DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **39**, Number **12**, December **2019**

doi:10.3934/dcds.2019296

pp. **7081–7099**

THE STRAUSS CONJECTURE ON NEGATIVELY CURVED BACKGROUNDS

Yannick Sire and Christopher D. Sogge

Department of Mathematics
Johns Hopkins University
Baltimore, MD 21218, USA

Chengbo Wang*

School of Mathematical Sciences
Zhejiang University
Hangzhou 310027, China

To Luis Caffarelli on the occasion of his 70th birthday, with admiration and friendship

Abstract. This paper is devoted to several small data existence results for semi-linear wave equations on negatively curved Riemannian manifolds. We provide a simple and geometric proof of small data global existence for any power $p \in (1, 1+\frac{4}{n-1}]$ for the shifted wave equation on hyperbolic space $\mathbb{H}^n$ involving nonlinearities of the form $\pm|u|^p$ or $\pm|u|^{p-1}u$. It is based on the weighted Strichartz estimates of Georgiev-Lindblad-Sogge [9] (or Tataru [29]) on Euclidean space. We also prove a small data existence theorem for variably curved backgrounds which extends earlier ones for the constant curvature case of Anker and Pierfelice [1] and Metcalfe and Taylor [22]. We also discuss the role of curvature and state a couple of open problems. Finally, in an appendix, we give an alternate proof of dispersive estimates of Tataru [29] for $\mathbb{H}^3$ and settle a dispute, in his favor, raised in [21] about his proof. Our proof is slightly more self-contained than the one in [29] since it does not make use of heavy spherical analysis on hyperbolic space such as the Harish-Chandra c-function; instead it relies only on simple facts about Bessel potentials.

1. Introduction.

As is well-known, wave equations on hyperbolic space $\mathbb{H}^n$, $n \geq 2$, are closely related with wave equations on $\mathbb{R}^n \times \mathbb{R}$, see, e.g., Tataru [29]. This paper is devoted to several results concerning global well-posedness for small data on negatively curved Riemannian manifolds. It is well-known fact that small data existence for nonlinear wave equations with power-like nonlinearities is related to the so-called *Strauss conjecture* in $\mathbb{R}^n$. This paper is three-fold: we provide first a general small data existence result for some range of p's based on the use of Hamiltonian identities and avoiding the heavy machinery of Strichartz estimates; second, we provide a geometric alternative proof of the optimal global existence on

2010 *Mathematics Subject Classification.* 35L71, 35L05, 58J45, 35B33, 35B45, 35R01, 58C40.
Key words and phrases. Wave equations, curvature, Strauss conjecture, Strichartz estimates, weighted Strichartz estimates, Bessel potentials.
The second author was supported in part by NSF Grant DMS-1665373 and the first two authors were supported in part by a Simons Fellowship. The third author was supported in part by NSFC 11971428 and National Support Program for Young Top-Notch Talents.
* Corresponding author.

hyperbolic spaces for smooth data. Finally, we provide the same result for rough data, settling in the case $n=3$ a dispute concerning dispersive estimates by Tataru by providing an alternate argument, which is based on almost the same analytic interpolation scheme.

Let $n\ge 2$ and (M,g) be an n-dimensional complete Riemannian manifold, and let Δ_g be the standard Laplace-Beltrami operator on M. The problems under consideration are of the following form for a scalar unknown function $u:\mathbb{R}\times M\to\mathbb{R}$,

$$\begin{cases} (\partial_t^2-\Delta_g+k)u=F_p(u), & t>0, x\in M\\ u(0,x)=\varepsilon u_0(x), \partial_t u(0,x)=\varepsilon u_1(x), & (u_0,u_1)\in C_0^\infty(M), \end{cases} \tag{1}$$

where k is a constant such that $-\Delta_g+k\ge 0$, $p>1$, and $F_p\in C^1$ behaves like $\pm|u|^p$ or $\pm|u|^{p-1}u$, by which we mean that

$$|F_p(u)|+|u||F_p'(u)|\le C|u|^p, \tag{2}$$

for some constant $C>0$. When M is non-compact, $C_0^\infty(M)$ denotes the space of smooth functions with compact support. Otherwise $C_0^\infty(M)=C^\infty(M)$. For these Cauchy problems, the task is then to determine the range of p such that we have the following small data global existence: for any given data (u_0,u_1), there exists an $\varepsilon_0>0$ such that there is a global solution to (1) for any $\varepsilon\in(0,\varepsilon_0]$. We define the critical power $p_c(n)$ as the infimum of $p>1$ such that there is small data global existence.

These problems are of course closely related to the so called Strauss conjecture, when (M,g) is the Euclidean space and $k=0$, with $F_p(u)=|u|^p$. The first work in this direction is [14], where John determined the critical power, $1+\sqrt{2}$, for the problem when $n=3$, by proving global existence results for $p>1+\sqrt{2}$ and blow-up results for $p<1+\sqrt{2}$. It was known from Kato [15] that there is no small data global solution in general, for $n=1$ or $1<p<1+2/(n-1)$. Shortly afterward, Strauss [27] conjectured that the critical power $p_S(n)$ for other dimensions $n\ge 2$ should be the positive root of the quadratic equation

$$(n-1)p^2-(n+1)p-2=0.$$

The existence portion of the conjecture was verified in Glassey [10] ($n=2$), Zhou [34] ($n=4$), Lindblad-Sogge [17] ($n\le 8$), and Georgiev-Lindblad-Sogge [9], Tataru [29] (all n, $p_S<p\le p_{\rm conf}$), where

$$p_{\rm conf}(n)=1+\frac{4}{n-1}$$

is the conformal power. The necessity of $p>p_S(n)$ for small data global existence is due to Glassey [11] ($1<p<p_S(2)$), Sideris [25] ($1<p<p_S(n)$, $n\ge 4$), Schaeffer [23] ($p=p_S(n)$, $n=2,3$), and Yordanov-Zhang [33], Zhou [35] ($p=p_S(n)$, $n\ge 4$). See Wang-Yu [32] or Wang [31] for more references.

Another model of particular interest is the so-called Klein-Gordon equation on $\mathbb{R}^n$, with $k=m^2>0$:

$$(\partial_t^2-\Delta+m^2)u=|u|^p. \tag{3}$$

In view of the decay rate for the solutions to the homogeneous Klein-Gordon equation, it is natural to expect that the critical power to be given by the

$$p_F(n)=1+\frac{2}{n},$$

which is also known as the Fujita's exponent for the heat equation. Lindblad-Sogge [18] proved small data global existence for any $p > p_F(n) = 1 + 2/n$ with $n = 1, 2, 3$. Moreover, Keel-Tao [16] provided an example

$$(\partial_t^2 - \Delta + m^2)u = F(u_t) - |u|^{p-1}u,$$

for which they showed that there is no global solutions for any $1 < p \leq p_F(n)$. Here $F(v) \sim |v|^{p-1}v$ for $|v| \leq 1$, $F(v) \sim |v|^{q-1}v$ with some $1 < q < p$ for $|v| \geq 1$.

However, p_F is not the critical power for (3). Actually, it is known that it admits global energy solutions (with small energy data) for any energy subcritical powers, that is $p \in (1, 1 + 4/(n-2))$. See, e.g., Keel-Tao [16, pp. 631-632].

Real hyperbolic spaces, $(\mathbb{H}^n, h)$, are the first examples of rank 1 symmetric spaces of non-compact type and their spherical analysis is to a certain extent very parallel to the one in the Euclidean case. The problem (1) on hyperbolic spaces with $k = 0$ is

$$(\partial_t^2 - \Delta_h)u = |u|^p, \tag{4}$$

was first considered by Metcalfe and Taylor in [21], where they proved small data global existence for $p \in [5/3, 3]$ for dimension $n = 3$, by proving improved dispersive and Strichartz estimates. Then Anker and Pierfelice [1] proved global existence for the problem (1) on hyperbolic spaces with $k > -\rho^2$,

$$(\partial_t^2 - \Delta_h + k)u = |u|^p, \tag{5}$$

for any $p \in (1, p_{\text{conf}}]$ and $n \geq 2$, where $\rho = (n-1)/2$. Soon after, Metcalfe and Taylor [22] gave an alternative proof for $n = 3$ with $k = 0$. This shows that the critical power for this problem is actually $p_c = 1$.

Recall that the spectrum of $-\Delta_h$ is $[\rho^2, \infty)$. See, e.g., McKean [20]. This means that the equation (5) is more like a nonlinear Klein-Gordon equation instead of a nonlinear wave equation. Thus, at least heuristically, it is not so surprising that we have small data global existence for any $p > 1$ (with a certain upper bound on p for technical reasons). Actually, in the following general theorem, we prove that there is small data global existence for any $1 < p < 1 + 2/(n-2)$ (which is understood to be $p \in (1, \infty)$ for $n = 2$).

Theorem 1.1. *Let (M, g) be a smooth, complete Riemannian manifold of dimension $n \geq 2$ with Ricci curvature bounded from below and $\inf_{x\in M} \mathrm{Vol}_g(B(x)) > 0$, where $\mathrm{Vol}_g(B(x))$ denotes the volume of the geodesic ball of center x and radius 1 with respect to g. Assume that k is a constant such that $\mathrm{Spec}(-\Delta_g + k) \subset (c, \infty)$ for some $c > 0$. Then for any $p \in (1, 1 + 2/(n-2))$, there exists a constant $\varepsilon_0 > 0$ such that (1) with $\varepsilon \in (0, \varepsilon_0]$ admits global solutions, provided that the data (u_0, u_1) satisfy*

$$\|\sqrt{k - \Delta_g} u_0\|_{L^2(M)}^2 + \|u_1\|_{L^2(M)}^2 \leq 1. \tag{6}$$

Our proof is elementary and completely avoids the somewhat delicate dispersive and Strichartz estimates used in the aforementioned earlier works. We first note that it is easy to prove local well-posedness in $C_t H^1 \cap C_t^1 L^2$ for $p \in (1, 1+2/(n-2))$ by classical energy arguments. Then the basic observation is that the problem (1) is Hamiltonian and, for these types of "Klein-Gordon equations", the nonlinear part can be easily controlled by the linear part. Such arguments are also classical (see, e.g., Cazenave[6], Keel-Tao [16]). We remark also that the assumptions on Ricci curvature and $\mathrm{Vol}_g(B(x))$ are made to ensure the Sobolev estimates

$$\|f\|_{L^q(M)} \lesssim \|\sqrt{k - \Delta_g} f\|_{L^2(M)}, \quad 2 \leq q \leq 2n/(n-2), \tag{7}$$

where it is understood that $q \in [2, \infty)$ when $n = 2$.

As a simple application, we see that Theorem 1.1 applies for any manifold (M, g), with $k > 0$, since $-\Delta_g$ is nonnegative. The condition on k is sharp in general, as we have seen that the critical power $p_c = p_S(n) > 1$ for the Strauss conjecture on $\mathbb{R}^n$ $(k = 0)$. The worst situation occurs for compact manifolds, for which it is easy to see that, generically, there is no small data global existence results for (1) with $k = 0$ for any $p > 1$. Actually, the simplest examples for this are (1) with $F_p(u) = \pm|u|^p$ or $|u|^{p-1}u$ and constant data, which reduces to the ODE $u_{tt} = F_p(u)$ which has the property that generic solutions blow up in finite time. In particular, there is no small data global existence for (1) with $k = 0$ and $F_p(u) = \pm|u|^p$ or $|u|^{p-1}u$, for any complete Riemannian manifolds (M, g) with positive lower bound on the Ricci curvature, which is compact due to the Bonnet-Myers theorem (see e.g. [7, p. 84]).

An important class of manifolds with the property $\mathrm{Spec}(-\Delta_g) \subset (0, \infty)$ is a simply connected, complete, Riemannian n-manifold with sectional curvature $K \le -\kappa$ for some constant $\kappa > 0$, for which it is known that $\mathrm{Spec}(-\Delta_g) \subset [\rho^2\kappa, \infty)$, see [20]. Recall also that a lower bound of sectional curvature implies that for the Ricci curvature and that an upper bound ensures that $\mathrm{Vol}_g(B(x)) > \delta > 0$ for some $\delta > 0$ by the Günther comparison theorem (see e.g. [7, p. 129]). Consequently, Theorem 1.1 yields the following result for simply connected complete manifolds with negatively pinched curvature:

Corollary 1. *Let (M, g) be a simply connected, complete, Riemannian manifold of dimension $n \ge 2$ with sectional curvature $K \in [-\kappa_2, -\kappa_1]$ for some $\kappa_2 \ge \kappa_1 > 0$. Then for any $k > -\rho^2\kappa_1$ and $p \in (1, 1 + 2/(n-2))$, there exists a constant $\varepsilon_0 > 0$ such that the problem*

$$(\partial_t^2 - \Delta_g + k)u = F_p(u), \ u(0) = \varepsilon u_0, \ u_t(0) = \varepsilon u_1 \tag{8}$$

with $\varepsilon \in (0, \varepsilon_0]$ admits global solutions, provided that the data (u_0, u_1) satisfy (6).

We remark that this Corollary could be strengthened a bit by using, say, the results in [24] and [5] which involve slightly weaker curvature assumptions that also ensure that $\mathrm{Spec}\ (-\Delta_g + k) \subset (c, \infty)$, some $c > 0$.

To state another corollary recall that (M, g) is said to be asymptotically hyperbolic, in the sense of Mazzeo-Melrose [19], if there is a compact Riemannian manifold with boundary $(X, \tilde{g})$, such that M could be realized as the interior of X, with metric $g = f^{-2}\tilde{g}$, where f is a smooth boundary defining function [1] with $\|df\|_{\tilde{g}} = 1$ on ∂X. It is known[2] that

$$\mathrm{Spec}(-\Delta_g) = [\rho^2, \infty) \cup \sigma_{pp}, \ \ \sigma_{pp} \subset (0, \rho^2),$$

where the pure point spectrum, σ_{pp} (the set of L^2 eigenvalues), is finite. See, e.g., Graham and Zworski [12, page 95-96]. In particular, we see that $\mathrm{Spec}(-\Delta_g) \subset (c, \infty)$ for some $c > 0$ and so Theorem 1.1 applies with $k = 0$ in this setting. Consequently we have the following:

Corollary 2. *Let (M, g) be an n-dimensional asymptotically hyperbolic manifold. Then the problem*

$$(\partial_t^2 - \Delta_g)u = F_p(u), \ u(0) = \varepsilon u_0, \ u_t(0) = \varepsilon u_1 \tag{9}$$

[1] Here $f \ge 0$ on X, $\partial X = f^{-1}(0)$, and $df \ne 0$ on ∂X.

[2] The third author would like to thank Fang Wang and Meng Wang for helpful information on the spectrum.

admits small data global solutions for any $p \in (1, 1 + 2/(n-2))$.

As we see from Theorem 1.1, in the Klein-Gordon case, the problem is relatively simple and the machinery of the Strichartz estimates could be avoided. As we have seen from the Strauss conjecture, the case of wave equations is much more delicate. To handle this case, one expects to have to develop space-time estimates that are specifically well-adapted to the problem.

In the case of the existence problem on hyperbolic spaces, that is, (1) with $(M, g) = (\mathbb{H}^n, h)$ and $k = -\rho^2$, $\rho = (n-1)/2$, such that $\mathrm{Spec}(-\Delta_h + k) = [0, \infty)$,

$$\Box_{\mathbb{H}^n} u := (\partial_t^2 - \Delta_h - \rho^2) u = F_p(u), \quad u(0) = \varepsilon u_0,\ \partial_t u(0) = \varepsilon u_1, \tag{10}$$

we expect that the critical power $p_c(n)$ satisfies $p_c(n) \le p_S(n)$, due to negative curvature and the resulting better decay behavior for the linear waves. For convenience of presentation, we set $D_0 = \sqrt{-\Delta_{\mathbb{H}^n} - \rho^2}$, $D = \sqrt{-\Delta_{\mathbb{H}^n}}$ and then we have

$$\Box_{\mathbb{H}^n} = \partial_t^2 + D_0^2.$$

It was considered earlier by Fontaine [8], where global existence with small data was proved for $n = 2, 3$ and $p \ge 2$. We note that Anker, Pierfelice and Vallarino [2], [3] proved dispersive and Strichartz estimates for linear (shifted) wave equations on hyperbolic spaces and more generally Damek-Ricci spaces, which behave better than ones in Euclidean space. With help of these estimates, it was shown that there is small data global existence for certain $p > 1$ arbitrarily close to 1, which shows that the critical power $p_c(n) = 1$.

The second aim of the present work is to provide a simple geometric proof of the small data global existence for the less favorable equation (10) with any power $p \in (1, 1 + 4/(n-1)]$. More precisely, we will prove the following result, based on the space-time weighted Strichartz estimates of Georgiev-Lindblad-Sogge [9] (see also Tataru [29] for the scale-invariant case).

Theorem 1.2. *Let* $p \in (1, p_{\mathrm{conf}}]$. *Assume further that* $F_p(u)$ *is a homogeneous function of* u, *of order* p, *i.e.,* $F_p(u) = c|u|^{p-1}u$ *or* $c|u|^p$ *for some* c. *Then, for any* $(u_0, u_1) \in C_0^\infty$, *there exists a constant* $\varepsilon_1 > 0$ *such that* (10) *with* $\varepsilon \in (0, \varepsilon_1]$ *admits global solutions.*

As already mentioned, the spherical analysis on $\mathbb{H}^n$ is very similar to the one of $\mathbb{R}^n$. Here we provide a very simple geometric argument based on the fact that, on real hyperbolic space, the conformal Laplacian is conformally covariant and that $\mathbb{H}^n$ is conformal to $\mathbb{R}^n$. Of course, this argument does not work, as far as we know, for other rank one symmetric spaces of non-compact type, and even less on Damek-Ricci spaces (for which the spherical analysis is actually similar to the one of the hyperbolic space).

In the statement of Theorem 1.2, we assume the data to be smooth with compact support. As usual, with some more effort, we could relax the condition to admit less regular data. Specifically, we have the following:

Theorem 1.3. *Under the same assumption as in Theorems 1.2. There exists a constant* $\varepsilon_2 > 0$ *such that* (10) *with* $\varepsilon \in (0, \varepsilon_2]$ *admits global solutions for any* (u_0, u_1), *provided that*

$$\|D^s u_0\|_{L^{(p+1)/p}(\mathbb{H}^n)} + \|D^{s-1} u_1\|_{L^{(p+1)/p}(\mathbb{H}^n)} \le 1\ , \tag{11}$$

where $s = (n+1)(\frac{1}{2} - \frac{1}{p+1})$.

As we have mentioned, the condition on the data could be replaced by using the Strichartz estimates (see, e.g., [2]) to conditions that

$$\|D^{s/2-1/2}D_0^{1/2}u_0\|_{L^2(\mathbb{H}^n)} + \|D^{s/2-1/2}D_0^{-1/2}u_1\|_{L^2(\mathbb{H}^n)} \le 1. \tag{12}$$

Here, instead of directly using Strichartz estimates, we present a proof, based on the dispersive estimates of Tataru [29] for the linear homogeneous waves on hyperbolic space. See also (34) for alternative conditions on the data.

In Theorems 1.2 and 1.3, we have restricted ourselves to the conformal or sub-conformal case, $p \le p_{\text{conf}}$. As usual, the idea of proof could be further exploited to prove results for certain larger powers. We thank the referee for drawing our attention to [3], where global existence for certain super-conformal powers has been discussed. Here, as illustration, we present a stronger result in the following

Theorem 1.4. *Let $p \in (p_{\text{conf}}, 1+\frac{4n}{n^2-3n-2}]$ (which is understood to be $p \in (p_{\text{conf}}, \infty)$ for $n = 2, 3$) and $s = 2 - \frac{1}{n+1} + \delta$, with $\delta \in (0, \frac{2}{(n^2-1)(p-1)})$. Then there exists a constant $\varepsilon_3 > 0$ such that (10) with $\varepsilon \in (0, \varepsilon_3]$ admits global solutions for any (u_0, u_1), provided that*

$$\|D^s u_0\|_{L^{2/(1+2\delta)}(\mathbb{H}^n)} + \|D^{s-1}u_1\|_{L^{2/(1+2\delta)}(\mathbb{H}^n)} \le 1. \tag{13}$$

Remark. In [3], global results were obtained for any $1 < p < p_1(n)$, where

$$p_1(n) = \begin{cases} 5/2 & n = 4, \\ \frac{6+\sqrt{21}}{5} & n = 5, \\ 1 + \frac{2}{(n-1)/2-1/(n-1)} & n \ge 6. \end{cases}$$

For comparison, if $n \ge 4$, and let $p_2(n) = 1 + \frac{4n}{n^2-3n-2}$ be as in Theorem 1.4, then $p_1(n) < p_2(n)$, which means that our results improve those in [3] somewhat. Moreover, there appears to be gaps in the proof of the super-conformal result given there. For example, (49) on [3, page 751] for the case $n = 6$, $\gamma = 2$ (which is p in our notation), could not be satisfied with their choice of $q = 14/3$ (see page 752, Case (D)), as

$$\frac{\gamma}{2} + \frac{n-5}{2(n-1)} - \frac{\gamma}{q} = 1 + \frac{1}{10} - \frac{3}{7} > \frac{1}{2}\ .$$

Outline. Our paper is organized as follows. In the next section, we present the proof of global existence for Klein-Gordon type equations, Theorem 1.1, for fairly general manifolds. In §3, we recall the relation between the wave equations on hyperbolic space $\mathbb{H}^n$, and Euclidean space and prove global existence results for wave equations on $\mathbb{H}^n$, with C_0^∞ data, Theorem 1.2, by using the space-time weighted Strichartz estimates of Georgiev-Lindblad-Sogge [9] and Tataru [29]. In §4 we prove Theorem 1.3, by removing the restriction of compact support and relaxing the regularity condition on the initial data imposed in Theorem 1.2. The idea is to exploit the dispersive estimates of Tataru [29], for the linear homogeneous waves on hyperbolic spaces. In addition, in §5, an alternate proof of Theorem 1.3 for $p \in (1, 1 + 2/(n-1))$, as well as another global result involving different conditions on the data, (34), are obtained after proving certain Strichartz type estimates. In §6, we present the proof of Theorem 1.4. Lastly, in an Appendix, we give an independent proof of dispersive estimates of Tataru [29] for $\mathbb{H}^3$ and explain how there is an incorrect assertion in [21] that there is a gap in Tataru's argument.

2. **Global existence for Klein-Gordon type equations on manifolds.** In this section, we shall present the proof of global existence for Klein-Gordon type equations, Theorem 1.1.

First, though, let us present the Sobolev estimates that we shall require.

Lemma 2.1. *Let $\|f\|_{H^1} = \|\sqrt{k-\Delta_g}u_0\|_{L^2(M)}$ be the natural Sobolev norm for the positive operator $k-\Delta_g$, then we have the Sobolev estimates (7).*

Proof. As $k-\Delta_g > 0$, we know from spectral theorem that

$$\|f\|_{L^2(M)} + \|\sqrt{-\Delta_g}f\|_{L^2(M)} \le C\|f\|_{H^1(M)},$$

for some constant $C > 0$. Here we see that the left hand side is just the standard H^1 norm on (M, g), for which the standard Sobolev embedding is available, for smooth complete manifolds with Ricci curvature bounded from below and $\inf_{x\in M} \mathrm{Vol}_g(B(x)) > 0$. See, e.g., Hebey [13, Theorem 3.2] for $n \ge 3$. When $n = 2$, the result $H^1 \subset L^q$ for any $q \in [2,\infty)$ could be derived from [13, Theorem 3.2] with $q = 1$ using a similar argument in [13, Lemma 2.1]. □

Proof of Theorem 1.1. If we let

$$E(t) = \|u_t\|^2_{L^2} + \|u\|^2_{H^1}$$

be the energy functional, we see that, if $u_{tt} - \Delta_g u + ku = F$, then

$$\begin{aligned}\frac{d}{dt}E(t) = 2\langle u_t, u_{tt}\rangle + 2\langle\sqrt{k-\Delta_g}u, \sqrt{k-\Delta_g}u_t\rangle &= 2\langle u_t, u_{tt} - \Delta_g u + ku\rangle\\ &= 2\langle u_t, F\rangle \le 2E^{1/2}\|F\|_{L^2}.\end{aligned}$$

This yields the natural energy estimates for $t \ge 0$

$$E(t)^{1/2} \le E(0)^{1/2} + \int_0^t \|F(\tau)\|_{L^2} d\tau. \tag{14}$$

With help of the Sobolev embedding and energy estimates, we are able to prove local well-posedness in $CH^1\cap C^1L^2$. Observe that for any given $p \in (1, 1+2/(n-2))$, we know from Hölder's inequality, the Sobolev embedding ($H^1 \subset L^{2p}$) and (2) that there exist constants C_1 and C_2 such that

$$\begin{aligned}&\|F_p(u)-F_p(v)\|_{L^1([0,T];L^2)}\\ &\qquad \le C_1T(\|u\|_{L^\infty([0,T];L^{2p})} + \|v\|_{L^\infty([0,T];L^{2p})})^{p-1}\|u-v\|_{L^\infty([0,T];L^{2p})}\\ &\qquad \le C_2T(\|u\|^{p-1}_{C([0,T];H^1)} + \|v\|^{p-1}_{C([0,T];H^1)})\|u-v\|_{C([0,T];H^1)},\end{aligned}$$

for any $u, v \in C([0,T];H^1)\cap C^1([0,T];L^2) \subset L^\infty([0,T];L^{2p})$. Combined with (14), a standard contraction mapping argument yields local well-posedness for (1) in $C([0,T_*);H^1)\cap C^1([0,T_*);L^2)$, for some

$$T_* \ge \frac{E(0)^{-(p-1)/2}}{2^{p+1}C_2} \ge \frac{\varepsilon^{-(p-1)/2}}{2^{p+1}C_2},$$

where we have used the assumption (6). Moreover, if T_* is the maximal time of existence, with $T_* < \infty$, we have

$$\sup_{t\in[0,T_*)} E(t) = \infty.$$

To prove the theorem, it remains to give a uniform a priori control on the energy of the solution, for small ε. Observe that the problem (1) is Hamiltonian with the Hamiltonian functional given by

$$H[u(t),u_t(t)]=\int\left(\frac{u_t^2+|\sqrt{k-\Delta_g}u|^2}{2}-G_p(u)\right)dV_g,$$

where G_p is the primitive function of F_p with $G_p(0)=0$, and dV_g is the standard volume form for (M,g). Applying this fact to the solution $u\in C([0,T_*);H^1)\cap C^1([0,T_*);L^2)$ for (1), we see that

$$H[u(t),u_t(t)]=H[\varepsilon u_0,\varepsilon u_1]\le C_3\varepsilon^2,\quad \forall t\in[0,T_*), \tag{15}$$

for some $C_3>0$ and any $\varepsilon\le 1$. Then we have

$$\begin{aligned} E(t) &= 2H[u(t),u_t(t)]+2\int G_p(u)dV_g\\ &\le 2H[u(t),u_t(t)]+C\|u\|_{L^{p+1}}^{p+1}\\ &\le 2H[u(t),u_t(t)]+\tilde{C}\|u(t)\|_{H^1}^{p+1}\\ &\le 2C_3\varepsilon^2+C_4E(t)^{(p+1)/2}, \end{aligned}$$

where we have used the fact that $|G_p(u)|\le C|u|^{p+1}/(p+1)$, by (2). Therefore, a continuity argument implies that

$$E(t)\le 4C_3\varepsilon^2,\quad \forall t\in[0,T_*), \tag{16}$$

as long as

$$\varepsilon\le\varepsilon_0:=(4C_4)^{-1/(p-1)}(4C_3)^{-1/2}. \tag{17}$$

In view of the local well-posed results, we see that (16) is sufficient to conclude $T_*=\infty$ and so is the proof of global existence with $\varepsilon\in(0,\varepsilon_0]$, where ε_0 is given by (17). □

3. The Strauss conjecture on hyperbolic space.

In this section, we first recall the relation between the wave equations on the hyperbolic space-time $\mathbb{H}^n\times\mathbb{R}$, $n\ge 2$, and the wave equations on $\mathbb{R}^n\times\mathbb{R}$. With help of this fact, we present the proof of Theorem 1.2, by using the space-time weighted Strichartz estimates of Georgiev-Lindblad-Sogge [9] and Tataru [29].

Recall that inside the forward light cone, $\Lambda=\{(x,t)\in\mathbb{R}^n\times\mathbb{R}: t>|x|\}$, we may introduce coordinates

$$r=|x|,\ \ t=e^\tau\cosh s,\ \ r=e^\tau\sinh s,\ \ s\in[0,\infty),\ \ \tau\in\mathbb{R}.$$

Here, with $\omega\in\mathbb{S}^{n-1}$, we may view (s,ω), as natural polar coordinates in hyperbolic space $\mathbb{H}^n:=\Lambda_{\tau=0}$, with the natural metric, $ds^2+(\sinh s)^2d\omega^2$, induced from the Minkowski metric $g=-dt^2+dx^2=-dt^2+dr^2+r^2d\omega^2$ to $\mathbb{H}^n$. In the new coordinates, a simple computation leads to

$$\Box=-\partial_t^2+\Delta=e^{-2\tau}(-\partial_\tau^2+\Delta_{\mathbb{H}^n}-(n-1)\partial_\tau)=e^{-(2+\rho)\tau}(-\partial_\tau^2+\Delta_{\mathbb{H}^n}+\rho^2)e^{\rho\tau},$$

with $\rho=(n-1)/2$. That is, with $\Box_{\mathbb{H}^n}:=-\partial_\tau^2+\Delta_{\mathbb{H}^n}+\rho^2$, we have

$$\Box=e^{-(2+\rho)\tau}\Box_{\mathbb{H}^n}e^{\rho\tau}. \tag{18}$$

Let $u_0, u_1 \in C_0^\infty(\mathbb{H}^n)$ and consider the Cauchy problem (10) with $p > 1$ and small data $(\varepsilon u_0, \varepsilon u_1)$. By (18), we know this problem is equivalent to solving, with $u = e^{\rho\tau} w$,

$$\Box w = e^{-(2+\rho)\tau}\Box_{\mathbb{H}^n} e^{\rho\tau} w = e^{-(2+\rho)\tau}\Box_{\mathbb{H}^n} u = e^{-(2+\rho)\tau} F_p(u) \\ = e^{-(2-\rho(p-1))\tau} F_p(w) = (t^2 - r^2)^{-\sigma} F_p(w) \quad (19)$$

with C_0^∞ data of form $\varepsilon(w_0, w_1)$ on $t = \sqrt{1+r^2}$, where we have use the assumption that F_p is homogeneous and

$$\sigma = 1 - \frac{\rho}{2}(p-1).$$

To solve the Cauchy problem (19), we recall two facts about wave equations. The first is a weighted Strichartz estimates of Tataru [29, Theorem 5]. See also Georgiev-Lindblad-Sogge [9, Theorem 1.2] for an earlier version, which is sufficient to prove results for compactly supported data.

Lemma 3.1 (Weighted Strichartz estimates)**.** *Let $n \geq 2$ and w be a solution of the equation $\Box w = F$ which is supported inside the forward light cone. Then the following estimate holds:*

$$\|(t^2 - r^2)^{\gamma_1} w\|_{L^q(\mathbb{R}^{n+1})} \leq C_{q,\gamma_1,\gamma_2} \|(t^2 - r^2)^{\gamma_2} F\|_{L^{q'}(\mathbb{R}^{n+1})}, \quad (20)$$

provided that $2 \leq q \leq 2(n+1)/(n-1)$ and

$$\gamma_1 < \frac{n-1}{2} - \frac{n}{q},\ \gamma_2 = \gamma_1 - \frac{n-1}{2} + \frac{n+1}{q}.$$

In addition, it is well-known that the solutions of the homogeneous wave equation with compactly supported smooth data, of size ε, satisfy

$$|w(t,x)| \lesssim \varepsilon (t^2 - r^2)^{-\rho}. \quad (21)$$

With help of (20) and (21), it is not hard to show that (19) admits global solutions for any $p \in (1, p_{\text{conf}}]$. Actually, by setting $q = p+1$ and $\gamma_2 = -\gamma_1 = \frac{\sigma}{p+1}$, such that we have

$$\gamma_2 - \sigma = \gamma_1 p,\ \gamma_2 = \gamma_1 - \frac{n-1}{2} + \frac{n+1}{q},$$

we can solve (19) by iteration. Let $w^{(-1)} = 0$, we define inductively

$$\Box w^{(m)} = (t^2 - r^2)^{-\sigma} |w^{(m-1)}|^p, \quad m = 1, 2, \ldots, \quad (22)$$

with C_0^∞ data $\varepsilon(w_0, w_1)$ on $t = \sqrt{1+r^2}$.

Let $\|w\|_X := \|(t^2 - r^2)^{\gamma_1} w\|_{L^q(t > \sqrt{1+r^2})}$, then, by a routine calculation, we see from (21) that

$$\|w^{(0)}\|_X \leq C_0 \varepsilon. \quad (23)$$

As a result by (20), for $m \geq 1$ we have

$$\|w^{(m)}\|_X \leq \|w^{(0)}\|_X + \|w^{(m)} - w^{(0)}\|_X \leq C_0\varepsilon + C_1 \|w^{(m-1)}\|_X^p.$$

Based on these estimates, a standard continuity argument ensures that

$$\|w^{(m)}\|_X \leq 2C_0\varepsilon,$$

provided $\varepsilon \leq \varepsilon_0$ with $\varepsilon_0 \ll 1$. Moreover, with a possibly smaller $\varepsilon_1 \ll 1$, we have the convergence of $w^{(m)}$ in X, which proves the global existence of weak solutions for (19), with sufficiently small data of size $\varepsilon \leq \varepsilon_1$. This completes the proof of Theorem 1.2.

4. **General data: Proof of Theorem 1.3.** In this section, we present a proof of the Strauss conjecture on hyperbolic spaces with general data, Theorem 1.3, based on the dispersive estimates of Tataru [29] for the linear homogeneous wave equation on hyperbolic spaces. In addition, an alternative proof of Theorem 1.3 for $p \in (1, 1+2/(n-1))$ will be presented in §5, by proving certain Strichartz type estimates.

From the proof of global results in Section 3, we see that we need only to ensure the first iteration $w^{(0)} \in X$, for general data. To achieve this goal, we would like to translate it back to hyperbolic space.

Observing that

$$\begin{aligned} \|(t^2-r^2)^{\gamma_1} w\|^q_{L^q(\Lambda)} &= \int (t^2-r^2)^{\gamma_1 q}|w|^q dtdx \\ &= \int e^{2\gamma_1 q\tau}|w|^q e^{(n+1)\tau} d\tau dV_{\mathbb{H}^n} \\ &= \int e^{(2\gamma_1-\rho)q\tau}|u|^q e^{(n+1)\tau} d\tau dV_{\mathbb{H}^n} \\ &= \|e^{(2\gamma_1-\rho+\frac{n+1}{q})\tau}u\|^q_{L^q(d\tau dV_{\mathbb{H}^n})}, \end{aligned}$$

we see that what we need is to find an estimate for

$$\|e^{(2\gamma_1-\rho+\frac{n+1}{q})\tau}u^{(0)}\|_{L^q(d\tau dV_{\mathbb{H}^n})}, \tag{24}$$

where $\Box_{\mathbb{H}^n} u^{(0)} = 0$ with data $\varepsilon(u_0, u_1)$ on $\tau = 0$. Recall that

$$u^{(0)} = \varepsilon C(\tau)u_0 + \varepsilon S(\tau)u_1,$$

where $S(\tau) = D_0^{-1}\sin(\tau D_0)$, $C(\tau) = \cos(\tau D_0)$, with $D_0 = \sqrt{-\Delta_{\mathbb{H}^n} - \rho^2}$.

To control (24), we recall the following dispersive estimate of Tataru [29, Theorem 3]:

Lemma 4.1 (Dispersive estimates)**.** *Let $D = \sqrt{-\Delta_{\mathbb{H}^n}}$. Then the following estimate holds:*

$$\|S(\tau)f\|_{L^q} \lesssim \frac{(1+\tau)^{\frac{2}{q}}}{(\sinh\tau)^{(n-1)(\frac12-\frac1q)}}\|D^{(n+1)(\frac12-\frac1q)-1}f\|_{L^{q'}}, \quad 2 \le q < \infty, \tag{25}$$

$$\|C(\tau)f\|_{L^q} \lesssim \frac{1}{(\sinh\tau)^{(n-1)(\frac12-\frac1q)}}\|D^{(n+1)(\frac12-\frac1q)}f\|_{L^{q'}}, \quad 2 \le q < \infty. \tag{26}$$

If we recall that $\gamma_1 = -\sigma/q$, $q = p+1$, $\sigma = 1-\rho(p-1)/2$, $\rho = (n-1)/2$, it is easy to check that

$$2\gamma_1 - \rho + \frac{n+1}{q} < (n-1)\left(\frac12 - \frac1q\right) \Leftrightarrow \gamma_1 + \frac{n}{p+1} < \rho \Leftrightarrow p > 1. \tag{27}$$

Consequently we see from Lemma 4.1 that for any $q = p+1 > 2$ there exists a constant $C > 0$ such that

$$\|w^{(0)}\|_X = \|e^{(2\gamma_1-\rho+\frac{n+1}{q})\tau}u^{(0)}\|_{L^q(d\tau dV_{\mathbb{H}^n})} \le C\varepsilon(\|D^s u_0\|_{L^{\frac{p+1}{p}}} + \|D^{s-1}u_1\|_{L^{\frac{p+1}{p}}}),$$

with $s = (n+1)(\frac12 - \frac{1}{p+1})$. Since this is (23), we obtain Theorem 1.3 as before.

Remark. It was pointed out by the referee that Metcalfe and Taylor [21] assert that when $n = 3$ the proof in Tataru [29] of (25)–(26) has a gap. As a result, we felt the need to present an independent proof of these dispersive estimates for $\mathbb{H}^3$. Our proof makes use of elementary properties of Bessel potentials. Unlike the proof in [29], it is a bit more self-contained in the sense that it does not rely on the use of the Harish-Chandra c-function.

The claim in [21] rests on the fact that Bessel functions of order one have a logarithmic singularity at the origin (and blow up logarithmically at infinity). We shall present our proof of Tataru's dispersive estimates (25)–(26) for $n = 3$ in such a way that we can highlight the oversight in [21] which lead the authors to make their incorrect assertion about this "gap". We thank the referee for bringing this issue to our attention so that we may hopefully settle this minor simple misunderstanding and use these dispersive estimates. We are also grateful to Michael Taylor for helpful comments.

5. **Strichartz type estimates and an alternative proof of global existence for the shifted wave equation on $\mathbb{H}^n$.** As a side remark, let us now show how we could use Lemma 4.1 to prove inhomogeneous Strichartz type estimates that are sufficient to give an alternative proof of Theorem 1.3 for $p \in (1, 1 + 2/(n-1))$.

First, let us observe that when $p \in (1, p_{\text{conf}}]$ we have $s \le 1$, and so

$$\|S(\tau)f\|_{L^{p+1}(\mathbb{H}^n)} \lesssim K_{p+1}(\tau)\|f\|_{L^{\frac{p+1}{p}}(\mathbb{H}^n)}, \tag{28}$$

by (25), where

$$K_q(\tau) = \frac{(1+\tau)^{\frac{2}{q}}}{(\sinh\tau)^{(n-1)(\frac{1}{2}-\frac{1}{q})}}. \tag{29}$$

Based on Duhamel's principle and (28), we see that

$$\|u(\tau)\|_{L^{p+1}(\mathbb{H}^n)} \le C \int_0^\tau K_{p+1}(\tau - s)\|F(s)\|_{L^{\frac{p+1}{p}}(\mathbb{H}^n)} ds$$

for solutions to $\Box_{\mathbb{H}^n} u = F$ with vanishing data at $\tau = 0$. Since

$$K_{p+1}(\tau)\chi_{\tau>0} \lesssim |\tau|^{-2/(p+1)}, \quad 1 < p \le p_{\text{conf}},$$

we obtain the inhomogeneous Strichartz estimates

$$\|u\|_{L^{p+1}(\mathbb{R}^+\times\mathbb{H}^n)} \le C\|F\|_{L^{(p+1)/p}(\mathbb{R}^+\times\mathbb{H}^n)}, \ 1 < p \le p_{\text{conf}}, \tag{30}$$

by the Hardy-Littlewood-Sobolev inequality.

Concerning the homogeneous solutions, we observe that if

$$(n-1)\left(\frac{1}{2} - \frac{1}{p+1}\right) < \frac{1}{p+1},$$

that is, $1 < p < 1 + \frac{2}{n-1}$, we have $K_{p+1}(\tau) \in L^{p+1}(\mathbb{R}^+)$ and so we have the homogeneous estimates

$$\|u\|_{L^{p+1}(\mathbb{R}^+\times\mathbb{H}^n)} \lesssim \|D^s u(0)\|_{L^{(p+1)/p}(\mathbb{H}^n)} + \|D^{s-1}u_\tau(0)\|_{L^{(p+1)/p}(\mathbb{H}^n)}$$

for any solutions to $\Box_{\mathbb{H}^n} u = 0$, in view of Lemma 4.1, where $s = (n+1)(\frac{1}{2} - \frac{1}{p+1})$.

For general p, the argument still works, if we impose other conditions on the data. To state these we require the following homogeneous estimates.

Lemma 5.1 (Homogeneous estimates)**.** *Let $q \in [1, \infty)$ and $r \in (2, \infty)$, then for any $r_0 \in (2, r]$ such that*

$$\frac{1}{q} > (n-1)\left(\frac{1}{2} - \frac{1}{r_0}\right), \tag{31}$$

we have

$$\|u\|_{L^q([0,\infty);L^r(\mathbb{H}^n))} \lesssim \|D^{s_0}u(0)\|_{L^{r_0'}(\mathbb{H}^n)} + \|D^{s_0-1}u_\tau(0)\|_{L^{r_0'}(\mathbb{H}^n)}, \tag{32}$$

if $\Box_{\mathbb{H}^n} u = 0$ *and* $s_0 = \frac{n+1}{2} - \frac{1}{r_0} - \frac{n}{r}$. *In particular, for any* $p \in (1, \infty)$, *we have*

$$\|u\|_{L^{p+1}(\mathbb{R}^+\times\mathbb{H}^n)} \lesssim \|D^{s_1}u(0)\|_{L^{\frac{2}{1+2\delta}}(\mathbb{H}^n)} + \|D^{s_1-1}u_\tau(0)\|_{L^{\frac{2}{1+2\delta}}(\mathbb{H}^n)} \tag{33}$$

with $s_1 = n(\frac{1}{2} - \frac{1}{p+1}) + \delta$, *for any* $\delta > 0$ *sufficiently small such that* $\delta < \frac{1}{(n-1)(p+1)}$ *and* $\delta \le \frac{1}{2} - \frac{1}{p+1}$.

To prove this lemma, we need only to prove (32). By Sobolev embedding and Lemma 4.1, we have

$$\|u(\tau)\|_{L^r(\mathbb{H}^n)} \lesssim \|D^{s_2}u(\tau)\|_{L^{r_0}(\mathbb{H}^n)} \lesssim K_{r_0}(\tau)(\|D^{s_0}u(0)\|_{L^{r_0'}(\mathbb{H}^n)} + \|D^{s_0-1}u_\tau(0)\|_{L^{r_0'}(\mathbb{H}^n)}),$$

with $s_2 = n(1/r_0 - 1/r)$ and $s_0 - s_2 = (n+1)(1/2 - 1/r_0)$. Notice that (31) ensures $K_{r_0} \in L^q$, and so we obtain (32), which completes the proof of Lemma 5.1.

With help of (33) and (30), it is standard to conclude the proof of Theorem 1.3, with the condition on the data replaced by

$$\|D^{s_1}u_0\|_{L^{\frac{2}{1+2\delta}}(\mathbb{H}^n)} + \|D^{s_1-1}u_1\|_{L^{\frac{2}{1+2\delta}}(\mathbb{H}^n)} \le 1, \tag{34}$$

where $s_1 = n(\frac{1}{2} - \frac{1}{p+1}) + \delta$, and $\delta > 0$ is sufficiently small such that $\delta < \frac{1}{(n-1)(p+1)}$ and $\delta \le \frac{1}{2} - \frac{1}{p+1}$.

Actually, under the assumption of (34), we can solve (10), with sufficiently small ε, by a contraction mapping argument. For any $u \in L^{p+1}(\mathbb{R}^+ \times \mathbb{H}^n)$, we define $w = T[u]$ as the solution of

$$\Box_{\mathbb{H}^n} w = F_p(u), \tag{35}$$

with initial data $(\varepsilon u_0(x), \varepsilon u_1(x))$. With help of (33) and (30), we know that there exists a constant $C > 0$ such that

$$\|T[u]\|_{L^{p+1}(\mathbb{R}^+\times\mathbb{H}^n)} \le C\varepsilon + C\|F_p(u)\|_{L^{(p+1)/p}(\mathbb{R}^+\times\mathbb{H}^n)} \le C\varepsilon + C'\|u\|^p_{L^{p+1}(\mathbb{R}^+\times\mathbb{H}^n)},$$

$$\begin{aligned} \|T[u] - T[v]\|_{L^{p+1}(\mathbb{R}^+\times\mathbb{H}^n)} &\le C\|F_p(u) - F_p(v)\|_{L^{(p+1)/p}(\mathbb{R}^+\times\mathbb{H}^n)} \\ &\le C''(\|u\|_{L^{p+1}} + \|v\|_{L^{p+1}})^{p-1}\|u - v\|_{L^{p+1}}. \end{aligned}$$

Thus T is a contraction mapping on the complete set

$$\{u \in L^{p+1}(\mathbb{R}^+ \times \mathbb{H}^n),\ \|u\|_{L^{p+1}(\mathbb{R}^+\times\mathbb{H}^n)} \le 2C\varepsilon\},$$

provided that $C'(2C\varepsilon)^p \le C\varepsilon$ and $C''(4C\varepsilon)^{p-1} \le 1/2$, which are be ensured if we assume

$$\varepsilon \le (2^pC' + 2C'')^{-1/(p-1)}C^{-1}.$$

Remark. It would be interesting to see if there were an analog of Theorem 1.2 for spaces of variable curvature. Specifically, if (M, g) is a simply connected and complete Riemannian manifold of dimension $n \ge 2$ and has sectional curvatures satisfying $K \in [-\kappa_1, -\kappa_0]$ for some $\kappa_1 > \kappa_0 > 0$ and if $p > 1$, are there always global solutions to the equation

$$(\partial_t^2 - \Delta_g + \rho^2\kappa_0)u = F_p(u), \quad \rho = (n-1)/2,$$

for sufficiently small initial data with fixed compact support? Note that Corollary 1 says that such a result is true if $\rho^2\kappa_0$ is replaced by any larger constant k. Also, as we mentioned before, one needs $p > p_S(n)$ for this to be true for $\kappa_0 = 0$ due to what happens for the standard d'Alembertian in Minkowski space, and thus the assumption that (M, g) be negatively curved is needed.

In practice we can always take $\kappa_0 = 1$. In this case, a perhaps harder problem would be whether one has dispersive estimates as in Lemma 4.1 assuming that

$K \le -1$ and $\inf K > -\infty$. Due to properties of the leading term in the Hadamard parametrix (see e.g., [26]), this problem seems to be related to classical Riemannian volume comparison theorems and the Cartan-Hadamard conjecture.

6. **Super-conformal case.** In this section, we use Strichartz type estimates and Sobolev embedding to prove Theorem 1.4.

First, by (28) with $p = p_{\text{conf}}$, we know that for $q = 2(n+1)/(n-1)$ with $q = q' p_{\text{conf}}$,

$$\|S(\tau)f\|_{L^q(\mathbb{H}^n)} \lesssim K_q(\tau)\|f\|_{L^{q'}(\mathbb{H}^n)}, \tag{36}$$

where $K_q(\tau)$ is given in (29).

Based on Duhamel's principle and (36), we see that

$$\|u(\tau)\|_{L^q(\mathbb{H}^n)} \le C\int_0^\tau K_q(\tau - s)\|F(s)\|_{L^{q'}(\mathbb{H}^n)} ds$$

for solutions to $\Box_{\mathbb{H}^n} u = F$ with vanishing data at $\tau = 0$. Since for any $N > 0$, we have

$$K_q(\tau)\chi_{\tau>0} \lesssim |\tau|^{-(n-1)(1/2-1/q)}(1+|\tau|)^{-N} = |\tau|^{-(n-1)/(n+1)}(1+|\tau|)^{-N},$$

by the Hardy-Littlewood-Sobolev inequality and Young's inequality, we obtain the inhomogeneous Strichartz estimates

$$\|u\|_{L^{p_0}_\tau L^q_x(\mathbb{R}^+\times\mathbb{H}^n)} \le C\|F\|_{L^{p_1'}_\tau L^{q'}_x(\mathbb{R}^+\times\mathbb{H}^n)}, \tag{37}$$

provided that

$$p_0, p_1 \in [1,\infty],\ \frac{1}{p_0} + \frac{1}{p_1} \in [\frac{n-1}{n+1}, 1]\ . \tag{38}$$

If we combine it with the homogeneous estimates, Lemma 5.1, we get

$$\|u\|_{L^{p_0}_\tau L^q_x(\mathbb{R}^+\times\mathbb{H}^n)} \lesssim \|D^{s_0}u(0)\|_{L^{r_0'}(\mathbb{H}^n)} + \|D^{s_0-1}u_\tau(0)\|_{L^{r_0'}(\mathbb{H}^n)} + \|\Box_{\mathbb{H}^n}u\|_{L^{p_1'}_\tau L^{q'}_x(\mathbb{R}^+\times\mathbb{H}^n)}, \tag{39}$$

provided that $r_0 \in (2,q]$, $p_0 \in [1,\infty)$, $p_1 \in [1,\infty]$ and

$$s_0 = \frac{n+1}{2} - \frac{1}{r_0} - \frac{n}{q},\ \frac{1}{p_0} + \frac{1}{p_1} \in [\frac{n-1}{n+1}, 1],\ \frac{1}{p_0} > (n-1)\left(\frac{1}{2} - \frac{1}{r_0}\right)\ . \tag{40}$$

To prove Theorem 1.4 where $p \in (p_{\text{conf}}, 1 + \frac{4n}{n^2-3n-2}]$, we choose

$$p_0 = \frac{(n+1)(p-1)}{2}, p_1' = \frac{p_0}{p}, r_0 = \frac{2}{1-2\delta}, s_0 = \frac{n}{n+1} + \delta,$$

with $\delta \in (0, \frac{2}{(n^2-1)(p-1)})$, and we have

$$\|Du\|_{L^{p_0}_\tau L^q_x} \lesssim \|D^{s_0+1}u(0)\|_{L^{2/(1+2\delta)}} + \|D^{s_0}u_\tau(0)\|_{L^{2/(1+2\delta)}} + \|DF_p(u)\|_{L^{p_0/p}_\tau L^{q'}_x}. \tag{41}$$

Thus we see that to finish the proof as in Section 5, we need only to prove the following nonlinear inequality

$$\|DF_p(u)\|_{L^{q'}_x} \lesssim \|Du\|^p_{L^q_x}\ . \tag{42}$$

Actually, by (2), we have the chain rule

$$\|DF_p(u)\|_{L^{q'}_x} \lesssim \|Du\|_{L^q_x}\|u\|^{p-1}_{L^{(n+1)(p-1)/2}_x}\ . \tag{43}$$

Moreover, the Sobolev embedding gives us

$$\|u\|_{L^{(n+1)(p-1)/2}_x} \lesssim \|Du\|_{L^q_x}$$

provided that

$$1 \geq \frac{n}{q} - \frac{n}{(n+1)(p-1)/2}$$

which is ensured by our assumption that $p \in (p_{\rm conf}, 1 + \frac{4n}{n^2-3n-2}]$.

7. **Appendix.** It was pointed out by the referee that, owing to the fact that Bessel potentials of order one in $\mathbb{R}$ have a logarithmic singularity at the origin (and infinity), Metcalfe and Taylor in [21] assert that Tataru's [29] proof of the dispersive estimates (25) and (26) has a small gap when $n = 3$. As a result, we felt it necessary to give an independent proof of these results here. As we shall point out at the end of this Appendix, our proof involves essentially the same analytic family of operators as was employed by Tataru. We choose our normalization to highlight how the unboundedness of this particular Bessel potential does not lead to the gap in Tataru's [29] proof that was asserted in [21].

For simplicity (and since the assertion about the "gap" was for $n = 3$) we shall just treat the three-dimensional case of $\mathbb{H}^3$ for the dispersive estimates in Lemma 4.1. After the proof we shall explain that the lack of boundedness of Bessel potentials of certain critical orders does not cause any problems for $n \geq 2$. Incidentally, Tataru's [29] analytic family of operators is set up in a way that is very similar to the ones used by Strichartz [28] in the proof of the original "Strichartz estimates".

As before, $D_0 = \sqrt{-\Delta_{\mathbb{H}^3} - \rho^2}$ denotes the square root of minus the shifted Laplacian, with $\rho = (n-1)/2$ being equal to one in this case since we are working in $\mathbb{H}^n$ with $n = 3$. We then make a slight modification of Tataru's argument by introducing the two analytic family of operators

$$S_z(t) = (z+1)e^{z^2} D^z \, \frac{\sin tD_0}{D_0}, \tag{44}$$

and

$$C_z(t) = (z+1)e^{z^2} D^{-1+z} \, \cos tD_0, \tag{45}$$

where, as before, $D = \sqrt{-\Delta_{\mathbb{H}^n}}$. We have included the crucial factor $(z+1)$ in the definition of S_z to compensate for the aforementioned unboundedness of the Bessel potential of order one.

To prove the dispersive estimates for $\mathbb{H}^3$ we then require the following bounds which follow from well known properties of Bessel potentials and the spectral theorem.

Proposition 1. *There is a uniform constant C so that for $t > 0$*

$$\|S_z\|_{L^1(\mathbb{H}^3)\to L^\infty(\mathbb{H}^3)}, \; \|C_z\|_{L^1(\mathbb{H}^3)\to L^\infty(\mathbb{H}^3)} \leq C/\sinh t, \; \text{if } \; Re\; z = -1, \tag{46}$$

and also

$$\|S_z(t)\|_{L^2(\mathbb{H}^3)\to L^2(\mathbb{H}^3)} \leq C(1+t), \;\; \|C_z(t)\|_{L^2(\mathbb{H}^3)\to L^2(\mathbb{H}^3)} \leq C, \quad \text{if } \; Re\; z = 1. \tag{47}$$

By analytic interpolation, (46) and (47) yield (25) and (26).

Since (47) trivially follows from the spectral theorem, we just need to prove (46), which amounts to showing that the kernels of $S_z(t)$ and $C_z(t)$ are $O(1/\sinh t)$ when $\text{Re}\; z = -1$.

To prove these bounds we shall make use of the following simple lemma.

Lemma 7.1. *Let*

$$G_z(r) = (z+1)e^{z^2} \int_{-\infty}^{\infty} (1+\eta^2)^{z/2} \, e^{i\eta r} \, d\eta,$$

and

$$H_z(r) = (z+1)e^{z^2} \int_{-\infty}^{\infty} \eta\,(1+\eta^2)^{-1/2+z/2}\, e^{i\eta r}\, d\eta.$$

Then if $r \neq 0$

$$|G_z(r)|,\ |H_z(r)| \le Ce^{-|r|}, \quad \text{if } \ Re\ z = -1. \tag{48}$$

To prove this, we will use the well known formula for Bessel potentials (see e.g., [4]),

$$\begin{aligned}
&\int_{-\infty}^{\infty} (1+\eta^2)^{-z/2} e^{i\eta r}\, d\eta \qquad (49)\\
&= \frac{2\pi e^{-|r|}}{2^{z/2}\Gamma(z/2)\Gamma(1-z/2)} \int_0^{\infty} e^{-|r|\tau}\big(\tau+\tau^2/2\big)^{-z/2}\, d\tau\\
&= \frac{2\pi e^{-|r|}}{2^{z/2}\Gamma(z/2)} \times \frac{1}{\Gamma(1-z/2)} \int_0^{\infty} \tau^{-z/2}\Big((1+\tau/2)^{-z/2} e^{-|r|\tau}\Big)\, d\tau, \ \ r \neq 0.
\end{aligned}$$

This would leads to the bound for G_z in (48) if we could show that

$$\frac{se^{-s^2}}{2^{(1+is)/2}\Gamma((1+is)/2)\,\Gamma((1-is)/2)} \int_0^{\infty} e^{-|r|\tau}\big(\tau+\tau^2/2\big)^{-(1+is)/2}\, d\tau = O(1). \tag{7.5$'$}$$

Note that if $r \neq 0$ is fixed, the first and last terms in (49) are entire functions of $z \in \mathbb{C}$. There is no problem interpreting the first term, since it is a standard oscillatory integral, while, because of the $1/\Gamma(1-z/2)$ factor, the last term in (49) is well defined as a standard Riemann-Liouville integral.

In [4], (49) is stated for $z \in (1,2)$ (i.e., (2, 10)–(3, 6) in [4]), which, by analytic continuation, of course implies the formula for Re $z \in (1,2)$. If $r \neq 0$ is fixed, then, as we just mentioned, the first term in this formula and the last term are entire functions of z, and so the formula is valid for all $z \in \mathbb{C}$.

To verify (7.5′), fix $\beta \in C^\infty(\mathbb{R})$ satisfying $\beta(\tau) = 0$ for $\tau \le 1$ and $\beta(\tau) = 1$ for $\tau \ge 2$. Then clearly the left side of (7.5′) is of the form

$$\begin{aligned}
&\frac{se^{-s^2}}{2^{(1+is)/2}\Gamma((1+is)/2)\,\Gamma((1-is)/2)} \int_0^{\infty} \beta(\tau)e^{-|r|\tau}\big(\tau+\tau^2/2\big)^{-(1+is)/2}\, d\tau + O(1)\\
&= \frac{se^{-s^2}}{\Gamma((1+is)/2)\,\Gamma((1-is)/2)} \int_0^{\infty} \beta(\tau)e^{-|r|\tau}\tau^{-1-is}\, d\tau + O(1)\\
&= \frac{ie^{-s^2}}{\Gamma((1+is)/2)\,\Gamma((1-is)/2)} \int_0^{\infty} \beta(\tau)e^{-|r|\tau}\frac{d}{d\tau}\tau^{-is}\, d\tau + O(1) = O(1),
\end{aligned}$$

as claimed in (7.5′).

To prove the bounds for $H_z(r)$ in (48) , we first note that

$$H_z(r) = \tfrac{-ie^{-2z-1}}{z+2} rG_{z+1}(r), \ \ r \neq 0, \tag{50}$$

due to the fact that if $m \in \mathcal{S}'(\mathbb{R})$, then the Fourier transform of $\frac{\partial}{\partial\eta}m(\eta)$ is $ir\hat{m}(r)$, where $\hat{m} \in \mathcal{S}'(\mathbb{R})$ is the Fourier transform of m. Consequently, we would obtain the bounds for $H_z(r)$ in (48) if

$$e^{-s^2} r \int_{-\infty}^{\infty} (1+\eta^2)^{is/2} e^{i\eta r} = O(e^{-|r|}).$$

This clearly follows from (49)and the fact that

$$\Big| r\int_0^\infty e^{-|r|\tau}\big(\tau+\tau^2/2\big)^{-is/2}\,d\tau\,\Big| \le \int_0^\infty |r|\,e^{-|r|\tau}\,d\tau=1,$$

which completes the proof.

We can also give another simple proof of the bounds for H_z in (48) using well-known formulae for Bessel potentials in $\mathbb{R}^3$. We first observe that

$$\begin{aligned}H_z(r)&=2i(z+1)e^{z^2}\int_0^\infty \rho(1+\rho^2)^{-(1-z)/2}\,\sin(r\rho)\,d\rho\\ &=(2\pi)^{-1}i(z+1)e^{z^2}r\int_{\mathbb{R}^3}(1+|\xi|^2)^{-(1-z)/2}\,e^{i\langle\xi,(r,0,0)\rangle}\,d\xi,\end{aligned}$$

due to the fact that, if $d\sigma$ denotes surface measure on the two-sphere, we have

$$\frac{|x|}{4\pi}\int_{S^2}e^{ix\cdot\omega}\,d\sigma(\omega)=\sin|x|.$$

Thus, by standard formulae for Bessel potentials (i.e., (2, 10)–(3, 6) in [4])), we have, as before,

$$\begin{aligned}H_z(r)=2\pi i(z+1)e^{z^2}&\frac{e^{-|r|}}{2^{(1-z)/2}\Gamma((1-z)/2)\Gamma((3+z)/2)}\\ &\times r\int_0^\infty e^{-|r|\tau}\big(\tau+\tau^2/2\big)^{(1+z)/2}\,d\tau\ .\end{aligned}\tag{51}$$

Using this formula, we can easily obtain another proof of the bounds for H_z in (48).

Remark. Note that the $(z+1)$ factor is needed in the definition of G_z to ensure that (48) holds near $z=-1$. In particular, at the value of $z=-1$ we have $G_{-1}\equiv 0$. Additionally, since $G_0(r)=2\pi\delta_0(r)$ and $r\delta_0(r)\equiv 0$, (50) yields $H_{-1}(r)\equiv 0$, which (by what follows) is necessary as $C_{-1}(t)\equiv 0$ in (45). On the other hand, even though we included the $(z+1)$ factor in the definition of H_z for consistency, (51) shows that it is superfluous.

End of proof of Proposition 1. If $m(\tau)$ is an even function of τ, the operator $m(D_0)$ has kernel

$$\frac{-1}{4\pi^2\sinh r}\,\frac{\partial}{\partial\tau}\hat m(\tau)|_{\tau=r},\quad r=d_g(x,y),$$

where $d_g(x,y)$ is the hyperbolic distance and $\hat m$ is the Fourier transform of m. (See [30].)

Thus, the kernel of $S_z(t)$ equals

$$\frac{-(z+1)e^{z^2}}{4\pi^2 i\sinh r}\int_{-\infty}^\infty(1+\eta^2)^{z/2}\sin t\eta\,e^{-ir\eta}\,d\eta.$$

By Lemma 7.1 this is $O((\sinh t)^{-1})$ as desired when Re $z=-1$ if either $r\ge 1$ or $r\ge t/2$. For the remaining case where $0<r<1$ and $2r<t$ we claim that

$$\Big|\ \frac{se^{-s^2}}{r}\int_{-\infty}^\infty(1+\eta^2)^{-1/2+is/2}\ \sin t\eta\,e^{-i\eta r}\,d\eta\Big|\le C(\sinh t)^{-1},\quad s\in\mathbb{R},\tag{52}$$

which would finish the proof of the bounds for S_z in (46).

Using (49), Euler's formula and the mean value theorem shows that this bound is valid in this case when $t\ge 2$. So we would be done with our $L^1(\mathbb{H}^3)\to L^\infty(\mathbb{H}^3)$

bounds for $S_z(t)$, Re $z=-1$, if we could show that the left side of (52) is $O(1/t)$ when $0<2r<t<2$.

To prove this fix an even nonnegative function $\rho\in C^\infty(\mathbb{R})$ which vanishes for $\eta\in(-1,1)$ and equals one when $|\eta|\ge 2$. Then since $\eta\to(1+\eta^2)^{-1/2+is/2}\sin t\eta$ is odd the left side of (52) equals

$$\Big|\frac{se^{-s^2}}{r}\int_{-\infty}^{\infty}(1+\eta^2)^{-1/2+is/2}\sin t\eta\,\sin r\eta\,d\eta\Big|$$
$$=\Big|\frac{se^{-s^2}}{r}\int_{-\infty}^{\infty}\rho(t\eta)(1+\eta^2)^{-1/2+is/2}\,\sin r\eta\,\sin t\eta\,d\eta\Big|+O(1/t).$$

If we integrate by parts, we find that the first term in the right is majorized by

$$\Big|\frac{se^{-s^2}}{t}\int_{-\infty}^{\infty}\rho(t\eta)\,(1+\eta^2)^{-1/2+is/2}\,\cos r\eta\,\cos t\eta\,d\eta\Big|$$
$$+\Big|\frac{se^{-s^2}}{tr}\int_{-\infty}^{\infty}\frac{d}{d\eta}\big(\rho(t\eta)(1+\eta^2)^{-1/2+is/2}\big)\,\sin r\eta\,\cos t\eta\,d\eta\Big|.$$

By Euler's formula and a simple integration by parts argument, since $0<2r<t<2$, the first term is $O(1/t)$ as desired. If we integrate by parts one more time, we find that the remaining term is

$$\Big|\frac{se^{-s^2}}{t^2r}\int_{-\infty}^{\infty}\frac{d^2}{d\eta^2}\big(\rho(t\eta)(1+\eta^2)^{-1/2+is/2}\big)\,\sin r\eta\sin t\eta\,d\eta\Big|+O(1/t).$$

Since the first term here is also clearly $O(1/t)$, this finishes the proof of our bounds for $S_z(t)$ in (46).

Since the proof of the bounds for $C_z(t)$ in (46) follows from the same argument, the proof of Proposition 1 is complete. □

Remarks. To prove the dispersive estimates in Lemma 4.1 when $n=3$, Tataru [29] used the analytic family of operators

$$a(z)\big(D_0^2+\beta^2\big)^{z/2}\,\frac{\sin tD_0}{D_0},\tag{53}$$

for fixed $\beta>\rho$ (which is more favorable than the case $\beta=\rho$ treated above), with, *crucially*,

$$a(z)=\frac{e^{z^2}}{\Gamma(z+\rho)},\quad \rho=(n-1)/2.\tag{54}$$

In [21] it was noted that when $n=3$ the operators

$$\big(D_0^2+\beta^2\big)^{z/2}\,\frac{\sin tD_0}{D_0},\quad t\neq 0,\tag{55}$$

do not map $L^1(\mathbb{H}^3)\to L^\infty(\mathbb{H}^3)$ if $z=-1$. This is due to the fact that for any fixed $\beta>0$ the Bessel potential

$$r\to\int_{-\infty}^{\infty}\big(\beta^2+\eta^2\big)^{z/2}\,e^{i\eta r}\,d\eta\tag{56}$$

is not in $L^\infty(\mathbb{R})$, if $z=-1$. Indeed, for instance, this particular Bessel potential is $\approx|\ln r|$ for $r>0$ near the origin.

We have to point out that inequality (27) in Tataru [29], which is disputed on p. 3496 of [21], does *not* lead to their (3.26), which is the assertion that the operators in (55) do map $L^1(\mathbb{H}^3) \to L^\infty(\mathbb{H}^3)$ with norm $O(1/\sinh|t|)$. It is this *incorrect inequality* (i.e., (3.26) in [21]) that lead the authors in [21] to say that there is a gap in Tataru's argument.

To be more specific, Tataru *never* claims that this inequality is valid, and, *moreover*, his proof does *not* imply this fallacious inequality. Indeed, what Tataru proves is that the operators defined in (53)–(54) satisfy

$$\| \, a(z) \left(D_0^2+\beta^2\right)^{z/2} \frac{\sin tD_0}{D_0} \, \|_{L^1(\mathbb{H}^3)\to L^\infty(\mathbb{H}^3)} \le C/\sinh|t|, \quad \text{if } \ \mathrm{Re}\, z=-1. \tag{57}$$

It seems the authors in [21] overlooked the fact that $z \to 1/\Gamma(z+1)$ behaves like $(z+1)$ near $z=-1$, and, consequently

$$a(-1)=0.$$

Thus, since $0\cdot \ln r\equiv 0$, $r>0$, Tataru's assertion (57) when $z=-1$ is *not* (3.26) in [21] since

$$a(z)\left(D_0^2+\beta^2\right)^{z/2}\frac{\sin tD_0}{D_0}\equiv 0, \quad \text{if } \ z=-1, \ \text{ and } \ t\neq 0,$$

which means that Tataru's estimate (57) is trivial in this disputed case.

We proved (57) above when $\beta=\rho=1$ and $n=3$, with, in our case

$$a(z)=(z+1)e^{z^2}.$$

Just as Tataru's proof relies crucially on the holomorphic damping factor $1/\Gamma(z+1)$, ours used, in a critical way, the damping factor $(z+1)$ so that, in our case, $a(z)=0$ for $z=-1$.

In any dimension $n\ge 2$, Tataru's normalizing factor $a(z)$ vanishes when $z=-\rho$ and thus, like the related original argument of Strichartz [28], avoids possible problems that could arise from Bessel potentials of order one in $\mathbb{R}$ being unbounded.

As we said before, we are grateful to the referee for bringing to our attention the potential (but non-existent) problems of our use of Lemma 4.1 so that we could address this issue in hopes that the minor oversight in [21] about the (non-existent) "gap" in Tataru's proof of this estimate is not propagated further.

REFERENCES

[1] J.-P. Anker and V. Pierfelice, Wave and Klein-Gordon equations on hyperbolic spaces, *Anal. PDE*, **7** (2014), 953–995.

[2] J.-P. Anker, V. Pierfelice and M. Vallarino, The wave equation on hyperbolic spaces, *J. Differential Equations*, **252** (2012), 5613–5661.

[3] J.-P. Anker, V. Pierfelice and M. Vallarino, The wave equation on Damek-Ricci spaces, *Ann. Mat. Pura Appl. (4)*, **194** (2015), 731–758.

[4] N. Aronszajn and K. T. Smith, Theory of Bessel potentials. I., *Ann. Inst. Fourier*, **11** (1961), 385–475.

[5] A. Borbély, On the spectrum of the Laplacian in negatively curved manifolds, *Studia Sci. Math. Hungar.*, **30** (1995), 375–378.

[6] T. Cazenave, Uniform estimates for solutions of nonlinear Klein-Gordon equations, *J. Funct. Anal.*, **60** (1985), 36–55.

[7] I. Chavel, *Riemannian Geometry*, volume 98 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, second edition, 2006. A modern introduction.

[8] J. Fontaine, A semilinear wave equation on hyperbolic spaces, *Comm. Partial Differential Equations*, **22** (1997), 633–659.

[9] V. Georgiev, H. Lindblad and C. D. Sogge, Weighted Strichartz estimates and global existence for semilinear wave equations, *Amer. J. Math.*, **119** (1997), 1291–1319.

[10] R. T. Glassey, Existence in the large for $\Box u = F(u)$ in two space dimensions, *Math. Z.*, **178** (1981), 233–261.

[11] R. T. Glassey, Finite-time blow-up for solutions of nonlinear wave equations, *Math. Z.*, **177** (1981), 323–340.

[12] C. R. Graham and M. Zworski, Scattering matrix in conformal geometry, *Invent. Math.*, **152** (2003), 89–118.

[13] E. Hebey, *Nonlinear analysis on manifolds: Sobolev spaces and inequalities*, volume 5 of *Courant Lecture Notes in Mathematics*, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1999.

[14] F. John, Blow-up of solutions of nonlinear wave equations in three space dimensions, *Manuscripta Math.*, **28** (1979), 235–268.

[15] T. Kato, Blow-up of solutions of some nonlinear hyperbolic equations, *Comm. Pure Appl. Math.*, **33** (1980), 501–505.

[16] M. Keel and T. Tao, Small data blow-up for semilinear Klein-Gordon equations, *Amer. J. Math.*, **121** (1999), 629–669.

[17] H. Lindblad and C. D. Sogge, Long-time existence for small amplitude semilinear wave equations, *Amer. J. Math.*, **118** (1996), 1047–1135.

[18] H. Lindblad and C. D. Sogge, Restriction theorems and semilinear Klein-Gordon equations in $(1 + 3)$-dimensions, *Duke Math. J.*, **85** (1996), 227–252.

[19] R. R. Mazzeo and R. B. Melrose, Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature, *J. Funct. Anal.*, **75** (1987), 260–310.

[20] H. P. McKean, An upper bound to the spectrum of Δ on a manifold of negative curvature, *J. Differential Geometry*, **4** (1970), 359–366.

[21] J. Metcalfe and M. Taylor, Nonlinear waves on 3D hyperbolic space, *Trans. Amer. Math. Soc.*, **363** (2011), 3489–3529.

[22] J. Metcalfe and M. Taylor, Dispersive wave estimates on 3D hyperbolic space, *Proc. Amer. Math. Soc.*, **140** (2012), 3861–3866.

[23] J. Schaeffer, The equation $u_{tt} - \Delta u = |u|^p$ for the critical value of p, *Proc. Roy. Soc. Edinburgh Sect. A*, **101** (1985), 31–44.

[24] A. G. Setti, A lower bound for the spectrum of the Laplacian in terms of sectional and Ricci curvature, *Proc. Amer. Math. Soc.*, **112** (1991), 277–282.

[25] T. C. Sideris, Nonexistence of global solutions to semilinear wave equations in high dimensions, *J. Differential Equations*, **52** (1984), 378–406.

[26] C. D. Sogge, *Hangzhou Lectures on Eigenfunctions of the Laplacian*, volume 188 of *Annals of Mathematics Studies*, Princeton University Press, Princeton, NJ, 2014.

[27] W. A. Strauss, Nonlinear scattering theory at low energy, *J. Funct. Anal.*, **41** (1981), 110–133.

[28] R.t S. Strichartz, Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations, *Duke Math. J.*, **44** (1977), 705–714.

[29] D. Tataru, Strichartz estimates in the hyperbolic space and global existence for the semilinear wave equation, *Trans. Amer. Math. Soc.*, **353** (2001), 795–807 (electronic).

[30] M. Taylor, *Partial Differential Equations II. Qualitative Studies of Linear Equations. Second Edition*, Applied Mathematical Sciences, 116. Springer, New York, 2011.

[31] C. Wang, Recent progress on the strauss conjecture and related problems, *Scientia Sinica Mathematica*, **48** (2018), 111–130.

[32] C. Wang and X. Yu, Recent works on the Strauss conjecture, In *Recent Advances in Harmonic Analysis and Partial Differential Equations*, volume 581 of *Contemp. Math.*, pages 235–256, Amer. Math. Soc., Providence, RI, 2012.

[33] B. T. Yordanov and Q. S. Zhang, Finite time blow up for critical wave equations in high dimensions, *J. Funct. Anal.*, **231** (2006), 361–374.

[34] Y. Zhou, Cauchy problem for semilinear wave equations in four space dimensions with small initial data, *J. Partial Differential Equations*, **8** (1995), 135–144.

[35] Y. Zhou, Blow up of solutions to semilinear wave equations with critical exponent in high dimensions, *Chin. Ann. Math. Ser. B*, **28** (2007), 205–212.

Received November 2018; 1st revision July 2019; 2nd revision August 2019.

E-mail address: sire@math.jhu.edu
E-mail address: sogge@math.jhu.edu
E-mail address: wangcbo@zju.edu.cn