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On $K-P$ sub-Riemannian Problems and their Cut Locus†

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***Abstract*— The problem of finding minimizing geodesics for a manifold M with a sub-Riemannian structure is equivalent to the time optimal control of a driftless system on M with a bound on the control. We consider here a class of sub-Riemannian problems on the classical Lie groups G where the dynamical equations are of the form $\dot{x} = \sum_j X_j(x)u_j$ and the $X_j = X_j(x)$ are right invariant vector fields on G and $u_j := u_j(t)$ the controls. The vector fields X_j are assumed to belong to the P part of a Cartan K-P decomposition. These types of problems admit a group of symmetries K which act on G by conjugation. Under the assumption that the minimal isotropy group in K is discrete, we prove that we can reduce the problem to a Riemannian problem on the regular part of the associated quotient space G/K. On this part we define the corresponding quotient metric. For the special cases of the K-P decomposition of $SU(n)$ of type AIII we prove that the assumption on the minimal isotropy group is verified. As an example of application of the techniques discussed we find the cut locus of a K-P optimal control problem on $SU(2)$.**

Keywords: Minimum time geometric control, Sub-Riemannian geometry, Symmetry reduction, Cut locus.

List of Symbols

T_xM – Tangent space at x for a manifold M
TM– Tangent bundle of M, i.e., $TM := \cup_{x\in M} T_xM$
Δ – sub-bundle of the tangent bundle TM
$\pi_\Delta : \Delta \to M$ – restriction to Δ of the standard projection map from TM to M
$f_*|_x : T_xM \to T_{f(x)}N$ – push-forward of a map $f : M \to N$ at the point $x \in M$
$f_* : TM \to TN$ – f_* restricted to T_xM is equal to $f_*|_x$
G – compact semisimple finite-dimensional real Lie group with corresponding Lie algebra $\mathfrak{g}$ identified with the tangent space at the identity.
$\mathbf{1}$ – identity element of G
$K = e^{\mathcal{K}}$ – connected component containing $\mathbf{1}$ of the Lie group associated to the Lie algebra $\mathcal{K}$
$R_p : G \to G$ – right multiplication by p, $R_p(x) = xp$
$L_p : G \to G$ – left multiplication by p, $L_p(x) = px$
$\text{ad}_P : \mathfrak{g} \to \mathfrak{g}$ – adjoint map of a Lie algebra $\mathfrak{g}$ at $P \in \mathfrak{g}$, $\text{ad}_P(Q) := [P, Q]$ for every $Q \in \mathfrak{g}$
$\langle\cdot|\cdot\rangle$ – inner product induced by the Killing form[1] of a Lie algebra $\mathfrak{g}$, $\langle P|Q\rangle := -\text{tr}(\text{ad}_P \circ \text{ad}_Q)$
$\langle\cdot,\cdot\rangle_x$ – Riemannian metric at $x \in G$ or its sub-Riemannian restriction, $\langle R_{x*}P, R_{x*}Q\rangle_x := \langle P|Q\rangle$ for $P, Q \in \mathfrak{g}$,
$\pi : G \to G/K$ – quotient map associated to the action of the Lie group K acting on the Lie group G
G_{reg} – set of points in G having minimal isotropy type.
G_{sing} – set of non-regular points in G, i.e. $G_{\text{sing}} := G - G_{\text{reg}}$
$g_{\pi(x)}(\cdot,\cdot)$ – Riemannian metric at $\pi(x) \in G_{\text{reg}}/K$.
$d(p, q)$ – sub-Riemannian distance from p to q in G.
$d_Q(\pi(p), \pi(q))$ – Riemannian distance from $\pi(p)$ to $\pi(q)$ in G_{reg}/K

I. Introduction

Sub-Riemannian problems are equivalent to optimal control problems for driftless control systems when we want to minimize time with bounded energy or vice-versa [1], [3], [15]. In these problems, one has a set of allowed directions at each point p of a manifold M, with a given metric. One wants to transfer the state between two points by moving at each point following only the allowed directions and minimizing the corresponding distance (see next section for formal definitions). In the paper [12], V. Jurdjević introduced a class of sub-Riemannian problems on matrix Lie groups G for which he was able to find an explicit expression of the optimal candidates. Such a class of problems, which were named *K-P problems*, was then reconsidered in [8], [2], [3], because of their interest in quantum control. In particular in [2], [3] an approach to their study was used based on considering the symmetry action of a Lie subgroup of G, $K \subseteq G$, on G. This allowed the reduction of the number of unknown parameters in the optimal control law in several cases of interest. The action of K on G considered in [2], [3] is the *conjugation* (or *adjoint*) action where a matrix $x \in G$, is transformed by a matrix $k \in K$ according to $x \to kxk^{-1}$. With this action the corresponding orbit space G/K can often be mathematically described and visualized [2] [4]. However, in general, such a space is not a manifold but it has the more general structure of a *stratified space* on which one has to generalize the standard notions of differential geometry [20]. The strata are (connected components of) the orbit types (see, e.g., [7]). Among them, the *minimal orbit type*, i.e., the orbit type corresponding to points with a minimal isotropy group, is, according to a theorem in the theory of Lie transformation groups, an open and dense manifold in G/K, which is called the *regular part* or *principal part* of G/K (or of G) [7]. The remaining part of G/K (or of G) is called the *singular part.*

In this paper, we explore the possibility of studying K-P sub-Riemannian problems as Riemannian problems on the orbit space G/K. Sub-Riemannian geodesics on G can be

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[1]Or any positive scalar multiple of it.

obtained from Riemannian geodesics on G/K as inverse images of the natural projection. We restrict ourselves to the regular part of G/K and define a metric which allows us to obtain this reduction under the assumption that the minimum isotropy group in K is discrete.

Once the correspondence between a sub-Riemannian problem and a Riemannian one is established, one can use the powerful machinery of Riemannian geometry to answer questions for sub-Riemannian manifolds. We illustrate this, in particular, for the determination of the *sub-Riemannian cut locus*, that is, the set of points where geodesics lose optimality. We do this for the particular example of a K-P problem on $SU(2)$ but we believe the technique can be used for more general K-P problems and will be discussed in more generality elsewhere. In the optimal control context, the knowledge of the cut locus (from a given initial state) is the first step to obtain the *complete optimal synthesis*, i.e, the knowledge of all optimal trajectories. In fact, *all* the optimal trajectories are the ones with final point in the cut locus.

The paper is organized as follows: In section II, we give a more precise description of K-P models and their symmetries. In section III we describe, in general, the choice of the metric on the quotient space G/K that reduces the sub-Riemannian problem on G to a Riemannian problem on G/K. Such a metric is well defined if the minimal isotropy group in K is discrete, and we identify a subclass of K-P problems for which this assumption is verified. In the last section IV, we apply this reduction from sub-Riemannian to Riemannian problem to determine the cut locus of a problem on $SU(2)$. This problem was also considered in [2] but a formal explicit description of the sub-Riemannian cut locus was not given there.

II. $K-P$ SUB-RIEMANNIAN PROBLEMS AND THEIR SYMMETRIES

A. Sub-Riemannian manifolds

Given a Riemannian manifold M, a *sub-Riemannian structure* on M is a subbundle Δ of the tangent bundle TM. Denoting by Δ_x the fiber at $x \in M$, (i.e., if π_Δ is the natural projection $\pi_\Delta : \Delta \to M$, $\Delta_x := \pi_\Delta^{-1}(x)$) we assume that $\dim(\Delta_x) = m$ is constant, independent of x. In control theory Δ is often specified by giving a *distribution*, or *frame*, that is, a set of m vector fields $\mathcal{F} := \{X_1, ..., X_m\}$ such that, for every $x \in M$, $\texttt{span}\{X_1(x), ..., X_m(x)\} = \Delta_x$. For simplicity we shall assume that $\{X_1, ..., X_m\}$ is an orthonormal frame, that is, $\forall x \in M$, $\langle X_j(x), X_k(x)\rangle = \delta_{j,k}$, where $\langle \cdot, \cdot \rangle$ denotes an underlying Riemannian metric of M, and $\delta_{j,k}$ is the Kronecker delta. It is also assumed that the frame $\mathcal{F}$ is *bracket generating*, that is, if $Lie\,\mathcal{F}$ is the Lie algebra generated by $\mathcal{F}$, $Lie\mathcal{F}(x) = T_x M$, for every $x \in M$. The Δ in this definition is frequently called a *horizontal distribution*.

A curve $\gamma : [a, b] \to M$ is said to be a *horizontal curve* if the tangent vector $\dot{\gamma}(t) \in \Delta_{\gamma(t)}$ for every $t \in [a, b]$. For a manifold M, with a sub-Riemannian structure Δ, one may define a metric structure with a distance between points p and q in M defined as

$$d(p, q) := \inf_{\gamma} \int_a^b \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle} dt \qquad (1)$$

where the infimum is taken over all horizontal curves $\gamma : [a, b] \to M$ such that $\gamma(a) = p$, $\gamma(b) = q$. A horizontal curve γ which is distance minimizing is called a *sub-Riemannian geodesic*. The above condition of $\mathcal{F}$ being bracket generating guarantees, under the assumption that M is a connected and complete metric space, that, for any two points p and q, the distance is realized by a horizontal curve (cf. the Chow-Raschevskii theorem in [1], [15]). In the context of geometric control theory, sub-Riemannian manifolds arise in finite-dimensional smooth control systems with constraints. The horizontal distribution $\mathcal{F}$ describes the directions which may be taken at each point [1]. The sub-Riemannian geodesics γ parametrized by arclength ($\|\dot{\gamma}(t)\| = c$, $\forall t$) are the trajectories which, for a given state transfer $p \to q$, minimize time for the system

$$\dot{x} = \sum_{j=1}^{m} X_j(x) u_j(t), \qquad x(0) = p,$$

subject to the constraint $\|u\|^2 \leq c^2$ (cf., e.g., [1], [3]).

B. $K - P$ problems

K-P sub-Riemannan problems were introduced in [12] and studied in [2], [3], [8] in the context of quantum control. Such problems provide, in some sense, the simplest class of examples of sub-Riemannian problems (after the Riemannian case) since they are systems of *non-holonomy degree one* [16]: It is sufficient to take *one* Lie bracket of the available vector fields to have a distribution which at every point spans the whole tangent space.

The basic setup is as follows: Let G be a finite-dimensional, real, connected, compact semisimple Lie group with corresponding Lie algebra $\mathfrak{g}$. A *K-P Cartan decomposition* is a decomposition $\mathfrak{g} = \mathcal{K} \oplus \mathcal{P}$ into vector spaces $\mathcal{K}$ and $\mathcal{P}$ such that:

$$[\mathcal{K}, \mathcal{K}] \subseteq \mathcal{K} \qquad [\mathcal{K}, \mathcal{P}] \subseteq \mathcal{P} \qquad [\mathcal{P}, \mathcal{P}] \subseteq \mathcal{K}. \qquad (2)$$

A bi-invariant, positive-definite, symmetric inner product $\langle \cdot | \cdot \rangle$ on $\mathfrak{g}$ (cf. [13], chapter III section 7, and chapter X) can be defined as follows: For G compact,

$$\langle P|Q\rangle := -B(P, Q) := -\text{tr}(\text{ad}_P \circ \text{ad}_Q), \qquad (3)$$

the negative of the *Killing form*, or any positive multiple of it. The K-P problem is the minimum time problem for systems of the form

$$\dot{X} = \sum_j u_j B_j X, \qquad X(0) = \mathbf{1}, \qquad (4)$$

with $X \in G$ and u_j the controls, with $\|u\| \leq 1$. The elements B_j form an orthonormal basis of $\mathcal{P}$ and the sub-Riemannian structure is given by the frame of right invariant vector fields $\mathcal{F} := \{B_1 X, ..., B_m X\}$. In other terms, the sub-bundle Δ is given by $\cup_{x \in G} R_{x*}\mathcal{P}$. Here we identify the tangent space of G at the identity with the Lie algebra $\mathfrak{g}$, and let R_x (L_x) denote the right (left) translation; The fiber at x is given by $R_{x*}\mathcal{P}$ and it is spanned $\{R_{x*}B_j\}$. The Riemannian metric $\langle \cdot, \cdot \rangle$ on G is derived from the above Killing form B on $\mathfrak{g}$, again by identifying $\mathfrak{g}$ with the tangent space of G at the identity: if V_1 and V_2 are tangent vectors at x then $\langle V_1, V_2 \rangle := -B(R_{x^{-1}*}V_1, R_{x^{-1}*}V_2)$. By definition the above metric is right-invariant, i.e., $\forall x \in G$, V_1, V_2 in some tangent space of G, $\langle R_{x*}V_1, R_{x*}V_2 \rangle = \langle V_1, V_2 \rangle$. By the properties of the Killing form, it also follows that it is *left invariant* (same definition as before with L_{x*} replacing R_{x*}) therefore the metric thus defined is *bi-invariant*.

Applying the Pontryagin maximum principle for the minimum time problem for system (4) one finds, [8], [12], that the optimal control has the form $\sum_j B_j u_j(t) = e^{At}Pe^{-At}$ for some $A \in \mathcal{K}$ and $P \in \mathcal{P}$, and the corresponding optimal trajectory has the form $X(t) = e^{At}e^{(-A+P)t}$.

C. Symmetries

One of the main features of K-P problems is the existence of a *group of symmetries*. Consider the connected Lie group $K := e^{\mathcal{K}}$ associated to the Lie algebra $\mathcal{K}$, and assume this Lie group to be compact. This Lie group has a (left, proper) action Φ_k, on G by conjugation, i.e, for $x \in G$, $k \in K$,

$$\Phi_k x := kxk^{-1}. \tag{5}$$

Such an action gives a *symmetry* for the sub-Riemannian optimal control problem (4) as it satisfies the following conditions: 1) For the initial condition, which is the identity $\mathbf{1}$ in (4), and every k, $\Phi_k \mathbf{1} = \mathbf{1}$, that is, the action leaves the initial condition unchanged 2) (invariance) For any $k \in K$, and given the distribution Δ defining the sub-Riemannian structure, we have $\Phi_{k*}\Delta_x = \Delta_{\Phi_k x}$. This property is a consequence of the fact that the B_j's in (4) form a basis in $\mathcal{P}$ and from the second one in (2), the set $\{kB_jk^{-1} \,|\, j = 1, 2, ..., m\}$ is a basis in $\mathcal{P}$ as well.[2] 3) For any $k \in K$, Φ_k is an *isometry*, that is, for any two tangent vectors V and W at $x \in G$,

$$\langle \Phi_{k*}V, \Phi_{k*}W \rangle_{\Phi_k x} = \langle V, W \rangle_x,$$

where $\langle \cdot, \cdot \rangle_x$ is the Riemannian metric on G calculated at x. In our case, the Riemannian metric is given by the Killing metric above described. We have $\langle \Phi_{k*}V, \Phi_{k*}W \rangle_{\Phi_k(x)} = \langle L_{k*}R_{k^{-1}*}V, L_{k*}R_{k^{-1}*}W \rangle_{\Phi_k(x)} = \langle V, W \rangle_x$ because of the left and right invariance (bi-invariance) of the metric.

A consequence of these properties is that (cf., [3]) if $\gamma(t)$ is a minimizing sub-Riemannian geodesic from the identity $\mathbf{1}$ to $p \in G$, for any $k \in K$, $\Phi_k(\gamma(t))$ is a minimizing geodesic from $\mathbf{1}$ to $\Phi_k(p)$. Therefore optimal geodesics are the 'lifts' (cf. next section) of appropriate curves on the quotient space G/K which, we will see, are also geodesics corresponding to an appropriate Riemannian metric.

[2]More in detail $\Phi_{k*}R_{x*}B_j := L_{k*}R_{k^{-1}*}R_{x*}B_j = L_{k*}R_{k^{-1}*}R_{x*}R_{k*}R_{k^{-1}*}B_j = L_{k*}R_{kxk^{-1}*}R_{k^{-1}*}B_j = R_{kxk^{-1}*}L_{k*}R_{k^{-1}*}B_j$, since L_{k*} and $R_{kxk^{-1}*}$ commute. However this is equal to $R_{\Phi_k(x)*}L_{k*}R_{k^{-1}*}B_j$ and since $L_{k*}R_{k^{-1}*}B_j \in \mathcal{P}$ it belongs to $\Delta_{\Phi_k(x)}$.

The action of K on G by conjugation is proper (since K is assumed to be compact) but it is not a *free* action since, for instance, the isotropy group of the identity is the full group K. Therefore G/K is not guaranteed to be a manifold and it is in fact a *stratified space*. To understand the stratified structure of G/K, following the theory of Lie transformation groups (see, e.g., [7]), one considers all the possible subgroups of K which are isotropy groups for some elements in G. Groups H which can be obtained one from the other by a similarity transformation ($H_2 = kH_1k^{-1}$, for $k \in K$) are placed in equivalence classes (H) called *isotropy types*. Elements in G which have isotropy groups with the same isotropy type (H) are placed in the same set $G_{(H)}$, and points on the same orbit must be in the same set $G_{(H)}$. Therefore it makes sense to consider the quotient spaces $G_{(H)}/K$. The full quotient space G/K is the disjoint union of the *orbit types* $G_{(H)}/K$'s over all possible isotropy types (H). The connected components of $G_{(H)}/K$ are manifolds which are the strata in the *stratified space* G/K. Among the various isotropy types, one can introduce a *partial ordering* by saying that $(H_1) \leq (H_2)$ if and only if there exists a group in (H_1) which is conjugate to a subgroup of a group in (H_2). The *minimum isotropy type theorem* (cf., e.g., [7]) states that there exists a minimum isotropy type (H_{min}) which is $\leq$ any isotropy type and the corresponding orbit type $G_{(H_{min})}/K$ is a *connected open and dense* manifold in G/K. It is called the *regular part* of G/K and we shall denote it by G_{reg}/K. The remaining part $G/K - G_{\text{reg}}/K =: G_{\text{sing}}/K$ is called the *singular part*. The pre-images in G, under the natural projection $\pi : G \to G/K$ are called the regular part G_{reg} and singular part G_{sing} of G, respectively. The following example which was also treated in [2] clarifies these ideas and will be the object of the analysis in section IV.

Example 2.1: Consider $G = SU(2)$ and the decomposition of $su(2)$ into diagonal matrices and antidiagonal matrices which give the $\mathcal{K}$ and $\mathcal{P}$ part of the Cartan K-P decomposition, respectively. The Lie group $K := e^{\mathcal{K}}$ is the (one dimensional) Lie group of diagonal matrices in $SU(2)$. Matrices that are diagonal in $SU(2)$ have as isotropy group the whole K while matrices that are not diagonal have as isotropy group $\{\pm\mathbf{1}\}$. Therefore there exist only two isotropy types and the minimum isotropy type is given by the discrete group $\{\pm\mathbf{1}\}$. Writing a general element $X \in SU(2)$ as

$$X := \begin{pmatrix} z & w \\ -w^* & z^* \end{pmatrix}, \qquad |z|^2 + |w|^2 = 1, \tag{6}$$

conjugation by an element of K does not modify the diagonal element z, while it may arbitrarily change the phase of the antidiagonal element w. Therefore, the orbits in $SU(2)/K$ are parametrized by the $(1,1)$ element, z, i.e., a point in the closed unit disc of the complex plane. If $|z| = 1$ the isotropy group is K. This is the singular part of the orbit space. If $|z| < 1$ then the isotropy group is $\{\pm\mathbf{1}\}$. This is the regular part corresponding to the interior of the unit disc.

Symmetry reduction has a long history in control theory and we refer to [10], [11], [14], [18], as entry points to an extensive literature. The novelty here is the application to K-P systems, the fact that the quotient space has a more general structure than the one of a manifold and the extensive use of Riemannian geometry.

III. Riemannian metric on the quotient space

We define a Riemannian metric on the regular part of the quotient space, $G_{\rm reg}/K$ as follows: Suppose the minimal isotropy type is discrete, and recall that $G_{\rm reg}$ is an open and dense submanifold of G (c.f. [7], Chapter IV Theorem 3.1). For $V, W \in T_{\pi(x)}(G_{\rm reg}/K)$, let $P, Q \in \mathcal{P}$ such that $\pi_* R_{x*} P = V$ and $\pi_* R_{x*} Q = W$, where π is the natural projection $\pi : G_{\rm reg} \to G_{\rm reg}/K$ and define the metric g on $G_{\rm reg}/K$,

$$g_{\pi(x)}(V, W) := \langle R_{x*}P, R_{x*}Q\rangle_x := -{\rm tr}({\rm ad}_P \circ {\rm ad}_Q) := \langle P|Q\rangle, \quad (7)$$

(cf., (3)). For this definition to be well posed, we must prove that the 'lifts' P and Q of V and W, respectively, exist and that the metric is independent of the choice of such lifts and the choice of the basepoint x in the fiber corresponding to $\pi(x)$. We first observe that, for $x \in G_{\rm reg}$, $\pi_* : T_x G_{\rm reg} \to T_{\pi(x)}(G_{\rm reg}/K)$ is defined since $G_{\rm reg}$ is an open dense submanifold of G and therefore $T_x G_{\rm reg}$ may be identified with $T_x G$. The projection π in the definition and in the following is meant to be restricted to $G_{\rm reg}$, so that π_* is restricted to $T_x G_{\rm reg}$. The following theorem whose proof is presented in [19] gives the conditions for the metric above defined to be well defined

Theorem 1: The metric in (7) is defined if and only if the minimal isotropy type in K is discrete. Moreover at every $x \in G_{reg}$, the map $\pi_*|_x : R_{x*}\mathcal{P} \to T_{\pi(x)} G_{reg}/K$ is an isomorphism.

Remark 3.1: An alternative to the definition (7) could have been to see the projection $\pi : G_{\rm reg} \to G_{\rm reg}/K$ as a Riemannian submersion [9] and define a *vertical* distribution given by $\ker_x \pi_*$ at any point x and the *horizontal* distribution given by the orthogonal space (to the vertical one) in the given Riemannian metric. A metric is defined analogously to what we have done here taking for each tangent vector in the quotient space its 'lift' to the horizontal space. This however does not coincide with the $\mathcal{P}$ space of the sub-Riemannian structure.

An important feature of the above defined metric is that the length is preserved going from horizontal curves γ in $G_{\rm reg}$ to the corresponding curves $\pi(\gamma)$ in $G_{\rm reg}/K$. If γ is a horizontal curve in $G_{\rm reg}$, then from (7) and since $\dot\gamma(t) \in R_{\gamma(t)*}\mathcal{P}$ we have $g_{\pi(\gamma(t))}(\pi_*\dot\gamma(t), \pi_*\dot\gamma(t)) = \langle \dot\gamma(t), \dot\gamma(t)\rangle_{\gamma(t)}$ and therefore from (1) the length is preserved. If $\gamma = \gamma[0, T]$ has one of the endpoints in $G_{\rm sing}$, the length is preserved on any sub-interval of $[0, T]$. This is the case of interest for us since our initial point, the identity $\mathbf{1}$, belongs to the singular part of G.

In the following, we shall denote by $d(\cdot, \cdot)$ the sub-Riemannian distance on G and therefore $G_{\rm reg}$ defined by (1) and by $d_Q(\cdot, \cdot)$ the Riemannian distance on $G_{\rm reg}/K$ with the metric defined in section III. If two points p and q are in $G_{\rm reg}$, since the length is preserved by the projection π under the adopted metric, we have

$$d_Q(\pi(p), \pi(q)) \leq d(p, q). \quad (8)$$

The following theorem gives the connection between Riemannian geodesics in G_{reg}/K with the given metric and sub-Riemannian geodesics in G starting from $\mathbf{1}$.

Theorem 2: Assume $\gamma = \gamma(t)$ is a sub-Riemannian geodesic defined in $[0, T]$ optimally connecting $\mathbf{1}$ and $q \in G_{\rm reg}$. Then $\pi(\gamma)$ is a (optimal) Riemannian geodesic from $\pi(\gamma(t_0))$ to $\pi(\gamma(T)) = \pi(q)$, for any $t_0 \in (0, T)$.

Moreover

$$\lim_{t_0 \to 0^+} d_Q(\pi(\gamma(t_0)), \pi(q)) = d(\mathbf{1}, q). \quad (9)$$

Proof: Assume that there exists a $t_0 \in (0, T)$ such that the geodesic between $\pi(\gamma(t_0))$ and $\pi(q)$ is not $\pi(\gamma)$. Denote such a geodesic by Γ. Let $\bar t$ be the infimum among the values of $t \geq t_0$ such that $\Gamma(t) \neq \pi(\gamma(t))$. By continuity of geodesics we have $\Gamma(\bar t) = \pi(\gamma(\bar t))$. However we also have for some $t \in (\bar t, \bar t + \bar\epsilon)$

$$\Gamma(t) \neq \pi(\gamma(t)), \quad (10)$$

for any $\bar\epsilon > 0$.

Recall from Theorem 1 that for any $x \in G$, $\pi_*|_x$ restricted to $R_{x*}\mathcal{P}$ is an isomorphism from $R_{x*}\mathcal{P}$ to $T_{\pi(x)} G_{\rm reg}/K$, denoting by $\pi_*|_x^{-1}$ its inverse, consider (in local coordinates) the differential equation

$$\dot\gamma_1 = \pi_*|_{\gamma_1(t)}^{-1} \dot\Gamma(t), \qquad \gamma_1(\bar t) = \gamma(\bar t), \quad (11)$$

which has a unique solution γ_1 in $[\bar t - \epsilon, \bar t + \epsilon]$ for appropriate ϵ, choosing $\epsilon < \bar\epsilon$. Moreover $\pi(\gamma_1) = \Gamma$.

Denote by L the length of γ_1 between $\gamma_1(\bar t - \epsilon)$ and $\gamma_1(\bar t + \epsilon)$, which is $\geq$ the (sub-Riemannian) distance $d(\gamma_1(\bar t - \epsilon), \gamma_1(\bar t + \epsilon))$. Since π preserves the distance we have $L = d_Q(\pi(\gamma_1(\bar t - \epsilon), \pi(\gamma_1(\bar t + \epsilon)) \leq d(\gamma_1(\bar t - \epsilon), \gamma_1(\bar t + \epsilon))$ because of (8). Therefore γ_1 is a sub-Riemannian geodesic between $\gamma_1(\bar t - \epsilon)$ and $\gamma_1(\bar t + \epsilon)$. Moreover $\gamma(t)$ coincides with $\gamma_1(t)$, for $t \in [\bar t - \epsilon, \bar t]$. Since they are both geodesics and coincide on an open interval $(\bar t - \epsilon, \bar t)$, because of analyticity of geodesics, they must coincide, which contradicts (10).

The above proof also shows that for every t_0

$$d(\gamma(t_0), q) = d_Q(\pi(\gamma(t_0)), \pi(q))$$

Taking the limit when $t_0 \to 0$ and using the continuity of the distance function d from the Chow-Rashevski theorem we obtain (9). ■

The theorem suggests a way to calculate the sub-Riemannian geodesics to points q in $G_{\rm reg}$ using Riemannian geometry. One calculates Riemannian geodesics Γ leading to $\pi(q)$ in $G_{\rm reg}/K$ and then calculate the 'lift' i.e. the sub-Riemannian geodesic γ_1 leading to q such that $\pi(\gamma_1) = \Gamma$ (cf. (11)). Our main use of this correspondence is in the determination of the sub-Riemannian cut locus in G, and we shall show an example of this in the next section. We first give a proof that for certain classes of K-P problems the condition of discrete minimal isotropy group is verified.

A K-P Cartan decomposition of $su(n)$ of the type **AIII** is a decomposition $su(n) = \mathcal{K} \oplus \mathcal{P}$ satisfying (2) where $\mathcal{K}$ consists of block diagonal matrices in $su(n)$ and $\mathcal{P}$ are block anti-diagonal matrices. More specifically let $q \leq n-q$. Then the matrices in $\mathcal{K}$ have the form $\begin{pmatrix} A_{q\times q} & \mathbf{0} \\ \mathbf{0} & B_{(n-q)\times(n-q)} \end{pmatrix}$ with $Tr(A_{q\times q})+Tr(B_{(n-q)\times(n-q)}) = 0$, while the matrices in $\mathcal{P}$ have the form $\begin{pmatrix} \mathbf{0}_{q\times q} & C_{q\times(n-q)} \\ -C^\dagger_{q\times(n-q)} & \mathbf{0}_{(n-q)\times(n-q)} \end{pmatrix}$. The Lie group K is the Lie subgroup of $SU(n)$ of block diagonal matrices with blocks of dimension $q\times q$ and $(n-q)\times(n-q)$. If we consider the left (or right) multiplication action of K on $SU(n)$, the quotient space is one of the symmetric spaces classified by Cartan [13]. If we consider the conjugation action as we do in this paper, the quotient space is one of the stratified spaces discussed above. We show here that in this case we can define a metric as in Theorem 1 of the above section.

Theorem 3: Consider a K-P Cartan decomposition of the type **AIII**. The minimum isotropy group in K for the conjugation action on $SU(n)$ is the discrete (Abelian) group $H := \{\mathbf{1}, \omega\mathbf{1}, ..., \omega^{n-1}\mathbf{1}\}$, where $\omega := e^{i\frac{2\pi}{n}}$.
In particular, the K-P problem of the type **AIII** satisfies the condition of Theorem 1 to define a quotient metric on the regular part.

Proof: Recall that there is some neighborhood U of $0 \in \mathfrak{su}(n)$ such that $\exp|_U : U \to \exp(U)$ is a diffeomorphism. Therefore, for a sufficiently small open ball $B \subseteq U$ centered around 0 and $X \in B$, $e^{KXK^\dagger} = Ke^X K^\dagger = e^X$ if and only if $KXK^\dagger = X$. Writing $X = A+P$ with $A \in \mathcal{K}$ and $P \in \mathcal{P}$ implies $KAK^\dagger = A$ and $KPK^\dagger = P$. Now, take such an X with A a diagonal matrix with distinct entries. Then K commutes with A if and only if K is a diagonal matrix. Call the diagonal entries of K $k_1, ..., k_n$ and the entries of P p_{ij}. Then $KPK^\dagger = P$ if and only if $\lambda_i p_{ij} = \lambda_j p_{ij}$ for $1 \leq i \leq q$, $q+1 \leq j \leq n$. Choosing sufficiently many entries of P to be nonzero implies K is a scalar matrix, hence $K \in H$. Note that every element of H commutes with every element of $SU(n)$, hence the minimal isotropy group is equal to H. ■

IV. Example: The cut locus for $SU(2)$

K-P problems on $SU(n)$ are particularly interesting because $SU(n)$ may represent quantum mechanical evolutions of $n-$level quantum systems. In this context, the time optimal control problem is especially motivated because of the need to obtain fast computations in quantum information and because fast evolution is a way to avoid the degrading of the quantum state due to the effect of the environment, the so-called *decoherence*. Furthermore geometric time optimal control theory gives a method to study the fundamental limitations of quantum evolution, the so-called *quantum speed limit*, and the related time-energy uncertainty relations (see, e.g., [21] and the references therein). The case of $SU(2)$ is the simplest one but also a very important one, as it models the evolution of two-level quantum systems, quantum bits, which are the basic building blocks of quantum information in the circuit based model [17]. This example has been treated in detail in [2] where a method to find the time optimal control law from the identity to any final condition was described. The description of the cut locus was somehow implicit in [2]. We derive it here with the methods of this paper.

For $G = SU(2)$ and $K = e^{\mathcal{K}}$, the (Abelian) Lie subgroup of diagonal matrices in $SU(2)$, we have seen in Example 2.1 that G/K is homeomorphic to the closed unit disc in the complex plane, and that G_{reg}/K is the open unit disc. Explicitly, the homeomorphism is given by mapping a matrix in $SU(2)$ to its $(1,1)$-entry z as in (6). By setting $z = x + iy$, we will use x, y as coordinates in the open unit disc G_{reg}/K. Now, in order to compute the components g_{ij} of the metric (7) in these coordinates, we need to know, at a point z in the open disc, what $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ may be lifted to in the fiber $\pi^{-1}(z)$. So, lifting to a point (cf. (6)) $q = \begin{pmatrix} z & w \\ -w^* & z^* \end{pmatrix} \in SU(2)$, we would like to find a matrix $P = \begin{pmatrix} 0 & a+bi \\ -a+bi & 0 \end{pmatrix} \in \mathcal{P}$ such that:

$$\pi_* R_{q*} P = \pi_* \left(\begin{pmatrix} 0 & a+bi \\ -a+bi & 0 \end{pmatrix} \begin{pmatrix} z & w \\ -w^* & z^* \end{pmatrix} \right) = \frac{\partial}{\partial x} \tag{12}$$

Letting $w = w_R + iw_I$, this implies:

$$\begin{pmatrix} -aw_R - bw_I \\ aw_I - bw_R \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{13}$$

Therefore, $a = \frac{-w_R}{1-|z|^2}$, $b = \frac{-w_I}{1-|z|^2}$. Similarly, one may find a $Q = \begin{pmatrix} 0 & c+di \\ -c+di & 0 \end{pmatrix} \in \mathcal{P}$ so that $\pi_* R_{q*} Q = \frac{\partial}{\partial y}$; in this case $c = \frac{w_I}{1-|z|^2}$, $d = \frac{-w_R}{1-|z|^2}$. Using this and the definition (7) (with the Killing metric $-\text{tr}(\text{ad}_P \circ \text{ad}_Q) = -\frac{1}{2}\text{Tr}(AB)$) we have that the components of the metric on the regular part of the quotient space are given by $g_{ij}(z) = \frac{1}{1-|z|^2}\delta_{ij}$ with $i, j \in \{x, y\}$. Recalling that $z = x + iy$ and letting $r^2 = |z|^2 = x^2 + y^2$, we may compute the Christoffel symbols $\Gamma^i_{kl} = \frac{1}{2}\sum_m g^{im}(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m})$, using the standard formulas (in the case of a Riemannian connection; cf., e.g., [9] formula (10) Chapter 2, section 3) at the point (x, y):

$$\begin{pmatrix} \Gamma^x_{xx} = \frac{x}{1-r^2}, & \Gamma^x_{xy} = \Gamma^x_{yx} = \frac{y}{1-r^2}, & \Gamma^x_{yy} = \frac{-x}{1-r^2} \\ \Gamma^y_{xx} = \frac{-y}{1-r^2}, & \Gamma^y_{xy} = \Gamma^y_{yx} = \frac{x}{1-r^2}, & \Gamma^y_{yy} = \frac{y}{1-r^2} \end{pmatrix} \tag{14}$$

Recall that, in general, the curvature (written with respect to a coordinate system $X_i = \frac{\partial}{\partial x_i}$) is defined as $\hat{R}(X_i, X_j)X_k := \nabla_{X_j}\nabla_{X_i}X_k - \nabla_{X_i}\nabla_{X_j}X_k + \nabla_{[X_i,X_j]}X_k = \sum_l R^l_{ijk}X_l$, with $R^s_{ijk} = \sum_l \Gamma^l_{ik}\Gamma^s_{jl} - \sum_l \Gamma^l_{jk}\Gamma^s_{il} + \frac{\partial}{\partial x_j}\Gamma^s_{ik} - \frac{\partial}{\partial x_i}\Gamma^s_{jk}$. Moreover, the sectional curvature of a two-dimensional subspace of the tangent space at a point which is spanned by X, Y is given by $\hat{K}(X, Y) = \frac{g(R(X,Y)X,Y)}{|X|^2|Y|^2 - g(X,Y)^2}$ and is independent of the choice of X, Y (c.f. [9], Chapter 4, Section 3, Proposition 3.1). In our case, letting $X = \frac{\partial}{\partial x}$ and $Y = \frac{\partial}{\partial y}$, $g_z(X, Y) = 0$ (since the metric is diagonal), and $|X|^2 = |Y|^2 = \frac{1}{1-r^2}$. We compute

$$\hat{K}(X, Y) = (1-r^2)^2 g_z(\hat{R}(X, Y)X, Y) \tag{15}$$

$$= (1-r^2)^2 (g_z(R^x_{xyx}\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) + g_z(R^y_{xyx}\frac{\partial}{\partial y}, \frac{\partial}{\partial y})) =$$

$$(1-r^2)R^y_{xyx},$$

where

$$R^y_{xyx} = \tag{16}$$

$$\Gamma^x_{xx}\Gamma^y_{yx} + \Gamma^y_{xx}\Gamma^y_{yy} - (\Gamma^x_{yx}\Gamma^y_{xx} + \Gamma^y_{yx}\Gamma^y_{xy}) + \frac{\partial}{\partial y}\Gamma^y_{xx} - \frac{\partial}{\partial x}\Gamma^y_{yx}$$

$$= \frac{-1}{(1-r^2)^2}(x^2 - y^2 - (-y^2 + x^2) + 1 + y^2 - x^2 + 1 + x^2 - y^2) =$$

$$\frac{-2}{(1-r^2)^2}$$

Therefore (15) becomes

$$\hat{K}(X,Y) = (1-r^2)\frac{-2}{(1-r^2)^2} = \frac{-2}{1-r^2} \tag{17}$$

The open disc is two-dimensional, and so this shows that the the sectional curvature of $SU(2)_{\text{reg}}/K$ is nonpositive, which will be utilized in the following result:

Theorem 4: The cut locus of $\mathbf{1}$ in $SU(2)$ is equal to $K = SU(2)_{\text{sing}}$.

Proof: First, note that every optimal trajectory in $SU(2)$ must pass through $SU(2)_{\text{reg}}$.[3] Therefore, since optimal trajectories lose optimality when they pass to a lower-dimensional stratum [3] [5], it is sufficient to prove that there are no cut points in $SU(2)_{\text{reg}}$. Assume q is a cut point in $SU(2)_{\text{reg}}$ and γ the corresponding sub-Riemannian geodesic (parametrized by constant speed) defined in $[0,T]$, with $\gamma(0) = \mathbf{1}$ and $\gamma(T) = q$. Then according to Theorem 2 $\pi(\gamma(t))$ is a minimimizing geodesic from $\pi(\gamma(t_0))$ to $\pi(q)$, for every $t_0 \in (0,T)$. Let $t_1 > T$ sufficiently small so that $p := \gamma(t_1)$ is still in $SU(2)_{\text{reg}}$ and the extension of $\pi(\gamma)$ to $(0,t_1]$ is still a *locally minimizing* geodesic in $SU(2)_{\text{reg}}/K$ (with constant speed).[4] Since $q = \gamma(T)$ is a cut point there exists another sub-Riemannian optimal geodesic η joining $\mathbf{1}$ with $p = \gamma(t_1)$, and clearly $\pi(\eta)$ is a (locally) minimizing geodesic in $(0,t_1]$. Now, the theorem is proved if we prove the following claim:

Claim *Two locally minimizing geodesics $\hat{\gamma}$ and $\hat{\eta}$ in $SU(2)_{reg}/K$ such that $\lim_{t\to 0^+}\hat{\gamma}(t) = \lim_{t\to 0^+}\hat{\eta}(t) = \pi(\mathbf{1})$ cannot intersect in $SU(2)_{reg}/K$.*

Suppose $\hat{\gamma}(t_1) = \hat{\eta}(t_1) = \hat{p} \in SU(2)_{\text{reg}}/K$ (one may always reparametrize one of the geodesics so that they intersect at the same time t_1). Define the continuous, nonnegative, function $f : [0,t_1] \to \mathbb{R}$ by $f(t) := d_Q(\hat{\gamma}(t), \hat{\eta}(t))$ for $t \in (0,t_1]$ and $f(0) := 0$; Since $SU(2)_{\text{reg}}/K$ is a simply connected, complete smooth Riemannian manifold with nonpositive sectional curvature, it is a *Hadamard manifold*.[5]

[3]Otherwise we would have that the trajectory $e^{At}e^{(-A+P)t} = e^{Ft}$ for $F \in \mathcal{K}$ which would imply $P = 0$, a contradiction.

[4]It coincides with $\pi(\gamma)$ in $(0,T]$ and therefore it satisfies the same geodesic equations [9] and initial conditions at T.

[5]We take this to be the definition of a Hadamard manifold. Other equivalent definitions exist (see [6], Proposition 5.1).

Therefore, $f(t)$ is a convex function (see [6], Proposition 5.4). Therefore, for every $t \in [0,1]$, we have:

$$0 \le f(t\cdot t_1) = f(t\cdot t_1 + (1-t)\cdot 0) \le tf(t_1) + (1-t)f(0) = 0 \tag{18}$$

So, $f(t\cdot t_0) = 0$ for every $t \in [0,1]$, implying $\hat{\gamma}(t) = \hat{\eta}(t)$ for every $t \in (0,t_1]$. Therefore, two different geodesics starting from $\pi(\mathbf{1})$ cannot intersect in $SU(2)_{\text{reg}}/K$, and the Claim and therefore the theorem are proved. ■

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