Dynamical Decomposition of Bilinear Control Systems subject to Symmetries

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Abstract

We describe a method to analyze and decompose the dynamics of a bilinear control system subject to symmetries. The method is based on the concept of *generalized Young symmetrizers* of representation theory. It naturally applies to the situation where the system evolves on a tensor product space and there exists a finite group of symmetries for the dynamics which interchanges the various factors. This is the case for *quantum mechanical multipartite* systems, such as spin networks, where each factor of the tensor product represents the state of one of the component systems. We present several examples of application.

Keywords: Decomposition of Dynamics; Symmetries; Applications of Representation Theory to Control; Bilinear Systems on Lie groups; Control of Quantum Mechanical Systems.

1 Introduction

In geometric control theory, one often considers bilinear systems of the form

$$\dot{X} = AX + \sum_{j=1}^{m} B_j u_j X, \qquad X(0) = \mathbf{1}, \tag{1}$$

where X varies in a matrix Lie group and A and B_j's belong to the corresponding Lie algebra, with u_j the controls, and $\mathbf{1}$ the identity of the group. It is a well known fact [20] that the *reachable set* for (1) is the connected Lie group $e^{\mathcal{L}}$, containing the identity $\mathbf{1}$, corresponding to the Lie algebra $\mathcal{L}$ generated by A and B_j's, assuming that $e^{\mathcal{L}}$ is compact. Therefore system (1) is called *controllable* if $e^{\mathcal{L}}$ is some 'natural' Lie group where the system is supposed to evolve. Common examples are the special orthogonal group $SO(N)$ and the unitary group $U(N)$ which appears in applications of control theory to quantum mechanics. If the system of interest has the form

$$\dot{\psi} = A\psi + \sum_{j=1}^{m} B_j u_j \psi, \qquad \psi(0) = \psi_0, \tag{2}$$

where ψ belongs to a *vector space* $\tilde{V}$, the reachable set from ψ_0 is $\{X\psi_0 \,|\, X \in e^{\mathcal{L}}\}$. This fact has had many applications. In particular, for controlled *quantum mechanical systems*, in finite dimensions, the equation (1), (2) is the *Schrödinger equation* incorporating a semiclassical control field $\vec{u}(t) := (u_1, ..., u_m)$ (see, e.g., [6], [8] for examples of modeling). In this case, the matrices A and B_j in (1), (2) belong to the Lie algebra $u(N)$ of skew-Hermitian, $N \times N$, matrices, so that $\mathcal{L}$ is a Lie *subalgebra* of $u(N)$. The matrix X in (1) is called the (quantum mechanical) *evolution operator* and ψ is the *state* of the quantum system belonging to a Hilbert space $\tilde{V}$. In this case, controllability is said to be verified if $e^{\mathcal{L}}$ is the full unitary ($U(N)$) or special unitary ($SU(N)$) Lie group.

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Although controllability is a generic property (see, e.g., [4], [15], [19] (Chapter 6, Sec. 4), [21]) often, in reality, symmetries of the physical system, and a too small number of control functions as compared to the dimension of the system, cause the *dynamical Lie algebra* $\mathcal{L}$, generated by A and B_j's, to be only a *proper* Lie subalgebra of the natural Lie algebra associated to the model (for example $u(N)$). The problem therefore arises to analyze the structure of this Lie algebra and to understand how this impacts the dynamics of the system (1), (2).

In the context of control of quantum systems, which is the main area of application we have in mind, this problem has been tackled in several references with tools of Lie algebras and representation theory (see, e.g., [23], [31], [32]). One sees the vector space $\tilde{V}$, where ψ in (2) lives, as the space associated to a *representation* (see basic definitions of representation theory in the next section) of the Lie group $e^{\mathcal{L}}$ or the Lie algebra $\mathcal{L}$. In the paper [9], one assumes to have a basis of the dynamical Lie algebra $\mathcal{L}$. Algorithms are given to decompose such a Lie algebra into Abelian and simple ideals which are its elementary components (Lie sub-algebras). Such algorithms are, for the most part, simplified and adapted versions of general algorithms presented for Lie algebras over arbitrary fields in the book [10]. The paper [23] identifies *two* causes of uncontrollability for quantum systems. On one hand, the presence of symmetries, i.e., operators commuting with the full dynamical Lie algebra $\mathcal{L}$, implies that the given representation of $\mathcal{L}$ is not *irreducible*, that is, the vector space $\tilde{V}$, where ψ in (2) lives, *splits* into a number of invariant subspaces each carrying an irreducible representation of the dynamical Lie algebra $\mathcal{L}$. Transitions from one subspace to the other are forbidden for the dynamics which results in uncontrollability. The second cause of uncontrollability is the fact that, even within the invariant subspaces, the system might be not controllable because of lack of control power. In fact, the paper [23] presents a list of possible Lie subalgebra that might appear as irreducible restrictions of $\mathcal{L}$ to invariant subspaces. In view of the recalled decomposition of the dynamical Lie algebra into irreducible components, a new, weaker, notion of controllability was introduced for quantum systems called *subspace controllability.* This is verified when controllability is verified on all of the invariant subspaces. Subspace controllability was recently investigated for a number of quantum control systems, most notably networks of spins [7], [28], [29]. It was shown [28] that, in some cases, the dimension of the largest invariant subspace grows exponentially with the number of particles in the network so, subspace controllability gives the opportunity of doing universal quantum computation on a restricted subspace even in the absence of full controllability.

From a practical point of view, for a quantum control system with a group of symmetries G, the question arises of *how* to obtain the decomposition of the dynamics into invariant subspaces. This is the topic of this paper. We focus on a specific method to obtain this which exploits the *duality* between representations of $\mathcal{L}$ and representations of G (this is some times referred to as Schur-Weyl duality (cf., e.g., [16])). Methods of decomposition such as the ones described in [9], [10], although very general, require the explicit solution of linear systems of equations of possibly very high dimension. For example, in the case of a network of n spin $\frac{1}{2}$ particles, the dimension of the state space increases as 2^n and therefore these computations involve matrices in $u(2^n)$, a space of dimension 4^n. Moreover, in these algorithms, the role of the group of symmetries G is hidden when we transform the problem into a (high dimensional) linear algebra problem. For example, if the system is a network of spin $\frac{1}{2}$'s and the symmetry group is some subgroup of the symmetric group (the permutations which leave the matrices appearing in (1), (2) unchanged) such a symmetry group is suggested by the topology of the network.

This paper is devoted to presenting methods of dynamical decomposition based on the representation theory of finite groups. The representation theory of these groups is a topic for which much is known (see, e.g., [11], [14], [16], [17], [24], [25], [27], [30]). From the knowledge of the representations of the relevant group of symmetries G one obtains the change of coordinates which places the Lie subalgebra of all elements of $u(N)$ which commute with G, $u(N)^G$, in a block diagonal form, where each block corresponds to an irreducible representation. Since the dynamical Lie algebra $\mathcal{L}$ is a Lie subalgebra of $u(N)^G$, it will also be placed in the same block diagonal form.

This paper is a *survey paper* or, perhaps more appropriately, an *application paper* aimed at presenting known results in representation theory in a self-contained fashion so that they can be used by control theorists dealing with systems of the form (1), (2), and in particular for quantum systems.

The paper is organized as follows: In Section 2, we give some background notions from representation theory including the definition and properties of *Generalized Young Symmetrizers (GYS)*, which play a crucial role in the method described. The method for dynamical decomposition is described in Section 3. It requires identifying certain GYS's and, in Section 4, we discuss how these are obtained in two special cases: the case of the full symmetric group S_n and the case of Abelian groups. In Section 5, we present two examples of applications to spin networks where we use the above techniques to obtain the GYS's and the decomposition. These results, in particular extend the results of [3] for *fully symmetric* spin networks to the case of an arbitrary number n of spins, with the computations for the case $n = 4$ presented in detail.

2 Background and Statement of the Problem

2.1 Representation theory and statement of the problem

We shall be interested in representations, $(\pi, \tilde{V})$, of groups, G, algebras, $\mathcal{A}$, or Lie algebras, $\mathcal{R}$, on a finite dimensional complex inner product space $\tilde{V}$ of dimensions N which we can identify with $\mathbb{C}^N$. The space $\tilde{V}$ is often called a *G-module* (or *$\mathcal{A}$-module*, or an *$\mathcal{R}$-module*). Representations are group, algebra or Lie algebra homomorphisms from G, $\mathcal{A}$ or $\mathcal{R}$ to $\mathtt{End}(\tilde{V})$ the space of endomorphisms on $\tilde{V}$, which if $\tilde{V} \simeq \mathbb{C}^N$ can be identified with the space of $N \times N$ matrices with complex entries. Given representations of G, $\mathcal{A}$ and $\mathcal{R}$, on the same space $\tilde{V}$, we shall denote by $\mathcal{A}^G$ or $\mathcal{R}^G$ the (Lie) subalgebra of elements in $\mathcal{A}$ or $\mathcal{R}$, or more precisely their representation, which commute with the representation of G. For example, for a quantum control system (1), (2), we are given a representation of the dynamical Lie algebra $\mathcal{L}$ generated by the "Hamiltonians" A and B_j's which is a subalgebra of $u(N)$, where, for $u(N)$, we take the standard representation given by skew-Hermitian matrices of dimension N. We are also given on the same space a representation of a group of symmetries G which commute with the elements of $\mathcal{L}$. Therefore $\mathcal{L} \subseteq u(N)^G$ the subalgebra of $u(N)$ commuting with G. This is the scenario we shall deal with in the following.

We fix some notations. We shall denote by $\mathtt{End}_G(\tilde{V})$ the space of all endomorphisms of $\tilde{V}$ commuting with G. Given two representations $(\pi_1, \tilde{V_1})$ and $(\pi_2, \tilde{V_2})$, $\mathtt{Hom}(\tilde{V_1}, \tilde{V_2})$ denotes the space of homomorphisms $\phi : \tilde{V_1} \to \tilde{V_2}$, $\mathtt{Hom}_G(\tilde{V_1}, \tilde{V_2})$ denotes the subspace of $\mathtt{Hom}(\tilde{V_1}, \tilde{V_2})$ of elements $\phi \in \mathtt{Hom}(\tilde{V_1}, \tilde{V_2})$ such that $\phi\pi_1(g) = \pi_2(g)\phi$ for every $g \in G$. Such a type of map is called a *G-map.* Analogously one can consider $\mathcal{A}$-maps and $\mathcal{R}$-maps, for algebra ($\mathcal{A}$) and Lie algebra ($\mathcal{R}$) representations. If the two representations coincide $\mathtt{Hom}_G(\tilde{V}, \tilde{V})$ coincides with $\mathtt{End}_G(\tilde{V})$. Two representations $(\pi_1, \tilde{V_1})$ and $(\pi_2, \tilde{V_2})$ are called *G-isomorphic* if there exists an element in $\mathtt{Hom}_G(\tilde{V_1}, \tilde{V_2})$, i.e., a G-map, which is also an isomorphism, a *G-isomorphism.*

Representations of groups are called *unitary* if their images are unitary matrices. Representations of Lie algebras are called unitary if their images are skew-Hermitian matrices. A representation $(\pi, \tilde{V})$ is called *reducible* if there exists a proper nonzero subspace of $\tilde{V}$ which is invariant under the representation, *irreducible* if there is no such subspace. Representations $(\pi, \tilde{V})$, of finite groups as well as those of unitary groups or Lie algebras, are *completely reducible*, i.e., they can be decomposed into the direct sum of irreducible representations (see, e.g., [14], [30]). In these cases, $\tilde{V}$ is the direct sum of invariant subspaces for π, so that the restriction of π to each invariant subspace is an irreducible representation. In this case, in appropriate coordinates, the matrices $\pi(x)$, for x element in the group, algebra, or Lie algebra, take a block diagonal form. The finite group case and the case of unitary representations are the cases that will be of interest for us in this paper.

In view of these notions, the problem to be solved in this paper, that we have outlined in the introduction, is as follows:

Problem:

Given a unitary representation of a Lie algebra $\mathcal{R}$, and a unitary representation of a finite symmetry group G, on a finite dimensional Hilbert space $\tilde{V}$, find a decomposition of $\mathcal{R}^G$ into its irreducible components and the associated change of coordinates in $\tilde{V}$. In the case of quantum control, the Lie algebra $\mathcal{R}$ is $u(N)$ in its standard representation and if the dynamical Lie algebra $\mathcal{L} \subseteq u(N)$ commutes with a group of symmetries G (in a given representation), then we look for a decomposition in irreducible representations of $u(N)^G$. Since $\mathcal{L} \subseteq u(N)^G$, in the coordinates we find, $\mathcal{L}$ also takes a block diagonal form.

A fundamental tool in representation theory is the following *Schur's Lemma* (see, e.g., [30], Section 2.1).

Theorem 1. *(Schur's Lemma) Let B be a group or an algebra or a Lie algebra.*

1. *If a complex representation $(\pi, \tilde{V})$ of B is irreducible, all B-maps $\tilde{V} \to \tilde{V}$ are multiples of the identity map.*

2. *Two irreducible representations $(\pi_1, \tilde{V_1})$ and $(\pi_2, \tilde{V_2})$ are such that the space of B-maps is either $1-$dimensional or 0-dimensional according to whether the two representations are isomorphic or not.*

Proof. The two statements are equivalent. If statement 2 holds, than taking $(\pi_1, \tilde{V_1}) = (\pi_2, \tilde{V_2}) = (\pi, \tilde{V})$ and noticing that the identity map is a B-map, we obtain statement 1. Now we prove first statement 1 and then show that statement 2 follows from it.

For any B-map ϕ between two representations $(\pi_1, \tilde{V}_1)$ and $(\pi_2, \tilde{V}_2)$, the Kernel of ϕ and the Image of ϕ are invariant subspaces for the representations $(\pi_1, \tilde{V}_1)$ and $(\pi_2, \tilde{V}_2)$, respectively. Consider now a B-map, ϕ for the representation $(\pi, \tilde{V})$ and let α be an eigenvalue of ϕ. Then, if $\mathbf{1}$ is the identity map, $\hat{\phi}_\alpha := \phi - \alpha\mathbf{1}$ is a B-map as well, and the Kernel of $\hat{\phi}_\alpha$ is not zero. Since $(\pi, \tilde{V})$ is irreducible, the Kernel must be all of $\tilde{V}$, that is $\phi - \alpha\mathbf{1} = 0$, which proves the first statement.

As for the second statement, assume $\phi : \tilde{V}_1 \to \tilde{V}_2$ is a B-map. Then because of irreducibility $Ker(\phi) = 0$ or $Ker(\phi) = \tilde{V}_1$ and $Im(\phi) = 0$ or $Im(\phi) = \tilde{V}_2$. If $Ker(\phi) = 0$ and $Im(\phi) = \tilde{V}_2$ then ϕ is an isomorphism. In all other cases it is zero. If ϕ and γ are two isomorphisms, from $\phi\pi_1 = \pi_2\phi$ and $\gamma\pi_1 = \pi_2\gamma$, we obtain $\phi\gamma^{-1}\pi_2 = \pi_2\phi\gamma^{-1}$ which using the first statement implies that ϕ is a multiple of γ, which proves the second statement.

□

We remark that Schur's Lemma applies to both real and complex (Lie) algebras as long as the considered representations are complex, i.e., $\tilde{V}$, (or $\tilde{V}_{1,2}$) are complex vector spaces. We need, in fact, the underlying field to be algebraically closed in order to be able to always find an eigenvalue for the B-map of part 1. More general and abstract formulations of Schur's Lemma exist (see, e.g., [16] and references therein).

2.2 Group algebra, regular representation and Generalized Young Symmetrizers

Given a finite group G, the group algebra $\mathbb{C}[G] := \bigoplus_{\Pi \in G} \mathbb{C}\Pi$ is the complex vector space with basis given by the elements of G equipped with multiplication given by bilinearly extending the group operation. For example, for $G = S_3$, the symmetric group of three elements,

$$(12) \cdot \big(\lambda \cdot (1) + \mu \cdot (13)\big) = \lambda \cdot (12) \cdot (1) + \mu \cdot (12) \cdot (13) = \lambda \cdot (12) + \mu \cdot (132),$$

for $\lambda, \mu \in \mathbb{C}$.[1] If $\tilde{V}$ is a G-module then it is also $\mathbb{C}[G]$-module where $\mathbb{C}[G]$ acts on $\tilde{V}$ by linearly extending the action of the group G. If we take as $\tilde{V}$ exactly $\mathbb{C}[G]$, the action of G on $\tilde{V}$ gives a representation of G called the *regular representation.* The regular representation is, in general, *not* irreducible and it contains, as irreducible components, all the irreducible representations of the finite group G. More precisely, the following fundamental fact holds (cf., e.g., [14]):

Theorem 2. *Every irreducible representation of a finite group G on a vector space $\tilde{V}$ is G-isomorphic to one irreducible representation contained in the regular representation.*

Irreducible representations may be contained (up to $G-$isomorphism) more than once in $\mathbb{C}[G]$. Their multiplicity is equal to the dimension of the representation. That is, we have (cf., e.g., [14] § 3.4)

$$\mathbb{C}[G] = \bigoplus_j (\mathcal{C}_j)^{\oplus \dim \mathcal{C}_j}, \tag{3}$$

for the irreducible representations $\mathcal{C}_j \subseteq \mathbb{C}[G]$ of G which, in particular, implies that

$$\sum_j (\dim \mathcal{C}_j)^2 = \dim \mathbb{C}[G] = |G|,$$

the number of elements in the group G.

Definition 2.1. (**Generalized Young Symmetrizers (*GYS*)**) Given a finite group G, a *complete set of Generalized Young Symmetrizers* is a set of elements $\{P_j\}$, $j = 1, \ldots, q$, of the associated group algebra $\mathbb{C}[G]$ satisfying the following properties:

[1] Here and in the following the computation of the product of permutations is performed from right to left. This guarantees that they have the correct *left* action on the underlying space. For example consider S_3 acting naturally on $\mathbb{C}^3$ by permuting the components of a vector. Then the action on $(a, b, c)^T$ of (13)(12) is as follows. The permutation (12) switches positions 1 and 2. After that, the permutation (13) switches positions 1 and 3. The final result is $(c, a, b)^T$. The product (13)(12) is calculated (right to left) $1 \to 2 \to 2$, then $2 \to 1 \to 3$, $3 \to 3 \to 1$, so as to obtain (123). Here the first '→' refers to the permutation (12), the second '→' refers to the permutation (13). Consistently, the action of (123) on $(a, b, c)^T$ also gives $(c, a, b)^T$.

1. (*Completeness*)
$$\mathbf{1} = \sum_{j=1}^{q} P_j; \tag{4}$$
where $\mathbf{1}$ is the identity of the group.

2. (*Orthogonality*)
$$P_j P_k = \delta_{j,k} P_j, \quad \forall j, k; \tag{5}$$
where $\delta_{j,k}$ is the Kronecker delta.

3. (*Primitivity*) Fix a given P_j. For every $g \in G$
$$P_j g P_j = \lambda_g P_j, \tag{6}$$
with λ_g a scalar that depends on g.

Generalized Young symmetrizers are called a *complete set of primitive orthogonal idempotents* in ring theory. Their significance in representation theory is that they generate left ideals in the group algebra $\mathbb{C}[G]$ which correspond to irreducible sub-representations of the regular representation of G. In particular given a set of GYS's, we can write $\mathbb{C}[G]$ as
$$\mathbb{C}[G] = \mathbb{C}[G]\mathbf{1} = \mathbb{C}[G](\sum_j P_j) = \mathcal{C}_1 + \mathcal{C}_2 + \cdots + \mathcal{C}_q, \tag{7}$$
where $\mathcal{C}_j := \mathbb{C}[G]P_j$, $j = 1, \ldots, q$, is a left ideal of $\mathbb{C}[G]$ and, in particular, an invariant subspace of G in $\mathbb{C}[G]$, i.e., a sub-representation of the regular representation. Fix $j \geq 2$ and let $x \in \mathcal{C}_j \cap \mathcal{C}_1 + \mathcal{C}_2 + \cdots + \mathcal{C}_{j-1}$. Then there exist $A_1, A_2, ..., A_j$ in $\mathbb{C}[G]$ such that $x = A_j P_j = A_1 P_1 + A_2 P_2 + \cdots + A_{j-1} P_{j-1}$. Multiplying on the right by P_j and using (5) we obtain $x = 0$. Therefore the sum in (7) is a *direct* sum of sub-modules, i.e., $\mathbb{C}[G] = \bigoplus_{j=1}^{q} \mathcal{C}_j$. The following fact is important for the development that follows. It is proved in Theorem III.3 of the Appendix III of [27].

Proposition 2.2. Let $P_j \in \mathbb{C}[G]$ be such that $P_j^2 = P_j$. Condition (6) is necessary and sufficient so that the ideal $\mathcal{C}_j := \mathbb{C}[G]P_j$ is *minimal* which means that it does not properly contain any other ideal. In terms of representations this means that the *representation associated with $\mathcal{C}_j$ is irreducible.*

According to Theorem III.2 in Appendix III of [27], a complete set of GYS's, $\{P_j\}$, always exists, so that the irreducible sub-modules $\mathcal{C}_j$ of $\mathbb{C}[G]$ can always be written as $\mathcal{C}_j = \mathbb{C}[G]P_j$.

Primitive, orthogonal idempotents are called *Young Symmetrizers* in the context of the symmetric group S_n and therefore we used here the terminology 'Generalized Young Symmetrizers' to refer to the case of a general finite group. In the case of the symmetric group, Young symmetrizers are obtained from Young tableaux as summarized in many textbooks such as [14], [16], and [27]. We shall review the main points in Subsection 4.

Another property of GYS's which we shall require is of being *Hermitian.* To define this property define a conjugate-linear map on $\mathbb{C}[G]$, denoted by †. This is defined on elements of G, by $g^\dagger := g^{-1}$ and extended to $\mathbb{C}[G]$ by conjugate linearity, that is, $\left(\sum_j a_j g_j\right)^\dagger = \sum_j \bar{a}_j g_j^\dagger$, for $g_j \in G$ and $a_j \in \mathbb{C}$. With this definition we may require that the GYS's are Hermitian, i.e.,
$$P_j = P_j^\dagger, \qquad j = 1, 2, ..., q. \tag{8}$$

In our context, we have a G-module, $\tilde{V}$, which is extended by linearity to be a $\mathbb{C}[G]$-module. We shall see elements in the group algebra $\mathbb{C}[G]$ as operators on the vector space $\tilde{V}$. We can view, in particular any GYS, P_j, as an operator on $\tilde{V}$. For $a := \sum_j a_j g_j$, we have $\pi(a) = \sum_j a_j \pi(g_j)$ and $\pi(a^\dagger) = \sum_j \bar{a}_j \pi(g_j^\dagger) = \sum_j \bar{a}_j (\pi(g_j))^{-1}$. If the representation (π, G) is unitary, with the standard inner product, $(\pi(g_j))^{-1} = (\pi(g_j))^\dagger$ so that $\pi(a^\dagger) = \sum_j \bar{a}_j (\pi(g_j))^\dagger$, so that $\pi(a^\dagger) = (\pi(a))^\dagger$. So if a is Hermitian ($a = a^\dagger$), its image under a unitary representation will also be Hermitian in the standard sense of Hermitian matrices.

The Hermiticity property will be important in our treatment of representations of Lie subalgebras of $u(N)$, in applications to quantum mechanical systems. We will take advantage of recent results of [2] and [18] which show how to modify the standard procedure to obtain Young Symmetrizers in order to obtain *Hermitian* Young Symmetrizers, for the case of the symmetric group.

3 Decomposition of the Dynamics

We now assume that, for a group G, we have a complete set of GYS's. We show how this information can be used to decompose a Lie algebra $\mathcal{R}^G$, i.e., the subalgebra of a Lie algebra $\mathcal{R}$ consisting of all elements of $\mathcal{R}$ commuting with G. This gives the decomposition of the dynamics induced by the symmetries in G, and in the associated coordinates, the system (2) (and (1)) can be put in a block diagonal form. We shall discuss in the following section how GYS's can, in certain cases, be obtained.

When we are given a system (2) with ψ varying in a complex vector space $\tilde{V}$, the space $\tilde{V}$ simultaneously carries representations of the dynamical Lie algebra $\mathcal{L}$, a natural Lie algebra $\mathcal{R}$ (for example $u(N)$), with $\mathcal{L} \subseteq \mathcal{R}$, a finite group of symmetries G, its group algebra $\mathbb{C}[G]$, as well as $\mathtt{End}(\tilde{V})$ and $\mathtt{End}_G(\tilde{V})$. We are ultimately interested in $\mathcal{R}^G$, since $\mathcal{L} \subseteq \mathcal{R}^G$, but we describe the representation of $\mathtt{End}_G(\tilde{V})$ first. Since $\mathcal{R}^G = \mathcal{R} \cap \mathtt{End}_G(\tilde{V})$, the representation of $\mathcal{R}^G$ is obtained by restricting the elements of the representation of $\mathtt{End}_G(\tilde{V})$ to the ones that also belong to the representation of $\mathcal{R}$ (for example skew-Hermitian matrices if $\mathcal{R} = u(N)$). Given the complete set of GYS's, $\{P_j\}$ and their representations (as elements of the group algebra $\mathbb{C}[G]$), which, with some abuse of notation, we still denote by $\{P_j\}$, we consider a decomposition of $\tilde{V}$ as

$$\tilde{V} = \oplus_j^q P_j \tilde{V}. \tag{9}$$

To see that this holds, first notice that for every $\vec{y} \in \tilde{V}$, $\vec{y} = (\sum_j P_j)\vec{y} = \sum_j P_j \vec{y}$, because of the completeness property (4). Moreover, for $j \geq 2$, if $\vec{x} \in P_j \tilde{V} \cap (P_1 \tilde{V} + P_2 \tilde{V} + \cdots P_{j-1} \tilde{V})$, i.e., $\vec{x} = P_j \vec{x}_j = P_1 \vec{x}_1 + P_2 \vec{x}_2 + \cdots P_{j-1} \vec{x}_{j-1}$, applying P_j to both sides and using the orthogonality relation (5), we obtain that $\vec{x} = 0$, and therefore the sum (9) is a direct sum (cf. (7)). We choose a basis of $\tilde{V}$ by putting together bases of $P_1 \tilde{V}$, $P_2 \tilde{V}$,...,$P_q \tilde{V}$. Furthermore, we group together bases corresponding to GYS's, P_j, which give isomorphic ideals, $\mathcal{C}_j$, in the group algebra $\mathbb{C}[G]$.

We now analyze the matrix representation of elements in $\mathtt{End}_G(\tilde{V})$ in this basis. If $F \in \mathtt{End}_G(\tilde{V})$, then, for every j, $FP_j = P_j F$, and therefore $P_j \tilde{V}$ is invariant under F. This implies that, in the chosen basis, F has a block diagonal form

$$F = \left(\begin{array}{c|c|c|c}
\begin{matrix} A_1 & & \\ & \ddots & \\ & & A_{m_A} \end{matrix} & & & \\ \hline
& \begin{matrix} B_1 & & \\ & \ddots & \\ & & B_{m_B} \end{matrix} & & \\ \hline
& & \begin{matrix} C_1 & & \\ & \ddots & \\ & & C_{m_C} \end{matrix} & \\ \hline
& & & \ddots
\end{array}\right) \tag{10}$$

where we denoted with the same letter blocks corresponding to isomorphic ideals in the group algebra. We remark that, depending on the representation at hand, some ideals $\mathcal{C}_j$ and corresponding GYS's P_j might not be present in the above decomposition meaning that some $P_j \tilde{V}$, might be zero.

Now, we want to obtain more information on the nature of the submatrices of F in (10) and we want to study the form of the representation of G in the same basis. From this, the *duality* between the representation of G and $\mathtt{End}_G(\tilde{V})$ will be apparent. This will rely on the following three propositions whose proofs are presented in the next subsection.

Proposition 3.1. Two left ideals in $\mathbb{C}[G]$, $\mathcal{C}_1$ and $\mathcal{C}_2$, generated by GYS's P_1 and P_2, respectively, are G-isomorphic if and only if there exists an $r \in G$ such that

$$P_1 r P_2 \neq 0. \tag{11}$$

Proposition 3.2. For each GYS, P_j, $P_j \tilde{V}$ is either zero or it is an irreducible representation of $\mathtt{End}_G(\tilde{V})$.

Proposition 3.3. Two $\mathtt{End}_G(\tilde{V})$-modules $P_j \tilde{V}$ and $P_k \tilde{V}$ are $\mathtt{End}_G(\tilde{V})$-isomorphic if and only if $\mathcal{C}_j$ and $\mathcal{C}_k$ are isomorphic as G-modules. In this case, an $\mathtt{End}_G(\tilde{V})$-isomorphism, $P_j \tilde{V} \to P_k \tilde{V}$, is given by $P_k r P_j$, for any $r \in G$ such that $P_k r P_j \neq 0$, which exists because of Proposition 3.1.

We recall that we are omitting the reference to the particular representation. Here $P_k r P_j$ acts on $\tilde{V}$ through the representation π. In the proof we show that $P_k r P_j \neq 0$ implies $\pi(P_k r P_j) \neq 0$ and this gives the desired isomorphism.

Before proving these facts, we see how they impact the form of the representation of $\mathtt{End}_G(\tilde{V})$ in (10). Sub-blocks of the matrix F corresponding to isomorphic ideals $\mathcal{C}_j$, must have the same dimension, according to Proposition 3.3. Therefore the blocks $A_1, ..., A_{m_A}$ have all the same dimension in (10), and the same is true for $B_1, ..., B_{m_B}$, and so on. Moreover, we can refine our choice of the basis as follows. Let $P_1\tilde{V}$,...,$P_m\tilde{V}$ be a maximal set of subspaces in the decomposition isomorphic to each-other. Choose a basis of for $P_1\tilde{V}$, $\{\vec{x}_1, ..., \vec{x}_{d_1}\}$, and using the isomorphism of Proposition 3.3, choose a basis for $P_2\tilde{V}$, given by $\{P_2 r P_1 \vec{x}_1, ..., P_2 r P_1 \vec{x}_{d_1}\}$, for $r \in G$ such that $P_2 r P_1 \neq 0$. If, for $b = 1, ..., d_1$, $F\vec{x}_b = \sum_{l=1}^{d_1} a_{lb}\vec{x}_l$ for some coefficients a_{lb}, then

$$F(P_2 r P_1 \vec{x}_b) = P_2 r P_1 F \vec{x}_b = P_2 r P_1 \sum_{l=1}^{d_1} a_{lb}\vec{x}_l = \sum_{l=1}^{d_1} a_{lb}(P_2 r P_1 \vec{x}_l).$$

Therefore, the coefficients of the matrix corresponding to F, $\{a_{lb}\}$ are the same for the actions on $P_1\tilde{V}$ and $P_2\tilde{V}$, and the matrix representations are the same. We can repeat this argument for the remaining $P_3\tilde{V}$,...,$P_m\tilde{V}$, if any, and show that all the matrices A_1,...,A_{m_A}, in (10) are equal, i.e., $A_1 = A_2 = \cdots = A_{m_A} = A$. Repeating the same argument for all other sets of isomorphic spaces, we find that, in the given basis, the matrices of representations of $\mathtt{End}_G(\tilde{V})$ have the form

$$F = \begin{pmatrix} \mathbf{1}_{m_A} \otimes A & & & \\ & \mathbf{1}_{m_B} \otimes B & & \\ & & \mathbf{1}_{m_C} \otimes C & \\ & & & \ddots \end{pmatrix}, \tag{12}$$

where the numbers m_A, m_B,m_C,.... describe how many times isomorphic representations enter the given representation of $\mathtt{End}_G(\tilde{V})$. Moreover the matrices A, B,C,... correspond to irreducible representations of $\mathtt{End}_G(\tilde{V})$.

We now study the form of the representation of G in the above basis. Fix one GYS, P_1 and let P_2,...,P_m the GYS's corresponding to isomorphic $\mathtt{End}_G(\tilde{V})$-modules and isomorphic G-submodules in $\mathbb{C}[G]$ (cf. Proposition 3.3). If $g \in G$ and $\vec{y} := P_1\vec{x} \in P_1\tilde{V}$, we have

$$g\vec{y} = gP_1\vec{x} = \left(\sum_{j=1}^{m} P_j + \sum_{j \notin \{1,...,m\}} P_j \right) gP_1\vec{x} = \sum_{j=1}^{m} P_j g P_1 \vec{x},$$

where we used the completeness relation (4) and Proposition 3.1. This shows that $\bigoplus_{j=1}^{m} P_j\tilde{V}$ is invariant under g and therefore (repeating this argument for every set of isomorphic spaces) that the matrix corresponding to g takes a block diagonal form where each block corresponds to a (large) block in (12) and it is of dimension $m_A d_A \times m_A d_A$, $m_B d_B \times m_B d_B$,$m_C d_C \times m_C d_C$,.... Here the integers $m_{A,B,C,...}$ indicate how many times isomorphic subspaces $P_j\tilde{V}$ appear in the representation of $\mathtt{End}_G(\tilde{V})$ (cf. formula (10)), the integers $d_{A,B,C,...}$ denote their dimensions. Let us focus on the first large block and indicate the number of occurrences simply by m and the dimension simply by d. If $\vec{x}_1, ..., \vec{x}_d$ is a basis of $P_1\tilde{V}$, the chosen basis is $\vec{x}_1, ..., \vec{x}_d, \Phi_{2,1}\vec{x}_1, ..., \Phi_{2,1}\vec{x}_d, ..., \Phi_{m,1}\vec{x}_1, ..., \Phi_{m,1}\vec{x}_d$, where $\Phi_{j,1}$, $j = 2, ..., m$, is the $\mathtt{End}_G(\tilde{V})$-isomorphism, $P_1\tilde{V} \to P_j\tilde{V}$ chosen above. Therefore the basis for $\bigoplus_{j=1}^{m} P_j\tilde{V}$, is given by $P_j\Phi_{j,1}\vec{x}_k$, $j = 1, ..., m$, $k = 1, ..., d$, ordered first by j and then by k, where we set $\Phi_{1,1}$ equal to the identity. Now, for $g \in G$, calculate $gP_j\Phi_{j,1}\vec{x}_k$. This gives

$$gP_j\Phi_{j,1}\vec{x}_k = \sum_{l=1}^{m} P_l g P_j \Phi_{j,1}\vec{x}_k.$$

The element $P_l g P_j$ is either zero or, according to Proposition 3.3, is an $\mathtt{End}_G(\tilde{V})$-isomorphism $P_j\tilde{V} \to P_l\tilde{V}$. Therefore, $P_l g P_j \Phi_{j,1}$ is an $\mathtt{End}_G(\tilde{V})$-isomorphism $P_1\tilde{V} \to P_l\tilde{V}$. According to Schur's Lemma, Theorem 1, the space of $\mathtt{End}_G(\tilde{V})$-isomorphisms $P_1\tilde{V} \to P_l\tilde{V}$ is one-dimensional. Therefore $P_l g P_j \Phi_{j,1} = \lambda_{l,j}(g)\Phi_{l,1}$, and we have

$$gP_j\Phi_{j,1}\vec{x}_k = \sum_{l=1}^{m} \lambda_{l,j}(g)\Phi_{l,1}\vec{x}_k,$$

with $\lambda_{l,j}(g)$ possibly zero for some $g \in G$. Therefore, by defining $\Lambda_m = \Lambda(g)$, the $m \times m$ matrix $\{\lambda_{l,j}(g)\}$, the matrix corresponding to g in the given basis of $\bigoplus_{l=1}^m P_l\tilde{V}$ has the form $\Lambda_m(g) \otimes \mathbf{1}_d$. Repeating this for every set of isomorphic representations, we find that the representation of g on $\tilde{V}$ has the form

$$g = \begin{pmatrix} \Lambda^A_{m_A} \otimes \mathbf{1}_{d_A} & & & \\ & \Lambda^B_{m_B} \otimes \mathbf{1}_{d_B} & & \\ & & \Lambda^C_{m_C} \otimes \mathbf{1}_{d_C} & \\ & & & \ddots \end{pmatrix}. \tag{13}$$

Comparing formula (13) with formula (12) reveals the duality of the representations of $\mathtt{End}_G(\tilde{V})$ and G. The commutativity of the two representations is also made clear in the given basis. Moreover, the dual roles of the integers $m_{A,B,C,...}$ and $d_{A,B,C,...}$ is also apparent. In the representation of $\mathtt{End}_G(\tilde{V})$, m is the number of isomorphic copies of a certain $P_j\tilde{V}$ in $\tilde{V}$ of dimension d. In the representation of G, the roles of m and d are reversed. The number m represents the *dimension* of a sub-representation of G and d represents *how many times it occurs.*

If the Lie algebra $\mathcal{R}$, for which we want to study the representation of $\mathcal{R}^G$, is not the full $\mathtt{End}(\tilde{V})$, we can take $\mathcal{R}^G = \mathcal{R} \cap \mathtt{End}_G(\tilde{V})$ and take in (12) the matrices $A, B, C, ...$, so that the full matrices give the representation of $\mathcal{R}^G$. Let us consider in detail the case $\mathcal{R} = u(N)$ where we take for $u(N)$ the standard representation for matrices which are skew-Hermitian in one orthonormal basis (and therefore all). In addition to properties (4), (5) and (6), we also use the Hermitian property (8). For consistency we will need that the chosen basis is orthonormal. This can be obtained by specializing a little bit more the basis of $\tilde{V}$ chosen above. First notice that two vector spaces corresponding to different GYS's, $P_j\tilde{V}$ and $P_k\tilde{V}$, are necessarily orthogonal to each other since $\vec{y}^\dagger P_j^\dagger P_k\vec{x} = \vec{y}^\dagger P_j P_k \vec{x} = 0$, for every $\vec{x}$ and $\vec{y}$ in $\tilde{V}$, because of (5) and (8). Now consider a set of isomorphic spaces, $P_1\tilde{V}$,...,$P_m\tilde{V}$, and choose first the basis of $P_1\tilde{V}$, $\{\vec{x}_1, ..., \vec{x}_d\}$ as an orthonormal basis. Modify the basis of $P_2\tilde{V}$ chosen above by multiplying all terms by a nonzero scalar μ, i.e., as $\{\mu P_2 r P_1 \vec{x}_1, ..., \mu P_2 r P_1 \vec{x}_d\}$. Then we have $(\mu P_2 r P_1 \vec{x}_u)^\dagger (\mu P_2 r P_1 \vec{x}_v) = |\mu|^2 \vec{x}_u^\dagger P_1^\dagger r^\dagger P_2^\dagger P_2 r P_1 \vec{x}_v = |\mu|^2 \vec{x}_u^\dagger (P_1 r^\dagger P_2) P_2 r P_1 \vec{x}_v$ Since $P_1 r^\dagger P_2$ is a nonzero $G-$map $P_2 \to P_1$, from Schur's lemma (Theorem 1) $P_1 r^\dagger P_2$ is a multiple of $(P_2 r P_1)^{-1}$ so we have $P_1 r^\dagger P_2 = \lambda (P_2 r P_1)^{-1}$. This gives $(\mu P_2 r P_1 \vec{x}_u)^\dagger (\mu P_2 r P_1 \vec{x}_v) = \lambda |\mu|^2 \vec{x}_u^\dagger \vec{x}_v = \lambda |\mu|^2 \delta_{u,v}$ and choosing $|\mu|^2 = \frac{1}{\lambda}$, we obtain an orthonormal basis.

3.1 Proofs of Propositions 3.1, 3.2, and 3.3.

For the proof of Proposition 3.1, we follow [27], Theorem III.4 in Appendix III. For the proofs of Propositions 3.2 and 3.3, we combine the treatment of [16] (cf. Theorem 4.2.1) which gives isomorphisms between $\mathtt{Hom}_G(\mathcal{C}_j, \tilde{V})$ and $\mathtt{Hom}_G(\mathcal{C}_k, \tilde{V})$ with Theorem 9.7 of [25] which says that $\mathtt{Hom}_G(\mathcal{C}_j, \tilde{V})$ and $P_j\tilde{V}$ are isomorphic $\mathtt{End}_G(\tilde{V})$-modules.

3.1.1 Proof of Proposition 3.1

Proof. First assume that (11) holds and consider the G-map $\Phi(x) := xP_1rP_2$, $\mathcal{C}_1 \to \mathcal{C}_2$. The fact that this is a G-map follows easily since, $\forall g \in G$, $\Phi(gx) = (gx)P_1rP_2 = g(xP_1rP_2) = g\Phi(x)$. We remark that since $P_1rP_2 \neq 0$ the map Φ is not zero on $\mathcal{C}_1$. In fact, $\Phi(\mathcal{C}_1) = \Phi(\mathbb{C}[G]P_1) = \mathbb{C}[G]P_1rP_2$, which in particular contains P_1rP_2. Therefore according to Schur lemma, Theorem 1, $\mathcal{C}_1$ and $\mathcal{C}_2$ are G-isomorphic.

Viceversa assume that there is a G-isomorphism, $\Phi : \mathcal{C}_1 \to \mathcal{C}_2$. Then $\Phi(P_1)$ must be different from zero otherwise we would have $\Phi(\mathcal{C}_1) = \Phi(\mathbb{C}[G]P_1) = \mathbb{C}[G]\Phi(P_1) = 0$. Moreover $\Phi(P_1) = \Phi(P_1)P_2 = \Phi(P_1P_1)P_2 = P_1\Phi(P_1)P_2 \neq 0$, where the first equality is due to the fact that $\Phi(P_1) \in \mathcal{C}_2$ and the last one to the fact that Φ is a G-map. Therefore since there exists an element S in $\mathbb{C}[G]$ ($S = \Phi(P_1)$) such that $P_1SP_2 \neq 0$, there must exist $r \in G$ such that $P_1rP_2 \neq 0$. Otherwise we would have $P_1SP_2 = 0$ for any $S \in \mathbb{C}[G]$.

□

3.1.2 Proof of Proposition 3.2

Proof. Assume $\vec{x} \in P_j\tilde{V}$ and $\vec{y} \in P_j\tilde{V}$ both different from zero (we are assuming $P_j\tilde{V} \neq 0$). We shall find an element $R \in \mathtt{End}_G(\tilde{V})$ such that $R\vec{x} = \vec{y}$. Since $\vec{x}$ and $\vec{y}$ are arbitrary, this will imply irreducibility of the $\mathtt{End}_G(\tilde{V})$-module, $P_j\tilde{V}$. Consider $\mathcal{C}_j\vec{x}$ which is a G-module. The map $\Phi_x : \mathcal{C}_j \to \mathcal{C}_j\vec{x}$, given by $\Phi_x(a) = a\vec{x}$ is a G-map. Moreover it is injective since $\mathcal{C}_j$ is irreducible (the Kernel would be a sub-representation (cf. Theorem 1)). Since Φ_x is surjective by definition $\mathcal{C}_j$ and $\mathcal{C}_j\vec{x}$ are G-isomorphic. The same can be said for $\mathcal{C}_j$ and $\mathcal{C}_j\vec{y}$, with a map Φ_y. We have therefore a G-isomorphism $\Phi_y \circ \Phi_x^{-1}$ from $\mathcal{C}_j\vec{x}$ to $\mathcal{C}_j\vec{y}$. In particular, $\Phi_x^{-1}(\vec{x}) = P_j$, so that $\Phi_y \circ \Phi_x^{-1}(\vec{x}) = P_j\vec{y} = \vec{y}$. Let Φ be any linear extension of $\Phi_y \circ \Phi_x^{-1}$ to $\tilde{V}$. The map

$$R := \frac{1}{|G|}\sum_{g\in G} g\Phi g^{-1}, \tag{14}$$

is in $\texttt{End}_G(\tilde{V})$ and coincides with Φ on $\mathcal{C}_j\vec{x}$. Applying R to $\vec{x}$, we get $\vec{y}$. □

3.1.3 Proof of Proposition 3.3

Proof. First consider GYS's P_j and P_k corresponding to G-isomorphic modules $\mathcal{C}_j$ and $\mathcal{C}_k$. Then, according to Proposition 3.1 there exists $r \in G$ such that $P_k r P_j \neq 0$. There also exists an $r_1 \in G$ such that $P_k r_1 P_j \neq 0$. The right multiplication by $P_j r P_k$ ($P_k r_1 P_j$) is a G-isomorphism $\mathcal{C}_j \to \mathcal{C}_k$ ($\mathcal{C}_k \to \mathcal{C}_j$). Moreover from Schur's Lemma (Theorem 1), except for a nonzero multiple λ, these maps must be inverse of each other. Applying them in sequence to $P_j \in \mathcal{C}_j$ we have

$$P_j r P_k P_k r_1 P_j = \lambda P_j.$$

Applying the representation π to both terms, we have

$$\pi(P_j r P_k)\pi(P_k r_1 P_j) = \lambda\pi(P_j), \tag{15}$$

which shows that if $\pi(P_j) \neq 0$ then $\pi(P_j r P_k) \neq 0$. Dropping again the notation π, because of the irreducibility of $P_j\tilde{V}$ and $P_k\tilde{V}$, from Schur's lemma, it follows that $P_j\tilde{V}$ and $P_k\tilde{V}$, are $\texttt{End}_G(\tilde{V})$-isomorphic. We remark that formula (15), written as $\pi(P_j r P_k)\pi(P_k)\pi(r_1 P_j) = \lambda\pi(P_j)$, also shows that if $\pi(P_j) \neq 0$ then $\pi(P_k) \neq 0$ for any k corresponding to isomorphic modules. In other terms, the $\pi(P_j)\tilde{V}$'s are all different from zero or all equal to zero.

Viceversa, assume that $P_j\tilde{V}$ and $P_k\tilde{V}$, are $\texttt{End}_G(\tilde{V})$-isomorphic, and both non-zero. Let Ψ be an $\texttt{End}_G(\tilde{V})$-isomorphism, $\Psi : P_j\tilde{V} \to P_k\tilde{V}$. Assume by contradiction that $\mathcal{C}_j$ and $\mathcal{C}_k$ are not G-isomorphic. We show that Ψ must be necessarily equal to zero, which gives the desired contradiction. Consider $\vec{x} \neq 0$ in $P_j\tilde{V}$ and the corresponding $\Psi(\vec{x}) \neq 0$ in $P_k\tilde{V}$. Consider the (non-zero) spaces $\mathcal{C}_j\vec{x}$ and $\mathcal{C}_k\Psi(\vec{x})$, and consider the G-map between G-modules $\mathcal{C}_j$ and $\mathcal{C}_j\vec{x}$, Φ_x, defined by $\Phi_x(a) = a\vec{x}$. Let $T := \mathcal{C}_j\vec{x} \cap \mathcal{C}_k\Psi(\vec{x})$. The pre-image of T under Φ_x is a G-invariant subspace of $\mathcal{C}_j$ and since $\mathcal{C}_j$ is an irreducible G-module it must be zero or the whole $\mathcal{C}_j$. It cannot be the whole $\mathcal{C}_j$, because that would imply (repeating the same argument for $\mathcal{C}_k$) that $\mathcal{C}_j\vec{x} = \mathcal{C}_k\Psi(\vec{x})$. In particular, it would imply $P_j\vec{x} = aP_k\Psi(\vec{x})$ with $a \in \mathbb{C}[G]$. However, since $\mathcal{C}_j$ and $\mathcal{C}_k$ are assumed to be not isomorphic, from Proposition 3.1 we obtain $P_j\vec{x} = P_j^2\vec{x} = P_j a P_k \Psi(\vec{x}) = 0$, which is a contradiction. Therefore the subspace $T \subseteq \tilde{V}$ is zero and we have a direct sum, $W = \mathcal{C}_j\vec{x} \oplus \mathcal{C}_k\Psi(\vec{x})$. Let Π be an element in $\texttt{End}(\tilde{V})$ which, when restricted to W gives the projection onto $\mathcal{C}_k\Psi(\vec{x})$. We can define $R \in \texttt{End}_G(\tilde{V})$, by $R := \frac{1}{|G|}\sum_{g\in G} g\Pi g^{-1}$. The endomorphism R is equal to Π when restricted to W. In particular, it is zero on $\mathcal{C}_j\vec{x}$ and the identity on $\mathcal{C}_k\Psi(\vec{x})$. We have

$$0 = \Psi(R\vec{x}) = R\Psi(\vec{x}) = \Psi(\vec{x}),$$

which gives the desired contradiction. □

3.2 Examples

Example 3.4. Let the group G be the group Q_8, of *unit quaternions* $\{\pm 1, \pm i, \pm j, \pm k\}$ with the standard multiplication between unit quaternions $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$. Since it has order 8, in the regular representation there are two isomorphic $2-$dimensional representations, and four non-isomorphic $1-$dimensional representations. This fact can be inferred using the formula $\sum_j (\dim \mathcal{C}_j)^2 = \dim \mathbb{C}[G] = |G|$ (cf. (3)), along with the known fact that the number of non-isomorphic representations in the regular representation is equal to the number of conjugacy classes in the group (cf., e.g., [26] Theorem 7 in Section 2.5), which is equal to 5 in the case of Q_8. Denote by χ^2 the $2-$dimensional representation and by $\chi_1^1, \chi_2^1, \chi_3^1, \chi_4^1$ the four $1-$dimensional representations in the regular representation. Consider now, for instance, as $\tilde{V}$ a $7-$dimensional space and assume that the representation of Q_8 on $\tilde{V}$ has one $2-$dimensional representation isomorphic to χ^2, three isomorphic $1-$dimensional representations isomorphic to χ_1^1 and two isomorphic 1−dimensional representations isomorphic to χ_3^1. We assume therefore that, in the coordinates given by the GYS's, the representation of $g \in Q_8$ is given as

$$g = \begin{pmatrix} A_{2\times 2} & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & 0 & c \end{pmatrix}, \tag{16}$$

for scalar b and c, and 2×2 matrix $A_{2\times 2}$. Comparing (16) with formula (13) (and (12)), we see that, in this case, $m_A = 2$ and $d_A = 1$, $m_B = 1$ and $d_B = 3$, and $m_C = 1$ and $d_C = 2$. The matrices that commute with the matrices in (16) have the form

$$F = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & B_{3\times 3} & 0 \\ 0 & 0 & 0 & C_{2\times 2} \end{pmatrix}, \tag{17}$$

for scalar a, and 3×3 general matrix $B_{3\times 3}$, and 2×2 general matrix $C_{2\times 2}$. The irreducible representation χ^2 has dimension $m_A = 2$ but it enters one time ($d_A = 1$) the representation of Q_8. Dually, there are $m_A = 2$ isomorphic representations of $\mathtt{End}_{Q_8}(\tilde{V})$ which have dimension $d_A = 1$. There are $m_A = 2$ GYS's corresponding to the representation χ^2. They are given by the block diagonal matrices

$$P_1 := \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{0}_6 \end{pmatrix}, \qquad P_2 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathbf{0}_5 \end{pmatrix}.$$

The irreducible representation χ_1^1 has dimension $m_B = 1$ but enters three times ($d_B = 3$) the representation of Q_8. Dually, there is only one ($m_B = 1$) isomorphic representation of $\mathtt{End}_{Q_8}(\tilde{V})$ which has dimension $d_B = 3$. There is only one GYS corresponding to the representation χ_1^1, which, in the chosen coordinates, is given by

$$P_3 := \begin{pmatrix} \mathbf{0}_2 & 0 & 0 \\ 0 & \mathbf{1}_3 & 0 \\ 0 & 0 & \mathbf{0}_2 \end{pmatrix}.$$

Analogously, the irreducible representation χ_3^1 has dimension $m_C = 1$ but enters two times ($d_C = 2$) the representation of Q_8. Dually, there is only one ($m_C = 1$) isomorphic representation of $\mathtt{End}_{Q_8}(\tilde{V})$ which has dimension $d_C = 2$. There is only one GYS corresponding to the representation χ_3^1, which, in the chosen coordinates, is given by

$$P_4 := \begin{pmatrix} \mathbf{0}_5 & 0 \\ 0 & \mathbf{1}_2 \end{pmatrix}.$$

In the above example, we assumed that the representation of the group is already given in the 'natural' basis from which the expression of the GYS's was immediately deduced. Our goal was to illustrate the duality between the representation of the group G and the representation of $\mathtt{End}_G(\tilde{V})$. In practice, one is given a representation of G, and therefore of $\mathbb{C}[G]$. From the knowledge of the GYS's and from their images under the given representation, one obtains the change of coordinates which transforms the dynamics in the desired form.

We now present a simple example of application to a quantum spin network with symmetries. More examples of applications to this type of setting will be given in Section 5. We recall the definition of the Pauli matrices $\sigma_{x,y,z}$ which will also be used in Section 5.

$$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y := \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \qquad \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{18}$$

Example 3.5. In recent years there has been a large interest for the controllability of *central spin networks* (see, e.g., [5], [32]), i.e., networks of spin $\frac{1}{2}$ particles where one (central) spin of a given type is connected in various ways to spins of a different type, which may represent a bath. The control may be local, on the central spin, or global on

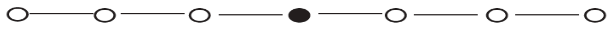


Figure 1: Example of a symmetric spin network with a central spin

all the spins. One possible topology of the network, which we consider here, is a linear chain with the central spin in the middle and connected with two strings of (bath) spins, of the same length. All spins are interacting with each other via next neighbor interaction which we assume of the Ising type. Figure 1 describes the configuration of such a spin network:

Denote by σ_k^j for $k = x, y, z$ the tensor product of $2n$ identities, with positions numbered from $-n$ to n, and with only the j-th position occupied by σ_k, so that, for example, for $n = 2$, $\sigma_x^1 = \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \sigma_x \otimes \mathbf{1}$. The Hamiltonians describing the dynamics of such a system, i.e., A and B_j's in (1), are

$$iA = \sum_{j=0}^{n-1} \sigma_z^j \sigma_z^{j+1} + \sum_{j=0}^{-n+1} \sigma_z^j \sigma_z^{j-1}, \qquad iB_{x,y,z} = i\sigma_{x,y,z}^0, \tag{19}$$

with controls $u_{x,y,z}$ representing local x, y, z-components of electromagnetic fields acting on the central spin only.

For every n, such a system presents a *reflection* symmetry $\hat{R}$ since the transformation $j \leftrightarrow -j$ does not modify the Hamiltonians in (19). Together with the identity, $\mathbf{1}$, $\hat{R}$ forms a group of symmetries for the system (1),(19). The two operators $P_S := \frac{1}{2}(\mathbf{1} + \hat{R})$, $P_A := \frac{1}{2}(\mathbf{1} - \hat{R})$ form a complete sets of GYS's for this group of symmetries. $P_S\tilde{V}$ ($P_A\tilde{V}$) gives all the states which are symmetric (antisymmetric) with respect to the group $\{\mathbf{1}, \hat{R}\}$. In this basis the Hamiltonians in (19) are written in block diagonal form.

4 Determination of the GYS's

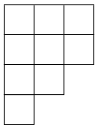
The above method assumes that we are able to obtain, for a given group of symmetries G, the corresponding (Hermitian) GYS's in the associated group algebra $\mathbb{C}[G]$, without knowing the irreducible modules of $\mathbb{C}[G]$ in advance. To the best of our knowledge, there is no general method to achieve this and it has to be done on a case by case basis. After one finds the GYS's, their image in the given representation of G applied to $\tilde{V}$ gives the desired change of coordinates which puts the dynamics in block diagonal form.

We now discuss two cases where it is possible to find the GYS's. In both cases, we assume that the space $\tilde{V}$ is the tensor product of a number n of identical vector spaces V, i.e., $\tilde{V} = V^{\otimes n}$ and G is a subgroup of the symmetric group S_n, which permutes the various factors in $V^{\otimes}$. The representation of G is unitary in these cases. The situations we shall treat are when G is the *full* symmetric group, $G = S_n$, and when G is *Abelian*.

4.1 GYS's for the symmetric group $G := S_n$

The construction of the GYS is classical in the case where $G = S_n$ (see, e.g., [14], [27]) and we survey here the theory. We shall apply it to a system in quantum control in the following section.

Conjugacy classes within S_n are determined by the *cycle type* of a permutation, i.e., the number of cycles of a certain length. For example for $n = 9$, the permutation $(123)(546)(78)(9)$ has cycle type: 2 for cycles of length 3, 1 for length 2 and 1 for length 1. Cycle types also correspond to *partitions* of n, i.e., sets of positive integer numbers $\lambda := \{\lambda_1, ..., \lambda_k\}$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1$, and $\lambda_1 + \lambda_2 + \cdots \lambda_k = n$. For example, the cycle type of $(123)(546)(78)(9)$ corresponds to the partition of $n = 9$, $(3, 3, 2, 1)$ meaning that the permutations (in the given conjugacy class) have a cycle of length 3 another cycle of length 3, a cycle of length 2 and a cycle of length 1. Partitions are encoded by *Young diagrams* which are diagrams composed of boxes in rows of non-decreasing lengths corresponding to the numbers in the partitions. For example, the partition of 9, $(3, 3, 2, 1)$ is encoded in the Young diagram



.

As we have recalled in Example 3.4, it is a known fact in the theory of representations of finite groups that the number of non-isomorphic irreducible representations of a finite group G in the regular representation is equal to the number of conjugacy classes in G. Therefore, in the case of the symmetric group, S_n, the number of irreducible representations is equal to the number of Young diagrams. In fact, there is a stronger correspondence between Young diagrams and irreducible sub-representations of the regular representation. If λ is a partition of n, a *standard Young tableaux of shape* λ is obtained from the corresponding Young diagram by distributing the numbers $1, 2, \ldots, n$ over the boxes in such a way that each row and column contains a strictly increasing sequence. For example,

$$T := \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 6 & 7 \\ \hline 4 & 8 \\ \cline{1-2} 9 \\ \cline{1-1} \end{array} \tag{20}$$

is a standard Young tableaux of shape $\lambda := (3, 3, 2, 1)$. The set of all standard Young tableaux of shape λ is denoted by $\mathrm{SYT}(\lambda)$. Then there is a correspondence between irreducible sub-representations of the regular representation, corresponding to the partition λ (which are all isomorphic), and elements in $\mathrm{SYT}(\lambda)$. Each representation is given by $\mathbb{C}[G]P_T$ where P_T is the GYS associated to the tableaux T in $\mathrm{SYT}(\lambda)$. The GYS P_T corresponding to a standard Young tableaux T in $\mathrm{SYT}(\lambda)$ is obtained as follows: Let R_T be the subgroup of S_n consisting of all permutations Π which preserve the rows of T. Similarly, let C_T be the subgroup of S_n of all permutations preserving the columns of T. For example:

$$T := \begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 7 \\ \hline 3 & 6 \\ \cline{1-2} 4 & 9 \\ \cline{1-2} 8 \\ \cline{1-1} \end{array} \qquad R_T = S_{\{1,2,5,7\}} \times S_{\{3,6\}} \times S_{\{4,9\}} \quad , \qquad C_T = S_{\{1,3,4,8\}} \times S_{\{2,6,9\}},$$

where we omitted the singleton symmetric groups such as $S_{\{5\}}$ because they are the trivial group. Here, for instance, $S_{\{1,2,5,7\}}$ is the subgroup of permutations over the elements $\{1, 2, 5, 7\}$. The *row symmetrizer r_T and column anti-symmetrizer c_T* are elements of $\mathbb{C}[S_n]$ defined as follows:

$$r_T = \sum_{\sigma \in R_T} \sigma, \qquad c_T = \sum_{\sigma \in C_T} (\mathtt{sgn}\,(\sigma))\sigma \tag{21}$$

The Young symmetrizer associated with T, $P_T^{'}$, is defined as

$$P_T^{'} := r_T \cdot c_T \qquad .$$

Let us consider, for example, $n = 3$ and the Standard Young Tableaux

$$T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}.$$

Then $R_T = S_{\{1,2\}}$ and $C_T = S_{\{1,3\}}$ and

$$r_T = \mathbf{1} + (12), \qquad c_T = \mathbf{1} - (13),$$

$$P_T^{'} := r_T \cdot c_T = (\mathbf{1} + (12))(\mathbf{1} - (13)) = \mathbf{1} - (13) + (12) - (12)(13) = \mathbf{1} - (13) + (12) - (132).$$

Young symmetrizers defined this way satisfy, after being divided by a normalization factor, the completeness property (4) and the primitivity property (6). Therefore they give irreducible sub-representations of the regular representation. They satisfy the orthogonality property (5), in general, only for small values of n ($n \leq 4$). The recent paper [18], motivated by applications in quantum chromodynamics, shows how to modify the procedure above so that the resulting Young symmetrizers also satisfy properties (5) and (8). These recent results make the treatment in the present paper possible since we need properties (5) and (8). In particular, property (8) guarantees that the in the block diagonal decomposition of $u(2^n)^G$, every block is also skew-Hermitian. The procedure of [18] has been then modified in [2] to make it significantly more efficient, in particular for large values of n. For our purposes however it is enough to use the original recursive algorithm of [18]. We shall call the modified Hermitian Young Symmetrizers of [18] the *KS-Young symmetrizers* (from the last names of the authors of [18]). Given a Young Tableaux T corresponding to a partition of n, let $\texttt{Pre}(T)$ be the Young tableaux obtained from T by removing the box containing the highest number and therefore corresponding to a partition of $n-1$. For example, for the tableau T in (20),

$$\texttt{Pre}(T) := \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 6 & 7 \\ \hline 4 & 8 & \\ \cline{1-2} \end{array}. \tag{22}$$

The KS-Young symmetrizer P_T associated with a tableaux T coincides with the standard Young symmetrizer $P_T^{'}$, if $n \leq 2$. If $n > 2$, it is obtained recursively as

$$P_T = (P_{\texttt{Pre}(T)} \otimes \mathbf{1}) P_T^{'} (P_{\texttt{Pre}(T)} \otimes \mathbf{1}). \tag{23}$$

It is proved in [18] that this definition satisfies the requirements (4),(5), (6) and (8).

More information can be obtained from the Young tableau T even without calculating the corresponding KS-Young symmetrizer P_T. For instance, the dimension of $\texttt{Im}(P_T)$ is equal to (cf. Lemma 3 in [18])

$$\dim(\texttt{Im}P_T) = \frac{\prod_{l=1}^{r} \prod_{k=1}^{\lambda_l} (N - l + k)}{\texttt{Hook}(T)}. \tag{24}$$

Here $N = \dim(V)$, r is the number of rows of the Young tableaux, λ_l the number of boxes in the l-th row, $\texttt{Hook}(T)$ is the *Hook length* of the Young diagram associated with T. It is calculated by considering, for each box of the Young diagram, the number of boxes directly to the right + the number of boxes directly below + 1 and then taking the product of all the numbers obtained. For example the Hook length of the Young tableau in (20) is 2160. It follows from formula (24) that if the number of rows of the tableaux is greater than the dimension N of the vector space V, then $\dim(\texttt{Im}(P_T)) = 0$.

4.2 GYS's for finite Abelian groups

Let G be a finite Abelian group. It follows from Schur's Lemma that every irreducible representation is one dimensional.[2] In the following, we shall use some concepts concerning the *character* χ of a representation ρ (cf., e.g., Lecture 2 in [14]). This is a function $G \rightarrow \mathbb{C}$ defined as $\chi(g) = \texttt{Tr}\rho(g)$, for $g \in G$. Various properties of characters of representations can be found in the representation theory texts we have cited. One property that we will use, and that directly follows from the definition, is that the character of the direct sum of two representations is the sum of the characters (cf. Proposition 2.1 in [14]). Characters corresponding to irreducible (and therefore one dimensional) representations are called *irreducible characters*. There is a one to one correspondence between irreducible characters and irreducible representations. Every irreducible character is a group homomorphism $\chi : G \rightarrow \mathbb{C}^{\times}$, whose image is contained in the unit circle S^1, in $\mathbb{C}^{\times}$, the complex plane without the origin. Recall that from formula (3) (with $\dim(\mathcal{C}_j) = 1$) there are $|G|$ different irreducible representations in the regular representation and therefore $|G|$ different characters. To each such character χ we associate an element P_χ of the group algebra $\mathbb{C}[G]$ as follows:

[2] Since any element of the representation acts as a multiple of the identity, irreducibility can only occur in dimension 1.

$$P_\chi := \frac{1}{|G|}\sum_{g\in G}\chi(g)g. \tag{25}$$

Proposition 4.1. The set $\{P_\chi\}$ where χ ranges over the set of all possible irreducible characters, in the regular representation, forms a complete set of Hermitian GYS for the group G, i.e., it satisfies properties (4)-(6) and (8).

Proof. Consider the following calculation.

$$\begin{aligned}
P_\chi P_{\chi'} &= \frac{1}{|G|^2}\sum_{g,g'\in G}\chi(g)\chi'(g')gg' \\
&= \frac{1}{|G|^2}\sum_{h\in G}\Big(\sum_{\substack{g,g'\in G\\ gg'=h}}\chi(g)\chi'(g')\Big)h \\
&= \frac{1}{|G|}\sum_{h\in G}\Big(\frac{1}{|G|}\sum_{g\in G}\chi(g)\chi'(g^{-1})\Big)\chi'(h)h \\
&= \frac{1}{|G|}\delta_{\chi\chi'}\sum_{h\in G}\chi'(h)h \\
&= \delta_{\chi\chi'}P_{\chi'},
\end{aligned}$$

where we used the character orthogonality condition $\frac{1}{|G|}\sum_{g\in G}\chi(g)\chi'(g^{-1}) = \delta_{\chi\chi'}$ (cf. formula (2.10) in [14]), with the property $\bar{\chi}(g) = \chi(g^{-1})$. This gives (5).

To see that P_χ is Hermitian, we calculate

$$P_\chi^\dagger = \sum_{g\in G}\overline{\chi(g)}g^{-1} = \sum_{g\in G}\chi(g^{-1})g^{-1} = \sum_{h\in G}\chi(h)h = P_\chi.$$

In the last equality, we used the substitution $h = g^{-1}$.

Next, we have

$$\sum_\chi P_\chi = \frac{1}{|G|}\sum_\chi\sum_{g\in G}\chi(g)g = \frac{1}{|G|}\sum_{g\in G}\Big(\sum_\chi\chi(g)\Big)g \tag{26}$$

The function $\sum_\chi \chi$, as a function of g, is the character of the regular representation (being the sum of all its irreducible characters). The matrix associated with g (as a linear transformation on $\mathbb{C}[G]$) is a permutation matrix which transforms the basis $\{h \,|\, h\in G\}$ to $\{gh \,|\, h\in G\}$. Such a permutation has trace zero for any $g\in G$, except when g is the identity. In that case $\sum_\chi \chi(g) = |G|$, and the right hand side of (26) is equal to the identity.

Lastly, we need to show that $P_\chi g P_\chi = \lambda_g P_\chi$, for some λ_g depending on g, i.e., property (6). In fact, we have, since the group G is Abelian,

$$P_\chi g P_\chi = P_\chi P_\chi g = P_\chi g = \frac{1}{|G|}\sum_{h\in G}\chi(h)hg = \frac{1}{|G|}\sum_{m\in G}\chi(mg^{-1})m =$$

$$\frac{1}{|G|}\sum_{m\in G}\chi(m)\chi(g^{-1})m = \chi(g^{-1})\left(\frac{1}{|G|}\sum_{m\in G}\chi(m)m\right) = \chi(g^{-1})P_\chi,$$

as desired. □

5 Application to spin networks subject to symmetries

We now apply the above described method to the analysis of the dynamics of two examples concerning networks of spins.

5.1 Completely symmetric spin networks

Consider a network of n *identical* spin $\frac{1}{2}$ particles under the control action of a common magnetic field and exhibiting identical *Ising* interaction with each other [7].[3] We denote by $|0\rangle$ and $|1\rangle$ the states of the spin $\frac{1}{2}$ particle, i.e., the two possible eigenstates when measuring the spin in a given direction (e.g., the z-direction). Since every spin interacts with every other spin in the same way, we call such networks *completely symmetric*. The state space is $V^{\otimes n}$ where $V = \mathbb{C}^2$ with the standard inner product $\langle\phi|\psi\rangle := \phi^*\psi$. Schrödinger equation for the dynamics is given by (2) with $A = -iH_{zz}$ and $\sum B_j u_j := -iH_x u_x - iH_y u_y$, where the *quantum mechanical Hamiltonians*, H_{zz}, H_x and H_y, acting on $V^{\otimes n}$, are given by

$$H_x = \sum \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \sigma_x \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}, \tag{27}$$

$$H_y = \sum \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \sigma_y \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}, \tag{28}$$

$$H_{zz} = \sum \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \sigma_z \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \sigma_z \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}, \tag{29}$$

u_x and u_y represent x and y components of the external (semi-classical) control magnetic field and $\sigma_{x,y,z}$ are the Pauli matrices defined in (18). In (27), (28) the sum is taken over all the spins, which are assumed identical, while in (29) it is taken over all the $\binom{n}{2}$ pairs of spins. The group of all permutations on n objects, i.e., the symmetric group S_n, acts as a group of symmetries for this system by permuting the factors in the tensor products:

$$\Pi(v_1 \otimes \cdots \otimes v_n) = v_{\Pi(1)} \otimes \cdots \otimes v_{\Pi(n)}, \qquad \forall \Pi \in S_n. \tag{30}$$

Let $u^{S_n}(2^n) := \big(u(2^n)\big)^{S_n}$. The three Hamiltonians (27), (28), (29) commute with the action of the symmetric group S_n. Therefore the dynamical Lie algebra $\mathcal{L}$ is a subalgebra of $u^{S_n}(2^n)$. The dimension of $u^{S_n}(2^n)$ was calculated in [3] to be $\binom{n+3}{n}$. In fact, it was shown in [3], that the dynamical Lie algebra $\mathcal{L}$ in this case is *exactly equal* to $u^{S_n}(2^n) \cap su(2^n)$, i.e., $u^{S_n}(2^n)$ with the restriction that the trace is equal to 0.

Models of this type often represent crystals of identical equidistant particles. The fact that the particles have the same distance from each other implies that they have the same interaction with each other.

The GYS's and the associated change of coordinates can be calculated with the method of Young tableaux described in Subsection 4.1. Here we calculate the explicit change of coordinates for the case $n = 4$. This case is not only the simplest case that was not treated in [3] but also the highest dimension physically relevant when we consider spin networks, since symmetry often requires that the spins are equidistant. Therefore in 3−dimensional space there are at most 4 of them. In the following we denote by $S_{a_1,a_2,...,a_r}$ the symmetrizer of positions $a_1, a_2, ..., a_r$ and by $A_{a_1,a_2,...,a_r}$ the anti-symmetrizer of positions $a_1, a_2, ..., a_r$, i.e., (cf. (21))

$$S_{a_1,a_2,...,a_r} := \sum_{\sigma \in S_{\{a_1,a_2,...,a_r\}}} \sigma, \qquad A_{a_1,a_2,...,a_r} := \sum_{\sigma \in S_{\{a_1,a_2,...,a_r\}}} \mathtt{sgn}(\sigma)\sigma, \tag{31}$$

where $S_{\{a_1,a_2,...,a_r\}}$ is the permutation group of the symbols $\{a_1, a_2, ..., a_r\}$. We also denote by V_j, $j = 0, 1, 2, 3, 4$, the subspaces of $V^{\otimes 4}$ spanned by states with j, 1's, so that, for instance, $V_0 = \mathtt{span}\{|0000\rangle\}$.

5.1.1 Young diagram corresponding to the partition (4)

There is only one Standard Young Tableaux (SYT) corresponding to such a partition given by

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline \end{array} \quad .$$

The corresponding KS-Young Symmetrizer $P_{\boxed{1}\boxed{2}\boxed{3}\boxed{4}}$ coincides with the standard Young symmetrizer $P'_{\boxed{1}\boxed{2}\boxed{3}\boxed{4}}$ (this can be shown by induction to be true for every KS-symmetrizer corresponding to partition (n) for every n). The image of $P_{\boxed{1}\boxed{2}\boxed{3}\boxed{4}}$ is spanned by the symmetric orthogonal states (for simplicity we omit the normalization factor).

$$\varphi_0 = |0000\rangle, \tag{32}$$

$$\varphi_1 = |1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle, \tag{33}$$

$$\varphi_2 = |1100\rangle + |0110\rangle + |0011\rangle + |1001\rangle + |0101\rangle + |1010\rangle, \tag{34}$$

$$\varphi_3 = |0111\rangle + |1011\rangle + |1101\rangle + |1110\rangle, \tag{35}$$

$$\varphi_4 = |1111\rangle. \tag{36}$$

[3] See also [12] and [13] for interesting quantum states possibly generated by these systems.

5.1.2 Young diagram corresponding to the partition $(3, 1)$

There are three SYT's corresponding to a partition $(3, 1)$. They are:

$$\ytableaushort{123,4}\,, \qquad \ytableaushort{124,3}\,, \qquad \ytableaushort{134,2}\,.$$

Using the recursive method of [18] described in Subsection 4.1 we compute the KS-Young symmetrizers and the corresponding bases. All the Young symmetrizers described below need to be multiplied by a normalizing constant which ensures that they are projections, i.e., $P^2 = P$ according to property (5). This constant is irrelevant for our purposes as we are mostly interested in the images of such operators. We therefore omit it.

- For $P_{\ytableaushort{123,4}}$, we get,

  $$P_{\ytableaushort{123,4}} = P_{\ytableaushort{123}} P'_{\ytableaushort{123,4}} P_{\ytableaushort{123}} = P'_{\ytableaushort{123}} P'_{\ytableaushort{123,4}} P'_{\ytableaushort{123}} = S_{1,2,3}A_{1,4}S_{1,2,3},$$

  which applied to V_0 and V_4 gives zero, while applied to $V_{1,2,3}$ gives the span of $\psi_{1,2,3}$ with

  $$\psi_1 = |1000\rangle + |0100\rangle + |0010\rangle - 3|0001\rangle$$

  $$\psi_2 = |1100\rangle + |1010\rangle + |0110\rangle - |1001\rangle - |0101\rangle - |0011\rangle$$

  $$\psi_3 = |0111\rangle + |1011\rangle + |1101\rangle - 3|1110\rangle.$$

  Notice that ψ_3 is obtained from ψ_1 by exchanging the 1's with the 0's.

- For $P_{\ytableaushort{124,3}}$, we get

  $$P_{\ytableaushort{124,3}} = P_{\ytableaushort{12,3}} P'_{\ytableaushort{124,3}} P_{\ytableaushort{12,3}} = P_{\ytableaushort{12}} P'_{\ytableaushort{12,3}} P_{\ytableaushort{12}} P'_{\ytableaushort{124,3}} P_{\ytableaushort{12}} P'_{\ytableaushort{12,3}} P_{\ytableaushort{12}} =$$
  $$S_{1,2}A_{1,3}S_{1,2}S_{1,2,4}A_{1,3}S_{1,2}A_{1,3}S_{1,2} = S_{1,2}A_{1,3}S_{1,2,4}A_{1,3}S_{1,2}A_{1,3}S_{1,2}$$

  which applied to V_0 and V_4 gives zero, while applied to $V_{1,2,3}$ gives the span of $\chi_{1,2,3}$ with

  $$\chi_1 = |1000\rangle + |0100\rangle - 2|0010\rangle$$

  $$\chi_2 = 2|1100\rangle - 2|0011\rangle + |1001\rangle + |0101\rangle - |0110\rangle - |1010\rangle$$

  $$\chi_3 = |0111\rangle + |1011\rangle - 2|1101\rangle.$$

- For $P_{\ytableaushort{134,2}}$, we get

  $$P_{\ytableaushort{134,2}} = P_{\ytableaushort{13,2}} P'_{\ytableaushort{134,2}} P_{\ytableaushort{13,2}} = P'_{\ytableaushort{1,2}} P'_{\ytableaushort{13,2}} P'_{\ytableaushort{1,2}} P'_{\ytableaushort{134,2}} P'_{\ytableaushort{1,2}} P'_{\ytableaushort{13,2}} P'_{\ytableaushort{1,2}} =$$
  $$A_{1,2}S_{1,3}A_{1,2}A_{1,2}S_{1,3,4}A_{1,2}A_{1,2}S_{1,3}A_{1,2}A_{1,2} = A_{1,2}S_{1,3}A_{1,2}S_{1,3,4}A_{1,2}S_{1,3}A_{1,2},$$

  which applied to V_0 and V_4 gives zero, while applied to $V_{1,2,3}$ gives the span of $\eta_{1,2,3}$ with

  $$\eta_1 = |1000\rangle - |0100\rangle,$$

  $$\eta_2 = |1010\rangle + |1001\rangle - |0110\rangle - |0101\rangle.$$

  $$\eta_3 = |0111\rangle - |1011\rangle.$$

5.1.3 Young diagram corresponding to the partition $(2, 2)$

There are two SYT's corresponding to a partition $(2, 2)$. They are

$$\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}, \qquad \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array}.$$

Using the algorithm in [18], we compute the KS-Young symmetrizers and the corresponding bases.

- For $P_{\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}}$, we get

$$P_{\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}} = P_{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}} P^{'}_{\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}} P_{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}} = P^{'}_{\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}} P_{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}} P^{'}_{\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}} P^{'}_{\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}} P^{'}_{\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}} P_{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}} P^{'}_{\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}} =$$

$$S_{1,2}A_{1,3}S_{1,2}S_{3,4}A_{1,3}A_{2,4}S_{1,2}A_{1,3}S_{1,2}.$$

which applied to $V_{0,1,3,4}$ gives zero, while applied to V_2 gives the span of

$$\mu_2 = 2|1100\rangle + 2|0011\rangle - |0110\rangle - |1010\rangle - |1001\rangle - |0101\rangle$$

- For $P_{\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array}}$, we get

$$P_{\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array}} = P_{\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}} P^{'}_{\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array}} P_{\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}} = P^{'}_{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}} P^{'}_{\begin{array}{|c|c|}\hline 1&3\\\hline\end{array}} P^{'}_{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}} P^{'}_{\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array}} P^{'}_{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}} P^{'}_{\begin{array}{|c|c|}\hline 1&3\\\hline\end{array}} P^{'}_{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}} =$$

$$A_{1,2}S_{1,3}A_{1,2}S_{1,3}S_{2,4}A_{1,2}A_{3,4}A_{1,2}S_{1,3}A_{1,2} = A_{1,2}S_{1,3}A_{1,2}S_{1,3}S_{2,4}A_{3,4}A_{1,2}S_{1,3}A_{1,2},$$

which applied to $V_{0,1,3,4}$ gives zero, while applied to V_2 gives the span of

$$\nu_2 = |1010\rangle + |0101\rangle - |0110\rangle - |1010\rangle.$$

5.1.4 Structure of the dynamical Lie algebra $\mathcal{L}$

According to the theory developed in this paper, the above change of coordinates transforms the matrices in $u^{S_4}(2^4)$ into a block diagonal form with one copy of $u(5)$ acting on $\mathtt{span}\{\varphi_0, \varphi_1, \varphi_2, ..., \varphi_4\}$, the so called *symmetric states*, three copies of $u(3)$ acting respectively on $\mathtt{span}\{\psi_1, \psi_2, \psi_3\}$, $\mathtt{span}\{\chi_1, \chi_2, \chi_3\}$, or $\mathtt{span}\{\eta_1, \eta_2, \eta_3\}$ and two copies of $u(1)$ acting, respectively, on $\mathtt{span}\{\mu_2\}$ or $\mathtt{span}\{\nu_2\}$. Therefore, in the given coordinates, matrices in $\mathcal{L} = u^{S_n}(2^4) \cap su^{S_n}(2^4)$ (recall that from the results of [3] the dynamical Lie algebra $\mathcal{L}$ is *equal* to $u^{S_n}(2^n)$ except for the requirement that the matrices have zero trace) have the form (cf. (12))

$$\begin{pmatrix} A_{5\times5} & 0 & 0 & 0 & 0 & 0 \\ 0 & B_{3\times3} & 0 & 0 & 0 & 0 \\ 0 & 0 & B_{3\times3} & 0 & 0 & 0 \\ 0 & 0 & 0 & B_{3\times3} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{1\times1} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{1\times1} \end{pmatrix}$$

where $A_{5\times5}$ is an arbitrary matrix in $u(5)$, $B_{3\times3}$ is an arbitrary matrix in $u(3)$ and $C_{1\times1}$ is an arbitrary number in $u(1)$ (i.e., a purely imaginary number), with $Tr(A_{5\times5}) + 3Tr(B_{3\times3}) + 2C_{1\times1} = 0$. The system is state controllable on each of the invariant subspaces, that is, it is subspace controllable. We may calculate the matrices of the restrictions of $-iH_{zz}$, $-iH_x$ and $-iH_y$ to the various invariant subspaces and consider control theoretic problems in each subspace.

5.2 Circularly symmetric spin networks

Consider a circular network of identical spin $\frac{1}{2}$ particles interacting via Ising *z-z* interaction but with *nearest neighbor interaction* only. The Hamiltonians modeling the interaction with the external magnetic (control) field in the x and y direction are again given by (27) and (28). However the Hamiltonian modeling the interaction between the particles, H_{zz} in (29), has to be replaced by

$$H_{zz}^{NN} = \sigma_z \otimes \sigma_z \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} + \mathbf{1} \otimes \sigma_z \otimes \sigma_z \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} + \cdots + \mathbf{1} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \sigma_z \otimes \sigma_z + \sigma_z \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \sigma_z. \quad (37)$$

The relevant group of symmetries here is the Abelian subgroup C_n of S_n, defined as the group generated by the circular shift $\{1, 2, ..., n\} \to \{n, 1, 2, ..., n-1\}$,[4] i.e., the permutation $Z := (123 \cdots n)$, with $Z^n = \mathbf{1}$. The dynamical Lie algebra $\mathcal{L}$ is a subalgebra of $u^{C_n}(2^n) := (u(2^n))^{C_n}$. The dimension of $u^{C_n}(2^n)$ is derived in Appendix A and it is given by

$$\dim u^{C_n}(2^n) = \frac{1}{n} \sum_{m|n} 4^{\frac{n}{m}} \phi(m), \tag{38}$$

where $\sum_{m|n}$ means we sum over all positive integers m which divide n, and $\phi(m)$ is the *Euler's totient function* (see, e.g., [1]) defined as $\phi(1) = 1$ and $\phi(m)$ equal to the number of positive integers less than m which are relatively prime to m, if $m > 1$. It is interesting to note that, contrary to what happens in the example of the previous subsection, the dynamical Lie algebra $\mathcal{L}$ in this case may be a *proper* Lie subalgebra of $u^{C_n}(2^n)$ (modulo the requirement of zero trace). Consider, for instance, the case $n = 3$. From formula (38) since $\phi(1) = 1$ and $\phi(3) = 2$, we have

$$\dim u^{C_3}(2^3) = \frac{1}{3}\left(4^3 \times 1 + 4^1 \times 2\right) = 24.$$

Therefore $u^{C_n}(2^n)$ is larger than $u^{S_n}(2^n)$, since the latter has dimension $\binom{n+3}{n} = 20$. On the other hand, for $n = 3$, the dynamical Lie algebra generated by iH_{zz}^{NN} in (37) and iH_x and iH_y in (27), (28) is the same as the one generated by (27), (28) and (29) since the Hamiltonian H_{zz}^{NN} in (37) coincides with the Hamiltonian H_{zz} in (29) in this case. So the dynamical Lie algebra is $\mathcal{L} = u^{S_n}(2^n) \cap su(2^n)$ in this case because of the result of [3]. This has dimension 19 while $u^{C_n}(2^n) \cap su(2^n)$ has dimension 23.

Since C_n is an Abelian group, every finite-dimensional irreducible representation is 1-dimensional. There are exactly n not equivalent such representations (in the regular representation) which we denote by: $\rho_0, \rho_1, \ldots, \rho_{n-1}$. They are given by

$$\rho_k : C_n \to GL(1, \mathbb{C}) = \mathbb{C}^{\times} \tag{39}$$

$$\rho_k(Z^j) = \varepsilon^{kj} \tag{40}$$

where $\varepsilon := \varepsilon_n := e^{2\pi i/n}$ is the n-th root of the identity.[5] The character associated to the representation ρ_k, $k = 0, 1, 2, ..., n-1$ is $\chi_k(Z^j) := \mathtt{Tr}\rho_k(Z^j) = \varepsilon^{kj}$. Using Proposition 4.1, a complete set of Hermitian GYS's is then given by the following n Fourier sums in the group algebra $\mathbb{C}[C_n]$:

$$P_k = \frac{1}{n} \sum_{j=0}^{n-1} \chi_k(Z^j) Z^j = \frac{1}{n} \sum_{j=0}^{n-1} \varepsilon^{kj} Z^j, \qquad k = 0, 1, \ldots, n-1. \tag{41}$$

5.2.1 States and decomposition of the dynamical Lie algebra

We now want to decompose the Lie algebra $u^{C_n}(2^n)$, which has dimension given in formula (38), using the GYS's (41). From this, we deduce the decomposition of the dynamical Lie algebra $\mathcal{L}$ for the system of n interacting spin with circular symmetry.

Let $V = \mathbb{C}^2$ modeling the state of spin $\frac{1}{2}$ systems. States in a basis of $V^{\otimes n}$ are labeled by binary words $\underline{a} = a_1 a_2 \ldots a_n \in \{0, 1\}^n$ as follows:

$$|\underline{a}\rangle = \vec{a}_1 \otimes \vec{a}_2 \otimes \cdots \otimes \vec{a}_n, \tag{42}$$

where $\vec{0} = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$, $\vec{1} = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$. According to the method of this paper, we need to describe $\mathtt{Im}(P_k)$, for a complete set of GYS's $\{P_k\}$. We notice that the space $V_T^{\otimes n}$ defined as the span of states $|\underline{a}\rangle$ with $\underline{a}$ a word of period T (necessarily) dividing n, is invariant under C_n and therefore it is invariant under the action of any element of the group algebra $\mathbb{C}[C_n]$ including the GYS's $\{P_k\}$. The period T is the smallest positive integer such that $Z^T(\underline{a}) = \underline{a}$. We have

$$(P_k V^{\otimes n}) = P_k \bigoplus_{T|n} V_T^{\otimes n} = \bigoplus_{T|n} (P_k V_T^{\otimes n}). \tag{43}$$

[4] Here the sets are meant to be *ordered* sets.

[5] For a representation ρ, we can write $\rho(Z)$ in Jordan canonical form, in appropriate coordinates. From $\rho(Z)^n = \rho(\mathbf{1})^n = \mathbf{1}$ it follows that each Jordan block must be a multiple of the identity, $\lambda\mathbf{1}$ with λ an n-th root of the identity.

Consider a general vector $|\underline{a}\rangle$ in the standard basis of $V^{\otimes n}$ and belonging to $V_T^{\otimes n}$. With a GYS, P_k, defined in (41), we have

$$P_k\big(\,|\underline{a}\rangle\,\big) = \frac{1}{n}\sum_{j=0}^{n-1} \varepsilon^{kj}\cdot|a_{1+j}a_{2+j}\cdots a_{n+j}\rangle \tag{44}$$

where the indices of a_l are considered modulo n. Since the word $a_1a_2\cdots a_n$ is periodic of period T, that is, $a_{1+T}a_{2+T}\cdots a_{n+T} = a_1a_2\cdots a_n$. or $Z^T(\underline{a}) = \underline{a}$, in the right hand side of (44), we can divide the summation variable j by T to get

$$j = Tq + r, \qquad 0 \le r < T,\ 0 \le q < \frac{n}{T}. \tag{45}$$

Thus

$$P_k\big(\,|\underline{a}\rangle\,\big) = \frac{1}{n}\sum_{r=0}^{T-1}\Big(\sum_{q=0}^{\frac{n}{T}-1}\varepsilon^{kTq+kr}\Big)\cdot|a_{1+r}a_{2+r}\cdots a_{n+r}\rangle = \frac{1}{n}\sum_{r=0}^{T-1}\varepsilon^{kr}\Big(\sum_{q=0}^{\frac{n}{T}-1}\varepsilon^{kTq}\Big)\cdot|a_{1+r}a_{2+r}\cdots a_{n+r}\rangle\,. \tag{46}$$

The quantity in parenthesis can be computed as a geometric series to give

$$\sum_{q=0}^{\frac{n}{T}-1}\varepsilon^{kTq} = \begin{cases}\frac{n}{T}, & \text{if } \varepsilon^{kT}=1,\\ \frac{(\varepsilon^{kT})^{n/T}-1}{\varepsilon^{kT}-1} = 0, & \text{otherwise,}\end{cases} \tag{47}$$

because $\varepsilon^n = 1$, since by definition $\varepsilon := e^{i\frac{2\pi}{n}}$. Using this, we get

$$P_k\big(\,|\underline{a}\rangle\,\big) = \begin{cases}\frac{1}{T}\sum_{r=0}^{T-1}\varepsilon^{kr}\cdot|a_{1+r}a_{2+r}\cdots a_{n+r}\rangle\,, & \text{if } \varepsilon^{kT}=1,\\ 0, & \text{otherwise.}\end{cases} \tag{48}$$

Then $P_k\big(\,|\underline{a}\rangle\,\big)$ is non-zero if and only if $\varepsilon^{kT} = 1$, which happens if and only n/T divides k. Therefore $P_kV_T^{\otimes n}$ in (43) is nonzero only if n/T divides k.

Example 5.1. Consider $n = 4$, so that $V^{\otimes n}$ is 16-dimensional. In general the possible values of period (dividing $n = 4$) are $T = 1, T = 2$, and $T = 4$. Let us calculate $\mathtt{Im}(P_0)$. All $T = 1$, $T = 2$, and $T = 4$ are such that $n/T = 4/T$, divide $k = 0$. We have one state for each orbit of C_4, which gives 6 states $\frac{1}{4}\sum_{j=0}^{3}Z^j|0000\rangle$, $\frac{1}{4}\sum_{j=0}^{3}Z^j|1111\rangle$, $\frac{1}{4}\sum_{j=0}^{3}Z^j|1000\rangle$, $\frac{1}{4}\sum_{j=0}^{3}Z^j|1100\rangle$, $\frac{1}{4}\sum_{j=0}^{3}Z^j|1010\rangle$, and $\frac{1}{4}\sum_{j=0}^{3}Z^j|0111\rangle$, which span $\mathtt{Im}P_0$. For $k = 1$ the only possibility is $T = 4$, so that $n/T = 1$. We have the three states: $\frac{1}{4}\sum_{j=0}^{3}\epsilon^jZ^j|1000\rangle$, $\frac{1}{4}\sum_{j=0}^{3}\epsilon^jZ^j|0111\rangle$, $\frac{1}{4}\sum_{j=0}^{3}\epsilon^jZ^j|1100\rangle$. For $k = 3$ the only possibility is also $T = 4$, and we also have three states: $\frac{1}{4}\sum_{j=0}^{3}\epsilon^{3j}Z^j|1000\rangle$, $\frac{1}{4}\sum_{j=0}^{3}\epsilon^{3j}Z^j|0111\rangle$, and $\frac{1}{4}\sum_{j=0}^{3}\epsilon^{3j}Z^j|1100\rangle$. For $k = 2$ the possibilities are $T = 4$ and $T = 2$. For $T = 4$, we also have three states:$\frac{1}{4}\sum_{j=0}^{3}\epsilon^{2j}Z^j|1000\rangle$, $\frac{1}{4}\sum_{j=0}^{3}\epsilon^{2j}Z^j|0111\rangle$, and $\frac{1}{4}\sum_{j=0}^{3}\epsilon^{2j}Z^j|1100\rangle$. For $T = 2$, we have one state $\frac{1}{4}\sum_{j=0}^{3}\epsilon^{2j}Z^j|1010\rangle$. Therefore we have $\dim(\mathtt{Im}P_0) = 6$, $\dim(\mathtt{Im}P_1) = 3$, $\dim(\mathtt{Im}P_2) = 4$, $\dim(\mathtt{Im}P_3) = 3$, so that $u^{C_n}(2^4) = u(6)\oplus u(3)\oplus u(4)\oplus u(3)$, since all the irreducible representations associated to the GYS, P_k, are inequivalent. The dimension of $u^{C_n}(2^4)$ which is equal to $6^2 + 3^2 + 4^2 + 3^2 = 70$ can also be calculated using formula (38), which gives $\frac{1}{4}(4^4 + 4^2 + 2\times 4^2) = 70$.

Generalizing the previous example, we now want to calculate the dimension of $\mathtt{Im}(P_j)$, which we denote by $m_j := \dim(\mathtt{Im}(P_j))$, so that,

$$u^{C_n}(2^n) = u(m_0)\oplus u(m_1)\oplus\cdots\oplus u(m_{n-1}). \tag{49}$$

Consider the set X_k of binary words $\underline{a}$ of length n and with a period T such that n/T divides k. Since the cyclic group C_n preserves the period, X_k is invariant under C_n. The cyclic group C_n acts on X_k by cyclic permutations of the letters. Moreover, as we have seen above, P_k is non zero only on the vector subspace of $V^{\otimes n}$ spanned by the vectors corresponding to the words in X_k. Similarly to what done in Proposition 5.2 in the Appendix A, there is a one to one correspondence between the orbits of C_n in X_k and elements in a basis of $\mathtt{Im}(P_k)$ given, using (48), by

$$[(a_1a_2\cdots a_n)] \in X_k/C_n \leftrightarrow \frac{1}{T}\sum_{r=0}^{T-1}\varepsilon^{kr}\cdot|a_{1+r}a_{2+r}\cdots a_{n+r}\rangle\,, \tag{50}$$

which is independent of the representative chosen for $[(a_1 a_2 \cdot a_n)]$. In particular $m_k = \dim \mathtt{Im}(P_k) = |X_k/C_n|$. Using this, we obtain in Appendix A

$$m_k = \frac{1}{n} \sum_{m|\gcd(n,k)} \mathsf{w}(n,k,m) \cdot \phi(m). \tag{51}$$

Here $\mathsf{w}(n,k,m)$ is the number of binary words $\underline{a}$ of length n which have a period T such that m divides n/T and n/T divides k. Consider as an example $\mathsf{w}(6,4,2)$. Since $n=6$, possible values for the periods T are $T=1,2,3,6$. For $T=1$, $\frac{n}{T}=6$, which does not divide $k=4$. For $T=2$, $\frac{n}{T}=3$ but $m=2$ does not divide $\frac{n}{T}=3$. For $T=6$, $\frac{n}{T}=1$, but $m=2$ does not divide $\frac{n}{T}=1$. However for $T=3$, we have $\frac{n}{T}=2$. $m=2$ divides $\frac{n}{T}=2$ and $\frac{n}{T}=2$ divides $k=4$. We count the number of binary words of period 3 with 6 elements which are 6. Therefore $\mathsf{w}(6,4,2)=6$. In formula (51) again, as in formula (38), $\phi(m)$ denotes the Euler's totient function computed at m.

The following is a case where we are able to calculate the dynamical decomposition of $u^{C_n}(2^n)$ explicitly.

5.2.2 The case where n is a prime number

Suppose $n=p$ where p is a prime number. If $k=0$ there are two terms in the sum (51), the one corresponding to $m=1$ and the one corresponding to $m=p$. For $m=1$ we can take words of period $T=1$ and $T=p$ which represent all possible 2^p words. So we have a term $2^p\phi(1)=2^p$ in the sum. For $m=p$ we can only take words of period $T=1$, since words of period $T=p$ are such that $n/T=1$ and $m=p$ does not divide 1. There are only 2 such words $(000\cdots 0)$ and $(111\cdots 1)$. Thus we have a term $2\phi(p)=2(p-1)$ in the sum. Therefore, we have

$$m_0 = \frac{1}{p}\left(2^p + 2(p-1)\right) = 2 + \frac{(2^p-2)}{p}.$$

Notice that, for any integer a and prime number p, the quantity $a^p - a$ is divisible by p, by Fermat's Little Theorem (see, e.g., [22]). If $k>0$ then, independently of the value of k, the only possible period in the sum (51) is $T=p$ and the only possible value of m is $m=1$. Thus there is only one term in the sum corresponding to all words except the two of period $T=1$. We obtain

$$m_k = m_0 - 2 = \frac{1}{p}(2^p - 2), \qquad 1 \le k < p. \tag{52}$$

Consequently,

$$u^{C_p}(2^n) = \begin{bmatrix} u\big(2+(2^p-2)/p\big) & & & \\ & u\big((2^p-2)/p\big) & & \\ & & \ddots & \\ & & & u\big((2^p-2)/p\big) \end{bmatrix} \tag{53}$$

. The dimension is equal to

$$\dim u^{C_p}(2^n) = \frac{(2^p+2p-2)^2 + (p-1)(2^p-2)^2}{p^2} = 4 + (4^p - 4)/p \tag{54}$$

after simplification. This also agrees with the formula (38) for $n=p$, a prime number.

5.2.3 The dynamical Lie algebra for a circularly symmetric spin network

As we have discussed above, the dynamical Lie algebra $\mathcal{L}$ associated with a circularly symmetric network of spin $\frac{1}{2}$ particles may, in general, be a *proper* subalgebra of $u^{C_n}(2^n)$. Nevertheless the change of coordinates we have obtained in this section places $\mathcal{L}$ in a block diagonal form from which its structure is easier to understand. We illustrate this for the case $n=3$.

Since $n=3$ is a prime number, we can use the simplified formula (54) for $m_0 = \dim(\mathtt{Im}(P_0))$, $m_1 = \dim(\mathtt{Im}(P_1))$, $m_2 = \dim(\mathtt{Im}(P_2))$, and we get $m_0=4$, $m_1=2$, $m_2=2$. From (48) we obtain a formula for an orthogonal basis of $\texttt{Im}(P_0)$ which, after normalization, is given by

$$\begin{aligned}\varphi_0 &:= |000\rangle;\\ \varphi_1 &:= |111\rangle;\\ \varphi_2 := \frac{1}{\sqrt{3}}&(|100\rangle + |010\rangle + |001\rangle)\\ \varphi_3 := \frac{1}{\sqrt{3}}&(|011\rangle + |101\rangle + |110\rangle).\end{aligned} \tag{55}$$

We also obtain a formula for an orthonormal basis of $\texttt{Im}(P_1)$ ($\epsilon := e^{\frac{i2\pi}{3}}$)

$$\begin{aligned}\psi_1 &:= \frac{1}{\sqrt{3}}(|100\rangle + \epsilon|010\rangle + \epsilon^2|001\rangle)\\ \psi_2 &:= \frac{1}{\sqrt{3}}(|011\rangle + \epsilon|101\rangle + \epsilon^2|110\rangle),\end{aligned} \tag{56}$$

and a formula for an orthonormal basis of $\texttt{Im}(P_2)$,

$$\begin{aligned}\eta_1 &:= \frac{1}{\sqrt{3}}(|100\rangle + \epsilon^2|010\rangle + \epsilon|001\rangle)\\ \eta_2 &:= \frac{1}{\sqrt{3}}(|011\rangle + \epsilon^2|101\rangle + \epsilon|110\rangle).\end{aligned} \tag{57}$$

By calculating the action of $-iH_{zz}^{NN}$, $-iH_x$ and $-iH_y$ in (27), (28), (37) on the above basis, using the fact that $1+\epsilon+\epsilon^2 = 0$, we obtain, the expression of these operators in the new basis. This is, $-i\hat{H}_{zz}^{NN} = \texttt{diag}(-3i, -3i, i, i, i, i, i, i)$, and

$$-i\hat{H}_x := \left[\begin{array}{c|c|c} \begin{matrix} 0 & 0 & -i\sqrt{3} & 0\\ 0 & 0 & 0 & -i\sqrt{3}\\ -i\sqrt{3} & 0 & 0 & -2i\\ 0 & -\sqrt{3}i & -2i & 0 \end{matrix} & 0 & 0\\ \hline 0 & \begin{matrix} 0 & i\\ i & 0\end{matrix} & 0\\ \hline 0 & 0 & \begin{matrix} 0 & i\\ i & 0\end{matrix} \end{array}\right], \quad -i\hat{H}_y := \left[\begin{array}{c|c|c} \begin{matrix} 0 & 0 & \sqrt{3} & 0\\ 0 & 0 & 0 & -\sqrt{3}\\ -\sqrt{3} & 0 & 0 & 2\\ 0 & \sqrt{3} & -2 & 0 \end{matrix} & 0 & 0\\ \hline 0 & \begin{matrix} 0 & -1\\ 1 & 0\end{matrix} & 0\\ \hline 0 & 0 & \begin{matrix} 0 & -1\\ 1 & 0\end{matrix} \end{array}\right].$$

The upper left blocks generates any possible 4×4 skew-Hermitian block, while the 2×2 blocks are required to be equal, something which is not true for general matrices in $u^{C_3}(2^3)$ (cf. equation (53)). Therefore the dimension of the dynamical Lie algebra is $4^2 + 2^2 - 1 = 20 - 1$, where the -1 is due to the fact that the trace has to be equal to zero. In fact such a Lie algebra coincides with the one we would have obtained had we considered the full symmetric group S_3 as the symmetry group of the model. From this decomposition we can infer further properties concerning the *subspace controllability* of the system under consideration. We know that the subsystems identified by the vectors $\{\varphi_0, \varphi_1, \varphi_2, \varphi_3\}$, $\{\psi_1, \psi_2\}$, $\{\eta_1, \eta_2\}$, are all state controllable. Therefore we have controllability for any invariant subspace, i.e., subspace controllability.

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Appendix A: Proofs of Formula (38) and of Formula (51)

Consider a Lie algebra $\mathcal{R}$ which has a basis $\mathcal{B} := \{E_j\}$ which is invariant, as a set, under the action of the group G, i.e., if $E \in \mathcal{B}$, $gEg^{-1} \in \mathcal{B}$, $\forall g \in G$. Then we can derive a basis for $\mathcal{R}^G$: Let $\mathcal{O}$ the set of orbits of G in $\mathcal{B}$ under the above action.

Proposition 5.2. The set of elements

$$\{\sum_{E_j \in O} E_j \,|\, O \in \mathcal{O}\}, \tag{58}$$

is a basis of $\mathcal{R}^G$. In particular, the dimension of $\mathcal{R}^G$ is equal to the number of orbits in $\mathcal{B}$ under the above described action of G on $\mathcal{B}$.

Proof. Since the $E_j \in \mathcal{B}$ form a basis and the orbits are disjoint, then elements $\sum_{E_j \in O} E_j$ in (58) for different orbits O are linearly independent. Moreover write $F \in \mathcal{R}^G$ as $F = \sum_{O \in \mathcal{O}} F_O$ where $\mathcal{O}$ is the set of orbits and F_O is a linear combination of elements in $\mathcal{B}$ in the orbit O. Since $gFg^{-1} = F$ for each $g \in G$, and each orbit is invariant, we have

$$gFg^{-1} = \sum_{O \in \mathcal{O}} gF_Og^{-1} = F = \sum_{O \in \mathcal{O}} F_O,$$

which implies that, for every orbit O, and every $g \in G$, $gF_Og^{-1} = F_O$. Write $F_O = \sum_j \alpha_j E_j$ where E_j are the elements in the basis $\mathcal{B}$ which also belong to the orbit O, and for some coefficients α_j. Fix j and k and a $g \in G$ so that g maps E_j to E_k. Such a g always exists because, by definition, the action of G is transitive on its orbits. By imposing $gF_Og^{-1} = F_O$, using the fact that the map associated with g is a bijection from the orbit to itself, we find that $\alpha_j = \alpha_k$. Since, this is valid for arbitrary j and k, we find that F_O must be proportional to the elements $\sum_{E_j \in O} E_j$ in the set (58). □

According to the proposition, the dimension of $\mathcal{R}^G$ can be calculated using the *Burnside's orbit counting theorem* (see, e.g., [24]),

$$\#\texttt{orbits} = \frac{1}{|G|} \sum_{g \in G} |\texttt{Fix}^g|, \tag{59}$$

where $\texttt{Fix}^g$ denotes the set of elements fixed by g, in $\mathcal{B}$.

5.3 Proof of Formula (38)

Proof. By Proposition 5.2 we have

$$\dim u^{C_n}(2^n) = \#\text{orbits} \tag{60}$$

where #orbits is the number of orbits with respect to the action of C_n on the set of all words of length n in the four symbols $\mathbf{1}, \sigma_x, \sigma_y, \sigma_z$.

Recall Burnside counting theorem (59) which applied to our case gives:

$$\#\text{orbits} = \frac{1}{|C_n|} \sum_{g \in C_n} |\text{Fix}^g| \,. \tag{61}$$

The cyclic group[6] $C_n = \langle Z \rangle$ has a unique subgroup H_m of order m for every positive divisor m of n, namely $H_m = \langle Z^{n/m} \rangle$. Since every element g of C_n generates some subgroup, we can partition C_n into subsets corresponding to which subgroup they generate. Then we get

$$\#\text{orbits} = \frac{1}{n} \sum_{m|n} \sum_{\substack{g \in C_n \\ \langle g \rangle = H_m}} |\text{Fix}^g|, \tag{62}$$

where $\sum_{m|n}$ means we sum over all positive integers m which divide n. Next we use the fact that a word is fixed by g if and only if it is fixed by the cyclic subgroup $\langle g \rangle$. Thus we get from (62)

$$\#\text{orbits} = \frac{1}{n} \sum_{m|n} \sum_{\substack{g \in C_n \\ \langle g \rangle = H_m}} |\text{Fix}^{H_m}| \,. \tag{63}$$

Now recall that any cyclic group has many possible generators. In particular if g generates a group G of order m, g^a generates G if and only if $\gcd(a, m) = 1$. Applying this to $G = H_m$, which is cyclic of order m, $(Z^{\frac{n}{m}})^a$ generates H_m if and only if $\gcd(a, m) = 1$. The *Euler's totient function* $\phi(m)$ counts the number of positive integers a less than or equal to m having greatest common divisor 1 with m. Therefore H_m has $\phi(m)$ generators. This means that we can rewrite (63) as follows:

$$\#\text{orbits} = \frac{1}{n} \sum_{m|n} |\text{Fix}^{H_m}| \cdot \phi(m) \tag{64}$$

If m is a positive integer that divides n then the number of words of length n in 4 letters that are fixed by H_m (equivalently, by $Z^{n/m}$) is $4^{n/m}$ because such words are uniquely determined by the first n/m positions, which can be arbitrarily chosen. This gives us the formula we wanted to show

$$\#\text{orbits} = \dim u^{C_n}(2^n) = \frac{1}{n} \sum_{m|n} 4^{n/m} \phi(m).$$

□

5.4 Proof of Formula (51)

With the same steps as in the previous proof applied to X_k rather then the whole set of words we arrive at (cf., formula (64))

$$|X_k/C_n| = \frac{1}{n} \sum_{m|n} |\text{Fix}^{H_m}| \cdot \phi(m), \tag{65}$$

where now the set fixed by H_m, Fix^{H_m} is considered in X_k rather than in the space of all 2^n binary words. Recall that H_m is the subgroup generated by $Z^{n/m}$. A word $\underline{a}$ in X_k is fixed by H_m if and only if $Z^{n/m}(\underline{a}) = \underline{a}$. This in turn holds if and only n/m is a multiple of the period T of $\underline{a}$. Therefore the words in Fix^{H_m} have period T such that n/T divides k and m divides n/T. Their number by definition is $\mathsf{w}(n, m, k)$. Moreover in the sum (65) m has to divide n/T and therefore n, and n/T has to divide k, so that m also has to divide k. Therefore the nonzero terms are obtained for m at most equal to the greatest common divisor of n and k, i.e., $\gcd(n, k)$ which gives formula (51).

[6] We use the standard convention in group theory denoting by $\langle F_1, F_2, ..., F_s \rangle$ the group generated by the set $\{F_1, F_2, ..., F_s\}$.