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Convergence as period goes to infinity of spectra of periodic traveling waves toward essential spectra of a homoclinic limit

Zhao Yang [1], Kevin Zumbrun [*,2]

Indiana University, Bloomington, IN 47405, United States of America

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A B S T R A C T

We revisit the analysis by R.A. Gardner of convergence of spectra of periodic traveling waves in the homoclinic, or infinite-period limit, extending his results to the case of essential rather than point spectra of the limiting homoclinic wave. Notably, convergence to essential spectra is seen to be of algebraic rate with respect to period as compared to the exponential rate of convergence to point spectra. In the course of the analysis, we show not only convergence of spectrum but also convergence of an appropriate renormalization of the associated periodic Evans function to the Evans function for the limiting homoclinic wave, a fact that is useful for numerical investigations.

R É S U M É

Nous revenons sur l'analyse de R.A. Gardner de convergence de spectres de périodique ondes progressives dans le limite de période infinie, etendant ses résultats au cas de spectres essentiels. Notamment, la convergence vers les spectres essentiels a un taux algébrique par rapport à la période. Au cours de l analyse, nous montrons non seulement la convergence du spectre mais aussi la convergence d'une renormalisation appropriée de la fonction d'Evans périodique à la fonction d'Evans pour l'onde homocline limite, un fait qui est utile pour les enquêtes numériques.

1. Introduction

In this note, using asymptotic Evans function techniques like those introduced for the study of homoclinic and heteroclinic traveling waves in [26,31,32], we build on the pioneering analysis of R.A. Gardner [10,11] of convergence of spectra of periodic traveling waves in the infinite-period, or "homoclinic", limit, extending his results to the case that the limiting homoclinic spectra are of essential rather than point spectrum type.

* Corresponding author.
E-mail addresses: yangzha@indiana.edu (Z. Yang), kzumbrun@indiana.edu (K. Zumbrun).
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Under quite general conditions, Gardner showed that loops of essential periodic spectra bifurcate from isolated point spectra λ_0 of the limiting homoclinic wave. Indeed, it is readily seen that, on compact sets bounded away from regions of essential homoclinic spectrum, periodic spectra converge as period $X \to \infty$ at exponential rate $O(e^{-\eta X})$, $\eta > 0$ to the point spectra of the limiting homoclinic; see [23,28], or Section 4 below.

In the standard case arising generically for reaction diffusion systems of a limiting homoclinic wave with strictly stable essential spectrum, or "spectral gap", and a single isolated eigenvalue at $\lambda = 0$ associated with translational invariance of the underlying equations, this reduces the study of periodic stability in the large-period limit to asymptotic analysis of the loop of "critical" periodic spectra bifurcating from the neutral eigenvalue $\lambda = 0$. For, recall that linearized and nonlinear stability have been shown in quite general circumstances to follow from the "dissipative spectral stability" condition of Schneider: that periodic spectra move into the stable half plane at quadratic rate in the associated Bloch-Floquet number as the Bloch number is varied about zero [15,16,18,19,29,30]. This problem was resolved definitively by Sandstede and Scheel in [28], essentially closing the question of large-period periodic stability in the case of a spectral gap.

However, there are interesting cases arising in systems with conservation laws, notably for models of elasticity and thin film flow [6,18,23,24] of families of periodic waves for which the spectral gap condition is not satisfied in the homoclinic limit, $\lambda = 0$ being an eigenvalue embedded in the essential spectrum. In particular, for thin film flows, the homoclinic limit typically has unstable essential spectrum branching from the origin, and it is spectra bifurcating from this essential spectrum rather than the embedded eigenvalue at $\lambda = 0$ that appears to dominate the stability behavior of nearby periodic waves; see the discussion of [2,3]. This motivates our study here of convergence in the vicinity of essential spectra, both to essential spectra themselves and to eigenvalues embedded in essential spectrum, to neither of which cases Gardner's original analysis applies.

Recall [12,14] that the essential spectrum of a homoclinic traveling wave is given by the union of algebraic curves $\lambda = \lambda_j(k)$ obtained from the dispersion relation of the (constant-coefficient) linearization of the governing evolution equation about the endstate $u_\infty = \lim_{x\to\pm\infty} \bar{u}(x)$ of the homoclinic profile $\bar{u}(\cdot)$, where $k \in \mathbb{R}$ denotes Fourier frequency, corresponding to the (entirely essential) spectra of the constant solution $u(x,t) \equiv u_\infty$. Thus, the generic situation in the context of essential spectrum, analogous to an isolated eigenvalue in the point spectrum context considered by Gardner, is a point λ lying on a single curve λ_j, corresponding to a single nondegenerate root k_* of $\lambda_j(k) = 0$. This is closer in nature to the (also entirely essential) spectra of periodic waves than is the case of an isolated eigenvalue; indeed, it is the spectra of the constant periodic solution $u(x,t) \equiv u_0$, with Fourier frequency k corresponding to γ-value $\gamma = e^{ikX^\varepsilon}$ in the notation of Section 2.

Similarly as in [11], our analysis is carried out by examination of the associated periodic Evans functions $E^\varepsilon(\lambda,\gamma)$ introduced by Gardner [10], $\lambda, \gamma \in \mathbb{C}$, $|\gamma| = 1$, an analytic function whose zeroes λ coincide with the spectrum of the linearized operator about the wave, where $\varepsilon \to \mathbb{R}$ indexes the family of periodic waves converging as $\varepsilon \to 0$ to a homoclinic, or solitary wave, profile. However, differently from the approach of [10,11], our results are obtained not by topological considerations, but, similarly in [17,31,32], by demonstration of convergence, at exponential rate $O(e^{-\eta X^\varepsilon})$, of a suitably rescaled version of the sequence of periodic Evans functions $E^\varepsilon(\lambda,\gamma)$ and a sequence of functions interpolating between different versions of the homoclinic Evans function $D^0(\lambda)$ defined on various components of the complement of the union of curves $\lambda_j(\cdot)$ composing the homoclinic essential spectrum, with transition zones of scale $\sim 1/X^\varepsilon \to 0$ around isolated points $\lambda = \lambda_j(k)$.

Away from the homoclinic essential spectrum, this reduces to the simpler computation an appropriate renormalization $D^\varepsilon(\lambda,\gamma)$ of E^ε converges to the homoclinic Evans function $D^0(\lambda)$ at exponential rate, recovering and further illuminating the original result of Gardner [11] that the zero-set of $E^\varepsilon(\cdot,\xi)$ converges for each ξ to the zero-set of D^0, at exponential rate; see Section 4. This convergence is potentially useful in numerical investigations, as the basis of numerical convergence studies in this singular, hence numerically sensitive, limit. See, e.g., the applications in [5], as discussed in [5, Appendix D, pp. 70–72].

Near isolated arcs of curves λ_j of the homoclinic essential spectrum, as described above, for which the bordering homoclinic Evans functions do not vanish (in particular precluding embedded eigenvalues), we find that the zero-set of E^ε, comprising curves of periodic essential spectrum, converges *not to a single point but to a full arc of* λ_j, and at *algebraic rather than exponential rate*; see Section 5. In the case of an isolated arc with a single embedded eigenvalue, we find as might be guessed that the periodic spectra comprise two curves: a loop converging exponentially to the isolated eigenvalue, and a curve converging algebraically to the arc λ_j; see Section 5.2. The method of analysis is general, and should extend to other, more degenerate cases, at the expense of further effort/computation.

Our results apply in particular to the Saint Venant equations of inclined shallow water flow studied in [2,3], verifying instability of periodic waves in the homoclinic limit by consideration of spectra bifurcating from unstable essential spectrum of the limiting homoclinic. A very interesting open problem is to carry out an analysis like that of [28] determining separately the stability of spectra bifurcating from the embedded eigenvalue at $\lambda = 0$. An interesting related problem is to verify the heuristic picture of "metastable" behavior conjectured in [3], deducing stability for large but not infinite-period waves based on properties of an essentially unstable homoclinic limit with stable point spectrum.

We note finally a close relation between the results obtained here on approximation by periodic waves of a homoclinic limit and those obtained in [27] for approximation by truncation of the homoclinic to a large but finite interval, in particular the similarity between reduced equation [27, Eq. (5.6)] and the limiting equation (5.5) obtained here in describing convergence to essential spectra. In their concluding discussion [27, p. 276], the authors mention the case of approximation by periodic waves as a direction for generalization, stating that "... the results" [for approximation by domain truncation] "remain true so long as the periodic waves remain $O(e^{-\delta L})$ close to the limiting solitary wave."

Our results confirm this statement in particularly transparent fashion by the introduction of a periodic conjugation lemma (Lemma 3.1) converting both limiting and approximant eigenvalue problems to constant-coefficient, and associated transitional Evans functions (5.2) and (5.1) toward which rescaled versions of homoclinic and periodic Evans functions converge. The latter observation if of interest in its own right, both as an illuminating theoretical framework and a test (as in, e.g., [5]) of numerical convergence.

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2. Preliminaries

Following Gardner [11], we consider a family of periodic traveling-wave solutions

$$u(x,t) = \bar{u}^\varepsilon(x - c^\varepsilon t), \quad \bar{u}^\varepsilon(x + X^\varepsilon) = \bar{u}^\varepsilon(x) \tag{2.1}$$

of a family of PDEs $u_t = \mathcal{F}^\varepsilon(\partial_x, u)$ with smooth coefficients, converging as $\varepsilon \to 0$ to a solitary-wave solution $\bar{u}^0$, or homoclinic orbit of the associated traveling-wave ODE $-c^0\partial_x u = \mathcal{F}^0(u)$, as meanwhile $X^\varepsilon \to \infty$. Taking without loss of generality $c^\varepsilon \equiv 0$ (changing to co-moving coordinates $\tilde{x} = x - c^\varepsilon t$), we investigate stability of the equilibria $\bar{u}^\varepsilon$, $\mathcal{F}^\varepsilon(\bar{u}^\varepsilon) = 0$, through the study of the spectra λ of the associated family of eigenvalue ODEs

$$\lambda u = L^\varepsilon u := d\mathcal{F}(\bar{u}^\varepsilon)u, \tag{2.2}$$

with an eye toward relating the spectral properties of periodic waves $\bar{u}^\varepsilon$ as $\varepsilon \to 0$ to those of the limiting homoclinic $\bar{u}^0$.

Assume as in [10,11] that (2.2) may be written as a first-order system

$$W' = A^\varepsilon(x, \lambda)W \tag{2.3}$$

in an appropriate phase variable W, where A^ε is analytic in λ, C^1 in x, and continuous in ε for $\varepsilon > 0$. Then, the spectrum of the periodic waves $\bar{u}^\varepsilon$, $\varepsilon > 0$ is made up of essential spectra given [10] by the union of γ-eigenvalues λ consisting of zeroes of the *periodic Evans function*

$$E^\varepsilon(\lambda, \gamma) := \det(\Psi^\varepsilon(X^\varepsilon, \lambda) - \gamma \mathrm{Id}), \tag{2.4}$$

where $\Psi^\varepsilon(x, \lambda)$ denotes the solution operator of (2.3), with $\Psi^\varepsilon(0, \lambda) = \mathrm{Id}$ and $\gamma \in \mathbb{C}$ with $|\gamma| = 1$. A γ-eigenvalue of particular importance is the 1-eigenvalue $\lambda = 0$ associated with eigenfunction $\partial_x \bar{u}$ corresponding to instantaneous translation, arising through translation-invariance of the underlying PDE. In the case that $\mathcal{F}^\varepsilon$ is divergence-form, there exist other important 1-eigenvalues corresponding to variations along the manifold of nearby X^ε-periodic solutions, which in this case has dimension $\dim u + 1 > 1$ [15,23].

As shown in a variety of settings (see [4,5,15,16,18,19,29,30] and references therein), linearized and nonlinear modulational stability are implied by the properties:

(D1) the multiplicity of the 1-eigenvalue $\lambda = 0$ is equal to the dimension d of the manifold of nearby X^ε-periodic solutions (in the typical case considered by Gardner [11], $d = 1$).
(D2) other than the 1 eigenvalue $\lambda = 0$, there are no other γ-eigenvalues with $\Re\lambda \geq 0$.
(D3) parametrizing $\gamma = e^{ikX}$, $\Re\lambda \leq -\eta k^2$ for $0 \leq kX \leq 2\pi$, for some $\eta > 0$.

Accordingly, these are the spectral properties that we wish to investigate. In particular, note that (D3) concerns not only location, but curvature of the spectral loop through $\lambda = 0$.

Conditions (D1)–(D2) are easily seen to be necessary for linearized modulational stability, while condition (D3) implies a Gaussian rate of time-algebraic decay sufficient to close a nonlinear iteration; see [15,16,24,29,30] for further discussion.

2.1. Assumptions

Loosely following [11], we assume, for $|\lambda| \leq M$:

(H1) $X^\varepsilon \to \infty$ as $\varepsilon \to 0$.
(H2) $|A^0(x, \lambda) - A^0_\infty(\lambda)| \leq C(M)e^{-\nu|x|}$, for some $C(M), \nu > 0$.
(A3) $|\bar{u}^\varepsilon(x) - \bar{u}^0(x)| \leq \delta(\varepsilon)$ for $|x| \leq \frac{X^\varepsilon}{2}$, with $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

In order to obtain the quantitative estimates we require, we augment (A3) with

$$\delta(\varepsilon) \leq Ce^{-\bar{\theta}X^\varepsilon/2} \text{ for some } \bar{\theta} > 0. \tag{2.5}$$

Remark 2.1. Condition (2.5) is an additional assumption beyond those made in [11]. However, it follows from (A3) in the standard case that the vertex $\bar{u}^0_\infty = \bar{u}(\pm\infty)$ of the limiting homoclinic is a hyperbolic rest point of the traveling-wave ODE, under the generically satisfied transversality condition that the associated Melnikov separation function be full rank with respect to ε, as do (H1)–(H2) as well, with $\nu = \bar{\theta} = \alpha$, were α is the minimum growth/decay rate of the linearized equations about u^0_∞; see [28, Prop. 5.1 pp. 166–167]. Thus, Gardner's original condition (A3) is the main assumption in practical terms.

Remark 2.2. In the planar Hamiltonian traveling-wave ODE setting, for which all periodics and the limiting homoclinic lie in the same phase portrait of a single traveling-wave ODE, setting ε to be the distance of $\bar{u}^\varepsilon(\cdot)$ from the saddle-point $\bar{u}^0_\infty$, one may compute more or less explicitly that $X^\varepsilon \sim c\log\varepsilon^{-1}$ and $\delta(\varepsilon) \le C\varepsilon$, with $|\bar{u}^\varepsilon - \bar{u}^0| \le C\varepsilon^2$ away from $\bar{u}^0_\infty$. This gives another class of interesting examples to which our assumptions apply.

As a consequence of (A3) we obtain for $|\lambda| \le M$, $|A^\varepsilon(x,\lambda) - A^0(x,\lambda)| \le C(M)\delta(\varepsilon)$ as in assumption (iii) of [11], p. 152, yielding together with (2.5):

(H3) $|A^\varepsilon(x,\lambda) - A^0(x,\lambda)| \le C(M)e^{-\bar{\theta}X^\varepsilon/2}$ for $|\lambda| \le M$, $|x| \le \frac{X^\varepsilon}{2}$, and $\bar{\theta} > 0$.

Hereafter, we drop the motivating assumption (A3) and work similarly as in [11] with hypotheses (H1)–(H3) on the first-order eigenvalue system (2.3) alone.

3. The homoclinic and rescaled periodic Evans functions

We begin by formulating the homoclinic and periodic Evans functions following the approach of [22,31,32], in a way that is particularly convenient for their comparison.

3.1. Reduction to constant coefficients

Adapting the asymptotic ODE techniques developed in [22,26,31,32] for problems on the half-line (see Appendix A), we obtain the following quantitative description relating (2.3) to a constant-coefficient version of the homoclinic eigenvalue problem $W' = A^0(x,\lambda)W$.

Lemma 3.1. *Assuming (H1)–(H3), for each $\varepsilon \ge 0$, there exist in a neighborhood of any $|\lambda_0| \le M$ bounded and uniformly invertible linear transformations $P^\varepsilon_+(x,\lambda)$ and $P^\varepsilon_-(x,\lambda)$ defined on $x \ge 0$ and $x \le 0$, respectively, analytic in λ as functions into $L^\infty[0,\pm\infty)$, such that, for any $0 < \bar{\eta} < min(\bar{\theta},\nu)$, $\bar{\theta}$, ν as in (H2)–(H3), some $C > 0$, and $|x| \le \frac{X^\varepsilon}{2}$,*

$$P^\varepsilon_\pm(\pm X^\varepsilon/2) = \mathrm{Id}, \tag{3.1}$$

$$|(P^\varepsilon - P^0)_\pm| \le Ce^{-\bar{\eta}X^\varepsilon/2} \quad \textit{for} \;\; x \gtrless 0, \tag{3.2}$$

and the change of coordinates $W =: P^\varepsilon_\pm Z$ reduces (2.3) to the constant-coefficient system

$$Z' = A^0_\infty Z, \quad \textit{for} \;\; x \gtrless 0 \; \textit{and} \;\; |x| \le \frac{X^\varepsilon}{2}. \tag{3.3}$$

Proof. Extending $A^\varepsilon(x,\lambda)$ by value A^0_∞ for $|x| > \frac{X^\varepsilon}{2}$, we obtain a modified family of coefficient matrices agreeing with A^ε on $|x| \le \frac{X^\varepsilon}{2}$ and satisfying

$$|(A^\varepsilon(x,\lambda) - A^\varepsilon_\infty) - (A^0(x,\lambda) - A^0_\infty)| = |A^\varepsilon(x,\lambda) - A^0(x,\lambda)| \le C(M)e^{-\bar{\theta}X^\varepsilon/2}$$

for $|x| \le X^\varepsilon/2$, and

$$|(A^\varepsilon(x,\lambda) - A^\varepsilon_\infty) - (A^0(x,\lambda) - A^0_\infty)| = |A^0(x,\lambda) - A^0_\infty| \le C(M)e^{-\nu|x|}$$

for $|x| \ge X^\varepsilon/2$, yielding for all x the estimate

$$|(A^\varepsilon(x,\lambda) - A^\varepsilon_\infty) - (A^0(x,\lambda) - A^0_\infty)| \le C(M)\delta_2(\varepsilon)e^{-\sigma|x|} \tag{3.4}$$

for $\delta_2(\varepsilon) := e^{-\bar\eta X^\varepsilon/2}$, $0 < \sigma < \min\{\nu, \bar\theta\} - \bar\eta$, and, trivially,

$$|A^\varepsilon_\infty - A^0_\infty| = 0 \le C(M)\delta_2(\varepsilon). \tag{3.5}$$

Likewise, we have

$$\begin{aligned} |A^\varepsilon(x,\lambda) - A^\varepsilon_\infty(\lambda)| &= |A^\varepsilon(x,\lambda) - A^0_\infty(\lambda)| \\ &\le |A^\varepsilon(x,\lambda) - A^0(x,\lambda)| + |A^0(x,\lambda) - A^0_\infty(\lambda)| \\ &\le 2C(M)e^{-\min\{\nu,\bar\theta\}|x|}, \end{aligned} \tag{3.6}$$

hence also, by $\sigma < \min\{\nu, \bar\theta\}$, evidently

$$|A^\varepsilon(x,\lambda) - A^\varepsilon_\infty(\lambda)| = |A^\varepsilon(x,\lambda) - A^0_\infty(\lambda)| \le 2C(M)e^{-\sigma|x|}. \tag{3.7}$$

Using (3.6) and applying Lemma A.1, Appendix A, with $p = \varepsilon$ and $\theta = \min\{\nu, \bar\theta\}$, we obtain $|P^\varepsilon_\pm - \mathrm{Id}| \le Ce^{-\bar\eta|x|}$ for $x \gtrless 0$. Moreover, by Remark A.2, $(P^\varepsilon_\pm)' = A^p P^\varepsilon_\pm - P^\varepsilon_\pm A^\varepsilon_\pm$ and $P^\varepsilon_\infty = \mathrm{Id}$, yielding $(P^\varepsilon_\pm)' = 0$ for $|x| \ge X^\varepsilon/2$, and therefore (3.1). Finally, using (3.4)–(3.5), (3.7), we obtain (3.2) by Lemma A.3, Appendix A, with $p = \varepsilon$, $\delta(p) = \delta_2(\varepsilon)$, and $\theta = \sigma$. □

3.2. The homoclinic Evans function

Away from a finite set of curves $\lambda_j(k)$ determined by the dispersion relation $ik \in \sigma(A^0_\infty(\lambda))$, $k \in \mathbb{R}$, where σ denotes spectrum, the eigenvalues of A^0_∞ have nonvanishing real part. Denote by Λ_r the open components of $\mathbb{C} \setminus \{\lambda_j(k)\}$. We refer to Λ_r as the *domains of hyperbolicity* of A^0_∞. Denote by n_r the number of negative real part eigenvalues of A^0_∞.

Definition 3.2 *([3,12]).* On each domain of hyperbolicity Λ_r, the homoclinic Evans function is defined as

$$D^0_r(\lambda) := \frac{\det(R^-, R^+)|_{x=0}}{\det(R^-_\infty, R^+_\infty)} = \frac{\det(P^0_- R^-_\infty, P^0_+ R^+_\infty)}{\det(R^-_\infty, R^+_\infty)}, \tag{3.8}$$

where R^-_∞ is any matrix whose columns are a basis for the unstable subspace of A^0_∞, R^+_∞ is any matrix whose columns are a basis for the stable subspace of A^0_∞, and $R^-(x) := P^0_-(x)e^{A^0_\infty x}R^-_\infty$ and $R^+(x) := P^0_+(x)e^{A^0_\infty x}R^+_\infty$ are matrices whose columns are bases for the subspaces of solutions of (2.3) decaying as $x \to -\infty$ and $x \to +\infty$, respectively.

Evidently, each D^0_r is analytic in λ on Λ_r, and vanishes at $\lambda_0 \in \Lambda_r$ if and only if λ_0 is an eigenvalue of L^0. Moreover, it can be shown with in great generality that its zeros correspond in multiplicity with the eigenvalues of L^0; see [12,21], and references therein.

3.3. The rescaled periodic Evans function

Following [11], we note that, by Abel's formula, $E^\varepsilon(\lambda, \gamma)$ may be written alternatively as

$$E^\varepsilon(\lambda,\gamma) = \tilde E^\varepsilon(\lambda,\gamma)e^{\int_0^{X^\varepsilon/2} \mathrm{tr}A^\varepsilon(\lambda,y)dy},$$

where

$$\tilde{E}^\varepsilon(\lambda,\gamma) := \det(\Psi^\varepsilon(0,\lambda)\Psi^\varepsilon(-X^\varepsilon/2,\lambda)^{-1} - \gamma\Psi^\varepsilon(0,\lambda)\Psi^\varepsilon(X^\varepsilon/2,\lambda)^{-1}) \tag{3.9}$$

is a "balanced" periodic Evans function defined symmetrically about $x = 0$ similarly as the homoclinic Evans function.

Definition 3.3. On Λ_r, we define the *rescaled balanced periodic Evans function* as

$$D_r^\varepsilon(\lambda,\gamma) := e^{-\operatorname{tr} A^0_\infty \Pi_u X^\varepsilon/2} e^{\operatorname{tr} A^0_\infty \Pi_s X^\varepsilon/2}(-\gamma)^{-n_r}\tilde{E}^\varepsilon(\lambda,\gamma), \tag{3.10}$$

where Π_u and Π_s denote the unstable and stable eigenprojections associated with $A^0_\infty(\lambda)$.

4. Convergence to isolated point spectra

We begin by recovering in a particularly direct and simple fashion the basic result of Gardner [11] on bifurcation from isolated point spectra; for related arguments, see [23,28,31].

4.1. Convergence as $X^\varepsilon \to \infty$

To show convergence, we first reformulate the homoclinic Evans function as a Jost-function type determinant such as appears in the definition of the periodic Evans function, involving the difference of two matrix-valued solutions. See [13,33] for related discussion.

Lemma 4.1. *Assuming (H1)–(H3), for* $\begin{pmatrix} L^-_\infty \\ L^+_\infty \end{pmatrix} := (R^-_\infty, R^+_\infty)^{-1}$,

$$D_r^0(\lambda) = (-1)^{n_r}\det(R^- L^-_\infty - R^+ L^+_\infty)|_{x=0}. \tag{4.1}$$

Proof. Factoring $(R^- L^-_\infty - R^+ L^+_\infty) = (R^-, -R^+)\begin{pmatrix} L^-_\infty \\ L^+_\infty \end{pmatrix} = (R^-, -R^+)(R^-_\infty, R^+_\infty)^{-1}$, taking determinants, and comparing to (3.8), we obtain the result. □

Proposition 4.2. *Assuming (H1)–(H3), on each compact $K \subset \Lambda_r$, there exist $C, \eta > 0$ such that*

$$|D_r^\varepsilon(\lambda,\gamma) - D_r^0(\lambda)| \le Ce^{-\eta X^\varepsilon/2} \text{ for all } \lambda \in K, |\gamma| = 1, \tag{4.2}$$

for any η less than the minimum of $\bar{\eta}$, given in Lemma 3.1, and the spectral gap of $A^0_\infty(\lambda)$, defined as the minimum absolute value of the real parts of the eigenvalues of A^0_∞.

Remark 4.3. In the generic case discussed in Remark 2.1, the restrictions on η in Proposition 4.2 reduce to $0 < \eta <$ spectral gap of $A^0_\infty(\lambda)$, by the estimates of [28, Prop. 5.1].

Proof. Using the description of Lemma 3.1, and the spectral expansion formula

$$e^{A^0_\infty x} = e^{A^0_\infty \Pi_u x} R^-_\infty L^-_\infty + e^{A^0_\infty \Pi_s x} R^+_\infty L^+_\infty,$$

we find, using $P^\varepsilon_\pm(\pm X^\varepsilon/2) = \mathrm{Id}$ (see (3.1)) and $|e^{A^0_\infty \Pi_s X^\varepsilon/2}| \le Ce^{-\eta X^\varepsilon/2}$, that

$$\begin{aligned}
\Psi^\varepsilon(0,\lambda)\Psi^\varepsilon(-X^\varepsilon/2,\lambda)^{-1} &= P^\varepsilon_-(0)\Big(P^\varepsilon_-(-X^\varepsilon/2)e^{-A^0_\infty X^\varepsilon/2}\Big)^{-1} \\
&= P^\varepsilon_-(0)e^{A^0_\infty X^\varepsilon/2} \\
&= P^\varepsilon_-(0)e^{A^0_\infty \Pi_u X^\varepsilon/2} R^-_\infty L^-_\infty + O(e^{-\eta X^\varepsilon/2})
\end{aligned}$$

and, likewise, $\Psi^\varepsilon(0,\lambda)\Psi^\varepsilon(X^\varepsilon/2,\lambda)^{-1} = P_+^\varepsilon(0)e^{-A_\infty^0\Pi_s X^\varepsilon/2}R_\infty^+L_\infty^+ + O(e^{-\eta X^\varepsilon/2})$, from which we find, factoring similarly as in the proof of Lemma 4.1, that

$$\begin{aligned} D_r^\varepsilon(\lambda,\gamma) &= e^{-\operatorname{tr}A_\infty^0\Pi_u X^\varepsilon/2}e^{\operatorname{tr}A_\infty^0\Pi_s X^\varepsilon/2}(-\gamma)^{-n_r}\det\begin{pmatrix}P_-^\varepsilon(0)R_\infty^- & P_+^\varepsilon(0)R_\infty^+\end{pmatrix} \\ &\quad\times\det\left(\begin{pmatrix}L_\infty^- e^{A_\infty^0\Pi_u X^\varepsilon/2}R_\infty^- & 0\\ 0 & -\gamma L_\infty^+ e^{-A_\infty^0\Pi_s X^\varepsilon/2}R_\infty^+\end{pmatrix} + O(e^{-\eta X^\varepsilon/2})\right) \\ &\quad\times\det\begin{pmatrix}L_\infty^-\\ L_\infty^+\end{pmatrix} \\ &= \det\begin{pmatrix}P_-^\varepsilon(0)R_\infty^- & P_+^\varepsilon(0)R_\infty^+\end{pmatrix}\det(\mathrm{Id}+O(e^{-\eta X^\varepsilon}))\det\begin{pmatrix}L_\infty^-\\ L_\infty^+\end{pmatrix} \\ &= D_r^0(\lambda) + O(e^{-\eta X^\varepsilon/2}), \end{aligned} \tag{4.3}$$

by $\begin{pmatrix}L_\infty^-\\ L_\infty^+\end{pmatrix} := \begin{pmatrix}R_\infty^- & R_\infty^+\end{pmatrix}^{-1}$ and the definition of D_r^0 in (3.8). □

Corollary 4.1 *([11,23,28]). Assuming (H1)–(H3), on compact $K \subset \Lambda_r$ such that D_r^0 does not vanish on ∂K, the spectra of L^ε for X^ε sufficiently large consists of loops of spectra $\lambda_{r,k}^\varepsilon(\gamma)$, $k = 1,\dots,m_r$, within $O(e^{-\eta X^\varepsilon/2m_r})$ of the eigenvalues λ_r of L^0, where m_r denotes the multiplicity of λ_r and η is as in (4.2).*

Proof. Immediate by properties of analytic functions (Rouchés Theorem). □

Remark 4.4. Note that different rescalings of $\tilde{E}^\varepsilon$ converge as $X^\varepsilon \to \infty$ to different versions D_r^0 of the homoclinic Evans function on different components Λ_r.

4.2. A flip-type stability index and behavior near $\lambda = 0$

We mention in passing the important special case of an isolated eigenvalue at $\lambda = 0$ of the limiting homoclinic wave, corresponding with translational invariance of the underlying PDE. Similar translational ($\gamma = 1$)-eigenvalues occur at $\lambda = 0$ for periodic waves of all periods X^ε. As shown in [28] by rather different Melnikov integral/Lyapunov-Schmidt computations, this exact correspondence for $\gamma = 1$ implies cancellation in $D_r^\varepsilon - D_r^0$, yielding convergence at faster exponential rate $O(e^{-\alpha\eta X^\varepsilon/2})$, where $\alpha > 1$, and also an asymptotic description of the location of ($\gamma \neq 1$)-eigenvalues near $\lambda = 0$, deciding diffusive spectral stability of spectral loops passing through the origin. In typical cases, these loops are to lowest order in $|\lambda - \lambda_*|$ ellipses with axes parallel to real and imaginary coordinate axes, hence their diffusive stability or instability is decided by whether the ($\gamma = -1$)-eigenvalue lies in the stable ($\Re\lambda < 0$) or unstable ($\Re\lambda > 0$) half-space [28, Discussion, p. 182, par. 2].

These conclusions do not follow by our straightforward computations above, and indeed would appear to be cumbersome to reproduce by such an Evans function approach. However, a related *necessary* condition $\sigma \geq 0$ based on the stability index

$$\sigma := \operatorname{sgn}E^\varepsilon(0,-1)\operatorname{sgn}E^\varepsilon(\infty_{real},-1), \tag{4.4}$$

is readily obtained from the Evans function formulation by the observation that $E^\varepsilon(\lambda,-1)$ is real-valued for λ real, hence $\delta \leq 0$ by the intermediate value theorem implies existence of a -1-eigenvalue with nonnegative real part, violating diffusive stability conditions (D2)–(D3).

The necessary condition (4.4) is valid in much more general contexts than is the necessary and sufficient condition obtained in [28], in particular to systems with conservation laws for which $\lambda = 0$ is an embedded eigenvalue of the limiting homoclinic wave. See, for example, [20, Thm. 1.9] for an important application of this principle in the case of the Saint Venant equations of inclined shallow water flow. A change of sign in δ corresponds to passage of a -1-eigenvalue through $\lambda = 0$, or "flip" bifurcation in the periodic traveling-wave ODE.

5. Convergence to essential spectra

We now turn to our main object, of bifurcation from essential homoclinic spectra of periodic spectra in the large-period limit. Recall that the essential spectrum of the homoclinic limit is given by the union of curves $\lambda_j(k)$ determined by the dispersion relation $ik \in \sigma(A^0_\infty(\lambda))$, $k \in \mathbb{R}$, bounding domains of hyperbolicity Λ_r with Evans functions D^0_r.

5.1. Convergence to an isolated arc of essential spectra

Consider the generic situation of a point of homoclinic essential spectrum $\lambda_* = \lambda_j(k_*)$ lying on a single curve λ_j, for which k_* is a nondegenerate root and unique solution of $\lambda_j(k) = \lambda_*$. Without loss of generality, suppose that λ_j separates domains of hyperbolicity Λ_1 and Λ_2, on which are defined homoclinic Evans functions D^0_1 and D^0_2, as described in (3.8).

By assumption, $\mu_* := ik_*$ is a simple imaginary eigenvalue of $A^0_\infty(\lambda_*)$, with all other eigenvalues of strictly positive or negative real part. By standard matrix perturbation theory, therefore, there exist near λ_* an analytic eigenvalue $\mu_c(\lambda)$ and associated eigenprojecton $\Pi_c(\lambda)$ with $\mu_c(\lambda_*) = ik_*$ and analytic strongly stable and unstable eigenprojections Π_{ss} and Π_{su}, corresponding at $\lambda = \lambda_*$ to center, stable, and unstable projections of $A^0_\infty(\lambda_*)$. Without loss of generality, suppose that $\Re\mu_c < 0$ on Λ_2 and $\Re\mu_c > 0$ on Λ_1. By analyticity of Π_c, Π_{ss}, and Π_{su}, the homoclinic Evans functions D^0_1 and D^0_2 extend analytically to a neighborhood of λ_*; denote their values at λ_* as $d_1 = D^0_1(\lambda_*)$ and $d_2 = D^0_2(\lambda_*)$.

Definition 5.1. Near λ_*, we define the *transitional periodic Evans function* as

$$\tilde{D}^\varepsilon(\lambda,\gamma) := e^{-\operatorname{tr} A^0_\infty \Pi_{su} X^\varepsilon/2} e^{\operatorname{tr} A^0_\infty \Pi_{ss} X^\varepsilon/2} (-\gamma)^{-n_1} \tilde{E}(\lambda,\gamma), \tag{5.1}$$

where Π_{su} and Π_{ss} denote strongly unstable and stable eigenprojections of $A^0_\infty(\lambda)$ (n_1 here corresponding to $\dim \operatorname{Range}\Pi_{ss}$), and the transitional homoclinic Evans function as

$$\tilde{H}^\varepsilon(\lambda,\gamma) := e^{\mu_c(\lambda)X^\varepsilon/2} D^0_1(\lambda) - \gamma e^{-\mu_c(\lambda)X^\varepsilon/2} D^0_2(\lambda). \tag{5.2}$$

Proposition 5.2. *Assuming (H1)–(H3), let $\lambda_* = \lambda_j(k_*)$ lie on an isolated arc of homoclinic essential spectra as described above. Then, there exists $C_1 > 0, \eta > 0$ such that for any $C > 0$, for X^ε sufficiently large,*

$$|\tilde{D}^\varepsilon(\lambda,\gamma) - \tilde{H}^\varepsilon(\lambda,\gamma)| \le C_1 e^{-\eta X^\varepsilon/2}, \text{ for all } \lambda \in B(\lambda_*, C/X^\varepsilon), |\gamma| = 1. \tag{5.3}$$

Proof. Letting R^-_∞, R^c_∞, and R^+_∞ denote bases of $\operatorname{Range}\Pi_{su}$, $\operatorname{Range}\Pi_c$, and $\operatorname{Range}\Pi_{ss}$, set

$$\begin{pmatrix} L^-_\infty \\ L^c_\infty \\ L^+_\infty \end{pmatrix} = \begin{pmatrix} R^-_\infty & R^c_\infty & R^+_\infty \end{pmatrix}^{-1}.$$

Expanding as in the proof of Proposition 4.2 via spectral resolution of A^0_∞ gives

$$\begin{aligned}\tilde{D}^\varepsilon(\lambda,\gamma) &= e^{-\operatorname{tr} A^0_\infty \Pi_{su} X^\varepsilon/2} e^{\operatorname{tr} A^0_\infty \Pi_{ss} X^\varepsilon/2}(-\gamma)^{-n_1} \\ &\quad \times \det \Big(P^\varepsilon_-(0) e^{A^0_\infty \Pi_{su} X^\varepsilon/2} R^-_\infty L^-_\infty + P^\varepsilon_-(0) e^{\mu_c(\lambda) X^\varepsilon/2} R^c_\infty L^c_\infty \\ &\quad - \gamma P^\varepsilon_+(0) e^{-\mu_c(\lambda) X^\varepsilon/2} R^c_\infty L^c_\infty - \gamma P^\varepsilon_+(0) e^{-A^0_\infty \Pi_{ss} X^\varepsilon/2} R^+_\infty L^+_\infty + O(e^{-\eta X^\varepsilon/2}) \Big).\end{aligned}$$

Using $\Re \mu_c X^\varepsilon = O(1)$ on $B(\lambda_*, C/X^\varepsilon)$ and factoring as in the proof of Proposition 4.2 gives

$$\begin{aligned}\tilde{D}^\varepsilon(\lambda,\gamma) &= \det \big(P^\varepsilon_-(0) R^-_\infty \quad \big(e^{\mu_c(\lambda) X^\varepsilon/2} P^\varepsilon_-(0) R^c_\infty - \gamma e^{-\mu_c(\lambda) X^\varepsilon/2} P^\varepsilon_+(0) R^c_\infty\big) \quad P^\varepsilon_+(0) R^+_\infty \big) \\ &\quad \times \det \begin{pmatrix} L^-_\infty \\ L^c_\infty \\ L^+_\infty \end{pmatrix} + O(e^{-\eta X^\varepsilon/2}),\end{aligned}$$

or, expanding the first determinant with respect to the middle column and recalling the definitions of D^0_1 and D^0_2, we have

$$\tilde{D}^\varepsilon(\lambda,\gamma) = e^{\mu_c(\lambda) X^\varepsilon/2} D^0_1(\lambda) - \gamma e^{-\mu_c(\lambda) X^\varepsilon/2} D^0_2(\lambda) + O(e^{-\eta X^\varepsilon/2}) =: \tilde{H}^\varepsilon(\lambda,\gamma) + O(e^{-\eta X^\varepsilon/2}). \quad \square$$

Corollary 5.1. *Assuming (H1)–(H3), let $\lambda_* = \lambda_j(k_*)$ lie on an isolated arc of homoclinic essential spectra as above, with $D^0_1(\lambda_*)$, $D^0_2(\lambda_*) \neq 0$. Then, denoting $\gamma = e^{ikX^\varepsilon}$, there exists $C_1 > 0$ such that for any $C > 0$, for X^ε sufficiently large, the γ-eigenvalues of $\bar{u}^\varepsilon$ in $B(\lambda_*, C/X^\varepsilon)$, corresponding to zeros of $E^\varepsilon(\cdot,\gamma)$, lie within C_1/X^ε of the set*

$$\{\lambda_j(\kappa) : \kappa = k \bmod (2\pi/X^\varepsilon)\}. \tag{5.4}$$

Proof. From (5.2)–(5.3) and Taylor expansion $\mu_c(\lambda_* + z/X^\varepsilon) = ik_* + \mu_c'(\lambda_*) z/X^\varepsilon + O(|X^\varepsilon|^{-2})$, we see immediately that for $z \in B(0,C)$, $|\hat{\gamma}| = 1$ fixed

$$d_1^{-1} e^{-ik_* X^\varepsilon/2 + \mu_c'(\lambda_*) z/2} \tilde{D}^\varepsilon(\lambda_* + z/X^\varepsilon, e^{ik_* X^\varepsilon}\hat{\gamma}) \to e^{\mu_c'(\lambda_*) z} - \hat{\gamma} d_2/d_1$$

as $\varepsilon \to 0$, at rate $O(|X^\varepsilon|^{-1})$, from which we find by properties of analytic functions (Rouchés Theorem) that zeros of $\tilde{D}^\varepsilon(\lambda_* + z/X^\varepsilon, e^{ik_* X^\varepsilon}\hat{\gamma})$ converge at rate $O(|X^\varepsilon|^{-1})$ to solutions of

$$\mu_c'(\lambda_*) z = [\ln(\hat{\gamma}) + \ln(d_2/d_1)] \bmod (2\pi i). \tag{5.5}$$

Converting back to λ coordinates, we have that $\lambda - \lambda_* = z/X^\varepsilon$ converges at rate $O(|X^\varepsilon|^{-2})$ to

$$(1/\mu_c'(\lambda_*))(\ln(\hat{\gamma}) + \ln(d_2/d_1))/X^\varepsilon \bmod (2\pi i/X^\varepsilon), \tag{5.6}$$

whence, using $\hat{\gamma} = \gamma e^{-ik_* X^\varepsilon} = e^{ikX^\varepsilon - ik_* X^\varepsilon}$, or $\ln(\hat{\gamma})/X^\varepsilon = ik - ik_*$ (mod $2\pi i/X^\varepsilon$), we find that $\mu_c(\lambda) = ik_* + \mu_c'(\lambda_*)(\lambda - \lambda_*) + O(|X^\varepsilon|^{-2})$ converges at rate $O(|X^\varepsilon|^{-2})$ to

$$[ik + \ln(d_2/d_1)/X^\varepsilon] \bmod (2\pi i/X^\varepsilon),$$

and thus at $O(|X^\varepsilon|^{-1})$ rate to $ik \bmod (2\pi i/X^\varepsilon)$. By definition of λ_j, this is equivalent to convergence of λ at rate $O(|X^\varepsilon|^{-1})$ to $\lambda_j(\kappa)$ for $\kappa = k \bmod (2\pi/X^\varepsilon)$. Noting that convergence at each step of the argument is uniform with respect to $\hat{\gamma}$, we obtain the result. $\square$

Remark 5.3. Estimate (5.6) shows that the rate of convergence $O(|X^\varepsilon|^{-1})$ is sharp unless $d_1 = d_2$, giving an explicit corrector $\ln(d_2/d_1)/\mu_c'(\lambda_*) X^\varepsilon$ valid to order $O(|X^\varepsilon|^{-2})$.

5.2. Convergence to embedded point spectra

Finally, we consider the case of an embedded homoclinic eigenvalue λ_* of multiplicity m contained in an isolated arc of essential spectrum λ_j dividing regions of hyperbolicity Λ_1 and Λ_2 as described in Section 5.1. By multiplicity m eigenvalue, we mean a value with an m-dimensional subspace of decaying generalized eigenfunctions. This implies that both homoclinic Evans functions D_1^0 and D_2^0 have a zero of multiplicity at least m at $\lambda = \lambda_*$, since they differ only with respect to the nondecaying mode μ_c. We make the additional nondegeneracy assumption that D_1^0 and D_2^0 have zeros at λ_* of exactly multiplicity m.

Corollary 5.2. *Assuming (H1)–(H3), let $\lambda_* = \lambda_j(k_*)$ be a point lying on an isolated arc of homoclinic essential spectra at which D_1^0, D_2^0 possess zeros of degree m. Then, denoting $\gamma = e^{ikX}$, there exists $C_1 > 0$ such that for any $C > 0$, $\eta > \tilde\eta > 0$, for X^ε sufficiently large, the γ-eigenvalues of $\bar u^\varepsilon$ in $B(\lambda_*, C/X^\varepsilon)$, or zeros of $E^\varepsilon(\cdot, \gamma)$, consist of points lying within C_1/X^ε of $\{\lambda_j(\kappa) : \kappa = k \mod(2\pi/X^\varepsilon)\}$ plus m points lying within $C_1 e^{-\tilde\eta X^\varepsilon/2(m+1)}$ of λ_*.*

Proof. Factoring $D_1^0(\lambda) = (\lambda - \lambda_*)^m \hat D_1^0(\lambda)$, $D_2^0(\lambda) = (\lambda - \lambda_*)^m \hat D_2^0(\lambda)$, setting $\hat d_1 := \hat D_1^0(\lambda_*)$, $\hat d_2 := \hat D_2^0(\lambda_*)$, and applying again (5.3), we find that for $z \in B(0, C)$, $|\hat\gamma| = 1$ fixed,

$$(X^\varepsilon)^m \hat d_1^{-1} e^{-ik_* X^\varepsilon/2 + \mu_c'(\lambda_*)z/2} \tilde D^\varepsilon(\lambda_* + z/X^\varepsilon, e^{ik_* X^\varepsilon}\hat\gamma) \to z^m \big(e^{\mu_c'(\lambda_*)z} - \hat\gamma \hat d_2/\hat d_1\big)$$

as $\varepsilon \to 0$, at rate $O(|X^\varepsilon|^{-1})$, from which we find that the zeros of $\tilde D^\varepsilon(\lambda_* + z/X^\varepsilon, e^{ik_* X^\varepsilon}\hat\gamma)$ converge at rate $O(|X^\varepsilon|^{-1/(m+1)})$ to solutions of

$$\mu_c'(\lambda_*) z = [\ln(\hat\gamma) + \ln(\hat d_2/\hat d_1)] \mod(2\pi i), \tag{5.7}$$

and to the m-tuple root at $z = 0$. Converted back to λ-coordinates, this gives the desired result of $O(|X^\varepsilon|^{-1})$ convergence to $\mathcal{L}_k = \{\lambda_j(\kappa) : \kappa = k \mod(2\pi/X^\varepsilon)\}$ along with the suboptimal result of $O(|X^\varepsilon|^{-1+1/(m+1)})$ convergence to the m-tuple root at $\lambda = \lambda_*$.

To obtain the optimal $O(e^{-\tilde\eta X^\varepsilon/2(m+1)})$ rate of convergence stated for the m-fold eigenvalue λ_*, we may go back again to (5.3) to obtain the sharper result that

$$(X^\varepsilon)^m (\hat D_1^0)^{-1} e^{-ik_* X^\varepsilon/2 + (\mu_c(\lambda_* + z/X^\varepsilon) - \mu_c(\lambda_*))X^\varepsilon/2} \tilde D^\varepsilon(\lambda_* + z/X^\varepsilon, e^{ik_* X^\varepsilon}\hat\gamma)$$

lies within $O(e^{-\tilde\eta X^\varepsilon/2})$ of $z^m\big(e^{(\mu_c(\lambda_* + z/X^\varepsilon) - \mu_c(\lambda_*))X^\varepsilon} - \hat\gamma \hat D_2^0/\hat D_1^0\big)$, from which we may obtain the result by direct application of Rouchés Theorem, on a case-by-case basis depending whether or not $\hat d_2/\hat d_1 = e^{ikX^\varepsilon}$, i.e., whether or not $\lambda_* \in \mathcal{L}_k$. We omit the details. □

5.3. Behavior near an embedded eigenvalue at $\lambda = 0$

In the case of a multiplicity-one embedded "translational" homoclinic eigenvalue at $\lambda = 0$, it often transpires that, besides the corresponding translational ($\gamma = 1$)-eigenvalues of nearby periodic waves at $\lambda = 0$, the ($\gamma = 1$)-eigenvalue at $\lambda = 0$ has additional multiplicity equal to the number of arcs λ_j of homoclinic essential spectra on which it lies. See in particular the case of hyperbolic and parabolic balance laws discussed in [18,19]. In this case we may deduce from (5.7) that $\hat d_1 = \hat d_2$, giving an improved convergence rate of $(C_1/X^\varepsilon)^2$ of periodic to homoclinic essential spectra as described in Corollary 5.2.

6. Application to Saint Venant equations

The Saint Venant equations for inclined shallow-water flow are, in nondimensional form,

$$\partial_t h + \partial_x(hu) = 0, \quad \partial_t(hu) + \partial_x\left(hu^2 + \frac{h^2}{2F^2}\right) = h - |u|u + \nu\partial_x(h\partial_x u), \tag{6.1}$$

where h is fluid height, u vertical fluid velocity average, x longitudinal distance, t time, F a Froude number given by the ratio between (a chosen reference) speed of the fluid and speed of gravity waves, and $\nu = R_e^{-1}$, with R_e the Reynolds number of the fluid. Terms h and $|u|\,u$ on the right represent opposing forces of gravity and turbulent bottom friction.

Well-known solutions of (6.1) are periodic *roll* (traveling) *waves* [7–9] advancing with constant speed down a canal or spillway. For fixed Froude number $F > 2$, these appear in families indexed by period X and average height u over one period, arising through a classical Hopf to homoclinic bifurcation scenario [2]. In particular, they feature a homoclinic limit as studied in this note, with a single embedded eigenvalue at $\lambda = 0$, contained in a single arc of unstable essential spectra, and all other spectra strictly stable; see [3] for further details. As noted in the introduction, this is a case to which the results of [10,11,28] do not apply but that can be treated by our analysis here.

Specifically, Corollary 5.2 verifies the intuitive conclusion that periodic roll waves are unstable in the large-period limit due to convergence of periodic spectra to unstable homoclinic essential spectra, settling the question of large-period stability. However, there is a much more interesting phenomenon involved with the homoclinic limit, worthy of further investigation. Namely, as noted in [25] more generally, mathematical models of inclined thin film flow appear to share the feature that homoclinic waves are essentially unstable; yet, both experiment and models of inclined thin film flow yield asymptotic behavior consisting of the approximate superposition of well-separated homoclinic waves.

A heuristic explanation of this paradox given in [2,3] is that sufficiently closely arrayed homoclinic waves can stabilize each others convective essential instabilities, manifested as exponentially growing perturbations traveling with a nonzero group velocity, through de-amplifying properties of the localized homoclinic pulses encoded in their strictly stable point spectrum. Indeed, this appears to match well with observed onset of stability of periodic waves at periods suggested by this proposed mechanism [3, Section 6].

However, up to now, there is lacking a rigorous explanation at the level of spectral stability relating the observed behavior to properties of the homoclinic wave. To carry out such an analysis by asymptotic techniques like those used here and in [10,11,28] seems an outstanding open problem in periodic stability theory and dynamics of thin-film flow.

Appendix A. Asymptotic ODE theory

Here, we recall the asymptotic Evans function results cited earlier; for proofs, see, e.g., [32].

A.1. The conjugation lemma

Consider a general first-order system

$$W' = A^p(x,\lambda)W \tag{A.1}$$

with asymptotic limits $A^p_\pm$ as $x \to \pm\infty$, where $p \in \mathbb{R}^m$ denote model parameters.

Lemma A.1 *([22,26]). Suppose for fixed $\theta > 0$ and $C > 0$ that*

$$|A^p - A^p_\pm|(x,\lambda) \le Ce^{-\theta|x|} \tag{A.2}$$

for $x \gtrless 0$ uniformly for (λ, p) in a neighborhood of (λ_0), p_0 and that A varies analytically in λ and continuously in p as a function into $L^\infty(x)$. Then, there exist in a neighborhood of (λ_0, p_0) invertible linear transformations $P_+^p(x,\lambda) = I + \Theta_+^p(x,\lambda)$ and $P_-^p(x,\lambda) = I + \Theta_-^p(x,\lambda)$ defined on $x \geq 0$ and $x \leq 0$, respectively, analytic in λ and continuous in p as functions into $L^\infty[0, \pm\infty)$, such that

$$|\Theta_\pm^p| \leq C_1 e^{-\bar\theta|x|} \quad \text{for } x \gtrless 0, \tag{A.3}$$

for any $0 < \bar\theta < \theta$, some $C_1 = C_1(\bar\theta, \theta) > 0$, and the change of coordinates $W =: P_\pm^p Z$ reduces (A.1) to the constant-coefficient limiting systems

$$Z' = A_\pm^p Z \quad \text{for } x \gtrless 0. \tag{A.4}$$

Remark A.2. As shown in the proof (e.g., [32]), necessarily also $(P_\pm^p)' = A^p P_\pm - P_\pm A_\pm^p$.

A.2. The convergence lemma

Consider a family of first-order equations

$$W' = A^p(x,\lambda)W \tag{A.5}$$

indexed by a parameter p, and satisfying exponential convergence condition (A.2) uniformly in p. Suppose further that, for some $\delta(p) \to 0$ as $p \to 0$,

$$|(A^p - A_\pm^p) - (A^0 - A_\pm^0)| \leq C\delta(p)e^{-\theta|x|}, \qquad \theta > 0 \tag{A.6}$$

and

$$|(A^p - A^0)_\pm)| \leq C\delta(p). \tag{A.7}$$

Lemma A.3 *([1,26]). Assuming (A.2) and (A.6)–(A.7), for $|p|$ sufficiently small, there exist invertible linear transformations $P_+^p(x,\lambda) = I + \Theta_+^p(x,\lambda)$ and $P_-^0(x,\lambda) = I + \Theta_-^p(x,\lambda)$ defined on $x \geq 0$ and $x \leq 0$, respectively, analytic in λ as functions into $L^\infty[0, \pm\infty)$, such that*

$$|(P^p - P^0)_\pm(x)| \leq C_1\delta(p)e^{-\bar\theta|x|} \quad \text{for } x \gtrless 0, \tag{A.8}$$

for any $0 < \bar\theta < \theta$, some $C_1 = C_1(\bar\theta, \theta) > 0$, and the change of coordinates $W =: P_\pm^p Z$ reduces (A.5) to the constant-coefficient limiting systems

$$Z' = A_\pm^p(\lambda) Z \quad \text{for } x \gtrless 0. \tag{A.9}$$

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