J. reine angew. Math., Ahead of Print
DOI 10.1515/crelle-2019-0044

Journal für die reine und angewandte Mathematik
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Curved Rickard complexes and link homologies

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Abstract. Rickard complexes in the context of categorified quantum groups can be used to construct braid group actions. We define and study certain natural deformations of these complexes which we call curved Rickard complexes. One application is to obtain deformations of link homologies which generalize those of Batson–Seed [3] [A link-splitting spectral sequence in Khovanov homology, Duke Math. J. **164** (2015), no. 5, 801–841] and Gorsky–Hogancamp [Hilbert schemes and y-ification of Khovanov–Rozansky homology, preprint 2017] to arbitrary representations/partitions. Another is to relate the deformed homology defined algebro-geometrically in [S. Cautis and J. Kamnitzer, Knot homology via derived categories of coherent sheaves IV, colored links, Quantum Topol. **8** (2017), no. 2, 381–411] to categorified quantum groups (this was the original motivation for this paper).

1. Introduction

Knot homologies have had many significant applications in knot theory and low dimensional topology. Rasmussen [34] gave a combinatorial proof of Milnor's conjecture (originally proved by Kronheimer and Mrowka [23]) about the slice genus of torus knots by utilizing a spectral sequence coming from the work of Lee [27, 28]. In another direction, Kronheimer and Mrowka proved that Khovanov homology detects the unknot using a spectral from Khovanov homology to instanton Floer homology [24].

One of the main ideas from [13] is that the braiding necessary to construct a link homology can be defined, as a by-product of skew Howe duality, using the theory of Rickard complexes (or Chuang–Rouquier complexes [15]). These are complexes which live in the homotopy category of categorified quantum groups and give rise to braid group actions.

1.1. Curved Rickard complexes. A typical Rickard complex has the following form:

$$\mathcal{F}_i^{(\lambda_i)}\mathbb{1}_\lambda \xrightarrow{d^+} \mathcal{E}_i^{(1)}\mathcal{F}_i^{(\lambda_i+1)}\mathbb{1}_\lambda\langle 1\rangle \xrightarrow{d^+} \cdots \xrightarrow{d^+} \mathcal{E}_i^{(k)}\mathcal{F}_i^{(\lambda_i+k)}\mathbb{1}_\lambda\langle k\rangle \xrightarrow{d^+} \cdots$$

Sabin Cautis is supported by an NSERC Discovery grant. Aaron D. Lauda is partially supported by the NSF grants DMS-1255334 and DMS-1664240. Joshua Sussan is partially supported by the NSF grant DMS-1807161, PSC-CUNY Award 61028-00 49, and Simons Foundation Collaboration Grant 516673.

(see Section 2.4). Motivated by the construction in [17] we deform such complexes by adding maps back in the other direction

$$\mathcal{F}_i^{(\lambda_i)}\mathbb{1}_\lambda \underset{ud^-}{\overset{d^+}{\rightleftarrows}} \mathcal{E}_i^{(1)}\mathcal{F}_i^{(\lambda_i+1)}\mathbb{1}_\lambda\langle 1\rangle \underset{ud^-}{\overset{d^+}{\rightleftarrows}} \cdots \underset{ud^-}{\overset{d^+}{\rightleftarrows}} \mathcal{E}_i^{(k)}\mathcal{F}_i^{(\lambda_i+k)}\mathbb{1}_\lambda\langle k\rangle \underset{ud^-}{\overset{d^+}{\rightleftarrows}} \cdots, \tag{1.1}$$

where u is a formal parameter of homological and internal degree $[2]\langle -2\rangle$. The fact that these are "curved" complexes is a consequence of a detailed computation in the categorified quantum group (cf. Proposition 4.9). We then adapt some facts in [17] to show that these complexes braid. The deformed complexes from [17] correspond to two term Rickard complexes of the form $\mathbb{1}_\lambda \to \mathcal{E}_i\mathcal{F}_i\mathbb{1}_\lambda\langle 1\rangle$.

1.2. Deformed link homologies. Starting with [10] an algebro-geometric construction of $\mathfrak{sl}_m$-link homologies was obtained using certain convolution varieties in the affine Grassmannian of PGL_m. Subsequent work of [7, 8, 13] related this construction to categorified quantum groups.

A deformation of this construction was obtained in [12] using the geometry of the Beilinson–Drinfeld Grassmannian. This deformed $\mathfrak{sl}_m$-link homology generalized the deformed $\mathfrak{sl}_2$-link homology defined earlier by Batson and Seed [3]. One of the motivations of the current paper is to give an interpretation of the construction from [12] in terms of categorified quantum groups (cf. Section 5.4).

As an application of curved Rickard complexes we obtain the following:

- A deformation of (colored) $\mathfrak{sl}_m$-link homology (Theorem 5.4).
- A deformation of (colored) HOMFLYPT link homology (Theorem 6.3).
- A deformation of clasps in the contexts of $\mathfrak{sl}_m$ and HOMFLYPT link homologies (Corollaries 5.6 and 6.4).

Recall that clasps were used in [8] (resp. [9]) to obtain $\mathfrak{sl}_m$-link homologies corresponding to arbitrary representations (resp. HOMFLYPT homologies of links labeled by arbitrary partitions). Our HOMFLYPT deformation extends that of Gorsky and Hogancamp [17] to arbitrary partitions.

These deformations, apart from (likely) playing a fundamental role in the structure of knot homologies, also have potential applications. For instance, [17] shows how to use these deformations to partially resolve some conjectures relating HOMFLYPT homology to the geometry of the Hilbert scheme of points on $\mathbb{C}^2$ (cf. [18, 31–33]). These results depend upon a spectral sequence from HOMFLYPT homology converging to the homology of the unlinked components as well as some earlier results about the HOMFLYPT homology of torus links [16, 19, 30].

1.3. Further remarks and directions. There are two ways to view curved Rickard complexes. The first, which is the focus in this paper, is as deformations of Rickard complexes. The second is as an action of the braid group on degree two endomorphisms of the identity. We expect that this latter interpretation will have further applications in the theory of categorified quantum groups. One such application that comes to mind, but which we do not pursue in this paper, is proving a categorical analogue of the classical isomorphism between the Kac–Moody and loop presentations of quantum affine algebras.

The braid group action mentioned above actually extends to an action on higher degree endomorphisms of the identity. From the point of view of Rickard complexes this corresponds to studying what one might call “higher degree homotopies”. In this paper we only consider the simplest possible (and by many accounts the most natural) homotopy, namely the maps ud^- from (1.1). We hope that the study of these higher homotopies will also lead to various applications (particularly in the context of knot homologies).

1.4. Outline of paper. We review foundational material on categorified quantum groups and the definition of Rickard complexes in Section 2. In Section 3 we prove some key technical results about so-called bubbles in the categorified quantum group. Some aspects of the general theory of curved complexes are reviewed in Section 4 followed by a discussion of curved Rickard complexes. Sections 5 and 6 contain our main applications: deformations of $\mathfrak{sl}_m$-link homologies and HOMFLYPT homologies, respectively.

Acknowledgement. The authors are grateful to Matt Hogancamp for many illuminating discussions on y-ification. Sabin Cautis thanks Eugene Gorsky for taking the time to elaborate on his recent work and Joshua Sussan would like to thank Shotaro Makisumi for explaining his work related to [17].

2. The categorified quantum group $\mathcal{U}_Q$

2.1. Conventions. By a graded category we will mean a category equipped with an auto-equivalence $\langle 1\rangle$. We denote by $\langle l\rangle$ the auto-equivalence obtained by applying $\langle 1\rangle$ l times. If A, B are two objects then $\mathrm{Hom}^l(A,B)$ will be short-hand for $\mathrm{Hom}(A, B\langle l\rangle)$. A graded additive $\Bbbk$-linear 2-category is a category enriched over graded additive $\Bbbk$-linear categories, that is, a 2-category $\mathcal{K}$ such that the Hom categories $\mathrm{Hom}_{\mathcal{K}}(A,B)$ between objects A and B are graded additive $\Bbbk$-linear categories and the composition maps

$$\mathrm{Hom}_{\mathcal{K}}(A,B)\times \mathrm{Hom}_{\mathcal{K}}(B,C)\to \mathrm{Hom}_{\mathcal{K}}(A,C)$$

form a graded additive $\Bbbk$-linear functor.

Given a 1-morphism A in an additive 2-category $\mathcal{K}$, we let $\bigoplus_{[n]} A$ denote the direct sum $\bigoplus_{k=0}^{n-1} A\langle n-1-2k\rangle$.

Given an additive category $\mathcal{C}$, we let $\mathsf{Kom}(\mathcal{C})$ denote the homotopy category of complexes in $\mathcal{C}$. Write $\mathsf{Kom}^+(\mathcal{C})$, respectively $\mathsf{Kom}^-(\mathcal{C})$ for the corresponding subcategory of bounded below, respectively above, complexes. By convention, we work with *cochain* complexes, so an object (X,d) of $\mathsf{Kom}(\mathcal{C})$ is a collection of objects X^i in $\mathcal{C}$ together with maps

$$\cdots \xrightarrow{d_{i-2}} X^{i-1} \xrightarrow{d_{i-1}} X^i \xrightarrow{d_i} X^{i+1} \xrightarrow{d_{i+1}} \cdots$$

such that $d_{i+1}d_i = 0$ and only finitely many of the objects X^i are non-zero. A morphism $f\colon (X,d)\to (Y,d')$ in $\mathsf{Kom}(\mathcal{C})$ consists of a collection of morphisms $f_i\colon X^i\to Y^i$ in $\mathcal{C}$ such that

$$f_{i+1}d_i = d'_i f_i$$

modulo null-homotopic maps. Recall that morphisms $f,g\colon (X,d)\to (Y,d')$ in $\mathsf{Kom}(\mathcal{C})$ are called homotopic if there exist morphisms $h^i \colon X^i \to Y^{i-1}$ such that

$$f_i - g_i = h^{i+1} d_i + d'_{i-1} h^i$$

for all i. A morphism of complexes is said to be null-homotopic if it is homotopic to the zero map. We let $X[n]$ denote the complex obtained from X by shifted each object X^i down by n.

Given an additive 2-category $\mathcal{K}$, define $\mathsf{Kom}^+(\mathcal{K})$ to be the additive 2-category with the same objects as $\mathcal{K}$ and additive hom categories $\mathrm{Hom}_{\mathsf{Kom}^+(\mathcal{K})}(A,B) := \mathsf{Kom}^+(\mathrm{Hom}_{\mathcal{K}}(A,B))$. The horizontal composition in $\mathsf{Kom}(\mathcal{K})$ is given using the horizontal composition from $\mathcal{K}$ together with the tensor product of complexes. The 2-category $\mathsf{Kom}^-(\mathcal{K})$ can be defined analogously. When no confusion is likely to arise, we often write $\mathsf{Kom}(\mathcal{K})$ in place of $\mathsf{Kom}^{\pm}(\mathcal{K})$.

2.2. Categorified quantum group. For this article we restrict our attention to simply-laced Kac–Moody algebras. These algebras are associated to a symmetric Cartan data consisting of

- a free $\mathbb{Z}$-module X (the weight lattice),
- for $i \in I$ (I is an indexing set) there are elements $\alpha_i \in X$ (simple roots) and $\Lambda_i \in X$ (fundamental weights),
- for $i \in I$ an element $h_i \in X^\vee = \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})$ (simple coroots),
- a bilinear form $(\cdot,\cdot)$ on X.

Write $\langle\cdot,\cdot\rangle \colon X^\vee \times X \to \mathbb{Z}$ for the canonical pairing. This data should satisfy:

- $(\alpha_i, \alpha_i) = 2$ for any $i \in I$,
- $(\alpha_i, \alpha_j) \in \{0, -1\}$ for $i, j \in I$ with $i \neq j$,
- $\langle i, \lambda\rangle := \langle h_i, \lambda\rangle = (\alpha_i, \lambda)$ for $i \in I$ and $\lambda \in X$,
- $\langle h_j, \Lambda_i\rangle = \delta_{ij}$ for all $i, j \in I$.

Hence $(a_{ij})_{i,j\in I}$ is a symmetrizable generalized Cartan matrix, where

$$a_{ij} = \langle h_i, \alpha_j\rangle = (\alpha_i, \alpha_j).$$

We will sometimes denote the bilinear pairing (α_i, α_j) by $i \cdot j$ and abbreviate $\langle i, \lambda\rangle$ to λ_i. We denote by $X^+ \subset X$ the dominant weights which are of the form $\sum_i \lambda_i \Lambda_i$ where $\lambda_i \geq 0$.

We write $W = W_{\mathfrak{g}}$ for the Weyl group of type $\mathfrak{g}$ and $\mathrm{Br}_{\mathfrak{g}}$ for the corresponding braid group. The Weyl group W acts on the weight lattice X via

$$s_i(\lambda) = \lambda - \alpha_i^\vee(\lambda)\lambda = \lambda - \langle i, \lambda\rangle\lambda$$

for each simple transposition $s_i \in W$.

Definition 2.1. Associated to a symmetric Cartan datum, define a *choice of scalars* Q consisting of:

- $\{t_{ij} \mid \text{for all } i, j \in I\}$

such that

- $t_{ii} = 0$ for all $i \in I$ and $t_{ij} \in \Bbbk^\times$ for $i \neq j$,
- $t_{ij} = t_{ji}$ when $a_{ij} = 0$.

The choice of scalars Q controls the form of the KLR algebra R_Q that governs the upward oriented strands. The 2-category $\mathcal{U}_Q(\mathfrak{g})$ is controlled by the products $v_{ij} = t_{ij}^{-1} t_{ji}$ taken over all pairs $i, j \in I$. When the underlying graph of the simply-laced Kac–Moody algebra $\mathfrak{g}$ is a tree, in particular a Dynkin diagram, all choices of Q lead to isomorphic KLR-algebras and these isomorphisms extend to isomorphisms of categorified quantum groups

$$\mathcal{U}_Q(\mathfrak{g}) \to \mathcal{U}_{Q'}(\mathfrak{g})$$

for scalars Q and Q', see [26].

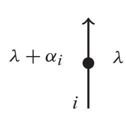
Let $\mathcal{U}_Q(\mathfrak{g})$ denote the non-cyclic form of the categorified quantum group from [14]. Though a cyclic form of the categorified quantum group has been defined [4], for our purposes the non-cyclic variant is the most natural version. By [4, Theorem 2.1] the cyclic and non-cyclic variant are isomorphic as 2-categories.

Definition 2.2. The 2-category $\mathcal{U}_Q := \mathcal{U}_Q(\mathfrak{g})$ is the graded linear 2-category consisting of:

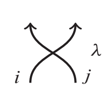
- **Objects** λ for $\lambda \in X$.
- **1-morphisms** are formal direct sums of (shifts of) compositions of

  $$\mathbb{1}_\lambda, \quad \mathbb{1}_{\lambda+\alpha_i}\mathcal{E}_i = \mathbb{1}_{\lambda+\alpha_i}\mathcal{E}_i\mathbb{1}_\lambda, \quad \text{and} \quad \mathbb{1}_{\lambda-\alpha_i}\mathcal{F}_i = \mathbb{1}_{\lambda-\alpha_i}\mathcal{F}_i\mathbb{1}_\lambda$$

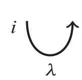
  for $i \in I$ and $\lambda \in X$. We denote the grading shift by $\langle 1 \rangle$, so that for each 1-morphism x in $\mathcal{U}_Q$ and $t \in \mathbb{Z}$ we a 1-morphism $x\langle t \rangle$.
- **2-morphisms** are $\Bbbk$-vector spaces spanned by compositions of colored, decorated tangle-like diagrams illustrated below:



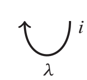
$: \mathcal{E}_i\mathbb{1}_\lambda \to \mathcal{E}_i\mathbb{1}_\lambda\langle i \cdot i \rangle,$



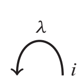
$: \mathcal{E}_i\mathcal{E}_j\mathbb{1}_\lambda \to \mathcal{E}_j\mathcal{E}_i\mathbb{1}_\lambda\langle -i \cdot j \rangle,$



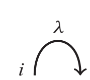
$: \mathbb{1}_\lambda \to \mathcal{F}_i\mathcal{E}_i\mathbb{1}_\lambda\langle 1 + \lambda_i \rangle,$



$: \mathbb{1}_\lambda \to \mathcal{E}_i\mathcal{F}_i\mathbb{1}_\lambda\langle 1 - \lambda_i \rangle,$



$: \mathcal{F}_i\mathcal{E}_i\mathbb{1}_\lambda \to \mathbb{1}_\lambda\langle 1 + \lambda_i \rangle,$



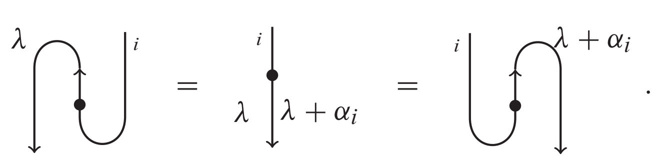
$: \mathcal{E}_i\mathcal{F}_i\mathbb{1}_\lambda \to \mathbb{1}_\lambda\langle 1 - \lambda_i \rangle.$

In this 2-category (and those throughout the paper) we read diagrams from bottom to top and right to left. That is, in a diagram representing a 1-morphism from λ to μ, the region on the right will be labeled λ and the region on the left will be labeled μ. The identity 2-morphism of the 1-morphism $\mathcal{E}_i\mathbb{1}_\lambda$ is represented by an upward oriented line labeled by i and the identity 2-morphism of $\mathcal{F}_i\mathbb{1}_\lambda$ is represented by a downward such line.

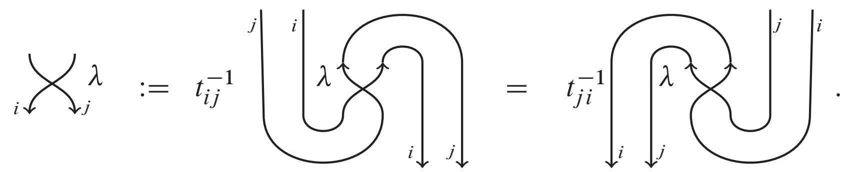
The 2-morphisms satisfy the following relations:

(1) The 1-morphisms $\mathcal{E}_i\mathbb{1}_\lambda$ and $\mathcal{F}_i\mathbb{1}_\lambda$ are biadjoint (up to a specified degree shift).

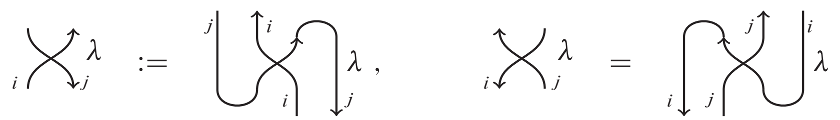
(2) The dot 2-morphisms are cyclic with respect to this biadjoint structure:



The Q-cyclic relations for crossings are given by

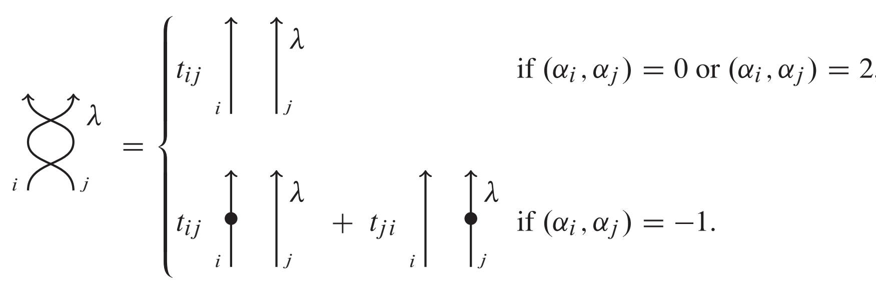


Sideways crossings are equivalently defined by the following identities:

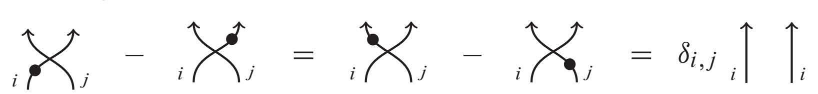


(3) The 1-morphisms $\mathcal{E}$ (respectively the 1-morphisms $\mathcal{F}$) carry an action of the KLR algebra for a fixed choice of parameters Q. The KLR algebra R associated to a fixed set of parameters Q is defined by finite $\Bbbk$-linear combinations of braid–like diagrams in the plane, where each strand is labeled by a vertex $i \in I$. Strands can intersect and can carry dots, but triple intersections are not allowed. Diagrams are considered up to planar isotopy that do not change the combinatorial type of the diagram. We recall the local relations.

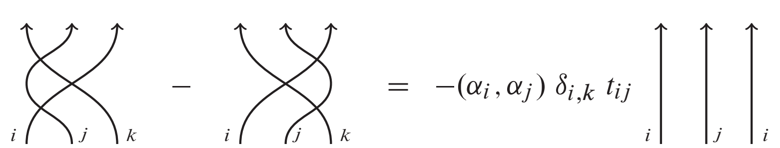
(i) The quadratic KLR relations are



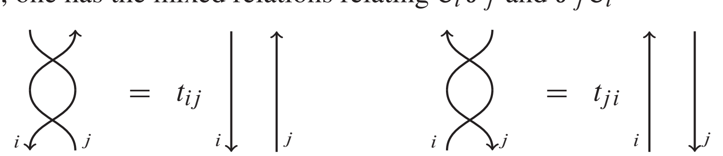
(ii) The dot sliding relations are



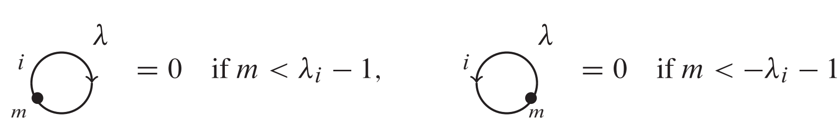
(iii) The cubic KLR relations are



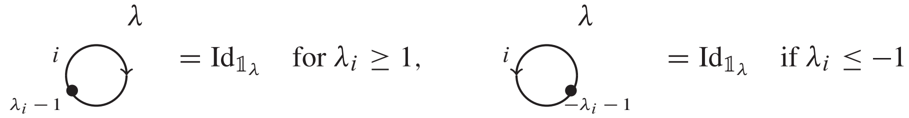
(4) When $i \neq j$, one has the mixed relations relating $\mathcal{E}_i\mathcal{F}_j$ and $\mathcal{F}_j\mathcal{E}_i$



(5) Negative degree bubbles are zero. That is for all $m \in \mathbb{Z}_{>0}$ one has

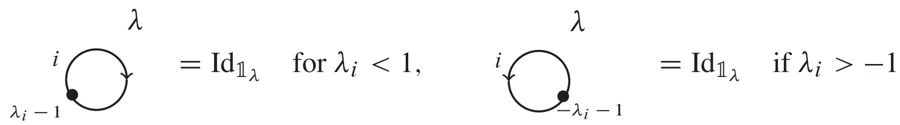


Furthermore, dotted bubbles of degree zero are scalar multiples of the identity 2-morphisms

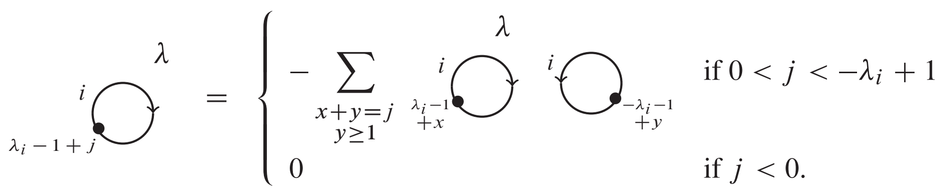


We introduce formal symbols called *fake bubbles*. These are positive degree endomorphisms of $\mathbb{1}_\lambda$ that carry a formal label by a negative number of dots.

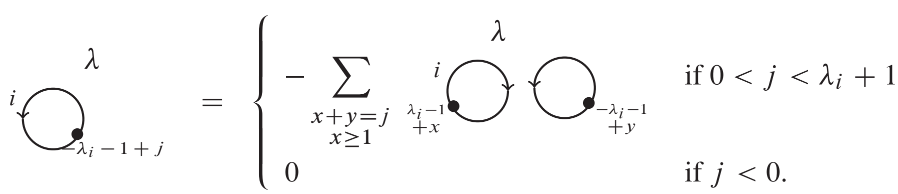
- Degree zero fake bubbles are normalized by



- Higher degree fake bubbles for $\lambda_i < 0$ are defined inductively as

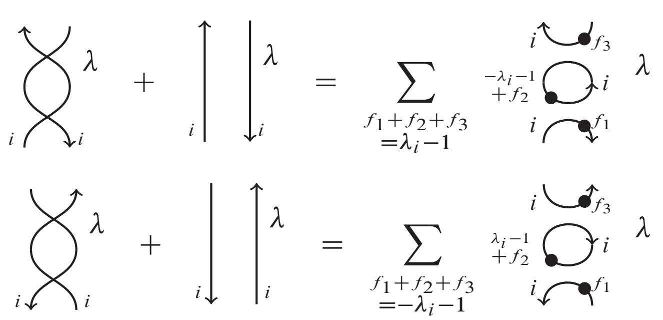


- Higher degree fake bubbles for $\lambda_i > 0$ are defined inductively by

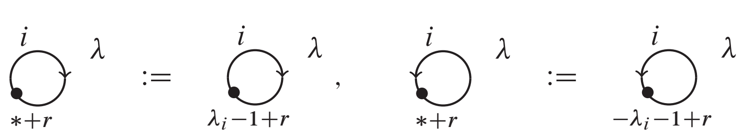


The above relations are sometimes referred to as the *infinite Grassmannian relations*.

(6) The $\mathfrak{sl}_2$ relations (which we also refer to as the $\mathcal{E}\mathcal{F}$ and $\mathcal{F}\mathcal{E}$ decompositions) are



It is sometimes convenient to use a shorthand notation for the bubbles that emphasizes their degrees:



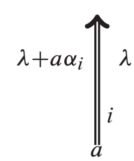
Definition 2.3. A 2-representation of $\mathcal{U}_Q(\mathfrak{g})$ is a graded additive $\Bbbk$-linear 2-functor $\mathcal{U}_Q(\mathfrak{g}) \to \mathcal{K}$ for some graded, additive 2-category $\mathcal{K}$.

2.3. Thick calculus. The idempotent completion, or Karoubi envelope $\dot{\mathcal{C}}$, of an additive category $\mathcal{C}$ can be viewed as a minimal enlargement of the category $\mathcal{C}$ so that idempotents split. The idempotent completion $\dot{\mathcal{K}}$ of a 2-category $\mathcal{K}$ is the 2-category with the same objects as $\mathcal{K}$, but with Hom categories given by the usual Karoubi envelope of $\mathrm{Hom}_{\mathcal{K}}(A, B)$. Any additive 2-functor $\mathcal{K} \to \mathcal{K}'$ that has splitting of idempotent 2-morphisms in $\mathcal{K}'$ extends uniquely to an additive 2-functor $\dot{\mathcal{K}} \to \mathcal{K}'$; see [21, Section 3.4] for more details.

In order to define the Rickard complexes lifting the braid group action on integrable modules, we must first introduce the augmented graphical calculus for $\dot{\mathcal{U}}_Q$. This so-called 'thick calculus' describes 2-morphisms in the Karoubi envelope of the 2-category $\mathcal{U}_Q$. For more details in the $\mathfrak{sl}_2$ case see [22]. For the simply laced case see [36,37], although care must be taken because all of the formulas in these references are for the unsigned version of the KLR-algebra where all $t_{ij} = 1$. Those formulas can be transferred to our conventions using the rescaling functors from [26, Section 3.3].

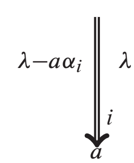
In the Karoubi envelope $\dot{\mathcal{U}}_Q$ we can define divided power 1-morphisms $\mathcal{E}_i^{(a)}\mathbb{1}_\lambda$ and $\mathcal{F}_i^{(b)}\mathbb{1}_\lambda$ by

$$\mathcal{E}_i^{(a)}\mathbb{1}_\lambda\langle t\rangle := \left(\mathcal{E}_i^{a}\mathbb{1}_\lambda\left\langle t - \frac{a(a-1)}{2}\right\rangle, e_a\right) =:$$



and

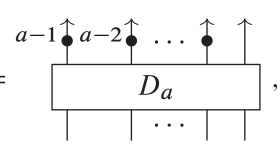
$$\mathcal{F}_i^{(a)}\mathbb{1}_\lambda\langle t\rangle := \left(\mathcal{F}_i^{a}\mathbb{1}_\lambda\left\langle t + \frac{a(a-1)}{2}\right\rangle, e_a'\right) =:$$



,

where the idempotent e_a is defined as follows:

$$e_a := \delta_a D_a =$$



,

where all strands are labeled i. Here D_a is the longest braid on a-strands. The idempotents e_a' are obtained from e_a by a 180° rotation. We have

$$\mathcal{E}_i^{a}\mathbb{1}_\lambda \cong \bigoplus_{[a]!} \mathcal{E}_i^{(a)}\mathbb{1}_\lambda$$

and

$$\mathcal{F}_i^{b}\mathbb{1}_\lambda \cong \bigoplus_{[b]!} \mathcal{F}_i^{(b)}\mathbb{1}_\lambda.$$

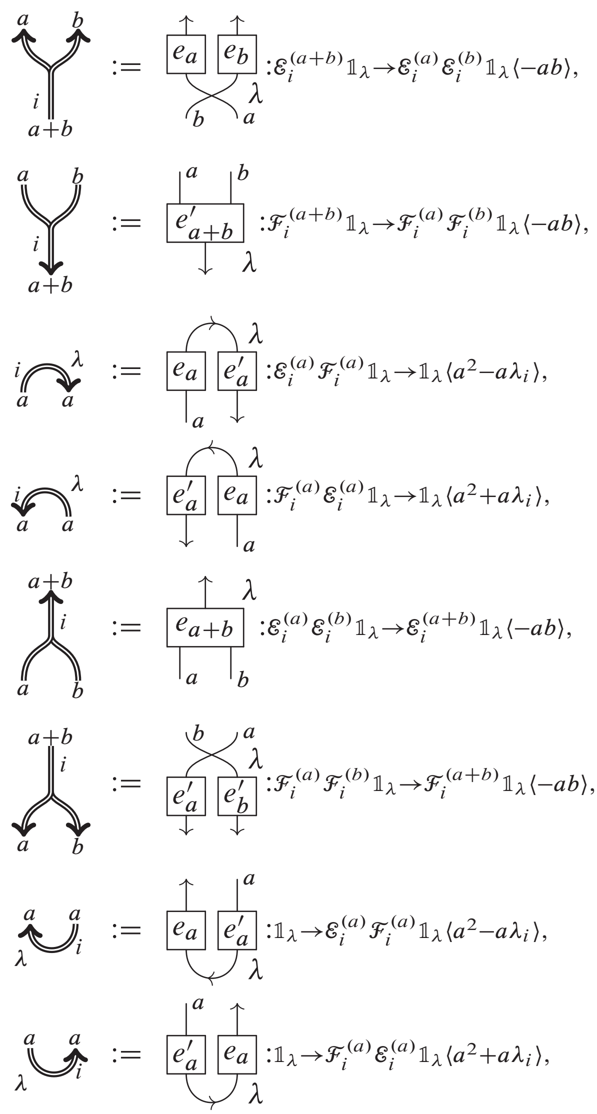
Here we use the standard notation

$$[a] := \frac{q^a - q^{-a}}{q - q^{-1}} = q^{a-1} + q^{a-3} + \cdots + q^{-a+1}, \quad [a]! = \prod_{i=1}^{a}[i],$$

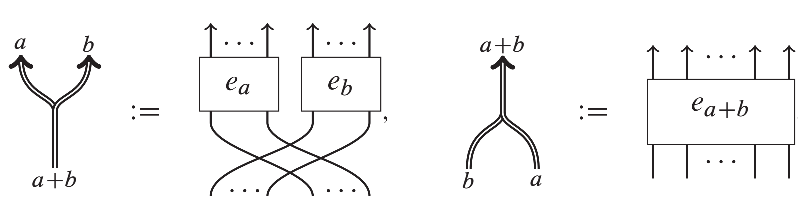
and

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{[a]!}{[b]![a-b]!}.$$

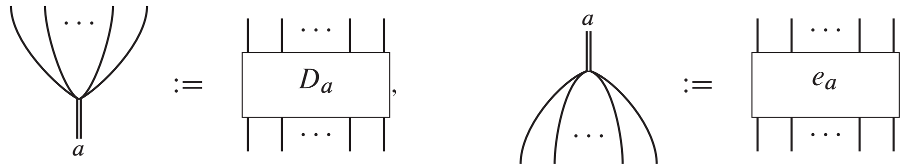
We define here some additional 2-morphisms in $\dot{\mathcal{U}}_Q$, whose degrees can be read from the shift on the right-hand side. In this section we will assume that all strings are colored by the Dynkin node i that we omit for simplicity unless explicitly indicated in the diagrams:



where we use the short hand notation of thin strands labeled a corresponds to a thin strands labeled by $i \in I$. For example,

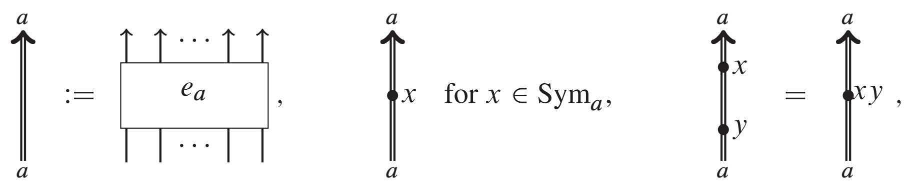


The thick cap and cups satisfy zig-zag equations so that diagrams in $\dot{\mathcal{U}}_Q$ related by isotopy define identical 2-morphisms in $\dot{\mathcal{U}}_Q$. Furthermore, the splitters and mergers defined above satisfy associativity and coassociativity relations [22, Proposition 2.4] making it possible to define



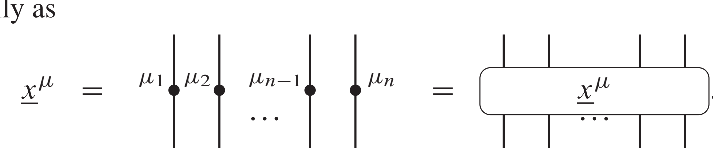
unambiguously.

The center of the nilHecke algebra NH_a is isomorphic to the ring of symmetric functions $\mathbb{Z}[x_1,\dots,x_a]^{S_a}$. Hence, any $x \in \mathrm{Sym}_a$ defines an endomorphism of $\mathcal{E}^{(a)}\mathbb{1}_\lambda$ (respectively $\mathcal{F}^{(a)}\mathbb{1}_\lambda$) in $\dot{\mathcal{U}}_Q$ since multiplication by symmetric functions $x \in \mathrm{Sym}_a$ commutes with the idempotent e_a:

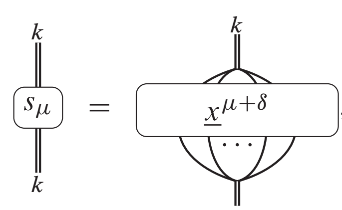


where the product xy is well defined since $xe_a ye_a = xye_a$.

For any composition $\mu = (\mu_1,\dots,\mu_n)$ write $\underline{x}^\mu := x_1^{\mu_1}x_2^{\mu_2}\dots x_n^{\mu_n}$. We depict these diagrammatically as



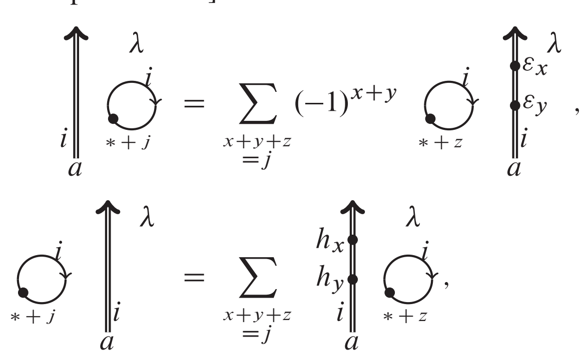
For any Schur polynomial s_μ corresponding to the partition $\mu = (\mu_1,\dots,\mu_k)$ one can show that

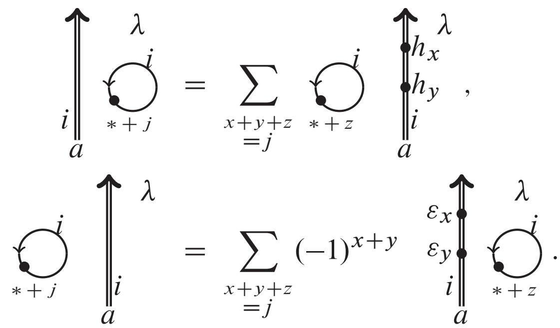


where $\mu + \delta$ is the partition $(\mu_1 + k - 1, \mu_2 + k - 2, \dots, \mu_k + 0)$. We denote by $h_m = s_{(m)}$ and $\varepsilon_m = s_{(1^m)}$ the complete and elementary symmetric function, respectively.

We now present some important identities holding in the thick calculus. Note that by [22, Corollary 4.7 and Proposition 4.8]

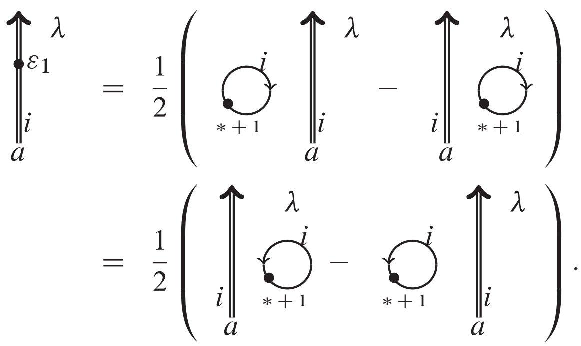
(2.1)





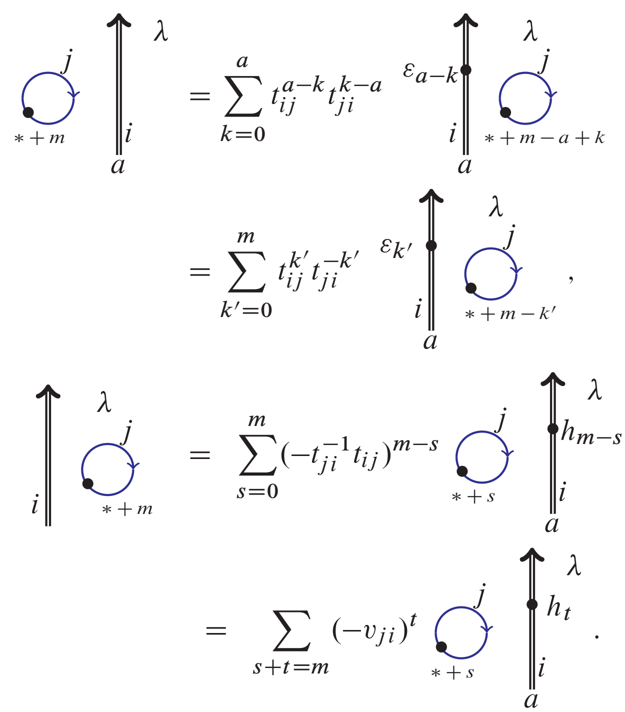
This implies

(2.2)



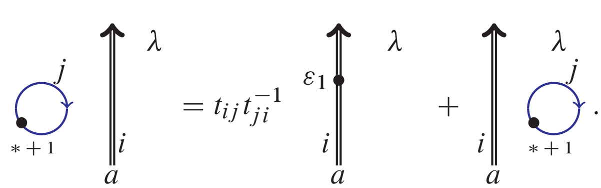
For $i \cdot j = -1$, one can calculate in a similar way that the mixed bubble slides have the form

(2.3)

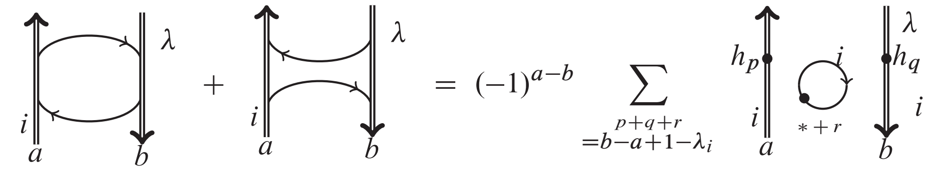


Equation (2.3) implies

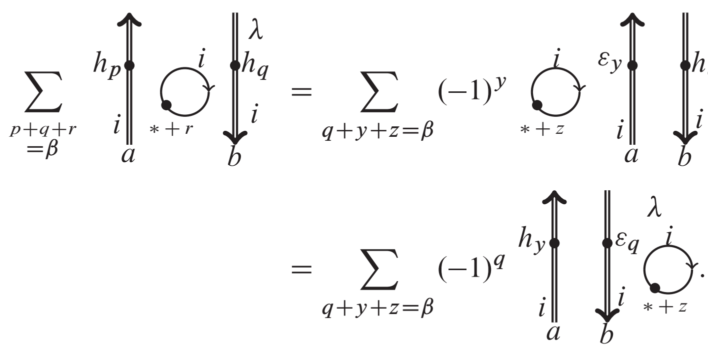
(2.4)



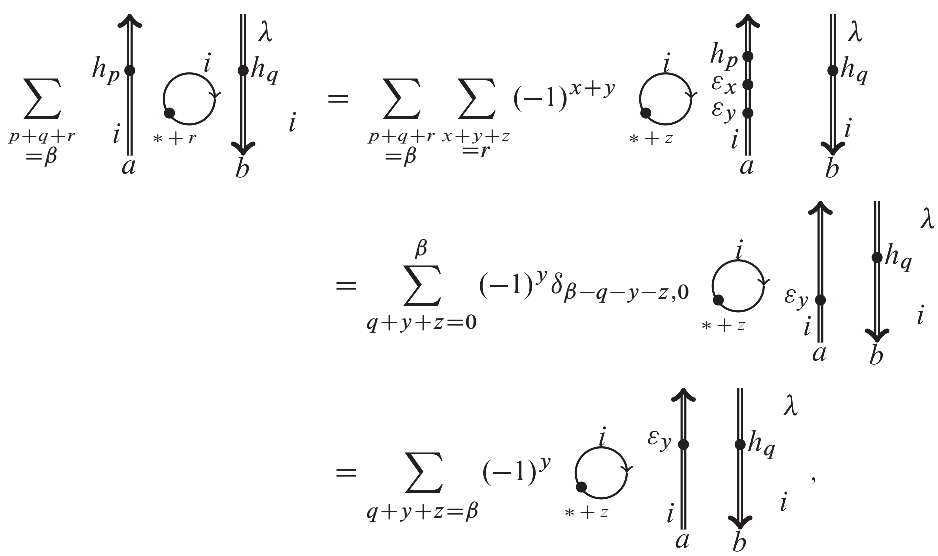
Lemma 2.4 ([22, Lemma 4.16]). *There is an equality of diagrams*



Lemma 2.5. *For any $a, b, \beta \geq 0$ the following identities hold:*

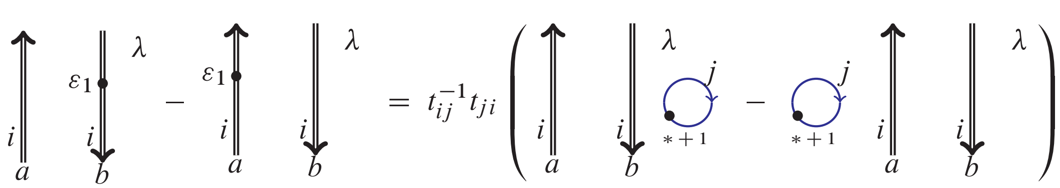


Proof. The first identity is proved using (2.1) as follows:



where we used the fundamental relation between complete and elementary symmetric functions. The second identity is proven similarly by pushing the interior bubble to the right. □

Lemma 2.6. *We have*

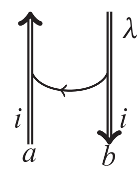


Proof. Using (2.4) the result follows. □

2.4. Braid group actions. We now recall how a categorical $\mathcal{U}_Q$ gives rise to a braid group action in the homotopy category (cf. [11]). To do this we consider the following complexes (usually called Rickard complexes):

$$\tau_i \mathbb{1}_\lambda = \begin{cases} \left[\mathcal{E}_i^{(-\lambda_i)} \mathbb{1}_\lambda \xrightarrow{d^+} \cdots \xrightarrow{d^+} \mathcal{E}_i^{(-\lambda_i+k)} \mathcal{F}_i^{(k)} \mathbb{1}_\lambda \langle k \rangle \xrightarrow{d^+} \cdots \right] & \text{if } \lambda_i \leq 0, \\ \left[\mathcal{F}_i^{(\lambda_i)} \mathbb{1}_\lambda \xrightarrow{d^+} \cdots \xrightarrow{d^+} \mathcal{E}_i^{(k)} \mathcal{F}_i^{(\lambda_i+k)} \mathbb{1}_\lambda \langle k \rangle \xrightarrow{d^+} \cdots \right] & \text{if } \lambda_i \geq 0, \end{cases} \tag{2.5}$$

where the left most term is in cohomological degree zero and with

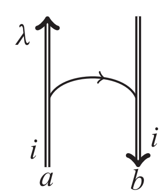
$$d^+ = d^+_{(a,b)} := (-1)^{\lambda_i+1}$$ 

,

where $b - a = \lambda_i$. Similarly we define

$$\mathbb{1}_\lambda \tau_i' := \begin{cases} \left[\cdots \xrightarrow{d^-} \mathbb{1}_\lambda \mathcal{E}_i^{(k)} \mathcal{F}_i^{(-\lambda_i+k)} \langle -k \rangle \xrightarrow{d^-} \cdots \xrightarrow{d^-} \mathbb{1}_\lambda \mathcal{F}^{(-\lambda_i)} \right] & \text{if } \lambda_i \leq 0, \\ \left[\cdots \xrightarrow{d^-} \mathbb{1}_\lambda \mathcal{E}_i^{(\lambda_i+k)} \mathcal{F}_i^{(k)} \langle -k \rangle \xrightarrow{d^-} \cdots \xrightarrow{d^-} \mathbb{1}_\lambda \mathcal{E}^{(\lambda_i)} \right] & \text{if } \lambda_i \geq 0, \end{cases}$$

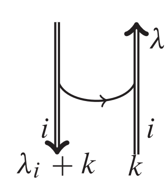
where the right most term is in cohomological degree zero and with

$$d^- = d^-_{(a,b)} =$$ 

,

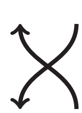
where $a - b = \lambda_i$.

Remark 2.7. In [8] a slightly different form of the complex $\tau_i \mathbb{1}_\lambda$ is used when $\lambda_i \geq 0$, namely

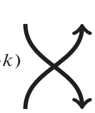
$$\left[\mathcal{F}_i^{(\lambda_i)} \mathbb{1}_\lambda \xrightarrow{\widehat{d}^+} \cdots \xrightarrow{\widehat{d}^+} \mathcal{F}_i^{(\lambda_i+k)} \mathcal{E}_i^{(k)} \mathbb{1}_\lambda \langle k \rangle \xrightarrow{\widehat{d}^+} \cdots \right], \quad \widehat{d}^+ =$$ 

.

However, this version of the complex is isomorphic to the complex in (2.5) by [22, Lemma 5.3 and Proposition 5.10] with the isomorphism in homological degree k given by



$$\mathcal{F}_i^{(\lambda_i+k)} \mathcal{E}_i^{(k)} \mathbb{1}_\lambda \longrightarrow \mathcal{E}_i^{(k)} \mathcal{F}_i^{(\lambda_i+k)} \mathbb{1}_\lambda \xrightarrow{(-1)^{k(\lambda_i+k)}} \mathcal{F}_i^{(\lambda_i+k)} \mathcal{E}_i^{(k)} \mathbb{1}_\lambda.$$



Definition 2.8. Let $\mathcal{K}$ be a graded $\Bbbk$-linear 2-category. A 2-representation $F\colon \mathcal{U}_Q \to \mathcal{K}$ is said to be *integrable* if it satisfied the following additional properties:

(1) The objects $\boldsymbol{\lambda} := F(\lambda)$ of $\mathcal{K}$ are zero for all but finitely many λ,

(2) The hom categories $\mathrm{Hom}_{\mathcal{K}}(\boldsymbol{\lambda}, \boldsymbol{\lambda}')$ are idempotent complete with finite dimensional hom spaces in each degree, in other words, hom categories have the Krull-Schmidt property,

(3) For any weight λ, $\mathrm{Hom}(1_{\boldsymbol{\lambda}}, 1_{\boldsymbol{\lambda}}\langle \ell \rangle) = 0$ if $\ell < 0$ and is one-dimensional if $\ell = 0$ and $1_{\boldsymbol{\lambda}} \neq 0$, where $1_{\boldsymbol{\lambda}}$ denotes the identity morphism on $F(\lambda)$.

Note that since $\mathcal{K}$ is idempotent complete, a 2-representation $F\colon \mathcal{U}_Q \to \mathcal{K}$ extends uniquely to a 2-representations $\dot{F}\colon \dot{\mathcal{U}}_Q \to \mathcal{K}$. Furthermore, passing to homotopy categories of complexes, this induces a 2-functor $\dot{F}\colon \mathsf{Kom}(\dot{\mathcal{U}}_Q) \to \mathsf{Kom}(\mathcal{K})$.

Proposition 2.9 ([11, Theorem 6.3]). *Let $F\colon \mathcal{U}_Q \to \mathcal{K}$ be an integrable 2-representation. Then the complexes τ_i, τ_i' satisfy the following braid group relations in $\mathsf{Kom}(\mathcal{K})$:*

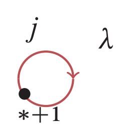
$$\begin{aligned} \tau_i \tau_i' \mathbb{1}_\lambda &\cong \mathbb{1}_\lambda \cong \tau_i' \tau_i \mathbb{1}_\lambda, & & \\ \tau_i \tau_j \mathbb{1}_\lambda &\cong \tau_j \tau_i \mathbb{1}_\lambda & & \text{if } \langle i, j \rangle = 0, \\ \tau_i \tau_j \tau_i \mathbb{1}_\lambda &\cong \tau_j \tau_i \tau_j \mathbb{1}_\lambda & & \text{if } \langle i, j \rangle = -1. \end{aligned}$$

Remark 2.10. The results in [11, Section 6] actually require something much weaker than a full action of the 2-category $\mathcal{U}_Q$. However, results of [7] together with [5], see also [14], give the full action of $\mathcal{U}_Q$.

3. Bubbles and homotopies

3.1. Notation. For $j \in I$ we write

$$b_j = b_j(\lambda) :=$$



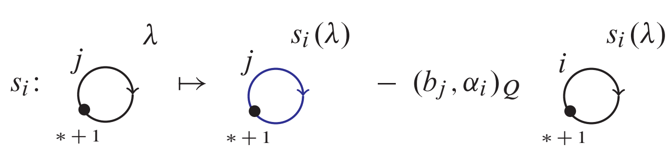
Note that any degree two 2-morphism in $\mathrm{End}^2_{\mathcal{U}_Q(\mathfrak{g})}(\mathbb{1}_\lambda)$ can be written uniquely as a linear combination of such $b_j(\lambda)$, cf. [21, 39].

Definition 3.1. For any $\lambda \in X$, define a bilinear pairing $(\cdot,\cdot)_Q$ between $\mathrm{End}^2_{\mathcal{U}_Q(\mathfrak{g})}(\mathbb{1}_\lambda)$ and $\mathfrak{h}^*_{\Bbbk}$ by

$$(b_j, \alpha_i)_Q = (-1)^{(\alpha_i, \alpha_j)} v_{ji} (\alpha_j, \alpha_i),$$

where $v_{ji} = t_{ji}^{-1} t_{ij}$.

It is an easy exercise to check that the Weyl group $W_{\mathfrak{g}}$ acts on $\bigoplus_\lambda \mathrm{End}^2_{\mathcal{U}_Q(\mathfrak{g})}(\mathbb{1}_\lambda)$ via



or

$$s_i(b_j(\lambda)) := b_j(s_i(\lambda)) - (b_j, \alpha_i)_Q b_i(s_i(\lambda)).$$

Note that we can simplify this equation by omitting the domains to obtain

$$s_i(b_j) := b_j - (b_j, \alpha_i)_Q b_i.$$

Observe that for all choices of Q, we have $s_i(b_i) := -b_i$.

Remark 3.2. Consider the choice of coefficients for $\mathcal{U}_Q(\mathfrak{g})$, where

$$t_{ij} = \begin{cases} -1 & \text{if } i \cdot j = -1 \text{ and } i \to j, \\ 1 & \text{otherwise.} \end{cases}$$

This choice of parameters Q corresponds is the natural choice of KLR algebra R_Q that describes Ext-algebras between perverse sheaves on the quiver variety [38]. In this case we have $v_{ij} = t_{ij}^{-1} t_{ji} = -1$ whenever $(\alpha_i, \alpha_j) = -1$ and subsequently $(b_j, \alpha_i)_Q = (\alpha_j, \alpha_i)$ for all $i, j \in I$.

Subsequently, the Weyl group action above gives an action on $\bigoplus_\lambda \mathrm{End}^2_{\mathcal{U}_Q(\mathfrak{g})}(\mathbb{1}_\lambda)$ via

$$s_i(b_j) := b_j(s_i) - \alpha_i^\vee(\alpha_j) b_i.$$

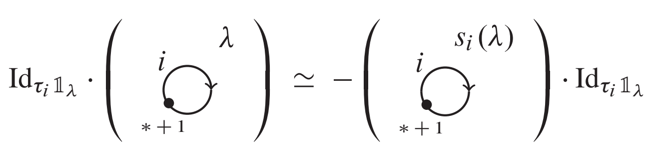
In this case, the natural map $\mathfrak{h}^* \to \mathrm{End}^2_{\mathcal{U}_Q(\mathfrak{g})}(\mathbb{1}_\lambda)$ given by $\alpha_j \mapsto b_j$ intertwines the Weyl group actions.

3.2. The homotopies. The following is the main result of this section. It is an immediate corollary of the subsequent three propositions.

Theorem 3.3. *For all $i \in I$ and $b \in \mathrm{End}^2_{\mathcal{U}_Q(\mathfrak{g})}(\mathbb{1}_\lambda)$ we have*

$$(b, \alpha_i)_Q [d^- d^+ + d^+ d^-] = s_i(b) \cdot \mathrm{Id}_{\tau_i \mathbb{1}_\lambda} - \mathrm{Id}_{\tau_i \mathbb{1}_\lambda} \cdot b. \tag{3.1}$$

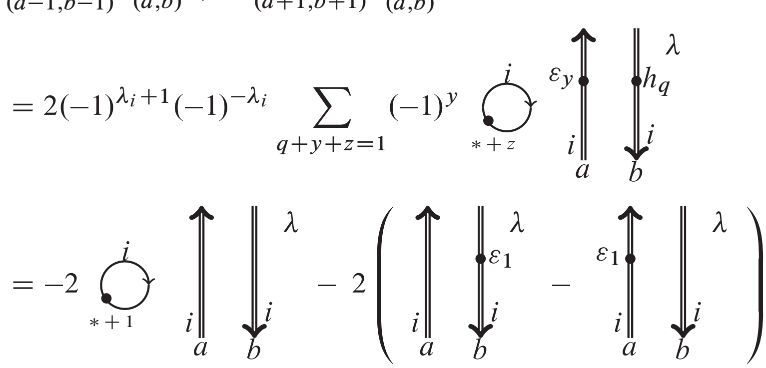
Proposition 3.4. *For all λ and $i \in I$ there is a homotopy equivalence*



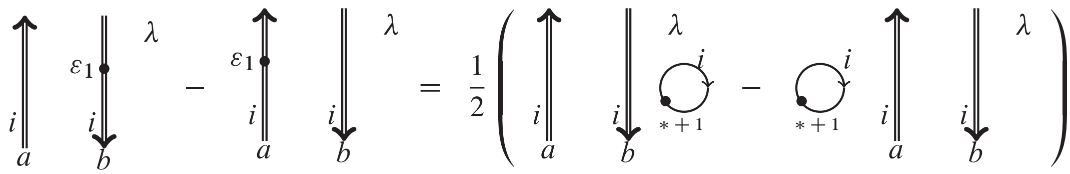
given by $2d^-$.

Proof. Lemmas 2.4 and 2.5 with $b - a = \lambda_i$ imply

$$2d^+_{(a-1,b-1)} d^-_{(a,b)} + 2d^-_{(a+1,b+1)} d^+_{(a,b)} \tag{3.2}$$

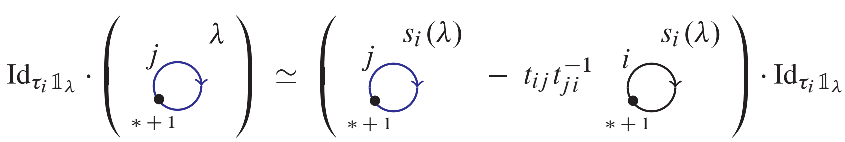


(with $(-1)^{\lambda_i+1}$ coming from the differential and $(-1)^{-\lambda_i}$ coming from (the square flop) Lemma 2.4). The result then follows using (2.2) since



so that (3.2) is equal to $(s_i(b_i))\mathrm{Id}_{\mathcal{E}_i^{(a)}\mathcal{F}_i^{(b)}\mathbb{1}_\lambda} - \mathrm{Id}_{\mathcal{E}_i^{(a)}\mathcal{F}_i^{(b)}\mathbb{1}_\lambda} b_i$. □

Proposition 3.5. *For all λ and $i, j \in I$ such that $i \cdot j = -1$ there is a homotopy equivalence*

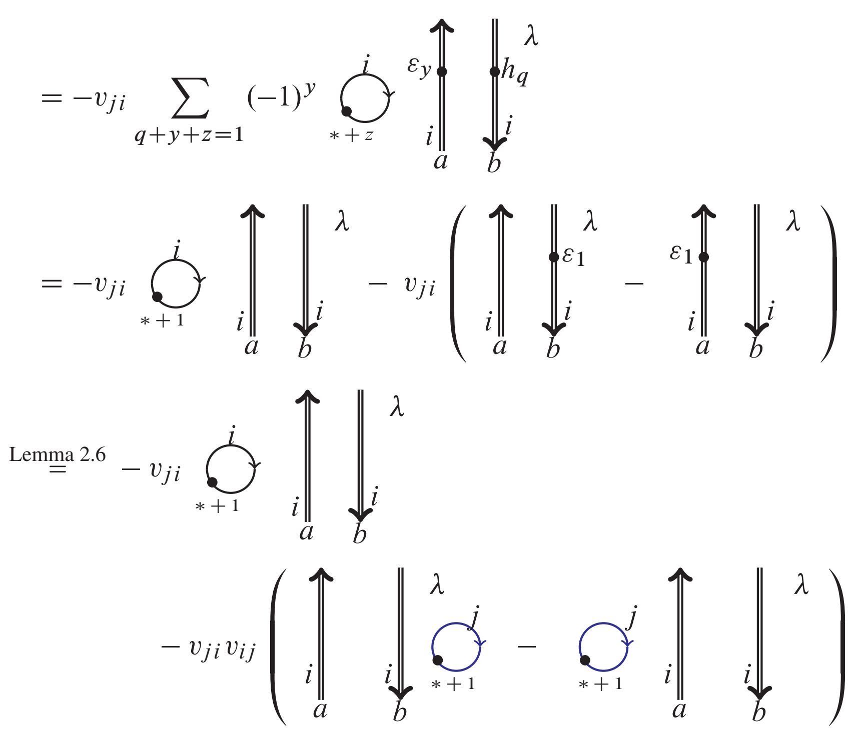


given by

$$t_{ji}^{-1} t_{ij} d^- = v_{ji} d^-.$$

Proof. By Lemmas 2.4 and 2.5,

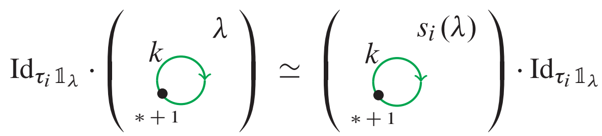
$$v_{ji} d^+_{(a-1,b-1)} d^-_{(a,b)} + v_{ji} d^-_{(a+1,b+1)} d^+_{(a,b)}$$



$$= (s_i(b_j))\mathrm{Id}_{\mathcal{E}_i^{(a)}\mathcal{F}_i^{(b)}\mathbb{1}_\lambda} - \mathrm{Id}_{\mathcal{E}_i^{(a)}\mathcal{F}_i^{(b)}\mathbb{1}_\lambda} b_j,$$

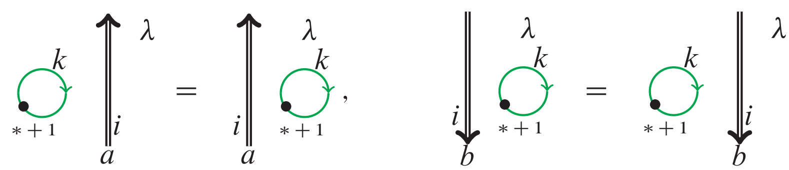
where we used that $v_{ij} v_{ji} = 1$ in the second to last equality. □

Proposition 3.6. *For all λ and $i, k \in I$ such that $i \cdot k = 0$ the following maps are equal (and in particular homotopic)*



Proof. This is immediate from the bubble slide rules

(3.3)



so that no homotopy is required. □

3.3. Some remarks. There is a natural internal braid group action on the quantum group that interacts with the braid group action on integrable modules that is categorified by Rickard complexes and which generalizes the action on bubbles discussed above. In [1] 2-functors $\mathcal{T}_i$ categorifying the internal braid group action were defined from $\mathcal{U}_Q$ to its homotopy category of complexes $\mathsf{Kom}(\mathcal{U}_Q)$. It was shown in [2] that these 2-functors interact with Rickard complexes giving 2-natural isomorphisms

$$\beth \colon \tau_i \mathbb{1}_{\lambda'}(-)\mathbb{1}_\lambda \Rightarrow \mathcal{T}_i(-)\tau_i\mathbb{1}_\lambda,$$

where $(-)$ denotes an input from $\mathcal{U}_Q$. This amounts to defining for each one morphism $\mathbb{1}_{\lambda'}x\mathbb{1}_\lambda \in \mathsf{Kom}(\mathcal{U}_Q)$ a chain homotopy equivalence

$$\beth_{x\mathbb{1}_\lambda} \colon \tau_i \mathbb{1}_{\lambda'}x\mathbb{1}_\lambda \Rightarrow \mathcal{T}_i(x\mathbb{1}_\lambda)\tau_i\mathbb{1}_\lambda,$$

and for each 2-morphism $f \colon \mathbb{1}_{\lambda'}x\mathbb{1}_\lambda \to \mathbb{1}_{\lambda'}y\mathbb{1}_\lambda$ in $\mathsf{Kom}(\mathcal{U}_Q)$ a chain map

$$\mathcal{T}_i(f) \colon \mathcal{T}_i(\mathbb{1}_{\lambda'}x\mathbb{1}_\lambda) \to \mathcal{T}_i(\mathbb{1}_{\lambda'}y\mathbb{1}_\lambda)$$

giving a commutative diagram

$$\begin{array}{ccc} \tau_i y\mathbb{1}_\lambda & \xrightarrow{\ \beth_{y\mathbb{1}_\lambda}\ } & \mathcal{T}_i(y\mathbb{1}_\lambda)\tau_i\mathbb{1}_\lambda \\ {\scriptstyle \mathrm{Id}_{\tau_i} f}\uparrow & & \uparrow{\scriptstyle \mathcal{T}_i(f)\mathrm{Id}_{\tau_i\mathbb{1}_\lambda}} \\ \tau_i x\mathbb{1}_\lambda & \xrightarrow[\ \beth_{x\mathbb{1}_\lambda}\ ]{} & \mathcal{T}_i(x\mathbb{1}_\lambda)\tau_i\mathbb{1}_\lambda \end{array} \tag{3.4}$$

in $\mathsf{Kom}(\mathcal{U}_Q)$. For identity 1-morphisms $\mathbb{1}_\lambda$, the maps $\beth_{\mathbb{1}_\lambda}$ are identities since $\mathcal{T}_i(\mathbb{1}_\lambda) = \mathbb{1}_{s_i(\lambda)}$ and $\tau_i\mathbb{1}_\lambda = \mathbb{1}_{s_i(\lambda)}\tau_i$. One can show that the braid group action via 2-functors $\mathcal{T}_i$ on the space of endomorphisms $\bigoplus_\lambda \mathrm{End}^2(\mathbb{1}_\lambda)$ factors through the corresponding Weyl group, so that $\mathcal{T}_i^2 = \mathrm{Id}$ on this space. A bubble $b_j(\lambda)$ defines a degree 2 endomorphism of $\mathbb{1}_\lambda$ and the homotopy for the naturality square above defines a chain homotopy from $\tau_i\mathbb{1}_\lambda b_j(\lambda)$ to $\mathcal{T}_i(b_j(\lambda))\tau_i\mathbb{1}_\lambda$. This homotopy equivalence agrees with the homotopies defined in Theorem 3.3.

From this perspective, there is nothing special about degree 2 bubbles. One can similarly show that the braid group action induced by the 2-functors $\mathcal{T}_i$ on $\bigoplus_\lambda \mathrm{End}^{2k}(\mathbb{1}_\lambda)$ factors through the Weyl group and the commutative square (3.4) gives homotopies from $\tau_i \mathbb{1}_\lambda x \mathbb{1}_\lambda$ to $\mathcal{T}_i(x\mathbb{1}_\lambda)\tau_i \mathbb{1}_\lambda$ for any 2-morphisms $x\mathbb{1}_\lambda \in \mathrm{End}^{2k}(\mathbb{1}_\lambda)$. Further, the homotopies for degree $2k$ bubbles can be chosen so that they square to zero, so they naturally fit into the framework of curved complexes studied in the next section.

4. Curved Rickard complexes

4.1. 2-categories and curved complexes. In this subsection we review some definitions and properties of curved complexes in the context of 2-categories.

Let us fix a $\mathbb{k}$-linear 2-category $\mathcal{K}$. Recall that $\mathsf{Kom}(\mathcal{K})$ is used to denote the homotopy category of complexes in $\mathcal{K}$. The objects here are the same but the 1-morphisms are complexes (V, Δ) of 1-morphisms in $\mathcal{K}$

$$V_1 \xrightarrow{\Delta} V_2 \xrightarrow{\Delta} \cdots \xrightarrow{\Delta} V_n$$

which we consider up to homotopy equivalence. One can rewrite this more compactly as

$$V = \bigoplus_i V_i[-i],$$

where $[\cdot]$ denotes a shift in homological grading, and $\Delta : V \to V[1]$. Notice that, by definition, $\Delta^2 = 0$ in this case. We now explain how one can relax this condition to obtain the homotopy category $\widetilde{\mathsf{Kom}}(\mathcal{K})[u]$ of curved complexes.

First, for two objects $A, B \in \mathcal{K}$ we denote by ${}_B\mathcal{K}_A := \mathrm{Hom}_{\mathcal{K}}(A, B)$ the 1-category of maps between A and B. This category is naturally an $(\mathrm{End}(\mathbb{1}_B), \mathrm{End}(\mathbb{1}_A))$-bilinear category. Here $\mathbb{1}_A$ is the identity functor of A and $\mathrm{End}(\mathbb{1}_A)$ is its endomorphism algebra (and similarly for $\mathbb{1}_B$).

For $A, B \in \mathcal{K}$ as above, let $(z_B, z_A) \in (\mathrm{End}^2(\mathbb{1}_B), \mathrm{End}^2(\mathbb{1}_A))$. We would like to consider a sequence of maps

$$V_1 \underset{\Delta^-}{\overset{\Delta^+}{\rightleftarrows}} V_2 \underset{\Delta^-}{\overset{\Delta^+}{\rightleftarrows}} \cdots \underset{\Delta^-}{\overset{\Delta^+}{\rightleftarrows}} V_n \tag{4.1}$$

such that $(\Delta^+ + \Delta^-)^2 = z_B 1_V - 1_V z_A$. We will often abuse notation and write $z_B 1_V - 1_V z_A$ simply as $z_B - z_A$. This is all fine except that the maps Δ^- have homological degree -1 instead of $+1$.

To fix this we enlarge our 2-category by introducing a formal variable u of homological degree 2. More precisely, for any $\mathbb{k}$-linear 2-category $\mathcal{D}$ we can consider the 2-category $\mathcal{D}[u]$ where the objects and 1-morphisms are the same but where the 2-morphisms are formally tensored with $\mathbb{k}[u]$. In other words, if V, V' are 1-morphisms, then

$$\mathrm{Hom}_{\mathcal{D}[u]}(V, V') = \mathrm{Hom}_{\mathcal{D}}(V, V') \otimes_{\mathbb{k}} \mathbb{k}[u].$$

In particular, any degree zero map in $\mathcal{D}[u]$ can be written as a sum $\phi = \sum_{i \geq 0} \phi_i u^i$ for some maps ϕ_i of degree $-2i$ in $\mathcal{D}$. In this language, we can combine the maps in (4.1) as

$$\Delta := \Delta^+ + u\Delta^- : V \mapsto V[1].$$

Definition 4.1. By using the setup above, a (z_B, z_A)-factorization in ${}_B\mathcal{K}_A$ is a map $\Delta : V \mapsto V[1]$ such that $\Delta^2 = (z_B - z_A)u$. Such a pair (V, Δ) is called a curved complex in ${}_B\mathcal{K}_A$ with curvature (z_B, z_A) and connection Δ.

Remark 4.2. In [17], the construction analogous to Definition 4.1 would be called a *strict* y-ification.

Definition 4.3. For two (z_B, z_A)-factorizations (V, Δ) and (V', Δ'), a morphism

$$f : (V, \Delta) \to (V', \Delta')$$

is a morphism (which we always assume to be degree zero) $f : V \to V'$ such that

$$f \circ \Delta = \Delta' \circ f.$$

Two morphisms

$$f, g : (V, \Delta) \to (V', \Delta')$$

of (z_B, z_A)-factorizations are homotopic if there exists $H : (V, \Delta) \to (V', \Delta')$ such that

$$f - g = H \circ \Delta + \Delta' \circ H.$$

For objects $A, B, C \in \mathcal{K}$ we have natural maps

$${}_C\mathcal{K}_B \otimes {}_B\mathcal{K}_A \to {}_C\mathcal{K}_A.$$

For a (z_B, z_A)-factorization $(V, \Delta) \in {}_B\mathcal{K}_A$ and a (z_C, z_B)-factorization $(V', \Delta') \in {}_C\mathcal{K}_B$ let us denote the image by $V' * V \in {}_C\mathcal{K}_A$ (although later we will drop the $*$ in order to simplify notation). It has a connection $\Delta' * \Delta$ defined by $(\Delta' * \Delta)(x' * x) = \Delta'(x') * x + (-1)^i x' * \Delta(x)$, where x' is in homological degree i of V'.

Lemma 4.4. *Suppose* $(V, \Delta) \in {}_B\mathcal{K}_A$ *is a* (z_B, z_A)*-factorization and* $(V', \Delta) \in {}_C\mathcal{K}_B$ *is a* (z_C, z_B)*-factorization. Then* $(V' * V, \Delta' * \Delta)$ *is a* (z_C, z_A)*-factorization.*

Proof. Assume that x' lies in homological degree i of V'. Then we have the following chain of equalities:

$$\begin{aligned}(\Delta' * \Delta)^2(x' * x) &= (\Delta' * \Delta)(\Delta'(x') * x + (-1)^i x' * \Delta(x)) \\ &= (\Delta')^2(x') * x + (-1)^{i+1}\Delta'(x') * \Delta(x) \\ &\qquad + (-1)^i \Delta'(x') * \Delta(x) + x' * \Delta^2(x) \\ &= (z_C - z_B)ux' * x + (z_B - z_A)ux' * x \\ &= (z_C - z_A)ux' * x.\end{aligned}$$

□

We will denote the homotopy category of factorizations in $\mathcal{K}$ by $\widetilde{\mathsf{Kom}}(\mathcal{K})[u]$. The following result is a straightforward extension of the analogous classical fact in homological algebra.

Proposition 4.5 ([17, Lemma 2.5]). *If* $f : (V_1, \Delta_1) \to (V_2, \Delta_2)$ *is an isomorphism of* (z_B, z_A)*-factorizations in* $\widetilde{\mathsf{Kom}}(\mathcal{K})[u]$*, then* $\mathrm{Cone}(f) \cong 0$ *in* $\widetilde{\mathsf{Kom}}(\mathcal{K})[u]$.

The following is adapted from [17, Lemma 2.19].

Proposition 4.6. *Suppose* $(V, \Delta^+) \in \mathsf{Kom}^-(\mathcal{K})$ *is contractible. Then any deformation* $(V, \Delta^+ + u\Delta^-)$ *is contractible in* $\widetilde{\mathsf{Kom}}(\mathcal{K})[u]$.

Proof. Let $(V, d) \in \mathsf{Kom}^-({}_B\mathcal{K}_A)$ be a contractible complex and suppose that it deforms to a factorization (V, Δ). Then $\Delta^2 = uz$ for some central element z and $\Delta^+\Delta^- + \Delta^-\Delta^+ = z$. Fix a nullhomotopy $h \in \mathrm{End}^{-1}(X)$ satisfying $dh + hd = \mathrm{Id}_V$. It suffices to assume that $h^2 = 0$ since we can always replace h by $h' = hdh$ which is still a homotopy and satisfies $h'^2 = 0$.[1] We claim that the map

$$\Phi := \mathrm{Id}_V + uh\Delta^- \in \mathrm{End}^0(V[u])$$

is invertible. To see this note that $(h\Delta^-) \in \mathrm{End}^{-2}(V)$ must be nilpotent for a bounded below complex $V \in \mathsf{Kom}^-(\mathcal{K})[u]$. Hence, the map

$$\Phi^{-1} := \sum_{j \geq 0} (-h\Delta^-)^j u^j$$

is a well-defined endomorphism of V. Similarly, the map $\Phi' := \mathrm{Id}_V + u\Delta^- h \in \mathrm{End}^0(V[u])$ is also invertible with inverse

$$\Phi'^{-1} := \sum_{j \geq 0} (-\Delta^- h)^j u^j.$$

It follows that $\Phi^{-1}h = h\Phi'^{-1}$. We show that $H := \Phi^{-1}h = h\Phi'^{-1}$ is a nullhomotopy for the factorization (V, Δ), in particular, that

$$\Delta H + H\Delta = \mathrm{Id}_{(V,\Delta)}, \tag{4.2}$$

for $\Delta = \Delta^+ + u\Delta^-$. Since Φ and Φ' are invertible maps, (4.2) is equivalent to proving

$$\Phi(\Delta H + H\Delta)\Phi' = \Phi\Phi' = \mathrm{Id}_V + u(h\Delta^- + h\Delta^-),$$

since $(\Delta^-)^2 = 0$. Expanding out we have

$$\Phi(\Delta H + H\Delta)\Phi' = \Delta^+ h + h\Delta^+ + u(\Delta^- h + h\Delta^-) + uh(\Delta^-\Delta^+ + \Delta^+\Delta^-)h$$

and the result follows since $\Delta^-\Delta^+ + \Delta^+\Delta^- = z$ is central and $h^2 = 0$. □

Proposition 4.7 ([17, Proposition 2.20]). *Let* $(C, \Delta^+), (C', \Delta'^+) \in \mathsf{Kom}({}_B\mathcal{K}_A)$ *be two invertible complexes and let* $\phi_0 \colon C \to C'$ *be a homotopy equivalence. Then* ϕ_0 *extends to a homotopy equivalence* $\phi \colon (C, \Delta) \to (C', \Delta')$ *of* (z_B, z_A)*-factorizations, where* (C, Δ) *and* (C', Δ') *are deformations of* C *and* C'*, respectively.*

Proof. Since C and C' are invertible and homotopically equivalent, we have

$$\mathrm{Hom}(C, C') \cong \mathrm{Hom}(\mathrm{Id}, C' \otimes C^{-1}) \cong \mathrm{End}(\mathbb{1}_B).$$

Thus any chain map in $\mathrm{Hom}(C, C')$ of non-zero homological degree is null-homotopic. Now we will show how to extend ϕ_0 to a chain map $\phi = \sum_{i \geq 0} \phi_i u^i$ between deformations where the homological degree of ϕ_i is $-2i$. Note that $\Delta = \Delta^+ + \Delta^- u$ and $\Delta' = \Delta'^+ + \Delta'^- u$.

[1] We thank Matt Hogancamp for pointing this out.

Assume that we have constructed ϕ_i for all $i < k$ and thus that for such i we may assume

$$\Delta' \phi_i = \phi_i \Delta,$$

which is equivalent to assuming that for all $i < k$ that

$$\Delta'^{+} \phi_i - \phi_i \Delta^{+} + \Delta'^{-} \phi_{i-1} - \phi_{i-1} \Delta^{-} = 0. \tag{4.3}$$

Then consider the element

$$\theta_k = \Delta'^{-} \phi_{k-1} - \phi_{k-1} \Delta^{-}.$$

We will now check that θ_k is a chain map between C and C'. Since the homological degree of θ_k is $1 - 2k$, and the homological degrees of Δ^{+} and Δ'^{+} are odd as well, we must verify $\Delta'^{+} \theta_k = -\theta_k \Delta^{+}$. We have

$$\Delta'^{+} \theta_k + \theta_k \Delta^{+} = \Delta'^{+} \Delta'^{-} \phi_{k-1} - \phi_{k-1} \Delta^{-} \Delta^{+} - \Delta'^{+} \phi_{k-1} \Delta^{-} + \Delta'^{-} \phi_{k-1} \Delta^{+}.$$

Using (4.3) on each of the last two terms in (4.1), we get

$$\begin{aligned} \Delta'^{+} \theta_k + \theta_k \Delta^{+} = {} & (\Delta'^{+} \Delta'^{-} \phi_{k-1} - \phi_{k-1} \Delta^{-} \Delta^{+}) \\ & + (\Delta'^{-} \Delta'^{+} \phi_{k-1} + \Delta'^{-} \Delta'^{-} \phi_{k-2} - \Delta'^{-} \phi_{k-2} \Delta^{-}) \\ & + (-\phi_{k-1} \Delta^{+} \Delta^{-} + \Delta'^{-} \phi_{k-2} \Delta^{-} - \phi_{k-2} \Delta^{-} \Delta^{-}). \end{aligned} \tag{4.4}$$

Using the fact that $\Delta^2 = \Delta'^2 = (z_B - z_A)$, we get that (4.4) is equal to

$$(z_B - z_A)\phi_{k-1} - \phi_{k-1}(z_B - z_A) = 0$$

since ϕ_{k-1} is comprised of $(\mathrm{End}(\mathbb{1}_B), \mathrm{End}(\mathbb{1}_A))$-bimodule homomorphisms.

Since the homological degree of θ_k is $1 - 2k$ and it is a chain map between invertible, equivalent complexes, by the above reasoning, θ_k must be null-homotopic. Thus there exists a map $h_k \colon C \to C'$ of homological degree $-2k$ such that

$$\theta_k = \Delta'^{+} h_k - h_k \Delta^{+}.$$

Now define $\phi_k = -h_k$. Then the part of $\Delta' \circ \phi - \phi \circ \Delta$ of homological degree $1 - 2k$ is

$$\begin{aligned} \Delta'^{+} \phi_k - \phi_k \Delta^{+} + \Delta'^{-} \phi_{k-1} - \phi_{k-1} \Delta^{-} &= \Delta'^{+} \phi_k - \phi_k \Delta^{+} + \theta_k \\ &= -(\Delta'^{+} h_k - h_k \Delta^{+}) + \theta_k = 0. \end{aligned}$$

Thus building ϕ in this way, we see that it is a chain map between deformations.

By Proposition 4.6, Cone(ϕ) is contractible so ϕ is a homotopy equivalence of deformations. □

4.2. Curved Rickard complexes. We take our 2-category $\mathcal{K}$ to be $\mathcal{U}_Q$ and consider the corresponding homotopy category of curved complexes $\widetilde{\mathsf{Kom}}(\mathcal{U}_Q)[u]$ where u is a formal indeterminate of bi-degree $[2]\langle -2 \rangle$.

Remark 4.8. In Section 4.1, u has degree just [2] but now, since our underlying 2-category has an extra grading, we need to impose an additional grading on u.

For a parameter $c \in \Bbbk$ we define $\tau_{i,c}\mathbb{1}_\lambda$ by

$$\left[\mathcal{E}_i^{(-\lambda_i)}\mathbb{1}_\lambda \underset{cud^-}{\overset{d^+}{\rightleftarrows}} \mathcal{E}_i^{(-\lambda_i+1)}\mathcal{F}_i^{(1)}\mathbb{1}_\lambda\langle 1\rangle \underset{cud^-}{\overset{d^+}{\rightleftarrows}} \cdots \underset{cud^-}{\overset{d^+}{\rightleftarrows}} \mathcal{E}_i^{(-\lambda_i+k)}\mathcal{F}_i^{(k)}\mathbb{1}_\lambda\langle k\rangle \underset{cud^-}{\overset{d^+}{\rightleftarrows}} \cdots\right]$$

if $\lambda_i \le 0$ and

$$\left[\mathcal{F}_i^{(\lambda_i)}\mathbb{1}_\lambda \underset{cud^-}{\overset{d^+}{\rightleftarrows}} \mathcal{E}_i^{(1)}\mathcal{F}_i^{(\lambda_i+1)}\mathbb{1}_\lambda\langle 1\rangle \underset{cud^-}{\overset{d^+}{\rightleftarrows}} \cdots \underset{cud^-}{\overset{d^+}{\rightleftarrows}} \mathcal{E}_i^{(k)}\mathcal{F}_i^{(\lambda_i+k)}\mathbb{1}_\lambda\langle k\rangle \underset{cud^-}{\overset{d^+}{\rightleftarrows}} \cdots\right]$$

if $\lambda_i \ge 0$ so that

$$\Delta = d^+ + cud^-.$$

In particular, taking $c = 0$ recovers the original Rickard complex. Note that for any value of c, $\tau_{i,c}\mathbb{1}_\lambda$ is a curved complex with connection $\Delta = d^+ + ucd^-$. To see this, choose any $j \in I$ with $(b_j, \alpha_i)_Q \neq 0$ and use (3.1) to conclude that

$$(d^+ + cd^-)^2 = c\left(d^+d^- + d^-d^+\right) = \frac{c}{(b_j,\alpha_i)_Q}\left(s_i(b_j)\cdot \mathrm{Id}_{\tau_i} - \mathrm{Id}_{\tau_i}\cdot b_j\right). \tag{4.5}$$

Similarly we define $\mathbb{1}_\lambda\tau'_{i,c}$ by

$$\left[\cdots \underset{cud^+}{\overset{d^-}{\rightleftarrows}} \mathbb{1}_\lambda\mathcal{E}_i^{(k)}\mathcal{F}_i^{(-\lambda_i+k)}\langle -k\rangle \underset{cud^+}{\overset{d^-}{\rightleftarrows}} \cdots \underset{cud^+}{\overset{d^-}{\rightleftarrows}} \mathbb{1}_\lambda\mathcal{E}_i^{(1)}\mathcal{F}_i^{(-\lambda_i+1)}\langle -1\rangle \underset{cud^+}{\overset{d^-}{\rightleftarrows}} \mathbb{1}_\lambda\mathcal{F}_i^{(-\lambda_i)}\right]$$

if $\lambda_i \le 0$ and

$$\left[\cdots \underset{cud^+}{\overset{d^-}{\rightleftarrows}} \mathbb{1}_\lambda\mathcal{E}_i^{(\lambda_i+k)}\mathcal{F}_i^{(k)}\langle -k\rangle \underset{cud^+}{\overset{d^-}{\rightleftarrows}} \cdots \underset{cud^+}{\overset{d^-}{\rightleftarrows}} \mathbb{1}_\lambda\mathcal{E}_i^{(\lambda_i+1)}\mathcal{F}_i^{(1)}\langle -1\rangle \underset{cud^+}{\overset{d^-}{\rightleftarrows}} \mathbb{1}_\lambda\mathcal{E}_i^{(\lambda_i)}\right]$$

if $\lambda_i \ge 0$ so that

$$\Delta = d^- + cud^+.$$

Proposition 4.9. *For any $b \in \mathrm{End}^2_{\mathcal{U}_Q(\mathfrak{g})}(\mathbb{1}_\lambda)$, $\tau_{i,(b,\alpha_i)_Q}\mathbb{1}_\lambda$ is an $(s_i(b), b)$-factorization and likewise $\mathbb{1}_\lambda\tau'_{i,(b,\alpha_i)_Q}$ is a $(b, s_i(b))$-factorization.*

Proof. This follows immediately from Theorem 3.3 and the subsequent calculation in (4.5). □

Remark 4.10. We will write $\tau_{i,(b,\alpha_i)_Q}\mathbb{1}_\lambda$ to implicitly mean the curved complex which is a $(s_i(b), b)$-factorization.

Remark 4.11. One can always specialize the formal variable u to a scalar. However, this requires one to identify $[1]\langle -1\rangle$ with the trivial shift (which has the effect of killing one of the gradings).

Lemma 4.12. *If $c \neq 0$, then*

$$\tau_{i,c}\mathbb{1}_\lambda \cong \tau'_{i,c}\mathbb{1}_\lambda$$

inside the localized category $\widetilde{\mathsf{Kom}}(\mathcal{U}_Q)[u^{\pm}]$.

Proof. Suppose $\lambda_i \geq 0$ (the case $\lambda_i \leq 0$ is similar). If we ignore the differentials, then

$$\tau_{i,c}\mathbb{1}_\lambda \cong \bigoplus_{k\geq 0} \mathcal{E}_i^{(k)}\mathcal{F}_i^{(\lambda_i+k)}\mathbb{1}_\lambda[-k]\langle k\rangle$$

and

$$\tau'_{i,c}\mathbb{1}_\lambda \cong \bigoplus_{k\geq 0} \mathcal{E}_i^{(k)}\mathcal{F}_i^{(\lambda_i+k)}\mathbb{1}_\lambda[k]\langle -k\rangle.$$

We define a map $\widetilde{\eta}_i : \tau_{i,c}\mathbb{1}_\lambda \to \tau'_{i,c}\mathbb{1}_\lambda$ by using, for $k \geq 0$, the maps

$$(cu)^k : \mathcal{E}_i^{(k)}\mathcal{F}_i^{(\lambda_i+k)}[-k]\langle k\rangle \to \mathcal{E}_i^{(k)}\mathcal{F}_i^{(\lambda_i+k)}[k]\langle -k\rangle.$$

It is not hard to check that this defines a map of curved complexes. Similarly we can define $\widetilde{\eta}'_i : \tau'_{i,c}\mathbb{1}_\lambda \to \tau_{i,c}\mathbb{1}_\lambda$ by using

$$(cu)^{-k} : \mathcal{E}_i^{(k)}\mathcal{F}_i^{(\lambda_i+k)}[k]\langle -k\rangle \to \mathcal{E}_i^{(k)}\mathcal{F}_i^{(\lambda_i+k)}[-k]\langle k\rangle.$$

This is also easily seen to be a map of curved complexes. Moreover, $\widetilde{\eta}_i$ and $\widetilde{\eta}'_i$ are clearly inverses of each other. □

Proposition 4.13. *If $\mathcal{K}$ is an integrable 2-representation of $\mathcal{U}_Q(\mathfrak{g})$, then inside $\widetilde{\mathsf{Kom}}(\mathcal{K})[u]$ the curved complexes $\tau_{i,c}$ and $\tau'_{i,c}$ satisfy the braid group relations of* $\mathrm{Br}_{\mathfrak{g}}$. *More precisely, for any $b \in \mathrm{End}^2(\mathbb{1}_\lambda)$, we have*

$$\begin{aligned}
&\tau'_{i,(s_i(b),\alpha_i)}\tau_{i,(b,\alpha_i)}\mathbb{1}_\lambda \cong \mathbb{1}_\lambda \cong \tau_{i,(s_i(b),\alpha_i)}\tau'_{i,(b,\alpha_i)}\mathbb{1}_\lambda \\
&\tau_{i,(s_j(b),\alpha_i)}\tau_{j,(b,\alpha_j)}\mathbb{1}_\lambda \cong \tau_{j,(s_i(b),\alpha_j)}\tau_{i,(b,\alpha_i)}\mathbb{1}_\lambda \qquad \text{if } \langle i,j\rangle = 0,\\
&\tau_{i,(s_js_i(b),\alpha_i)}\tau_{j,(s_i(b),\alpha_j)}\tau_{i,(b,\alpha_i)}\mathbb{1}_\lambda \cong \tau_{j,(s_is_j(b),\alpha_j)}\tau_{i,(s_j(b),\alpha_i)}\tau_{j,(b,\alpha_j)}\mathbb{1}_\lambda \quad \text{if } \langle i,j\rangle = -1,
\end{aligned}$$

where we suppress the subscript of Q from the pairing $(\cdot,\cdot)_Q$ for readability.

Proof. Proposition 2.9 gives homotopy equivalences of undeformed complexes for each braid relation. By Proposition 4.7 these homotopies extend uniquely to homotopies of the corresponding curved complexes. □

Finally, we have the following relatively straightforward result which we will use later.

Lemma 4.14. *For $b \in \mathrm{End}^2_{\mathcal{U}_Q(\mathfrak{g})}(\mathbb{1}_\lambda)$, the 1-morphisms $\mathcal{E}_i^{(k)}\mathbb{1}_\lambda$ and $\mathbb{1}_\lambda\mathcal{F}_i^{(k)}$ (thought of as complexes with only one term) are (b,b)-factorizations if $(b,\alpha_i)_Q = 0$.*

Proof. One must check that acting on the left by b is the same as acting on the right by b. The condition $(b,\alpha_i)_Q = 0$ guarantees this, see (3.3). □

5. Application: Deformed $\mathfrak{sl}_m$ homology

Since the 2-categories $\mathcal{U}_Q(\mathfrak{sl}_m)$ and $\mathcal{U}_{Q'}(\mathfrak{sl}_m)$ are isomorphic for all choice of scalars Q and Q' (see [26, Theorem 3.5]), throughout this section and the next we fix the choice of scalars from Remark 3.2 so that $(b_j,\alpha_i)_Q = (\alpha_j,\alpha_i)$.

5.1. Background. We briefly review the construction of $\mathfrak{sl}_m$ homology from [8]. The starting point is the 2-category $\mathcal{U}_Q$ where the Cartan data is that of $\mathfrak{sl}_{2N}$ for some large N. The N will depend on the link (the more complicated the presentation of the link the bigger the N) or we can consider the limit and take the more canonical choice $N = \infty$.

For any fixed $d \in \mathbb{Z}$ a weight $\lambda = (\lambda_1, \dots, \lambda_{2N-1})$ in $\mathcal{U}_Q(\mathfrak{sl}_{2N})$ corresponds to a sequences $\underline{k} = (k_1, \dots, k_{2N})$ of integers $k_i \in \mathbb{Z}$ determined by

$$\lambda_i = k_{i+1} - k_i, \quad \sum_i k_i = d$$

(when such a solution exists). We will use λ and $\underline{k}$ interchangeably.

For our purposes we set $d = mN$ and take $\mathcal{U}_Q^m$ to be the quotient category which kills any weight $\underline{k}$ where either $k_i < 0$ or $k_i > m$ for some i. This is equivalent to killing any weight which is zero in the $\mathfrak{sl}_{2N}$ irreducible representation $V_{m\Lambda_N}$ with highest weight $m\Lambda_N$.

In this notation, the roots correspond to

$$\alpha_i = (0, \dots, 0, -1, 1, 0, \dots, 0),$$

where the -1 occurs in position i and $m\Lambda_N = (\underline{0}, \underline{m}) = (0, \dots, 0, m, \dots, m)$ where there are a total of N 0's and N m's. The Weyl group, which can be identified with S_{2N}, then acts by permuting these sequences in the usual way. As usual we denote by $\dot{\mathcal{U}}_Q^m$ the Karoubi envelope of $\mathcal{U}_Q^m$. In this way $\dot{\mathcal{U}}_Q^m$ is an integrable 2-representation in the sense of Definition 2.8.

Lemma 5.1 ([8, Lemma 7.1]). *Suppose $\underline{k}$ and $\underline{k}'$ are non-zero weights which differ by only applying transpositions that involve 0 or m. Then $\mathbb{1}_{\underline{k}}$ and $\mathbb{1}_{\underline{k}'}$ in $\dot{\mathcal{U}}_Q^m$ are canonically isomorphic in the sense that if $\tau, \tau' : \mathbb{1}_{\underline{k}} \to \mathbb{1}_{\underline{k}'}$ are isomorphisms induced by a sequence of transpositions as above then $\tau \cong \tau'$.*

Proof. It suffices to assume $\underline{k} = \underline{k}'$ and to show that $\tau\mathbb{1}_{\underline{k}}$ is isomorphic to the identity for τ associated to a sequence of transpositions each of which involve 0 or m. Now, if k_i or k_{i+1} is in $\{0, m\}$, then $\tau_i\mathbb{1}_{\underline{k}}$ is either $\mathcal{E}_i^{(-\lambda_i)}1_{\underline{k}}$ if $\lambda_i \leq 0$ or $\mathcal{F}_i^{(\lambda_i)}\mathbb{1}_{\underline{k}}$ if $\lambda_i \geq 0$ and one immediately checks that $\tau_i^2\mathbb{1}_{\underline{k}} \cong \mathbb{1}_{\underline{k}}$. Moreover, if $k_i = k_{i+1} \in \{0, m\}$, then $\tau_i\mathbb{1}_{\underline{k}}$ is the identity. The result now follows since any pure braid element can be simplified to the identity using the braid relation and these two observations above. □

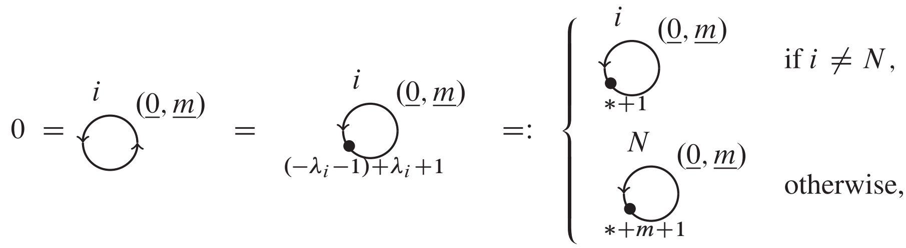
Lemma 5.2. *The space of* 1*-morphisms* $\mathrm{Hom}_{\dot{\mathcal{U}}_Q^m}(\mathbb{1}_{(\underline{0},\underline{m})}, \mathbb{1}_{(\underline{0},\underline{m})})$ *is spanned by direct sums of* $\mathbb{1}_{(\underline{0},\underline{m})}$ *(together with shifts). Moreover,*

$$\mathrm{End}^*_{\dot{\mathcal{U}}_Q^m}(\mathbb{1}_{(\underline{0},\underline{m})}) \cong \Bbbk[e_1, \dots, e_m],$$

where e_j is the degree $2j$ fake bubble labeled by N.

Proof. By [21, Lemma 6.15] any endomorphism of $\mathbb{1}_{(\underline{0},\underline{m})}$ can be expressed as a linear combination of products of non-nested dotted bubbles of the same orientation. Here we consider the counterclockwise dotted bubbles.

The first claim is a consequence of the PBW theorem and the fact that $(\underline{0}, \underline{m})$ is a highest weight in $\dot{\mathcal{U}}_Q^m$, which means that all $\mathcal{E}_i$ act by zero. From this it also follows that all counterclockwise bubbles labeled by $i \in I$ vanish in $\mathrm{End}^*_{\dot{\mathcal{U}}_Q^m}(\mathbb{1}_{(\underline{0},\underline{m})})$. Note that the i-labeled counterclockwise bubble with no dots has degree two if $i \neq N$ and has degree $2(m+1)$ if $i = N$. Hence, in the quotient $\dot{\mathcal{U}}_Q^m$, we have



so that all positive degree i labeled bubbles with $i \neq N$ vanish and all N-labeled bubbles of degree greater than $2m$ also vanish. The N-labeled counterclockwise bubbles of degrees 0 to $2m$ are all fake bubbles, so the vanishing of weights $\mathbb{1}_{\underline{k}} = 0$ if $k_i \notin [0, m]$ does not require these to be zero; linear independence of products of these fake bubbles follows from the equivariant 2-representation [21, Proof of Lemma 6.15]. □

Now consider an oriented link L whose components are colored by elements of $[0, m]$. A color k should be thought of as a labeling of the strand by the fundamental representation V_{Λ_k} of $\mathfrak{sl}_m$. We now explain how L induces a 1-morphism

$$\Psi(L) \in \mathrm{Hom}_{\mathsf{Kom}(\dot{\mathcal{U}}_Q^m)}(\mathbb{1}_{(\underline{0},\underline{m})}, \mathbb{1}_{(\underline{0},\underline{m})}).$$

To obtain this 1-morphism from L, we decompose the link into a composition of caps, cups and crossings as shown in Figure 1. At each level of this decomposition we can associate an object $\underline{k}$ of $\dot{\mathcal{U}}_Q^m$ where we add in 0's or m's if needed so that $\underline{k} \in \mathbb{Z}^{2N}$ with $\sum_i k_i = mN$. By Lemma 5.1 there is a canonical isomorphism between any two objects $\underline{k}$ associated to a given level of the decomposition of L. To a cap/cup we associate the 1-morphisms

$$\mathcal{E}_i^{(k_i)} : (\dots, k_i, m - k_i, \dots) \to (\dots, 0, m, \dots), \tag{5.1}$$

$$\mathcal{F}_i^{(k_i)} : (\dots, 0, m, \dots) \to (\dots, k_i, m - k_i, \dots), \tag{5.2}$$

where i denotes the position of the cap/cup. To the four over crossings in Figure 1 we associate maps $\tau_i \mathbb{1}_{\underline{k}}$, $\tau_i \mathbb{1}_{\underline{k}}[k_i]\langle -k_i\rangle$, $\tau_i \mathbb{1}_{\underline{k}}[m - k_{i+1}]\langle -m + k_{i+1}\rangle$ and $\tau_i \mathbb{1}_{\underline{k}}[-k_{i+1} + k_i]\langle k_{i+1} - k_i\rangle$ respectively. The corresponding four under crossings are the associated inverse maps.

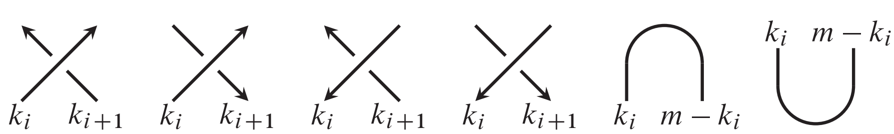


Figure 1. The cap and cup can have either orientation.

Composing these morphisms together gives us $\Psi(L)$. Note that, by Lemma 5.2, this is a complex with terms direct sums of $\mathbb{1}_{(\underline{0},\underline{m})}$. To obtain the link invariant $H^{*,*}_{\mathfrak{sl}_m}(L)$ we apply the functor $\mathrm{Hom}_{\dot{\mathcal{U}}_Q^m}(\mathbb{1}_{(\underline{0},\underline{m})}, \bullet)$. By Lemma 5.2 this has the effect of replacing each summand $\mathbb{1}_{(\underline{0},\underline{m})}$ with $\Bbbk$. For more details see [8, Section 7].

Remark 5.3. If we apply the functor $\mathrm{Hom}^*_{\dot{\mathcal{U}}_Q^m}(\mathbb{1}_{(\underline{0},\underline{m})}, \bullet)$ (i.e. the direct sum over all degrees) then by Lemma 5.2 this has the effect of replacing each $\mathbb{1}_{(\underline{0},\underline{m})}$ with $\Bbbk[e_1, \dots, e_m]$ with

e_j corresponding to the fake bubble of degree $2j$, see Lemma 5.2. The resulting homology is a deformation of $H^{*,*}_{\mathfrak{sl}_m}(L)$ over $\Bbbk[e_1, \dots, e_m]$, cf. [35, 40].

5.2. Deformations. To obtain the desired deformation of the construction above, we will label crossings with curved complexes. More precisely, given $\underline{y} = (y_1, \dots, y_{2N})$,[2)] we will denote by $b_{\underline{y}} \in \mathfrak{h}^*_{\Bbbk} \cong \mathrm{End}^2_{\dot{\mathcal{U}}_Q}(\mathbb{1}_{\underline{k}})$ the 2-morphism determined by the conditions

$$(b_{\underline{y}}, \alpha_i) = -y_i + y_{i+1}.$$

In this notation, we have

$$(b_{\underline{y}}, \underline{k}) = \underline{y} \cdot \underline{k}.$$

We now work in the quotient $\dot{\mathcal{U}}^m_Q$. We denote by $B_{\underline{y}}$ the image of $b_{\underline{y}}$ in the 2-representation $\dot{\mathcal{U}}^m_Q$. Given a link L we decompose it as before into cups, caps and crossings and associate to each the corresponding curved complexes. More precisely, to the first crossing in Figure 1 we associate $\tau_{i,-y_i+y_{i+1}}\mathbb{1}_{\underline{k}}$ which is a $(B_{s_i(\underline{y})}, B_{\underline{y}})$-factorization (and similarly for the other crossings). Whenever we see a cap/cup, we associate the curved complexes as in (5.1) and (5.2) *and* we impose the condition that $\underline{y} \cdot \alpha_i = 0$ (namely $y_i = y_{i+1}$, cf. Lemma 4.14).

Theorem 5.4. *Suppose $L = L_1 \cup \cdots \cup L_c$ is an oriented, colored link with c components. Then the construction above defines a family of link homologies $\widetilde{H}^{*,*}_{\mathfrak{sl}_m}(L)$ parameterized by $\underline{z} \in \Bbbk^c$. This homology has the usual splitting properties from Batson–Seed* [6]. *In particular, for distinct values of $\underline{z}$ the deformed homology of L is isomorphic to the homology of the disjoint union of its components, and moreover, there exists a spectral sequence starting with $H^{*,*}_{\mathfrak{sl}_m}(L)$ and converging to $\widetilde{H}^{*,*}_{\mathfrak{sl}_m}(L)$.*

Proof. Consider a presentation $[L]$ of the link L and a choice of $\underline{y}$. The construction above imposes a set of linear relations that the $\underline{y}$ should satisfy (one condition for each cap and cup). We say that $\underline{y}$ is compatible with $[L]$ if it satisfies all these linear conditions.

In this way, for any compatible $\underline{y}$, we obtain a 1-morphism

$$\Psi_{\underline{y}}(L) : \mathrm{Hom}_{\widetilde{\mathsf{Kom}}(\dot{\mathcal{U}}^m_Q)[u]}(\mathbb{1}_{(0,\underline{m})}, \mathbb{1}_{(0,\underline{m})}).$$

Since, by Lemma 5.2, $\mathrm{End}^2(\mathbb{1}_{(0,\underline{m})})$ is spanned by e_1, it follows that this curved complex is a $(ce_1, c'e_1)$-factorization for some $c, c' \in \Bbbk$. The fact that the description of L begins with a cup and ends with a cap means that we impose the condition that $(ce_1, e_1) = 0 = (c'e_1, e_1)$. Thus $c = c' = 0$ and $\Psi_{\underline{y}}(L)$ is an actual (non-curved) complex.

Next we need to understand why the set of $\underline{y}$ compatible with $[L]$ is indexed by $\Bbbk^c$. This is more easily realized if we present L as a composition of simple cups, a braid β and then simple caps. In the construction above this means a composition of the form

$$\mathbb{1}_{(0,m,0,m,\dots,0,m)}(\mathcal{E}_1^{(\ell_1)}\mathcal{E}_3^{(\ell_2)}\dots\mathcal{E}_{2N-1}^{(\ell_N)})\Psi(\beta)(\mathcal{F}_1^{(k_1)}\mathcal{F}_3^{(k_2)}\dots\mathcal{F}_{2N-1}^{(k_N)})\mathbb{1}_{(0,m,0,m,\dots,0,m)},$$

where $k_1, \dots, k_N$ and $\ell_1, \dots, \ell_N$ are the colors of the strands of β at the bottom and top respectively while $\Psi(\beta)$ is the curved complex associated to β (a composition of $\tau_{i,c}$'s).

[2)] Here our parameters y_i are deformation parameters rather than formal variables as in [17]

We start with a general deformation parameter $\underline{y} = (y_1, \dots, y_{2N})$. The cups impose linear relations

$$\underline{y} \cdot \alpha_1 = 0, \quad \underline{y} \cdot \alpha_3 = 0, \quad \dots, \quad \underline{y} \cdot \alpha_{2\ell-1} = 0$$

which are equivalent to

$$y_1 = y_2, \quad y_3 = y_4, \quad \dots, \quad y_{2\ell-1} = y_{2\ell}.$$

Now $\Psi(\beta)$ is a $(B_{\beta(\underline{y})}, B_{\underline{y}})$-factorization. This means that the caps impose the linear relations

$$\beta(\underline{y}) \cdot \alpha_1 = 0, \quad \beta(\underline{y}) \cdot \alpha_3 = 0, \quad \dots, \quad \beta(\underline{y}) \cdot \alpha_{2N-1} = 0.$$

It is easy to see from this description that $\Psi_{\underline{y}}(L)$ is a $(B_{\beta(\underline{y})}, B_{\underline{y}})$-factorization and that the linear relations on $\underline{y}$ cut out exactly a family indexed by the components of L.

Finally, the splitting properties of $\widetilde{H}^{*,*}_{\mathfrak{sl}_m}(L)$ as in Batson–Seed [6] are a consequence of Lemma 4.12. □

5.3. Clasps. The construction of $\widetilde{H}^{*,*}_{\mathfrak{sl}_m}(L)$ above is for links L colored by fundamental representations V_{Λ_k} of $\mathfrak{sl}_m$. One way to generalize this to arbitrary representations $V_{\sum_i \Lambda_{k_i}}$ is by cabling and using clasps (projectors) $\mathcal{P}$ corresponding to the composition

$$V_{\Lambda_{k_1}} \otimes \cdots \otimes V_{\Lambda_{k_s}} \to V_{\sum_i \Lambda_{k_i}} \to V_{\Lambda_{k_1}} \otimes \cdots \otimes V_{\Lambda_{k_s}},$$

where the first map is the natural projection while the second is the natural inclusion.

This approach was followed in [8, Theorems 2.2, 2.3] where $\mathcal{P}$ was defined as a limit $\lim_{\ell \to \infty} (\tau'_\omega)^{2\ell}$, where

$$\tau'_\omega = (\tau'_1 \dots \tau'_{s-1})(\tau'_1 \dots \tau'_{s-2}) \dots (\tau'_1 \tau'_2)(\tau'_1)$$

is a full twist. Here, for notational simplicity, we assume that the clasp $\mathcal{P}$ consists of cabling the first s strands (otherwise the indices $1, \dots, s-1$ would need to be shifted). Also note that our notation is such that τ'_i in the current paper corresponds to T_i in [8].

Proposition 5.5. *For any $B_{\underline{y}} \in \mathrm{End}^2_{\dot{\mathcal{U}}^m_Q}(\mathbb{1}_{\underline{k}})$ the clasp $\mathcal{P}\mathbb{1}_{\underline{k}} \in \mathsf{Kom}^-(\dot{\mathcal{U}}^m_Q)$ deforms to a $(B_{\underline{y}}, B_{\underline{y}})$-factorization*

$$\widetilde{\mathcal{P}} \in \widetilde{\mathsf{Kom}^-}(\dot{\mathcal{U}}^m_Q)[u].$$

Proof. We will prove that the corresponding limit $\lim_{\ell \to \infty} (\tau'_\omega)^{2\ell} \mathbb{1}_{\underline{k}}$ in $\widetilde{\mathsf{Kom}^-}(\dot{\mathcal{U}}^m_Q)[u]$ converges. The argument from [8] that this limit exists in $\mathsf{Kom}^-(\dot{\mathcal{U}}^m_Q)$ relies on a connecting map $\eta : \mathbb{1}_{\underline{k}} \to (\tau'_\omega)^2 \mathbb{1}_{\underline{k}}$ (cf. [8, Section 5.2]) such that

$$(\tau'_\omega)^{2\ell}\, \mathrm{Cone}(\eta) \mathbb{1}_{\underline{k}}$$

is isomorphic to a complex supported in cohomological degrees $< -d(\ell)$, where $d(\ell) \to \infty$ as $\ell \to \infty$. So we just need to deform this story.

The map η is constructed from simpler $\mathbb{1}_{\underline{k}} \to (\tau'_i)^2 \mathbb{1}_{\underline{k}}$ or, equivalently, from maps

$$\eta_i : \tau_i \mathbb{1}_{\underline{k}} \to \tau'_i \mathbb{1}_{\underline{k}}.$$

It turns out that this rather simple map is just the map $\widetilde{\eta}_i$ from Lemma 4.12, where we take $u = 0$. Thus we already have a natural deformation for η_i which leads to a natural deformation $\widetilde{\eta}$ of η.

Now consider the truncation of $(\tau'_\omega)^{2\ell}\,\mathrm{Cone}(\widetilde{\eta})\mathbb{1}_{\underline{k}}$ to degrees greater than or equal to $-d(\ell)$. When restricted to $u=0$, this is homotopic to zero. It follows by Proposition 4.6 that the same is true of the whole truncation $(\tau'_\omega)^{2\ell}\,\mathrm{Cone}(\widetilde{\eta})\mathbb{1}_{\underline{k}}$. This implies that $\lim_{\ell\to\infty}(\tau'_\omega)^{2\ell}$ converges in $\widetilde{\mathsf{Kom}}^-(\dot{\mathcal{U}}^m_Q)[u]$. □

The deformed clasps satisfy the usual properties of clasps by Proposition 4.6. Once we have this deformation $\widetilde{\mathcal{P}}$ the construction from [8] can be repeated to yield a deformed homology for links L labeled by arbitrary representations of $\mathfrak{sl}_m$.

Corollary 5.6. *Suppose $L = L_1 \cup \cdots \cup L_c$ is an oriented link whose components are labeled by $V_{\mu_1}, \dots, V_{\mu_c}$, where $\mu_i = \sum_j \Lambda_{k_{i,j}}$. Then the construction above yields a family of link homologies $\widetilde{H}^{*,*}_{\mathfrak{sl}_m}(L)$ parametrized by $\underline{z} = (z_{i,j})$. This homology satisfies the usual/expected splitting properties. In particular, if $c=1$ and $\mu = \Lambda_{k_{1,1}} + \Lambda_{k_{1,2}}$, then restricting to $\underline{z}$, where $z_{1,1} \neq z_{1,2}$ recovers the homology of L labeled by the (reducible) representation $V_{\Lambda_{k_{1,1}}} \otimes V_{\Lambda_{k_{1,2}}}$.*

5.4. Comparison with [12]. Recall that in [10] and subsequent papers an algebro-geometric definition of $\mathfrak{sl}_m$ link homology was developed. This construction was "deformed" in [12] to yield a deformed theory with the same properties as the link homologies $\widetilde{H}^{*,*}_{\mathfrak{sl}_m}(L)$ from Theorem 5.4. Without going into too many details we would like to compare the construction of these two deformations.

In [12] the role of $\dot{\mathcal{U}}^m_Q$ is played by a 2-representation $\mathcal{K}_{\mathrm{Gr},m}$. The objects in $\mathcal{K}_{\mathrm{Gr},m}$ are the derived categories of coherent sheaves on certain convolution varieties $Y(\underline{k})$ associated to the affine Grassmannian of type PGL_m. These categories are $\mathbb{Z}$-graded with the grading shift denoted $\{1\}$. The 2-functor $\Phi : \dot{\mathcal{U}}_Q \to \mathcal{K}_{\mathrm{Gr},m}$ takes $\langle 1\rangle \mapsto [1]\{-1\}$.

The source of the deformation in [12] are certain deformations $\mathbb{Y}(\underline{k}) \to \mathbb{A}^{2N}$ of the varieties $Y(\underline{k})$ (these deformations are very natural from the perspective of the Beilinson–Drinfeld Grassmannian). Here points of $\mathbb{A}^{2N}$ can be identified with sequences $\underline{y}$. In general, a geometric deformation gives us degree two classes in the Hochschild cohomology of the variety. More precisely, this class is the product of the Atiyah and Kodaira-Spencer classes (cf. [20]). In our case this yields a linear map

$$\mathbb{C}^{2N} \to \mathrm{End}^2_{\mathcal{K}_{\mathrm{Gr},m}}(\mathbb{1}_{\underline{k}}), \quad \underline{y} \mapsto B_{\underline{y}}. \tag{5.3}$$

Now consider the image of a complex $\tau_i \mathbb{1}_{\underline{k}} \in \dot{\mathcal{U}}_Q$ under the 2-functor Φ. This be identified with a kernel $\Phi(\tau_i \mathbb{1}_{\underline{k}})$ (a sheaf) living on $Y(s_i(\underline{k})) \times Y(\underline{k})$. The main (technical) result of [12] is that $\Phi(\tau_i \mathbb{1}_{\underline{k}})$ deforms along

$$\{(s_i(\underline{y}), \underline{y}) : \underline{y} \in \mathbb{A}^{2N}\} = \mathbb{A}^{2N} \subset \mathbb{A}^{2N} \times \mathbb{A}^{2N}.$$

The proof requires a more detailed geometric understanding of $\Phi(\tau_i \mathbb{1}_{\underline{k}})$. One should compare this result with Proposition 4.9 which states that $\tau_i \mathbb{1}_{\underline{k}}$ deforms along

$$\{(s_i(b), b) : b \in \mathrm{End}^2_{\dot{\mathcal{U}}^m_Q}(\mathbb{1}_{\underline{k}})\} \subset \mathrm{End}^2_{\dot{\mathcal{U}}^m_Q}(\mathbb{1}_{s_i(\underline{k})}) \times \mathrm{End}^2_{\dot{\mathcal{U}}^m_Q}(\mathbb{1}_{\underline{k}}).$$

The geometric deformation of $\Phi(\tau_i \mathbb{1}_{\underline{k}})$ has the property that its fiber over any $(s_i(\underline{y}), \underline{y})$ with $y_i \neq y_{i+1}$ is the graph of an isomorphism

$$\mathbb{Y}(\underline{k})|_{\underline{y}} \xrightarrow{\sim} \mathbb{Y}(s_i(\underline{k}))|_{s_i(\underline{y})}.$$

This isomorphism is actually an involution (both these facts are easy to see). Hence, under the hypothesis that $y_i \neq y_{i+1}$, this deformation is isomorphic to its inverse. This result should be compared to Lemma 4.12.

Another remark involves the formal indeterminate u. Recall that u in $\dot{\mathcal{U}}_Q$ has bi-degree $[2]\langle -2\rangle$. Since the 2-functor Φ sends $\langle 1\rangle \mapsto [1]\{-1\}$ and $[1] \mapsto [1]$, it follows that the image of u has bi-degree $\{2\}$ in $\mathcal{K}_{\mathrm{Gr},m}$. This agrees with the fact that the parameter space $\mathbb{A}^{2N}$ of the deformation $\mathbb{Y}(\underline{k})$ is equipped with a dilating $\mathbb{C}^\times$ action of weight $\{2\}$.

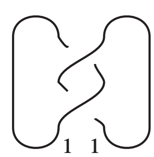
In the undeformed case a procedure for comparing link homologies constructed via skew Howe duality was introduced in [8, Section 7.5]. The rough idea is that any two link homologies constructed from a categorical action on a 2-category $\mathcal{K}$ whose (non-zero) weight spaces can be identified with those of $\bigwedge_q^{m\infty}(\mathbb{C}^m \otimes \mathbb{C}^{2\infty})$ are automatically isomorphic. This can be used to show that the (underformed) link homologies defined using the affine Grassmannian [8] must agree with (for example) the link homology $H^{*,*}_{\mathfrak{sl}_m}(L)$ defined in Section 5.1 using the 2-category $\dot{\mathcal{U}}^m_Q$.

We expect the same is true in the deformed setting – namely, that the deformed homology from [12] is isomorphic to $\widetilde{H}^{*,*}_{\mathfrak{sl}_m}(L)$. To prove this given the results mentioned above it remains to identify the deformation in [12] and, more precisely, the induced map (5.3) with the map

$$\mathbb{C}^{2N} \to \mathrm{End}^2_{\dot{\mathcal{U}}^m_Q}(\mathbb{1}_{\underline{k}}), \quad \underline{y} \mapsto B_{\underline{y}}$$

from Section 5.2. Although this is not terribly difficult, it is a bit technical and beyond the scope of the current paper.

5.5. Example. Consider the Hopf link L where both components are labeled by the standard representation V_{Λ_1} of $\mathfrak{sl}_m$. Label the components by the deformation parameters y_1 and y_2:

$L =$ 

.

Following the calculation in [8, Section 10.3], we get that the complex for the Hopf link is homotopic to

$$\bigoplus_{[m]}\bigoplus_{[m-1]} \Bbbk\langle -3\rangle \underset{cu\mathrm{Id}}{\overset{0}{\rightleftarrows}} \bigoplus_{[m]}\bigoplus_{[m-1]} \Bbbk\langle -1\rangle \xrightarrow{f} \bigoplus_{[m]}\bigoplus_{[m]} \Bbbk,$$

where f is an injective map and $c = y_1 - y_2$.

Thus, if $c = 0$, we get the following for the $\mathfrak{sl}_m$ homology:

$$\bigoplus_j \widetilde{H}^{i,j}_{\mathfrak{sl}_m}(L) = \begin{cases} \bigoplus_{[m]} \Bbbk\langle m-1\rangle & \text{if } i = 0, \\ \bigoplus_{[m][m-1]} \Bbbk\langle -3\rangle & \text{if } i = -2, \\ 0 & \text{otherwise.} \end{cases}$$

If $c \neq 0$, then the homology has rank m^2. This is the rank of the homology of the unlink with two components which is what we expect from Lemma 4.12. Compare these results with the analogous HOMFLYPT homology computation in [17, Example 3.7].

6. Application: Deformed colored HOMFLYPT homology

In this section we continue to work with the choice of scalars Q from Remark 3.2.

6.1. Background. We now explain an analogue of Section 5 for colored HOMFLYPT homology. Let us suppose that we have a link L which is presented as the closure of a braid β on N strands. We color the components of this link with arbitrary non-negative integers so that the bottom and top of β is colored $\underline{k} = (k_1, \dots, k_N)$. We denote $d := \sum_i k_i$.

The starting point of our construction is the 2-category $\mathcal{U}_Q$ where the Cartan data is that of $\mathfrak{sl}_N$ and the choice of scalars Q has been fixed as in Remark 3.2. As in Section 5, we identify the weights of $\mathcal{U}_Q$ with sequences $\underline{k}$ of integers such that $\sum_i k_i = d$. We will now define an integrable 2-representation $\mathcal{K}_N^d$ of $\mathcal{U}_Q$.

First we fix some notation following [9, Section 3.1]. For $k \geq 0$ denote

$$A_k := \Bbbk[x_1, \dots, x_k]^{S_k} = \Bbbk[\varepsilon_1, \dots, \varepsilon_k],$$

where the symmetric group S_k acts naturally and where the ε_i are elementary symmetric functions. These algebras are naturally $\mathbb{Z}$-graded with x_i having degree 2. The convention we use is that multiplication by x_i induces a map $x_i : A_k \to A_k\{2\}$, where $\{\cdot\}$ denotes a shift in grading.

For a sequence $\underline{k}$ we define

$$A_{\underline{k}} := A_{k_1} \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} A_{k_N}.$$

We will denote the element $1 \otimes \cdots \otimes \varepsilon_i \otimes \cdots \otimes 1$ by $\varepsilon_i^{(j)}$ where the ε_i occurs in the jth factor of $A_{\underline{k}}$.

The (non-zero) objects in $\mathcal{K}_N^d$ are indexed by sequences $\underline{k}$, where all k_i are non-negative and $\sum_i k_i = d$. The 1-morphisms consist of $(A_{\underline{k}}, A_{\underline{k}'})$-bimodules which are flat as $A_{\underline{k}}$ and also $A_{\underline{k}'}$-modules. Composition of these 1-morphisms is tensor product. The 2-morphisms are then morphisms of bimodules.

Notice that we have natural inclusion maps

$$A_{(\dots,k_i+k_{i+1},\dots)} \to A_{(\dots,k_i,k_{i+1},\dots)}.$$

Subsequently, the algebra $A_{(\dots,k_i-1,1,k_{i+1},\dots)}$ is both an $A_{\underline{k}}$-module as well as an $A_{\underline{k}+\alpha_i}$-module.

Proposition 6.1. *There exists a 2-functor* $\Gamma_N^d : \mathcal{U}_Q \to \mathcal{K}_N^d$ *which sends* $\langle 1 \rangle$ *to* $\{1\}$ *and*

$$\begin{aligned} \mathcal{E}_i \mathbb{1}_{\underline{k}} &\mapsto A_{(\dots,k_i-1,1,k_{i+1},\dots)}\{k_i - 1\}, \\ \mathbb{1}_{\underline{k}} \mathcal{F}_i &\mapsto A_{(\dots,k_i-1,1,k_{i+1},\dots)}\{k_{i+1}\}, \\ \mathrm{End}^2_{\mathcal{U}_Q}(\mathbb{1}_{\underline{k}}) \ni b_j &\mapsto -\varepsilon_1^{(j)} + \varepsilon_1^{(j+1)} \in \mathrm{End}^2_{\mathcal{K}_N^d}(\mathbb{1}_{\underline{k}}). \end{aligned}$$

Proof. This 2-functor is (up to a grading shift) the equivariant flag 2-representation [21, Section 6.3.3], see also [25] and [29, Section 4.3]. The image of bubbles in this 2-representation are given in [21, equation (6.47)] or [29, equation (4.12)]. □

Next let us define the following shifted complexes:

$$\widehat{\tau}_i \mathbb{1}_{\underline{k}} := \begin{cases} \tau_i \mathbb{1}_{\underline{k}}[-k_{i+1}]\{k_{i+1} + k_i k_{i+1}\} & \text{if } \langle \underline{k}, \alpha_i \rangle \leq 0, \\ \tau_i \mathbb{1}_{\underline{k}}[-k_i]\{k_i + k_i k_{i+1}\} & \text{if } \langle \underline{k}, \alpha_i \rangle \geq 0. \end{cases}$$

Using these shifted complexes is not crucial but it does simplify the grading/cohomological shifts in the long run and it does agree with the notation/definition from [9].

Given a braid β, we denote by $\widehat{\tau}_\beta$ the corresponding composition of $\widehat{\tau}_i$. Then the 2-functor Γ_N^d from Proposition 6.1 gives us a complex of $(A_{\underline{k}}, A_{\underline{k}})$-bimodules $\Gamma_N^d(\widehat{\tau}_\beta)$. Taking Hochschild cohomology of this complex of bimodules defines a triply graded homology $HH(L)$ (see [9, Theorem 4.1]).

6.2. Deformations. To obtain the desired deformation of the construction above, we will label crossings with curved complexes. Notice that we have

$$\mathrm{End}^*_{\mathcal{K}_N^d}(\mathbb{1}_{\underline{k}}) = \mathrm{Hom}^*_{(A_{\underline{k}}, A_{\underline{k}})}(A_{\underline{k}}, A_{\underline{k}}) = A_{\underline{k}}.$$

Given $\underline{y} = (y_1, \dots, y_N)$, we denote

$$B_{\underline{y}} := \sum_i y_i \varepsilon_1^{(i)} \in \mathrm{End}^*_{\mathcal{K}_N^d}(\mathbb{1}_{\underline{k}}).$$

Lemma 6.2. *The complex $\Gamma_N^d(\tau_{i,-y_i+y_{i+1}}\mathbb{1}_{\underline{k}})$ is a $(B_{s_i(\underline{y})}, B_{\underline{y}})$-factorization inside $\widetilde{\mathsf{Kom}}(\mathcal{K}_N^d)[u]$.*

Proof. As in Section 5.2, we define $b_{\underline{y}} \in \mathrm{End}^2_{\dot{\mathcal{U}}_Q}(\mathbb{1}_{\underline{k}})$ as the linear combination of bubbles determined by the relations $(b_{\underline{y}}, \alpha_i) = -y_i + y_{i+1}$. Then we know that $\tau_{i,-y_i+y_{i+1}}\mathbb{1}_{\underline{k}}$ is a $(b_{s_i(\underline{y})}, b_{\underline{y}})$-factorization.

On the other hand, the third relation in Proposition 6.1 implies that $\Gamma_N^d(b_{\underline{y}}) = B_{\underline{y}}$ if $\sum_i y_i = 0$. It follows that $\Gamma_N^d(\tau_{i,-y_i+y_{i+1}}\mathbb{1}_{\underline{k}})$ is a $(B_{s_i(\underline{y})}, B_{\underline{y}})$-factorization if $\sum_i y_i = 0$.

More generally, we can write any $\underline{y}$ as $\underline{y}' + (c, \dots, c)$, where $\sum_i y_i' = 0$. Then $B_{\underline{y}} = B_{\underline{y}'}$ and so, by the above, it remains to show that

$$\begin{aligned}
&\mathrm{Id}_{\mathbb{1}_{s_i(\underline{k})}} \mathrm{Id}_{\Gamma_N^d(\tau_{i,-y_i+y_{i+1}})} B_{(c,\dots,c)} \\
&\quad = B_{(c,\dots,c)} \mathrm{Id}_{\Gamma_N^d(\tau_{i,-y_i+y_{i+1}})} \mathrm{Id}_{\mathbb{1}_{\underline{k}}} \in \mathrm{End}^2_{\mathcal{K}_N^d}(\mathbb{1}_{s_i(\underline{k})}\Gamma_N^d(\tau_{i,-y_i+y_{i+1}})\mathbb{1}_{\underline{k}}).
\end{aligned}$$

But this is clear since it is not difficult to check that $B_{(c,\dots,c)}$ commutes with all $\mathcal{E}_i$ and $\mathcal{F}_i$ in $\mathcal{K}_N^d$. □

Repeating the construction from Section 6.1, we start with a braid β and apply Lemma 6.2 repeatedly to obtain a $(B_{\beta(\underline{y})}, B_{\underline{y}})$-factorization $\Gamma_N^d(\tau_\beta')$. To finish, we apply the functor $\mathrm{Hom}^*_{(A_{\underline{k}}, A_{\underline{k}})}(A_{\underline{k}}, \cdot)$. Since elements of $\mathrm{End}^2_{(A_{\underline{k}}, A_{\underline{k}})}(A_{\underline{k}})$ (such as $B_{\underline{y}}$) commute with any 1-morphism of $(A_{\underline{k}}, A_{\underline{k}})$-bimodules, it follows that the resulting curved complex has curvature $B_{\beta(\underline{y})} - B_{\underline{y}}$. Thus, if we choose $\underline{y}$ so that $\underline{y} = \beta(\underline{y})$, then we get an actual complex (with zero curvature). We denote the resulting triply graded vector space $\widetilde{HH}(L)$.

Theorem 6.3. *Suppose $L = L_1 \cup \dots \cup L_c$ is an oriented, colored link with c components. Then the construction above defines a family of link homologies $\widetilde{HH}(L)$ parametrized by $\underline{z} \in \Bbbk^c$. This homology has the same splitting properties as $\widetilde{H}^{*,*}_{\mathfrak{sl}_m}(L)$ from Theorem* 5.4. *In particular, for distinct values of $\underline{z}$, the deformed homology of L is isomorphic to the homology of the disjoint union of its components, and moreover, there exists a spectral sequence starting with $HH(L)$ and converging to $\widetilde{HH}(L)$.*

Proof. We just need to verify that $\widetilde{HH}(L)$ is indeed a link invariant (the rest follows from Lemma 4.12 as it does in the proof of Theorem 5.4). The fact that it is a link invariant can be verified as in the proof of [8, Theorem 4.1] where the analogous result is proved for the undeformed complex $HH(L)$. □

6.3. Clasps. In [9] clasps were constructed (just like in [8]) as a limit of twists. This allowed one to define a HOMFLYPT homology for links colored by arbitrary partitions. The discussion from Section 5.3 now repeats word for word to give us a deformation of these clasps in the context of HOMFLYPT homology. This leads to the following analogue of Corollary 5.6.

Corollary 6.4. *Suppose $L = L_1 \cup \cdots \cup L_c$ is an oriented link whose components are colored by partitions $(\mu_1, \dots, \mu_c)$, where $\mu_i = (\mu_i^{(1)} \geq \cdots \geq \mu_i^{(s_i)})$ is the decomposition into parts. Then the constructions above yield a family of link homologies $\widetilde{HH}(L)$ parametrized by $\underline{z} = (z_{i,1}, \dots, z_{i,s_i})$. This homology satisfies the usual/expected splitting properties. In particular, if $c = 1$ and $\mu_1 = (\mu_1^{(1)} \geq \mu_1^{(2)})$, then restricting to $\underline{z}$ where $z_{1,1} \neq z_{1,2}$ recovers the homology of two unlinked copies of L colored by the partitions $\mu_1^{(1)}$ and $\mu_2^{(2)}$.*

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Eingegangen 22. März 2019, in revidierter Fassung 15. November 2019