{x}_{\boldsymbol{p}}^{\boldsymbol{\sigma}} e^{-\mathbf{i}\boldsymbol{x}_{\boldsymbol{p}}^{\boldsymbol{\sigma}}\cdot\boldsymbol{\xi}_{\boldsymbol{p}}^{\boldsymbol{\sigma}}} G(t-s_1^\sigma, x-x_1^\sigma) \prod_{i=1}^{p-1} G(s_i^\sigma - s_{i+1}^\sigma, x_i^\sigma - x_{i+1}^\sigma) \\
&= \mathbf{1}_{\Delta_p(t)}(\boldsymbol{s}_{\boldsymbol{p}}^{\boldsymbol{\sigma}}) e^{-\mathbf{i}x\cdot\tau(\boldsymbol{\xi}_{\boldsymbol{p}})} \mathbb{E}\left[\prod_{j=1}^{p} \exp\left(-\mathbf{i}(X_t^1 - X_{s_j^\sigma}^1)\cdot\xi_j^\sigma\right)\right] \\
&= \mathbf{1}_{\Delta_p(t)}(\boldsymbol{s}_{\boldsymbol{p}}^{\boldsymbol{\sigma}}) e^{-\mathbf{i}x\cdot\tau(\boldsymbol{\xi}_{\boldsymbol{p}})} \mathbb{E}\left[\prod_{j=1}^{p} \exp\left(-\mathbf{i}(X_t^1 - X_{s_j}^1)\cdot\xi_j\right)\right], \qquad (3.13)
\end{aligned}$$

so that

$$\mathscr{F} f_{t,x,p}(\boldsymbol{s}_{\boldsymbol{p}}, \boldsymbol{\xi}_{\boldsymbol{p}}) = \frac{1}{p!} e^{-\mathbf{i}x\cdot\tau(\boldsymbol{\xi}_{\boldsymbol{p}})} \mathbb{E}\left[\prod_{j=1}^{p} \exp\left(-\mathbf{i}(X_t^1 - X_{s_j}^1)\cdot\xi_j\right)\right],$$

for $\boldsymbol{s}_{\boldsymbol{p}} \in [0,t]^p$ and

$$\mathscr{F} f_{s,y,p}(\widetilde{\boldsymbol{s}}_{\boldsymbol{p}}, -\boldsymbol{\xi}_{\boldsymbol{p}}) = \frac{1}{p!} e^{\mathbf{i}y\cdot\tau(\boldsymbol{\xi}_{\boldsymbol{p}})} \mathbb{E}\left[\prod_{j=1}^{p} \exp\left(\mathbf{i}(X_s^1 - X_{\widetilde{s}_j}^1)\cdot\xi_j\right)\right] \text{ for } \widetilde{\boldsymbol{s}}_{\boldsymbol{p}} \in [0,s]^p.$$

Keeping in mind the above expressions and making the time changes in (3.12) (from s_j to $t - s_j$ and from $\widetilde{s}_j$ to $s - \widetilde{s}_j$, for $j = 1, \dots, p$) yields

$$\begin{aligned}
\left\langle f_{s,x,p}, f_{t,y,p}\right\rangle_{\mathscr{H}^{\otimes p}} &= \frac{1}{(p!)^2} \int_{[0,s]^p\times[0,t]^p} d\boldsymbol{s}_{\boldsymbol{p}} d\boldsymbol{r}_{\boldsymbol{p}} \prod_{j=1}^{p} \gamma_0(t - s_j - s + r_j) \int_{\mathbb{R}^{pd}} \mu_1(\boldsymbol{\xi}_{\boldsymbol{p}}) e^{-\mathbf{i}(x-y)\cdot\tau(\boldsymbol{\xi}_{\boldsymbol{p}})} \\
&\times \mathbb{E}\left[\prod_{j=1}^{p} \exp\left(-\mathbf{i}X_{s_j}^1\cdot\xi_j\right)\right] \cdot \mathbb{E}\left[\prod_{j=1}^{p} \exp\left(-\mathbf{i}X_{r_j}^1\cdot\xi_j\right)\right], \qquad (3.14)
\end{aligned}$$

since $\{X_t^1 - X_{t-u}^1, u \in [0,t]\}$ and $\{X_s^1 - X_{s-u}^1, u \in [0,s]\}$ have the same law as $\{X_u^1, u \in [0,t]\}$ and $\{X_u^1, u \in [0,s]\}$ respectively. So the expression (3.12) is indeed a function that depends only on the difference $x - y$. Furthermore, a quick comparison between (3.2) and (3.14) reveals that the only difference is that the variables inside the temporal covariance kernel are $\gamma_0(s_j - r_j)$ in (3.2) and $\gamma_0(t - s_j - s + r_j)$ in (3.14). Going through the same arguments that lead to (3.6) and (3.7), we get (with $s \leq t$)

$$p! \int_{\mathbb{R}^d} \langle f_{t,z,p}, f_{s,0,p}\rangle_{\mathscr{H}^{\otimes p}} dz \leq (2\pi)^d \|\varphi_1\|_\infty \Gamma_t^p t (4C_N)^{p-1} \exp\left(\frac{tD_N}{2C_N}\right)$$

and

$$\mathbb{E}\Big[I_p^W\big(g_{p,R}(t)\big) I_p^W\big(g_{p,R}(s)\big)\Big] = \frac{p!}{R^d} \int_{B_R^2} dxdy \left\langle f_{t,x,p}, f_{s,y,p}\right\rangle_{\mathscr{H}^{\otimes p}}$$

$$= p!\omega_d \int_{\mathbb{R}^d} dz \big\langle f_{t,0,p}, f_{s,z,p} \big\rangle_{\mathscr{H}^{\otimes p}} \frac{\text{vol}\big(B_R \cap B_R(-z)\big)}{\omega_d R^d}$$
$$\xrightarrow{R\to+\infty} p!\omega_d \int_{\mathbb{R}^d} dz \big\langle f_{t,0,p}, f_{s,z,p} \big\rangle_{\mathscr{H}^{\otimes p}} = \sigma_p(t,s),$$

with

$$\sup_{R>0} \mathbb{E}\big[I_p^W(g_{p,R}(t)) I_p^W(g_{p,R}(s)) \big] \le \sigma_p(t,s). \tag{3.15}$$

This completes the verification of condition (a). Notice that

$$\sigma_p(t,t) = \frac{\omega_d}{p!} \int_{\mathbb{R}^d} \mathbb{E}[\beta_{t,t}(z)^p] dz,$$

so condition (b) follows from (3.8) and (3.7). To see condition (d), it is enough to use (3.15) and condition (b).

Proof of condition (c): Given $t > 0$ and $1 \le r \le p-1$, we need to prove that

$$\lim_{R\to+\infty} \big\| g_{p,R}(t) \otimes_r g_{p,R}(t) \big\|_{\mathscr{H}^{\otimes(2p-2r)}} = 0.$$

We follow the same routine that leads to (2.17). We put

$$\mathfrak{f}(\boldsymbol{s_p}, \boldsymbol{y_p}) = f_{t,0,p}(\boldsymbol{s_p}, \boldsymbol{y_p}),$$

and in this way, we have $f_{t,x,p} = \mathfrak{f}^x$, with $\mathfrak{f}^x$ being the spatially shifted version of $\mathfrak{f}$. Now we write (notice that we have the extra temporal variables now)

$$\begin{aligned}
&(2\pi)^{-2d} \big\| g_{p,R}(t) \otimes_r g_{p,R}(t) \big\|^2_{\mathscr{H}^{\otimes(2p-2r)}} \\
&= \int_{[0,t]^{4p}} d\boldsymbol{s_r} d\widetilde{\boldsymbol{s}}_{\boldsymbol{r}} d\boldsymbol{v_r} d\widetilde{\boldsymbol{v}}_{\boldsymbol{r}} d\boldsymbol{t_{p-r}} d\widetilde{\boldsymbol{t}}_{\boldsymbol{p-r}} d\boldsymbol{w_{p-r}} d\widetilde{\boldsymbol{w}}_{\boldsymbol{p-r}} \left(\prod_{i=1}^r \gamma_0(s_i - \widetilde{s}_i)\gamma_0(v_i - \widetilde{v}_i) \right) \\
&\quad \times \left(\prod_{j=1}^{p-r} \gamma_0(t_j - \widetilde{t}_j)\gamma_0(w_j - \widetilde{w}_j) \right) \widetilde{\mathcal{J}}_R,
\end{aligned}$$

with $\widetilde{\mathcal{J}}_R = \widetilde{\mathcal{J}}_R\big(\boldsymbol{s_r}, \widetilde{\boldsymbol{s}}_{\boldsymbol{r}}, \boldsymbol{v_r}, \widetilde{\boldsymbol{v}}_{\boldsymbol{r}}, \boldsymbol{t_{p-r}}, \widetilde{\boldsymbol{t}}_{\boldsymbol{p-r}}, \boldsymbol{w_{p-r}}, \widetilde{\boldsymbol{w}}_{\boldsymbol{p-r}}\big)$ given by

$$\begin{aligned}
\widetilde{\mathcal{J}}_R = \int_{\mathbb{R}^{2pd}} &\mu_1(d\boldsymbol{\xi_r})\, \mu_1(d\widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}}) \mu_1(d\boldsymbol{\eta_{p-r}}) \mu_1(d\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}}) \\
&\times (\mathscr{F}\mathfrak{f})(\boldsymbol{s_r}, \boldsymbol{t_{p-r}}, \boldsymbol{\eta_{p-r}}, \boldsymbol{\xi_r}) (\mathscr{F}\mathfrak{f})(\widetilde{\boldsymbol{s}}_{\boldsymbol{r}}, \boldsymbol{w_{p-r}}, \boldsymbol{\eta_{p-r}}, \widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}}) \|a+b\|^{-d/2} \|\widetilde{b}+a\|^{-d/2} \\
&\times (\mathscr{F}\mathfrak{f})(\boldsymbol{v_r}, \widetilde{\boldsymbol{t}}_{\boldsymbol{p-r}}, \widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}}, \boldsymbol{\xi_r}) (\mathscr{F}\mathfrak{f})(\widetilde{\boldsymbol{v}}_{\boldsymbol{r}}, \widetilde{\boldsymbol{w}}_{\boldsymbol{p-r}}, \widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}}, \widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}}) \|\widetilde{a}+b\|^{-d/2} \|\widetilde{a}+\widetilde{b}\|^{-d/2} \\
&\times J_{d/2}\big(R\|a+b\|\big) J_{d/2}\big(R\|\widetilde{b}+a\|\big) J_{d/2}\big(R\|\widetilde{a}+b\|\big) J_{d/2}\big(R\|\widetilde{a}+\widetilde{b}\|\big),
\end{aligned}$$

where $\mathscr{F}\mathfrak{f}$ stands for the Fourier transform with respect to the spatial variables and we have used the short-hand notation $a = \tau(\boldsymbol{\xi_r}), b = \tau(\boldsymbol{\eta_{p-r}}), \widetilde{a} = \tau(\widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}})$ and $\widetilde{b} = \tau(\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}})$.

Recall from previous steps that, with X^1 standard Brownian motion on $\mathbb{R}^d$,

$$(\mathscr{F}\mathfrak{f})(\boldsymbol{s_p},\boldsymbol{\xi_p}) = (\mathscr{F} f_{t,0,p})(\boldsymbol{s_p},\boldsymbol{\xi_p}) = \frac{1}{p!}\mathbb{E}\left[\exp\Big(-\mathbf{i}\sum_{j=1}^{p}(X_t^1 - X_{s_j}^1)\cdot\xi_j\Big)\right], \tag{3.16}$$

which is a positive, bounded and uniformly continuous function in $\boldsymbol{\xi_p}$. As in the proof of Theorem 2.11 (Step 4), we decompose the integral in the spatial variable into two parts, that is, we write for any given $\delta > 0$,

$$\widetilde{\mathcal{J}}_R = \widetilde{\mathcal{J}}_{1,R} + \widetilde{\mathcal{J}}_{2,R} := \int_{\mathbb{R}^{2pd}} \mathbf{1}_{\{\|a+b\|\geq\delta\}} + \int_{\mathbb{R}^{2pd}} \mathbf{1}_{\{\|a+b\|<\delta\}}.$$

Similar to the arguments in Step 4 of the proof of Theorem 2.11, by using Cauchy-Schwarz inequality several times, we can write

$$\begin{aligned}\widetilde{\mathcal{J}}_{1,R} \leq \omega_d^2 &\left(\int_{\{\|\tau(\boldsymbol{\xi_p})\geq\delta\}} \ell_R(\tau(\boldsymbol{\xi_p}))\big|\mathscr{F}\mathfrak{f}\big|^2(\boldsymbol{s_r},\boldsymbol{t_{p-r}},\boldsymbol{\xi_p})\mu_1(d\boldsymbol{\xi_p})\right)^{1/2}\\
&\times\left(\int_{\mathbb{R}^{pd}} \ell_R(\tau(\boldsymbol{\xi_p}))\big|\mathscr{F}\mathfrak{f}\big|^2(\widetilde{\boldsymbol{v}}_{\boldsymbol{r}},\widetilde{\boldsymbol{w}}_{\boldsymbol{p-r}},\boldsymbol{\xi_p})\mu_1(d\boldsymbol{\xi_p})\right)^{1/2}\\
&\times\left(\int_{\mathbb{R}^{pd}} \ell_R(\tau(\boldsymbol{\xi_p}))\big|\mathscr{F}\mathfrak{f}\big|^2(\widetilde{\boldsymbol{s}}_{\boldsymbol{r}},\boldsymbol{w_{p-r}},\boldsymbol{\xi_p})\mu_1(d\boldsymbol{\xi_p})\right)^{1/2}\\
&\times\left(\int_{\mathbb{R}^{pd}} \ell_R(\tau(\boldsymbol{\xi_p}))\big|\mathscr{F}\mathfrak{f}\big|^2(\boldsymbol{v_r},\widetilde{\boldsymbol{t}}_{\boldsymbol{p-r}},\boldsymbol{\xi_p})\mu_1(d\boldsymbol{\xi_p})\right)^{1/2}.\end{aligned}$$

Therefore, by Cauchy-Schwarz inequality again applied to the integration in time, we get

$$\begin{aligned}&\int_{[0,t]^{4p}} d\boldsymbol{s_r}d\widetilde{\boldsymbol{s}}_{\boldsymbol{r}}d\boldsymbol{v_r}d\widetilde{\boldsymbol{v}}_{\boldsymbol{r}}d\boldsymbol{t_{p-r}}d\widetilde{\boldsymbol{t}}_{\boldsymbol{p-r}}d\boldsymbol{w_{p-r}}d\widetilde{\boldsymbol{w}}_{\boldsymbol{p-r}}\left(\prod_{i=1}^{r}\gamma_0(s_i-\widetilde{s}_i)\gamma_0(v_i-\widetilde{v}_i)\right)\\
&\quad\times\left(\prod_{j=1}^{p-r}\gamma_0(t_j-\widetilde{t}_j)\gamma_0(w_j-\widetilde{w}_j)\right)\widetilde{\mathcal{J}}_{1,R} \qquad\qquad (3.17)\\
&\leq \omega_d^2\Bigg\{\int_{[0,t]^{4p}} d\boldsymbol{s_r}d\widetilde{\boldsymbol{s}}_{\boldsymbol{r}}d\boldsymbol{v_r}d\widetilde{\boldsymbol{v}}_{\boldsymbol{r}}d\boldsymbol{t_{p-r}}d\widetilde{\boldsymbol{t}}_{\boldsymbol{p-r}}d\boldsymbol{w_{p-r}}d\widetilde{\boldsymbol{w}}_{\boldsymbol{p-r}}\left(\prod_{i=1}^{r}\gamma_0(s_i-\widetilde{s}_i)\gamma_0(v_i-\widetilde{v}_i)\right)\\
&\quad\times\left(\prod_{j=1}^{p-r}\gamma_0(t_j-\widetilde{t}_j)\gamma_0(w_j-\widetilde{w}_j)\right)\left(\int_{\mathbb{R}^{pd}} \ell_R(\tau(\boldsymbol{\xi_p}))\big|\mathscr{F}\mathfrak{f}\big|^2(\widetilde{\boldsymbol{v}}_{\boldsymbol{r}},\widetilde{\boldsymbol{w}}_{\boldsymbol{p-r}},\boldsymbol{\xi_p})\mu_1(d\boldsymbol{\xi_p})\right)\\
&\qquad\times\int_{\{\|\tau(\boldsymbol{\xi_p})\geq\delta\}} \ell_R(\tau(\boldsymbol{\xi_p}))\big|\mathscr{F}\mathfrak{f}\big|^2(\boldsymbol{s_r},\boldsymbol{t_{p-r}},\boldsymbol{\xi_p})\mu_1(d\boldsymbol{\xi_p})\Bigg\}^{1/2}\\
&\times\Bigg\{\int_{[0,t]^{4p}} d\boldsymbol{s_r}d\widetilde{\boldsymbol{s}}_{\boldsymbol{r}}d\boldsymbol{v_r}d\widetilde{\boldsymbol{v}}_{\boldsymbol{r}}d\boldsymbol{t_{p-r}}d\widetilde{\boldsymbol{t}}_{\boldsymbol{p-r}}d\boldsymbol{w_{p-r}}d\widetilde{\boldsymbol{w}}_{\boldsymbol{p-r}}\left(\prod_{i=1}^{r}\gamma_0(s_i-\widetilde{s}_i)\gamma_0(v_i-\widetilde{v}_i)\right)\end{aligned}$$

$$\times \left(\prod_{j=1}^{p-r} \gamma_0(t_j - \widetilde{t}_j)\gamma_0(w_j - \widetilde{w}_j)\right) \left(\int_{\mathbb{R}^{pd}} \ell_R(\tau(\boldsymbol{\xi_p}))\left|\mathscr{F}\mathfrak{f}\right|^2(\boldsymbol{\widetilde{s}_r}, \boldsymbol{w_{p-r}}, \boldsymbol{\xi_p})\mu_1(d\boldsymbol{\xi_p})\right)$$

$$\times \int_{\mathbb{R}^{pd}} \ell_R(\tau(\boldsymbol{\xi_p}))\left|\mathscr{F}\mathfrak{f}\right|^2(\boldsymbol{v_r}, \boldsymbol{\widetilde{t}_{p-r}}, \boldsymbol{\xi_p})\mu_1(d\boldsymbol{\xi_p})\Bigg\}^{1/2}$$

$$=: \omega_d^2 V_1^{1/2} V_2^{1/2}.$$

We will prove that $V_1 \to 0$ as $R \to +\infty$ and V_2 is uniformly bounded. For the term V_1, we have the estimate

$$V_1 \le \Gamma_t^{2p} \left[\int_{[0,t]^p} d\boldsymbol{t_p} \int_{\mathbb{R}^{pd}} \ell_R(\tau(\boldsymbol{\xi_p}))\left|\mathscr{F}\mathfrak{f}\right|^2(\boldsymbol{t_p}, \boldsymbol{\xi_p})\mu_1(d\boldsymbol{\xi_p})\right]$$

$$\times \int_{[0,t]^p} d\boldsymbol{s_p} \int_{\{\|\tau(\boldsymbol{\xi_p})\|\ge\delta\}} \ell_R(\tau(\boldsymbol{\xi_p}))\left|\mathscr{F}\mathfrak{f}\right|^2(\boldsymbol{s_p}, \boldsymbol{\xi_p})\mu_1(d\boldsymbol{\xi_p})$$

$$=: \Gamma_t^{2p} V_{11} V_{12}.$$

We claim that V_{11} is uniformly bounded and V_{12} vanishes asymptotically as $R \to +\infty$. In view of (3.16), making the change of variables $t_j = t - s_j$ and $\eta_j = \xi_1 + \cdots + \xi_j$ for each $j = 1, \ldots, p$, with $\eta_0 = 0$, we obtain, using (3.4)

$$\begin{aligned}
V_{11} &= \frac{1}{(p!)^2} \int_{[0,t]^p} d\boldsymbol{s_p} \int_{\mathbb{R}^{pd}} \mu_1(d\boldsymbol{\xi_p})\ell_R(\tau(\boldsymbol{\xi_p})) \left(\mathbb{E}\left[\exp\left(-\mathbf{i}\sum_{j=1}^{p} X_{s_j}^1 \cdot \xi_j\right)\right]\right)^2 \\
&= \frac{1}{p!} \int_{\Delta_p(t)} d\boldsymbol{s_p} \int_{\mathbb{R}^{pd}} \mu_1(d\boldsymbol{\xi_p})\ell_R(\tau(\boldsymbol{\xi_p})) \exp\left(-\sum_{j=1}^{p}(s_j - s_{j+1})\|\xi_1 + \cdots + \xi_j\|^2\right) \\
&= \frac{1}{p!} \int_{\mathbb{R}^d} d\eta_p \ell_R(\eta_p) \int_{\mathbb{R}^{pd-d}} d\boldsymbol{\eta_{p-1}} \int_{\mathrm{SIM}_p(t)} d\boldsymbol{w_p} \prod_{j=1}^{p} e^{-w_j\|\eta_j\|^2} \varphi_1(\eta_j - \eta_{j-1}) \\
&\le \frac{t}{p!}\|\varphi_1\|_\infty \sum_{j=0}^{p-1} \binom{p-1}{j} \frac{t^j}{j!} D_N^j C_N^{p-1-j} < +\infty.
\end{aligned}$$

In the same way, we have

$$V_{12} \le \left(\int_{\{\|\tau_1\|\ge\delta\}} d\tau_1 \ell_R(\tau_1)\right) \frac{t\|\varphi_1\|_\infty}{p!} \sum_{j=0}^{p-1} \binom{p-1}{j} \frac{t^j}{j!} D_N^j C_N^{p-1-j},$$

which converges to zero as R tends to infinity. By the same arguments, we can get the uniform boundedness of V_2 as R tends to infinity. Thus, the term (3.17) does not contribute to the limit of $\left\|g_{p,R}(t) \otimes_r g_{p,R}(t)\right\|^2_{\mathscr{H}^{\otimes(2p-2r)}}$ as $R \to +\infty$.

Now let us look at the second term and we need to prove that

$$\mathfrak{X}_R := \int_{[0,t]^{4p}} d\boldsymbol{s_r} d\boldsymbol{\widetilde{s}_r} d\boldsymbol{v_r} d\boldsymbol{\widetilde{v}_r} d\boldsymbol{t_{p-r}} d\boldsymbol{\widetilde{t}_{p-r}} d\boldsymbol{w_{p-r}} d\boldsymbol{\widetilde{w}_{p-r}} \left(\prod_{i=1}^{r} \gamma_0(s_i - \widetilde{s}_i)\gamma_0(v_i - \widetilde{v}_i)\right)$$

$$\times \left(\prod_{j=1}^{p-r} \gamma_0(t_j - \widetilde{t}_j)\gamma_0(w_j - \widetilde{w}_j) \right) \widetilde{J}_{2,R} \xrightarrow{R\to+\infty} 0.$$

We can first rewrite $\omega_d^{-2}\widetilde{J}_{2,R}$ as we did for $\int_{\mathbb{R}^{pd}\times D_\delta^c}$ in the proof of Theorem 2.11. In fact, using Cauchy-Schwarz multiple times, we obtain

$$\begin{aligned}
\omega_d^{-2}\widetilde{J}_{2,R} &\le \int_{\{\|a+b\|<\delta\}} \mu_1(d\boldsymbol{\xi_r})\mu_1(d\boldsymbol{\eta_{p-r}})\sqrt{\ell_R(a+b)}\, \mathscr{F}\mathfrak{f}(\boldsymbol{s_r}, \boldsymbol{t_{p-r}}, \boldsymbol{\eta_{p-r}}, \boldsymbol{\xi_r}) \\
&\times \left(\int_{\mathbb{R}^{pd}} \mu_1(d\widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}})\mu_1(d\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}})\ell_R(\widetilde{a}+\widetilde{b})\big|\mathscr{F}\mathfrak{f}\big|^2(\widetilde{\boldsymbol{v}}_{\boldsymbol{r}}, \widetilde{\boldsymbol{w}}_{\boldsymbol{p-r}}, \widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}}, \widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}}) \right)^{1/2} \Bigg\{ \int_{\mathbb{R}^{pd}} \mu_1(d\widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}}) \\
&\times \mu_1(d\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}})\ell_R(\widetilde{a}+b)\ell_R(a+\widetilde{b})\big|\mathscr{F}\mathfrak{f}\big|^2(\boldsymbol{v_r}, \widetilde{\boldsymbol{t}}_{\boldsymbol{p-r}}, \widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}}, \boldsymbol{\xi_r})\big|\mathscr{F}\mathfrak{f}\big|^2(\widetilde{\boldsymbol{s}}_{\boldsymbol{r}}, \boldsymbol{w_{p-r}}, \boldsymbol{\eta_{p-r}}, \widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}}) \Bigg\}^{1/2} \\
&\le \Bigg[\left(\int_{\mathbb{R}^{pd}} \mu_1(d\widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}})\mu_1(d\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}})\ell_R(\widetilde{a}+\widetilde{b})\big|\mathscr{F}\mathfrak{f}\big|^2(\widetilde{\boldsymbol{v}}_{\boldsymbol{r}}, \widetilde{\boldsymbol{w}}_{\boldsymbol{p-r}}, \widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}}, \widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}}) \right) \\
&\quad \times \left(\int_{\{\|a+b\|<\delta\}} \mu_1(d\boldsymbol{\xi_r})\mu_1(d\boldsymbol{\eta_{p-r}})\ell_R(a+b)\big|\mathscr{F}\mathfrak{f}\big|^2(\boldsymbol{s_r}, \boldsymbol{t_{p-r}}, \boldsymbol{\eta_{p-r}}, \boldsymbol{\xi_r}) \right) \Bigg]^{1/2} \\
&\quad \times \Bigg[\int_{\{\|a+b\|<\delta\}\times\mathbb{R}^{pd}} \mu_1(d\boldsymbol{\xi_r})\mu_1(d\boldsymbol{\eta_{p-r}})\mu_1(d\widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}})\mu_1(d\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}}) \\
&\quad \times \big|\mathscr{F}\mathfrak{f}\big|^2(\widetilde{\boldsymbol{s}}_{\boldsymbol{r}}, \boldsymbol{w_{p-r}}, \boldsymbol{\eta_{p-r}}, \widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}})\big|\mathscr{F}\mathfrak{f}\big|^2(\boldsymbol{v_r}, \widetilde{\boldsymbol{t}}_{\boldsymbol{p-r}}, \widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}}, \boldsymbol{\xi_r})\ell_R(\widetilde{a}+b)\ell_R(a+\widetilde{b}) \Bigg]^{1/2} \\
&:= \widetilde{V}_1^{1/2}\widetilde{V}_2^{1/2}.
\end{aligned}$$

Therefore,

$$\omega_d^{-2}\mathfrak{X}_R \le \sqrt{\mathfrak{X}_{1,R}\mathfrak{X}_{2,R}},$$

where

$$\begin{aligned}
\mathfrak{X}_{1,R} := \int_{[0,t]^{4p}} & d\boldsymbol{s_r} d\widetilde{\boldsymbol{s}}_{\boldsymbol{r}} d\boldsymbol{v_r} d\widetilde{\boldsymbol{v}}_{\boldsymbol{r}} d\boldsymbol{t_{p-r}} d\widetilde{\boldsymbol{t}}_{\boldsymbol{p-r}} d\boldsymbol{w_{p-r}} d\widetilde{\boldsymbol{w}}_{\boldsymbol{p-r}} \left(\prod_{i=1}^{r} \gamma_0(s_i - \widetilde{s}_i)\gamma_0(v_i - \widetilde{v}_i) \right) \\
&\times \left(\prod_{j=1}^{p-r} \gamma_0(t_j - \widetilde{t}_j)\gamma_0(w_j - \widetilde{w}_j) \right) \widetilde{V}_1
\end{aligned}$$

is uniformly bounded over $R > 0$, as one can verify by the same arguments as before, and

$$\begin{aligned}
\mathfrak{X}_{2,R} &:= \int_{[0,t]^{4p}} d\boldsymbol{s_r} d\widetilde{\boldsymbol{s}}_{\boldsymbol{r}} d\boldsymbol{v_r} d\widetilde{\boldsymbol{v}}_{\boldsymbol{r}} d\boldsymbol{t_{p-r}} d\widetilde{\boldsymbol{t}}_{\boldsymbol{p-r}} d\boldsymbol{w_{p-r}} d\widetilde{\boldsymbol{w}}_{\boldsymbol{p-r}} \left(\prod_{i=1}^{r} \gamma_0(s_i - \widetilde{s}_i)\gamma_0(v_i - \widetilde{v}_i) \right) \\
&\times \left(\prod_{j=1}^{p-r} \gamma_0(t_j - \widetilde{t}_j)\gamma_0(w_j - \widetilde{w}_j) \right) \int_{\{\|a+b\|<\delta\}\times\mathbb{R}^{pd}} \mu_1(d\boldsymbol{\xi_r})\mu_1(d\boldsymbol{\eta_{p-r}})\mu_1(d\widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}})\mu_1(d\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}})
\end{aligned}$$

$$\times \left|\mathscr{F}\mathfrak{f}\right|^2(\widetilde{\boldsymbol{s}}_{\boldsymbol{r}}, \boldsymbol{w}_{\boldsymbol{p-r}}, \boldsymbol{\eta}_{\boldsymbol{p-r}}, \widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}})\left|\mathscr{F}\mathfrak{f}\right|^2(\boldsymbol{v}_{\boldsymbol{r}}, \widetilde{\boldsymbol{t}}_{\boldsymbol{p-r}}, \widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}}, \boldsymbol{\xi}_{\boldsymbol{r}})\ell_R(\widetilde{a}+b)\ell_R(a+\widetilde{b})$$
$$\le \Gamma_t^{2p}\int_{[0,t]^{2p}} d\widetilde{\boldsymbol{s}}_{\boldsymbol{r}} d\widetilde{\boldsymbol{t}}_{\boldsymbol{p-r}} d\boldsymbol{v}_{\boldsymbol{r}} d\boldsymbol{w}_{\boldsymbol{p-r}} \int_{\{\|a+b\|<\delta\}\times\mathbb{R}^{pd}} \mu_1(d\boldsymbol{\xi}_{\boldsymbol{r}})\mu_1(d\boldsymbol{\eta}_{\boldsymbol{p-r}})\mu_1(d\widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}})\mu_1(d\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}})$$
$$\times \left|\mathscr{F}\mathfrak{f}\right|^2(\widetilde{\boldsymbol{s}}_{\boldsymbol{r}}, \boldsymbol{w}_{\boldsymbol{p-r}}, \boldsymbol{\eta}_{\boldsymbol{p-r}}, \widetilde{\boldsymbol{\xi}}_{\boldsymbol{r}})\left|\mathscr{F}\mathfrak{f}\right|^2(\boldsymbol{v}_{\boldsymbol{r}}, \widetilde{\boldsymbol{t}}_{\boldsymbol{p-r}}, \widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r}}, \boldsymbol{\xi}_{\boldsymbol{r}})\ell_R(\widetilde{a}+b)\ell_R(a+\widetilde{b})$$
$$= \Gamma_t^{2p}\int_{\mathbb{R}^{2pd}} \mu_1(d\boldsymbol{\xi}_{\boldsymbol{p}})\mu_1(d\widetilde{\boldsymbol{\xi}}_{\boldsymbol{p}})\mathbf{1}_{\{\|\xi_1+\cdots+\xi_r+\widetilde{\xi}_{r+1}+\cdots+\widetilde{\xi}_p\|<\delta\}}\ell_R\big(\tau(\boldsymbol{\xi}_{\boldsymbol{p}})\big)\ell_R\big(\tau(\widetilde{\boldsymbol{\xi}}_{\boldsymbol{p}})\big)$$
$$\times \left(\int_{[0,t]^p} d\boldsymbol{s}_{\boldsymbol{p}}\left|\mathscr{F}\mathfrak{f}\right|^2(\boldsymbol{s}_{\boldsymbol{p}}, \widetilde{\boldsymbol{\xi}}_{\boldsymbol{p}})\right)\left(\int_{[0,t]^p} d\boldsymbol{t}_{\boldsymbol{p}}\left|\mathscr{F}\mathfrak{f}\right|^2(\boldsymbol{t}_{\boldsymbol{p}}, \boldsymbol{\xi}_{\boldsymbol{p}})\right).$$

Using (3.16) and a change of variable in time, we can rewrite the last expression as follows

$$\mathfrak{X}_{2,R} \le \frac{\Gamma_t^{2p}}{(p!)^2}\int_{\mathbb{R}^{2pd}} \mu_1(d\boldsymbol{\xi}_{\boldsymbol{p}})\mu_1(d\widetilde{\boldsymbol{\xi}}_{\boldsymbol{p}})\mathbf{1}_{\{\|\xi_1+\cdots+\xi_r+\widetilde{\xi}_{r+1}+\cdots+\widetilde{\xi}_p\|<\delta\}}\ell_R\big(\tau(\boldsymbol{\xi}_{\boldsymbol{p}})\big)\ell_R\big(\tau(\widetilde{\boldsymbol{\xi}}_{\boldsymbol{p}})\big)$$
$$\times \int_{[0,t]^{2p}} d\boldsymbol{s}_{\boldsymbol{p}} d\boldsymbol{t}_{\boldsymbol{p}}\mathbb{E}\left[\exp\Big(-\mathbf{i}\sum_{j=1}^p X^1_{s_j}\cdot\widetilde{\xi}_j\Big)\right]\mathbb{E}\left[\exp\Big(-\mathbf{i}\sum_{j=1}^p X^2_{t_j}\cdot\xi_j\Big)\right].$$

For $\boldsymbol{s}_{\boldsymbol{p}} \in \Delta_p(t)$, we write

$$\mathbb{E}\left[\exp\Big(-\mathbf{i}\sum_{j=1}^p X^1_{s_j}\cdot\widetilde{\xi}_j\Big)\right] = \exp\left(-\sum_{j=1}^p \frac{s_{\sigma(j)}-s_{\sigma(j+1)}}{2}\|\widetilde{\xi}_{\sigma(1)}+\cdots+\widetilde{\xi}_{\sigma(j)}\|^2\right).$$

Then

$$\int_{[0,t]^p} d\boldsymbol{s}_{\boldsymbol{p}}\mathbb{E}\left[\exp\Big(-\mathbf{i}\sum_{j=1}^p X^1_{s_j}\cdot\widetilde{\xi}_j\Big)\right]$$
$$= \sum_{\sigma\in\mathfrak{S}_p}\int_{\mathrm{SIM}_p(t/2)} d\widetilde{\boldsymbol{w}}_{\boldsymbol{p}}\exp\left(-\sum_{j=1}^p \widetilde{w}_j\|\widetilde{\xi}_{\sigma(1)}+\cdots+\widetilde{\xi}_{\sigma(j)}\|^2\right)$$

and in the same way,

$$\int_{[0,t]^p} d\boldsymbol{t}_{\boldsymbol{p}}\mathbb{E}\left[\exp\Big(-\mathbf{i}\sum_{j=1}^p X^2_{t_j}\cdot\xi_j\Big)\right]$$
$$= \sum_{\pi\in\mathfrak{S}_p}\int_{\mathrm{SIM}_p(t/2)} d\boldsymbol{w}_{\boldsymbol{p}}\exp\left(-\sum_{j=1}^p w_j\|\xi_{\pi(1)}+\cdots+\xi_{\pi(j)}\|^2\right).$$

By a further change of variables $\xi_{\pi(1)}+\cdots+\xi_{\pi(j)} = \eta_j$ and $\widetilde{\xi}_{\sigma(1)}+\cdots+\widetilde{\xi}_{\sigma(j)} = \widetilde{\eta}_j$ for given σ, π, we can write

$$\mathbf{1}_{\{\|\xi_1+\cdots+\xi_r+\widetilde{\xi}_{r+1}+\cdots+\widetilde{\xi}_p\|<\delta\}} = \mathbf{1}_{\{\|L(\boldsymbol{\eta}_{\boldsymbol{p}},\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p}})\|<\delta\}},$$

where $L(\boldsymbol{\eta_p},\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p}})$ stands for linear combinations of $\eta_1,\dots,\eta_p,\widetilde{\eta}_1,\dots\widetilde{\eta}_p$ that depend on σ,π. With this notation, we have

$$\begin{aligned}\mathfrak{X}_{2,R} \le \frac{\Gamma_t^{2p}}{(p!)^2}\sum_{\sigma,\pi\in\mathfrak{S}_p}\int_{\mathbb{R}^{2d}} d\eta_p d\widetilde{\eta}_p \ell_R(\eta_p)\ell_R(\widetilde{\eta}_p)\int_{\mathrm{SIM}_p(t/2)^2} d\boldsymbol{w_p} d\widetilde{\boldsymbol{w}}_{\boldsymbol{p}}\int_{\mathbb{R}^{2pd-2d}} d\boldsymbol{\eta_{p-1}} d\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-1}}\\ \times\left(\prod_{j=1}^{p-1}\varphi_1(\eta_j-\eta_{j-1})e^{-w_j\|\eta_j\|^2}\varphi_1(\widetilde{\eta}_j-\widetilde{\eta}_{j-1})e^{-w_j\|\widetilde{\eta}_j\|^2}\right)\\ \times\varphi_1(\eta_p-\eta_{p-1})\varphi_1(\eta_p-\eta_{p-1})e^{-w_p\|\eta_p\|^2-\widetilde{w}_p\|\widetilde{\eta}_p\|^2}\mathbf{1}_{\{\|L(\boldsymbol{\eta_p},\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p}})\|<\delta\}}\\ =: \frac{\Gamma_t^{2p}}{(p!)^2}\sum_{\sigma,\pi\in\mathfrak{S}_p}\int_{\mathbb{R}^{2d}} d\eta_p d\widetilde{\eta}_p \ell_R(\eta_p)\ell_R(\widetilde{\eta}_p)\mathcal{E}_\delta^{\sigma,\pi}(\eta_p,\widetilde{\eta}_p)\end{aligned}$$

where $\mathcal{E}_\delta^{\sigma,\pi}$ is defined in an obvious way. By the arguments leading to (3.4), it is clear that $\mathcal{E}_\delta^{\sigma,\pi}$ is uniformly bounded. It follows that

$$\begin{aligned}&\limsup_{R\to+\infty}\int_{\mathbb{R}^{2d}} d\eta_p d\widetilde{\eta}_p \ell_R(\eta_p)\ell_R(\widetilde{\eta}_p)\mathcal{E}_\delta^{\sigma,\pi}(\eta_p,\widetilde{\eta}_p)\\ &=\limsup_{R\to+\infty}\int_{\mathbb{R}^{2d}} d\eta_p d\widetilde{\eta}_p \ell_R(\eta_p)\ell_R(\widetilde{\eta}_p)\mathcal{E}_\delta^{\sigma,\pi}(\eta_p,\widetilde{\eta}_p)\mathbf{1}_{\{\|\eta_p\|<\delta,\|\widetilde{\eta}_p\|<\delta\}}.\end{aligned}$$

For fixed $\sigma,\pi\in\mathfrak{S}_p$, we have the decomposition $L(\boldsymbol{\eta_p},\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p}}) = L_1(\eta_p,\widetilde{\eta}_p) + L_2(\boldsymbol{\eta_{p-1}},\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-1}})$, where $L_1(\eta_p,\widetilde{\eta}_p)$ stands for a linear combination of η_p and $\widetilde{\eta}_p$, while $L_2(\boldsymbol{\eta_{p-1}},\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-1}})$ stands for linear combinations of $\eta_1,\dots,\eta_{p-1},\widetilde{\eta}_1,\dots,\widetilde{\eta}_{p-1}$. Notice that L_1 and L_2 also depend on σ,π. If $\|\eta_p\|,\|\widetilde{\eta}_p\|<\delta$, then there exists some constant $K=K(\sigma,\pi)$ such that

$$\|L_1(\eta_p,\widetilde{\eta}_p)\| < K\delta,$$

thus $\mathbf{1}_{\{\|L(\boldsymbol{\eta_p},\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p}})\|<\delta\}} \le \mathbf{1}_{\{\|L_2(\boldsymbol{\eta_{p-1}},\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-1}})\|<(K+1)\delta\}}$. As a consequence,

$$\begin{aligned}&\int_{\mathbb{R}^{2d}} d\eta_p d\widetilde{\eta}_p \ell_R(\eta_p)\ell_R(\widetilde{\eta}_p)\mathcal{E}_\delta^{\sigma,\pi}(\eta_p,\widetilde{\eta}_p)\mathbf{1}_{\{\|\eta_p\|<\delta,\|\widetilde{\eta}_p\|<\delta\}}\\ &\le t^2\|\varphi_1\|_\infty^2\int_{\mathbb{R}^{2d}} d\eta_p d\widetilde{\eta}_p \ell_R(\eta_p)\ell_R(\widetilde{\eta}_p)\int_{\mathrm{SIM}_{p-1}(t)^2} d\boldsymbol{w_{p-1}} d\widetilde{\boldsymbol{w}}_{\boldsymbol{p-1}}\int_{\mathbb{R}^{2pd-2d}} d\boldsymbol{\eta_{p-1}} d\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-1}}\\ &\quad\times\left(\prod_{j=1}^{p-1}\varphi_1(\eta_j-\eta_{j-1})e^{-w_j\|\eta_j\|^2}\varphi_1(\widetilde{\eta}_j-\widetilde{\eta}_{j-1})e^{-w_j\|\widetilde{\eta}_j\|^2}\right)\mathbf{1}_{\{\|L_2(\boldsymbol{\eta_{p-1}},\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-1}})\|<(K+1)\delta\}}\\ &= t^2\|\varphi_1\|_\infty^2\int_{\mathrm{SIM}_{p-1}(t)^2} d\boldsymbol{w_{p-1}} d\widetilde{\boldsymbol{w}}_{\boldsymbol{p-1}}\int_{\mathbb{R}^{2pd-2d}} d\boldsymbol{\eta_{p-1}} d\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-1}}\\ &\quad\times\left(\prod_{j=1}^{p-1}\varphi_1(\eta_j-\eta_{j-1})e^{-w_j\|\eta_j\|^2}\varphi_1(\widetilde{\eta}_j-\widetilde{\eta}_{j-1})e^{-w_j\|\widetilde{\eta}_j\|^2}\right)\mathbf{1}_{\{\|L_2(\boldsymbol{\eta_{p-1}},\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-1}})\|<(K+1)\delta\}}\\ &=: t^2\|\varphi_1\|_\infty^2 T_\delta(\sigma,\pi).\end{aligned}$$

By previous arguments,

$$\int_{\mathrm{SIM}_{p-1}(t)^2} d\boldsymbol{w_{p-1}} d\widetilde{\boldsymbol{w}}_{\boldsymbol{p-1}} \int_{\mathbb{R}^{2pd-2d}} d\boldsymbol{\eta_{p-1}} d\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-1}} \times \left(\prod_{j=1}^{p-1} \varphi_1(\eta_j - \eta_{j-1}) e^{-w_j \|\eta_j\|^2} \varphi_1(\widetilde{\eta}_j - \widetilde{\eta}_{j-1}) e^{-w_j \|\widetilde{\eta}_j\|^2} \right) < \infty.$$

Therefore, taking into account that $L_2(\boldsymbol{\eta_{p-1}}, \widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-1}}) \neq 0$ for almost every $\boldsymbol{\eta_{p-1}}$ and $\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-1}}$, we obtain $T_\delta(\sigma, \pi) \to 0$, as $\delta \downarrow 0$ and

$$\limsup_{R\to+\infty} \mathfrak{X}_{2,R} \leq t^2 \|\varphi_1\|_\infty^2 \sum_{\sigma,\pi \in \mathfrak{S}_p} T_\delta(\sigma, \pi),$$

which converges to zero, as $\delta \downarrow 0$. This concludes the proof of condition (c).

Combing the above steps, we conclude that if $t_1, t_2, \dots, t_n \in \mathbb{R}_+$, then

$$R^{-d/2}\big(A_{t_1}(R), \dots, A_{t_n}(R)\big) \xrightarrow[R\to+\infty]{\text{law}} N\Big(0, \big(\Sigma_{t_i,t_j}\big)_{i,j=1}^n\Big),$$

where Σ_{t_i,t_j} is defined in (1.13).

3.3 Proof of tightness in Theorem 1.6

In this section, we are going to prove the tightness of $\left\{ \frac{A_t(R)}{R^{d/2}}, t \geq 0 \right\}$ under the extra condition (1.14). Under this condition, one can see easily that

$$\Gamma_{t,\alpha} := \int_0^t \int_0^t \gamma_0(r - v) r^{-\alpha} v^{-\alpha} dr dv < +\infty \tag{3.18}$$

for any $t > 0$.

Recall that $\alpha \in (0, 1/2)$ is fixed. For any $T > 0$, we will show for any $0 < s < t \leq T$ and any integer $k \in [2, \infty)$

$$R^{-d/2} \big\| A_t(R) - A_s(R) \big\|_{L^k(\Omega)} \leq C|t - s|^\alpha, \tag{3.19}$$

where $C = C_{T,k,\alpha}$ is a constant that depends on T, k and α. If we pick a large k such that $k\alpha > 2$, we get the desired tightness by Kolmogorov's criterion. To show (3.19), we first derive the Wiener chaos expansion of $A_t(R) - A_s(R)$ and apply the hypercontractivity property of the Ornstein-Uhlenbeck semigroup (see *e.g.* [21]) that allows us to estimate the $L^k(\Omega)$-norm by the $L^2(\Omega)$-norm on a fixed Wiener chaos.

We know that

$$u_{t,x} = 1 + \int_{\mathbb{R}_+ \times \mathbb{R}^d} G(t - s_1, x - y_1) \mathbf{1}_{[0,t)}(s_1) u_{s_1,y_1} W(ds_1, dy_1)$$

and if we put

$$d(s, t, x; s_1, y_1) = G(t - s_1, x - y_1)\mathbf{1}_{[0,t)}(s_1) - G(s - s_1, x - y_1)\mathbf{1}_{[0,s)}(s_1)$$

for $s<t$, we can write

$$u_{t,x}-u_{s,x}=\int_{\mathbb{R}_+\times\mathbb{R}^d} d(s,t,x;s_1,y_1)u_{s_1,y_1}W(ds_1,dy_1).$$

We can write $d(s,t,x;s_1,y_1)=d_1(s,t,x;s_1,y_1)+d_2(s,t,x;s_1,y_1)$ with

$$d_1(s,t,x;s_1,y_1)=\mathbf{1}_{[0,s)}(s_1)\big[G(t-s_1,x-y_1)-G(s-s_1,x-y_1)\big] \tag{3.20}$$

and

$$d_2(s,t,x;s_1,y_1)=\mathbf{1}_{[s,t)}(s_1)G(t-s_1,x-y_1). \tag{3.21}$$

According to [5, Lemma 3.1], there exists some constant C_α that depends on α such that

$$\big|d_1(s,t,x;s_1,y_1)\big|\le C_\alpha(t-s)^\alpha(s-s_1)^{-\alpha}G(4t-4s_1,x-y_1)\mathbf{1}_{[0,s)}(s_1). \tag{3.22}$$

Now we can express $A_t(R)-A_s(R)$ as a sum of two chaos expansions that correspond to d_1 and d_2:

$$\begin{aligned}A_t(R)-A_s(R)&=\sum_{p\ge1}\int_{B_R}I_p^W\big(\mathfrak{g}_{1,p,x}\big)dx+\sum_{q\ge1}\int_{B_R}I_q^W\big(\mathfrak{g}_{2,q,x}dx\big)\\&=:\sum_{p\ge1}J_{1,p,R}+\sum_{q\ge1}J_{2,q,R},\end{aligned}$$

where $J_{i,p,R}=\int_{B_R}I_p^W\big(\mathfrak{g}_{i,p,x}\big)dx$ for $i\in\{1,2\}$ and

$$\begin{aligned}\mathfrak{g}_{1,p,x}(\boldsymbol{s_p},\boldsymbol{y_p})&=\frac{1}{p!}\sum_{\sigma\in\mathfrak{S}_p}\mathbf{1}_{\Delta_p(s)}(\boldsymbol{s_p^\sigma})d_1(s,t,x;s_1^\sigma,y_1^\sigma)\prod_{j=1}^{p-1}G(s_j^\sigma-s_{j+1}^\sigma,y_j^\sigma-y_{j+1}^\sigma)\\ \mathfrak{g}_{2,p,x}(\boldsymbol{s_p},\boldsymbol{y_p})&=\frac{1}{p!}\sum_{\sigma\in\mathfrak{S}_p}\mathbf{1}_{\Delta_p(s,t)}(\boldsymbol{s_p^\sigma})G(t-s_1^\sigma,x-y_1^\sigma)\prod_{j=1}^{p-1}G(s_j^\sigma-s_{j+1}^\sigma,y_j^\sigma-y_{j+1}^\sigma),\end{aligned}$$

with $\Delta_p(s,t)=\{t>s_1>\cdots>s_p>s\}$.

Let us first estimate the $L^2(\Omega)$-norm of $J_{2,p,R}$ in several familiar steps. As in (3.12), (3.13) and (3.14), we write for $p\ge1$, with X^1,X^2 independent standard Brownian motions on $\mathbb{R}^d$,

$$\begin{aligned}\big\langle\mathfrak{g}_{2,p,x},\mathfrak{g}_{2,p,y}\big\rangle_{\mathscr{H}^{\otimes p}}&=\frac{1}{(p!)^2}\int_{[0,t-s)^{2p}}d\boldsymbol{s_p}d\boldsymbol{r_p}\prod_{j=1}^p\gamma_0(s_j-r_j)\int_{\mathbb{R}^{pd}}\mu_1(d\boldsymbol{\xi_p})e^{-\mathbf{i}(x-y)\cdot\tau(\boldsymbol{\xi_p})}\\&\quad\times\mathbb{E}\left[\exp\left(-\mathbf{i}\sum_{j=1}^p\xi_j\cdot X_{s_j}^1\right)\right]\mathbb{E}\left[\exp\left(-\mathbf{i}\sum_{j=1}^p\xi_j\cdot X_{r_j}^2\right)\right],\end{aligned}$$

which is a nonnegative function in x, y that only depends on the difference $x-y$. Observe that this inner product coincides with $\frac{1}{(p!)^2}\mathbb{E}\big[\beta_{t-s,t-s}(x-y)^p\big]$ for every $p \geq 1$, see (3.2). Therefore, for $p \geq 2$, we can write by using (3.6)

$$\begin{aligned}\big\|J_{2,p,R}\big\|^2_{L^2(\Omega)} &= p!\int_{B_R^2} dxdy\big\langle \mathfrak{g}_{2,p,x}, \mathfrak{g}_{2,p,y}\big\rangle_{\mathscr{H}^{\otimes p}} \leq p!\omega_d R^d \int_{\mathbb{R}^d} dz\big\langle \mathfrak{g}_{2,p,0}, \mathfrak{g}_{2,p,z}\big\rangle_{\mathscr{H}^{\otimes p}} \\ &= \frac{\omega_d R^d}{p!}\int_{\mathbb{R}^d} dz\mathbb{E}\big[\beta_{t-s,t-s}(z)^p\big] \\ &\leq \omega_d R^d \|\varphi_1\|_\infty (2\pi)^d \Gamma^p_{t-s}(t-s)(4C_N)^{p-1}\exp\left(\frac{(t-s)D_N}{2C_N}\right) \\ &\leq (t-s)R^d\Big\{(2\pi)^d\omega_d\|\varphi_1\|_\infty \exp\left(\frac{TD_N}{2C_N}\right)\Big\}\Gamma_T^p(4C_N)^{p-1}.\end{aligned}$$

Hence, as a consequence of the hypercontractivity property (see *e.g.* [20, Corollary 2.8.14]), we have for $k \geq 2$

$$\begin{aligned}&\frac{1}{R^{d/2}}\Bigg\|\sum_{p\geq 2} J_{2,p,R}\Bigg\|_{L^k(\Omega)} \leq \frac{1}{R^{d/2}}\sum_{p\geq 2}\|J_{2,p,R}\|_{L^k(\Omega)} \leq \frac{1}{R^{d/2}}\sum_{p\geq 2}(k-1)^{p/2}\,\|J_{2,p,R}\|_{L^2(\Omega)} \\ &\leq \sqrt{t-s}\Big\{(2\pi)^d\omega_d\|\varphi_1\|_\infty \exp\left(\frac{TD_N}{2C_N}\right)/(4C_N)\Big\}^{1/2}\sum_{p\geq 1}\Big[4(k-1)\Gamma_T C_N\Big]^{p/2} \\ &= \sqrt{t-s}\Big\{(2\pi)^d\omega_d\|\varphi_1\|_\infty \exp\left(\frac{TD_N}{2C_N}\right)\Big\}^{1/2}\frac{\sqrt{(k-1)\Gamma_T}}{1-2\sqrt{(k-1)\Gamma_T C_N}}, \qquad (3.23)\end{aligned}$$

provided $0 < 4(k-1)\Gamma_T C_N < 1$, which is always valid for some $N > 0$. For $p = 1$, we have, in view of (3.1),

$$\begin{aligned}R^{-d/2}\|J_{2,1,R}\|_{L^k(\Omega)} &= c_k R^{-d/2}\|J_{2,1,R}\|_{L^2(\Omega)} \leq c_k\left(\int_{\mathbb{R}^d}\mathbb{E}[\beta_{t-s,t-s}(z)]dz\right)^{1/2} \\ &\leq c_k\sqrt{t-s}(\Gamma_T\|\gamma_1\|_{L^1(\mathbb{R}^d)})^{1/2},\end{aligned}$$

where $c_k = (\mathbb{E}[|Z|^k])^{1/k}$, with $Z \sim N(0,1)$.

Now let us estimate the $L^2(\Omega)$-norm of $J_{1,p,R}$. Put

$$\widehat{d}_1(s,t,x;s_1,y_1) = (s-s_1)^{-\alpha}G(4t-4s_1, x-y_1)\mathbf{1}_{[0,s)}(s_1)$$

and

$$\widehat{\mathfrak{g}}_{1,p,x}(\boldsymbol{s_p},\boldsymbol{y_p}) = \frac{1}{p!}\sum_{\sigma\in\mathfrak{S}_p}\mathbf{1}_{\Delta_p(s)}(\boldsymbol{s_p^\sigma})\widehat{d}_1(s,t,x;s_1^\sigma,y_1^\sigma)\prod_{j=1}^{p-1}G(s_j^\sigma - s_{j+1}^\sigma, y_j^\sigma - y_{j+1}^\sigma).$$

From (3.22) we deduce that

$$\Big|\big\langle \mathfrak{g}_{1,p,x}, \mathfrak{g}_{1,p,y}\big\rangle_{\mathscr{H}^{\otimes p}}\Big| \leq C_\alpha^2(t-s)^{2\alpha}\big\langle \widehat{\mathfrak{g}}_{1,p,x}, \widehat{\mathfrak{g}}_{1,p,y}\big\rangle_{\mathscr{H}^{\otimes p}}.$$

Similarly as before, we can write

$$\big(\mathscr{F}\widehat{\mathfrak{g}}_{1,p,x}\big)(\boldsymbol{s_p},\boldsymbol{\xi_p}) = \frac{1}{p!}\sum_{\sigma\in\mathfrak{S}_p}\mathbf{1}_{\Delta_p(s)}(\boldsymbol{s_p^\sigma})e^{-\mathbf{i}x\cdot\tau(\boldsymbol{\xi_p})}(s-s_1^\sigma)^{-\alpha} \times \mathbb{E}\left[e^{-\mathbf{i}\sum_{j=1}^p(X^1_{4t}-X^1_{4s_1^\sigma}+X^1_{s_1^\sigma}-X^1_{s_j^\sigma})\cdot\xi_j^\sigma}\right],$$

from which we see that $\big\langle\widehat{\mathfrak{g}}_{1,p,x},\widehat{\mathfrak{g}}_{1,p,y}\big\rangle_{\mathscr{H}^{\otimes p}}$ is a nonnegative function that depends only on the difference $x-y$ and is given by

$$\begin{aligned}
&\big\langle\widehat{\mathfrak{g}}_{1,p,x},\widehat{\mathfrak{g}}_{1,p,y}\big\rangle_{\mathscr{H}^{\otimes p}}\\
&=\int_{[0,s]^{2p}}d\boldsymbol{s_p}d\boldsymbol{r_p}\prod_{j=1}^p\gamma_0(s_j-r_j)\int_{\mathbb{R}^{pd}}\mu_1(d\boldsymbol{\xi_p})\big(\mathscr{F}\widehat{\mathfrak{g}}_{1,p,x}\big)(\boldsymbol{s_p},\boldsymbol{\xi_p})\big(\mathscr{F}\widehat{\mathfrak{g}}_{1,p,y}\big)(\boldsymbol{r_p},-\boldsymbol{\xi_p})\\
&=\frac{1}{(p!)^2}\sum_{\sigma,\pi\in\mathfrak{S}_p}\int_{\Delta_p(s)^2}d\boldsymbol{s_p^\sigma}d\boldsymbol{r_p^\pi}\frac{\prod_{j=1}^p\gamma_0(s_j-r_j)}{(s-s_1^\sigma)^\alpha(s-r_1^\pi)^\alpha}\int_{\mathbb{R}^{pd}}\mu_1(d\boldsymbol{\xi_p})e^{-\mathbf{i}(x-y)\cdot\tau(\boldsymbol{\xi_p})}\\
&\times\mathbb{E}\left[e^{-\mathbf{i}\sum_{j=1}^p(X^1_{4t}-X^1_{4s_1^\sigma}+X^1_{s_1^\sigma}-X^1_{s_j^\sigma})\cdot\xi_j^\sigma}\right]\mathbb{E}\left[e^{-\mathbf{i}\sum_{j=1}^p(X^1_{4t}-X^1_{4r_1^\pi}+X^1_{r_1^\pi}-X^1_{r_j^\pi})\cdot\xi_j^\pi}\right].
\end{aligned}\tag{3.24}$$

Then, we can write for $p\geq 2$,

$$\begin{aligned}
\big\|J_{1,p,R}\big\|^2_{L^2(\Omega)} &= p!\int_{B_R^2}dxdy\big\langle\mathfrak{g}_{1,p,x},\mathfrak{g}_{1,p,y}\big\rangle_{\mathscr{H}^{\otimes p}}\\
&\leq C_\alpha^2(t-s)^{2\alpha}p!\int_{B_R^2}dxdy\big\langle\widehat{\mathfrak{g}}_{1,p,x},\widehat{\mathfrak{g}}_{1,p,y}\big\rangle_{\mathscr{H}^{\otimes p}}\\
&\leq C_\alpha^2(t-s)^{2\alpha}p!\omega_dR^d\int_{\mathbb{R}^d}dz\big\langle\widehat{\mathfrak{g}}_{1,p,0},\widehat{\mathfrak{g}}_{1,p,z}\big\rangle_{\mathscr{H}^{\otimes p}}.
\end{aligned}\tag{3.25}$$

By the same trick of inserting $\exp\big(-\frac{\varepsilon}{2}\|z\|^2\big)$, we have

$$\int_{\mathbb{R}^d}dz\big\langle\widehat{\mathfrak{g}}_{1,p,0},\widehat{\mathfrak{g}}_{1,p,z}\big\rangle_{\mathscr{H}^{\otimes p}} = \lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^d}dz\big\langle\widehat{\mathfrak{g}}_{1,p,0},\widehat{\mathfrak{g}}_{1,p,z}\big\rangle_{\mathscr{H}^{\otimes p}}e^{-\frac{\varepsilon}{2}\|z\|^2} =: \lim_{\varepsilon\downarrow 0}\widehat{T}_{p,\varepsilon},\tag{3.26}$$

where $\widehat{T}_{p,\varepsilon}$ is equal to

$$\begin{aligned}
&\int_{[0,s]^{2p}}d\boldsymbol{s_p}d\boldsymbol{r_p}\prod_{j=1}^p\gamma_0(s_j-r_j)\int_{\mathbb{R}^{pd+d}}dz\mu_1(d\boldsymbol{\xi_p})\big(\mathscr{F}\widehat{\mathfrak{g}}_{1,p,0}\big)(\boldsymbol{s_p},\boldsymbol{\xi_p})\big(\mathscr{F}\widehat{\mathfrak{g}}_{1,p,z}\big)(\boldsymbol{r_p},-\boldsymbol{\xi_p})\\
&=\frac{(2\pi)^d}{(p!)^2}\sum_{\sigma,\pi\in\mathfrak{S}_p}\int_{\Delta_p(s)^2}d\boldsymbol{s_p^\sigma}d\boldsymbol{r_p^\pi}\frac{\prod_{j=1}^p\gamma_0(s_j-r_j)}{(s-s_1^\sigma)^\alpha(s-r_1^\pi)^\alpha}\int_{\mathbb{R}^{pd}}\mu_1(d\boldsymbol{\xi_p})G(\varepsilon,\tau(\boldsymbol{\xi_p}))\\
&\times\mathbb{E}\left[e^{-\mathbf{i}\sum_{j=1}^p(X^1_{4t}-X^1_{4s_1^\sigma}+X^1_{s_1^\sigma}-X^1_{s_j^\sigma})\cdot\xi_j^\sigma}\right]\mathbb{E}\left[e^{-\mathbf{i}\sum_{j=1}^p(X^1_{4t}-X^1_{4r_1^\pi}+X^1_{r_1^\pi}-X^1_{r_j^\pi})\cdot\xi_j^\pi}\right].
\end{aligned}\tag{3.27}$$

Note that for $\boldsymbol{s_p^\sigma}\in\Delta_p(s)$, $2t-2s_1^\sigma>2s-2s_1^\sigma>\frac{1}{2}(s-s_1^\sigma)$ so that

$$\mathbb{E}\left[e^{-\mathbf{i}\sum_{j=1}^p(X^1_{4t}-X^1_{4s_1^\sigma}+X^1_{s_1^\sigma}-X^1_{s_j^\sigma})\cdot\xi_j^\sigma}\right] = e^{-(2t-2s_1^\sigma)\|\tau(\boldsymbol{\xi_p})\|^2}e^{-\frac{1}{2}\sum_{j=1}^{p-1}(s_j^\sigma-s_{j+1}^\sigma)\|\xi_{j+1}^\sigma+\cdots+\xi_p^\sigma\|^2}$$

$$\begin{aligned}
&\leq e^{-\frac{1}{2}(s-s_1^\sigma)\|\tau(\boldsymbol{\xi_p})\|^2} e^{-\frac{1}{2}\sum_{j=1}^{p-1}(s_j^\sigma - s_{j+1}^\sigma)\|\xi_{j+1}^\sigma+\cdots+\xi_p^\sigma\|^2} \\
&= \mathbb{E}\left[e^{-\mathbf{i}\sum_{j=1}^p (X_s^1 - X_{s_j^\sigma}^1)\cdot\xi_j^\sigma}\right] = \mathbb{E}\left[e^{-\mathbf{i}\sum_{j=1}^p (X_s^1 - X_{s_j}^1)\cdot\xi_j}\right] \\
&= \exp\left(-\frac{1}{2}\mathrm{Var}\sum_{j=1}^p (X_s^1 - X_{s_j}^1)\cdot\xi_j\right). \qquad (3.28)
\end{aligned}$$

Therefore, we can write

$$\begin{aligned}
\widehat{T}_{p,\varepsilon} &\leq \frac{(2\pi)^d}{(p!)^2}\int_{[0,s]^{2p}} d\boldsymbol{s_p} d\boldsymbol{r_p} \frac{\prod_{j=1}^p \gamma_0(s_j - r_j)}{(s-s_1)^\alpha (s-r_1)^\alpha}\int_{\mathbb{R}^{pd}}\mu_1(d\boldsymbol{\xi_p})G(\varepsilon,\tau(\boldsymbol{\xi_p})) \\
&\quad \times \mathbb{E}\left[e^{-\mathbf{i}\sum_{j=1}^p (X_s^1 - X_{s_j}^1)\cdot\xi_j}\right]\mathbb{E}\left[e^{-\mathbf{i}\sum_{j=1}^p (X_s^1 - X_{r_j}^1)\cdot\xi_j}\right]\mathbf{1}_{\{s_1>s_2\vee\cdots\vee s_p\}}\mathbf{1}_{\{r_1>r_2\vee\cdots\vee r_p\}} \\
&\leq \frac{(2\pi)^d\Gamma_s^{p-1}}{(p!)^2}\int_{[0,s]^{p+1}} dr_1 ds_1\cdots ds_p \frac{\gamma_0(s_1-r_1)}{(s-r_1)^\alpha (s-s_1)^\alpha}\mathbf{1}_{\{s_1>s_2\vee\cdots\vee s_p\}} \\
&\quad \times \int_{\mathbb{R}^{pd}}\mu_1(d\boldsymbol{\xi_p})G(\varepsilon,\tau(\boldsymbol{\xi_p}))\exp\left(-\frac{1}{2}\mathrm{Var}\sum_{j=1}^p (X_s^1 - X_{s_j}^1)\cdot\xi_j\right).
\end{aligned}$$

By the usual time change $(r_1, s_j)\to(s-r_1, s-s_j)$, we have

$$\begin{aligned}
\widehat{T}_{p,\varepsilon} &\leq \frac{(2\pi)^d\Gamma_s^{p-1}}{(p!)^2}\int_{[0,s]^{p+1}} dr_1 ds_1\cdots ds_p \frac{\gamma_0(s_1-r_1)}{r_1^\alpha s_1^\alpha}\mathbf{1}_{\{s_1<s_2\wedge\cdots\wedge s_p\}} \\
&\quad \times \int_{\mathbb{R}^{pd}}\mu_1(d\boldsymbol{\xi_p})G(\varepsilon,\tau(\boldsymbol{\xi_p}))\exp\left(-\frac{1}{2}\mathrm{Var}\sum_{j=1}^p X_{s_j}^1\cdot\xi_j\right).
\end{aligned}$$

Note that for $s_1 < s_2\wedge\cdots\wedge s_p$

$$\begin{aligned}
e^{-\frac{1}{2}\mathrm{Var}\sum_{j=1}^p X_{s_j}^1\cdot\xi_j} &= e^{-\frac{1}{2}s_1\|\tau(\boldsymbol{\xi_p})\|^2} e^{-\frac{1}{2}\mathrm{Var}\sum_{j=2}^p (X_{s_j}^1 - X_{s_1}^1)\cdot\xi_j} \\
&= e^{-\frac{1}{2}s_1\|\tau(\boldsymbol{\xi_p})\|^2} e^{-\frac{1}{2}\mathrm{Var}\sum_{j=2}^p X_{s_j-s_1}^1\cdot\xi_j}.
\end{aligned}$$

Then, by another time change $(s_j - s_1\to s_j)$ for $j\geq 2$, we can write

$$\begin{aligned}
\widehat{T}_{p,\varepsilon} &\leq \frac{(2\pi)^d\Gamma_s^{p-1}}{(p!)^2}\int_0^s\int_0^s dr_1 ds_1 \frac{\gamma_0(s_1-r_1)}{r_1^\alpha s_1^\alpha}\int_{[0,s-s_1]^{p-1}} ds_2\cdots ds_p \\
&\quad \times \int_{\mathbb{R}^{pd}}\mu_1(d\boldsymbol{\xi_p})G(\varepsilon,\tau(\boldsymbol{\xi_p}))e^{-\frac{1}{2}s_1\|\tau(\boldsymbol{\xi_p})\|^2} e^{-\frac{1}{2}\mathrm{Var}\sum_{j=2}^p X_{s_j}^1\cdot\xi_j} \\
&\leq \frac{(2\pi)^d\Gamma_s^{p-1}}{(p!)^2}\left(\int_0^s\int_0^s dr_1 ds_1 \frac{\gamma_0(s_1-r_1)}{r_1^\alpha s_1^\alpha}\right) \\
&\quad \times \int_{[0,s]^{p-1}} ds_2\cdots ds_p \int_{\mathbb{R}^{pd}}\mu_1(d\boldsymbol{\xi_p})G(\varepsilon,\tau(\boldsymbol{\xi_p}))e^{-\frac{1}{2}\mathrm{Var}\sum_{j=2}^p X_{s_j}^1\cdot\xi_j} \\
&= \frac{(2\pi)^d\Gamma_s^{p-1}}{(p!)^2}\left(\int_0^s\int_0^s dr_1 ds_1 \frac{\gamma_0(s_1-r_1)}{r_1^\alpha s_1^\alpha}\right)(p-1)!\int_{\mathrm{SIM}_{p-1}(s)} dw_2\cdots dw_p
\end{aligned}$$

$$\times \int_{\mathbb{R}^{pd}} \mu_1(d\boldsymbol{\xi_p}) G(\varepsilon, \tau(\boldsymbol{\xi_p})) \exp\left(-\frac{1}{2}\sum_{j=2}^{p} w_j \|\xi_2 + \cdots + \xi_j\|^2\right). \tag{3.29}$$

Now making the change of variables $\eta_j = \xi_1 + \cdots + \xi_j$ yields

$$\begin{aligned}
&\int_{\mathrm{SIM}_{p-1}(s)} dw_2 \cdots dw_p \int_{\mathbb{R}^{pd}} \mu_1(d\boldsymbol{\xi_p}) G(\varepsilon, \tau(\boldsymbol{\xi_p})) \exp\left(-\frac{1}{2}\sum_{j=2}^{p} w_j \|\xi_2 + \cdots + \xi_j\|^2\right) \\
&= \int_{\mathrm{SIM}_{p-1}(s)} dw_2 \cdots dw_p \int_{\mathbb{R}^d} d\eta_p G(\varepsilon, \eta_p) \int_{\mathbb{R}^{pd-d}} d\boldsymbol{\eta_{p-1}} \left(\varphi_1(\eta_1) e^{-\frac{1}{2}w_p\|\eta_p-\eta_1\|^2}\right) \\
&\quad \times \left(\varphi_1(\eta_2-\eta_1)\varphi_1(\eta_3-\eta_2) e^{-\frac{1}{2}w_2\|\eta_2-\eta_1\|^2}\right)\left(\varphi_1(\eta_4-\eta_3) e^{-\frac{1}{2}w_3\|\eta_3-\eta_1\|^2}\right) \\
&\quad \times \cdots \times \left(\varphi_1(\eta_p-\eta_{p-1}) e^{-\frac{1}{2}w_{p-1}\|\eta_{p-1}-\eta_1\|^2}\right).
\end{aligned}$$

Moreover, we can apply (4.3) and (4.2) to the integral with respect to the variables $d\eta_2, d\eta_3, \ldots, d\eta_{p-1}, d\eta_1$ in order to get

$$\begin{aligned}
\int_{\mathbb{R}^d} d\eta_2 \varphi_1(\eta_2-\eta_1)\varphi_1(\eta_3-\eta_2) e^{-\frac{1}{2}w_2\|\eta_2-\eta_1\|^2} &\le \int_{\mathbb{R}^d} \varphi_1(\xi)^2 e^{-\frac{1}{2}w_2\|\xi\|^2} d\xi \\
\int_{\mathbb{R}^d} d\eta_3 \varphi_1(\eta_4-\eta_3) e^{-\frac{1}{2}w_3\|\eta_3-\eta_1\|^2} &\le \int_{\mathbb{R}^d} \varphi_1(\xi) e^{-\frac{1}{2}w_3\|\xi\|^2} d\xi \\
&\cdots\cdots \\
\int_{\mathbb{R}^d} d\eta_{p-1} \varphi_1(\eta_p-\eta_{p-1}) e^{-\frac{1}{2}w_{p-1}\|\eta_{p-1}-\eta_1\|^2} &\le \int_{\mathbb{R}^d} \varphi_1(\xi) e^{-\frac{1}{2}w_{p-1}\|\xi\|^2} d\xi \\
\int_{\mathbb{R}^d} d\eta_1 \varphi_1(\eta_1) e^{-\frac{1}{2}w_p\|\eta_p-\eta_1\|^2} &\le \int_{\mathbb{R}^d} \varphi_1(\xi) e^{-\frac{1}{2}w_p\|\xi\|^2} d\xi.
\end{aligned}$$

Thus, with $\Gamma_{s,\alpha} = \int_0^s \int_0^s dr_1 ds_1 \gamma_0(s_1 - r_1) r_1^{-\alpha} s_1^{-\alpha}$, we have

$$\begin{aligned}
\widehat{T}_{p,\varepsilon} &\le \frac{(2\pi)^d \Gamma_s^{p-1} \|\varphi_1\|_\infty \Gamma_{s,\alpha}}{p!p} \int_{\mathrm{SIM}_{p-1}(s)} dw_2 \cdots dw_p \int_{\mathbb{R}^{pd-d}} \prod_{j=2}^{p} \varphi_1(\xi_j) e^{-\frac{1}{2}w_j\|\xi_j\|^2} \\
&\le \frac{(2\pi)^d \Gamma_s^{p-1} \|\varphi_1\|_\infty \Gamma_{s,\alpha}}{p!p} \sum_{j=1}^{p-1} \binom{p-1}{j} \frac{s^j}{j!} D_N^j (2C_N)^{p-1-j} \quad \text{by (3.4)} \\
&\le \frac{(2\pi)^d \|\varphi_1\|_\infty \Gamma_{s,\alpha} \exp\left(sD_N/(2C_N)\right)}{p!p} (4C_N\Gamma_s)^{p-1}.
\end{aligned}$$

Therefore, for $p \ge 2$,

$$\begin{aligned}
&\left\|J_{1,p,R}\right\|_{L^2(\Omega)}^2 \\
&\le (t-s)^{2\alpha} R^d \left\{(2\pi)^d C_\alpha^2 \omega_d \|\varphi_1\|_\infty \Gamma_{s,\alpha} \exp\left(sD_N/(2C_N)\right)\right\} (4C_N\Gamma_s)^{p-1}.
\end{aligned}$$

For $p=1$, it is easier to get the desired bound. Indeed, from (3.27), it follows that

$$\begin{aligned}\widehat{T}_{1,\varepsilon} &= (2\pi)^d \int_0^s \int_0^s ds_1 dr_1 \gamma_0(s_1-r_1)(s-s_1)^{-\alpha}(s-r_1)^{-\alpha} \int_{\mathbb{R}^d} d\xi \varphi_1(\xi) G(\varepsilon,\xi) \\ &\qquad \times \mathbb{E}\left[e^{-\mathbf{i}(X^1_{4t}-X^1_{4s_1})\cdot\xi}\right] \mathbb{E}\left[e^{-\mathbf{i}(X^1_{4t}-X^1_{4r_1})\cdot\xi}\right] \\ &\le (2\pi)^d \|\varphi_1\|_\infty \Gamma_{s,\alpha},\end{aligned}$$

so that

$$\left\|J_{1,1,R}\right\|^2_{L^2(\Omega)} \le (t-s)^{2\alpha} R^d \Big\{(2\pi)^d C^2_\alpha \omega_d \|\varphi_1\|_\infty \Gamma_{s,\alpha}\Big\}.$$

Hence,

$$\begin{aligned}\frac{1}{R^{d/2}}\left\|\sum_{p\ge1} J_{1,p,R}\right\|_{L^k(\Omega)} &\le \frac{1}{R^{d/2}} \sum_{p\ge1} (k-1)^{p/2} \left\|J_{1,p,R}\right\|_{L^2(\Omega)} \\ \le (t-s)^\alpha \Big\{(2\pi)^d C^2_\alpha \omega_d \|\varphi_1\|_\infty \big[1+\exp(TD_N C_N^{-1})\big]\Gamma_{s,\alpha}\Big\}^{1/2} &\sum_{p\ge0}\Big[4(k-1)\Gamma_T C_N\Big]^{p/2} \\ = (t-s)^\alpha \frac{\Big\{(2\pi)^d C^2_\alpha \omega_d \|\varphi_1\|_\infty \big[1+\exp(TD_N C_N^{-1})\big]\Gamma_{s,\alpha}\Big\}^{1/2}}{1-2\sqrt{(k-1)\Gamma_T C_N}}, &\qquad (3.30)\end{aligned}$$

provided $0<4(k-1)\Gamma_T C_N<1$, which is always valid for some $N>0$.

Combing (3.23) and (3.30), we get (3.19) and hence the desired tightness. □

3.4 Proof of Theorem 1.7

We are going to show that, under the hypotheses of Theorem 1.7, the first chaos dominates and, as a consequence, the proof of the central limit theorem reduces to the computation of the limit variance of the first chaos. The proof will be done in several steps.

Step 1. We have shown in the proof of Theorem 1.6 that, if γ_0 is locally integrable, γ_1 is integrable and Dalang's condition (1.10) is satisfied, then for any integer $p\ge2$,

$$\mathrm{Var}\Big(\Pi_p A_t(R)\Big) \sim \sigma_p(t,t) R^d \text{ as } R\to+\infty \text{ and } \sum_{p\ge2}\sigma_p(t,t)<\infty. \qquad (3.31)$$

The above results also hold true, provided γ_0 is locally integrable and the modified version of Dalang's condition (1.15) is satisfied. To see the latter point, it is enough to proceed with the same arguments but replacing the estimate (3.3) by

$$\int_{\mathbb{R}^d} \varphi_1(\eta_1)\varphi_1(\eta_2-\eta_1) h_1(\eta_1) d\eta_1 \le \int_{\mathbb{R}^d} \varphi_1(\eta_1)^2 h_1(\eta_1) d\eta_1,$$

obtained by applying (4.2). Then, we can use the same arguments as in the proof of [9, Lemma 3.3], with C_N, D_N replaced by

$$C'_N = \int_{\{\|\xi\|\ge N\}} \frac{\varphi_1(\xi)+\varphi_1(\xi)^2}{\|\xi\|^2} d\xi \quad \text{and} \quad D'_N = \int_{\{\|\xi\|\le N\}} \big(\varphi_1(\xi)+\varphi_1(\xi)^2\big) d\xi.$$

In this way, instead of the inequality (3.4), we can get

$$Q_p(\eta_p) \le t \sum_{j=0}^{p-1} \binom{p-1}{j} \frac{t^j}{j!} (D'_N)^j (2C'_N)^{p-1-j} \tag{3.32}$$

and by choosing large N such that $0 < 4\Gamma_t C'_N < 1$, we can get instead of (3.6)

$$\int_{\mathbb{R}^d} \mathbb{E}\big[\beta_{s,t}(z)^p\big] dz \le (2\pi)^d \Gamma_t^p p! t (4C'_N)^{p-1} \exp\left(\frac{tD'_N}{2C'_N}\right) < \infty \tag{3.33}$$

and as a result,

$$\sum_{p\ge 2} \frac{1}{p!} \int_{\mathbb{R}^d} \mathbb{E}\big[\beta_{t,t}(z)^p\big] dz < +\infty,$$

which is equivalent to (3.31).

Step 2. For the first chaotic component, if $\gamma_1 \notin L^1(\mathbb{R}^d)$, then

$$R^{-d}\mathrm{Var}\Big(\Pi_1 A_t(R)\Big) \to \infty \text{ as } R \to +\infty.$$

This observation, together with Step 1, justifies part (1) of Theorem 1.7.

Step 3. When $\gamma_1(z) = \|z\|^{-\beta}$ for some $\beta \in (0, 2\wedge d)$, let us first compute the variance of $\Pi_1 A_t(R)$. We have

$$\begin{aligned}\mathrm{Var}\big(\Pi_1 A_t(R)\big) &= \int_0^t \int_0^t dudv \gamma_0(u-v) \\ &\quad \times \int_{\mathbb{R}^d} d\xi \int_{B_R^2} dxdy e^{-\mathbf{i}(x-y)\cdot\xi} c_{d,\beta} \|\xi\|^{\beta-d} e^{-\frac{1}{2}(u+v)\|\xi\|^2},\end{aligned}$$

for some constant $c_{d,\beta}$. Then by making change of variables $(x,y,\xi) \to (Rx, Ry, \xi/R)$, we get

$$\begin{aligned}&\mathrm{Var}\big(\Pi_1 A_t(R)\big) R^{-2d+\beta} \\ &= \int_0^t \int_0^t dudv \gamma_0(u-v) \int_{\mathbb{R}^d} d\xi \left[\int_{B_1^2} dxdy e^{-\mathbf{i}(x-y)\cdot\xi}\right] c_{d,\beta} \|\xi\|^{\beta-d} e^{-\frac{1}{2R^2}(u+v)\|\xi\|^2}.\end{aligned} \tag{3.34}$$

This expression is increasing in R and it converges, as $R \to +\infty$, to

$$\int_0^t \int_0^t dudv \gamma_0(u-v) \int_{\mathbb{R}^d} d\xi \int_{B_1^2} dxdy e^{-\mathbf{i}(x-y)\cdot\xi} \varphi_1(\xi) = \kappa_\beta \in (0,\infty).$$

Then, it suffices to show that

$$\sum_{p\ge 2} \mathrm{Var}\big(\Pi_p A_t(R)\big) = o(R^{2d-\beta}),$$

which implies the central limit theorem (1.16) immediately. For $p \geq 2$, we read from (3.12), (3.13) and (3.14) that

$$\mathrm{Var}\big(\Pi_p A_t(R)\big) = \frac{c_{d,\beta}^p}{p!} \int_{B_R^2} dxdy \int_{[0,t]^{2p}} d\boldsymbol{s_p} d\boldsymbol{r_p} \prod_{j=1}^p \gamma_0(s_j - r_j) \int_{\mathbb{R}^{pd}} d\boldsymbol{\xi_p} \left(\prod_{j=1}^p \|\xi_j\|^{\beta-d} \right) \times e^{-\mathbf{i}(x-y)\cdot\tau(\boldsymbol{\xi_p})} e^{-\frac{1}{2}\mathrm{Var}\sum_{j=1}^p \xi_j\cdot X_{s_j}^1} e^{-\frac{1}{2}\mathrm{Var}\sum_{j=1}^p \xi_j\cdot X_{r_j}^2}.$$

Note that

$$\int_{B_R^2} dxdy e^{-\mathbf{i}(x-y)\cdot\tau(\boldsymbol{\xi_p})} = (2\pi R)^d \omega_d \ell_R\big(\tau(\boldsymbol{\xi_p})\big) \geq 0.$$

Then by similar arguments as before, we obtain

$$\begin{aligned}
\mathrm{Var}\big(\Pi_p A_t(R)\big) &\leq \frac{c_{d,\beta}^p}{p!} \int_{B_R^2} dxdy \int_{[0,t]^{2p}} d\boldsymbol{s_p} d\boldsymbol{r_p} \prod_{j=1}^p \gamma_0(s_j - r_j) \int_{\mathbb{R}^{pd}} d\boldsymbol{\xi_p} \left(\prod_{j=1}^p \|\xi_j\|^{\beta-d} \right) \\
&\qquad \times e^{-\mathbf{i}(x-y)\cdot\tau(\boldsymbol{\xi_p})} \exp\left(-\frac{1}{2}\mathrm{Var}\sum_{j=1}^p \xi_j \cdot X_{s_j}^1 \right) \\
&\leq c_{d,\beta}^p \Gamma_t^p \int_{B_R^2} dxdy \int_{\mathrm{SIM}_p(t)} d\boldsymbol{w_p} \int_{\mathbb{R}^{pd}} d\boldsymbol{\xi_p} \left(\prod_{j=1}^p \|\xi_j\|^{\beta-d} \right) e^{-\mathbf{i}(x-y)\cdot\tau(\boldsymbol{\xi_p})} \\
&\qquad \times \exp\left(-\frac{1}{2} \sum_{j=1}^p w_j \|\xi_1 + \cdots + \xi_j\|^2 \right).
\end{aligned}$$

By the usual change of variables $\eta_j = \xi_1 + \cdots + \xi_j$, with $\eta_0 = 0$, and $(x, y, \eta_p) \to (Rx, Ry, \eta_p/R)$, we obtain

$$\begin{aligned}
\mathrm{Var}\big(\Pi_p A_t(R)\big) &\leq c_{d,\beta}^{p-1} \Gamma_t^p R^d \int_{\mathrm{SIM}_p(t)} d\boldsymbol{w_p} \int_{\mathbb{R}^{pd-d}} d\boldsymbol{\eta_{p-1}} \left(\prod_{j=1}^{p-1} \|\eta_j - \eta_{j-1}\|^{\beta-d} e^{-\frac{1}{2}w_j\|\eta_j\|^2} \right) \\
&\quad \times \int_{\mathbb{R}^d} d\eta_p \|\eta_p R^{-1} - \eta_{p-1}\|^{\beta-d} \int_{B_1^2} dxdy e^{-\mathbf{i}(x-y)\cdot\eta_p} e^{-w_p\|\eta_p\|^2/(2R^2)}. \qquad (3.35)
\end{aligned}$$

Let us first analyze the part in the display (3.35), which can be rewritten as

$$\begin{aligned}
&R^{d-\beta} \int_{\mathbb{R}^d} d\eta_p \|\eta_p - R\eta_{p-1}\|^{\beta-d} \int_{B_1^2} dxdy e^{-\mathbf{i}(x-y)\cdot\eta_p} e^{-w_p\|\eta_p\|^2/(2R^2)} \\
&\leq R^{d-\beta} \int_{B_1^2} dxdy \int_{\mathbb{R}^d} d\eta_p \|\eta_p - R\eta_{p-1}\|^{\beta-d} e^{-\mathbf{i}(x-y)\cdot\eta_p} \\
&= c_{d,\beta}^{-1} R^{d-\beta} \int_{B_1^2} dxdy e^{-\mathbf{i}(x-y)\cdot\eta_{p-1}R} \|x-y\|^{-\beta} =: R^{d-\beta} U_R(\eta_{p-1}). \qquad (3.36)
\end{aligned}$$

The function U_R defined above is uniformly bounded by $c_{d,\beta}^{-1}\int_{B_1^2} dxdy\|x-y\|^{-\beta}$ and for $\eta_{p-1}\neq 0$, by the Riemann-Lebesgue's Lemma, $0\leq U_R(\eta_{p-1})$ converges to zero as $R\to+\infty$. As a result,

$$\begin{aligned} R^{-2d+\beta}\sum_{p\geq 2}\mathrm{Var}\big(\Pi_p A_t(R)\big) &\leq \sum_{p\geq 2} t\Gamma_t^p c_{d,\beta}^p \int_{\mathrm{SIM}_{p-1}(t)} d\boldsymbol{w_{p-1}} \int_{\mathbb{R}^{pd-d}} d\boldsymbol{\eta_{p-1}} \\ &\quad\times\left(\prod_{j=1}^{p-1}\|\eta_j-\eta_{j-1}\|^{\beta-d} e^{-\frac12 w_j\|\eta_j\|^2}\right) U_R(\eta_{p-1}) \\ &\leq t\left(\int_{B_1^2} dxdy\|x-y\|^{-\beta}\right)\sum_{p\geq 2}\Gamma_t^p \int_{\mathrm{SIM}_{p-1}(t)} d\boldsymbol{w_{p-1}} \\ &\quad\times\int_{\mathbb{R}^{pd-d}} d\boldsymbol{\eta_{p-1}}\left(\prod_{j=1}^{p-1}\varphi_1(\eta_j-\eta_{j-1}) e^{-\frac12 w_j\|\eta_j\|^2}\right). \end{aligned}$$

By using (4.3) for the integration with respect to $d\eta_{p-1},\dots,d\eta_3,d\eta_2$ inductively, we get

$$\begin{aligned} &\sum_{p\geq 2}\Gamma_t^p \int_{\mathrm{SIM}_{p-1}(t)} d\boldsymbol{w_{p-1}} \int_{\mathbb{R}^{pd-d}} d\boldsymbol{\eta_{p-1}}\left(\prod_{j=1}^{p-1}\varphi_1(\eta_j-\eta_{j-1}) e^{-\frac12 w_j\|\eta_j\|^2}\right) \\ &\leq \sum_{p\geq 2}\Gamma_t^p \int_{\mathrm{SIM}_{p-1}(t)} d\boldsymbol{w_{p-1}} \int_{\mathbb{R}^{pd-d}} d\boldsymbol{\eta_{p-1}}\left(\prod_{j=1}^{p-1}\varphi_1(\eta_j) e^{-\frac12 w_j\|\eta_j\|^2}\right), \end{aligned}$$

which is a convergent series by previous discussion. Then by dominated convergence and the Riemann-Lebesgue's lemma, we have

$$\sum_{p\geq 2}\mathrm{Var}\big(\Pi_p A_t(R)\big) = o(R^{2d-\beta}).$$

This tells us that the first chaos is indeed dominant and we have the desired Gaussian fluctuation (1.16). This concludes the proof of Theorem 1.7. □

3.5 Proof of Theorem 1.9

Part (1): The proof of the functional CLT for $\widehat{A}_t(R)$ can be done exactly by the same arguments from Sections 3.1, 3.2 and 3.3 except for using (3.32) and (3.33) instead of (3.4) and (3.6). So we leave the details for interested readers and refer to the forthcoming work [24] for similar situation when dealing with parabolic Anderson model driven by rough noise.

Part (2): By results in part (2) of Theorem 1.7, $R^{-d+\frac{\beta}{2}}\widehat{A}_t(R)$ converges to the zero process in finite-dimensional distributions. So our proof consists in two parts:

(i) We prove $\left\{R^{-d+\frac{\beta}{2}}\Pi_1\big(A_t(R)\big) : t\in\mathbb{R}_+\right\}\xrightarrow[\text{law}]{R\to\infty}\widetilde{\mathcal{G}}$.

(ii) We prove $\{R^{-d+\frac{\beta}{2}}\widehat{A}_t(R): t\geq 0\}$ converges in law (hence in probability) to the zero process, as $R\to\infty$. This will follow from the tightness of $\{R^{-d+\frac{\beta}{2}}\widehat{A}_\bullet(R): R>0\}$.

Proof of (i): It is clear that $R^{-d+\frac{\beta}{2}}\Pi_1\big(A_t(R)\big) = R^{-d+\frac{\beta}{2}}\int_0^t\int_{\mathbb{R}^d} G_{t-r}(x-z)W(dr,dz)$, $t\in\mathbb{R}_+$ is a centered Gaussian process with

$$
\begin{aligned}
&R^{-2d+\beta}\mathbb{E}\big[\Pi_1\big(A_t(R)\big)\Pi_1\big(A_s(R)\big)\big]\\
&= \int_0^t\int_0^s dudv\gamma_0(u-v)\int_{\mathbb{R}^d} d\xi\left[\int_{B_1^2} dxdy e^{-\mathbf{i}(x-y)\cdot\xi}\right] c_{d,\beta}\|\xi\|^{\beta-d}e^{-\frac{(t-u+s-v)}{2R^2}\|\xi\|^2}
\end{aligned}
$$

by the same change of variables as in (3.34). By monotone convergence, we have

$$
R^{-2d+\beta}\mathbb{E}\big[\Pi_1\big(A_t(R)\big)\Pi_1\big(A_s(R)\big)\big] \xrightarrow{R\to\infty} \int_0^t\int_0^s dudv\gamma_0(u-v)\int_{B_1^2} dxdy\|x-y\|^{-\beta}.
$$

This implies easily the convergence in finite-dimensional distributions. As in section 3.3, we let $s<t$ and write

$$
\Pi_1\big(A_t(R)\big) - \Pi_1\big(A_s(R)\big) = J_{1,1,R} + J_{2,1,R}
$$

with $J_{1,1,R} := \int_0^s\int_{\mathbb{R}^d}\Big(\int_{B_R} d_1(s,t,x;s_1,y_1)dx\Big)W(ds_1,dy_1)$ and

$$
J_{2,1,R} := \int_0^t\int_{\mathbb{R}^d}\left(\int_{B_R} d_2(s,t,x;s_1,y_1)dx\right)W(ds_1,dy_1),
$$

where d_1, d_2 are introduced in (3.20), (3.21) and

$$
\big|d_1(s,t,x;s_1,y_1)\big| \leq C(t-s)^\alpha(s-s_1)^{-\alpha}G(4t-4s_1,x-y_1)\mathbf{1}_{[0,s)}(s_1).
$$

As before, we can write

$$
\begin{aligned}
\big\|J_{1,1,R}\big\|^2_{L^2(\Omega)} &= \int_0^s\int_0^s ds_1ds_2\gamma_0(s_1-s_2)\int_{\mathbb{R}^{2d}} dy_1dy_2\|y_1-y_2\|^{-\beta}\int_{B_R^2} dx_1dx_2\\
&\quad\times d_1(s,t,x_1;s_1,y_1)d_1(s,t,x_2;s_2,y_2)\\
&\leq C(t-s)^{2\alpha}\int_0^s\int_0^s ds_1ds_2\gamma_0(s_1-s_2)(s-s_1)^{-\alpha}(s-s_2)^{-\alpha}\int_{\mathbb{R}^{2d}} dy_1dy_2\|y_1-y_2\|^{-\beta}\\
&\quad\times\int_{B_R^2} dx_1dx_2G(4t-4s_1,x_1-y_1)G(4t-4s_2,x_2-y_2)\\
&= C(t-s)^{2\alpha}\int_0^s\int_0^s ds_1ds_2\gamma_0(s_1-s_2)(s-s_1)^{-\alpha}(s-s_2)^{-\alpha}\int_{\mathbb{R}^d} d\xi c_{d,\beta}\|\xi\|^{\beta-d}\\
&\qquad\times\int_{B_R^2} dx_1dx_2e^{-\mathbf{i}(x_1-x_2)\cdot\xi}e^{-(2t-2s_1+2t-2s_2)\|\xi\|^2}\\
&\leq C(t-s)^{2\alpha}\int_0^s\int_0^s ds_1ds_2\gamma_0(s_1-s_2)s_1^{-\alpha}s_2^{-\alpha}\int_{\mathbb{R}^d} d\xi c_{d,\beta}\|\xi\|^{\beta-d}
\end{aligned}
$$

$$\times \int_{B_R^2} dx_1 dx_2 e^{-\mathbf{i}(x_1-x_2)\cdot\xi}.$$

Making the change of variables $(x_1, x_2, \xi) \to (Rx_1, Rx_2, \xi/R)$ yields

$$\begin{aligned}\left\|J_{1,1,R}\right\|_{L^2(\Omega)}^2 &\le C(t-s)^{2\alpha} R^{2d-\beta} \int_0^s \int_0^s ds_1 ds_2 \gamma_0(s_1-s_2) s_1^{-\alpha} s_2^{-\alpha} \int_{\mathbb{R}^d} d\xi c_{d,\beta} \|\xi\|^{\beta-d} \\ &\times \left(\int_{B_1^2} dx_1 dx_2 e^{-\mathbf{i}(x_1-x_2)\cdot\xi} \right) = C(t-s)^{2\alpha} R^{2d-\beta} \Gamma_{s,\alpha} \int_{B_1^2} dx dy \|x-y\|^{-\beta},\end{aligned}$$

where $\Gamma_{s,\alpha}$ is given as in (3.18). Now let us estimate $\left\|J_{2,1,R}\right\|_{L^2(\Omega)}^2$:

$$\begin{aligned}\left\|J_{2,1,R}\right\|_{L^2(\Omega)}^2 &= \int_s^t \int_s^t ds_1 ds_2 \gamma_0(s_1-s_2) \int_{\mathbb{R}^{2d}} dy_1 dy_2 \|y_1-y_2\|^{-\beta} \int_{B_R^2} dx_1 dx_2 \\ &\quad \times G(t-s_1, x_1-y_1) G(t-s_2, x_2-y_2) \\ &= \int_s^t \int_s^t ds_1 ds_2 \gamma_0(s_1-s_2) \int_{\mathbb{R}^d} d\xi c_{d,\beta} \|\xi\|^{\beta-d} \int_{B_R^2} dx_1 dx_2 e^{-\mathbf{i}(x_1-x_2)\cdot\xi} e^{-\frac{(2t-s_1-s_2)}{2}\|\xi\|^2} \\ &\le R^{2d-\beta} \int_s^t \int_s^t ds_1 ds_2 \gamma_0(s_1-s_2) \int_{\mathbb{R}^d} d\xi c_{d,\beta} \|\xi\|^{\beta-d} \int_{B_1^2} dx_1 dx_2 e^{-\mathbf{i}(x_1-x_2)\cdot\xi} \\ &\le R^{2d-\beta}(t-s) \left(\int_{B_1^2} dx dy \|x-y\|^{-\beta} \right) \left(\int_{-t}^t \gamma_0(s_1) ds_1 \right).\end{aligned}$$

Hence given $T \in (0,\infty)$, we have for any $0 < s < t \le T$ and for any $k \in [2,\infty)$,

$$\left\|\Pi_1\big(A_t(R)\big) - \Pi_1\big(A_s(R)\big)\right\|_{L^k(\Omega)} = c_k \left\|\Pi_1\big(A_t(R)\big) - \Pi_1\big(A_s(R)\big)\right\|_{L^2(\Omega)} \le C(t-s)^\alpha,$$

where c_k is the $L^k(\Omega)$-norm of $Z \sim N(0,1)$ and the constant C does not depend on R, s or t. This gives us the desired tightness and hence leads to the functional CLT for $\big\{\Pi_1\big(A_t(R)\big) : t \in \mathbb{R}_+\big\}$.

Proof of (ii): Given $T \in (0,\infty)$, we consider any $0 < s < t \le T$ and as before, we write

$$\Pi_p(A_t(R)) - \Pi_p(A_s(R)) = J_{1,p,R} + J_{2,p,R}.$$

Then following the arguments that led to (3.35), we have

$$\begin{aligned}\left\|J_{2,p,R}\right\|_{L^2(\Omega)}^2 &\le C^p R^d \int_{\mathrm{SIM}_p(t-s)} d\boldsymbol{w_p} \int_{\mathbb{R}^{pd-d}} d\boldsymbol{\eta_{p-1}} \left(\prod_{j=1}^{p-1} \|\eta_j - \eta_{j-1}\|^{\beta-d} e^{-\frac{w_j\|\eta_j\|^2}{2}} \right) \\ &\quad \times \int_{\mathbb{R}^d} d\eta_p \|\eta_p R^{-1} - \eta_{p-1}\|^{\beta-d} \int_{B_1^2} dx dy e^{-\mathbf{i}(x-y)\cdot\eta_p} e^{-w_p\|\eta_p\|^2/(2R^2)} \\ &\le C^p R^{2d-\beta} \int_{\mathrm{SIM}_p(t-s)} d\boldsymbol{w_p} \int_{\mathbb{R}^{pd-d}} d\boldsymbol{\eta_{p-1}} \left(\prod_{j=1}^{p-1} \|\eta_j - \eta_{j-1}\|^{\beta-d} e^{-\frac{w_j\|\eta_j\|^2}{2}} \right), \text{ see (3.36)}\end{aligned}$$

$$\leq C^p R^{2d-\beta}(t-s)\int_{\mathrm{SIM}_{p-1}(t-s)} d\boldsymbol{w_{p-1}} \int_{\mathbb{R}^{pd-d}} d\boldsymbol{\eta_{p-1}} \prod_{j=1}^{p-1} \|\eta_j-\eta_{j-1}\|^{\beta-d} e^{-\frac{w_j\|\eta_j\|^2}{2}}.$$

By using (4.3) under the Dalang's condition, we have

$$\int_{\mathbb{R}^{pd-d}} d\boldsymbol{\eta_{p-1}} \prod_{j=1}^{p-1} \|\eta_j-\eta_{j-1}\|^{\beta-d} e^{-\frac{w_j\|\eta_j\|^2}{2}} \leq \prod_{j=1}^{p-1}\int_{\mathbb{R}^d} d\eta_j \|\eta_j\|^{\beta-d} e^{-\frac{w_j\|\eta_j\|^2}{2}}$$

so that by the same application of Lemma 3.3 in [9] as in (3.4), we deduce

$$\left\|J_{2,p,R}\right\|^2_{L^2(\Omega)} \leq C R^{2d-\beta}(t-s)(4C_N)^{p-1}.$$

where $C_N>0$ can be chosen arbitrarily small for large enough N, see (3.5).

Now let us estimate $\left\|J_{1,p,R}\right\|^2_{L^2(\Omega)}$: Following the arguments around (3.25), (3.26), (3.24), (3.28) and (3.29), we can write

$$\left\|J_{1,p,R}\right\|^2_{L^2(\Omega)} \leq C(t-s)^{2\alpha}\frac{1}{p!}\int_{B_R^2} dxdy \sum_{\sigma,\pi\in\mathfrak{S}_p}\int_{\Delta_p(s)^2} d\boldsymbol{s_p^\sigma} d\boldsymbol{r_p^\pi}\frac{\prod_{j=1}^p \gamma_0(s_j-r_j)}{(s-s_1^\sigma)^\alpha(s-r_1^\pi)^\alpha}$$
$$\times \int_{\mathbb{R}^{pd}} \mu_1(d\boldsymbol{\xi_p}) e^{-\mathbf{i}(x-y)\cdot\tau(\boldsymbol{\xi_p})}\exp\left(-\frac{1}{2}\mathrm{Var}\sum_{j=1}^p (X_s^1-X_{s_j}^1)\cdot\xi_j\right),$$

since $\int_{B_R^2} dxdy e^{-\mathbf{i}(x-y)\cdot\tau(\boldsymbol{\xi_p})}$ is nonnegative;

$$\leq \frac{C(t-s)^{2\alpha}\Gamma_{s,\alpha}\Gamma_s^{p-1}}{p}\int_{B_R^2} dxdy \int_{\mathrm{SIM}_{p-1}(s)} dw_2\cdots dw_p$$
$$\times \int_{\mathbb{R}^{pd}} \mu_1(d\boldsymbol{\xi_p}) e^{-\mathbf{i}(x-y)\cdot\tau(\boldsymbol{\xi_p})}\exp\left(-\frac{1}{2}\sum_{j=2}^p w_j\|\xi_2+\cdots+\xi_j\|^2\right).$$

Then by the usual change of variables $\eta_j=\xi_1+\cdots+\xi_j$ and $(x,y,\eta_p)\to(Rx,Ry,\frac{\eta_p}{R})$, we have

$$\int_{B_R^2} dxdy \int_{\mathbb{R}^{pd}} \mu_1(d\boldsymbol{\xi_p}) e^{-\mathbf{i}(x-y)\cdot\tau(\boldsymbol{\xi_p})}\exp\left(-\frac{1}{2}\sum_{j=2}^p w_j\|\xi_2+\cdots+\xi_j\|^2\right)$$
$$=\int_{B_R^2} dxdy\int_{\mathbb{R}^{pd}} d\boldsymbol{\eta_p}\|\eta_p-\eta_{p-1}\|^{\beta-d} e^{-\mathbf{i}(x-y)\cdot\eta_p} e^{-\frac{1}{2}\sum_{j=2}^p w_j\|\eta_j-\eta_1\|^2}\prod_{j=1}^{p-1}\|\eta_j-\eta_{j-1}\|^{\beta-d}$$
$$=R^{2d-\beta}\int_{\mathbb{R}^{pd-d}} d\boldsymbol{\eta_{p-1}} e^{-\frac{1}{2}\sum_{j=2}^{p-1} w_j\|\eta_j-\eta_1\|^2}\left(\prod_{j=1}^{p-1}\|\eta_j-\eta_{j-1}\|^{\beta-d}\right)$$
$$\times\left(\int_{B_1^2} dxdy\int_{\mathbb{R}^d} d\eta_p\|\eta_p-R\eta_{p-1}\|^{\beta-d} e^{-\mathbf{i}(x-y)\cdot\eta_p} e^{-\frac{w_p}{2}\|\eta_p R^{-1}-\eta_{p-1}\|^2}\right)$$

$$\leq \frac{\int_{B_1^2} dxdy\|x-y\|^{-\beta}}{c_{d,\beta}} R^{2d-\beta} \int_{\mathbb{R}^{pd-d}} d\boldsymbol{\eta_{p-1}} e^{-\frac{1}{2}\sum_{j=2}^{p-1} w_j\|\eta_j-\eta_1\|^2} \prod_{j=1}^{p-1} \|\eta_j-\eta_{j-1}\|^{\beta-d}$$
$$\leq CR^{2d-\beta} \prod_{j=1}^{p-1} \int_{\mathbb{R}^d} d\eta_j e^{-\frac{1}{2}w_j\|\eta_j\|^2} \|\eta_j\|^{\beta-d}$$

where the last inequality is a consequence of (4.3). So an application of Lemma 3.3 from [9] yields

$$\left\|J_{1,p,R}\right\|_{L^2(\Omega)}^2 \leq C(t-s)^{2\alpha}(4C_N\Gamma_s)^{p-1}.$$

Therefore, for large enough N, we deduce from the hypercontractivity property that for any $k \in [2,\infty)$

$$\left\|\widehat{A}_t(R)-\widehat{A}_s(R)\right\|_{L^k(\Omega)} \leq \sum_{p\geq 2}(k-1)^{p/2}\Big(\left\|J_{1,p,R}\right\|_{L^2(\Omega)}+\left\|J_{2,p,R}\right\|_{L^2(\Omega)}\Big)$$
$$\leq C(t-s)^{\alpha}R^{2d-\beta}\sum_{p\geq 2}\left(\left[4(k-1)C_N\Gamma_s\right]^{p/2}+\left[4(k-1)C_N\right]^{p/2}\right) \leq C(t-s)^{\alpha}R^{2d-\beta}.$$

This proves (ii), and hence concludes our proof. □

4 Proof of technical results

Proof of Proposition 2.7. Recall the definition of Ψ_p, which is defined a.e. by the following change of variables:

$$\int_{\mathbb{R}^{pd}} \|\tau(\boldsymbol{\xi_p})\|^{-d} J_{d/2}(R\|\tau(\boldsymbol{\xi_p})\|)^2 |\mathscr{F} f_p|^2(\boldsymbol{\xi_p})\mu(d\boldsymbol{\xi_p}) = \int_{\mathbb{R}^d} dx\|x\|^{-d}J_{d/2}(R\|x\|)^2\Psi_p(x)$$

with $\Psi_p(x)$ almost everywhere equal to

$$\int_{\mathbb{R}^{pd-d}} |\mathscr{F} f_p|^2(\boldsymbol{\xi_{p-1}}, x-\tau(\boldsymbol{\xi_{p-1}}))\varphi(x-\tau(\boldsymbol{\xi_{p-1}})) \prod_{j=1}^{p-1}\varphi(\xi_j)d\boldsymbol{\xi_{p-1}}\,.$$

We write

$$\sigma_{p,R}^2 R^{-d} = \omega_d p!(2\pi)^d \int_{\mathbb{R}^d} \ell_R(x)\Psi_p(x)dx \geq \omega_d p!(2\pi)^d \int_{\{\|x\|\leq R^{-1}\}} R^d\ell_1(Rx)\Psi_p(x)dx$$

and for $y = Rx \in B_1$, we have

$$(2\pi)^d\omega_d\ell_1(y) = \left(\int_{B_1} e^{-\mathbf{i}y\cdot u}du\right)^2 = \left(\int_{B_1}\cos(y\cdot u)du\right)^2 \in \left[\cos(1)^2\omega_d^2, \omega_d^2\right]. \tag{4.1}$$

As a consequence,

$$\sigma_{p,R}^2 R^{-d} \geq p!\omega_d^2\cos(1)^2 R^d \int_{\|x\|\leq R^{-1}} \Psi_p(x)dx$$

$$= p!\omega_d^2 \cos(1)^2 R^d \int_{\{\|\tau(\boldsymbol{\xi_p})\|\le R^{-1}\}} |\mathscr{F} f_p|^2(\boldsymbol{\xi_p})\mu(d\boldsymbol{\xi_p}) = p!\omega_d^2\cos(1)^2 R^d \widehat{\Psi}_p(R^{-1}).$$

This gives us

$$\liminf_{R\to+\infty} \sigma_{p,R}^2 R^{-d} \ge \omega_d \cos(1)^2 p! \liminf_{R\to+\infty} R^d \widehat{\Psi}_p(R^{-1}) > 0\,.$$

For the upper bound, we proceed as follows:

$$\begin{aligned}\sigma_{p,R}^2 R^{-d} &= \omega_d p!(2\pi)^d \int_{\mathbb{R}^d} \ell_R(x)\Psi_p(x)dx \\ &= \omega_d p!(2\pi)^d \int_{\|x\|\le R^{-1}} R^d \ell_1(Rx)\Psi_p(x)dx + \omega_d p!(2\pi)^d \int_{\|x\|>R^{-1}} \ell_R(x)\Psi_p(x)dx\,.\end{aligned}$$

It follows from (4.1) that

$$(2\pi)^d \int_{\|x\|\le R^{-1}} R^d \ell_1(Rx)\Psi_p(x)dx \le \omega_d R^d \int_{\|x\|\le R^{-1}} \Psi_p(x)dx = \omega_d R^d \widehat{\Psi}_p(R^{-1})\,.$$

By Lemma 2.1, there exists some absolute constant C such that $\ell_R(x) \le C(R/n)^d n^{-1}$ for $n \le R\|x\| < n+1$. Therefore,

$$\begin{aligned}&\int_{\|x\|>R^{-1}} \ell_R(x)\Psi_p(x)dx = \sum_{n=1}^{\infty} \int_{nR^{-1}\le\|x\|<(n+1)R^{-1}} \ell_R(x)\Psi_p(x)dx \\ &\le C\sum_{n=1}^{\infty} \int_{nR^{-1}\le\|x\|<(n+1)R^{-1}} (R/n)^d n^{-1}\Psi_p(x)dx \\ &= CR^d \sum_{n=1}^{\infty} n^{-d-1}\Big(\widehat{\Psi}_p(\frac{n+1}{R}) - \widehat{\Psi}_p(\frac{n}{R})\Big) \\ &= CR^d \sum_{n=2}^{\infty} \widehat{\Psi}_p(n/R)\big[(n-1)^{-d-1} - n^{-d-1}\big] \le CR^d \sum_{n=2}^{\infty} \widehat{\Psi}_p(n/R) n^{-1}(n-1)^{-d-1} \\ &= CR^d \sum_{2\le n\le R^\delta+1} \widehat{\Psi}_p(n/R)n^{-1}(n-1)^{-d-1} + CR^d \sum_{n>R^\delta+1} \widehat{\Psi}_p(n/R)n^{-1}(n-1)^{-d-1},\end{aligned}$$

where $\delta = d/(d+1)$. This implies

$$\begin{aligned}\int_{\|x\|>R^{-1}} \ell_R(x)\Psi_p(x)dx &\le C\left(\sup_{h\le R^{-1}+R^{\delta-1}} \frac{\widehat{\Psi}_p(h)}{h^d}\right)\left(\sum_{2\le n\le R^\delta+1} \frac{n^{d-1}}{(n-1)^{d+1}}\right) \\ &\quad + C\widehat{\Psi}_p(\infty) \sum_{n>R^\delta+1} \frac{n^{-1}R^d}{(n-1)^{d+1}} \\ &\le C\left(\sup_{h\le R^{-1}+R^{\delta-1}} \widehat{\Psi}_p(h)h^{-d}\right) + C\,.\end{aligned}$$

Therefore,

$$\limsup_{R\to+\infty} \sigma_{p,R}^2 R^{-d} \le C + C\limsup_{R\to+\infty} \widehat{\Psi}_p(h)h^{-d} < \infty\,.$$

This finishes our proof. ☐

Proof of Lemma 2.8. Notice that the condition $f_p \in L^1(\mathbb{R}^{pd})$ implies that $\mathscr{F} f_p$ is uniformly continuous and bounded. We fix a generic $z \in \mathbb{R}^d$, and we write

$$|\Psi_p(x) - \Psi_p(z)| \le \int_{\mathbb{R}^{pd-d}} \Big| |\mathscr{F} f_p|^2 \big(\boldsymbol{\xi_{p-1}}, x - \tau(\boldsymbol{\xi_{p-1}})\big) \varphi\big(x - \tau(\boldsymbol{\xi_{p-1}})\big) - |\mathscr{F} f_p|^2 \big(\boldsymbol{\xi_{p-1}}, z - \tau(\boldsymbol{\xi_{p-1}})\big) \varphi\big(z - \tau(\boldsymbol{\xi_{p-1}})\big) \Big| \prod_{i=1}^{p-1} \varphi(\xi_i) d\boldsymbol{\xi_{p-1}} \le A_1(x) + A_2(x),$$

where

$$A_1(x) := \int_{\mathbb{R}^{pd-d}} \Big| |\mathscr{F} f_p|^2 \big(\boldsymbol{\xi_{p-1}}, x - \tau(\boldsymbol{\xi_{p-1}})\big) - |\mathscr{F} f_p|^2 \big(\boldsymbol{\xi_{p-1}}, z - \tau(\boldsymbol{\xi_{p-1}})\big) \Big| \times \varphi\big(x - \tau(\boldsymbol{\xi_{p-1}})\big) \prod_{i=1}^{p-1} \varphi(\xi_i) d\boldsymbol{\xi_{p-1}}$$

and

$$A_2(x) := \int_{\mathbb{R}^{pd-d}} |\mathscr{F} f_p|^2 \big(\boldsymbol{\xi_{p-1}}, z - \tau(\boldsymbol{\xi_{p-1}})\big) \Big| \varphi\big(x - \tau(\boldsymbol{\xi_{p-1}})\big) - \varphi\big(z - \tau(\boldsymbol{\xi_{p-1}})\big) \Big| \times \prod_{i=1}^{p-1} \varphi(\xi_i) d\boldsymbol{\xi_{p-1}}.$$

Estimation of A_1: We write

$$A_1(x) \le \sup_{\boldsymbol{\eta_{p-1}} \in \mathbb{R}^{pd-d}} \Big| |\mathscr{F} f_p|^2 \big(\boldsymbol{\eta_{p-1}}, x - \tau(\boldsymbol{\eta_{p-1}})\big) - |\mathscr{F} f_p|^2 \big(\boldsymbol{\eta_{p-1}}, z - \tau(\boldsymbol{\eta_{p-1}})\big) \Big| \times \int_{\mathbb{R}^{pd-d}} \varphi\big(x - \tau(\boldsymbol{\xi_{p-1}})\big) \prod_{i=1}^{p-1} \varphi(\xi_i) d\boldsymbol{\xi_{p-1}}.$$

The first factor tends to zero as $x \to 0$, due to the uniform continuity of $\mathscr{F} f_p$. We rewrite the second factor as the p-convolution $\varphi^{*p}(x)$ and we deduce from (2.5) that

$$\big\|\varphi^{*p}\big\|_\infty \le \|\varphi\|^p_{L^q(\mathbb{R}^d)}.$$

Thus, we obtain that $A_1(x) \to 0$, as $x \to 0$. Moreover, the previous computations also lead to

$$A_1(x) \le \big\| |\mathscr{F} f_p|^2 \big\|_\infty \|\varphi\|^p_{L^q(\mathbb{R}^d)} < \infty\,.$$

Estimation of A_2: Using the boundedness of $\mathscr{F} f_p$, we write

$$A_2(x) \le C \int_{\mathbb{R}^{pd-d}} \Big| \varphi\big(x - \tau(\boldsymbol{\xi_{p-1}})\big) - \varphi\big(z - \tau(\boldsymbol{\xi_{p-1}})\big) \Big| \prod_{i=1}^{p-1} \varphi(\xi_i) d\boldsymbol{\xi_{p-1}}$$

$$
\begin{aligned}
&= C \int_{\mathbb{R}^d} dy \big|\varphi(x-y) - \varphi(z-y)\big| \left(\int_{\mathbb{R}^{pd-2d}} \varphi\big(y - \tau(\boldsymbol{\xi_{p-2}})\big) \prod_{i=1}^{p-2} \varphi(\xi_i) d\boldsymbol{\xi_{p-2}} \right) \\
&= C \int_{\mathbb{R}^d} \big|\varphi(x-y) - \varphi(z-y)\big| \varphi^{*p-1}(y)\, dy \\
&\le C \left(\int_{\mathbb{R}^d} \big|\varphi(x-y) - \varphi(z-y)\big|^q dy \right)^{1/q} \|\varphi^{*p-1}\|_{L^p(\mathbb{R}^d)},
\end{aligned}
$$

where we made the change of variables $\boldsymbol{\xi_{p-1}} \to (\boldsymbol{\xi_{p-2}}, y - \tau(\boldsymbol{\xi_{p-2}}))$ in the first equality. We know from the proof of (2.5) that $\|\varphi^{*p-1}\|_{L^p(\mathbb{R}^d)} \le \|\varphi\|_{L^q(\mathbb{R}^d)}^{p-1}$, so

$$
A_2(x) \le C \|\varphi\|_{L^q(\mathbb{R}^d)}^{p-1} \left(\int_{\mathbb{R}^d} \big|\varphi(x-y) - \varphi(z-y)\big|^q dy \right)^{1/q} \xrightarrow{x \to z} 0 .
$$

The above bound also indicates that A_2 is uniformly bounded.

Hence we conclude our proof by combining the above two estimates. □

Proof of Lemma 2.10. Let us first prove the boundedness. Since $f_p \in L^1(\mathbb{R}^{pd})$, $\mathscr{F} f_p$ is uniformly bounded, so that

$$
\big|\Psi_p^{(r,\delta)}(x,y)\big| \le C \varphi^{*p}(x) \varphi^{*p}(y) \le C \|\varphi\|_{L^q(\mathbb{R}^d)}^{2p} ,
$$

where the last inequality follows from (2.5). Now let us show the continuity. To ease the presentation, we define

$$
\begin{aligned}
&\mathbf{M}_{x,y} \equiv \mathbf{M}_{x,y}\big(\boldsymbol{\xi_r}, \widetilde{\boldsymbol{\xi}}_{\boldsymbol{r-1}}, \boldsymbol{\eta_{p-r}}, \widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r-1}}\big) \\
&= |\mathscr{F} f_p|^2 \big(\boldsymbol{\eta_{p-r}}, \widetilde{\boldsymbol{\xi}}_{\boldsymbol{r-1}}, x - \tau(\widetilde{\boldsymbol{\xi}}_{\boldsymbol{r-1}}) - \tau(\boldsymbol{\eta_{p-r}})\big) |\mathscr{F} f_p|^2 \big(\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r-1}}, y - \tau(\boldsymbol{\xi_r}) - \tau(\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r-1}}), \boldsymbol{\xi_r}\big).
\end{aligned}
$$

Suppose $x_n, y_n \in \mathbb{R}^d$ converge to x and y respectively, as $n \to +\infty$. Then

$$
\begin{aligned}
&\big|\Psi_p^{(r,\delta)}(x,y) - \Psi_p^{(r,\delta)}(x_n,y_n)\big| \\
&\le \int_{\mathbb{R}^{2pd-2d}} d\boldsymbol{\xi_r} d\widetilde{\boldsymbol{\xi}}_{\boldsymbol{r-1}} d\boldsymbol{\eta_{p-r}} d\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r-1}} \mathbf{1}_{\{\|\tau(\boldsymbol{\xi_r}) + \tau(\boldsymbol{\eta_{p-r}})\| < \delta\}} \left(\prod_{i=1}^{r-1} \varphi(\xi_i) \varphi(\widetilde{\xi}_i) \right) \varphi(\xi_r) \varphi(\eta_{p-r}) \\
&\quad \times \left(\prod_{j=1}^{p-r-1} \varphi(\eta_j) \varphi(\widetilde{\eta}_j) \right) \Big| \mathbf{M}_{x,y} \varphi\big(y - \tau(\boldsymbol{\xi_r}) - \tau(\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r-1}})\big) \varphi\big(x - \tau(\widetilde{\boldsymbol{\xi}}_{\boldsymbol{r-1}}) - \tau(\boldsymbol{\eta_{p-r}})\big) \\
&\quad - \mathbf{M}_{x_n,y_n} \varphi\big(y_n - \tau(\boldsymbol{\xi_r}) - \tau(\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r-1}})\big) \varphi\big(x_n - \tau(\widetilde{\boldsymbol{\xi}}_{\boldsymbol{r-1}}) - \tau(\boldsymbol{\eta_{p-r}})\big) \Big| \le A_{1,n} + A_{2,n} ,
\end{aligned}
$$

where

$$
A_{1,n} = \int_{\mathbb{R}^{2pd-2d}} d\boldsymbol{\xi_r} d\widetilde{\boldsymbol{\xi}}_{\boldsymbol{r-1}} d\boldsymbol{\eta_{p-r}} d\widetilde{\boldsymbol{\eta}}_{\boldsymbol{p-r-1}} \mathbf{1}_{\{\|\tau(\boldsymbol{\xi_r}) + \tau(\boldsymbol{\eta_{p-r}})\| < \delta\}} \left(\prod_{i=1}^{r-1} \varphi(\xi_i) \varphi(\widetilde{\xi}_i) \right) \varphi(\xi_r) \varphi(\eta_{p-r})
$$

$$\times \left(\prod_{j=1}^{p-r-1} \varphi(\eta_j)\varphi(\widetilde{\eta}_j)\right) \varphi\big(y - \tau(\boldsymbol{\xi_r}) - \tau(\boldsymbol{\widetilde{\eta}_{p-r-1}})\big)\varphi\big(x - \tau(\boldsymbol{\widetilde{\xi}_{r-1}}) - \tau(\boldsymbol{\eta_{p-r}})\big) \\ \times \left|\mathbf{M}_{x,y} - \mathbf{M}_{x_n,y_n}\right|$$

and

$$A_{2,n} = \int_{\mathbb{R}^{2pd-2d}} d\boldsymbol{\xi_r} d\boldsymbol{\widetilde{\xi}_{r-1}} d\boldsymbol{\eta_{p-r}} d\boldsymbol{\widetilde{\eta}_{p-r-1}} \mathbf{1}_{\{\|\tau(\boldsymbol{\xi_r})+\tau(\boldsymbol{\eta_{p-r}})\|<\delta\}} \left(\prod_{i=1}^{r-1} \varphi(\xi_i)\varphi(\widetilde{\xi}_i)\right) \varphi(\xi_r)\varphi(\eta_{p-r}) \\ \times \left(\prod_{j=1}^{p-r-1} \varphi(\eta_j)\varphi(\widetilde{\eta}_j)\right) \mathbf{M}_{x_n,y_n} \Big|\varphi\big(y - \tau(\boldsymbol{\xi_r}) - \tau(\boldsymbol{\widetilde{\eta}_{p-r-1}})\big)\varphi\big(x - \tau(\boldsymbol{\widetilde{\xi}_{r-1}}) - \tau(\boldsymbol{\eta_{p-r}})\big) \\ - \varphi\big(y_n - \tau(\boldsymbol{\xi_r}) - \tau(\boldsymbol{\widetilde{\eta}_{p-r-1}})\big)\varphi\big(x_n - \tau(\boldsymbol{\widetilde{\xi}_{r-1}}) - \tau(\boldsymbol{\eta_{p-r}})\big)\Big| .$$

It follows immediately from the first part of our proof that

$$A_{1,n} \le C\|\varphi\|_{L^q(\mathbb{R}^d)}^{2p} \sup\left\{|\mathbf{M}_{x_n,y_n} - \mathbf{M}_{x,y}| : \boldsymbol{\xi_r}, \boldsymbol{\widetilde{\xi}_{r-1}}, \boldsymbol{\eta_{p-r}}, \boldsymbol{\widetilde{\eta}_{p-r-1}}\right\} \xrightarrow{n\to+\infty} 0\,,$$

due to the uniform continuity of $\mathscr{F} f_p$. Now, using $\|\mathscr{F} f_p\|_\infty < \infty$, we write

$$A_{2,n} \le C \int_{\mathbb{R}^{2pd-2d}} d\boldsymbol{\xi_r} d\boldsymbol{\widetilde{\xi}_{r-1}} d\boldsymbol{\eta_{p-r}} d\boldsymbol{\widetilde{\eta}_{p-r-1}} \left(\prod_{i=1}^{r-1} \varphi(\xi_i)\varphi(\widetilde{\xi}_i)\right) \varphi(\xi_r)\varphi(\eta_{p-r}) \\ \times \left(\prod_{j=1}^{p-r-1} \varphi(\eta_j)\varphi(\widetilde{\eta}_j)\right) \Big|\varphi\big(y - \tau(\boldsymbol{\xi_r}) - \tau(\boldsymbol{\widetilde{\eta}_{p-r-1}})\big)\varphi\big(x - \tau(\boldsymbol{\widetilde{\xi}_{r-1}}) - \tau(\boldsymbol{\eta_{p-r}})\big) \\ - \varphi\big(y_n - \tau(\boldsymbol{\xi_r}) - \tau(\boldsymbol{\widetilde{\eta}_{p-r-1}})\big)\varphi\big(x_n - \tau(\boldsymbol{\widetilde{\xi}_{r-1}}) - \tau(\boldsymbol{\eta_{p-r}})\big)\Big| \le C(A_{21,n} + A_{22,n}),$$

with

$$A_{21,n} := \int_{\mathbb{R}^{2pd-2d}} d\boldsymbol{\xi_r} d\boldsymbol{\widetilde{\xi}_{r-1}} d\boldsymbol{\eta_{p-r}} d\boldsymbol{\widetilde{\eta}_{p-r-1}} \left(\prod_{i=1}^{r-1} \varphi(\xi_i)\varphi(\widetilde{\xi}_i)\right) \varphi(\xi_r)\varphi(\eta_{p-r}) \\ \times \left(\prod_{j=1}^{p-r-1} \varphi(\eta_j)\varphi(\widetilde{\eta}_j)\right) \Big|\varphi\big(y - \tau(\boldsymbol{\xi_r}) - \tau(\boldsymbol{\widetilde{\eta}_{p-r-1}})\big) - \varphi\big(y_n - \tau(\boldsymbol{\xi_r}) - \tau(\boldsymbol{\widetilde{\eta}_{p-r-1}})\big)\Big| \\ \times \varphi\big(x - \tau(\boldsymbol{\widetilde{\xi}_{r-1}}) - \tau(\boldsymbol{\eta_{p-r}})\big) \\ = \varphi^{*p}(x) \int_{\mathbb{R}^{pd-d}} d\boldsymbol{\xi_{p-1}} \left(\prod_{i=1}^{p-1} \varphi(\xi_i)\right) \Big|\varphi\big(y - \tau(\boldsymbol{\xi_{p-1}})\big) - \varphi\big(y_n - \tau(\boldsymbol{\xi_{p-1}})\big)\Big|$$

and smilarly,

$$A_{22,n} := \varphi^{*p}(y_n) \int_{\mathbb{R}^{pd-d}} d\boldsymbol{\xi_{p-1}} \left(\prod_{i=1}^{p-1} \varphi(\xi_i)\right) \Big|\varphi\big(x - \tau(\boldsymbol{\xi_{p-1}})\big) - \varphi\big(x_n - \tau(\boldsymbol{\xi_{p-1}})\big)\Big| .$$

Put $\varphi_y(x) = \varphi(x-y)$, so we can rewrite

$$\int_{\mathbb{R}^{pd-d}} d\boldsymbol{\xi_{p-1}} \left(\prod_{i=1}^{p-1} \varphi(\xi_i)\right) \left|\varphi\big(x - \tau(\boldsymbol{\xi_{p-1}})\big) - \varphi\big(x_n - \tau(\boldsymbol{\xi_{p-1}})\big)\right|$$

as $\big(\varphi^{*p-1} * |\varphi_{-x} - \varphi_{-x_n}|\big)(0)$, which is bounded by

$$\left\|\varphi^{*p-1}\right\|_{L^p(\mathbb{R}^d)} \|\varphi_{-x} - \varphi_{-x_n}\|_{L^q(\mathbb{R}^d)} \leq \|\varphi\|^{p-1}_{L^q(\mathbb{R}^d)} \|\varphi_{-x} - \varphi_{-x_n}\|_{L^q(\mathbb{R}^d)} \xrightarrow{n\to+\infty} 0\,,$$

that is, $A_{22,n} \to 0$, as $n \to +\infty$. The same arguments also imply that $A_{21,n} \to 0$, as $n \to +\infty$. This concludes our proof. □

Lemma 4.1. *Let φ_1 be given as in Theorem 1.6. Then for any $x, y \in \mathbb{R}^d$ and $s > 0$, we have*

$$\int_{\mathbb{R}^d} e^{-s\|\eta\|^2} \varphi_1(\eta - x)\varphi_1(y - \eta) d\eta \leq \int_{\mathbb{R}^d} e^{-s\|\eta\|^2} \varphi_1^2(\eta) d\eta \tag{4.2}$$

and

$$\int_{\mathbb{R}^d} e^{-s\|\eta\|^2} \varphi_1(\eta - x) d\eta \leq \int_{\mathbb{R}^d} e^{-s\|\eta\|^2} \varphi_1(\eta) d\eta. \tag{4.3}$$

Proof. It suffices to prove it for $x = y$, as the general case follows from the Cauchy-Schwarz inequality and symmetry of φ_1.

Put $h(\eta) = e^{-s\|\eta\|^2}$, then its Fourier transform $\mathscr{F}h$ is a nonnegative function. Then, we write, using Plancherel's identity and the fact $\varphi_1^2 = \frac{1}{(2\pi)^{2d}} \mathscr{F}(\gamma_1 * \gamma_1)$

$$\begin{aligned}
&\int_{\mathbb{R}^d} h(\eta)\varphi_1(\eta - x)^2 d\eta = \int_{\mathbb{R}^d} h(\eta + x) \frac{1}{(2\pi)^{2d}} \mathscr{F}(\gamma_1 * \gamma_1)(\eta) d\eta \\
&= \int_{\mathbb{R}^d} (\mathscr{F}h)(a) e^{\mathbf{i}ax} \frac{1}{(2\pi)^{2d}} (\gamma_1 * \gamma_1)(a) da \quad (\gamma_1 \text{ is also nonnegative}) \\
&\leq \int_{\mathbb{R}^d} (\mathscr{F}h)(a) \frac{1}{(2\pi)^{2d}} (\gamma_1 * \gamma_1)(a) da = \int_{\mathbb{R}^d} h(\eta)\varphi_1(\eta)^2 d\eta\,,
\end{aligned}$$

which proves (4.2). The same argument also leads easily to (4.3). □

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