



Classical limits of gauge-invariant states and the choice of algebra for strict quantization

Thomas L. Browning¹ · Benjamin H. Feintzeig²

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Abstract

We analyze the quantization of a system consisting of a particle in an external Yang–Mills field within a C^* -algebraic framework. We show that in both the classical and quantum theories of such a system, the kinematical algebra of physical quantities can be obtained by restricting attention to symmetry-invariant states on a C^* -algebra. We use this to show that symmetry-invariant quantum states correspond to symmetry-invariant classical states in the classical limit.

Keywords Strict quantization · Classical limit · Gauge invariance · Reduction

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1 Introduction

The purpose of this paper is to investigate the classical limits of quantum states of a particle moving in an external Yang–Mills (gauge) field. We use the framework for analyzing the classical limit of such a system provided by Landsman [23] in terms of a strict deformation quantization. Our results demonstrate a correspondence between gauge-invariant states in the quantum theory and gauge-invariant states in the classical limit. Furthermore, we show that in both the classical and quantum theories, one can understand the choice of appropriate algebra for formulating the theory in terms of analogous algebraic constructions for limiting attention to a particular collection of states—namely the gauge-invariant ones—when one has started from the full theory containing both gauge-variant and gauge-invariant states. This construction yields the kinematical algebras of both the universal phase-space formulation of the reduced classical system and the superselection structure of the quantum system. These results illustrate the success of a general procedure outlined in Feintzeig [10, 11] for choosing

✉ Benjamin H. Feintzeig
bfeintze@uw.edu

¹ Department of Mathematics, University of California, Berkeley, USA

² Department of Philosophy, University of Washington, Seattle, USA

an appropriate algebra for quantization by restricting attention to a privileged class of physical states whose classical limits are well behaved.

Our approach to the classical limit follows the tradition of strict deformation quantization initiated by Rieffel [39–41] and Landsman [26,30,31], in which one defines a C^* -algebra for each possible value of Planck's constant $\hbar$. This is in contrast to formal deformation quantization in which $\hbar$ is treated as a parameter in formal power series [7,9,46,47]. We hope that our work demonstrates the applicability of the tools of strict deformation quantization to a range of foundational issues.

Our results provide a different perspective on the well-known classical and quantum reduction procedures. (For classical reduction, see Marsden and Weinstein [33]; for quantum reduction, see Rieffel [37,38]; and for the relation between the two, see Landsman [24,25].) Our investigation is somewhat related to the question of whether quantization commutes with reduction [17], but this question is typically approached in the framework of geometric quantization (although see Landsman [28,29]). We will provide an analogous result for a particular model within the C^* -algebraic approach to quantization, but we will not comment further on the relation to geometric quantization.

We proceed as follows. In Sect. 2, we review the mathematical model for the theory of a particle moving in an external Yang–Mills field and the strict deformation quantization for this system as constructed by Landsman [23]. In Sects. 3 and 4, we investigate the classical and quantum kinematics separately, establishing the existence of an algebraic “reduction procedure” in each case that restricts attention to only gauge-invariant states. In Sect. 5, we then establish that the quantization maps commute with our reduction procedure, from which it follows as an immediate corollary that the classical limit of a gauge-invariant quantum state (understood in a certain sense) is gauge-invariant. In Sect. 6, we discuss some possible interpretations of the significance of these results. Our results contain little, if any, novel mathematics, but we believe they are interesting for the perspective they provide on quantization.

2 Model for a particle in a Yang–Mills field

The classical theory of a particle moving in an external Yang–Mills field is specified in the geometrical background of a principal fiber bundle (P, Q, H, pr) with total space given by a smooth, paracompact manifold P , base space given by a smooth manifold Q , typical fiber given by a compact Lie group H , and projection $pr : P \rightarrow Q$. A principal connection on this space represents a Yang–Mills field [49], and our particle is assumed to move through this background space in the presence of such a field. Further, we assume the external Yang–Mills field determines an H -invariant Riemannian metric on the total space P , which is assumed to be geodesically complete with respect to this structure.

We will work with two possible Hamiltonian formulations describing the phase space of particle positions and momenta. The first naïve formulation of such a phase space is T^*P , the cotangent bundle over P . In this phase space, pure states for the particle are represented by points in T^*P and we specify a class of bounded physical magnitudes by the C^* -algebra $C_0(T^*P)$, the continuous functions on T^*P that vanish at infinity. It follows then that the collection of states (including both pure and mixed

states) on this algebra consists precisely in the collection of countably additive probability measures on T^*P [14,31]. We call this a naïve formulation because it takes the total space at face value as the space of possible configurations for the particle rather than identifying points contained in the same fiber. As such, this first formulation contains a considerable redundancy.

Our second formulation of the phase space of the particle system is given by $(T^*P)/H$, the so-called universal phase space of the system. This manifold is well studied [45,48], and it is known to be a vector bundle over T^*Q . As before, we will take points in this phase space to represent pure states of the particle, and we will specify a class of bounded physical magnitudes in this formulation by the C^* -algebra $C_0((T^*P)/H)$. We interpret this phase space as representing the states of a particle moving through Q with the redundancy of the total space for particle configurations removed. Notice that this algebra is isomorphic to the algebra $C_0(T^*P)^H$ of H -invariant continuous functions on the phase space T^*P vanishing at infinity. So the universal phase space is a kind of subtheory of the naïve phase space in which we restrict attention to a particular class of magnitudes.

Next, we specify a kinematical framework for the quantum theories constructed from each of the above classical phase spaces. Actually, we will only need to specify this framework for the first naïve formulation above. The framework for the universal phase-space formulation will then follow once we have defined quantization maps sending classical magnitudes to quantum magnitudes. In the naïve formulation, the pure state space of the quantum theory corresponding to the naïve classical phase space T^*P is given by $L^2(P)$, the square integrable functions on P with respect to the metric. We define a corresponding class of bounded quantum magnitudes specified by the algebra $\mathcal{K}(L^2(P))$ of compact operators on $L^2(P)$. It follows then that the collection of all states on this algebra is the collection of all normalized density operators on $L^2(P)$ [36, §VI.6].

It is not universally agreed that we should use the C^* -algebra of compact operators to represent the physical magnitudes of a quantum system. Feintzeig et al. [12] and Feintzeig and Weatherall [13] have argued, however, that this is an appropriate algebra, at least for systems with finitely many degrees of freedom. The construction of the algebra can also be generalized to settings with constraints [20–22] or infinitely many degrees of freedom, see [15,16]. Still, it is worth mentioning that there are alternative options for C^* -algebras of physical magnitudes for quantum systems, including the so-called Weyl or CCR algebra [1,32,35] and, more recently, the resolvent algebra [3–6]. In what follows, we set aside these alternatives.

Now, we recall the strict deformation quantization provided by Landsman [23] for the particle moving in an external Yang–Mills field. For background on strict deformation quantization, see Landsman [26,30,31]. We note here that the construction we work with is closely related to the strict deformation quantization of systems with constraints [27].

Some preliminaries: In what follows, we let dh denote the normalized Haar measure on H [44]. Further, the H -invariant Riemannian metric on P induces H -invariant measures μ on P , μ_x on $T_x P$ and $\hat{\mu}_x$ on $T_x^* P$ for each $x \in P$. We define the fiberwise Fourier transform $\mathcal{F}$ as follows. If $f : T^*P \rightarrow \mathbb{C}$ is measurable and integrable over each fiber, then we define $\mathcal{F}f : TP \rightarrow \mathbb{C}$ by

$$(\mathcal{F}f)(X) := \int_{T_x^*P} e^{i\langle \theta, X \rangle} f(\theta) d\hat{\mu}_x(\theta)$$

for each $X \in T_x P$. Recall that the Paley–Weiner functions [36, p. 333] on T^*P , denoted by $\text{PW}(T^*P)$, are precisely those functions whose fiberwise Fourier transform is smooth and compactly supported on T^*P .

We define a generalized Weyl quantization map on the norm dense domain $\text{PW}(T^*P) \subseteq C_0(T^*P)$. For each $x \in P$, let $B(x)$ be the largest open ball on which the exponential map $\exp_x : T_x P \rightarrow P$ is a diffeomorphism. Let $V_1 = \cup_{x \in P} B(x)$, and let V_2 be the image of V_1 under the map $\varphi : V_1 \rightarrow P \times P$ defined by $\varphi(X) := (\exp_x(X/2), \exp_x(-X/2))$ for $X \in T_x P$. Consider $f \in \text{PW}(T^*P)$, where $\mathcal{F}f \in C_c^\infty(TP)$ has compact support K . Let $\hbar_0 > 0$ be such that $\hbar K \subseteq V_1$ for all $\hbar \in (0, \hbar_0)$. Then for each $\hbar \in (0, \hbar_0)$, we first define a kernel $k_\hbar(f) \in C_c^\infty(P \times P)$ for an integral operator by:

$$(k_\hbar(f))(Y) := \begin{cases} \hbar^{-n}(\mathcal{F}f)(X) & Y = \varphi(\hbar X), \\ 0 & Y \in (P \times P) \setminus V_2. \end{cases}$$

Then, we define an integral operator on $L^2(P)$ for each kernel through a quantization map $\mathcal{Q}_\hbar : \text{PW}(T^*P) \rightarrow \mathcal{K}(L^2(P))$ given by

$$(\mathcal{Q}_\hbar(f)\psi)(x) := \int_P (k_\hbar(f))(x, x') \psi(x') d\mu(x').$$

As we will note later, each integral operator $\mathcal{Q}_\hbar(f)$ is Hilbert–Schmidt and so is compact as claimed. Note that $\mathcal{Q}_\hbar$ is a linear $*$ -preserving map for each $\hbar$ on which it is defined. Landsman [23, p. 105] has shown that this family of quantization maps $\mathcal{Q}_\hbar$ provides a strict deformation quantization.

On this definition, if $f \in C_0(T^*P)^H$, then $\mathcal{Q}_\hbar(f) \in \mathcal{K}(L^2(P))^H$, where H acts by a unitary representation on $L^2(P)$ defined later and $\mathcal{K}(L^2(P))^H$ consists in the H -invariant compact operators. We thus take $\mathcal{K}(L^2(P))^H$ to specify the physical magnitudes of our quantum theory in the universal phase-space formulation [23, p. 99]. Furthermore, it is known that the algebra $\mathcal{K}(L^2(P))^H$ has unitarily inequivalent representations, which gives rise to the superselection structure of the quantum theory [23, p. 108].

It is worth mentioning that we make a substantive choice in using the quantization defined here. On the one hand, we have made the substantive choice to work within the framework of strict deformation quantization rather than, for example, geometric quantization. Others have treated this and related systems from the perspective of geometric quantization [42, 43, 50]. Further, we have made a choice by using these particular quantization maps; it is known that one can construct a strict deformation quantization using alternative C^* -algebras [2, 18, 19]. In the remainder of the paper, we will use the tools of the strict deformation quantization defined here to investigate the classical limits of gauge-invariant states.

3 Classical reduction by gauge-invariant states

In this section, we will work only in the framework of the classical theory defined by the domain of the quantization maps. Our naïve theory is given by the C^* -algebra $C_0(T^*P)$. Note that we have a smooth right action of H on T^*P defined by the pullback of the given right action of H on P . Further, we define a left action of H on $C_0(T^*P)$ by $(hf)(\theta) := f(\theta h)$. Then, $C_0(T^*P)^H$ is defined as the collection of all $f \in C_0(T^*P)$ such that $hf = f$. The reduced theory is modeled by the C^* -algebra $C_0((T^*P)/H) \cong C_0(T^*P)^H$. For most of our results in this section, we only need to use the fact that T^*P is a locally compact Hausdorff space with a continuous right action by the compact Hausdorff unimodular group H . At only a few stages will we need to know that T^*P is a smooth manifold, H is a Lie group, and the group action is smooth. We will not need to use the further structure of the cotangent bundle in this section. Our goal will be to understand the reduction process as a map from $C_0(T^*P)$ to $C_0(T^*P)^H$ that restricts the collection of physical states.

We define a privileged collection of H -invariant linear functionals on the naïve theory $C_0(T^*P)$ by

$$V_C = \{\omega \in C_0(T^*P)^* \mid \omega(hf) = \omega(f) \text{ for all } f \in C_0(T^*P) \text{ and } h \in H\}.$$

Clearly, V_C is a weak* closed subspace. We will show that restricting attention to the states in V_C by forming the quotient with the annihilator $N(V_C)$ of V_C leads to the reduced algebra.

In this case, we can explicitly define a projection $\pi_0 : C_0(T^*P) \rightarrow C_0(T^*P)^H$ by

$$(\pi_0(f))(\theta) := \int_H f(\theta h) dh$$

for each $f \in C_0(T^*P)$. We must first establish that the codomain of π_0 is as claimed and that it is indeed a projection. Then, we will use the projection to show how $C_0(T^*P)^H$ can be constructed as a quotient space.

Lemma 1 *If $f \in C_0(T^*P)$, then $\pi_0(f) \in C_0(T^*P)$.*

Proof Let $\epsilon > 0$. Then, there exists a compact set $K \subseteq T^*P$ such that $|f(\theta)| \leq \epsilon$ for all $\theta \in T^*P \setminus K$. Define $KH := \{\theta h \mid \theta \in K, h \in H\}$. Note that KH is the image of the compact set $K \times H$ under the map $T^*P \times H \rightarrow T^*P$. Since the action of H is continuous, we know KH is compact. If $\theta \in T^*P \setminus KH$ and $h \in H$, then $\theta h \in T^*P \setminus K$, so $|f(\theta h)| \leq \epsilon$. As a consequence, if $\theta \in T^*P \setminus KH$, then

$$|(\pi_0(f))(\theta)| = \left| \int_H f(\theta h) dh \right| \leq \int_H |f(\theta h)| dh \leq \int_H \epsilon dh = \epsilon.$$

Thus, $\pi_0(f)$ vanishes at infinity.

Next, we show $\pi_0(f)$ is continuous at each point $\theta_0 \in T^*P$. Again, let $\epsilon > 0$ and choose $h_0 \in H$. Define $g : T^*P \times H \rightarrow \mathbb{R}$ by $g(\theta, h) := |f(\theta h) - f(\theta_0 h)|$. We know g is continuous because the action of H is continuous, and we have $g(\theta_0, h_0) = 0$. It

follows that there are open neighborhoods $U_{h_0} \subseteq H$ of h_0 and $V_{h_0} \subseteq T^*P$ of θ_0 such that $|g(\theta, h)| \leq \epsilon$ for all $h \in U_{h_0}$ and $\theta \in V_{h_0}$. This holds for each $h_0 \in H$, so we can construct an open cover $\{U_{h_0}\}_{h_0 \in H}$ of H , and since H is compact, there is a finite subcover $\{U_{h_j}\}_{j=1}^n$ of H . Now, $V = \cap_{j=1}^n V_{h_j}$ is open in T^*P , and $|g(\theta, h)| \leq \epsilon$ for all $\theta \in V$ and $h \in H$. As a consequence, for any $\theta \in V$,

$$\begin{aligned} |(\pi_0(f))(\theta) - (\pi_0(f))(\theta_0)| &\leq \int_H |f(\theta h) - f(\theta_0 h)| dh \\ &= \int_H |g(\theta, h)| dh \leq \int_H \epsilon dh = \epsilon. \end{aligned}$$

Since this holds at each $\theta_0 \in T^*P$, $\pi_0(f)$ is continuous. $\square$

Proposition 1 *If $f \in C_0(T^*P)$, then $h\pi_0(f) = \pi_0(f)$ and $\pi_0(hf) = \pi_0(f)$ for all $h \in H$.*

Proof For any $f \in C_0(T^*P)$, $h_0 \in H$, and $\theta \in T^*P$, we have

$$(h_0\pi_0(f))(\theta) = (\pi_0(f))(\theta h_0) = \int_H f(\theta h_0 h) dh = \int_H f(\theta h) dh = (\pi_0(f))(\theta)$$

by the left invariance of the Haar measure. Further,

$$(\pi_0(h_0 f))(\theta) = \int_H (h_0 f)(\theta h) dh = \int_H f(\theta h h_0) dh = \int_H f(\theta h) dh = (\pi_0(f))(\theta)$$

by the right invariance of the Haar measure. $\square$

Proposition 2 *For any $f \in C_0(T^*P)^H$, $\pi_0(f) = f$.*

Proof Suppose $f \in C_0(T^*P)^H$. Then for any $\theta \in T^*P$,

$$(\pi_0(f))(\theta) = \int_H f(\theta h) dh = \int_H (hf)(\theta) dh = \int_H f(\theta) dh = f(\theta)$$

which shows that $\pi_0(f) = f$. $\square$

Notice that Proposition 1 shows that if $f \in C_0(T^*P)$, then $\pi_0(f) \in C_0(T^*P)^H$ as claimed. Then, Proposition 2 shows that π_0 is surjective and a projection, i.e., $\pi_0(\pi_0(f)) = \pi_0(f)$.

Proposition 3 *π_0 is linear, $*$ -preserving, and norm continuous (in the sup norm).*

Proof For any $f \in C_0(T^*P)$ and any $\theta \in T^*P$,

$$|(\pi_0(f))(\theta)| \leq \int_H |f(\theta h)| dh \leq \int_H \|f\| dh = \|f\|,$$

so $\|\pi_0(f)\| \leq \|f\|$. Further, π_0 is linear and $*$ -preserving because the Lebesgue integral is linear and $*$ -preserving. Since π_0 is a bounded linear operator, it is norm continuous. $\square$

Next, we establish that π_0 preserves the subalgebra of functions that are smooth and compactly supported.

Lemma 2 *If $K \subseteq T^*P$ is compact and $U \subseteq T^*P$ is open, then $\{h \in H \mid Kh \subseteq U\}$ is open.*

Proof Since the group action is continuous, we know that

$$\{(\theta, h) \in K \times H \mid \theta h \in U^c\}$$

is a closed subset of the compact space $K \times H$ and so is compact. The image of this set under the continuous projection $K \times H \rightarrow H$ is

$$\{h \in H \mid \theta h \in U^c \text{ for some } \theta \in K\} = \{h \in H \mid Kh \subseteq U\}^c$$

which is a compact subset of the Hausdorff space H and so is closed. $\square$

Lemma 3 *Consider $\theta_0 \in T^*P$, and define $f : H \rightarrow T$ by $f(h) = \theta_0 h$. For any open subsets $U, V \subseteq T^*P$ with $\overline{V} \subseteq U$, there is a neighborhood W of θ_0 such that $\theta h \in U$ for all $h \in f^{-1}(V)$ and $\theta \in W$.*

Proof Since T^*P is first countable, locally compact, and Hausdorff, we can consider a nested neighborhood basis $\{W_j\}_{j=1}^\infty$ at θ_0 consisting of precompact open sets with $\overline{W_{j+1}} \subseteq W_j$ for all $j \geq 1$. If $\theta_0 h \in U$, then $W_{j-1}h \subseteq U$ for some $j \geq 2$, so $\overline{W_j}h \subseteq U$ for some $j \geq 1$. By the previous lemma, we know $\{h \in H \mid \overline{W_j}h \subseteq U\}_{j=1}^\infty$ is an increasing open cover of $\{h \in H \mid \theta_0 h \in \overline{V}\} = f^{-1}(\overline{V})$, which is a closed subset of the compact space H and so is compact. Thus, $\{h \in H \mid \theta_0 h \in \overline{V}\} \subseteq \{h \in H \mid \overline{W_j}h \subseteq U\}$ for some integer $j \geq 1$. If $h \in f^{-1}(V)$ and $\theta \in W_j$, then $\theta h \in V \subseteq \overline{V}$, so $\theta h \in \overline{W_j}h \subseteq U$ as desired. $\square$

Proposition 4 *If $f \in C_c^\infty(T^*P)$, then $\pi_0(f) \in C_c^\infty(T^*P)$.*

Proof First, we show that $\pi_0(f)$ is compactly supported. Let $K \subseteq T^*P$ be the compact support of f . Let $KH = \{\theta h \mid \theta \in K, h \in H\}$. For any $\theta \in T^*P \setminus KH$ and $h \in H$, $\theta h \in T^*P \setminus K$, so $f(\theta h) = 0$. As a consequence, if $\theta \in T^*P \setminus KH$, then

$$(\pi_0(f))(\theta) = \int_H f(\theta h) dh = \int_H 0 dh = 0.$$

Since the group action is continuous, KH is compact, so $\pi_0(f)$ is compactly supported.

Next, we show that $\pi_0(f)$ is smooth at each $\theta_0 \in T^*P$. Let (U, φ) be a chart at θ_0 . Since the integral expression for $\pi_0(f)$ may depend on the value of f outside U , we proceed in two steps: First, we break down $\pi_0(f)$ into a sum of components depending only on regions no larger than U , and second, we show that each of these components varies smoothly.

To begin, consider a precompact set V with $\theta_0 \in V \subseteq \overline{V} \subseteq U$. Define $g : H \rightarrow T^*P$ by $g(h) = \theta_0 h$, and let $\theta_0 H = \{g(h) \mid h \in H\}$. Letting $S = g^{-1}(V)$, and

applying the previous lemma, there is a neighborhood W of θ_0 such that $\theta h \in U$ for all $\theta \in W$ and $h \in S$. Now, $\{Vh\}_{h \in H}$ is an open cover of the compact set $\theta_0 H$ since $\theta_0 \in V$, and so there exists a finite subcover $\{Vh_j\}_{j=1}^n$. We have

$$\bigcup_{j=1}^n Sh_j = \bigcup_{j=1}^n g^{-1}(V)h_j = \bigcup_{j=1}^n g^{-1}(Vh_j) = g^{-1}\left(\bigcup_{j=1}^n Vh_j\right) = H.$$

Thus, the inclusion–exclusion formula implies that for any $\theta \in T^*P$,

$$\begin{aligned} (\pi_0(f))(\theta) &= \int_H f(\theta h) dh \\ &= \int_{\bigcup_{j=1}^n Sh_j} f(\theta h) dh \\ &= - \sum_{J \subseteq \{1, \dots, n\}; J \neq \emptyset} (-1)^{|J|} \int_{\bigcap_{j \in J} Sh_j} f(\theta h) dh. \end{aligned}$$

We will show that for each non-empty $J \subseteq \{1, \dots, n\}$, letting $I = \bigcap_{j \in J} Sh_j$, the map $\theta \mapsto \int_I f(\theta h) dh$ is smooth on W . Since the group action is smooth, we know that for each $h \in H$, the map $\theta \mapsto f(\theta h)$ is smooth, which implies that the composition $t \mapsto f(\varphi^{-1}(t)h)$ is smooth for each $t \in \varphi(W)$. The Leibniz integral rule can be applied to the domain $\varphi(W) \subseteq \mathbb{R}^m$, which shows that $\theta \mapsto \int_I f(\theta h) dh$ is smooth on W . This implies along with the above formula that $\pi_0(f)$ is smooth on a neighborhood of θ_0 , and since this holds for each point $\theta_0 \in T^*P$, it follows that $\pi_0(f)$ is smooth. $\square$

Note that the previous proposition only relies on the fact that T^*P is a smooth manifold with a smooth right action by H . As such, we can apply this proposition even to functions on TP , which also carries a smooth right action by H , when these functions are obtained as the Fourier transform of functions on T^*P . Once we show that π_0 commutes with the Fourier transform, this will show that the map π_0 preserves the subalgebra of Paley–Weiner functions. We will use this fact later on.

We will have occasion to use a particular class of linear functionals in V_C . For any $\theta \in T^*P$, define

$$\omega_\theta^H(f) := (\pi_0(f))(\theta) = \int_H f(\theta h) dh$$

for all $f \in C_0(T^*P)$.

Proposition 5 For each $\theta \in T^*P$, $\omega_\theta^H \in V_C$.

Proof For all $h_0 \in H$ and $f \in C_0(T^*P)$, $\omega_\theta^H(h_0 f) = (\pi_0(h_0 f))(\theta) = (\pi_0(f))(\theta) = \omega_\theta^H(f)$. $\square$

Now, let $N(V_C)$ be the annihilator of V_C in $C_0(T^*P)$.

Theorem 1 *There is a $*$ -preserving vector space isomorphism between $C_0(T^*P)/N(V_C)$ and $C_0(T^*P)^H$.*

Proof We will show that $\ker(\pi_0) = N(V_C)$.

Suppose $f \in N(V_C)$. Then for each $\theta \in T^*P$,

$$(\pi_0(f))(\theta) = \int_H f(\theta h) dh = \omega_\theta^H(f) = 0,$$

so $f \in \ker(\pi_0)$.

On the other hand, suppose $f \in \ker(\pi_0)$ and consider $\omega \in V_C$. By the Riesz–Markov representation theorem [36, p. 107], there is a complete regular finite measure μ_ω on T^*P such that

$$\omega(g) = \int_{T^*P} g(\theta) d\mu_\omega(\theta)$$

for all $g \in C_0(T^*P)$. It follows that

$$\begin{aligned} 0 &= \int_{T^*P} (\pi_0(f))(\theta) d\mu_\omega(\theta) \\ &= \int_{T^*P} \int_H f(\theta h) dh d\mu_\omega(\theta) \\ &= \int_H \int_{T^*P} (hf)(\theta) d\mu_\omega(\theta) dh \\ &= \int_H \omega(hf) dh \\ &= \int_H \omega(f) dh \\ &= \omega(f). \end{aligned}$$

Here, the use of the Fubini–Tonelli theorem [36, p. 21] in the third line is justified because both measures in question are finite and f is bounded. This shows that $f \in N(V_C)$. We conclude that $\ker(\pi_0) = N(V_C)$.

It follows that π_0 induces a $*$ -preserving vector space isomorphism from $C_0(T^*P)/N(V_C)$ to $C_0(T^*P)^H$. $\square$

Notice that it follows immediately that the map $\omega \mapsto \omega \circ \pi_0$ is a vector space isomorphism between $(C_0(T^*P)^H)^*$ and V_C . This shows that π_0 can be thought of as implementing the classical reduction by restricting attention to the states in V_C .

It may be of interest to note that π_0 is not in general a $*$ -homomorphism. The following proposition, for example, establishes that π_0 fails to be a homomorphism when H is finite and acts non-trivially. For more on quotients and ideals in C^* -algebras, see [8, 34].

Proposition 6 *If π_0 is a $*$ -homomorphism, then for each $f \in N(V_C)$ and each $\theta \in T^*P$, the map $h \mapsto f(\theta h)$ is zero almost everywhere on H .*

Proof Suppose π_0 is a $*$ -homomorphism. Then, $N(V_C)$ is an ideal [31, p. 675]. Suppose $f \in N(V_C)$. It follows that $|f| = f^*f \in N(V_C)$. For any $\theta \in T^*P$, we have

$$\int_H |f(\theta h)| dh = \omega_\theta^H(f^*f) = 0.$$

Now, since $h \mapsto |f(\theta h)|$ and the Haar measure are both nonnegative, it follows that $f(\theta h)$ is zero almost everywhere on H . $\square$

4 Quantum reduction by gauge-invariant states

We work now with the framework of the quantum theory defined by the codomain of the quantization map $\mathcal{Q}_h$. That is, our naïve theory is given by the C^* -algebra $\mathcal{K}(L^2(P))$. In this section, we will only need to use the fact that H is a compact Hausdorff unimodular group acting continuously on P on the right and that P also has an H -invariant measure. For current purposes, H need not even be a Lie group. For each $h \in H$, define a unitary operator $U(h)$ on $L^2(P)$ by

$$(U(h)\psi)(x) := \psi(xh)$$

for all $\psi \in L^2(P)$. Let $\mathcal{K}(L^2(P))^H$ be the C^* -algebra of compact operators that commute with $U(h)$ for every $h \in H$. This will be the C^* -algebra of the reduced theory. We will proceed in analogy to the classical case. Our goal will be to understand the reduction process as a map from $\mathcal{K}(L^2(P))$ to $\mathcal{K}(L^2(P))^H$ that restricts the collection of physical states.

We define a privileged collection of H -invariant linear functionals on the naïve theory $\mathcal{K}(L^2(P))$ by

$$\begin{aligned} V_Q &:= \{\omega \in \mathcal{K}(L^2(P))^* \mid \omega(U(h)AU(h)^*) \\ &= \omega(A) \text{ for all } A \in \mathcal{K}(L^2(P)) \text{ and } h \in H\}. \end{aligned}$$

In analogy with the classical case, we will show that restricting attention to the states in V_Q by forming the quotient with the annihilator $N(V_Q)$ of V_Q leads to the reduced algebra.

We begin with some preliminary facts about V_Q . Clearly, V_Q is a weak* closed subspace. Further, if $\omega \in \mathcal{K}(L^2(P))^*$, then there is a unique trace class operator $\rho_\omega \in \mathcal{T}(L^2(P))$ such that $\omega(A) = \text{Tr}(\rho_\omega A)$ for all $A \in \mathcal{K}(L^2(P))$ [36, p. 213]. We can characterize the functionals belonging to V_Q by reference to their corresponding trace class operators as follows. Let $\mathcal{T}(L^2(P))^H$ be the collection of trace class operators that commute with $U(h)$ for every $h \in H$.

Lemma 4 $\omega \in V_Q$ iff $\rho_\omega \in \mathcal{T}(L^2(P))^H$.

Proof If we suppose $\omega \in V_Q$ and $h \in H$, we have

$$\begin{aligned} \text{Tr}(U(h)\rho_\omega U(h)^*A) &= \text{Tr}(\rho_\omega U(h)^*AU(h)) = \omega(U(h^{-1})AU(h^{-1})^*) \\ &= \omega(A) = \text{Tr}(\rho_\omega A) \end{aligned}$$

for all $A \in \mathcal{K}(L^2(P))$. It follows from the uniqueness clause above that $\rho_\omega = U(h)\rho_\omega U(h)^*$.

On the other hand, if $\rho_\omega = U(h)\rho_\omega U(h)^*$ for all $h \in H$, then

$$\begin{aligned} \omega(U(h)AU(h)^*) &= \text{Tr}(\rho_\omega U(h)AU(h)^*) = \text{Tr}(U(h^{-1})\rho_\omega U(h^{-1})^*A) \\ &= \text{Tr}(\rho_\omega A) = \omega(A), \end{aligned}$$

so $\omega \in V_Q$. □

Notice it follows that $\omega \in V_Q$ iff $\omega^* \in V_Q$, where $\omega^*(A) := \text{Tr}(\rho_\omega^*A)$ for all $A \in \mathcal{K}(L^2(P))$.

Now, consider the norm dense subspace of $\mathcal{K}(L^2(P))$ given by the collection $\text{HS}(L^2(P))$ of Hilbert–Schmidt operators with Hilbert–Schmidt inner product and norm denoted by $\langle \cdot, \cdot \rangle_{\text{HS}}$ and $\|\cdot\|_{\text{HS}}$, respectively. Let $\text{HS}(L^2(P))^H$ denote the collection of Hilbert–Schmidt operators that commute with $U(h)$ for every $h \in H$.

Recall that we have

$$\mathcal{T}(L^2(P)) \subseteq \text{HS}(L^2(P)) \subseteq \mathcal{K}(L^2(P))$$

which implies

$$\mathcal{T}(L^2(P))^H \subseteq \text{HS}(L^2(P))^H \subseteq \mathcal{K}(L^2(P))^H.$$

We now establish some basic facts concerning $\text{HS}(L^2(P))^H$.

Lemma 5 $\text{HS}(L^2(P))^H$ is a $\|\cdot\|_{\text{HS}}$ -closed subspace of $\text{HS}(L^2(P))$.

Proof Clearly, $\text{HS}(L^2(P))^H$ is a subspace. Consider a net $B_\beta \in \text{HS}(L^2(P))^H$ such that $\|B_\beta - B\|_{\text{HS}} \rightarrow 0$ for some $B \in \text{HS}(L^2(P))$. Then for any $h \in H$,

$$\begin{aligned} \|B - U(h)BU(h)^*\|_{\text{HS}} &= \|(B - B_\beta) + (B_\beta - U(h)BU(h)^*)\|_{\text{HS}} \\ &\leq \|B - B_\beta\|_{\text{HS}} + \|U(h)B_\beta U(h)^* - U(h)BU(h)^*\|_{\text{HS}} \\ &\leq \|B - B_\beta\|_{\text{HS}} \\ &\quad + [\text{Tr}(U(h)(B_\beta - B)^*U(h)^*U(h)(B_\beta - B)U(h)^*)]^{1/2} \\ &\leq \|B - B_\beta\|_{\text{HS}} + [\text{Tr}((B_\beta - B)^*(B_\beta - B))]^{1/2} \\ &\leq 2\|B - B_\beta\|_{\text{HS}} \end{aligned}$$

and since the latter approaches zero, it follows that $B = U(h)BU(h)^*$. □

We define $\pi_1 : \text{HS}(L^2(P)) \rightarrow \text{HS}(L^2(P))^H$ as the orthogonal projection (with respect to the Hilbert–Schmidt inner product) onto the subspace of H -invariant Hilbert–Schmidt operators. We first provide an explicit representation of the map π_1 . Recall that for any $A \in \text{HS}(L^2(P))$, there is a function $k_A \in L^2(P \times P)$, called the kernel of A , such that A can be expressed as the integral operator [36, p. 210]:

$$(A\psi)(x) = \int_P k_A(x, y)\psi(y)d\mu(y)$$

for any $\psi \in L^2(P)$.

Lemma 6 $k_A \in L^2(P \times P)$ is the kernel of $A \in \text{HS}(L^2(P))^H$ if and only if

$$k_A(xh, yh) = k_A(x, y)$$

almost everywhere on $P \times P$ for each $h \in H$.

Proof Suppose $k_A(xh, yh) = k_A(x, y)$ for all $h \in H$. Then for any $h \in H$ and $\psi \in L^2(P)$,

$$\begin{aligned} (AU(h)\psi)(x) &= \int_P k_A(x, y)(U(h)\psi)(y)d\mu(y) \\ &= \int_P k_A(xh, yh)\psi(yh)d\mu(y) \\ &= \int_P k_A(xh, y)\psi(y)d\mu(y) \\ &= (A\psi)(xh) = (U(h)A\psi)(x), \end{aligned}$$

so $A \in \text{HS}(L^2(P))^H$.

On the other hand, suppose that $A \in \text{HS}(L^2(P))^H$. Then for any $h \in H$ and $\psi \in L^2(P)$,

$$\begin{aligned} \int_P k_A(x, y)\psi(y)d\mu(y) &= (A\psi)(x) \\ &= (U(h)AU(h)^*\psi)(x) \\ &= (AU(h)^*\psi)(xh) \\ &= \int_P k_A(xh, y)\psi(yh^{-1})d\mu(y) \\ &= \int_P k_A(xh, yh)\psi(y)d\mu(y). \end{aligned}$$

It follows that for any $\psi \in L^2(P)$,

$$\int_P (k_A(x, y) - k_A(xh, yh))\psi(y)d\mu(y) = 0$$

and hence $k_A(x, y) = k_A(xh, yh)$ almost everywhere. $\square$

Proposition 7 *Let k_A be the kernel of $A \in \text{HS}(L^2(P))$. Then, $\pi_1(A)$ has kernel*

$$\tilde{k}(x, y) := \int_H k_A(xh, yh)dh,$$

i.e.,

$$(\pi_1(A)\psi)(x) = \int_P \tilde{k}(x, y)\psi(y)d\mu(y).$$

Proof Let us denote the integral operator with kernel $\tilde{k}$ by B , i.e.,

$$(B\psi)(x) := \int_P \tilde{k}(x, y)\psi(y)d\mu(y).$$

Our task is to show that $B = \pi_1(A)$. First, we show that B belongs to $\text{HS}(L^2(P))^H$. For any $h_0 \in H$, we have

$$\tilde{k}(xh_0, yh_0) = \int_H k_A(xh_0h, yh_0h)dh = \int_H k_A(xh, yh)dh = \tilde{k}(x, y)$$

and hence it follows from the previous lemma that $B \in \text{HS}(L^2(P))^H$.

Next, we show that $A - B$ is orthogonal to $\text{HS}(L^2(P))^H$ with respect to $\langle \cdot, \cdot \rangle_{\text{HS}}$. Consider any $C \in \text{HS}(L^2(P))^H$ with kernel k_C . We know from the previous lemma that

$$k_C(xh, yh) = k_C(x, y)$$

almost everywhere. Then, we have

$$\begin{aligned} \langle B, C \rangle_{\text{HS}} &= \int_{P \times P} \overline{\tilde{k}(x, y)} k_C(x, y) d\mu(x) d\mu(y) \\ &= \int_{P \times P} \int_H \overline{k_A(xh, yh)} k_C(x, y) dh d\mu(x) d\mu(y) \\ &= \int_H \int_{P \times P} \overline{k_A(xh, yh)} k_C(x, y) d\mu(x) d\mu(y) dh \\ &= \int_H \int_{P \times P} \overline{k_A(x, y)} k_C(xh^{-1}, yh^{-1}) d\mu(x) d\mu(y) dh \\ &= \int_H \int_{P \times P} \overline{k_A(x, y)} k_C(x, y) d\mu(x) d\mu(y) dh \\ &= \int_{P \times P} \overline{k_A(x, y)} k_C(x, y) d\mu(x) d\mu(y) = \langle A, C \rangle_{\text{HS}}. \end{aligned}$$

The use of the Fubini–Tonelli theorem in the third line is justified because the Cauchy–Schwarz inequality implies

$$\begin{aligned} & \int_H \int_{P \times P} |k_A(xh, yh) \overline{k_C(x, y)}| d\mu(x) d\mu(y) dh \\ & \leq \sup_{h \in H} \left(\int_{P \times P} |k_A(xh, yh)|^2 d\mu(x) d\mu(y) \int_{P \times P} |k_C(x, y)|^2 d\mu(x) d\mu(y) dh \right)^{1/2} \end{aligned}$$

and each of the factors on the right is finite because $k_A, k_C \in L^2(P \times P)$.

Hence, it follows that $\langle A - B, C \rangle_{\text{HS}} = 0$ for all $C \in \text{HS}(L^2(P))^H$. This implies that B is the orthogonal projection of A onto $\text{HS}(L^2(P))^H$ and hence $B = \pi_1(A)$. $\square$

We will now aim to extend π_1 continuously from the dense subset $\text{HS}(L^2(P))$ to all of $\mathcal{K}(L^2(P))$. To do so, we need to know that π_1 is already continuous in norm on its domain.

Proposition 8 π_1 is operator norm continuous.

Proof For any $A \in \text{HS}(L^2(P))$ with kernel $k_A \in L^2(P \times P)$, and for any $\psi \in L^2(P)$,

$$\begin{aligned} (\pi_1(A)\psi)(x) &= \int_P \int_H k_A(xh, yh) \psi(y) dh d\mu(y) \\ &= \int_H \int_P k_A(xh, yh) \psi(y) dh d\mu(y) \\ &= \int_H (U(h)AU(h)^*\psi)(x) dh. \end{aligned}$$

The use of the Fubini–Tonelli theorem in the second line is justified because the Cauchy–Schwarz inequality implies

$$\begin{aligned} & \int_H \int_P |k_A(xh, yh) \psi(y)| dh d\mu(y) \\ & \leq \sup_{h \in H} \left(\int_P |k_A(xh, yh)|^2 d\mu(y) \int_P |\psi(y)|^2 d\mu(y) \right)^{1/2} \end{aligned}$$

and each of the factors on the right-hand side is square integrable. It follows that

$$\begin{aligned} \|\pi_1(A)\psi\|^2 &= \int_P \int_H |(U(h)AU(h)^*\psi)(x)|^2 d\mu(x) dh \\ &= \int_H \int_P |(U(h)AU(h)^*\psi)(x)|^2 dh d\mu(x) \\ &\leq \sup_{h \in H} \|U(h)AU(h)^*\psi\|^2 \\ &\leq \|A\|^2 \|\psi\|^2. \end{aligned}$$

Thus, $\|\pi_1(A)\| \leq \|A\|$. $\square$

We now consider the norm-continuous extension of π_1 as a map $\mathcal{K}(L^2(P)) \rightarrow \mathcal{K}(L^2(P))^H$, which we will continue to denote simply by π_1 . We must establish that π_1 does indeed have the range just specified.

Lemma 7 $\mathcal{K}(L^2(P))^H$ is operator norm closed.

Proof Suppose $A_\beta \in \mathcal{K}(L^2(P))^H$ is a net converging to A in operator norm. Then,

$$\begin{aligned} \|U(h)AU(h)^* - A\| &= \|U(h)AU(h)^* - U(h)A_\beta U(h)^* + U(h)A_\beta U(h)^* - A\| \\ &\leq \|U(h)AU(h)^* - U(h)A_\beta U(h)^*\| + \|U(h)A_\beta U(h)^* - A\| \\ &= \|U(h)(A - A_\beta)U(h)^*\| + \|A_\beta - A\| \\ &\leq 2\|A - A_\beta\| \end{aligned}$$

and since the latter approaches zero, it follows that $A = U(h)AU(h)^*$. $\square$

Lemma 8 $\text{HS}(L^2(P))^H$ is operator norm dense in $\mathcal{K}(L^2(P))^H$.

Proof Suppose $A \in \mathcal{K}(L^2(P))^H$. We know that for each $h \in H$, $U(h)AU(h)^* = A$ and it follows since the Haar measure is normalized that for any $\psi \in L^2(P)$,

$$(A\psi)(x) = \int_H (A\psi)(x)dh = \int_H (U(h)AU(h)^*\psi)(x)dh.$$

Further, there is a net $A_\beta \in \text{HS}(L^2(P))$ such that $\|A_\beta - A\| \rightarrow 0$. Now, consider the net $\pi_1(A_\beta) \in \text{HS}(L^2(P))^H$. For any $\psi \in L^2(P)$, using the representation of $\pi_1(A)\psi$ from the above, it follows that

$$\begin{aligned} \|(A - \pi_1(A_\beta))\psi\|^2 &= \int_P |(A\psi)(x) - (\pi_1(A_\beta)\psi)(x)|^2 d\mu(x) \\ &= \int_P \int_H |(U(h)AU(h)^*\psi)(x) - (U(h)A_\beta U(h)^*\psi)(x)|^2 dh d\mu(x) \\ &= \int_P \int_H |(U(h)(A - A_\beta)U(h)^*\psi)(x)|^2 dh d\mu(x) \\ &= \int_H \int_P |(U(h)(A - A_\beta)U(h)^*\psi)(x)|^2 d\mu(x) dh \\ &= \int_H \|U(h)(A - A_\beta)U(h)^*\psi\|^2 dh \\ &\leq \|A - A_\beta\|^2 \|\psi\|^2. \end{aligned}$$

Here, the use of the Fubini–Tonelli theorem in the fourth line is justified because the integrand is nonnegative. Hence, it follows that $\|A - \pi_1(A_\beta)\| \leq \|A - A_\beta\| \rightarrow 0$. $\square$

Proposition 9 If $A \in \mathcal{K}(L^2(P))^H$, then $\pi_1(A) = A$.

Proof Suppose $A \in \mathcal{K}(L^2(P))^H$. Then, there is a net $A_\beta \in \text{HS}(L^2(P))^H$ converging to A in operator norm. It follows that $\pi_1(A) = \lim_\beta \pi_1(A_\beta) = \lim_\beta A_\beta = A$. $\square$

It follows that π_1 is surjective and hence a projection, i.e., $\pi_1(\pi_1(A)) = \pi_1(A)$.

We will have occasion to use a particular class of linear functionals in V_Q . For any $\varphi, \psi \in L^2(P)$, define

$$\omega_{\varphi, \psi}^H(A) := \int_H \langle \varphi, U(h)AU(h)^*\psi \rangle dh$$

for all $A \in \mathcal{K}(L^2(P))$.

Proposition 10 For each $\varphi, \psi \in L^2(P)$, $\omega_{\varphi, \psi}^H \in V_Q$.

Proof For any $h_0 \in H$ and $A \in \mathcal{K}(L^2(P))$,

$$\begin{aligned} \omega_{\varphi, \psi}^H(U(h_0)AU(h_0)^*) &= \int_H \langle \varphi, U(h)U(h_0)AU(h_0)^*U(h)^*\psi \rangle dh \\ &= \int_H \langle \varphi, U(hh_0)AU(hh_0)^*\psi \rangle dh \\ &= \int_H \langle \varphi, U(h)AU(h)^*\psi \rangle dh = \omega_{\varphi, \psi}^H(A). \end{aligned}$$

□

Now, let $N(V_Q)$ be the annihilator of V_Q in $\mathcal{K}(L^2(P))$.

Proposition 11 If $A \in N(V_Q)$, then $A^* \in N(V_Q)$.

Proof Suppose $A \in N(V_Q)$. Then for any $\omega \in V_Q$, since $\omega^* \in V_Q$, we have

$$\omega(A^*) = \text{Tr}(\rho_\omega A^*) = \overline{\text{Tr}(\rho_\omega^* A)} = 0,$$

so $A^* \in N(V_Q)$. □

Theorem 2 There is a $*$ -preserving vector space isomorphism between $\mathcal{K}(L^2(P))/N(V_Q)$ and $\mathcal{K}(L^2(P))^H$.

Proof We show $\ker(\pi_1) = N(V_Q)$.

Suppose $A \in \mathcal{K}(L^2(P))$, and let $A_\beta \in \text{HS}(L^2(P))$ be a net converging to A in operator norm. Suppose first that $\pi_1(A) = 0$. Then for any $\omega \in V_Q$, we have

$$\begin{aligned} 0 &= \omega(\pi_1(A)) = \lim_\beta \omega(\pi_1(A_\beta)) = \lim_\beta \text{Tr}(\pi_1(A_\beta)\rho_\omega) \\ &= \lim_\beta \langle \rho_\omega^*, \pi_1(A_\beta) \rangle_{\text{HS}} \\ &= \lim_\beta \int_{P \times P} \int_H k_{\rho_\omega}(x, y) k_{A_\beta}(xh, yh) dh d\mu(x) d\mu(y) \\ &= \lim_\beta \int_{P \times P} \int_H k_{\rho_\omega}(xh^{-1}, yh^{-1}) k_{A_\beta}(x, y) dh d\mu(x) d\mu(y) \end{aligned}$$

$$\begin{aligned}
 &= \lim_{\beta} \int_{P \times P} \int_H k_{\rho_{\omega}}(x, y) k_{A_{\beta}}(x, y) dh d\mu(x) d\mu(y) \\
 &= \lim_{\beta} \langle \rho_{\omega}^*, A_{\beta} \rangle_{\text{HS}} = \lim_{\beta} \text{Tr}(\rho_{\omega} A_{\beta}) = \lim_{\beta} \omega(A_{\beta}) = \omega(A).
 \end{aligned}$$

Hence, $A \in N(V_Q)$.

On the other hand, suppose $A \in N(V_Q)$. Then for any $\varphi, \psi \in L^2(P)$, using the form for $\pi_1(A)\psi$ from a previous lemma, we have that

$$\begin{aligned}
 \langle \varphi, \pi_1(A)\psi \rangle &= \int_P \int_H \overline{\varphi(x)} (U(h)AU(h)^*\psi)(x) d\mu(x) \\
 &= \int_H \int_P \overline{\varphi(x)} (U(h)AU(h)^*\psi)(x) d\mu(x) \\
 &= \int_H \langle \varphi, U(h)AU(h)^*\psi \rangle dh = \omega_{\varphi, \psi}^H(A) = 0.
 \end{aligned}$$

The use of the Fubini–Tonelli theorem in the second line is justified again by the Cauchy–Schwarz inequality because both vectors are square integrable. Hence, it follows that $\pi_1(A) = 0$, and we conclude that $\ker(\pi_1) = N(V_Q)$.

It follows that π_1 induces a $*$ -preserving vector space isomorphism from $\mathcal{K}(L^2(P))/N(V_Q)$ to $\mathcal{K}(L^2(P))^H$. $\square$

Notice that it follows immediately that the map $\omega \mapsto \omega \circ \pi_1$ is a vector space isomorphism between $(\mathcal{K}(L^2(P))^H)^*$ and V_Q . This shows that π_1 can be thought of as implementing the quantum reduction by restricting attention to the states in V_Q .

As in the classical case, it may be of interest to note that π_1 is not in general a $*$ -homomorphism. The following proposition, for example, establishes that π_1 fails to be a homomorphism when H is finite and acts non-trivially.

Proposition 12 *If π_1 is a $*$ -homomorphism, then for each $A \in N(V_Q)$ and $\psi \in L^2(P)$, the map $h \mapsto AU(h)\psi$ is zero almost everywhere on H .*

Proof Suppose π_1 is a $*$ -homomorphism. Then, $N(V_Q)$ is an ideal. Suppose $A \in N(V_Q)$. It follows that $A^*A \in N(V_Q)$. For any $\psi \in L^2(P)$, we have

$$\begin{aligned}
 \int_H \|AU(h)^*\psi\| dh &= \int_H \langle \psi, U(h)A^*AU(h)^*\psi \rangle dh \\
 &= \omega_{\psi, \psi}^H(A^*A) = 0,
 \end{aligned}$$

but since $h \mapsto \|AU(h)^*\psi\|$ and the Haar measure are both nonnegative, it follows that $\|AU(h)^*\psi\|$ is zero almost everywhere on H . The invariance of the Haar measure under inversion in H then yields the result. $\square$

5 Equivalence of quantizations and limits of states

Recall that in the strict deformation quantization of our system, we have a classical C^* -algebra $\mathfrak{A}_0 := C_0(T^*P)$ and a family of quantum C^* -algebras $\mathfrak{A}_{\hbar} := \mathcal{K}(L^2(P))$

for each $\hbar > 0$, each of which carries a group action by H . Our task now is to show that $\mathcal{Q}_\hbar \circ \pi_0 = \pi_1 \circ \mathcal{Q}_\hbar$ or in other words that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{A}_0 & \xrightarrow{\mathcal{Q}_\hbar} & \mathfrak{A}_\hbar \\ \pi_0 \downarrow & & \downarrow \pi_1 \\ \mathfrak{A}_0^H & \xrightarrow{\mathcal{Q}_\hbar} & \mathfrak{A}_\hbar^H \end{array}$$

We need to take some care in interpreting the diagram, however, because $\mathcal{Q}_\hbar$ is defined only on the dense subalgebra $\text{PW}(T^*P) \subseteq \mathfrak{A}_0$. Since $\mathcal{Q}_\hbar$ fails to be norm continuous, it cannot be extended uniquely to all of $\mathfrak{A}_0$. Further, for each $f \in \mathfrak{A}_0$, $\mathcal{Q}_\hbar(f)$ will only be defined for certain values of $\hbar$. So we aim to show that the diagram commutes when both $\mathcal{Q}_\hbar(f)$ and $\mathcal{Q}_\hbar(\pi_0(f))$ are well defined. Once this result is established, we will be able to show immediately that the classical limit, understood in a certain sense, of any quantum state in V_Q belongs to V_C .

We establish that the diagram commutes in two steps: first, we show the projection maps commute with the Fourier transform, and second, we show the projection maps commute with the construction of the kernel $k_\hbar$.

To begin, note that we have a smooth right action of H on TP given by the push-forward of the given right action of H on P .

Lemma 9 *Suppose $f : T^*P \rightarrow \mathbb{C}$ is measurable and integrable over each fiber. If $h \in H$ and $X \in T_x P$, then $(\mathcal{F}f)(Xh) = (\mathcal{F}(h^{-1}f))(X)$.*

Proof For any $h \in H$ and $X \in T_x P$, we know $Xh \in T_{xh} P$, so

$$\begin{aligned} (\mathcal{F}f)(Xh) &= \int_{T_{xh}^* P} e^{i\langle \theta, Xh \rangle} f(\theta) d\hat{\mu}_{Xh}(\theta) \\ &= \int_{T_x^* P} e^{i\langle \theta h^{-1}, Xh \rangle} f(\theta h^{-1}) d\hat{\mu}_x(\theta) \\ &= \int_{T_x^* P} e^{i\langle \theta, X \rangle} (h^{-1}f)(\theta) d\hat{\mu}_x(\theta) \\ &= (\mathcal{F}(h^{-1}f))(X) \end{aligned}$$

as desired. $\square$

Lemma 10 *Suppose $f \in \text{PW}(T^*P)$. Then for each $x \in P$, the map $h \mapsto \int_{T_{xh}^* P} |f(\theta)| d\hat{\mu}_{Xh}(\theta)$ is bounded.*

Proof Our strategy is to rehearse the proof of one direction of the classical Paley–Wiener theorem [36, Thm. IX.11, p. 333], which establishes bounds on the decay of f on each fiber. We show that these bounds vary continuously across fibers, which yields the result.

We work in a neighborhood U of $x \in P$ admitting an orthonormal frame so that we can identify T^*U and TU with $U \times \mathbb{R}^m$. For any $y \in U$ and $\theta \in T_y^*P$, since $\mathcal{F}f \in C_c^\infty(TP)$, integration by parts yields

$$\int_{T_y P} e^{-i\langle \theta, X \rangle} (\partial_j (\mathcal{F}f))(y, X) d\mu_y(X) = i\theta_j \int_{T_y P} e^{-i\langle \theta, X \rangle} \mathcal{F}f(y, X) d\mu_y(X).$$

Further, by induction we have that for any multi-index α ,

$$\int_{T_y P} e^{-i\langle \theta, X \rangle} (\partial^\alpha (\mathcal{F}f))(y, X) d\mu_y(X) = (i\theta)^\alpha \int_{T_y P} e^{-i\langle \theta, X \rangle} \mathcal{F}f(y, X) d\mu_y(X)$$

and so the Fourier inversion theorem [36, Thm. IX.1, p. 320] implies that

$$(\partial^\alpha (\mathcal{F}f))(y, X) = \mathcal{F}((i\theta)^\alpha f)(y, X)$$

for any $y \in U$ and $X \in T_y P$. The Hausdorff–Young inequality [36, Thm. IX.8, p. 328] implies that

$$\sup_{\theta \in T_y^* P} |(i\theta)^\alpha f(y, \theta)| \leq C(y) := \int_{T_y P} |(\partial^\alpha (\mathcal{F}f))(y, X)| d\mu_y(X).$$

Moreover, $C(y)$ is continuous on U by the dominated convergence theorem [36, Thm. I.16, p. 24], and hence, $C(y)$ is bounded on some open set $V \subseteq U$.

Furthermore, since the above argument does not depend on the choice of $x \in P$, it follows that there is similarly a neighborhood V_h of xh for each $h \in H$ on which

$$\sup_{y \in V_h} \sup_{\theta \in T_y^* P} |(i\theta)^\alpha f(y, \theta)| \leq \sup_{y \in V_h} |C(y)| < \infty.$$

Since $\{V_h\}_{h \in H}$ is an open cover of xH , which is compact, there is a finite subcover $\{V_k\}$ of xH , which implies

$$\sup_{h \in H} \sup_{\theta \in T_{xh}^* P} |(i\theta)^\alpha f(xh, \theta)| \leq \sum_k \sup_{y \in V_k} |C(y)| < \infty.$$

It now follows that $\sup_{h \in H} \int_{T_{xh}^* P} |f(\theta)| d\hat{\mu}_{xh}(\theta) < \infty$, as desired. $\square$

Proposition 13 Suppose $f \in \text{PW}(T^*P)$. Then for any $X \in T_x P$,

$$(\mathcal{F}(\pi_0(f)))(X) = \int_H (\mathcal{F}f)(Xh) dh.$$

Proof The preceding lemma implies that for each $x \in P$,

$$\int_H \int_{T_x^* P} |e^{i\langle \theta, X \rangle} f(\theta h^{-1})| d\hat{\mu}_x(\theta) dh < \infty.$$

This justifies applying the Fubini–Tonelli theorem in calculating

$$\begin{aligned}
 \int_H (\mathcal{F}f)(Xh)dh &= \int_H (\mathcal{F}(h^{-1}f))(X)dh \\
 &= \int_H \int_{T_x^*P} e^{i\langle \theta, X \rangle} f(\theta h^{-1}) d\hat{\mu}_x(\theta) dh \\
 &= \int_{T_x^*P} e^{i\langle \theta, X \rangle} \int_H f(\theta h^{-1}) dh d\hat{\mu}_x(\theta) \\
 &= \int_{T_x^*P} e^{i\langle \theta, X \rangle} \int_H f(\theta h) dh d\hat{\mu}_x(\theta) \\
 &= \int_{T_x^*P} e^{i\langle \theta, X \rangle} (\pi_0(f))(\theta) d\hat{\mu}_x(\theta) \\
 &= (\mathcal{F}(\pi_0(f)))(X)
 \end{aligned}$$

where the fourth line follows from the invariance of the Haar measure under inversion. $\square$

Note that we also have a smooth right action of H on $P \times P$ given by $(x, x')h = (xh, x'h)$ for all $x, x' \in P$ and $h \in H$. We know that for any $X \in T_x P$ and $h \in H$, we have $\varphi(Xh) = \varphi(X)h$ because $\exp_x(X)h = \exp_{xh}(Xh)$.

Proposition 14 *Suppose $f \in \text{PW}(T^*P)$ and $K \subseteq TP$ is the compact support of $\mathcal{F}f$. Let $\hbar > 0$ be such that $\hbar KH \subseteq V_1$. Then for any $Y \in (P \times P)$,*

$$(k_{\hbar}(\pi_0(f)))(Y) = \int_H k_{\hbar}(f)(Yh)dh.$$

Proof If $Y \in (P \times P) \setminus V_2$, then both sides of the equality are zero. Suppose, on the other hand, that $Y = \varphi(\hbar X)$ for some $X \in T_x P$. Then for any $h \in H$, we have $\hbar Xh \in V_1$, so

$$\begin{aligned}
 \int_H k_{\hbar}(f)(Yh)dh &= \int_H k_{\hbar}(f)(\varphi(\hbar X)h)dh \\
 &= \int_H k_{\hbar}(f)(\varphi(\hbar Xh))dh \\
 &= \int_H \hbar^{-n} (\mathcal{F}f)(Xh)dh \\
 &= \hbar^{-n} (\mathcal{F}(\pi_0(f)))(X) \\
 &= k_{\hbar}(\pi_0(f))(Y)
 \end{aligned}$$

as desired. $\square$

Theorem 3 *Suppose $f \in \text{PW}(T^*P)$ and $K \subseteq TP$ is the compact support of $\mathcal{F}f$. Let $\hbar > 0$ be such that $\hbar KH \subseteq V_1$. Then, $\mathcal{Q}_{\hbar}(\pi_0(f)) = \pi_1(\mathcal{Q}_{\hbar}(f))$.*

Proof For any $\psi \in L^2(P)$,

$$\begin{aligned} (\mathcal{Q}_{\hbar}(\pi_0(f))\psi)(x) &= \int_P (k_{\hbar}(\pi_0(f)))(x, x') \psi(x') d\mu(x') \\ &= \int_P \int_H (k_{\hbar}(f))(xh, x'h) \psi(x') d\mu(x') \\ &= (\pi_1(\mathcal{Q}_{\hbar}(f))\psi)(x) \end{aligned}$$

which shows that $\mathcal{Q}_{\hbar}(\pi_0(f)) = \pi_1(\mathcal{Q}_{\hbar}(f))$. $\square$

Now, we can consider classical limits of states in V_Q . We say that a family of functionals $\{\omega_{\hbar} \in \mathfrak{A}_{\hbar}^*\}_{\hbar \in [0, \epsilon]}$ is a *continuous field of linear functionals* if for each $f \in \text{PW}(T^*P)$, the map $\hbar \mapsto \omega_{\hbar}(\mathcal{Q}_{\hbar}(f))$ is continuous for all $\hbar \in [0, \epsilon]$ on which $\mathcal{Q}_{\hbar}(f)$ is defined (cf. Landsman's *classical germs* [23, p. 103]). The intended interpretation is that ω_0 may be thought of as a classical limit of $\{\omega_{\hbar}\}_{\hbar \in [0, \epsilon]}$. Of course, there may be many distinct continuous fields of functionals converging to the same classical state, so we understand these classical limits to only be specified relative to a given continuous field of functionals. We now show that each classical state in V_C can be thought of as a classical limit of quantum states in V_Q and that each continuous family of quantum states $\{\omega_{\hbar}\}_{\hbar \in [0, \epsilon]}$ that all belong to V_Q has a classical limit in V_C .

We will need to use the fact that for any $\omega_0 \in \mathfrak{A}_0^*$, we have $\omega_0 \circ \pi_0 \in V_C$, and similarly, for any $\omega_{\hbar} \in \mathfrak{A}_{\hbar}^*$, we have $\omega_{\hbar} \circ \pi_1 \in V_Q$. In the same vein, we will need the following result:

Proposition 15 *If $\omega_0 \in V_C$, then $\omega_0 \circ \pi_0 = \omega_0$. Similarly, if $\omega_{\hbar} \in V_Q$, then $\omega_{\hbar} \circ \pi_1 = \omega_{\hbar}$.*

Proof Suppose $\omega_0 \in V_C$. Then, the Riesz–Markov representation theorem entails the existence of a regular, finite measure μ_{ω_0} on T^*P such that

$$\omega_0(f) = \int_{T^*P} f(\theta) d\mu_{\omega_0}(\theta)$$

for all $f \in C_0(T^*P)$. It follows that for any $f \in C_0(T^*P)$,

$$\begin{aligned} \omega_0(\pi_0(f)) &= \int_{T^*P} \int_H f(\theta h) dh d\mu_{\omega_0}(\theta) \\ &= \int_H \int_{T^*P} (hf)(\theta) d\mu_{\omega_0}(\theta) dh \\ &= \int_H \omega_0(hf) dh \\ &= \int_H \omega_0(f) = \omega_0(f). \end{aligned}$$

Here, the application of the Fubini–Tonelli theorem in the second line is justified because both measures are finite and the integrand is bounded.

On the other hand, suppose $\omega_{\hbar} \in V_Q$. Then, there is a trace class operator $\rho_{\omega_{\hbar}}$ such that

$$\omega_{\hbar}(A) = \text{Tr}(\rho_{\omega_{\hbar}} A)$$

for all $A \in \mathcal{K}(L^2(P))$. It follows that for all $A \in \text{HS}(L^2(P))$,

$$\begin{aligned} \omega_{\hbar}(\pi_1(A)) &= \text{Tr}(\rho_{\omega_{\hbar}} \pi_1(A)) \\ &= \int_{P \times P} \int_H k_{\rho_{\omega_{\hbar}}}(x, y) k_A(xh, yh) dh d\mu(x) d\mu(y) \\ &= \int_H \int_{P \times P} k_{\rho_{\omega_{\hbar}}}(x, y) k_A(xh, yh) d\mu(x) d\mu(y) dh \\ &= \int_H \text{Tr}(\rho_{\omega_{\hbar}} U(h) A U(h)^*) dh \\ &= \int_H \omega_{\hbar}(U(h) A U(h)^*) dh \\ &= \int_H \omega_{\hbar}(A) = \omega_{\hbar}(A). \end{aligned}$$

Here, the application of the Fubini–Tonelli theorem in the third line is justified by a now familiar application of the Cauchy–Schwarz inequality. Since $\omega_{\hbar}$ agrees with $\omega_{\hbar} \circ \pi_1$ on a norm dense subset of $\mathcal{K}(L^2(P))$ and both are continuous linear functionals, $\omega_{\hbar} = \omega_{\hbar} \circ \pi_1$. $\square$

We now have the resources to prove our main result.

Theorem 4 $\omega_0 \in V_C$ if and only if there is a continuous field of linear functionals $\{\omega_{\hbar}\}_{\hbar \in [0, \epsilon)}$ such that $\omega_{\hbar} \in V_Q$ for all $\hbar \in (0, \epsilon)$.

Proof First, suppose that $\omega_0 \in V_C$. Then, it follows from a result of Landsman [23, p. 105] that there is a continuous field of functionals $\{\omega_{\hbar}\}_{\hbar \in [0, \epsilon)}$ converging to ω_0 . Now, consider the family of functionals $\omega'_0 = \omega_0$ and $\omega'_{\hbar} = \omega_{\hbar} \circ \pi_1$. Clearly, for each $\hbar \in (0, \epsilon)$, we have $\omega'_{\hbar} \in V_Q$. We claim that $\{\omega'_{\hbar}\}_{\hbar \in [0, \epsilon)}$ is a continuous field of functionals. This follows immediately from the previous theorem because for all $\hbar \in (0, \epsilon)$ and $f \in \text{PW}(T^*P)$ for which $\mathcal{Q}_{\hbar}(f)$ is defined, we have

$$\omega'_{\hbar}(\mathcal{Q}_{\hbar}(f)) = \omega_{\hbar} \circ \pi_1(\mathcal{Q}_{\hbar}(f)) = \omega_{\hbar}(\mathcal{Q}_{\hbar}(\pi_0(f)))$$

and so since $\omega_0(\mathcal{Q}_0(f)) = \omega_0(\mathcal{Q}_0(\pi_0(f)))$, the fact that $\{\omega_{\hbar}\}_{\hbar \in [0, \epsilon)}$ is a continuous field of functionals implies that $\hbar \mapsto \omega_{\hbar}(\mathcal{Q}_{\hbar}(\pi_0(f))) = \omega'_{\hbar}(\mathcal{Q}_{\hbar}(f))$ is continuous.

On the other hand, suppose $\{\omega_{\hbar}\}_{\hbar \in [0, \epsilon)}$ is a continuous field of linear functionals and $\omega_{\hbar} \in V_Q$ for all $\hbar \in (0, \epsilon)$. Then for any $f \in \text{PW}(T^*P)$ and sufficiently small $\epsilon' < \epsilon$ for which $\mathcal{Q}_{\hbar}(f)$ is defined for all $\hbar \in [0, \epsilon')$,

$$\omega_0(f) = \omega_0(\mathcal{Q}_0(f)) = \lim_{\hbar \rightarrow 0} \omega_{\hbar}(\mathcal{Q}_{\hbar}(f))$$

$$\begin{aligned}
 &= \lim_{\hbar \rightarrow 0} \omega_{\hbar} \circ \pi_1(Q_{\hbar}(f)) \\
 &= \lim_{\hbar \rightarrow 0} \omega_{\hbar}(Q_{\hbar}(\pi_0(f))) = \omega_0(Q_0(\pi_0(f))) = \omega_0(\pi_0(f)).
 \end{aligned}$$

Since ω_0 agrees with $\omega_0 \circ \pi_0$ on a norm dense subset, this implies that $\omega_0 \in V_C$. $\square$

Notice that we do not specify a particular way of taking the classical limit of any given quantum state $\omega_{\hbar} \in V_Q$ because we do not determine a unique continuous field of functionals for a given $\omega_{\hbar} \in V_Q$. Instead, we make just one interpretive assumption: we assume that one interprets the classical limit as “zooming out” or “looking at $\omega_{\hbar}$ on larger scales” without changing the physical content of the state $\omega_{\hbar}$. We implement this interpretation mathematically by requiring as a necessary condition for a continuous field of functionals $\{\omega_{\hbar}\}_{\hbar \in (0, \epsilon)}$ to represent the classical limit of $\omega_{\hbar} \in V_Q$ that for each $\hbar' \in (0, \epsilon)$, $\omega_{\hbar'} \in V_Q$. On this interpretation, then, the preceding theorem establishes that the classical limit of any quantum state in V_Q belongs to V_C .

6 Conclusion

In this paper, we have analyzed the classical limits of quantum states for particles moving in an external Yang–Mills field. We have shown in Theorem 4 by using the framework of C^* -algebraic strict quantization that the classical limit of a gauge-invariant quantum state is a gauge-invariant classical state. Further, we have shown how to construct reduction mappings in the C^* -algebraic formulation of both the classical and quantum theories. We established in Theorem 1 that the classical reduction mapping produces a C^* -algebra whose states consist precisely in the gauge-invariant classical states, corresponding to the universal phase-space formulation of the classical theory. We analogously established in Theorem 2 that the quantum reduction mapping produces a C^* -algebra whose states consist precisely in the gauge-invariant quantum states, corresponding to the superselection structure on the quantum theory. We conclude by commenting briefly on the significance of these results.

First, the results in this paper can be understood to provide a kind of justification for use of the universal phase-space formulation of the classical theory when one makes assumptions about the quantum theory. Suppose one already knows the quantum theory of such a system to have the superselection structure represented by the C^* -algebra $\mathcal{K}(L^2(P))^H$ with physical states belonging to V_Q . Suppose we make the interpretive assumption that a classical state is physical only if it is the classical limit of a physical quantum state. Under this assumption, it follows that the only physical classical states belong to V_C and hence are states on the C^* -algebra $C_0(T^*P)^H$ of the universal phase-space formulation. Thus, if one already believes the physical quantum states belong to V_Q , one will have reason to believe the physical classical states belong to V_C and so reason to choose the classical C^* -algebra $C_0(T^*P)^H$ of the universal phase-space formulation.

Second, and conversely, the results of this paper can be understood to provide a partial justification for the superselection structure of the quantum theory when one makes assumptions about the classical theory. Suppose one already knows the classical

theory of such a system to be represented by the universal phase-space formulation with the C^* -algebra $C_0(T^*P)^H$ and physical states belonging to V_C . Suppose we make the analogous interpretive assumption that a quantum state is physical only if its classical limit is a physical classical state. In this case, it does not follow that only quantum states in V_Q are physical because in general there may be distinct continuous fields of functionals having the same classical state as their classical limit. Rather, it follows only that the quantum states in V_Q meet a necessary condition for being physical states. Still, this might give some reason to prefer the quantum states in V_Q and hence to use the reduced quantum algebra $\mathcal{K}(L^2(P))^H$.

Finally, we make a more general remark about the strategy used in this paper. We believe the results obtained here demonstrate the usefulness of the classical limit in constraining the construction of quantum theories. Our results establish how to transfer from classical to quantum theories both (i) information about which states are physical and (ii) reduction procedures for limiting the state space of a theory. This promising avenue gives further reason to investigate the classical limit in the context of strict quantization.

Compliance with ethical standards

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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