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CONTINUOUS BREUER–MAJOR THEOREM: TIGHTNESS AND NONSTATIONARITY

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Let $Y = (Y(t))_{t\geq 0}$ be a zero-mean Gaussian stationary process with covariance function $\rho : \mathbb{R} \to \mathbb{R}$ satisfying $\rho(0) = 1$. Let $f : \mathbb{R} \to \mathbb{R}$ be a square-integrable function with respect to the standard Gaussian measure, and suppose the Hermite rank of f is $d \geq 1$. If $\int_{\mathbb{R}} |\rho(s)|^d \, ds < \infty$, then the celebrated Breuer–Major theorem (in its continuous version) asserts that the finite-dimensional distributions of $Z_\varepsilon := \sqrt{\varepsilon} \int_0^{\cdot/\varepsilon} f(Y(s))\, ds$ converge to those of σW as $\varepsilon \to 0$, where W is a standard Brownian motion and σ is some explicit constant. Since its first appearance in 1983, this theorem has become a crucial probabilistic tool in different areas, for instance in signal processing or in statistical inference for fractional Gaussian processes.

The goal of this paper is twofold. First, we investigate the tightness in the Breuer–Major theorem. Surprisingly, this problem did not receive a lot of attention until now, and the best available condition due to Ben Hariz [*J. Multivariate Anal.* **80** (2002) 191–216] is neither arguably very natural, nor easy-to-check in practice. In contrast, our condition very simple, as it only requires that $|f|^p$ must be integrable with respect to the standard Gaussian measure for some p *strictly* bigger than 2. It is obtained by means of the Malliavin calculus, in particular Meyer inequalities.

Second, and motivated by a problem of geometrical nature, we extend the continuous Breuer–Major theorem to the notoriously difficult case of self-similar Gaussian processes which are *not* necessarily stationary. An application to the fluctuations associated with the length process of a regularized version of the bifractional Brownian motion concludes the paper.

1. Introduction and statement of the main results. Let $Y = (Y(t))_{t\geq 0}$ be a zero-mean Gaussian stationary process, with covariance function $\mathbb{E}[Y(t)Y(s)] = \rho(|t-s|)$ such that $\rho(0) = 1$. Let $\gamma = N(0,1)$ be the standard Gaussian measure on $\mathbb{R}$. Consider a function $f \in L^2(\mathbb{R}, \gamma)$ of Hermite rank $d \geq 1$, that is, f has a series expansion given by

$$f(x) = \sum_{q=d}^{\infty} c_q H_q(x), \qquad c_d \neq 0, \tag{1.1}$$

where $H_q(x)$ is the qth Hermite polynomial.

It has become a central result in modern stochastic analysis that, under the condition $\int_{\mathbb{R}} |\rho(s)|^d \, ds < \infty$, the *finite-dimensional distributions* (f.d.d.) of the process

$$Z_\varepsilon(t) := \sqrt{\varepsilon} \int_0^{t/\varepsilon} f\big(Y(s)\big)\, ds, \qquad t \geq 0 \tag{1.2}$$

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converge, as ε tends to zero, to those of σW, where $W = (W(t))_{t \geq 0}$ is a standard Brownian motion and

$$\sigma^2 = \sum_{q=d}^{\infty} c_q^2 q! \int_{\mathbb{R}} \rho(s)^q \, ds. \tag{1.3}$$

(Observe that $|\rho(s)| = |\mathbb{E}[Y(s)Y(0)]| \leq \rho(0) = 1$ by Cauchy–Schwarz, and thus σ^2 is well defined under our integrability assumption on ρ and the square-integrability of f). This is a continuous version of the celebrated Breuer–Major theorem proved in [3], that can be found stated this way, for example, in the paper by Ben Hariz [1]. We also refer the reader to [13], Chapter 7, where a modern proof of the original discrete version[1] of the Breuer–Major theorem is given, by means of the recent Malliavin–Stein approach.

The condition $\int_{\mathbb{R}} |\rho(s)|^d \, ds < \infty$ turns out to be also necessary for the convergence of Z_ε to σW in the sense of f.d.d., because σ^2 is not properly defined when $\int_{\mathbb{R}} |\rho(s)|^d \, ds = \infty$. What about the *functional convergence*, that is, convergence in law of Z_ε to σW in $C(\mathbb{R}_+)$ endowed with the uniform topology on compact sets? First, let us note that Chambers and Slud ([4], page 328) provide a counterexample of a zero-mean Gaussian stationary process Y and a square-integrable function f satisfying $Z_\varepsilon \Rightarrow \sigma W$ in the sense of f.d.d., but *not* in the functional sense; as a consequence, we see that the mere condition $\int_{\mathbb{R}} |\rho(s)|^d \, ds < \infty$ does not imply tightness in general.

Before the present paper, the best sufficient condition ensuring tightness in the continuous Breuer–Major theorem was due to Ben Hariz [1]: more precisely, it is shown in [1], Theorem 1, that the functional convergence of Z_ε to σW holds true whenever either

$$\text{there exists } R > 1 \text{ such that } \quad \sum_{q=d}^{\infty} \frac{|c_q|}{\sqrt{q!}} \left(\int_{\mathbb{R}} |\rho(s)|^q \, ds \right)^{1/2} R^q < \infty, \tag{1.4}$$

or

$$\text{the } c_q \text{ are all positive and } f \in L^4(\mathbb{R}, \gamma). \tag{1.5}$$

The two conditions (1.4)–(1.5) proposed by Ben Hariz [1] were obtained thanks to moment inequalities à la Rosenthal; they are neither very natural, nor easy-to-check.

In the present paper, our first main objective is to remedy the situation and provide a simple sufficient condition for the convergence $Z_\varepsilon \Rightarrow \sigma W$ to hold in law in $C(\mathbb{R}_+)$ endowed with the uniform topology on compact sets. Surprisingly and compared to [1], our finding is that only a little more integrability of the function f is enough.

THEOREM 1.1. *Let $Y = (Y(t))_{t \geq 0}$ be a zero-mean Gaussian stationary process with covariance function $\mathbb{E}[Y(t)Y(s)] = \rho(|t-s|)$ such that $\rho(0) = 1$. Consider a function $f \in L^2(\mathbb{R}, \gamma)$ with expansion* (1.1) *and Hermite rank $d \geq 1$. Suppose that $\int_{\mathbb{R}} |\rho(s)|^d \, ds < \infty$. Then, if $f \in L^p(\mathbb{R}, \gamma)$ for some $p > 2$, the process Z_ε defined in* (1.2) *converges in law in $C(\mathbb{R}_+)$ to $\sigma(W(t))_{t \geq 0}$, where W is a Brownian motion and σ^2 is defined in* (1.3).

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The proof of Theorem 1.1 is based on the application of the techniques of Malliavin calculus and it has been inspired by the recent work of Jaramillo and Nualart [9] on the asymptotic behavior of the renormalized self-intersection local time of the fractional Brownian motion. The main idea to prove tightness is to use the representation of the random variable $Z_\varepsilon(t)$ as

$$Z_\varepsilon(t) = \delta^d (-DL^{-1})^d Z_\varepsilon(t),$$

[1]Note that the proof contained in [13], Chapter 7, can be easily extended (mutatis mutandis) to cover the continuous framework as well.

where δ, D and L are the basic operators in Malliavin calculus and then apply Meyer inequalities to upper bound $\mathbb{E}[|Z_\varepsilon(t) - Z_\varepsilon(s)|^p]$ by $C|t-s|^{p/2}$, where p is the exponent appearing in Theorem 1.1.

Then, as an application of the previous result we aim to solve the following problem. Let $X = (X(t))_{t\geq 0}$ be a self-similar continuous Gaussian centered process, and assume moreover that almost no path of X is rectifiable, that is, the length of X on any compact interval is *infinite*: in symbols, $\mathcal{L}(X;[0,t]) = +\infty$ for all $t > 0$. Examples of such processes include the fractional Brownian motion and relatives, such as the bifractional Brownian motion and the subfractional Brownian motion. Consider the C^1-regularization X^ε of X given by

$$X^\varepsilon(t) = \frac{1}{\varepsilon}\int_t^{t+\varepsilon} X(u)\,du. \tag{1.6}$$

Can we compute at which speed the length of X^ε on $[0,t]$ explodes? Stated in a different way, what is the asymptotic behavior of the family of processes indexed by ε:

$$\mathcal{L}\big(X^\varepsilon;[0,t]\big) = \int_0^t \big|\dot{X}^\varepsilon(u)\big|\,du, \quad t\geq 0, \tag{1.7}$$

when $\varepsilon \to 0$?

Let us first take a look at the simplest case, that is, where $X = B$ is a fractional Brownian motion (fBm) of index $H \in (0,1)$. We recall that the fBm $B = (B(t))_{t\geq 0}$ is a centered Gaussian process with covariance

$$\mathbb{E}\big[B(t)B(s)\big] = \frac{1}{2}\big(t^{2H} + s^{2H} - |t-s|^{2H}\big). \tag{1.8}$$

Making a change of variable and using the self-similarity of B, we observe that

$$\begin{aligned}\mathcal{L}\big(B^\varepsilon;[0,t]\big) &= \varepsilon^{-1}\int_0^t \big|B(u+\varepsilon) - B(u)\big|\,du = \int_0^{t/\varepsilon}\big|B\big(\varepsilon(v+1)\big) - B(\varepsilon v)\big|\,dv \\ &\overset{\text{law}}{=} \varepsilon^H \int_0^{t/\varepsilon}\big|B(v+1) - B(v)\big|\,dv =: \varepsilon^{H-\frac{1}{2}} Z_\varepsilon(t) \quad \text{(as a process in } t\text{)},\end{aligned}$$

so that we are left to study the asymptotic behavior of Z_ε as $\varepsilon \to 0$. Since the fractional Gaussian noise $(B(t+1) - B(t))_{t\geq 0}$ is *stationary*, to conclude it actually suffices to apply Theorem 1.1 to the process $Y(t) = B(t+1) - B(t)$. Indeed, if we choose for f the function $f(x) = |x| - \sqrt{\frac{2}{\pi}}$ of Hermite rank 2 (indeed, $f = \sqrt{\frac{2}{\pi}}\sum_{q=1}^\infty \frac{(-1)^{q-1}}{2^q q!(2q-1)} H_{2q}$, see Section 6.3), we have $f \in L^p(\mathbb{R},\gamma)$ for any $p \geq 2$. In this way, we obtain the following result:

(i) If $H < \frac{3}{4}$, then

$$\varepsilon^{\frac{1}{2}-H}\left(\mathcal{L}\big(B^\varepsilon;[0,t]\big) - t\varepsilon^{H-1}\sqrt{\frac{2}{\pi}}\right)_{t\geq 0} \Rightarrow \sigma_H\big(W(t)\big)_{t\geq 0} \tag{1.9}$$

in $C(\mathbb{R}_+)$ as $\varepsilon \to 0$,

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with W a standard Brownian motion and $\sigma_H^2 = \frac{1}{\pi}\sum_{q=1}^\infty \frac{(2q)!}{2^{2q-1}q!^2(2q-1)^2}\int_{-\infty}^\infty a_{2H}(h)^{2q}\,dh$, where, for any $\alpha > 0$,

$$a_\alpha(h) = \frac{1}{2}\big(|h-1|^\alpha + |h+1|^\alpha - 2|h|^\alpha\big), \quad h \in \mathbb{R}. \tag{1.10}$$

Furthermore, in the case $H \geq \frac{3}{4}$, it is known that (tightness in the case $H = \frac{3}{4}$ can be proved by the same techniques as in Theorem 1.1 and follows from Theorem 1.2 below):

(ii) If $H = \frac{3}{4}$, then

$$\frac{\varepsilon^{-\frac{1}{4}}}{\sqrt{|\log \varepsilon|}}\left(\mathcal{L}(B^{\varepsilon};[0,t]) - t\varepsilon^{-\frac{1}{4}}\sqrt{\frac{2}{\pi}}\right)_{t\geq 0} \Rightarrow \frac{3}{8\sqrt{\pi}}(W(t))_{t\geq 0} \tag{1.11}$$

in $C(\mathbb{R}_+)$ as $\varepsilon \to 0$.

(iii) If $H > \frac{3}{4}$, then

$$\varepsilon^{H-1}\left(\mathcal{L}(B^{\varepsilon};[0,t]) - t\varepsilon^{H-1}\sqrt{\frac{2}{\pi}}\right) \Rightarrow \text{“Rosenblatt process”} \tag{1.12}$$

in $C(\mathbb{R}_+)$ as $\varepsilon \to 0$.

The asymptotic behavior of (1.7) is therefore completely understood when $X = B$ is a fBm but are the previous convergences (1.9), (1.11) and (1.12) still true for any self-similar continuous Gaussian centered process? In this paper, our second main objective is to answer this question, which is particularly difficult because of the lack of stationarity of the increments of X in such a generality.

To have a better idea of what may happen, let us now consider the case where $X = \widetilde{B}$ is the bifractional Brownian motion with indices $H \in (0,1)$ and $K \in (0,1]$, meaning that the covariance of $\widetilde{B}$ is given by

$$\mathbb{E}[\widetilde{B}(t)\widetilde{B}(s)] = 2^{-K}\left((t^{2H}+s^{2H})^K - |t-s|^{2HK}\right). \tag{1.13}$$

When $K = 1$, $\widetilde{B}$ is nothing but a fBm with index H. In general, we can think of $\widetilde{B}$ as a perturbation of a fBm B with index HK. Indeed, set $Z(t) = \int_0^\infty (1-e^{-\theta t})\theta^{-\frac{1+K}{2}}\,dW(\theta)$, $t \geq 0$, where W stands for a standard Brownian motion independent of $\widetilde{B}$. As shown by Lei and Nualart [10], the process Z has absolutely continuous trajectories; moreover, with $Y(t) = Z(t^{2H})$,

$$\left(\sqrt{\frac{2^{-K}K}{\Gamma(1-K)}}Y(t) + \widetilde{B}(t)\right)_{t\geq 0} \stackrel{\text{law}}{=} (2^{\frac{1-K}{2}}B(t))_{t\geq 0}. \tag{1.14}$$

Recall definition (1.6). We immediately deduce from (1.14) that, for any $\varepsilon > 0$,

$$\left(\sqrt{\frac{2^{-K}K}{\Gamma(1-K)}}Y^{\varepsilon}(t) + \widetilde{B}^{\varepsilon}(t)\right)_{t\geq 0} \stackrel{\text{law}}{=} (2^{\frac{1-K}{2}}B^{\varepsilon}(t))_{t\geq 0}. \tag{1.15}$$

We can thus write, assuming that Y and $\widetilde{B}$ are independent and defined on the same probability space, and with $B := 2^{\frac{K-1}{2}}(\sqrt{\frac{2^{-K}K}{\Gamma(1-K)}}Y + \widetilde{B})$:

$$\begin{aligned}
&\mathcal{L}(\widetilde{B}^{\varepsilon};[0,t]) - 2^{\frac{1-K}{2}}t\varepsilon^{HK-1}\sqrt{\frac{2}{\pi}} \\
&= \varepsilon^{-1}\int_0^t |\widetilde{B}(u+\varepsilon) - \widetilde{B}(u)|\,du - 2^{\frac{1-K}{2}}t\varepsilon^{HK-1}\sqrt{\frac{2}{\pi}} \\
&= 2^{\frac{1-K}{2}}\int_0^t \left\{\left|\frac{B(u+\varepsilon)-B(u)}{\varepsilon}\right| - \varepsilon^{HK-1}\sqrt{\frac{2}{\pi}}\right\}du \\
&\quad + \int_0^t \left\{\left|2^{\frac{1-K}{2}}\frac{B(u+\varepsilon)-B(u)}{\varepsilon} - \sqrt{\frac{2^{-K}K}{\Gamma(1-K)}}\frac{Y(u+\varepsilon)-Y(u)}{\varepsilon}\right|\right. \\
&\quad \left. - 2^{\frac{1-K}{2}}\left|\frac{B(u+\varepsilon)-B(u)}{\varepsilon}\right|\right\}du \\
&=: a_{\varepsilon}(t) + b_{\varepsilon}(t).
\end{aligned}$$

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When $HK < \frac{1}{2}$, we deduce from (1.9) that

$$\varepsilon^{\frac{1}{2}-HK} a_\varepsilon \Rightarrow 2^{\frac{1-K}{2}} \sigma_{HK} W, \tag{1.16}$$

whereas

$$\begin{aligned} |b_\varepsilon(t)| &\le \sqrt{\frac{2^{-K}K}{\Gamma(1-K)}} \int_0^t \left| \frac{Y(u+\varepsilon)-Y(u)}{\varepsilon} \right| du \\ &\to \sqrt{\frac{2^{-K}K}{\Gamma(1-K)}} \int_0^t |\dot{Y}(u)|\, du. \end{aligned} \tag{1.17}$$

By combining (1.16) and (1.17) together, we eventually obtain that

$$\varepsilon^{\frac{1}{2}-HK} \left(\mathcal{L}(\widetilde{B}^\varepsilon; [0,t]) - 2^{\frac{1-K}{2}} t \varepsilon^{HK-1} \sqrt{\frac{2}{\pi}} \right)_{t \ge 0} \Rightarrow \left(2^{\frac{1-K}{2}} \sigma_{HK} W(t)\right)_{t\ge 0}, \tag{1.18}$$

which is analogous to (1.9). The situation where $HK \ge \frac{1}{2}$ looks more complicated at first glance because to conclude we not only need an upper bound as the one given by (1.17), but we have to understand the *exact* behavior of b_ε when $\varepsilon \to 0$. For all $t > 0$, one has almost surely that $\varepsilon^{-1}(Y(t+\varepsilon) - Y(t)) \to \dot{Y}(t) = 2Ht^{2H-1}\dot{X}(t^{2H})$, whereas $\varepsilon^{-1}|B(t+\varepsilon) - B(t)|$ diverges to $+\infty$. Hence, at a *heuristic* level, one has that

$$\begin{aligned} &\left| 2^{\frac{1-K}{2}} \frac{B(t+\varepsilon)-B(t)}{\varepsilon} - \sqrt{\frac{2^{-K}K}{\Gamma(1-K)}} \frac{Y(t+\varepsilon)-Y(t)}{\varepsilon} \right| \\ &\quad - 2^{\frac{1-K}{2}} \left| \frac{B(t+\varepsilon)-B(t)}{\varepsilon} \right| \\ &\overset{\text{a.s.}}{\to} -\sqrt{\frac{2^{-K}K}{\Gamma(1-K)}} \dot{Y}(t) \times \lim_{\varepsilon\to 0} \operatorname{sign}\big(B(t+\varepsilon) - B(t)\big) =: A(t). \end{aligned}$$

Although the previous reasoning is only heuristic (because $\lim_{\varepsilon\to 0} \operatorname{sign}(B(t+\varepsilon) - B(t))$ does not exist), it seems to indicate that b_ε may converge almost surely as $\varepsilon \to 0$ *without* further renormalization, to a random variable of the form $\int_0^t A(u)\,du$. If such a claim were true, we would deduce from it that

$$\begin{aligned} &\left(\mathcal{L}(\widetilde{B}^\varepsilon; [0,t]) - 2^{\frac{1-K}{2}} t \varepsilon^{HK-1} \sqrt{\frac{2}{\pi}} \right)_{t \ge 0} \\ &\to \begin{cases} 2^{\frac{1-K}{2}} \sigma_{1/2} W + \int_0^\cdot A(u)\,du & \text{in law if } HK = \dfrac{1}{2}, \\ \int_0^\cdot A(u)\,du & \text{a.s. if } HK > \dfrac{1}{2}, \end{cases} \end{aligned} \tag{1.19}$$

with W a Brownian motion independent of $\widetilde{B}$, a statement which would be very different compared to (1.9), (1.11) and (1.12).

Our first attempt to study the asymptotic behavior of (1.6) in the case where $X = \widetilde{B}$ is a bifractional Brownian motion was to check whether the reasoning leading to (1.19) can be made rigorous. We failed to then realize that the claim (1.19) is actually wrong. What is correct is that convergences (1.9), (1.11) and (1.12) continue to be valid for a wide class of self-similar centered Gaussian processes, containing not only the bifractional Brownian motion, but other perturbations of the fractional Brownian motion.

With this application in mind, the second goal of our paper is to generalize Theorem 1.1 to self-similar Gaussian processes which are *not* necessarily stationary. We will also consider

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the case where the integral $\int_{\mathbb{R}} |\rho(s)|^d \, ds$ is infinite but the limit is still Gaussian (in such a critical case, a logarithmic factor is required), or when a non-Gaussian limit appears.

Let us first present the class of processes under consideration. Assume that $X = (X(t))_{t \geq 0}$ is a centered Gaussian process that is self-similar of order $\beta \in (0, 1)$. We define $\phi : [1, \infty) \to \mathbb{R}$ by $\phi(x) = \mathbb{E}[X(1)X(x)]$, so that, for $0 < s \leq t$, we have

$$\mathbb{E}[X(s)X(t)] = s^{2\beta}\mathbb{E}\left[X(1)X\left(\frac{t}{s}\right)\right] = s^{2\beta}\phi\left(\frac{t}{s}\right). \tag{1.20}$$

Therefore, ϕ characterizes the covariance function of X. Moreover, let us also assume the following two hypotheses on ϕ, which were first introduced and considered in [7]:

(H.1) There exists $\alpha \in (0, 2\beta]$ such that ϕ has the form

$$\phi(x) = -\lambda(x-1)^\alpha + \psi(x),$$

where $\lambda > 0$ and $\psi(x)$ is twice-differentiable on an open set containing $[1, \infty)$ and there exists a constant $C \geq 0$ such that, for any $x \in (1, \infty)$:

(a) $|\psi'(x)| \leq Cx^{\alpha-1}$,
(b) $|\psi''(x)| \leq Cx^{-1}(x-1)^{\alpha-1}$,
(c) $\psi'(1) = \beta\psi(1)$ when $\alpha \geq 1$.

(H.2) There are constants $C > 0$, $c > 1$ and $1 < \nu \leq 2$ such that, for all $x \geq c$:

(d) $|\phi'(x)| \leq \begin{cases} Cx^{-\nu} & \text{if } \alpha < 1, \\ Cx^{\alpha-2} & \text{if } \alpha \geq 1. \end{cases}$

(e) $|\phi''(x)| \leq \begin{cases} Cx^{-\nu-1} & \text{if } \alpha < 1, \\ Cx^{\alpha-3} & \text{if } \alpha \geq 1. \end{cases}$

We refer to [7], Section 4, for explicit examples of processes X satisfying (H.1) and (H.2), among them the bifractional Brownian motion ([7], Section 4.1) and the subfractional Brownian motion ([7], Section 4.2).

Now, for $\varepsilon > 0$ and $t \geq 0$, let us define

$$\Delta_\varepsilon X(t) = X(t+\varepsilon) - X(t) \quad \text{and} \quad Y_\varepsilon(t) = \frac{\Delta_\varepsilon X(t)}{\|\Delta_\varepsilon X(t)\|_{L^2(\Omega)}}. \tag{1.21}$$

Finally, define the family of stochastic processes $\widetilde{F}_\varepsilon = (\widetilde{F}_\varepsilon(t))_{t \geq 0}$ by

$$\widetilde{F}_\varepsilon(t) = \frac{1}{\sqrt{\varepsilon}} \int_0^t f\big(Y_\varepsilon(u)\big)\, du. \tag{1.22}$$

By the self-similarity property of X, the process $\widetilde{F}_\varepsilon$ has the same law as F_ε, where

$$F_\varepsilon(t) = \sqrt{\varepsilon} \int_0^{t/\varepsilon} f\big(Y_1(u)\big)\, du. \tag{1.23}$$

The second contribution of this paper is the following theorem, which is an extension of Theorem 1.1 (central case) and the main results of Taqqu's seminal paper [18] (noncentral case) to a situation where the underlying Gaussian process X does *not* need to have stationary increments. The lack of stationarity is actually the main difficulty we will have to cope with.

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THEOREM 1.2. *In the above setting, assume that* (H.1) *and* (H.2) *are in order for a centered Gaussian process* $X = (X(t))_{t \geq 0}$, *self-similar of order* $\beta \in (0, 1)$ *and whose covariance function is given by* (1.20). *Let* $f \in L^2(\mathbb{R}, \gamma)$ *a function with Hermite rank* $d \geq 1$ *and expansion* (1.1) *and let* F_ε *be defined in* (1.23). *Then the following is true as* $\varepsilon \to 0$:

1. *If* $\alpha < 2 - \frac{1}{d}$, *then the finite-dimensional distributions of the family* $\{F_\varepsilon : \varepsilon > 0\}$ *converges in law to those of a Brownian motion with variance given by* (1.3) *with* $\rho(h) = a_\alpha(h)$ *defined in* (1.10).

2. *If* $\alpha = 2 - \frac{1}{d}$, *then the finite-dimensional distributions of the family* $\{F_\varepsilon/\sqrt{|\log \varepsilon|} : \varepsilon > 0\}$ *converges in law to those of a Brownian motion with variance*

$$\sigma^2_{1-\frac{2}{d}} = c_d^2 d!\Big(1 + \Big(\beta - \frac{\alpha}{2}\Big)d\Big)\Big(1 - \frac{1}{2d}\Big)^d \Big(1 - \frac{1}{d}\Big)^d. \tag{1.24}$$

Moreover, if $f \in L^p(\mathbb{R}, \gamma)$ *for some* $p > 2$, *then the convergences in* (1) *and* (2) *hold in law in* $C(\mathbb{R}_+)$.

Let us finally consider the case $\alpha > 2 - \frac{1}{d}$ and $d \geq 2$. We will show that, for all $t \geq 0$, the random variable $\varepsilon^{\frac{1}{2} - d(1 - \frac{\alpha}{2})} \widetilde{F}_\varepsilon(t)$ converges in $L^2(\Omega)$ to a random variable $c_d H_\infty(t)$ belonging to the dth Wiener chaos. The process $H_\infty = (H_\infty(t))_{t \geq 0}$ is a generalization of the Hermite process (see [6, 12, 18]) and it has a covariance given by

$$\begin{aligned} K_d(s,t) &= \mathbb{E}\big[H_\infty(s) H_\infty(t)\big] \\ &= \frac{d!}{(2\lambda)^d} \int_0^s \int_0^t \Big(\frac{\partial_u \partial_v \mathbb{E}[X(u)X(v)]}{(uv)^{\beta - \alpha/2}}\Big)^d \, du\, dv. \end{aligned} \tag{1.25}$$

This then leads to the following noncentral limit theorem in the case $\alpha > 2 - \frac{1}{d}$.

THEOREM 1.3. *Under the assumptions of Theorem* 1.2, *if* $\alpha > 2 - \frac{1}{d}$, *then the process* $\{\varepsilon^{\frac{1}{2} - d(1 - \frac{\alpha}{2})} F_\varepsilon : \varepsilon > 0\}$ *converges in law in* $C(\mathbb{R}_+)$ *to* $F_\infty = c_d H_\infty$.

We note that a *discrete* counterpart of point 1 in Theorem 1.2 was already obtained by Harnett and Nualart in [7], in exactly the same setting. However, we would like to offer the following comments to help the reader comparing our results with those contained in [7]. First, neither point 2 of Theorem 1.2 nor the tightness property and Theorem 1.3 have been considered in [7]. Second, and a little bit against common intuition, it turns out that it was more difficult to deal with the continuous setting; indeed, in the continuous case we have to handle the situation where $|t - s| < 1$, which does not appear in the discrete setting. Third, our original motivation of proving Theorems 1.1, 1.2 and 1.3 is of geometrical nature; in our mind, this work actually represents a first step toward a better understanding of the asymptotic behavior of functionals of the kind (1.7) (or more complicated ones) that arise very often in differential geometry.

To conclude this Introduction, let us go back to the case of the bifractional Brownian motion $X = \widetilde{B}$, and let us see what the conclusions of Theorems 1.2 and 1.3 become in this case, when for f we choose the function $f(x) = |x| - \sqrt{\frac{2}{\pi}}$. Since, on one hand, the bifractional Brownian motion defined by (1.13) satisfies (H.1) and (H.2) with $\alpha = 2\beta = 2HK$ and, on the other hand, one has $\|\Delta_\varepsilon \widetilde{B}(t)\|_{L^2(\Omega)} \sim 2^{\frac{1-K}{2}} \varepsilon^{HK}$ as $\varepsilon \to 0$ for any $t > 0$, we deduce from our Theorems 1.2 and 1.3 that:

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- if $HK < 3/4$, then the family $\{\varepsilon^{\frac{1}{2} - HK}(\mathcal{L}(\widetilde{B}^\varepsilon; [0,t]) - \mathbb{E}[\mathcal{L}(\widetilde{B}^\varepsilon; [0,t])]) : \varepsilon > 0\}$ converges in law in $C(\mathbb{R}_+)$ to a Brownian motion with variance

$$\frac{2^{1-K}}{\pi} \sum_{q=1}^\infty \frac{(2q)!}{2^{2q-1} q!^2 (2q-1)^2} \int_{-\infty}^\infty a_{2HK}(h)^{2q}\, dh,$$

see also (1.18) and compare with claim (1.19);

- if $HK = 3/4$, then the family $\{\frac{\varepsilon^{-\frac{1}{4}}}{\sqrt{\log \varepsilon}}(\mathcal{L}(\widetilde{B}^\varepsilon; [0,t]) - \mathbb{E}[\mathcal{L}(\widetilde{B}^\varepsilon; [0,t])]) : \varepsilon > 0\}$ converges in law in $C(\mathbb{R}_+)$ to a Brownian motion with variance $2^{-K} \times \frac{9}{64\pi}$;
- if $HK > 3/4$, then the family $\{\varepsilon^{HK-1}(\mathcal{L}(\widetilde{B}^\varepsilon; [0,t]) - \mathbb{E}[\mathcal{L}(\widetilde{B}^\varepsilon; [0,t])]) : \varepsilon > 0\}$ converges in law in $C(\mathbb{R}_+)$ toward a stochastic process F_∞ which lies in the second Wiener chaos.

The rest of the paper is organized as follows. Section 2 contains some preliminaries on Maliavin calculus and a basic multivariate chaotic central limit theorem. The proof of Theorem 1.1 is given in Section 3. Section 4 provides some useful properties satisfied by self-similar processes X under assumptions (H.1) and (H.2) and contains the proof of Theorem 1.2. The proof of Theorem 1.3 is then given in Section 5. Finally, Section 6 contains some technical lemmas that are used along the paper.

Throughout the paper, C denotes a generic positive constant whose value may change from line to line.

2. Preliminaries. In this section, we gather several preliminary results that will be used for proving the main results of this paper.

2.1. *Elements of Malliavin calculus.* We assume that the reader is already familiar with the classical concepts of Malliavin calculus as outlined, for example, in the three books [13–15].

To be in a position to use Malliavin calculus to prove the results of our paper, we shall adopt the following classical Hilbert space notation. Let $\mathfrak{H}$ be a real and separable Hilbert space. Let X be an *isonormal* Gaussian process indexed by $\mathfrak{H}$ and defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, that is, $X = \{X(h), h \in \mathfrak{H}\}$ is a family of jointly centered Gaussian random variables satisfying $\mathbb{E}[X(h)X(g)] = \langle h, g\rangle_{\mathfrak{H}}$ for all $h, g \in \mathfrak{H}$. We will also assume that $\mathcal{F}$ is the σ-field generated by X.

For integers $q \geq 1$, let $\mathfrak{H}^{\otimes q}$ denote the qth tensor product of $\mathfrak{H}$, and let $\mathfrak{H}^{\odot q}$ denote the subspace of symmetric tensors of $\mathfrak{H}^{\otimes q}$. Let $\{e_n\}_{n\geq 1}$ be a complete orthonormal system in $\mathfrak{H}$. For functions $f, g \in \mathfrak{H}^{\odot q}$ and $r \in \{1, \dots, q\}$ we define the rth-order contraction of f and g as the element of $\mathfrak{H}^{\otimes(2q-2r)}$ given by

$$f \otimes_r g = \sum_{i_1,\dots,i_r=1}^{\infty} \langle f, e_{i_1} \otimes \cdots \otimes e_{i_r}\rangle_{\mathfrak{H}^{\otimes r}} \langle g, e_{i_1} \otimes \cdots \otimes e_{i_r}\rangle_{\mathfrak{H}^{\otimes r}},$$

where $f \otimes_0 g = f \otimes g$ by definition and, if $f, g \in \mathfrak{H}^{\odot q}$, $f \otimes_q g = \langle f, g\rangle_{\mathfrak{H}^{\otimes q}}$.

The qth Wiener chaos is the closed linear subspace of $L^2(\Omega)$ that is generated by the random variables $\{H_q(X(h)), h \in \mathfrak{H}, \|h\|_{\mathfrak{H}} = 1\}$, where H_q stands for the qth Hermite polynomial. For $q \geq 1$, it is known that the map $I_q(h^{\otimes q}) = H_q(X(h))$ ($h \in \mathfrak{H}$, $\|h\|_{\mathfrak{H}} = 1$) provides a linear isometry between $\mathfrak{H}^{\odot q}$ (equipped with the modified norm $\sqrt{q!}\|\cdot\|_{\mathfrak{H}^{\otimes q}}$) and the qth Wiener chaos. By convention, $I_0(x) = x$ for all $x \in \mathbb{R}$.

It is well known that any $F \in L^2(\Omega)$ can be decomposed into Wiener chaos as follows:

$$F = \mathbb{E}[F] + \sum_{q=1}^{\infty} I_q(f_q), \tag{2.1}$$

where the kernels $f_q \in \mathfrak{H}^{\odot q}$ are uniquely determined by F.

For a smooth and cylindrical random variable $F = f(X(h_1), \dots, X(h_n))$, with $h_i \in \mathfrak{H}$ and $f \in C_b^\infty(\mathbb{R}^n)$ (f and all of its partial derivatives are bounded), we define its Malliavin derivative as the $\mathfrak{H}$-valued random variable given by

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(X(h_1), \dots, X(h_n))h_i.$$

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By iteration, one can define the kth derivative $D^k F$ as an element of $L^2(\Omega; \mathfrak{H}^{\otimes k})$. For any natural number k and any real number $p \geq 1$, we define the Sobolev space $\mathbb{D}^{k,p}$ as the closure of the space of smooth and cylindrical random variables with respect to the norm $\|\cdot\|_{k,p}$ defined by

$$\|F\|_{k,p}^p = \mathbb{E}(|F|^p) + \sum_{i=1}^{k} \mathbb{E}(\|D^i F\|_{\mathfrak{H}^{\otimes i}}^p).$$

The divergence operator δ is defined as the adjoint of the derivative operator D. An element $u \in L^2(\Omega; \mathfrak{H})$ belongs to the domain of δ, denoted by $\operatorname{Dom}\delta$, if there is a constant c_u depending on u such that

$$\left|\mathbb{E}(\langle DF, u\rangle_{\mathfrak{H}})\right| \leq c_u \|F\|_{L^2(\Omega)}$$

for any $F \in \mathbb{D}^{1,2}$. If $u \in \operatorname{Dom}\delta$, then the random variable $\delta(u)$ is defined by the duality relationship

$$\mathbb{E}(F\delta(u)) = \mathbb{E}(\langle DF, u\rangle_{\mathfrak{H}}), \tag{2.2}$$

which holds for any $F \in \mathbb{D}^{1,2}$. In a similar way, we can introduce the iterated divergence operator δ^k for each integer $k \geq 2$, defined by the duality relationship

$$\mathbb{E}(F\delta^k(u)) = \mathbb{E}(\langle D^k F, u\rangle_{\mathfrak{H}^{\otimes k}}), \tag{2.3}$$

for any $F \in \mathbb{D}^{k,2}$, where $u \in \operatorname{Dom}\delta^k \subset L^2(\Omega; \mathfrak{H}^{\otimes k})$.

The Ornstein–Uhlenbeck semigroup $(T_t)_{t\geq 0}$ is the semigroup of operators on $L^2(\Omega)$ defined by

$$T_t F = \sum_{q=0}^{\infty} e^{-qt} I_q(f_q),$$

if F admits the Wiener chaos expansion (2.1). Denote by L the infinitesimal generator of $(T_t)_{t\geq 0}$ in $L^2(\Omega)$. Let $L^{-1}F = -\sum_{q=1}^{\infty} \frac{1}{q} I_q(f_q)$ if F is given by (2.1).

The operators D, δ and L satisfy the relationship $L = -\delta D$, which leads to the representation

$$F = -\delta D L^{-1} F, \tag{2.4}$$

for any centered random variable $F \in L^2(\Omega)$.

Consider the isonormal Gaussian process $X(h) = h$ indexed by $\mathfrak{H} = \mathbb{R}$, defined in the probability space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \gamma)$. We denote the corresponding Sobolev spaces of functions by $\mathbb{D}^{k,p}(\mathbb{R}, \gamma)$. In this context, for any function g, we have $Dg = g'$, $\delta g = xg - g'$ and $Lg = g'' - xg'$ (see [13]). Let $f \in L^2(\mathbb{R}, \gamma)$ be a function of Hermite rank d, with expansion (1.1). Let us introduce the function f_d defined by a shift of d units in the coefficients, that is,

$$f_d(x) = \sum_{q=d}^{\infty} c_q H_{q-d}(x).$$

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We claim that f_d belongs to the Sobolev space $\mathbb{D}^{d,2}(\mathbb{R}, \gamma)$. In fact, using that $H_q'(x) = qH_{q-1}(x)$, we can write

$$f_d^{(d)}(x) = \sum_{q=2d}^{\infty} c_q (q-d)(q-d-1)\cdots(q-2d+1) H_{q-2d}(x)$$

and

$$\|f_d^{(d)}\|_{L^2(\mathbb{R},\gamma)}^2 = \sum_{q=2d}^{\infty} c_q^2 (q-d)^2 (q-d-1)^2 \cdots (q-2d+1)^2 (q-2d)!$$
$$\leq \sum_{q=2d}^{\infty} c_q^2 q! < \infty.$$

The function f_d has the following representation in terms of the Malliavin operators:

(2.6) $$f_d = (-DL^{-1})^d f.$$

Indeed, using that $H_q'(x) = qH_{q-1}(x)$, we have

$$-DL^{-1} f = \sum_{q=d}^{\infty} \frac{c_q}{q} H_q'(x) = \sum_{q=d}^{\infty} c_q H_{q-1}(x),$$

and iterating d times this formula, we get (2.6). Formula (2.6) implies that if $f \in L^p(\mathbb{R},\gamma)$ for some $p > 1$, then $f_d \in \mathbb{D}^{d,p}(\mathbb{R},\gamma)$, that is, f_d is d-times weakly differentiable with derivatives in $L^p(\mathbb{R},\gamma)$. In fact, by Meyer inequalities (see [14]), the operators D and $(-L)^{1/2}$ are equivalent in $L^p(\mathbb{R},\gamma)$ and we obtain, for any $k = 1,\dots,d$,

$$\|f_d^{(k)}\|_{L^p(\mathbb{R},\gamma)} = \|D^k[(-DL^{-1})^d f]\|_{L^p(\mathbb{R},\gamma)}$$
$$= \|D^{k+1}(-L^{-1})[(-DL^{-1})^{d-1} f]\|_{L^p(\mathbb{R},\gamma)}$$
$$\leq c_p \|(-L)^{(k-1)/2}(-DL^{-1})^{d-1} f\|_{L^p(\mathbb{R},\gamma)}$$
$$\leq c_p' \|D^{k-1}[(-DL^{-1})^{d-1} f]\|_{L^p(\mathbb{R},\gamma)}.$$

Iterating this inequality and taking into account that the operator $-DL^{-1}$ is bounded in $L^p(\mathbb{R},\gamma)$, we obtain

(2.7) $$\|f_d^{(k)}\|_{L^p(\mathbb{R},\gamma)} \leq c_p^{(2)} \|(-DL^{-1})^{d-k} f\|_{L^p(\mathbb{R},\gamma)} \leq c_p^{(3)} \|f\|_{L^p(\mathbb{R},\gamma)}.$$

2.2. *Multivariate chaotic central limit theorem.* Points 1 and 2 in Theorem 1.2 will be obtained by checking that the assumptions of the following theorem are satisfied. We assume that X is an isonormal Gaussian process indexed by $\mathfrak{H}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with $\mathcal{F}$ is the σ-field generated by X.

THEOREM 2.1. *Fix an integer $p \geq 1$, and let $\{G^\varepsilon : \varepsilon > 0\}$ be a family of p-dimensional vectors with components in $L^2(\Omega)$ and centered. According to (2.1), we can write each component G_i^ε of G^ε in the form*

$$G_i^\varepsilon = \sum_{q=1}^{\infty} I_q(g_{i,q}^\varepsilon).$$

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Let us suppose that the following conditions hold:

(a) *For each $i, j \in \{1,\dots,p\}$ and each $q \geq 1$, $\sigma_{i,j,q} = \lim_{\varepsilon\to 0} q!\langle g_{i,q}^\varepsilon, g_{j,q}^\varepsilon\rangle_{\mathfrak{H}^{\otimes q}}$ exists.*

(b) *For each $i \in \{1,\dots,p\}$, $\sum_{q=1}^{\infty} \sigma_{i,i,q} < \infty$.*

(c) *For each $i \in \{1,\dots,p\}$, each $q \geq 2$ and each $r = 1,\dots,q-1$, $\lim_{\varepsilon\to 0} \|g_{i,q}^\varepsilon \otimes_r g_{i,q}^\varepsilon\|_{\mathfrak{H}^{\otimes 2q-2r}} = 0$.*

(d) *For each $i \in \{1,\dots,p\}$, $\lim_{N\to\infty} \sup_{\varepsilon\in(0,1]} \sum_{q=N+1}^{\infty} q!\|g_{i,q}^\varepsilon\|_{\mathfrak{H}^{\otimes 2q}}^2 = 0$.*

Then $G^\varepsilon = (G^\varepsilon_1, \dots, G^\varepsilon_p)$ *converges in distribution to* $N_p(0,\Sigma)$ *as* ε *tends to zero, where* $\Sigma = (\sigma_{i,j})_{1\le i,j\le p}$ *is defined by* $\sigma_{i,j} = \sum_{q=1}^{\infty} \sigma_{i,j,q}$.

PROOF. This theorem is a multivariate counterpart of the chaotic central limit theorem proved by Hu and Nualart in [8]. First, notice that, by the results of Nualart and Peccati [16] and Peccati and Tudor [17], conditions (a) and (c) imply that, for any $N \ge 1$, the family of random vectors $(I_q(g^\varepsilon_{i,q}))_{1\le q\le N, 1\le i\le p}$ converges in law to a centered Gaussian vector $(Z_{i,q})_{1\le q\le N,1\le i\le p}$ with covariance $\mathbb{E}[Z_{i,q}Z_{j,q'}] = \sigma_{i,j,q}\delta_{q,q'}$. This implies that, for each $N \ge 1$, the family of p-dimensional random vectors $(\sum_{q=1}^{N} I_q(g^\varepsilon_{i,q}))_{1\le i\le p}$ converges in law to the Gaussian distribution $N_p(0,\Sigma^N)$, where $\Sigma^N = (\sigma^N_{i,j})_{1\le i,j\le p}$ is defined by $\sigma^N_{i,j} = \sum_{q=1}^{N}\sigma_{i,j,q}$. Finally, conditions (b) and (d) and a simple triangular inequality allows us to conclude the proof. □

3. Proof of Theorem 1.1. Since the convergence in the sense of f.d.d. follows from the classical Breuer–Major theorem (see, e.g., [1]), it remains to show that the family $\{Z_\varepsilon : \varepsilon > 0\}$ is tight. For this we need to show that for any $0 \le s < t$ and $\varepsilon > 0$ and for some $p > 2$, there exists a constant $C_p > 0$ such that

$$\|Z_\varepsilon(t) - Z_\varepsilon(s)\|_{L^p(\Omega)} \le C_p|t-s|^{1/2}.$$

To show this inequality, we will use an approach based on stochastic integral representations and Meyer's inequalities.

Let $\mathfrak{H}$ be the Hilbert space defined as the closure of the set of step functions with respect to the scalar product $\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]}\rangle_{\mathfrak{H}} = \mathbb{E}[Y(s)Y(t)]$, $s,t \ge 0$. By identifying $Y(t)$ with $Y(\mathbf{1}_{[0,t]})$, we can thus suppose that Y is an isonormal Gaussian process indexed by $\mathfrak{H}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We will assume that $\mathcal{F}$ is generated by Y.

The function f_d introduced in (2.5) leads to the following representation of $f(Y(u))$ as an iterated divergence:

$$f(Y(u)) = \delta^d\big(f_d(Y(u))\mathbf{1}^{\otimes d}_{[0,u]}\big).$$

Indeed,

$$\begin{aligned} f(Y(u)) &= \sum_{q=d}^{\infty} c_q H_q(Y(u)) = \sum_{q=d}^{\infty} c_q I_q\big(\mathbf{1}^{\otimes q}_{[0,u]}\big) \\ &= \sum_{q=d}^{\infty} c_q \delta^d\big(I_{q-d}\big(\mathbf{1}^{\otimes q-d}_{[0,u]}\big)\mathbf{1}^{\otimes d}_{[0,u]}\big) = \delta^d\left(\sum_{q=d}^{\infty} c_q H_{q-d}(Y(u))\mathbf{1}^{\otimes d}_{[0,u]}\right). \end{aligned}$$

Then, using the continuity of δ^d, we obtain

$$\begin{aligned} \|Z_\varepsilon(t) - Z_\varepsilon(s)\|_{L^p(\Omega)} &= \sqrt{\varepsilon}\left\|\int_{s/\varepsilon}^{t/\varepsilon} f(Y(u))\,du\right\|_{L^p(\Omega)} \\ &= \sqrt{\varepsilon}\left\|\int_{s/\varepsilon}^{t/\varepsilon} \delta^d\big(f_d(Y(u))\mathbf{1}^{\otimes d}_{[0,u]}\big)\,du\right\|_{L^p(\Omega)} \\ &\le c_p \sum_{k=0}^{d} \sqrt{\varepsilon}\left\|\int_{s/\varepsilon}^{t/\varepsilon} D^k\big(f_d(Y(u))\mathbf{1}^{\otimes d}_{[0,u]}\big)\,du\right\|_{L^p(\Omega;\mathfrak{H}^{\otimes k+d})} \\ &=: c_p\sum_{k=0}^{d} R_k. \end{aligned}$$

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Using Minkowski and Hölder inequalities, we can write, for any $k = 0, 1, \dots, d$,

$$R_k = \sqrt{\varepsilon}\left\| \int_{[s/\varepsilon, t/\varepsilon]^2} f_d^{(k)}(Y(u)) f_d^{(k)}(Y(v)) \langle \mathbf{1}_{[0,u]}, \mathbf{1}_{[0,v]} \rangle_{\mathfrak{H}}^d \, du\, dv \right\|_{L^{p/2}(\Omega)}^{1/2}$$

$$\leq \| f_d^{(k)} \|_{L^p(\mathbb{R},\gamma)} \left(\varepsilon \int_{s/\varepsilon}^{t/\varepsilon} \int_{s/\varepsilon}^{t/\varepsilon} |\rho(u-v)|^{d+k} \, du\, dv \right)^{1/2}.$$

Using the assumptions of Theorem 1.1 as well as (2.7), we deduce that $\| f_d^{(k)} \|_{L^p(\mathbb{R},\gamma)}$ is finite. Finally, the change of variable $(u, v) \to (u, u+h)$ leads to

$$\varepsilon \int_{s/\varepsilon}^{t/\varepsilon} \int_{s/\varepsilon}^{t/\varepsilon} |\rho(u-v)|^{d+k} \, du\, dv \leq C(t-s) \int_{\mathbb{R}} |\rho(h)|^{d+k} \, dh \leq C(t-s),$$

which provides the desired estimate.

4. Proof of Theorem 1.2.

In this section, X will be a self-similar Gaussian process with covariance (1.20). The proof of Theorem 1.2 is quite technical and is divided into two parts: (i) the proof of convergence of the finite-dimensional distributions (see Sections 4.2 and 4.3, corresponding to the two cases $\alpha < 2 - 1/q$ and $\alpha = 2 - 1/q$) and (ii) the proof of tightness of the sequence (see Section 4.4). In order to prove convergence of finite-dimensional distributions, we will need three technical lemmas which provide information on the variance and covariance of X under Hypotheses (H.1) and (H.2). They are stated next in Section 4.1. Further technical lemmas, used in Sections 4.1, 4.2 and 4.3 are proved in Section 6.

4.1. *A few useful properties satisfied by X.* The first lemma give the structure of the variance of an increment of length one, assuming Hypothesis (H.1).

LEMMA 4.1. *Assuming* (H.1), *there exists a continuous function* $u_1 : (0, \infty) \to \mathbb{R}$ *such that for* $s > 0$

$$\mathbb{E}[(X(s+1) - X(s))^2] = 2\lambda s^{2\beta-\alpha}(1 + u_1(s)).$$

Furthermore, given $\eta > 0$, *there exists a positive constant* C_η *such that for all* $s \geq \eta$ *one has*

(4.1) $$|u_1(s)| \leq C_\eta s^{-\delta_1} \quad \textit{where } \delta_1 = \begin{cases} 1-\alpha & \textit{if } \alpha < 1, \\ 2-\alpha & \textit{if } \alpha \geq 1. \end{cases}$$

PROOF. If $s \geq 1$, the assertion follows from [7], Lemma 3.1. Let us now assume that $0 < s < 1$. Proceeding as in the proof of [7], Lemma 3.1, we get that

$$\mathbb{E}[(X(s+1) - X(s))^2]$$

$$= 2\lambda s^{2\beta-\alpha} + \psi(1)((s+1)^{2\beta} - s^{2\beta}) - 2s^{2\beta} \int_1^{1+\frac{1}{s}} \psi'(y)\, dy$$

$$= 2\lambda s^{2\beta-\alpha}(1 + u_1(s)),$$

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where

$$u_1(s) = (2\lambda)^{-1}\psi(1) s^{\alpha-2\beta}((s+1)^{2\beta} - s^{2\beta}) - \lambda^{-1} s^{\alpha} \int_1^{1+\frac{1}{s}} \psi'(y)\, dy.$$

Then the bound (4.1) for $\eta \leq s < 1$ follows immediately from the fact that $u_1(s)$ is bounded for $\eta \leq s < 1$. $\square$

In the next two lemmas, we will show formulas and estimates for the covariance $\mathbb{E}[(X(t+1)-X(t))(X(s+1)-X(s))]$ in two different situations. First, we will assume Hypothesis (H.1) and consider the case where $|t-s| \le M_1(s \wedge t) + M_2$ for some constants M_1 and M_2, and the second lemma will handle the case $|t-s| \ge (c-1)(s \wedge t) + c$ under Hypotheses (H.1) and (H.2), where c is the constant appearing in (H.2).

LEMMA 4.2. *Assume* (H.1) *and let* $\eta > 0$. *Then, for all* $s, t > \eta$ *satisfying* $\eta \le |s-t| \le M_1(s \wedge t) + M_2$ *for some positive constants* M_1, M_2, *it holds that*

$$\mathbb{E}\big[\big(X(t+1)-X(t)\big)\big(X(s+1)-X(s)\big)\big] = \lambda(s \wedge t)^{2\beta-\alpha}\big(2a_\alpha(s-t) + u_2(s,t)\big),$$

where $a_\alpha(h)$ *is the function defined in* (1.10) *and* $u_2 : [0,\infty)^2 \to \mathbb{R}$ *is a continuous function satisfying the bounds*

$$|u_2(s,t)| \le C\big((s \wedge t)^{-1}|s-t|^{\alpha-1} + (s \wedge t)^{\alpha-2}\big) \quad \text{if } |s-t| \ge 1 \tag{4.2}$$

and

$$|u_2(s,t)| \le C\big((s \wedge t)^{\alpha-1}\mathbf{1}_{\{\alpha<1\}} + (s \wedge t)^{\alpha-2}\mathbf{1}_{\{\alpha\ge1\}}\big) \quad \text{if } |s-t| < 1. \tag{4.3}$$

PROOF. Assume without loss of generality that $t > s$, so that $t = s+h$ for some $h > 0$. Then the assertion becomes

$$\begin{aligned}
&\mathbb{E}\big[\big(X(s+h+1)-X(s+h)\big)\big(X(s+1)-X(s)\big)\big] \\
&\quad = s^{2\beta-\alpha}\lambda\big(2a_\alpha(h) + u_2(s,s+h)\big),
\end{aligned}$$

where $u_2(s,s+h)$ satisfies the bounds

$$|u_2(s,s+h)| \le C\big(s^{-1}h^{\alpha-1} + s^{\alpha-2}\big) \tag{4.4}$$

for all s, h such that $s \ge \eta$ and $1 \le h \le M_1 s + M_2$ and

$$|u_2(s,s+h)| \le C\big(s^{\alpha-1}\mathbf{1}_{\{\alpha<1\}} + s^{\alpha-2}\mathbf{1}_{\{\alpha\ge1\}}\big) \tag{4.5}$$

for all s, h such that $s \ge \eta$ and $\eta \le h < 1$.

Let us first show the claim (4.5). In this case,

$$\begin{aligned}
&\mathbb{E}\big[\big(X(s+h+1)-X(s+h)\big)\big(X(s+1)-X(s)\big)\big] \\
&\quad = (s+1)^{2\beta}\phi\Big(\frac{s+h+1}{s+1}\Big) - (s+h)^{2\beta}\phi\Big(\frac{s+1}{s+h}\Big) \\
&\qquad - s^{2\beta}\phi\Big(\frac{s+h+1}{s}\Big) + s^{2\beta}\phi\Big(\frac{s+h}{s}\Big) \\
&\qquad + \lambda\big((s+1)^{2\beta-\alpha}h^\alpha - (s+h)^{2\beta-\alpha}(1-h)^\alpha - s^{2\beta-\alpha}(h+1)^\alpha + s^{2\beta-\alpha}h^\alpha\big) \\
&\qquad + (s+1)^{2\beta}\psi\Big(\frac{s+h+1}{s+1}\Big) - (s+h)^{2\beta}\psi\Big(\frac{s+1}{s+h}\Big) \\
&\qquad - s^{2\beta}\psi\Big(\frac{s+h+1}{s}\Big) + s^{2\beta}\psi\Big(\frac{s+h}{s}\Big) \\
&\quad = s^{2\beta-\alpha}\big(2\lambda a_\alpha(h) + u_2(s,s+h)\big),
\end{aligned}$$

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where

$$
\begin{aligned}
&u_2(s, s+h) \\
&= \left(1 - \left(1+\frac{1}{s}\right)^{2\beta-\alpha}\right)h^\alpha + \left(\left(1+\frac{h}{s}\right)^{2\beta-\alpha} - 1\right)(1-h)^\alpha \\
&\quad + s^\alpha\left(\left(1+\frac{1}{s}\right)^{2\beta}\psi\left(1+\frac{h}{s+1}\right) - \left(1+\frac{h}{s}\right)^{2\beta}\psi\left(1+\frac{1-h}{s+h}\right)\right. \\
&\quad \left. - \psi\left(1+\frac{h+1}{s}\right) + \psi\left(1+\frac{h}{s}\right)\right) \\
&= \left(1 - \left(1+\frac{1}{s}\right)^{2\beta-\alpha}\right)h^\alpha + \left(\left(1+\frac{h}{s}\right)^{2\beta-\alpha} - 1\right)(1-h)^\alpha + s^\alpha v(s,h).
\end{aligned}
$$

For the first part on the right-hand side, we have, using the mean value theorem,

$$
\begin{aligned}
&\left|\left(1 - \left(1+\frac{1}{s}\right)^{2\beta-\alpha}\right)h^\alpha + \left(\left(1+\frac{h}{s}\right)^{2\beta-\alpha} - 1\right)(1-h)^\alpha\right| \\
&\quad \le C\left(s^{-1}h^\alpha + (s/h)^{-1}(1-h)^\alpha\right) \le Cs^{-1}
\end{aligned}
$$

and we obtain the desired inequality.

For $v(s,h)$, we first treat the case $\alpha < 1$. In this case, it follows straightforwardly from the mean value theorem that $|v(s,h)| \le Cs^{-1}$, which yields $s^\alpha|v(s,h)| \le Cs^{\alpha-1}$. In the case $\alpha \ge 1$, a Taylor expansion in s^{-1} around 0 yields that

$$
\begin{aligned}
&v(s,h) \\
&= \left(1+\frac{1}{s}\right)^{2\beta}\psi\left(1+\frac{h}{s+1}\right) - \left(1+\frac{h}{s}\right)^{2\beta}\psi\left(1+\frac{1-h}{s+h}\right) \\
&\quad - \psi\left(1+\frac{h+1}{s}\right) + \psi\left(1+\frac{h}{s}\right) \\
&= \left(1 + 2\beta\frac{1}{s} + O\left(\frac{1}{s^2}\right)\right)\left(\psi(1) + \psi'(1)\frac{h}{s+1} + O\left(\frac{1}{s^2}\right)\right) \\
&\quad - \left(1 + 2\beta\frac{h}{s} + O\left(\frac{1}{s^2}\right)\right)\left(\psi(1) + \psi'(1)\frac{1-h}{s+h} + O\left(\frac{1}{s^2}\right)\right) \\
&\quad - \psi(1) - \psi'(1)\frac{h+1}{s} - O\left(\frac{1}{s^2}\right) \\
&\quad + \psi(1) + \psi'(1)\frac{h}{s} + O\left(\frac{1}{s^2}\right) \\
&= \psi(1)2\beta\frac{1}{s} + \psi'(1)\frac{h}{s} - \psi'(1)\frac{1-h}{s} - \psi(1)2\beta\frac{h}{s} \\
&\quad - \psi'(1)\frac{h+1}{s} + \psi'(1)\frac{h}{s} + O\left(\frac{1}{s^2}\right) \\
&= 2(1-h)\left(\beta\psi(1) - \psi'(1)\right)\frac{1}{s} + O\left(\frac{1}{s^2}\right) \\
&= O\left(\frac{1}{s^2}\right),
\end{aligned}
$$

where we have used that $\beta\psi(1) = \psi'(1)$ to derive the last equality. This yields (4.5).

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Let us now show the claim (4.4). In this case, we have

$$\begin{aligned}
&\mathbb{E}\big[\big(X(s+h+1)-X(s+h)\big)\big(X(s+1)-X(s)\big)\big]\\
&\quad=(s+1)^{2\beta}\Big(\phi\Big(\frac{s+h+1}{s+1}\Big)-\phi\Big(\frac{s+h}{s+1}\Big)\Big)\\
&\qquad-s^{2\beta}\phi\Big(\frac{s+h+1}{s}\Big)+s^{2\beta}\phi\Big(\frac{s+h}{s}\Big)\\
&\quad=-\lambda\big((s+1)^{2\beta-\alpha}(h^\alpha-(h-1)^\alpha)-s^{2\beta-\alpha}((h+1)^\alpha-h^\alpha)\big)\\
&\qquad+(s+1)^{2\beta}\Big(\psi\Big(\frac{s+h+1}{s+1}\Big)-\psi\Big(\frac{s+h}{s+1}\Big)\Big)\\
&\qquad-s^{2\beta}\Big(\psi\Big(\frac{s+h+1}{s}\Big)-\psi\Big(\frac{s+h}{s}\Big)\Big)\\
&\quad=s^{2\beta-\alpha}\big(2\lambda a_\alpha(h)+u_2(s,s+h)\big),
\end{aligned}$$

where

$$\begin{aligned}
u_2(s,s+h)&=\Big(1-\Big(1+\frac{1}{s}\Big)^{2\beta-\alpha}\Big)(h^\alpha-(h-1)^\alpha)\\
&\quad+s^\alpha\Big(\Big(1+\frac{1}{s}\Big)^{2\beta}\Big(\psi\Big(1+\frac{h}{s+1}\Big)-\psi\Big(1+\frac{h-1}{s+1}\Big)\Big)\\
&\quad-\psi\Big(1+\frac{h+1}{s}\Big)+\psi\Big(1+\frac{h}{s}\Big)\Big)\\
&=\Big(1-\Big(1+\frac{1}{s}\Big)^{2\beta-\alpha}\Big)(h^\alpha-(h-1)^\alpha)+s^\alpha w(s,h).
\end{aligned}$$

By the mean value theorem, we have that

$$\Big|\Big(1-\Big(1+\frac{1}{s}\Big)^{2\beta-\alpha}\Big)(h^\alpha-(h-1)^\alpha)\Big|\le Cs^{-1}h^{\alpha-1},$$

which gives the desired estimate. Furthermore,

$$\begin{aligned}
w(s,h)&=\Big(1+\frac{1}{s}\Big)^{2\beta}\Big(\psi\Big(1+\frac{h}{s+1}\Big)-\psi\Big(1+\frac{h-1}{s+1}\Big)\Big)\\
&\quad-\psi\Big(1+\frac{h+1}{s}\Big)+\psi\Big(1+\frac{h}{s}\Big)\\
&=\Big(1+\frac{1}{s}\Big)^{2\beta}\int_{\frac{h-1}{s+1}}^{\frac{h}{s+1}}\psi'(1+y)\,dy-\int_{\frac{h}{s}}^{\frac{h+1}{s}}\psi'(1+y)\,dy\\
&=\Big(\Big(1+\frac{1}{s}\Big)^{2\beta}-1\Big)\int_{\frac{h-1}{s+1}}^{\frac{h}{s+1}}\psi'(1+y)\,dy\\
&\quad+\int_{\frac{h-1}{s+1}}^{\frac{h}{s+1}}\psi'(1+y)\,dy-\int_{\frac{h}{s}}^{\frac{h+1}{s}}\psi'(1+y)\,dy\\
&=\Big(\Big(1+\frac{1}{s}\Big)^{2\beta}-1\Big)\int_{\frac{h-1}{s+1}}^{\frac{h}{s+1}}\psi'(1+y)\,dy\\
&\quad+\int_0^{\frac{1}{s+1}}\psi'\Big(1+\frac{h-1}{s+1}+y\Big)\,dy-\int_1^{\frac{1}{s}}\psi'\Big(1+\frac{h}{s}+y\Big)\,dy
\end{aligned}$$

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$$= \left(\left(1+\frac{1}{s}\right)^{2\beta} - 1\right)\int_{\frac{h-1}{s+1}}^{\frac{h}{s+1}} \psi'(1+y)\,dy$$
$$+ \int_0^{\frac{1}{s+1}} \left(\psi'\left(1+\frac{h-1}{s+1}+y\right) - \psi'\left(1+\frac{h}{s}+y\right)\right)dy$$
$$\times \int_{\frac{1}{s+1}}^{\frac{1}{s}} \psi'\left(1+\frac{h}{s}+y\right)dy.$$

Therefore, using the bounds on the derivatives of ψ given by (H.1) and the fact that $h \leq M_1 s + M_2$, we get that

$$|w(s,h)| \leq C\left(s^{-2} + s^{-1-\alpha}h^{\alpha-1}\right).$$

This completes the proof. □

If X is fBm with Hurst parameter $H \in (0,1)$, we have that

$$\mathbb{E}\left[\left(X(t+1)-X(t)\right)\left(X(s+1)-X(s)\right)\right] = a_{2H}(s-t).$$

Therefore, heuristically speaking, Lemma 4.2 expresses that a process X satisfying (H.1) is a "perturbed" fBm with Hurst parameter $\beta = \alpha/2$.

For later reference, let us record here that the function a_α defined in (1.10) has the asymptotics

$$a_\alpha(h) = \frac{1}{2}\alpha(\alpha-1)|h|^{\alpha-2} + o\left(|h|^{\alpha-2}\right) \tag{4.6}$$

as $|h| \to \infty$. In particular, if $|h| > \eta$, there exists a constant C_η such that

$$\left|a_\alpha(h)\right| \leq C_\eta |h|^{\alpha-2}. \tag{4.7}$$

Hypothesis (H.2) implies the following bound for the covariance.

LEMMA 4.3. *Let $s, t > 0$ such that $s \wedge t \geq \eta > 0$ and $|s-t| \geq (c-1)(s\wedge t) + c$, where c is the constant appearing in hypothesis* (H.2). *Then, assuming* (H.2), *there exists a constant $C_\eta > 0$ (not depending on s or t), such that*

$$\left|\mathbb{E}\left[\left(X(s+1)-X(s)\right)\left(X(t+1)-X(t)\right)\right]\right|$$
$$\leq C_\eta \begin{cases} (s\wedge t)^{2\beta+\nu-2}|s-t|^{-\nu} & \text{if } \alpha < 1, \\ (s\wedge t)^{2\beta-\alpha}|s-t|^{\alpha-2} & \text{if } \alpha \geq 1, \end{cases} \tag{4.8}$$

and the exponent ν is defined in hypothesis (H.2).

PROOF. Without loss of generality, we assume that $s \geq t$ so that $|t-s| \geq (c-1)s\wedge t + c$ translates into $s \geq c(t+1)$. As $s \geq t$, we have by self-similarity that

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$$\mathbb{E}\left[\left(X(s+1)-X(s)\right)\left(X(t+1)-X(t)\right)\right]$$
$$= (t+1)^{2\beta}\left(\phi\left(\frac{s+1}{t+1}\right) - \phi\left(\frac{s}{t+1}\right)\right) - t^{2\beta}\left(\phi\left(\frac{s+1}{t}\right) - \phi\left(\frac{s}{t}\right)\right)$$
$$= \left((t+1)^{2\beta} - t^{2\beta}\right)\left(\phi\left(\frac{s+1}{t+1}\right) - \phi\left(\frac{s}{t+1}\right)\right)$$
$$+ t^{2\beta}\left(\phi\left(\frac{s+1}{t+1}\right) - \phi\left(\frac{s}{t+1}\right) - \phi\left(\frac{s+1}{t}\right) + \phi\left(\frac{s}{t}\right)\right).$$

As $s \geq c(t+1)$, we have that $s/(t+1) \geq c$ and, therefore, by (H.2), for each $x \in [\frac{s}{t+1}, \frac{s+1}{t+1}]$,

$$|\phi'(x)| \leq C \begin{cases} t^{\nu}(s-t)^{-\nu} & \text{if } \alpha < 1, \\ t^{2-\alpha}(s-t)^{\alpha-2} & \text{if } \alpha \geq 1, \end{cases} \tag{4.9}$$

and, for each $x \in [\frac{s}{t+1}, \frac{s+1}{t}]$,

$$|\phi''(x)| \leq C \begin{cases} t^{\nu+1}(s-t)^{-\nu-1} & \text{if } \alpha < 1, \\ t^{3-\alpha}(s-t)^{\alpha-3} & \text{if } \alpha \geq 1. \end{cases} \tag{4.10}$$

This yields the assertion, as by the mean value theorem,

$$\phi\left(\frac{s+1}{t+1}\right) - \phi\left(\frac{s}{t+1}\right) = \frac{1}{t+1}\phi'(x_1)$$

and

$$\begin{aligned} &\phi\left(\frac{s+1}{t+1}\right) - \phi\left(\frac{s}{t+1}\right) - \phi\left(\frac{s+1}{t}\right) + \phi\left(\frac{s}{t}\right) \\ &= \frac{1}{t+1}\phi'(x_2) - \frac{1}{t}\phi'(x_3) \\ &= \frac{1}{t+1}(\phi'(x_2) - \phi'(x_3)) + \left(\frac{1}{t+1} - \frac{1}{t}\right)\phi'(x_3) \\ &= \frac{1}{t+1}(x_2 - x_3)\phi''(x_4) - \frac{1}{t^2+t}\phi'(x_3), \end{aligned}$$

where the x_i are some appropriate values in the correct intervals for (4.9) and (4.10) to hold. □

We can now proceed to the proof of Theorem 1.2. In this section, $\mathfrak{H}$ will denote the Hilbert space defined as the closure of the set of step functions with respect to the scalar product $\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathfrak{H}} = \mathbb{E}[X(s)X(t)]$, $s, t \geq 0$ and, as before, we can consider that X as an isonormal Gaussian process indexed by $\mathfrak{H}$, and defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We will assume that $\mathcal{F}$ is generated by X.

First, we will prove the convergence of the finite dimensional distributions of F_ε, separately in the two cases $\alpha < 2 - \frac{1}{d}$ and $\alpha = 2 - \frac{1}{d}$ and later we will show tightness.

4.2. *Convergence of finite-dimensional distributions*: *The case* $\alpha < 2 - \frac{1}{d}$. Fix an integer $p \geq 2$, choose times $0 < t_1 < \cdots < t_p < \infty$, and consider the random vector $G^\varepsilon = (F_\varepsilon(t_1), \dots, F_\varepsilon(t_p))$, where F_ε has been defined in (1.23). We will make use of the notation

$$\xi_t = \|X(t+1) - X(t)\|_{L^2(\Omega)}$$

and

$$\Phi(s,t) = \mathbb{E}[Y_1(s)Y_1(t)] = \xi_s^{-1}\xi_t^{-1}\mathbb{E}[\Delta_1 X(s)\Delta_1 X(t)]. \tag{4.11}$$

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...pansion of $F_\varepsilon(t)$ is given by

$$F_\varepsilon(t) = \sum_{q=d}^{\infty} I_q(g_{t,q}^\varepsilon), \tag{4.12}$$

where, for each $t > 0$,

$$g_{t,q}^\varepsilon = c_q \sqrt{\varepsilon} \int_0^{t/\varepsilon} \xi_u^{-q} \partial_u^{\otimes q} \, du,$$

and $\partial_u = \mathbf{1}_{[u,u+1]}$. We will denote by $F_{q,\varepsilon}(t) = I_q(g^\varepsilon_{t,q})$ the projection of $F_\varepsilon(t)$ on the qth Wiener chaos. Using the relation between multiple stochastic integrals and Hermite polynomials, we can write

$$\begin{aligned} E\big(H_q(Y_1(u))H_q(Y_1(v))\big) &= E\big(I_q(\partial_u^{\otimes q})I_q(\partial_v^{\otimes q})\big) \\ &= \langle \partial_u^{\otimes q}, \partial_v^{\otimes q}\rangle = \langle \partial_u, \partial_v\rangle^q \\ &= \big(E(Y_1(u)Y_1(v))\big)^q = \Phi^q(u,v), \end{aligned}$$

so that

$$\begin{aligned} \mathbb{E}[F_{q,\varepsilon}(s)F_{q,\varepsilon}(t)] &= c_q^2\varepsilon \int_0^{s/\varepsilon}\int_0^{t/\varepsilon} \mathbb{E}[H_q(Y_1(u))H_q(Y_1(v))]\,du\,dv \\ &= c_q^2 q!\varepsilon \int_0^{s/\varepsilon}\int_0^{t/\varepsilon} \Phi^q(u,v)\,du\,dv. \end{aligned} \tag{4.13}$$

We are now going to check that assumptions (a), (b), (c) and (d) of Theorem 2.1 are satisfied by the family of p-dimensional vectors G^ε.

Proof of condition (a). Lemma 6.3 implies that, for every $q \ge d$ and for every $i, j \in \{1, \dots, p\}$, $q!\langle g^\varepsilon_{t_i,q}, g^\varepsilon_{t_j,q}\rangle_{\mathfrak{H}^{\otimes q}} \to \sigma^2_{\alpha,q}(t_i \wedge t_j)$ as $\varepsilon \to 0$, where $\sigma^2_{\alpha,q}$ is given by (6.18).

Proof of condition (b). This is straightforward.

Proof of condition (c). We have to show that for $r = 1, 2, \dots, q-1$ and for all $T > 0$,

$$\lim_{\varepsilon\to 0} \|g^\varepsilon_{T,q} \otimes_r g^\varepsilon_{T,q}\|^2_{\mathfrak{H}^{\otimes 2(q-r)}} = 0. \tag{4.14}$$

Using the notation (4.11), we see that

$$\begin{aligned} g^\varepsilon_{T,q} \otimes_r g^\varepsilon_{i,q} &= c_q^2\varepsilon \int_0^{T/\varepsilon}\int_0^{T/\varepsilon} \xi_s^{-q}\xi_t^{-q}\langle \partial_s, \partial_t\rangle^r_{\mathfrak{H}} \partial_s^{\otimes(q-r)} \otimes \partial_t^{\otimes(q-r)}\,ds\,dt \\ &= c_q^2\varepsilon \int_0^{T/\varepsilon}\int_0^{T/\varepsilon} \xi_s^{-(q-r)}\xi_t^{-(q-r)}\Phi^r(s,t)\partial_s^{\otimes(q-r)} \otimes \partial_t^{\otimes(q-r)}\,ds\,dt. \end{aligned}$$

Therefore,

$$\begin{aligned} &\|g^\varepsilon_{T,q} \otimes_r g^\varepsilon_{T,q}\|^2_{\mathfrak{H}^{\otimes 2(q-r)}} \\ &\quad = c_q^4\varepsilon^2 \int_{(0,T/\varepsilon)^4} \Phi^r(s,t)\Phi^r(l,m)\Phi^{q-r}(s,l)\Phi^{q-r}(t,m)\,ds\,dt\,dl\,dm. \end{aligned} \tag{4.15}$$

We claim that

$$\sup_{\varepsilon>0} \varepsilon \int_0^{T/\varepsilon}\int_0^{T/\varepsilon} |\Phi^q(s,t)|\,ds\,dt \le C, \tag{4.16}$$

where C is some constant not depending on q or ε. Taking into account that $|\Phi(s,t)| \le 1$, it suffices to show that

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$$\sup_{\varepsilon>0} \varepsilon \int_0^{T/\varepsilon}\int_0^{T/\varepsilon} |\Phi^d(s,t)|\,ds\,dt < \infty.$$

By Lemmas 6.1 and 6.2, it suffices to show that

$$\sup_{\varepsilon>0} \varepsilon \int_0^{T/\varepsilon}\int_0^{T/\varepsilon} |a^d_\alpha(s-t)|\,ds\,dt < \infty,$$

which is an immediate consequence of (4.7) and the fact that $\alpha < 2 - \frac{1}{d}$.

Let us write the integration domain $(0, T/\varepsilon)^4$ in the form $\bigcup_{i=1}^4 D_i$, where

$$D_1 = \{(s,t,l,m) \in (0, T/\varepsilon)^4 : |s-t| \geq (c-1)(s \wedge t) + c\},$$

$$D_2 = \{(s,t,l,m) \in (0, T/\varepsilon)^4 : |l-m| \geq (c-1)(l \wedge m) + c\},$$

$$D_3 = \{(s,t,l,m) \in (0, T/\varepsilon)^4 : |s-l| \geq (c-1)(s \wedge l) + c\},$$

$$D_4 = \{(s,t,l,m) \in (0, T/\varepsilon)^4 : |t-m| \geq (c-1)(t \wedge m) + c\}.$$

We claim that the integral over any of the sets D_i converges to zero. By Hölder's inequality, we have for nonnegative functions f_1, f_2, f_3, f_4 and real numbers $x_i \leq y_i$ for $i = 1, 2, 3, 4$ that

$$\begin{aligned}
&\varepsilon^2 \int_{x_1}^{y_1} \int_{x_2}^{y_2} \int_{x_3}^{y_3} \int_{x_4}^{y_4} f_1(s,t) f_2(l,m) f_3(s,l) f_4(t,m)\, ds\, dt\, dl\, dm \\
&\leq \left(\varepsilon \int_{x_1}^{y_1} \int_{x_2}^{y_2} f_1(s,t)^{q/r}\right)^{r/q} \left(\varepsilon \int_{x_3}^{y_3} \int_{x_4}^{y_4} f_2(l,m)^{q/r}\right)^{r/q} \\
&\quad \times \left(\varepsilon \int_{x_1}^{y_1} \int_{x_3}^{y_3} f_3(s,t)^{q/(q-r)}\right)^{(q-r)/q} \left(\varepsilon \int_{x_2}^{y_2} \int_{x_4}^{y_4} f_4(s,t)^{q/(q-r)}\right)^{(q-r)/q}.
\end{aligned}$$

The above inequality, together with (4.16) and Lemmas 6.1 and 6.3, implies that the integral over $\bigcup_{i=1}^4 D_i$ converges to zero. It therefore suffices to consider the integral over $\bigcap_{i=1}^4 D_i^c$. Using the decompositions,

$$\Phi^r(s,t) = R_{\alpha,r}(s,t) + a_\alpha^r(s,t)$$

and

$$\Phi^{q-r}(s,t) = R_{\alpha,q-r}(s,t) + a_\alpha^{q-r}(s,t)$$

provided by Lemma 6.2, and applying the above Hölder inequality and Lemma 6.2 with $M_1 = c - 1$ and $M_2 = c$, yields

$$\begin{aligned}
\lim_{\varepsilon \to 0} \| g_{T,q}^\varepsilon \otimes_r g_{T,q}^\varepsilon \|^2_{\mathfrak{H}^{\otimes 2(q-r)}} = \lim_{\varepsilon \to 0} c_q^4 \varepsilon^2 \int_{\bigcap_{i=1}^4 D_i^c} & a_\alpha^r(t-s) a_\alpha^r(m-l) \\
& \times a_\alpha^{q-r}(l-s) a_\alpha^{q-r}(m-t)\, ds\, dt\, dl\, dm.
\end{aligned}$$

It thus suffices to show that

$$\lim_{\varepsilon \to 0} \varepsilon^2 \int_{(0, \frac{T}{\varepsilon})^4} \left| a_\alpha^r(t-s) a_\alpha^r(m-l) a_\alpha^{q-r}(l-s) a_\alpha^{q-r}(m-t) \right| ds\, dt\, dl\, dm = 0.$$

As a_α is the covariance function of a fractional Brownian motion with Hurst parameter $\alpha/2$, this follows from the results in Breton–Nourdin [2] or Darses–Nourdin–Nualart [5].

Proof of condition (d). We have to show that, for each $T > 0$,

$$\begin{aligned}
&\lim_{N \to \infty} \sup_{\varepsilon > 0} \sum_{q=N+1}^{\infty} q! \| g_{T,q}^\varepsilon \|^2_{\mathfrak{H}^{\otimes q}} \\
&\quad = \lim_{N \to \infty} \sup_{\varepsilon > 0} \sum_{q=N+1}^{\infty} c_q^2 q! \varepsilon \int_0^{T/\varepsilon} \int_0^{T/\varepsilon} \Phi^q(s,t)\, ds\, dt = 0.
\end{aligned}$$

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As by assumption $\sum_{q=d}^\infty c_q^2 q! = \|f\|^2_{L^2(\mathbb{R},\gamma)} < \infty$, this follows from (4.16).

As conditions (a)–(d) are verified, it follows that the random vector $(G^\varepsilon(t_1), \ldots, G^\varepsilon(t_p))$ converges in distribution, as ε tends to zero, to $N_p(0, \Sigma)$, where $\Sigma = (\sigma_{i,j})_{1 \leq i,j \leq p}$ is the matrix given by

$$\sigma_{i,j} = \sigma^2 (t_i \wedge t_j).$$

Here, σ^2 is given by (1.3) with $\rho(h) = a_\alpha(h)$ defined in (1.10). This completes the proof.

4.3. *Convergence of finite-dimensional distributions*: *The case* $\alpha = 2 - \frac{1}{d}$. Fix an integer $p \geq 2$, choose times $0 < t_1 < \cdots < t_p < \infty$, and consider the random vector $G^\varepsilon = (F_\varepsilon(t_1)/\sqrt{|\log \varepsilon|}, \dots, F_\varepsilon(t_p)/\sqrt{|\log \varepsilon|})$, where F_ε has been defined in (1.23). As before, we need to show that assumptions (a), (b), (c) and (d) of Theorem 2.1 are satisfied by the family of p-dimensional vectors G^ε.

Condition (a) follows from Lemma 6.3, with $\sigma_{i,j,d} = \sigma^2_{1-2/d}(t_i \wedge t_j)$ and $\sigma_{i,j,q} = 0$ for $q > d$. Condition (b) is obvious. In order to show conditions (c) and (d), let us first remark that (4.16) is replaced here by

$$\sup_{\varepsilon > 0} \frac{\varepsilon}{|\log \varepsilon|} \int_0^{T/\varepsilon} \int_0^{T/\varepsilon} |\Phi^q(s,t)|\, ds\, dt \leq C, \tag{4.17}$$

which follows from Lemmas 6.1 and 6.2, and the fact that

$$\frac{\varepsilon}{|\log \varepsilon|} \int_0^{T/\varepsilon} \int_0^{T/\varepsilon} |a_\alpha^d(s-t)|\, ds\, dt \leq \frac{C\varepsilon}{|\log \varepsilon|} \int_0^{T/\varepsilon} \int_0^{T/\varepsilon} |t-s|^{-1}\, ds\, dt < \infty.$$

By the same arguments as in the case $\alpha < 2 - \frac{1}{d}$, condition (c) reduces to show that for any $r = 1, \dots, q-1$,

$$\lim_{\varepsilon \to 0} \frac{\varepsilon^2}{(\log \varepsilon)^2} \int_{(0,T/\varepsilon)^4} a_\alpha^r(t-s) a_\alpha^r(m-l) a_\alpha^{q-r}(l-s) a_\alpha^{q-r}(m-t)\, ds\, dt\, dl\, dm = 0,$$

and again this follows from the analogous result for the fractional Brownian motion. Finally, condition (d) is a consequence of (4.17).

4.4. *Proof of tightness*. Suppose first that $\alpha < 2 - \frac{1}{d}$. It suffices to show that for any $0 \leq s < t$ and $\varepsilon > 0$, there exists a constant $C_p > 0$ such that

$$\|F_\varepsilon(t) - F_\varepsilon(s)\|_{L^p(\Omega)} \leq C_p |t-s|^{1/2}.$$

To show this inequality, we will follow the methodology developed in the proof of Theorem 1.1. The starting point of the proof is the following representation of $f(Y_1(u))$ as an iterated divergence:

$$f(Y_1(u)) = \delta^d\big(f_d(Y_1(u)) \xi_u^{-d} \partial_u^{\otimes d}\big).$$

Then using Meyer's inequalities, we obtain

$$\begin{aligned}
\|F_\varepsilon(t) - F_\varepsilon(s)\|_{L^p(\Omega)} &= \sqrt{\varepsilon} \bigg\| \int_{s/\varepsilon}^{t/\varepsilon} f(Y_1(u))\, du \bigg\|_{L^p(\Omega)} \\
&= \sqrt{\varepsilon} \bigg\| \int_{s/\varepsilon}^{t/\varepsilon} \delta^d\big(f_d(Y_1(u)) \xi_u^{-d} \partial_u^{\otimes d}\big)\, du \bigg\|_{L^p(\Omega)} \\
&\leq c_p \sum_{k=0}^d \sqrt{\varepsilon} \bigg\| \int_{s/\varepsilon}^{t/\varepsilon} D^k\big(f_d(Y_1(u)) \xi_u^{-d} \partial_u^{\otimes d}\big)\, du \bigg\|_{L^p(\Omega; \mathfrak{H}^{\otimes k})} \\
&=: c_p \sum_{k=0}^d R_k.
\end{aligned}$$

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Using Minkowski and Hölder inequalities, we can write for any $k = 0, 1, \dots, d$,

$$\begin{aligned}
R_k &= \sqrt{\varepsilon} \bigg\| \int_{[s/\varepsilon, t/\varepsilon]^2} f_d^{(k)}(Y_1(u)) f_d^{(k)}(Y_1(v)) \bigg(\frac{\langle \partial_u, \partial_v \rangle_{\mathfrak{H}}}{\xi_u \xi_v} \bigg)^{d+k} du\, dv \bigg\|_{L^{p/2}(\Omega)}^{1/2} \\
&\leq \|f_d^{(k)}\|_{L^p(\mathbb{R},\gamma)} \bigg(\varepsilon \int_{s/\varepsilon}^{t/\varepsilon} \int_{s/\varepsilon}^{t/\varepsilon} |\Phi^{d+k}(u,v)|\, du\, dv \bigg)^{1/2},
\end{aligned}$$

where $\Phi(u, v)$ has been defined in (4.11). From the assumptions of Theorem 1.1 and (2.7), it follows that the quantity $\|f_d^{(d)}\|_{L^p(\mathbb{R},\gamma)}$ is finite. Then it suffices to show that for all $0 \le s \le t$

$$A_\varepsilon := \varepsilon \int_{s/\varepsilon}^{t/\varepsilon} \int_{s/\varepsilon}^{t/\varepsilon} \left|\Phi^d(u, v)\right| du\, dv \le t - s. \tag{4.18}$$

In order to show (4.18), notice first that on the region where $u \le M$, $v \le M$ or $|u - v| \le M$, we obtain the bound $CM(t - s)$. Therefore, it suffices to consider the integral over the region

$$\mathcal{D}_{\varepsilon,M} = \left\{(u, v) \in [s/\varepsilon, t/\varepsilon]^2 \cap [M, \infty)^2 : |u - v| \ge M\right\}.$$

We denote the corresponding term by

$$\widetilde{A}_\varepsilon := \varepsilon \int_{\mathcal{D}_{\varepsilon,M}} \left|\Phi^d(u, v)\right| du\, dv.$$

We are going to use two different estimates for $|\Phi^d(u, v)|$. First, on the region $\{(u, v) \in \mathcal{D}_{\varepsilon,M} : |u - v| \le (c - 1)(s \wedge t) + c\}$, using Lemmas 4.1 and 4.2, we have for large M,

$$\begin{aligned}\Phi^d(u, v) &= \frac{1}{2^d(1 + u_1(u))^d(1 + u_1(u + h))^d} \\ &\quad \times \left(\frac{u \wedge v}{u \vee v}\right)^{(\beta - \frac{\alpha}{2})d} \left(2a_\alpha\left(|u - v|\right) + u_2(u, v)\right)^d \\ &\le C|u - v|^{(\alpha-2)d}.\end{aligned}$$

Second, on the region $\{(u, v) \in \mathcal{D}_{\varepsilon,M} : |u - v| \ge (c - 1)(s \wedge t) + c\}$, using this time Lemmas 4.1 and 4.3, we can write, again for large M,

$$\begin{aligned}\left|\Phi^d(u, v)\right| &\le \frac{C_M}{2^d(1 + u_1(u))^d(1 + u_1(u + h))^d} \\ &\quad \times \left((u \wedge v)^{(\alpha+\nu-2)d}|u - v|^{-d\nu}\mathbf{1}_{\alpha<1} + |u - v|^{d(\alpha-2)}\mathbf{1}_{\alpha\ge1}\right) \\ &\le C|u - v|^{(\alpha-2)d}.\end{aligned}$$

These estimates and the change of variable $(u, v) \to (u, u + h)$ lead to

$$\widetilde{A}_\varepsilon \le C(t - s) \int_M^\infty h^{(\alpha-2)d}\, dh.$$

Under the condition $\alpha < 2 - \frac{1}{d}$, the integral $\int_M^\infty h^{(\alpha-2)2d}\, dh$ is finite and we obtain the desired estimate.

Suppose now that $\alpha = 2 - \frac{1}{d}$. We claim that for any $0 \le s < t$ and $\varepsilon \in (0, 1)$, there exists a constant $C_p > 0$ such that

$$\frac{1}{\sqrt{|\log \varepsilon|}} \left\|F_\varepsilon(t) - F_\varepsilon(s)\right\|_{L^p(\Omega)} \le C_p|t - s|^{1/2}.$$

The proof is analogous to the case $\alpha < 2 - \frac{1}{d}$, and can be completed using the estimate

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$$\sup_{\varepsilon\in(0,1)} \frac{1}{\sqrt{|\log \varepsilon|}} \left(\int_M^{t/\varepsilon} h^{-1}\, dh\right)^{1/2} < \infty.$$

5. Proof of Theorem 1.3. Recall the definition of $\widetilde{F}_\varepsilon$ given by (1.22). Denote $\widehat{F}_\varepsilon = \varepsilon^{1/2-d(1-\alpha/2)}\widetilde{F}_\varepsilon$ and let $\widehat{F}_{q,\varepsilon}$ be the projection of $\widehat{F}_\varepsilon$ on the qth Wiener chaos. Note that by assumption, the exponent $1/2 - d(1 - \alpha/2)$ is positive. As in the proof of Theorem 1.2, we are again first proving convergence of finite-dimensional distributions and then tightness.

5.1. *Convergence of finite-dimensional distributions.* We will show for $s, t \geq 0$ that

$$\lim_{\varepsilon,\delta\to 0} \mathbb{E}[\widehat{F}_{q,\varepsilon}(s)\widehat{F}_{q,\delta}(t))] = \begin{cases} c_d^2 K_d(s,t) & \text{if } q = d, \\ 0 & \text{if } q > d, \end{cases} \tag{5.1}$$

where $K_d(s,t)$ has been defined in (1.25), and also that

$$\lim_{N\to\infty} \limsup_{\varepsilon\to 0} \sum_{q=N}^{\infty} \mathbb{E}[\widehat{F}_{q,\varepsilon}(t)^2] = 0. \tag{5.2}$$

Then (5.1) for $q = d$ implies that for every sequence $\varepsilon_n \to 0$, and for each $t \geq 0$, the sequence of random variables $\widehat{F}_{q,\varepsilon_n}(t)$ is a Cauchy sequence in $L^2(\Omega)$. Therefore, $\widehat{F}_{d,\varepsilon}(t)$ converges in $L^2(\Omega)$ as ε tends to zero to $c_d H_\infty(t)$, where $H_\infty(t)$ is the generalized Hermite process with covariance given by (1.25). Also, for $q > d$, $\widehat{F}_{q,\varepsilon}$ converges to zero in $L^2(\Omega)$, as ε tends to zero. Together with (5.2), this implies that for any $t \geq 0$, $\widehat{F}_\varepsilon(t)$ converges in $L^2(\Omega)$, as ε tends to zero, to $c_d H_\infty(t)$. As a consequence, the finite-dimensional distributions of the process $\widehat{F}_\varepsilon$ converge in law to those of the process $c_d H_\infty$. This is also true for the finite-dimensional distributions of the process $\varepsilon^{1/2-d(1-\alpha/2)} F_\varepsilon$, because this process has the same law as $\widehat{F}_\varepsilon$.

We now proceed to the proof of (5.1) and (5.2). Taking into account that (see (4.12))

$$\widehat{F}_{q,\varepsilon}(t) = c_q \varepsilon^{-d(1-\frac{\alpha}{2})} \int_0^t \|\Delta_\varepsilon X(u)\|_{L^2(\Omega)}^{-q} I_q(\mathbf{1}_{[u,u+\varepsilon]}^{\otimes q})\, du,$$

we can write

$$\mathbb{E}[\widehat{F}_{q,\varepsilon}(s)\widehat{F}_{q,\delta}(t)] = c_q^2 q!(\varepsilon\delta)^{-d(1-\frac{\alpha}{2})} \int_0^s \int_0^t \Phi_{\varepsilon,\delta}^q(u,v)\, du\, dv,$$

where

$$\Phi_{\varepsilon,\delta}(u,v) = \frac{\mathbb{E}[(X(u+\varepsilon) - X(u))(X(v+\delta) - X)(v)]}{\sqrt{\mathbb{E}[(X(u+\varepsilon) - X(u))^2]\mathbb{E}[(X(v+\delta) - X(v))^2]}}. \tag{5.3}$$

Assuming $t \geq \varepsilon$, consider the decomposition

$$\begin{aligned} \mathbb{E}[\widehat{F}_{q,\varepsilon}(t)^2] &= c_q^2 q! \varepsilon^{d(\alpha-2)} \int_0^t \int_0^t \Phi_{\varepsilon,\varepsilon}^q(u,v)\, du\, dv \\ &= c_q^2 q! \varepsilon^{d(\alpha-2)} \int_{[0,t]^2 \cap \{u \wedge v \leq \varepsilon\}} \Phi_{\varepsilon,\varepsilon}^q(u,v)\, du\, dv \\ &\quad + c_q^2 q! \varepsilon^{d(\alpha-2)} \int_\varepsilon^t \int_\varepsilon^t \Phi_{\varepsilon,\varepsilon}^q(u,v)\, du\, dv. \end{aligned}$$

Then, using the bound (6.23) and the fact that $|\Phi_{\varepsilon,\varepsilon}(u,v)| \leq 1$, we can write for any $q \geq q^* \geq d$ such that $\alpha > 2 - \frac{1}{q^*}$,

$$\begin{aligned} \mathbb{E}[\widehat{F}_{q,\varepsilon}(t)^2] &\leq c_q^2 q! t \varepsilon^{1+d(\alpha-2)} + C c_q^2 q! \varepsilon^{(q^*-d)(2-\alpha)} \\ &\quad \times \int_\varepsilon^t \int_\varepsilon^t ((u \vee v)^{\alpha-2}(uv)^{-\beta+\frac{\alpha}{2}})^{q^*}\, du\, dv. \end{aligned} \tag{5.4}$$

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Notice that the integral appearing in the right-hand side of the above equation is finite because $(-\beta + \frac{\alpha}{2})q^* \geq (\frac{\alpha}{2} - 1)q^* > -\frac{1}{2}$ and $(\alpha - 2)q^* > -1$. As a consequence, if $q > d$ we can choose $q^* > d$ with $\alpha > 2 - \frac{1}{q^*}$ and we obtain

$$\lim_{\varepsilon\to 0} \mathbb{E}[\widehat{F}_{q,\varepsilon}(t)^2] = 0.$$

Furthermore, the estimate (5.4) also implies

$$\sup_{\varepsilon\in(0,1)}\sup_{q\geq d}\mathbb{E}\big[\widehat{F}_{q,\varepsilon}(t)^2\big]\leq Cc_q^2 q!, \tag{5.5}$$

which allows us to establish (5.2) because the series $\sum_{q=d}^{\infty} c_q^2 q!$ is convergent.

It remains to show (5.1) for $q=d$. For any $\varepsilon>0$, we define

$$\widehat{F}^{(1)}_{d,\varepsilon}(t)=c_d\varepsilon^{-d(1-\frac{\alpha}{2})}\int_0^{\varepsilon}\|\Delta_\varepsilon X(u)\|_{L^2(\Omega)}^{-d} I_d\big(\mathbf{1}_{[u,u+\varepsilon]}^{\otimes q}\big)\,du$$

and $\widehat{F}^{(2)}_{d,\varepsilon}(t)=\widehat{F}_{d,\varepsilon}(t)-\widehat{F}^{(1)}_{d,\varepsilon}(t)$. It is easy to show that

$$\lim_{\varepsilon\to 0}\mathbb{E}\big[\widehat{F}^{(1)}_{d,\varepsilon}(t)^2\big]=0.$$

Therefore,

$$\lim_{\varepsilon,\delta\to 0}\mathbb{E}\big[\widehat{F}_{d,\varepsilon}(s)\widehat{F}_{d,\delta}(t)\big]=\lim_{\varepsilon,\delta\to 0}\mathbb{E}\big[\widehat{F}^{(2)}_{d,\varepsilon}(s)\widehat{F}^{(2)}_{d,\delta}(t)\big].$$

We can write

$$\mathbb{E}\big[\widehat{F}^{(2)}_{d,\varepsilon}(s)\widehat{F}^{(2)}_{d,\delta}(t)\big]=c_d^2 d!\int_\varepsilon^s\int_\delta^t\big((\varepsilon\delta)^{\alpha/2-1}\Phi_{\varepsilon,\delta}(u,v)\big)^d\,du\,dv. \tag{5.6}$$

By Lemma 6.5, we have that

$$\begin{aligned}
&\lim_{\varepsilon,\delta\to 0}\int_\varepsilon^s\int_\delta^t\big((\varepsilon\delta)^{\alpha/2-1}\Phi_{\varepsilon,\delta}(u,v)\big)^d\,du\,dv\\
&=\int_0^s\int_0^t\Big(\lim_{\varepsilon,\delta\to 0}(\varepsilon\delta)^{\alpha/2-1}\Phi_{\varepsilon,\delta}(u,v)\Big)^d\,du\,dv\\
&=\int_0^s\int_0^t\bigg(\frac{\partial_{u,v}\mathbb{E}[X(u)X(v)]}{2\lambda(uv)^{\beta-\alpha/2}}\bigg)^d\,du\,dv,
\end{aligned}$$

where the interchange of integration and limit is justified by the bound (6.23), which yields an integrable bound since we have $d(\alpha-2)>-1$. This completes the proof of (5.1) and (5.2).

5.2. *Tightness*. To show tightness, it suffices to estimate the moment of order two of an increment. We can write, for $s\leq t$,

$$\begin{aligned}
\mathbb{E}\big[\big|\widehat{F}_\varepsilon(s)-\widehat{F}_\varepsilon(t)\big|^2\big]&=\varepsilon^{d(\alpha-2)}\sum_{q=d}^{\infty}c_q^2 q!\int_s^t\int_s^t\Phi_{\varepsilon,\varepsilon}^q(u,v)\,du\,dv\\
&\leq\Big(\sum_{q=d}^{\infty}c_q^2 q!\Big)\varepsilon^{d(\alpha-2)}\int_s^t\int_s^t\big|\Phi_{\varepsilon,\varepsilon}^d(u,v)\big|\,du\,dv.
\end{aligned}$$

If we integrate on the set where at least one of the variables is less than ε, using Hölder's inequality we obtain the bound

$$C\varepsilon^{d(\alpha-2)}(t-s)\int_{\mathbb{R}}\mathbf{1}_{[s,t]}(u)\mathbf{1}_{[0,\varepsilon]}(u)\,du\leq C\varepsilon^{d(\alpha-2)+\frac{1}{p_1}}(t-s)^{1+\frac{1}{p_2}},$$

where $1/p_1+1/p_2=1$. Choosing $p_1=1/(d(2-\alpha))>1$, we obtain a bound of the form $C(t-s)^{2-d(2-\alpha)}$. Notice that $2-d(2-\alpha)>1$.

On the other hand, if both variables are larger than ε, we can use the estimate (6.23) and we obtain the bound

$$C\int_s^t\int_s^t\big((u\vee v)^{\alpha-2}(uv)^{-\beta+\frac{\alpha}{2}}\big)^d\,du\,dv.$$

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Making the change of variables $(u, v) \to (s + x(t-s), s + y(t-s))$, the above integral can be bounded by

$$C(t-s)^{2+d(2\alpha-\beta-2)} \int_0^1 \int_0^1 \big((x \vee y)^{\alpha-2}(xy)^{-\beta+\frac{\alpha}{2}}\big)^d \, dx \, dy$$

and again we obtain the desired estimate because

$$2 + d(2\alpha - \beta - 2) \geq 2 + d(2\alpha - 3) > d \geq 1.$$

6. Technical lemmas.

6.1. *Lemmas for the case $\alpha \leq 2 - 1/q$.* Define

$$\varphi_{\alpha,q}(\varepsilon) = \begin{cases} \varepsilon & \text{if } \alpha < 2 - 1/q, \\ \varepsilon/|\log \varepsilon| & \text{if } \alpha = 2 - 1/q. \end{cases} \tag{6.1}$$

LEMMA 6.1. *Assume* (H.1) *and* (H.2). *If $\alpha \leq 2 - 1/q$, then for any $T > 0$,*

$$\lim_{\varepsilon \to 0} \varphi_{\alpha,q}(\varepsilon) \int_0^{T/\varepsilon} \int_0^{T/\varepsilon} \mathbf{1}_{D_2}(s,t) \big| \Phi^q(s,t) \big| \, ds \, dt = 0 \tag{6.2}$$

and

$$\lim_{\varepsilon \to 0} \varphi_{\alpha,q}(\varepsilon) \int_0^{T/\varepsilon} \int_0^{T/\varepsilon} \mathbf{1}_{D_2}(s,t) \big| \alpha_\alpha^q(s-t) \big| \, ds \, dt = 0, \tag{6.3}$$

where Φ and a_α are defined by (4.11) *and* (1.10), *respectively, the set D_2 contains all tuples $(s,t) \in \mathbb{R}_+^2$ such that $|s-t| \geq (c-1)(s \wedge t) + c$, and c is the constant appearing in* (H.2).

PROOF. The second statement (6.3) is a special case of (6.2) as $\Phi(s,t) = a_\alpha(s-t)$ when X is a fractional Brownian motion. Indeed, as in this case $\mathbb{E}((\Delta_1 X(t))^2) = 1$ (recall that $\Delta_1 X(t) = X(t+1) - X(t)$), we have $\Delta_1 Y = \Delta_1 X$ and, therefore,

$$\Phi(s,t) = \mathbb{E}\big(\Delta_1 Y(s) \Delta_1 Y(t)\big) = \mathbb{E}\big(\Delta_1 X(s) \Delta_1 X(t)\big) = a_\alpha(s-t).$$

Consequently, it suffices to show (6.2).

Note that for all $\varepsilon \in (0,1)$,

$$\int_0^{T/\varepsilon} \int_0^{T/\varepsilon} \mathbf{1}_{A_\delta}(s,t) \, ds \, dt \leq C_T \delta \varepsilon^{-1},$$

where A_δ contains all tuples (s,t) such that at least one of the variables s and t is bounded by δ. Together with the fact that $|\Phi(s,t)| \leq 1$, we get that

$$\varphi_{\alpha,q}(\varepsilon) \int_0^{T/\varepsilon} \int_0^{T/\varepsilon} \mathbf{1}_{A_\delta}(s,t) \big| \Phi^q(s,t) \big| \, ds \, dt \leq C_T \delta$$

for all $\delta > 0$. It therefore suffices to show that

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$$\lim_{\varepsilon \to 0} \varphi_{\alpha,q}(\varepsilon) \int_0^{T/\varepsilon} \int_0^{T/\varepsilon} \mathbf{1}_{D_2 \cap A_\delta^c}(s,t) \big| \Phi^q(s,t) \big| \, ds \, dt = 0.$$

By symmetry, it suffices to consider the integral over the set $\{t \leq s\}$, meaning that we are eventually left to show that

$$\lim_{\varepsilon \to 0} \varphi_{\alpha,q}(\varepsilon) \int_0^{T/\varepsilon} \int_0^{T/\varepsilon} \mathbf{1}_{D_{2,\delta}}(s,t) \big| \Phi^q(s,t) \big| \, ds \, dt = 0,$$

where $D_{2,\delta} = D_2 \cap A_\delta^c \cap \{t \le s\}$ contains all tuples (s,t) which are elements of D_2 and such that $s \ge t \ge \delta$. Then we can apply Lemma 4.3 and obtain that

$$\begin{aligned}&\big|\mathbb{E}\big[\big(X(s+1)-X(s)\big)\big(X(t+1)-X(t)\big)\big]\big|\\ &\quad\le C_\delta \begin{cases} t^{2\beta-2+\nu}(s-t)^{-\nu} & \text{if } \alpha<1,\\ t^{2\beta-\alpha}(s-t)^{\alpha-2} & \text{if } \alpha\ge 1,\end{cases}\end{aligned}$$

where C_δ is a positive constant depending on δ.

Moreover, we claim that

$$\inf_{\delta\le s<\infty}\big(1+u_1(s)\big) = b_\delta > 0. \tag{6.4}$$

In fact, for any $s \ge 0$ we have $\mathbb{E}[(X(s+1)-X(s))^2] > 0$ (this is a consequence of the self-similarity property) and the map $s \to \mathbb{E}[(X(s+1)-X(s))^2]$ is continuous. Then (6.4) follows from the fact that that $1+u_1(s)$ is a strictly positive continuous function on $[\delta,\infty)$, which is bounded by $1+K_\delta s^{-\delta_1}$, with $\delta_1>0$, for s large enough, and for some constant K_δ, by Lemma 4.1. Notice that $u_1(s)$ may blow up at zero if $\alpha<2\beta$. Therefore, we obtain

$$\big|\Phi(s,t)\big| \le C_\delta b_\delta^{-1}\begin{cases} t^{\alpha-2+\nu}(s-t)^{-\nu} & \text{if } \alpha<1,\\ (s-t)^{\alpha-2} & \text{if } \alpha\ge 1.\end{cases}$$

If $\alpha < 2-1/q$, we then get that

$$\int_0^{T/\varepsilon}\int_0^{T/\varepsilon} \mathbf{1}_{D_{2,\delta}}(s,t)\big|\Phi^q(s,t)\big|\,ds\,dt \le C_\delta b_\delta^{-1}\varepsilon^{-(\alpha-2)q-2}$$

and the assertion follows as

$$\lim_{\varepsilon\to 0}\varepsilon^{-(\alpha-2)q-2}\varphi_{\alpha,q}(\varepsilon) = 0.$$

In the case $\alpha = 2-1/q \ge 1$, we obtain

$$\int_0^{T/\varepsilon}\int_0^{T/\varepsilon} \mathbf{1}_{D_{2,\delta}}(s,t)\big|\Phi^q(s,t)\big|\,ds\,dt \le C_\delta b_\delta^{-1}\varepsilon^{-1},$$

and again

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}\varphi_{\alpha,q}(\varepsilon) = 0. \qquad \square$$

LEMMA 6.2. *Let $\alpha \le 2-1/q$, assume* (H.1) *and* (H.2) *and, for $r=1,2,\dots,q$, define*

$$R_{\alpha,r}(s,t) = \Phi^r(s,t) - a_\alpha^r(s-t), \tag{6.5}$$

where Φ and a_α are defined by (4.11) *and* (1.10), *respectively. Then, for all $M_1, M_2>0$, it holds that*

$$\lim_{\varepsilon\to 0}\varphi_{\alpha,q}(\varepsilon)\int_0^{T/\varepsilon}\int_0^{T/\varepsilon}\mathbf{1}_{D_{M_1,M_2}}(s,t)\big|R_{\alpha,r}(s,t)\big|^{q/r}\,ds\,dt = 0, \tag{6.6}$$

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where the set D_{M_1,M_2} is given by

$$D_{M_1,M_2} = \big\{(s,t)\in\mathbb{R}^2 : |s-t| \le M_1(s\wedge t)+M_2\big\}. \tag{6.7}$$

PROOF. From (6.5), we see that

$$\sup_{(s,t)\in D_{M_1,M_2}}\big|R_{\alpha,r}(s,t)\big| \le C_{\alpha,q}. \tag{6.8}$$

Indeed, by the Cauchy–Schwarz inequality $|\Phi(s,t)| \le 1$ and, by (4.6), $|a_\alpha(s-t)| \le C_\alpha$. Also note that for $\delta > 0$,

$$\int_0^{T/\varepsilon} \int_0^{T/\varepsilon} \mathbf{1}_{D_\delta}(s,t)\, ds\, dt \le C_T n \delta, \tag{6.9}$$

where D_δ consists of all tuples $(s,t) \in \mathbb{R}_+^2$ such that at least one of the quantities $s, t, |s-t|$ is less than δ, and C_T is some positive constant. The bounds (6.8) and (6.9) now yield that

$$\varphi_{\alpha,q}(\varepsilon) \int_0^{T/\varepsilon} \int_0^{T/\varepsilon} \mathbf{1}_{D_\delta \cap D_{M_1,M_2}}(s,t) \big| R_{\alpha,r}(s,t) \big|^{q/r}\, ds\, dt \le C\delta$$

for all $\delta > 0$. Also, by symmetry it suffices to consider the integral over the set $\{(s,t) \in D_{M_1,M_2} : s < t\}$. It therefore suffices to prove that

$$\lim_{\varepsilon \to 0} \varphi_{\alpha,q}(\varepsilon) \int_0^{T/\varepsilon} \int_0^{T/\varepsilon} \mathbf{1}_{D_{M_1,M_2,\delta}}(s,t) \big| R_{\alpha,r}(s,t) \big|^{q/r}\, ds\, dt = 0, \tag{6.10}$$

where

$$D_{M_1,M_2,\delta} = \big\{(s,t) : \delta \le s < t, \delta \le t - s \le M_1 s + M_2\big\}.$$

For $(s,t) \in D_{M_1,M_2,\delta}$, Lemmas 4.1 and 4.2 apply and yield that

$$\begin{aligned} \Phi^r(s,t) &= \left(\frac{s}{t}\right)^{(\beta-\alpha/2)r} \frac{2^{-r}}{(1+u_1(s))^{r/2}(1+u_1(t))^{r/2}} \big(2a_\alpha(t-s) + u_2(s,t)\big)^r \\ &= \left(\frac{s}{t}\right)^{(\beta-\alpha/2)r} \big(1+u_1(s)\big)^{-r/2} \big(1+u_1(t)\big)^{-r/2} \\ &\quad \times \left(a_\alpha(t-s)^r + \sum_{r'=1}^{r} \binom{r}{r'} a_\alpha(t-s)^{r-r'} 2^{-r'} u_2(s,t)^{r'} \right). \end{aligned}$$

Therefore, we have that

$$R_{\alpha,r}(s,t) = \Phi^r(s,t) - a_\alpha^r(s-t) = \sum_{l=1}^{3} R_{\alpha,r,l}(s,t),$$

where

$$\begin{aligned} R_{\alpha,r,1}(s,t) &= \left(\frac{s}{t}\right)^{(\beta-\alpha/2)r} \big(\big(1+u_1(s)\big)^{-r/2} \big(1+u_2(t)\big)^{-r/2} - 1 \big) \\ &\quad \times \big(a_\alpha(t-s) + 2^{-1} u_2(s,t)\big)^r, \\ R_{\alpha,r,2}(s,t) &= \left(\left(\frac{s}{t}\right)^{(\beta-\alpha/2)r} - 1 \right) a_\alpha(t-s)^r, \\ R_{\alpha,r,3}(s,t) &= \left(\frac{s}{t}\right)^{(\beta-\alpha/2)r} \sum_{r'=1}^{r} \binom{r}{r'} a_\alpha(t-s)^{r-r'} 2^{-r'} u_2(s,t)^{r'}. \end{aligned}$$

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We will now show that

$$\lim_{\varepsilon \to 0} \varphi_{\alpha,q}(\varepsilon) \int_0^{T/\varepsilon} \int_0^{t} \mathbf{1}_{D_{M_1,M_2,\delta}}(s,t) \big| R_{\alpha,r,l}(s,t) \big|^{q/r}\, ds\, dt = 0 \tag{6.11}$$

for $l = 1, 2, 3$. This then implies (6.10) as

$$\big| R_{\alpha,r}(s,t) \big|^{q/r} \le 3 \max_{l=1,2,3} \big| R_{\alpha,r,l}(s,t) \big|^{q/r} \le 3 \sum_{l=1}^{3} \big| R_{\alpha,r,l}(s,t) \big|^{q/r}.$$

If not otherwise specified, all formulas proved throughout the rest of this section are only claimed to be valid for $(s,t) \in D_{M_1,M_2,\delta}$. Furthermore, C in the following denotes a generic positive constant which may change from line to line and might depend on δ. Dependence on variables is indicated as parameters. Notice that on the set $D_{M_1,M_2,\delta}$ one has

$$\frac{2M_1}{t-s} \geq \frac{1}{s} \quad \text{if } s \geq \frac{M_2}{M_1}. \tag{6.12}$$

Let us begin by treating $R_{\alpha,r,1}$. By Lemma 4.1, we know that for some positive constant C_δ only depending on δ, it holds that

$$\left|u_1(s)\right| < C_\varepsilon s^{-\delta_1} \quad \text{and} \quad \left|u_1(t)\right| < C_\varepsilon t^{-\delta_1} \tag{6.13}$$

with $\delta_1 = 1-\alpha$ if $\alpha < 1$, $\delta_1 = 2-\alpha$ if $\alpha \geq 1$. The bound (6.13) and the mean value theorem imply that

$$\left|(1+u_1(s))^{-r/2}(1+u_1(t))^{-r/2} - 1\right| \leq C\left(\left|u_1(s)\right| + \left|u_1(t)\right|\right) \leq C s^{-\delta_1}. \tag{6.14}$$

Furthermore, taking into account the bounds (4.7), (4.2) and (4.3), we can write

$$\begin{aligned}
&\left|a_\alpha(t-s) + 2^{-1}u_2(s,t)\right|^r \\
&\quad \leq C\big((t-s)^{(\alpha-2)r} + \big(s^{(\alpha-2)r} + s^{-r}(t-s)^{(\alpha-1)r}\big)\mathbf{1}_{\{t-s\geq 1\}} \\
&\qquad + \big(s^{(\alpha-1)r}\mathbf{1}_{\{\alpha<1\}} + s^{(\alpha-2)r}\mathbf{1}_{\{\alpha\geq 1\}}\big)\mathbf{1}_{\{t-s<1\}}\big).
\end{aligned} \tag{6.15}$$

The bounds (6.14) and (6.15), together with (6.12), thus yield

$$\begin{aligned}
\left|R_{\alpha,r,1}(s,t)\right|^{q/r} \leq C\big((t-s)^{(\alpha-2)q-\frac{q}{r}\delta_1} & \\
&+ \big(s^{(\alpha-2)q-\frac{q}{r}\delta_1} + s^{-q-\frac{q}{r}\delta_1}(t-s)^{(\alpha-1)q-\frac{q}{r}\delta_1}\mathbf{1}_{\{t-s\geq 1\}} \\
&+ \big(s^{(\alpha-1)q-\frac{q}{r}\delta_1}\mathbf{1}_{\{\alpha<1\}} + s^{(\alpha-2)q-\frac{q}{r}\delta_1}\mathbf{1}_{\{\alpha\geq 1\}}\big)\mathbf{1}_{\{t-s<1\}}\big)
\end{aligned}$$

and therefore, after a straightforward calculation,

$$\int_0^{T/\varepsilon}\int_0^t \mathbf{1}_{D_{M_1,M_2,\delta}}(s,t)\left|R_{\alpha,r,1}(s,t)\right|^{q/r}\,ds\,dt \leq C\varepsilon^{-(\alpha-2)q+\frac{q}{r}\delta_1-2}.$$

Notice that $(\alpha-2)q - \frac{q}{r}\delta_1 + 2 \leq 1 - \frac{q}{r}\delta_1$, with equality only if $\alpha = 2-1/q$. Therefore, taking into account that $0 < \delta_1 \leq 1$, we obtain for $\varepsilon \leq 1$,

$$\varphi_{\alpha,q}(\varepsilon)\varepsilon^{-(\alpha-2)q+\frac{q}{r}\delta_1-2}\begin{cases} < \varepsilon^{\frac{q}{r}\delta_1} & \text{if } \alpha < 2-1/q, \\ = \dfrac{1}{|\log\varepsilon|}\varepsilon^{\frac{q}{r}\delta_1} & \text{if } \alpha = 2-1/q. \end{cases}$$

This shows (6.11) for $l=1$.

Let us turn to $R_{\alpha,r,2}$. We can write, using (4.7),

$$\begin{aligned}
&\int_0^{T/\varepsilon}\int_0^t \mathbf{1}_{D_{M_1,M_2,\delta}}(s,t)\left|R_{\alpha,r,2}(s,t)\right|^{q/r}\,ds\,dt \\
&\quad \leq C\int_0^{T/\varepsilon}\int_0^t \mathbf{1}_{D_{M_1,M_2,\delta}}(s,t)\left(1-\left(\frac{s}{t}\right)^{(\beta-\alpha/2)r}\right)^{q/r}(t-s)^{(\alpha-2)q}\,ds\,dt.
\end{aligned}$$

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Making the change of variables $s = x/\varepsilon$ and $t = y/\varepsilon$, the integral in the right-hand side of the above inequality can be written as

$$\begin{aligned}
&\varepsilon^{-(\alpha-2)q-2}\int_0^T\int_0^y \mathbf{1}_{\{\frac{\delta}{\varepsilon}\leq x<y,\,\frac{\delta}{\varepsilon}\leq y-x\leq M_1 x+\frac{M_2}{\delta}\}}\left(1-\left(\frac{x}{y}\right)^{(\beta-\alpha/2)r}\right)^{q/r} \\
&\quad \times (y-x)^{(\alpha-2)q}\,dx\,dy.
\end{aligned}$$

We claim that

$$\int_0^T \int_0^y \left(1-\left(\frac{x}{y}\right)^{(\beta-\alpha/2)r}\right)^{q/r} (y-x)^{(\alpha-2)q}\,dx\,dy<\infty.$$

Indeed, with the change of variables $x = zy$, we obtain

$$\int_0^T \int_0^y \left(1-\left(\frac{x}{y}\right)^{(\beta-\alpha/2)r}\right)^{q/r} (y-x)^{(\alpha-2)q}\,dx\,dy$$
$$=\int_0^T \int_0^1 (1-z^{(\beta-\alpha/2)r})^{q/r}(1-z)^{(\alpha-2)q}y^{(\alpha-2)q+1}dz\,dy<\infty.$$

Finally, this shows (6.11) for $l = 2$ as $\varepsilon^{-(\alpha-1)q-2}\varphi_{\alpha,q}(\varepsilon)$ converges to zero as $\varepsilon \to 0$.

It remains to study $R_{\alpha,q,3}$. In this case, using (4.7) and the bounds (4.2) and (4.3) for u_2, we get that

$$\begin{aligned}&\left|a_\alpha(s-t)^{r-r'}u_2(s,t)^{r'}\right|^{q/r}\\ &\quad\le C(t-s)^{(\alpha-2)q(r-r')/r}\\ &\qquad\times\left(\left(s^{(\alpha-2)qr'/r}+s^{-qr'/r}(t-s)^{(\alpha-1)qr'/r}\right)\mathbf{1}_{\{t-s\ge1\}}\right.\\ &\qquad\left.+\left(s^{(\alpha-1)qr'/r}\mathbf{1}_{\{\alpha<1\}}+s^{(\alpha-2)qr'/r}\mathbf{1}_{\{\alpha\ge1\}}\right)\mathbf{1}_{\{t-s<1\}}\right).\end{aligned}$$

Therefore, $|R_{\alpha,r,3}(s,t)|^{q/r}$ is also bounded by the above quantity, after a tedious but straightforward calculation, leads to

$$\begin{aligned}&\int_0^{T/\varepsilon}\int_0^t \mathbf{1}_{D_{M_1,M_2,\varepsilon}}(s,t)\left|R_{\alpha,r,3}(s,t)\right|^{q/r}ds\,dt\\ &\qquad\le C\left(|\log\varepsilon|\varepsilon^{(2-\alpha)qr'/r-1}\vee\varepsilon^{(2-\alpha)q-2}\right).\end{aligned}$$

Noting that $(\alpha-2)qr'/r+1\le 1-r'/r<1$, we obtain (6.11) for $l = 3$, completing the proof. $\square$

LEMMA 6.3. *In the setting introduced above, let $s,t\ge 0$ and let q be a positive integer. Recall that $F_\varepsilon(t)$ has been introduced in* (4.12) *and that we denote by $F_{q,\varepsilon}(t)$ the projection of $F_\varepsilon(t)$ on the qth Wiener chaos for any $q\ge d$. Assume* (H.1) *and* (H.2). *Then, for $\alpha<2-\frac{1}{q}$, it holds that*

$$\lim_{\varepsilon\to0}\mathbb{E}\left[F_{q,\varepsilon}(s)F_{q,\varepsilon}(t)\right]=\sigma^2_{\alpha,q}(s\wedge t), \tag{6.16}$$

and, for $\alpha=2-\frac{1}{q}$, it holds that

$$\lim_{\varepsilon\to0}\frac{1}{|\log\varepsilon|}\mathbb{E}\left[F_{q,\varepsilon}(s)F_{q,\varepsilon}(t)\right]=\sigma^2_{2-1/q}(s\wedge t), \tag{6.17}$$

where $\sigma^2_{\alpha,q}$ is given by

$$\sigma^2_{\alpha,q}=c_q^2q!\int_{\mathbb{R}}a_\alpha^q(h)\,dh \tag{6.18}$$

if $\alpha<2-1/q$, and by (1.24), *if $\alpha=2-1/q$.*

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PROOF. Recall the definition (6.1) of the helper function $\varphi_{\alpha,q}$. From (4.13), we know that

$$\mathbb{E}\left[F_{q,\varepsilon}(s)F_{q,\varepsilon}(t)\right]=c_q^2q!\varepsilon\int_0^{s/\varepsilon}\int_0^{t/\varepsilon}\Phi^q(u,v)\,du\,dv.$$

By Lemmas 6.1 and 6.2, we get that

$$\lim_{\varepsilon\to 0} \varphi_{\alpha,q}(\varepsilon) \int_0^{s/\varepsilon} \int_0^{t/\varepsilon} \Phi^q(u,v)\,du\,dv = \lim_{\varepsilon\to 0} \varphi_{\alpha,q}(\varepsilon) \int_0^{s/\varepsilon} \int_0^{t/\varepsilon} a_\alpha(u-v)^q\,du\,dv,$$

where a_α is defined in (1.10). Then the convergences follow from the proof of the classical Breuer–Major theorem; see, for example, [11], Theorem 7.2, for the details (the proof given in [11] can be extended *mutatis mutandis* to cover continuous framework as well). □

6.2. *Lemmas for the case* $\alpha > 2 - 1/d$.

LEMMA 6.4. *Assume* $\alpha > 2 - 1/d$ *and* (H.2). *Then one has for* $s, t > 0$ *that*

$$\big|\partial_{s,t}\mathbb{E}\big[X(s)X(t)\big]\big| \le C(s\wedge t)^{2\beta-\alpha}(s\vee t)^{\alpha-2}, \tag{6.19}$$

where C *is a positive constant only depending on* α *and* β. *In particular, for* $u, v > 0$ *it holds that*

$$\lim_{\varepsilon,\delta\to 0} \frac{1}{\varepsilon\delta}\mathbb{E}\big[\big(X(s+\varepsilon)-X(s)\big)\big(X(t+\delta)-X(t)\big)\big] = \partial_{s,t}\mathbb{E}\big[X(s)X(t)\big]. \tag{6.20}$$

PROOF. For $0 < s \le t$, we have by self-similarity that

$$\mathbb{E}\big[X(s)X(t)\big] = s^{2\beta}\phi\left(\frac{t}{s}\right)$$

with $\phi(x) = \mathbb{E}[X(1)X(x)]$. A routine calculation yields that

$$\partial_{s,t}\mathbb{E}\big[X(s)X(t)\big] = (2\beta-1)s^{2\beta-2}\phi'\left(\frac{t}{s}\right) - s^{2\beta-3}t\phi''\left(\frac{t}{s}\right).$$

Using (H.2) if $t/s \ge c$ and the fact that $\phi'(t/s)$ is bounded and $|\phi''(t/s)| \le C(t/s-1)^{\alpha-2}$ if $t/s \le c$, we obtain

$$\big|\partial_{s,t}\mathbb{E}\big[X(s)X(t)\big]\big| \le Cs^{2\beta-\alpha}t^{\alpha-2}, \tag{6.21}$$

which proves the asserted bound. As by assumption $\alpha - 2 > -1/d \ge -1$ and by definition $2\beta - \alpha > 0$, the derivative is therefore integrable on any interval $[0,a]\times[0,b]$ and we get that

$$\mathbb{E}\big[\big(X(s+\varepsilon)-X(s)\big)\big(X(t+\delta)-X(t)\big)\big] = \int_t^{t+\delta}\int_s^{s+\varepsilon} \partial_{u,v}\mathbb{E}\big[X(u)X(v)\big]\,du\,dv,$$

so that (6.20) follows. □

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LEMMA 6.5. *Assume* $\alpha > 2 - 1/d$ *and* (H.2) *and recall that* $\Phi_{\varepsilon,\delta}(s,t)$ *has been introduced in* (5.3). *Then it holds that*

$$\lim_{\varepsilon,\delta\to 0} (\varepsilon\delta)^{\alpha/2-1}\Phi_{\varepsilon,\delta}(s,t) = \frac{\partial_{s,t}\mathbb{E}[X(s)X(t)]}{2\lambda(st)^{\beta-\alpha/2}}. \tag{6.22}$$

Furthermore, there exists a positive constant C_T *such that for any* $s, t \in [0,T]$ *such that* $s \ge \varepsilon$ *and* $t \ge \delta$, *it holds that*

$$\big|\Phi_{\varepsilon,\delta}(s,t)\big| \le C_T(\varepsilon\delta)^{1-\frac{\alpha}{2}}(s\vee t)^{\alpha-2}(st)^{-\beta+\frac{\alpha}{2}}. \tag{6.23}$$

PROOF. By self-similarity and Lemma 4.1, we have that

$$\begin{aligned}\mathbb{E}\big[\big(X(s+\varepsilon)-X(s)\big)^2\big] &= \varepsilon^{2\beta}\mathbb{E}\big[\big(X(s/\varepsilon+1)-X(s/\varepsilon)\big)^2\big]\\ &= 2\lambda\varepsilon^{\alpha}s^{2\beta-\alpha}\big(1+u_1(s/\varepsilon)\big),\end{aligned}$$

where $u_1(s/\varepsilon) \le C(s/\varepsilon)^{\alpha-2}$ converges to zero as $\varepsilon \to 0$. Therefore, also using Lemma 6.4, we have that

$$\begin{aligned}\lim_{\varepsilon,\delta\to 0}(\varepsilon\delta)^{\alpha/2-1}\Phi_{\varepsilon,\delta}(s,t) &= \lim_{\varepsilon,\delta\to 0}(\varepsilon\delta)^{-1}\frac{\mathbb{E}[(X(s+\varepsilon)-X(s))(X(t+\delta)-X(t))]}{2\lambda(st)^{\beta-\alpha/2}}\\ &= \frac{\partial_{s,t}\mathbb{E}[X(s)X(t)]}{2\lambda(st)^{\beta-\alpha/2}}.\end{aligned}$$

In order to establish the bound (6.23), using Lemma 4.1 and the condition $s/\varepsilon \ge 1$ and $t/\delta \ge 1$, we obtain

$$\big|\Phi_{\varepsilon,\delta}(s,t)\big| \le C(\varepsilon\delta)^{-\frac{\alpha}{2}}(st)^{-\beta+\frac{\alpha}{2}}\int_t^{t+\delta}\int_s^{s+\varepsilon}\partial_{u,v}\mathbb{E}\big[X(u)X(v)\big]\,du\,dv.$$

Finally, in view of the estimate (6.19), we can write

$$\big|\Phi_{\varepsilon,\delta}(s,t)\big| \le C(\varepsilon\delta)^{1-\frac{\alpha}{2}}(st)^{-\beta+\frac{\alpha}{2}}(s\vee t)^{\alpha-2}.$$

This completes the proof of the lemma. □

6.3. *Chaos expansion of the absolute value.* The next statement has been used in the end of the introductory section, when we applied Theorems 1.2 and 1.3 in the case where $X = \widetilde{B}$ is a bifractional Brownian motion, and when for f we choose the function $f(x) = |x| - \sqrt{\frac{2}{\pi}}$. The proof is well known and standard, we include it here for completeness.

PROPOSITION 6.1 (Chaos expansion of the absolute value). *It holds that*

$$|x| = \sqrt{\frac{2}{\pi}} + \sqrt{\frac{2}{\pi}}\sum_{q=1}^{\infty}\frac{(-1)^{q-1}}{2^q q!(2q-1)}H_{2q}(x), \qquad x \in \mathbb{R}.$$

PROOF. The absolute mean of a standard Gaussian is $\sqrt{\frac{2}{\pi}}$. By symmetry and the fact that

$$H_q(x) = (-1)^q\phi(x)^{-1}\phi^{(q)}(x),$$

with $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ the Gaussian density, we get $\int_{\mathbb{R}}|u|H_{2q+1}(u)\phi(u)\,du = 0$ and

$$\begin{aligned}\int_{\mathbb{R}}|u|H_{2q}(u)\phi(u)\,du &= 2\int_0^{\infty}uH_{2q}(u)\phi(u)\,du = 2\int_0^{\infty}u\phi^{(2q)}(u)\,du\\ &= -2\int_0^{\infty}\phi^{(2q-1)}(u)\,du = 2\phi^{(2q-2)}(0)\\ &= \sqrt{\frac{2}{\pi}}H_{2q-2}(0) = \frac{\sqrt{\frac{2}{\pi}}(-1)^{q-1}(2q-2)!}{2^{q-1}(q-1)!}.\end{aligned}$$

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The desired conclusion follows. □

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