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Improved critical eigenfunction estimates on manifolds of nonpositive curvature

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We prove new improved endpoint, L^{p_c}, $p_c = \frac{2(n+1)}{n-1}$, estimates (the "kink point") for eigenfunctions on manifolds of nonpositive curvature. We do this by using energy and dispersive estimates for the wave equation as well as new improved L^p, $2 < p < p_c$, bounds of Blair and the author [4], [6] and the classical improved supnorm estimates of Bérard [3]. Our proof uses Bourgain's [7] proof of weak-type estimates for the Stein-Tomas Fourier restriction theorem [42]–[43] as a template to be able to obtain improved weak-type L^{p_c} estimates under this geometric assumption. We can then use these estimates and the (local) improved Lorentz space estimates of Bak and Seeger [2] (valid for all manifolds) to obtain our improved estimates for the critical space under the assumption of nonpositive sectional curvatures.

1. Introduction

Let (M, g) be a compact n-dimensional Riemannian manifold and let Δ_g be the associated Laplace-Beltrami operator. We shall consider L^2-normalized eigenfunctions of frequency λ, i.e.,

$$-\Delta_g e_\lambda = \lambda^2 e_\lambda, \quad \int_M |e_\lambda|^2 \, dV_g = 1,$$

with dV_g denoting the volume element.

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The author showed in [28] that one has the following bounds for a given $2 < p \le \infty$ and $\lambda \ge 1$:

$$(1) \quad \|e_\lambda\|_{L^p(M)} \le C\lambda^{\mu(p)}, \quad \mu(p) = \max\left(\tfrac{n-1}{2}\left(\tfrac{1}{2} - \tfrac{1}{p}\right), n\left(\tfrac{1}{2} - \tfrac{1}{p}\right) - \tfrac{1}{2}\right).$$

These estimates are saturated on the round sphere by zonal functions, Z_λ, for $p \ge \frac{2(n+1)}{n-1} = p_c$ and for $2 < p \le p_c$ by the highest weight spherical harmonics $Q_\lambda = \lambda^{\frac{n-1}{4}}(x_1 + ix_2)^k$, if $\lambda = \lambda_k = \sqrt{(k+n-1)k}$. See [27]. The zonal functions have the maximal concentration at points allowed by the sharp Weyl formula, while the highest weight spherical harmonics have the maximal concentration near periodic geodesics that is allowed by (1).

Over the years there has been considerable work devoted to determining when (1) can be improved. Although not explicitly stated, this started in the work of Bérard [3], which implies that for manifolds of nonpositive curvature the estimate for $p = \infty$ can be improved by a $(\log \lambda)^{-\frac{1}{2}}$ factor (see [31, Proposition 3.6.2]). By interpolation with the special case of $p = p_c$ in (1), one obtains improvement for all exponents $p_c < p \le \infty$, which was further recently improved by Hassell and Tacy [14]. The author and Zelditch [35] showed that for generic manifolds one can obtain $o(\lambda^{\mu(p)})$ bounds for $\|e_\lambda\|_{L^p}$ if $p_c < p \le \infty$. These results were improved in [34] and in [39] and [40]. In the latter two articles, a necessary and sufficient condition in the real analytic setting was obtained for such bounds for exponents larger than the critical one, p_c.

The estimate for the complementary range of $2 < p < p_c$ has also garnered much attention of late. In works of Bourgain [8] and the author [30] for $n = 2$, it was shown that improvements of (1) for this range is equivalent to improvements of the geodesic restriction estimates of Burq, Gérard and Tzvetkov [9], as well as natural Kakeya-Nikodym bounds introduced in [30] measuring L^2-concentration of eigenfunctions on $\lambda^{-\frac{1}{2}}$ tubes about unit-length geodesics. This is all very natural in view of the properties of the highest weight spherical harmonics (see [30] and [32] for further discussion). Using this equivalence and improved geodesic restriction estimates, the author and Zelditch showed in [38] that $\|e_\lambda\|_{L^p} = o(\lambda^{\mu(p)})$ for $2 < p < p_c$ if $n = 2$ under the assumption of nonpositive curvature, and similar improved bounds in higher dimensions and the equivalence of this problem and improved Kakeya-Nikodym estimates were obtained by Blair and the author in [5]. Very recently, in [4] and [6], we were able to obtain logarithmic improvements for this range of exponents in all dimensions under the assumption of nonpositive curvature using microlocal analysis and the classical Toponogov triangle comparison theorem in Riemannian geometry. In addition to relationships with geodesic concentration and quantum ergodicity, improvements of (1) for $2 < p \le p_c$ are of interest because of their connection with nodal problems for eigenfunctions (see, e.g., [5], [4], [11], [15], [17], [36] and [37]).

Despite the success in obtaining improvements of (1) for the ranges $2 < p < p_c$ and $p_c < p \le \infty$, improvements for the critical space where $p = p_c = \frac{2(n+1)}{n-1}$ have proven to be elusive. The special case of (1) for this exponent reads as follows:

$$\|e_\lambda\|_{L^{\frac{2(n+1)}{n-1}}(M)} \le C\lambda^{\frac{n-1}{2(n+1)}}, \tag{1'}$$

and by interpolating with the trivial L^2 estimate and the sup-norm estimate $\|e_\lambda\|_{L^\infty} = O(\lambda^{\frac{n-1}{2}})$, which is implicit in Avakumović [1] and Levitan [23], one obtains all of the other bounds in (1).

Improving (1′) has been challenging in part because it detects both point concentration and concentration along periodic geodesics (as we mentioned for the sphere). The techniques developed for improving (1) for $p > p_c$ focused on the former and the more recent ones for $2 < p < p_c$ focused on the latter. To date the only improvements of (1′) are recent ones of Hezari and Rivière [15] who used small-scale variants of the classical quantum ergodic results of Colin de Verdière [12], Snirelman [26] and Zelditch [44] (see also [45]) to show that for manifolds of strictly negative sectional curvature there is a density one sequence of eigenfunctions for which (1′) can be logarithmically improved. The L^2-improvements for small balls that were used had been obtained independently by Han [13] earlier, and, in a companion article [33] to [15], the author showed that, under the weaker assumption of ergodic geodesic flow, one can improve (1′) for a density one sequence of eigenfunctions.

Our main result here is that, under the assumption of nonpositive curvature, one can obtain improved L^{p_c} estimates for *all* eigenfunctions:

Theorem 1. *Assume that (M, g) is of nonpositive curvature. Then there is a constant $C = C(M, g)$ so that for $\lambda \gg 1$*

$$\|e_\lambda\|_{L^{\frac{2(n+1)}{n-1}}(M)} \le C\lambda^{\frac{n-1}{2(n+1)}} \big(\log\log\lambda\big)^{-\frac{2}{(n+1)^2}}. \tag{2}$$

Additionally,

$$\big\|\chi_{[\lambda,\lambda+(\log\lambda)^{-1}]}f\big\|_{L^{\frac{2(n+1)}{n-1}}(M)} \le C\lambda^{\frac{n-1}{2(n+1)}} \big(\log\log\lambda\big)^{-\frac{2}{(n+1)^2}} \|f\|_{L^2(M)}. \tag{3}$$

Here if $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots$ are the eigenvalues of $\sqrt{-\Delta_g}$ counted with respect to multiplicity and if $\{e_j\}$ is an associated orthonormal basis of eigenfunctions, if $I \subset [0,\infty)$

$$\chi_I f = \sum_{\lambda_j \in I} E_j f,$$

where

$$E_j f(x) = \Bigl(\int_M f\, \overline{e_j}\, dV_g \Bigr) \times e_j(x),$$

denotes the projection onto the jth eigenspace. Thus, (3) implies (2).

By interpolation and an application of a Bernstein inequality, this bound implies that for all exponents $p \in (2,\infty]$ one can improve (1) by a power of $(\log\log\lambda)^{-1}$. Although stronger log-improvements are in [3], [4], [6] and [14] for $p \ne p_c$, (2) represents the first improvement involving all eigenfunctions for the critical exponent. Also, besides the earlier improved geodesic eigenfunction restriction estimates for $n = 2$ of Chen and the author [10], this result seems to be the first improvement of estimates that are saturated by both the zonal functions and highest weight spherical harmonics on spheres.

The main step in proving these L^{p_c}-bounds will be to show that one has the following related weak-type estimates:

Proposition 2. *Assume, as above, that (M,g) is a fixed manifold of nonpositive curvature. Then there is a uniform constant C so that for $\lambda \gg 1$ we have*

$$(3') \quad \bigl|\bigl\{x \in M : \bigl|\chi_{[\lambda,\lambda+(\log\lambda)^{-1}]} f(x)\bigr| > \alpha\bigr\}\bigr| \le C\lambda\bigl(\log\log\lambda\bigr)^{-\frac{2}{n-1}} \alpha^{-\frac{2(n+1)}{n-1}}, \quad \alpha > 0, \quad \textit{if } \|f\|_{L^2(M)} = 1.$$

Here $|\Omega|$ denotes the dV_g measure of a subset Ω of M.

Note that, by Chebyshev's inequality (3) implies an inequality of the type (3′), but with a less favorable exponent for the $\log\log\lambda$ factor. The inequality says that $\chi_{[\lambda,\lambda+(\log\lambda)^{-1}]}$ sends $L^2(M)$ into $L^{p_c,\infty}(M)$, i.e., weak-L^{p_c}, with norm satisfying

$$(3'') \qquad \|\chi_{[\lambda,\lambda+(\log\lambda)^{-1}]}\|_{L^2(M)\to L^{\frac{2(n+1)}{n-1},\infty}(M)} = O\bigl(\lambda^{\frac{n-1}{2(n+1)}}/(\log\log\lambda)^{\frac{1}{n+1}}\bigr).$$

After we obtain this weak-type L^{p_c} estimate, we shall be able to obtain (3) by, in effect, interpolating it with another improved L^{p_c} estimate of Bak and Seeger [2], which says that the operators $\chi_{[\lambda,\lambda+1]}$ map $L^2(M)$ into the Lorentz space $L^{p_c,2}(M)$ (see §4 for definitions) with norm $O(\lambda^{\frac{n-1}{2(n+1)}})$. This "local" estimate holds for all manifolds—no curvature assumption is needed.

Before turning to the proofs, let us point out that the weak-type bound (3′) cannot hold for S^n. There there are two special values of α that cause problems. The zonal functions are sensitive to $\alpha \approx \lambda^{\frac{n-1}{2}}$, and

$$|\{x \in S^n : |Z_\lambda(x)| > \alpha\}| \approx \lambda^{-n} \approx \lambda\alpha^{-\frac{2(n+1)}{n-1}}, \quad \text{if } \alpha = c\lambda^{\frac{n-1}{2}},$$

with $c > 0$ fixed sufficiently small. Similarly, the highest weight spherical harmonics, Q_λ, are sensitive to $\alpha \approx \lambda^{\frac{n-1}{4}}$ in that

$$|\{x \in S^n : |Q_\lambda(x)| > \alpha\}| \approx \lambda^{-\frac{n-1}{2}} \approx \lambda\alpha^{-\frac{2(n+1)}{n-1}}, \quad \text{if } \alpha = c\lambda^{\frac{n-1}{4}},$$

and $c > 0$ fixed sufficiently small. Note that by (1′) and Chebyshev's inequality, we always have, on any (M, g),

$$|\{x \in M : |e_\lambda(x)| > \alpha\}| \lesssim \lambda\alpha^{-\frac{2(n+1)}{n-1}}, \tag{4}$$

and so the zonal functions and the highest weight spherical harmonics saturate this weak-type estimate. We shall give a simple proof of (4) in the next section that will serve as a model for the proof of the improved weak-type bounds in Proposition 2. It is based on a modification of Bourgain's [7] proof of a weak-type version of the critical Fourier restriction estimate of Stein and Tomas [42]–[43].

Let us give an overview of why are able to obtain (3′) and (3). As we mentioned before, the potentially dangerous values of α for the former are $\alpha \approx \lambda^{\frac{n-1}{2}}$ and $\alpha \approx \lambda^{\frac{n-1}{4}}$. The aforementioned sup-norm estimates of Bérard [3] provide log-improvements over (4) for $\alpha \geq \lambda^{\frac{n-1}{2}}/(\log\lambda)^{\frac{1}{2}}$, while the recent log-improved L^p estimates, $2 < p < p_c$, of Blair and the author [4], [6] yield log-improvements for α near the other dangerous value $\lambda^{\frac{n-1}{4}}$. Specifically, we are able to obtain improvements when $\alpha \leq \lambda^{\frac{n-1}{4}}(\log\lambda)^{\delta_n}$ for some $\delta_n > 0$. We can cut and paste these improvements into the aforementioned argument of Bourgain [7] to obtain (3′). We then can upgrade the weak-type estimates that we obtain (at the expense of less favorable powers of $(\log\log\lambda)^{-1}$) to a standard L^{p_c} estimate using the result of Bak and Seeger [2]. Thus, we combine the earlier "global" results of [3], [4], and [6] with "local" harmonic analysis techniques to obtain our main estimate (3).

The paper is organized as follows. In the next section we shall give the variation of the argument from [7] that yields (4). In §3 we shall show how we can use it along with the results of [3], [4] and [6] to obtain Proposition 2. Then in §4 we shall give the simple proof showing that we can use it and the aforementioned result of Bak and Seeger [2] to obtain the Theorem. Finally, in §5, we shall state some natural problems related to our approach. Also, in what follows whenever we write $A \lesssim B$, we mean that A is dominated by an unimportant constant multiplied by B.

2. The model local argument

In this section we shall present an argument that yields the weak-type estimate (4) and serves as a model for the argument that we shall use to prove Theorem 1.

Let us fix a real-valued function $\rho \in \mathcal{S}(\mathbb{R})$ satisfying

$$\rho(0) = 1, \quad |\rho(\tau)| \le 1, \quad \text{and} \quad \operatorname{supp} \hat{\rho} \subset (-1/2, 1/2). \tag{5}$$

If we set

$$P = \sqrt{-\Delta_g},$$

consider the operators

$$\rho(\lambda - P)f(x) = \sum_{j=0}^{\infty} \rho(\lambda - \lambda_j) E_j f(x), \tag{6}$$

where, as before, $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots$ are the eigenvalues counted with respect to multiplicity and E_j denotes projection onto the jth eigenspace.

The "local" analog of Proposition 2 then is the following result whose proof we shall modify in the next section to obtain the "global" weak-type estimates (3′).

Proposition 3. *For $\lambda \ge 1$ there is a constant C, depending only on (M, g), so that*

$$\bigl|\{x \in M : |\rho(\lambda - P)f(x)| > \alpha\}\bigr| \le C\lambda\alpha^{-\frac{2(n+1)}{n-1}} \|f\|_{L^2(M)}^{\frac{2(n+1)}{n-1}}, \quad \alpha > 0. \tag{7}$$

Consequently, (4) *is valid, and, moreover, if χ_λ denotes the unit-band spectral projection operators*

$$\chi_\lambda f = \sum_{\lambda_j \in [\lambda, \lambda+1]} E_j f,$$

we have

$$(7') \qquad \bigl|\{x \in M : |\chi_\lambda f(x)| > \alpha\}\bigr| \le C\lambda\alpha^{-\frac{2(n+1)}{n-1}}\|f\|_{L^2(M)}^{\frac{2(n+1)}{n-1}}, \quad \alpha > 0.$$

Since $\rho(0) = 1$ we have that $|\rho(\tau)| \ge 1/2$ for $|\tau| \le \delta$ for some $\delta > 0$. Thus, if one applies (7) with f replaced by $f_\lambda = \sum_{\lambda_j \in [\lambda,\lambda+\delta]}(\rho(\lambda-\lambda_j))^{-1}E_j f$, one deduces that

$$\left|\left\{\left|\sum_{\lambda_j\in[\lambda,\lambda+\delta]} E_j f(x)\right| > \alpha\right\}\right| \le C\lambda\alpha^{-\frac{2(n+1)}{n-1}}\|f\|_{L^2(M)}^{\frac{2(n+1)}{n-1}}, \quad \alpha > 0,$$

which implies (7′). So to prove Proposition 3, we just need to prove (7).

To prove (7), we require the following lemma which will be useful in the sequel. We shall assume, as we may, here and in what follows that the injectivity radius of M, Inj M, satisfies

$$\text{Inj } M \ge 10.$$

Also, $B(x,r)$, $r < \text{Inj } M$, denotes the geodesic ball of radius r about a point $x \in M$ with respect to the Riemannian distance function $d_g(\,\cdot\,,\,\cdot\,)$. The result we need then is the following.

Lemma 4. *Let $a \in C_0^\infty((-1,1))$. Then there is a constant C, depending only on (M,g) and the size of finitely many derivatives of a, so that for $\lambda^{-1} \le r \le \text{Inj}\, M$ we have*

$$(8) \qquad \left\|\int a(t)e^{it\lambda}\bigl(e^{-itP}f\bigr)dt\right\|_{L^2(B(x,r))} \le Cr^{\frac12}\|f\|_{L^2(M)},$$

and, also, if $\bigl(e^{-itP}\bigr)(x,y)$ denotes the kernel of the half-wave operators e^{-itP}, we have

$$(9) \qquad \begin{aligned}\bigl|\bigl(\hat{a}(P-\lambda)\bigr)(x,y)\bigr| &= \left|\int a(t)e^{it\lambda}\bigl(e^{-itP}\bigr)(x,y)\,dt\right| \\ &\le C\lambda^{\frac{n-1}{2}}\bigl(d_g(x,y)+\lambda^{-1}\bigr)^{-\frac{n-1}{2}}.\end{aligned}$$

We shall omit the proof of (9) since it is well known and follows easily from using stationary phase and parametrices for the half-wave equation. One can easily obtain (9) by adapting the proof of Lemma 5.1.3 in [29].

Even though (8) is in a recent article of the author [33], for the sake of completeness, we shall present a different simple proof here, which only uses energy estimates and quantitative propagation of singularities estimates for the half-wave operators.

We start by introducing a Littlewood-Paley bump function $\beta \in C_0^\infty(\mathbb{R})$ satisfying

$$\beta(\tau) = 1, \quad \tau \in [1/2, 2], \quad \text{and} \quad \operatorname{supp} \beta \subset (1/4, 4). \tag{10}$$

Then standard arguments using the aforementioned parametrix show that for any N, we have that

$$\Big\| \int a(t) e^{it\lambda} \big(I - \beta(P/\lambda)\big) \circ e^{-itP} \, dt \Big\|_{L^2(M) \to L^2(M)} = O(\lambda^{-N}),$$

where for each $N \in \mathbb{N}$ the constants depend only on finitely many derivatives of a. Thus, since we are assuming $\lambda^{-1} \le r$, to prove (8), it suffices to prove the variant where e^{-itP} is replaced by $\beta(P/\lambda) \circ e^{-itP}$. By a routine TT^* argument, this in turn is equivalent to showing that

$$\begin{gathered} \Big\| \int b(t) e^{it\lambda} \big(\beta(P/\lambda) \circ e^{-itP}\big) h \, dt \Big\|_{L^2(B(x,r))} \le Cr \|h\|_{L^2(B(x,r))}, \\ \text{if } \operatorname{supp} h \subset B(x,r) \text{ and } \lambda^{-1} \le r \le \operatorname{Inj} M, \end{gathered} \tag{8$'$}$$

with

$$b = a(\,\cdot\,) * \overline{a(-\,\cdot\,)}.$$

By Minkowski's inequality, the left side of (8′) is dominated by

$$\begin{aligned} &\int_{|t| \le 10r} |b(t)| \, \big\| \big(\beta(P/\lambda) \circ e^{-itP}\big) h \, \big\|_{L^2(B(x,r))} \, dt \\ &+ \int_{|t| \ge 10r} |b(t)| \, \big\| \big(\beta(P/\lambda) \circ e^{-itP}\big) h \, \big\|_{L^2(B(x,r))} \, dt = I + II. \end{aligned}$$

By energy estimates, we trivially have

$$I \lesssim r \|h\|_{L^2},$$

as desired, and we do not need to use our support assumptions in (8′) here.

To handle II, though, we do need to make use of them. We also need the routine dyadic estimates

$$\begin{aligned}(11)\qquad &\big|\big(\beta(P/\lambda)\circ e^{-itP}\big)(w,z)\big| = O\big(\lambda^n(1+\lambda|t|)^{-N}\big)\ \forall N,\\ &\text{if } \ d_g(w,z)\le |t|/2,\ \ t\in \operatorname{supp} b,\end{aligned}$$

which also follows easily from an integration by parts argument using the parametrix for e^{-itP}. From (11) we immediately get for $t \in \operatorname{supp} b$

$$\begin{aligned}&\big|\big(\beta(P/\lambda)\circ e^{-itP}\big)(w,z)\big| = O\big(\lambda^n(1+\lambda|t|)^{-N}\big)\,\forall N,\\ &\text{if } \ w,z\in B(x,r) \ \text{and} \ \ |t|\ge 10r.\end{aligned}$$

As a result, by Schwarz's inequality, we have that if, as in ($8'$), $\operatorname{supp} h \subset B(x,r)$,

$$II \lesssim (r\lambda)^n \left(\int_{|t|\ge 10r} \big(\lambda|t|\big)^{-n}\,dt\right) \times \|h\|_{L^2} \approx r\|h\|_{L^2},$$

as desired, completing the proof of ($8'$).

Proof of Proposition 3. To prove (7) it suffices to show that if Ω is a relatively compact subset of a coordinate patch Ω_0 for M then we have

$$(12)\qquad \big|\big\{x\in\Omega:\ |\rho(\lambda-P)f(x)|>\alpha\big\}\big| \le C\lambda\alpha^{-\frac{2(n+1)}{n-1}},\quad \alpha>0,$$

assuming that

$$(13)\qquad \|f\|_{L^2(M)} = 1.$$

We shall work in these local coordinates to make the decomposition we require.

Let

$$A = \{x\in\Omega : |\rho(\lambda-P)f(x)|>\alpha\}$$

denote the set in (12). Our decomposition will be based on the scale

$$(14)\qquad r = \lambda\alpha^{-\frac{4}{n-1}},$$

which is motivated by an argument in Bourgain [7]. Note that, since the sup-norm estimates of Avakumović [1] and Levitan [23] give

$$\|\rho(\lambda-P)f\|_{L^\infty} = O(\lambda^{\frac{n-1}{2}}),$$

the estimate (12) is trivial when r is smaller than a multiple of λ^{-1}, which allows us to use (8).

Write

$$A = \bigcup A_j,$$

where $A_j = A \cap Q_j$ and Q_j denote a nonoverlapping lattice of cubes of side-length r in our coordinates. At the expense of replacing A by a set of proportional measure, we may assume that

$$\text{dist }(A_j, A_k) > C_0 r, \quad j \neq k, \tag{15}$$

for a constant C_0 to be specified later. Also, let

$$\psi_\lambda(x) = \begin{cases} \rho(\lambda - P)f(x)/|\rho(\lambda - P)f(x)|, & \text{if } \rho(\lambda - P)f(x) \neq 0 \\ 1, & \text{otherwise,} \end{cases} \tag{16}$$

so that ψ_λ, of modulus one, is the signum function of $\rho(\lambda - P)f$.

We then have, by Chebyshev's inequality, (13) and the Cauchy-Schwarz inequality,

$$\alpha|A| \leq \left| \int \rho(\lambda - P)f \, \overline{\psi_\lambda \mathbb{1}_A} \, dV_g \right| \leq \left(\int \left| \sum_j \rho(\lambda - P)a_j \right|^2 dV_g \right)^{\frac{1}{2}},$$

where $\mathbb{1}_A$ denotes the indicator function of A and a_j denotes ψ_λ times the indicator function of A_j. As a result, if $S_\lambda = \big(\rho(\lambda - P)^* \circ \rho(\lambda - P)\big) = \rho^2(\lambda - P)$,

$$\begin{aligned} \alpha^2|A|^2 &\leq \sum_j \int |\rho(\lambda - P)a_j|^2 \, dV_g + \sum_{j \neq k} \int \rho(\lambda - P)a_j \, \overline{\rho(\lambda - P)a_k} \, dV_g \\ &= \sum_j \int |\rho(\lambda - P)a_j|^2 \, dV_g + \sum_{j \neq k} \int S_\lambda a_j \, \overline{a_k} \, dV_g \\ &= I + II. \end{aligned}$$

Since a_j is supported in a ball of radius $\approx r$, by (5) and the dual version of (8) with $a = \hat{\rho}$, we have

$$\int |\rho(\lambda - P)a_j|^2 \, dV_g \leq Cr \int |a_j|^2 \, dV_g = Cr|A_j|.$$

Whence, by (14)

$$I \lesssim r|A| = \lambda\alpha^{-\frac{4}{n-1}}|A|.$$

To estimate II, we note that by (9) with $a = \hat{\rho}(\,\cdot\,) * \overline{\hat{\rho}(-\,\cdot\,)}$, we have that the kernel $K_\lambda(x,y)$ of S_λ satisfies

$$|K_\lambda(x,y)| \le C\lambda^{\frac{n-1}{2}}\big(d_g(x,y)+\lambda^{-1}\big)^{-\frac{n-1}{2}}. \tag{17}$$

Therefore, by (16),

$$\begin{aligned} II &\lesssim \sum_{j\neq k}\iint |K_\lambda(x,y)|\,|a_j(x)|\,|a_k(y)|\,dV_g(x)dV_g(y) \\ &\lesssim \lambda^{\frac{n-1}{2}}\big(C_0 r\big)^{-\frac{n-1}{2}}\sum_{j\neq k}\|a_j\|_{L^1}\|a_k\|_{L^1} \\ &\le C_0^{-\frac{n-1}{2}}\alpha^2|A|^2. \end{aligned}$$

Thus,

$$\alpha^2|A|^2 \lesssim \lambda\alpha^{-\frac{4}{n-1}}|A| + C_0^{-\frac{n-1}{2}}\alpha^2|A|^2,$$

and so, if C_0 in (15) is large enough, the last term can be absorbed in the left side. We conclude that

$$|A| \lesssim \lambda\alpha^{-2-\frac{4}{n-1}} = \lambda\alpha^{-\frac{2(n+1)}{n-1}},$$

which is (12). □

3. Proof of improved weak-type estimates

We shall now prove Proposition 2. Repeating the arguments from the previous section shows that if $\rho \in \mathcal{S}(\mathbb{R})$ is as in (5) then it suffices to show that we have the following

Proposition 5. *Let (M,g) be an n-dimensional compact Riemannian manifold of nonpositive curvature. Then for $\lambda \gg 1$*

$$\big\|\rho(\log\lambda(\lambda-P))\big\|_{L^2(M)\to L^{\frac{2(n+1)}{n-1},\infty}(M)} = O\big(\lambda^{\frac{n-1}{2(n+1)}}/(\log\log\lambda)^{\frac{1}{n+1}}\big). \tag{18}$$

The earlier arguments show that (18) yields (3″) and hence Proposition 2 assuming, as in there and as we shall throughout this section, that the sectional curvatures of (M,g) are nonpositive.

To prove (18), as in (12), it suffices to show now that if Ω is a relatively compact subset of a coordinate patch Ω_0, then

$$\Big|\big\{x\in\Omega : |\rho(\log\lambda(\lambda-P))f(x)|>\alpha\big\}\Big| \le C\alpha^{-\frac{2(n+1)}{n-1}}\lambda/(\log\log\lambda)^{\frac{2}{n-1}}, \tag{19}$$

assuming that

$$\|f\|_{L^2(M)}=1. \tag{20}$$

To prove this, in addition to (8), we shall require the following two results.

Lemma 6. *Let (M,g) be as above. Then there is a $\delta_n>0$ so that for $\lambda\gg 1$ and $\mu(p)$ as in* (1)

$$\big\|\rho(\log\lambda(\lambda-P))\big\|_{L^2(M)\to L^{\frac{2n}{n-1}}(M)} = O\big(\lambda^{\mu\left(\frac{2n}{n-1}\right)}/(\log\lambda)^{\delta_n}\big). \tag{21}$$

Lemma 7. *If (M,g) is as above then there is a constant $C=C(M,g)$ so that for $T\ge 1$ and large λ we have the following bounds for the kernel of $\eta(T(\lambda-P))$, $\eta=\rho^2$,*

$$\big|\eta\big(T(\lambda-P)\big)(w,z)\big| \le CT^{-1}(\lambda/d_g(w,z))^{\frac{n-1}{2}} + C\lambda^{\frac{n-1}{2}}\exp(CT). \tag{22}$$

The first estimate, (21), is a simple consequence of the bounds

$$\big\|\chi_{[\lambda,\lambda+(\log\lambda)^{-1}]}\big\|_{L^2(M)\to L^p(M)} \le \lambda^{\mu(p)}/(\log\lambda)^{\delta(p,n)}, \quad 2<p<\tfrac{2(n+1)}{n-1}, \tag{21$'$}$$

with $\delta(p,n)>0$ from [4] for the special case of $p=\frac{2n}{n-1}$. Any other exponent between 2 and $\frac{2(n+1)}{n-1}$ in (21$'$) would work as well for us. We just chose $p=\frac{2n}{n-1}$ to simplify the calculations.

The other bound, (22), is well known and follows from the arguments in Bérard [3]. Indeed, it is a simple consequence of inequality (3.6.8) in [31].

Let us see how we can use these results to obtain (19).

We first note that by Lemma 6 and the Chebyshev inequality we have that since $\frac{2n}{n-1}\cdot\mu(\frac{2n}{n-1})=\frac{1}{2}$,

$$\begin{aligned} &\Big|\big\{x\in\Omega : |\rho(\log\lambda(\lambda-P))f(x)|>\alpha\big\}\Big| \\ &\le \alpha^{-\frac{2n}{n-1}}\int_M |\rho(\log\lambda(\lambda-P))f|^{\frac{2n}{n-1}}\,dV_g \\ &\lesssim \alpha^{-\frac{2n}{n-1}}\lambda^{\frac{1}{2}}(\log\lambda)^{-\frac{2n}{n-1}\delta_n}. \end{aligned} \tag{23}$$

To use this, we note that for large λ we have

$$\text{(24)}\qquad \begin{aligned}&\alpha^{-\frac{2n}{n-1}}\lambda^{\frac12}(\log\lambda)^{-\frac{2n}{n-1}\delta_n}\ll\alpha^{-\frac{2(n+1)}{n-1}}\lambda\big(\log\log\lambda\big)^{-\frac{2}{n-1}},\\ &\text{if }\ \alpha\le\lambda^{\frac{n-1}{4}}(\log\lambda)^{\delta_n}.\end{aligned}$$

Thus, by (23), we would obtain (19) if we could show that for $\lambda\gg 1$

$$\text{(25)}\quad \begin{aligned}&\big|\big\{x\in\Omega: |\rho(\log\lambda(\lambda-P))f(x)|>\alpha\big\}\big|\le C\alpha^{-\frac{2(n+1)}{n-1}}\lambda(\log\log\lambda)^{-\frac{2}{n-1}},\\ &\text{if }\ \alpha\ge\lambda^{\frac{n-1}{4}}(\log\lambda)^{\delta_n}.\end{aligned}$$

As we mentioned in the introduction, this step is key for us since it has allowed us to use our curvature assumptions and move well past the dangerous heights where α is comparable to $\lambda^{\frac{n-1}{4}}$.

At this stage, due to the nature of the pointwise estimates in Lemma 7, we need to change the frequency scale we are working with. Instead of effectively working with $(\log\lambda)^{-1}$ windows for frequencies as above, we shall work with wider windows of size T^{-1} where $T=c_0\log\log\lambda$, with c_0 chosen later to deal with the second term in the right side of (22).

We claim that we would have (25), and therefore be done, if we could show that

$$\text{(26)}\quad \begin{aligned}&\big|\big\{x\in\Omega: |\rho\big(c_0\log\log\lambda(\lambda-P)\big)h(x)|>\alpha\big\}\big|\lesssim\alpha^{-\frac{2(n+1)}{n-1}}\lambda(\log\log\lambda)^{-\frac{1}{n+1}},\\ &\text{if }\ \alpha\ge\lambda^{\frac{n-1}{4}}(\log\lambda)^{\delta_n},\ \text{ and }\ \|h\|_{L^2(M)}\le 1.\end{aligned}$$

To verify this claim, we note that since $\rho(0)=1$ and $\rho\in\mathcal{S}$, for $\tau\in\mathbb{R}$ and for $\lambda\gg1$ have

$$\big|\big[\rho(c_0\log\log\lambda(\lambda-\tau))-1\big]\rho(\log\lambda(\lambda-\tau))\big|\lesssim\frac{\log\log\lambda}{\log\lambda}(1+|\lambda-\tau|)^{-N},$$

for any $N=1,2,\dots$. Thus, by using the fact that by [28] the unit band spectral projection operators χ_λ satisfy

$$\|\chi_\lambda\|_{L^2(M)\to L^{\frac{2(n+1)}{n-1}}(M)}=O(\lambda^{\frac{n-1}{2(n+1)}}),$$

we deduce that

$$\big\|\big[\rho(c_0\log\log\lambda(\lambda-P))-I\big]\circ\rho(\log\lambda(\lambda-P))f\big\|_{L^{\frac{2(n+1)}{n-1}}(M)}\lesssim\frac{\log\log\lambda}{\log\lambda}\lambda^{\frac{n-1}{2(n+1)}},$$

and so, by Chebyshev, for all $\alpha > 0$ we have

$$\bigl|\bigl\{x \in M : |[\rho(c_0 \log\log\lambda(\lambda - P)) - I] \circ \rho(\log\lambda(\lambda - P))f(x)| > \alpha\bigr\}\bigr| \\ \lesssim \bigl(\tfrac{\log\log\lambda}{\log\lambda}\bigr)^{\frac{2(n+1)}{n-1}} \lambda\alpha^{-\frac{2(n+1)}{n-1}},$$

which is much better than the bounds posited in (25). If we take $h = \rho(\log\lambda(\lambda - P))f$ in (26), we deduce the claim from this since, by (5), $\|\rho(\log\lambda(\lambda - P))\|_{L^2(M)\to L^2(M)} \le 1$.

Following the arguments from the preceding section, to prove (26), let

$$A = \{x \in \Omega : |\rho(c_0 \log\log\lambda(\lambda - P))h(x)| > \alpha\},$$

and let ψ_λ be defined as in (16) but with $\rho(\lambda - P)$ replaced by $\rho(c_0 \log\log(\lambda - P))$. Note that for large λ

$$A = \emptyset \quad \text{if } \lambda^{\frac{n-1}{2}}(\log\log\lambda)^{-\frac{1}{2}} \lesssim \alpha,$$

since estimates of Bérard [3] (see also [31]) give

$$\|\rho(c_0 \log\log\lambda(\lambda - P))\|_{L^2(M)\to L^\infty(M)} \lesssim \lambda^{\frac{n-1}{2}} / \bigl(\log\log\lambda\bigr)^{\frac{1}{2}}.$$

This will allow us to apply (8).

Next, as in the proof of Proposition 3, we write $A = \cup A_j$ where $A_j = Q_j \cap A$, with the Q_j coming from a lattice of nonoverlapping cubes in our coordinate system, except now, instead of (5), we take

$$r = \lambda\alpha^{-\frac{4}{n-1}}(\log\log\lambda)^{-\frac{2}{n-1}}. \tag{27}$$

As before, at the expense of replacing A by a set of proportional measure, we may assume that

$$\text{dist } (A_j, A_k) > C_0 r, \quad j \neq k, \tag{28}$$

where C_0 will be specified momentarily.

Let us now collect the two estimates that we need for the proof of (26). First, if $S_\lambda = \eta(c_0 \log\log\lambda(\lambda - P))$, $\eta = \rho^2$, then by (22) if $c_0 > 0$ is fixed small enough, we have that its kernel, K_λ, satisfies

$$|K_\lambda(w, z)| \le C \left[(\log\log\lambda)^{-1} \left(\frac{\lambda}{d_g(w, z)} \right)^{\frac{n-1}{2}} + \lambda^{\frac{n-1}{2}} (\log\lambda)^{\frac{\delta_n}{10}} \right], \tag{29}$$

with C independent of $\lambda \gg 1$.

The other estimate that we require is that there is a uniform constant so that, for $T \geq 1$, we have

$$\|\rho(T(\lambda - P))f\|_{L^2(B(x,r))} \leq Cr^{\frac{1}{2}}\|f\|_{L^2(M)}, \quad \text{if } \lambda^{-1} \leq r \leq \text{Inj } M, \tag{30}$$

with C independent of $\lambda \gg 1$. Since

$$\rho(T(\lambda - P)) = \frac{1}{2\pi T}\int \hat{\rho}(t/T)e^{it\lambda}e^{-itP}\,dt,$$

and, by (5), $\hat{\rho}(t/T) = 0$ if $|t| \geq T$, we claim that this follows from (8) and the fact that the half-wave operators e^{-itP} are unitary.

To verify this, choose $\psi \in C_0^\infty((-1,1))$ satisfying $\sum_{j=-\infty}^{\infty}\psi(t-j) = 1$, $t \in \mathbb{R}$, and then write

$$\begin{aligned}\rho(T(\lambda - P))f &= (2\pi T)^{-1}\sum_j \int \hat{\rho}(t/T)\psi(t-j)e^{i\lambda t}e^{-itP}f\,dt\\ &= (2\pi T)^{-1}\sum_j \int a_j(t)e^{i\lambda t}e^{-itP}f_j\,dt,\end{aligned}$$

where

$$a_j(t) = \hat{\rho}((t+j)/T)\psi(t) \quad \text{and} \quad f_j = e^{i\lambda j}e^{-ijP}f.$$

Since $\|f_j\|_{L^2(M)} = \|f\|_{L^2(M)}$, and since derivatives of the a_j are bounded independent of j if $T \geq 1$, by (8), we have the uniform bounds

$$\left\|\int a_j(t)e^{i\lambda t}e^{-itP}f_j\right\|_{L^2(B(x,r))} \leq Cr^{\frac{1}{2}}\|f\|_{L^2(M)},$$

which yield (30) since $a_j \equiv 0$ if $|j| \geq T+1$.

We now use the proof of Proposition 3 to obtain (26). We argue as before to see that if $T_\lambda = \rho(c_0 \log\log\lambda(\lambda - P))$ and $a_j = \psi_\lambda \times \mathbb{1}_{A_j}$, then since $\|h\|_{L^2(M)} \leq 1$, we have

$$\alpha^2|A|^2 \leq \sum_j \int |T_\lambda a_j|^2\,dV_g + \sum_{j\neq k}\int S_\lambda a_j\,\overline{a_k}\,dV_g = I + II.$$

By the dual version of (30) and (27)

$$I \lesssim r\sum_j \int |a_j|^2\,dV_g = r|A| = \lambda(\log\log\lambda)^{-\frac{2}{n-1}}\alpha^{-\frac{4}{n-1}}|A|.$$

By (29)

$$II \lesssim \Big[(\log\log\lambda)^{-1}\lambda^{\frac{n-1}{2}}\big(C_0 r\big)^{-\frac{n-1}{2}} + \lambda^{\frac{n-1}{2}}(\log\lambda)^{\frac{\delta_n}{10}}\Big]\sum_{j\neq k}\|a_j\|_{L^1}\|a_k\|_{L^1}$$

$$\le C_0^{-\frac{n-1}{2}}\alpha^2|A|^2 + \lambda^{\frac{n-1}{2}}(\log\lambda)^{\frac{\delta_n}{10}}|A|^2.$$

Since we are assuming that $\alpha \ge \lambda^{\frac{n-1}{4}}(\log\lambda)^{\delta_n}$, the last term is $\ll \alpha^2|A|^2$ if λ is large. This means that we can fix C_0 in (28) so that for large λ we have

$$II \le \frac{1}{2}\alpha^2|A|^2.$$

Hence

$$\alpha^2|A|^2 \le C\lambda(\log\log\lambda)^{-\frac{2}{n-1}}\alpha^{-\frac{4}{n-1}}|A| + \frac{1}{2}\alpha^2|A|^2,$$

which of course yields the desired estimate

$$|A| \lesssim \lambda(\log\log\lambda)^{-\frac{2}{n-1}}\alpha^{-2-\frac{4}{n-1}} = \lambda(\log\log\lambda)^{-\frac{2}{n-1}}\alpha^{-\frac{2(n+1)}{n-1}},$$

assuming, as we are, that $\alpha \ge \lambda^{\frac{n-1}{4}}(\log\lambda)^{\delta_n}$.

This concludes the proof of (26), Proposition 5 and Proposition 2.

4. Proof of Theorem 1

Even though (3), and hence Theorem 1, follows directly from interpolating between the weak-type estimate (3′) and the estimate,

$$\|\chi_{[\lambda,\lambda+1]}\|_{L^2(M)\to L^{p_c,2}(M)} = O(\lambda^{\frac{1}{p_c}}), \quad p_c = \tfrac{2(n+1)}{n-1}, \tag{31}$$

of Bak and Seeger [2], for the sake of completeness, we shall give the simple argument here.

Let us start by recalling some basic facts about Lorentz spaces. See §3 in Chapter 5 of Stein and Weiss [41] for more details.

First, given a function u on M, we let

$$\omega(\alpha) = \big|\{x\in M : |u(x)|>\alpha\}\big|, \quad \alpha>0,$$

denote its distribution function, and

$$u^*(t) = \inf\{\alpha : \omega(\alpha)\le t\}, \quad t>0,$$

the nonincreasing rearrangement of u.

Then the Lorentz spaces $L^{p,q}(M)$ for $1 \le p < \infty$ and $1 \le q < \infty$ are defined as all u so that

$$\|u\|_{L^{p,q}(M)} = \Big(\frac{q}{p} \int_0^\infty \big[\, t^{\frac{1}{p}} u^*(t) \,\big]^q \, \frac{dt}{t} \Big)^{\frac{1}{q}} < \infty. \tag{32}$$

By equation (3.9) in Chapter 5 of [41], we then have

$$\|u\|_{L^{p,p}(M)} = \|u\|_{L^p(M)}, \tag{33}$$

and by Lemma 3.8 there we also have

$$\begin{aligned} \sup_{t>0} t^{\frac{1}{p}} u^*(t) &= \sup_{\alpha>0} \alpha \big[\, \omega(\alpha) \,\big]^{\frac{1}{p}} \\ &= \sup_{\alpha>0} \alpha \,\big| \big\{ x \in M : \, |u(x)| > \alpha \big\} \big|^{\frac{1}{p}}. \end{aligned}$$

If we take $u = \chi_{[\lambda, \lambda+(\log\lambda)^{-1}]} f$ and assume from now on that $\|f\|_{L^2(M)} = 1$, we therefore have, by our improved weak-type estimates $(3')$,

$$\sup_{t>0} t^{\frac{1}{p_c}} u^*(t) \le C \lambda^{\frac{1}{p_c}} \big(\log\log\lambda \big)^{-\frac{1}{n+1}}. \tag{34}$$

Also, for this u we have $\chi_{[\lambda,\lambda+1]} u = u$, and so, by (31),

$$\|u\|_{L^{p_c,2}(M)} \le C \lambda^{\frac{1}{p_c}} \|u\|_{L^2(M)} \le C \lambda^{\frac{1}{p_c}} \|f\|_{L^2(M)} = C \lambda^{\frac{1}{p_c}}. \tag{35}$$

By (32)–(33) and (34)–(35), we therefore get

$$\begin{aligned} \|u\|_{L^{p_c}(M)} &= \Big(\int_0^\infty \big[\, t^{\frac{1}{p_c}} u^*(t) \,\big]^{p_c} \, \frac{dt}{t} \Big)^{\frac{1}{p_c}} \\ &\le (p_c/2)^{\frac{1}{p_c}} \Big(\sup_{t>0} t^{\frac{1}{p_c}} u^*(t) \Big)^{\frac{p_c-2}{p_c}} \Big(\frac{2}{p_c} \int_0^\infty \big[\, t^{\frac{1}{p_c}} u^*(t) \,\big]^2 \, \frac{dt}{t} \Big)^{\frac{1}{p_c}} \\ &\lesssim \big[\, \lambda^{\frac{1}{p_c}} \big(\log\log\lambda \big)^{-\frac{1}{n+1}} \,\big]^{\frac{p_c-2}{p_c}} \, \|u\|_{L^{p_c,2}(M)}^{\frac{2}{p_c}} \\ &\lesssim \big[\, \lambda^{\frac{1}{p_c}} \big(\log\log\lambda \big)^{-\frac{1}{n+1}} \,\big]^{\frac{p_c-2}{p_c}} \, \big(\lambda^{\frac{1}{p_c}} \big)^{\frac{2}{p_c}} \\ &= \lambda^{\frac{1}{p_c}} \big(\log\log\lambda \big)^{-\frac{2}{(n+1)^2}}, \end{aligned}$$

as $(p_c - 2)/(n+1)p_c = 2/(n+1)^2$. Since $u = \chi_{[\lambda,\lambda+(\log\lambda)^{-1}]} f$ and we are assuming that $\|f\|_{L^2(M)} = 1$, we conclude that (3) must be valid, which completes the proof of Theorem 1. □

5. Concluding remarks

First of all, we were only able to get endpoint results with gains of powers of $\log\log\lambda$ instead of powers of $\log\lambda$ due to the estimate (22) for the smoothed out spectral projection kernels. Ideally, one would want to be able to use a variant of (22) where the exponential factor is not present for the second term in the right. Lower bounds of Jakobson and Polterovich [20]–[21] show that this error term cannot be $O(\lambda^{\frac{n-1}{2}})$, but their bounds do not rule out some improvement over (22), which would lead to more favorable estimates.

A better avenue for improvement, though, might be to try to improve the ball-localized estimates (8), where the operators $\hat{a}(P-\lambda)$ are replaced by $\rho(T(\lambda-P)))$ for appropriate $T=T(r)$. A seemingly modest improvement where $r^{\frac12}$ is replaced by $r^{\frac12}/(\log\lambda)^{\varepsilon}$, for some $\varepsilon>0$, if $\lambda^{-1}\le r\le(\log\lambda)^{-\delta}$, for some $\delta>0$ could be of use. The author in [32] obtained such improvements with $\varepsilon=\frac12$ if $\lambda^{-1}\le r\ll\lambda^{-\frac12}$, but this does not seem very useful. On the other hand, assuming that the curvature is strictly negative, Han [13] and Hezari and Rivière [15] obtained these types of bounds with $\varepsilon=n/2$ and δ depending on the dimension for a density one sequence of eigenfunctions. For toral eigenfunctions, Lester and Rudnick [22] did even better for a density one sequence of eigenfunctions by showing, for instance, that in when $n=2$ one can replace $r^{\frac12}$ in (8) by $r^{\frac{n}{2}}$ all the way down to r being equal to the essentially the wavelength, i.e., $\lambda^{-1+o(1)}$ as $\lambda\to\infty$. (See also [16] for earlier work.)

Finally, the arguments we have given could possibly prove new sharp bounds for eigenfunctions on manifolds with boundary. Sharp estimates in the two-dimensional case were obtained by Smith and the author [25], but sharp estimates in higher dimensions are only known for certain exponents. It turns out that the critical exponent for manifolds with boundary should be $\frac{6n+4}{3n-4}$, which is larger than the one for the boundaryless case, $\frac{2(n+1)}{n-1}$.

If one could obtain the analog of (9) in this setting with the right hand side replaced by

$$(\lambda/\text{dist }(x,y))^{\frac{n-1}{2}+\frac16},$$

then one would likely be able to obtain sharp weak-type estimates for $p=\frac{6n+4}{3n-4}$, which by interpolation would yield sharp L^p estimates for all other $p\in(2,\infty]$. One would also need analogs of (8), but these are probably much easier and likely follow from stretching arguments of Ivriĭ [19] and Seeley [24]. In the model case involving the Friedlander model, recently Ivanovici, Lebeau and Planchon [18] obtained dispersive estimates for wave equations which have similarities with the types of spectral projection kernel estimates we just described.

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