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Subspace controllability of bipartite symmetric spin networks

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A B S T R A C T

We consider a class of spin networks where each spin in a certain set interacts, via Ising coupling, with a set of *central* spins, and the control acts simultaneously on all the spins. Due to the permutation symmetries of the network, the system is not globally controllable but it displays invariant subspaces of the underlying Hilbert space. The system is said to be *subspace controllable* if it is controllable on each of these subspaces. We characterize the given invariant subspaces and the dynamical Lie algebra of this class of systems and prove subspace controllability in every case.

1. Introduction

Controllability of finite dimensional quantum systems, described by a Schrödinger equation of the form

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$$|\dot{\psi}\rangle = (A + \sum_j B_j u_j(t))|\psi\rangle, \tag{1}$$

is usually assessed by computing the Lie algebra $\mathcal{G}$ generated by the matrices in $u(N)$, A and B_j (see, e.g., [7], [12], [16]). The Lie algebra $\mathcal{G}$ is called the *dynamical Lie algebra*. Here $u_j = u_j(t)$ are the (semiclassical) control electromagnetic fields and $|\psi\rangle$ is the quantum mechanical state varying in a finite dimensional Hilbert space $\mathcal{H}$. If $e^{\mathcal{G}}$ denotes the connected component containing the identity of the Lie group associated with $\mathcal{G}$, then the set of states reachable from $|\psi_0\rangle$ by choosing the control fields is (dense in) $\{|\psi\rangle := X|\psi_0\rangle \in \mathcal{H} \,|\, X \in e^{\mathcal{G}}\}$. In particular if $\mathcal{G} = u(N)$ or $\mathcal{G} = su(N)$, the system is said to be *(completely) controllable* and every unitary operation, or special unitary operation in the $su(n)$ case, can be performed on the quantum state. This is important in quantum information processing [15] when we want to ensure that every quantum operation can be obtained for a certain physical experiment (universal quantum computation). Although controllability is a generic property (see, e.g., [4], [14]), often symmetries of the physical system prevent it and the dynamical Lie algebra $\mathcal{G}$ is a *proper* subalgebra of $su(N)$. In this case, the given representation of the Lie algebra $\mathcal{G}$ splits into its irreducible components which all act on an invariant subspace of the full Hilbert space $\mathcal{H}$ on which the system state $|\psi\rangle$ is defined. It is therefore of interest to study whether, on each subspace, controllability is verified, so that, in particular, one can perform universal quantum computation and-or generate interesting states on a smaller portion of the Hilbert space (see, e.g., [9], [11]). This situation has recently been studied in detail for networks of particles with spin in the papers [18], [19]. In particular, in [19], various topologies of the spin network were considered for various possible interactions among the spins and results were proven concerning the controllability of the *first excitation space*, that is, the invariant subspace of the network of states of the form $\sum_j a_j|000\cdots 00100\cdots 000\rangle$, i.e., superpositions of states where only one spin is in the excited state. In [18], only *chains* with next neighbor interactions were considered (instead of general networks) but more comprehensive controllability results were given on *all* the invariant subspaces of this type of systems. In both these papers, the control affects *only one* of the spins in the network, which may be placed in various places in the network.

In this paper, we shall consider situations where the control acts on all the spins *simultaneously*. We want to study the structure of the dynamical Lie algebra and subspace controllability in this case. The symmetries of the network originate from the following physical fact: The spins are arranged in two sets, a set P and a set C. The set C is called of *central* spins. Spins in the set P (C) interact in the same (Ising) way with the set of spins in the set C (P). We shall initially assume that no (symmetric) interaction exists among spins in the set P or C although we shall later see (in Section 5) how to relax such an assumption. The systems we have in mind may be networks of spins arranged in a molecule where the distances from spins in the set P from one or two central spins are equal so that spins in the set C interact in the same way with a bath of surrounding spins as in Fig. 1. The interaction between spins is physically a function of the type of

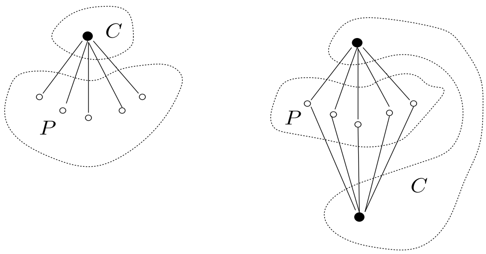


Fig. 1. Schematic representation of a spin network with one (a) and two (b) central spins C depicted with black bullets as opposed to empty circles (spins in P).

spins and of the *distance* between the spins. It is therefore impossible to have three or more spins in both sets C and P and therefore we assume that the set with smaller cardinality, which we assume to be C, has at most two spins. Systems of this type admit *symmetries*. In particular, by permuting the spins in the set C and-or the spins in the set P, the Hamiltonian describing the dynamics of the system as in (1) is left unchanged (see next section for details). Then, if n_c is the cardinality of the set C and n_p is the cardinality of the set P, the group of symmetries is the product between the symmetry group on n_c elements, S_{n_c}, and the symmetry group on n_p elements, S_{n_p}. In the mathematical study of controllability of spin networks, the bipartite case considered here can be seen as the first intermediate case between the *fully asymmetric* case studied in [1] where all the spins are distinct and the *fully symmetric* case studied in [2], [8], [6] where the spins are all equal and interact in the same way with each-other. In this context, the results of this paper are the first step towards developing a theory for controllability of spin networks where symmetries are 'localized' within certain subsets of the network. Furthermore our results contribute to the recently growing literature on the dynamical analysis and control of a central spin surrounded by a number of *bath spins* (see, e.g., [5], [20] and references therein). This model is of practical interest in applications and serve as a simplified example for the study of controlled quantum dynamics in the presence of decoherence.

In general terms, if there is a discrete group G of symmetries for a quantum mechanical system, the dynamical Lie algebra $\mathcal{G}$ associated with the system will be a subalgebra of $\mathcal{L}^G$, the largest subalgebra of $u(N)$ (N being the dimension of the system) which commutes with G. If $\mathcal{G}$ is *equal* to $\mathcal{L}^G$, subspace controllability is satisfied for each of the invariant subspaces of the system (cf. Theorem 2 in [8]). However $\mathcal{G}$ might be a *proper* Lie subalgebra of $\mathcal{L}^G$ and subspace controllability may not be satisfied. For the systems we consider in this paper we will see that $\mathcal{G}$ is not exactly equal to $\mathcal{L}^G$. However, this does not affect the subspace controllability of the system for each of its invariant subspaces which, we will prove, is still verified.

The paper is organized as follows. In the next section, we set up the notations and the basic definitions, so that we can precisely describe the model we want to treat and the problem we want to consider. We also prove a number of preliminary results which will be used later in the paper. The main results are given in section 3 where we describe the dynamical Lie algebra for Ising networks of spins with one or two central spin under global control. Subspace controllability will come as a consequence of this in section 4. Some concluding remarks are given in section 5 where we also give generalizations of our results to 1) the case of different interactions among the spins 2) nonzero interactions among the spins in the set C and-or in the set P. We also discuss how the results are affected by small changes in the dynamical Hamiltonian which break the symmetries of the model.

2. Preliminaries

2.1. Notations, basic definitions and properties

In the following, we will have to compute a basis for a Lie algebra generated by a given set of matrices. In these calculations, it is not important whether we obtain a matrix A or a matrix kA with $k \in \mathbb{R}$, $k \neq 0$. Therefore we shall use the notation $[A, B] \vDash D$ to indicate that the commutator of A and B ($[A, B] := AB - BA$) is kD for some $k \neq 0$ and therefore D belongs to the Lie algebra that contains A and B. We shall also often use the formula

$$[A \otimes B, C \otimes D] = \frac{1}{2}\{A, B\} \otimes [B, D] + \frac{1}{2}[A, C] \otimes \{B, D\},$$

where $\{A, B\}$ denotes the *anti-commutator* of A and B, i.e., $\{A, B\} := AB + BA$. We will do this routinely without explicitly referring to this formula. In $u(n)$ we shall use the *inner product* $\langle A, B\rangle := Tr(AB^{\dagger})$. One property of this inner product which will be useful is given by the following:

Lemma 2.1. *If A commutes with B and C, then it is also orthogonal to and commutes with $[B, C]$.*

Proof. Commutativity follows from the Jacobi identity. Moreover, $Tr(A[B, C]^{\dagger}) = -Tr(A[B, C]) = -Tr(ABC - ACB) = -Tr(BAC - CAB) = -Tr(BAC - BCA) = -Tr(B[A, C]) = 0$. $\square$

The *Pauli matrices* $\sigma_{(x,y,z)}$ are defined as

$$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y := \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

If $\mathbf{1_n}$ denotes the $n \times n$ identity matrix, the Pauli matrices satisfy

$$\begin{gathered} \sigma_x\sigma_x = \sigma_y\sigma_y = \sigma_z\sigma_z = \mathbf{1}_2, \\ \sigma_x\sigma_y = -i\sigma_z, \quad \sigma_y\sigma_z = -i\sigma_x, \quad \sigma_z\sigma_x = -i\sigma_y \ , \\ \sigma_y\sigma_x = i\sigma_z, \qquad \sigma_z\sigma_y = i\sigma_x, \qquad \sigma_x\sigma_z = i\sigma_y \end{gathered} \tag{3}$$

which give the commutation relations

$$[i\sigma_x, i\sigma_y] = 2i\sigma_z,\ [i\sigma_y, i\sigma_z] = 2i\sigma_x,\ [i\sigma_z, i\sigma_x] = 2i\sigma_y. \tag{4}$$

We shall often use just the symbol $\mathbf{1}$ for the identity matrix in different dimensions as the dimensions will be clear from the context. In the most general setting, our model consists of $n_c + n_p$ spin $\frac{1}{2}$ particles, with n_c of a type C (for example n_c nuclei) and n_p of the type P (for example n_p electrons). In our conventions, the first n_c positions in a tensor product refer to operators on the spins in the set C, while the following n_p refer to operators on the set P. We shall assume without loss of generality $n_c \leq n_p$. Our main results on the characterization of the dynamical Lie algebra and subspace controllability will concern the physical case of $n_c = 1$ and $n_c = 2$. We start giving some general results valid for arbitrary n_c.

We denote by $S^{C(P)}_{(x,y,z)}$ the sum of $n_{c(p)}$ tensor products $\sum_{j=1}^{n_{c(p)}} \mathbf{1} \otimes \cdots \otimes \sigma_{(x,y,z)} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}$ where the Pauli matrix $\sigma_{(x,y,z)}$ varies among all the possible $n_{c(p)}$ positions. For example, if $n_c = 2$, $S^C_x := \sigma_x \otimes \mathbf{1} + \mathbf{1} \otimes \sigma_x$. When it is not important or it is clear whether we refer to the set C or the set P, we shall simply denote this type of matrices by $S_{(x,y,z)}$. We notice that $S_{(x,y,z)}$ satisfy the same commutation relations as $\sigma_{(x,y,z)}$ and therefore $iS_{(x,y,z)}$ give a representation of $su(2)$ in the appropriate dimensions.

We shall denote the 3-dimensional Lie algebra spanned by $iS_{(x,y,z)}$ with $\mathcal{S}$. We shall also denote by $I^{C(P)}_{(x,y,z)(x,y,z)}$ matrices which are sum of the tensor products of 2×2 identities, $\mathbf{1}$, except in all possible pairs of positions which are occupied by $\sigma_{(x,y,z)}$ and $\sigma_{(x,y,z)}$. For example, if $n_c = 3$, we have

$$\begin{aligned} I^C_{xx} &:= \sigma_x \otimes \sigma_x \otimes \mathbf{1} + \sigma_x \otimes \mathbf{1} \otimes \sigma_x + \mathbf{1} \otimes \sigma_x \otimes \sigma_x, \\ I^C_{xy} &:= \sigma_x \otimes \sigma_y \otimes \mathbf{1} + \sigma_y \otimes \sigma_x \otimes \mathbf{1} + \sigma_x \otimes \mathbf{1} \otimes \sigma_y + \\ &\quad + \sigma_y \otimes \mathbf{1} \otimes \sigma_x + \mathbf{1} \otimes \sigma_x \otimes \sigma_y + \mathbf{1} \otimes \sigma_y \otimes \sigma_x. \end{aligned}$$

As before, when it is not important, or it is clear in the given context, whether we refer to the set C or P, we omit the superscript C or P. $\mathcal{I}^{C(P)}$ denotes the 6-dimensional span of $I^{C(P)}_{(x,y,z)(x,y,z)}$, while $\mathcal{I}^{C(P)}_0$ denotes the 5-dimensional subspace of $\mathcal{I}^{C(P)}$ spanned by $\{I^{C(P)}_{xy}, I^{C(P)}_{xz}, I^{C(P)}_{yz}, I^{C(P)}_{xx} - I^{C(P)}_{yy}, I^{C(P)}_{yy} - I^{C(P)}_{zz}\}$. Generalizing this notation, we shall denote by $I_{x(n_x \text{times})y(n_y \text{times})z(n_z \text{times})}$ the sum of symmetric tensor products with n_x σ_x's, n_y σ_y's and n_z, σ_z's. We omit the zeros. Therefore, for instance, $S_x := I_x$ and, for $n = 3$, $I_{xxx} := \sigma_x \otimes \sigma_x \otimes \sigma_x$.

Lemma 2.2.

$$[\mathcal{S}, i\mathcal{I}] = [\mathcal{S}, i\mathcal{I}_0] = i\mathcal{I}_0. \tag{5}$$

Furthermore, if $A := iI_{zz}$ *or* iI_{xx} *or* iI_{yy}*,*

$$[\mathcal{S}, \mathtt{span}\{A\}] \oplus [\mathcal{S}, [\mathcal{S}, \mathtt{span}\{A\}]] = i\mathcal{I}_0. \tag{6}$$

Proof. Formula (5) follows by direct verification using the indicated bases for $\mathcal{I}$, $\mathcal{I}_0$ and $\mathcal{S}$. For the second property take for example iI_{zz}. We have

$$[iS_x, iI_{zz}] \vDash iI_{yz}, \qquad [iS_y, iI_{zz}] \vDash iI_{xz},$$

in $[\mathcal{S}, \mathtt{span}\{A\}]$, and

$$[iS_x, iI_{yz}] \vDash iI_{zz} - iI_{yy}, \ [iS_y, iI_{xz}] \vDash iI_{zz} - iI_{xx},$$
$$[iS_x, iI_{xz}] \vDash iI_{xy},$$

which are in $[\mathcal{S}\,, [\mathcal{S}, \mathtt{span}\{A\}]]$. $\square$

With $\mathcal{L}^G$, we denote the full Lie algebra of matrices in $u(n)$, which commute with the symmetric group S_n. The dimension of $\mathcal{L}^G$ was calculated in [2] and it is given by $M(n) := \binom{n+3}{n}$. With $\mathcal{L}$, we shall denote the Lie algebra generated by $i\{S_x, S_y, S_z, I_{xx} - I_{yy}, I_{yy} - I_{zz}\}$. The dimensions of the matrices in $\mathcal{L}^G$ (and $\mathcal{L}$) are $2^n \times 2^n$. The number n will be often clear from the context. In some places $n = n_c$. In some other cases $n = n_p$. The following fact was one of the main results of [2].

Theorem 1. *Consider n spin $\frac{1}{2}$ particles and I_{zz}, $S_{(x,y,z)}$ matrices of the corresponding dimension 2^n. Then iI_{zz}, $iS_{(x,y,z)}$, generate all $\mathcal{L}^G \cap su(2^n)$.*

The matrix $J := I_{xx} + I_{yy} + I_{zz}$, will be important in our description of the dynamical Lie algebra. The Lie algebra $\mathcal{L}$ above defined is the same as $\mathcal{L}^G \cap su(2^n)$ except for iJ. More precisely:

Proposition 2.3.

$$\mathcal{L}^G \cap su(2^{\hat{n}}) = \mathcal{L} \oplus \mathtt{span}\{iJ\}. \tag{7}$$

Proof. The inclusion $\supseteq$ follows from the fact that iJ commutes with every permutation and so do the generators of $\mathcal{L}$, which are $iS_{(x,y,z)}$ and $\{i(I_{xx} - I_{yy}), i(I_{yy} - I_{zz})\}$, and therefore all of $\mathcal{L}$. Moreover both J and the generators of $\mathcal{L}$ are in $su(2^n)$. To show the inclusion $\subseteq$ it is enough to show that a set of generators of $\mathcal{L}^G \cap su(2^n)$ belong to $\mathcal{L} \oplus \mathtt{span}\{iJ\}$. For this, we use Theorem 1, and take as generators $iS_{(x,y,z)}$ and iI_{zz}. The matrices $iS_{(x,y,z)}$ are already in $\mathcal{L}$ by definition of $\mathcal{L}$. Since

$$iI_{zz} = -\frac{1}{3} i(I_{xx} - I_{yy}) - \frac{2}{3} i(I_{yy} - I_{zz}) + \frac{1}{3} iJ$$

and $\{i(I_{xx} - I_{yy}), i(I_{yy} - I_{zz})\}$ are also in $\mathcal{L}$, we have that iI_{zz} belongs to $\mathcal{L} \oplus \mathtt{span}\{iJ\}$. $\square$

We shall also use the following property of the matrix J.

Lemma 2.4. *The matrix iJ commutes with $\mathcal{L}$.*

Proof. We only need to prove that iJ commutes with the generators of $\mathcal{L}$. We start with $iS_{(x,y,z)}$. By symmetry we only need to consider one among $iS_{(x,y,z)}$. Take iS_x, and calculate $[iS_x, iJ] = [iS_x, i(I_{xx} + I_{yy} + I_{zz})] = [iS_x, i(I_{yy} + I_{zz})] = [iS_x, iI_{yy}] + [iS_x, iI_{zz}]$. The first term, using (4), gives iI_{zy} (it is clear that it contains sum of matrices with all identities except in two positions, one occupied by σ_z and one occupied by σ_y; moreover it has to be invariant under permutations and the only matrices with this property are proportional to iI_{yz}; the fact that the proportionality factor is 1 follows from the fact that σ_z in the first place can only occur once). Using again (4), the second term gives $-iI_{zy}$, thus these two terms sum up to zero.

As for $i(I_{xx} - I_{yy})$ and $i(I_{yy} - I_{zz})$, again by symmetry, we need to consider only one of them. We consider $i(I_{xx} - I_{yy})$. We have $[i(I_{xx} - I_{yy}), iJ] = [i(I_{xx} - I_{yy}), i(I_{xx} + I_{yy} + I_{zz})] = [iI_{xx}, iI_{yy}] + [iI_{xx}, iI_{zz}] - [iI_{yy}, iI_{xx}] - [iI_{yy}, iI_{zz}] =$

$$2[iI_{xx}, iI_{yy}] + [iI_{xx}, iI_{zz}] - [iI_{yy}, iI_{zz}]. \tag{8}$$

In the commutator $[iI_{xx}, iI_{yy}]$, writing I_{xx} and I_{yy} as symmetric sums of tensor products the only terms that do not give zero are the ones where the two positions in I_{xx} occupied by σ_x and the two positions of I_{yy} occupied by σ_y have only one index in common (e.g., positions $(1, 2)$ and position $(2, 3)$). The commutator gives a term with a single σ_x, a single σ_y and a single σ_z. Using the fact that the Lie bracket has to be permutation invariant, we obtain that $[iI_{xx}, iI_{yy}]$ must be proportional to iI_{xyz}. The proportionality factor is in fact 1. This can be seen by writing I_{xx} as $I_{xx} = \sigma_x \otimes S_x + \mathbf{1} \otimes I_{xx}^{n-1}$, where I_{xx}^{n-1} is I_{xx} but on $n-1$ positions, and, analogously $I_{yy} = \sigma_y \otimes S_y + \mathbf{1} \otimes I_{yy}^{n-1}$. Taking the commutator one can see that the coefficients of the terms having σ_z in the first place is 1, and therefore, by permutation symmetry this is the coefficient if iI_{xyz} as well. With an analogous reasoning, the commutator $[iI_{xx}, iI_{zz}]$ in (8) gives $-iI_{xyz}$ and the commutator $[iI_{yy}, iI_{zz}]$ in (8) gives iI_{xyz}, so that the sum in (8) gives zero. □

Using Proposition 2.3, we have

Corollary 2.5. *The matrix iJ commutes with $\mathcal{L}^G$.*

Lemma 2.6. *If $n = 2$, for each $A \in \mathcal{L}$,*

$$JA = AJ = A \tag{9}$$

Proof. Formula (9) can be directly verified for the generators of $\mathcal{L}$ using (3), and it is extended to commutators by $J[A, B] = J(AB - BA) = (JA)B - (JB)A = AB - BA = [A, B]$. □

2.2. The model

Spin $\frac{1}{2}$ particles in a network are divided into two sets, C and P. Each spin in the set C interacts via Ising interaction with each spin in the set P but there is no (significant) interaction within spins in the set C (P) (see, Section 5 for generalizations). The system is controlled by a common electro-magnetic field which is arbitrary in the x and y direction. Up to a proportionality factor, the quantum mechanical Hamiltonian of the system can be written as

$$\begin{aligned} H = \quad & S_z^C \otimes S_z^P + u_x(\gamma_C S_x^C \otimes \mathbf{1} + \gamma_P \mathbf{1} \otimes S_x^P) + \\ & + u_y(\gamma_C S_y^C \otimes \mathbf{1} + \gamma_P \mathbf{1} \otimes S_y^P). \end{aligned} \tag{10}$$

Here the term $S_z^C \otimes S_z^P$ models the Ising interaction of each spin of the set C with each spin of the set P. The functions $u_x := u_x(t)$ and $u_y := u_y(t)$ are control electromagnetic fields in the x and y directions. The parameters γ_C and γ_P are (proportional to) the *gyromagnetic ratios* of the spins in set C and set P, respectively. The dimensions of the identity matrices $\mathbf{1}$ in (10) are 2^{n_c} or 2^{n_p}, according to whether $\mathbf{1}$ is on the left or on the right, respectively, of the tensor product. The Schrödinger equation for the system takes the form (1) where $A + \sum_j B_j u_j = -iH$ with H in (10).

2.3. Dynamical Lie algebra and subspace controllability

We want to describe the possible evolutions that can be obtained by changing the controls in (10) and therefore we want to describe [12] the dynamical Lie algebra $\mathcal{G}$ generated by

$$\{iS_z^C \otimes S_z^P, i(\gamma_C S_x^C \otimes \mathbf{1} + \gamma_P \mathbf{1} \otimes S_x^P), i(\gamma_C S_y^C \otimes \mathbf{1} + \gamma_P \mathbf{1} \otimes S_y^P)\}.$$

Once $\mathcal{G}$ is determined, its elements will take, in appropriate coordinates, a block diagonal form which describes the *sub-representations* of $\mathcal{G}$. The Hilbert space $\mathcal{H}$ for the quantum state is accordingly decomposed into invariant subspaces. Subspace controllability is verified if, on each subspace, $\mathcal{G}$ acts as $u(m)$ or $su(m)$ where m is the dimension of the given subspace. Our problem is to determine the Lie algebra $\mathcal{G}$ and then find all its sub-representations and prove subspace controllability.

As a preliminary step, we remark that, letting

$$W = [i\gamma_C S_x^C \otimes \mathbf{1} + i\gamma_P \mathbf{1} \otimes S_x^P, i\gamma_C S_y^C \otimes \mathbf{1} + i\gamma_P \mathbf{1} \otimes S_y^P],$$

then

$$[i\gamma_C S_x^C \otimes \mathbf{1} + i\gamma_P \mathbf{1} \otimes S_x^P, W] = i\gamma_C^3 S_y^C \otimes \mathbf{1} + i\gamma_P^3 \mathbf{1} \otimes S_y^P.$$

Therefore, since the Lie algebra contains $i\gamma_C S_y^C \otimes \mathbf{1} + i\gamma_P \mathbf{1} \otimes S_y^P$ also, assuming $|\gamma_C| \neq |\gamma_P|$, we have that $iS_y^C \otimes \mathbf{1}$ and $i\mathbf{1} \otimes S_y^P$ belong to $\mathcal{G}$. Taking the Lie brackets of $i\gamma_C S_x^C \otimes$

$\mathbf{1} + i\gamma_P \mathbf{1} \otimes S_x^P$ with $iS_y^C \otimes \mathbf{1}$ and $i\mathbf{1} \otimes S_y^P$, we obtain that $iS_z^C \otimes \mathbf{1}$ and $i\mathbf{1} \otimes S_z^P$ are in $\mathcal{G}$, and taking the Lie bracket between $iS_y^C \otimes \mathbf{1}$ ($i\mathbf{1} \otimes S_y^P$) and $iS_z^C \otimes \mathbf{1}$ ($i\mathbf{1} \otimes S_y^P$) we obtain $iS_x^C \otimes \mathbf{1}$ ($i\mathbf{1} \otimes S_x^P$). Therefore $\mathcal{G}$ contains the 3-dimensional subspaces

$$\mathcal{A}^C := \mathtt{span}\{iS_{(x,y,z)}^C \otimes \mathbf{1}\}, \ \mathcal{A}^P := \mathtt{span}\{i\mathbf{1} \otimes S_{(x,y,z)}^P\}, \tag{11}$$

under the assumption that $|\gamma_C| \neq |\gamma_P|$. We shall assume this to be the case in the following. Therefore the dynamical Lie algebra $\mathcal{G}$ is the Lie algebra generated by $\mathcal{A}^C$, $\mathcal{A}^P$ and $iS_z^C \otimes S_z^P$.

3. Description of the dynamical Lie algebra

3.1. Results for general $n_c \geq 1$

Consider the group $\hat{G}$, $\hat{G} := S_{n_c} \otimes S_{n_p}$, where S_{n_c} is the group of permutation matrices (symmetric group) on the first n_c positions, corresponding to spins of the type C and S_{n_p} is the group of permutation matrices (symmetric group) on the second n_p positions, corresponding to spins of the type P. This means, for C (and analogously for P) that if Q is a matrix in S_{n_c} and A belongs to $u(2^{n_c})$ QAQ^{-1} is obtained from A by (possibly) permuting certain positions in the tensor products which appear once one expands A in the standard (tensor product type) of basis in $u(2^{n_c})$. This is a *group of symmetries* for the system described by the Hamiltonian (10) since for every element $Q_C \otimes Q_P \in S_{n_c} \otimes S_{n_p}$, we have

$$[iS_z^C \otimes S_z^P, Q_C \otimes Q_P] = 0,$$
$$[i(\gamma_C S_x^C \otimes \mathbf{1} + \gamma_P \mathbf{1} \otimes S_x^P), Q_C \otimes Q_P] = 0,$$
$$[i(\gamma_C S_y^C \otimes \mathbf{1} + \gamma_P \mathbf{1} \otimes S_y^P), Q_C \otimes Q_P] = 0.$$

The generators of $\mathcal{G}$ all commute with $\hat{G}$ and therefore all of $\mathcal{G}$ commutes with $\hat{G}$. This implies that the dynamical Lie algebra $\mathcal{G}$ must be a Lie subalgebra of the maximal subalgebra $\mathcal{L}^{\hat{G}}$ of $u(2^{n_c+n_p})$ which commutes with $\hat{G}$. We have $\mathcal{L}^{\hat{G}} = i\mathcal{L}^G \otimes \mathcal{L}^G$. Notice that here the first $i\mathcal{L}^G$ is in $u(2^{n_c})$, while the second is in $u(2^{n_p})$. A basis of $\mathcal{L}^{\hat{G}}$ can be obtained by taking tensor products of the basis of the two $\mathcal{L}^G$, and the dimension of $\mathcal{L}^{\hat{G}}$ is $M(n_c)M(n_p)$.

In fact, $\mathcal{G}$ is a Lie subalgebra of a slightly smaller Lie algebra.

Lemma 3.1. *The Lie algebra*

$$\hat{\mathcal{L}} = \left(i\mathcal{L} \otimes \mathcal{L}^G\right) + \left(i\mathcal{L}^G \otimes \mathcal{L}\right), \tag{12}$$

is a super Lie algebra of $\mathcal{G}$.

Proof. To see that (12) is a Lie algebra, we can notice that it is the orthogonal complement in $\mathcal{L}^G \otimes \mathcal{L}^G$ to the Abelian Lie algebra

$$\mathcal{J} := \mathtt{span}\{i\mathbf{1} \otimes \mathbf{1}, i\mathbf{1} \otimes J, iJ \otimes \mathbf{1}, iJ \otimes J\},$$

for the appropriate dimensions of the identity $\mathbf{1}$ and J on the left and on the right (this in the case $n_c = 1$ reduces to $\mathcal{J} := \mathtt{span}\{i\mathbf{1}\otimes\mathbf{1}, i\mathbf{1}\otimes J\}$) and commutes with it because of Lemma 2.4 and Corollary 2.5. Therefore is closed under commutation from Lemma 2.1. Moreover all generators of $\mathcal{G}$ belong to $\hat{\mathcal{L}}$. $\square$

We shall see that in the case $n_c = 1$, $\mathcal{G} = \hat{\mathcal{L}}$, while for $n_c = 2$ $\mathcal{G} \neq \hat{\mathcal{L}}$. We now identify certain subspaces of $\hat{\mathcal{L}}$ which belong to the dynamical Lie algebra $\mathcal{G}$.

Proposition 3.2. *The following vector spaces belong to $\mathcal{G}$:*

$$\begin{aligned} \mathcal{B} \quad &:= \quad \mathtt{span}\{iS^C_{x,y,z} \otimes S^P_{x,y,z}\}, \\ \mathcal{D}_1 \quad &:= \quad \mathtt{span}\{iS^C_{(x,y,z)} \otimes I^P_{(x,y,z)(x,y,z)}\}, \\ \mathcal{D}_2 \quad &:= \quad \mathtt{span}\{iI^C_{(x,y,z)(x,y,z)} \otimes S^P_{(x,y,z)}\}. \end{aligned} \tag{13}$$

Remark 3.3. Notice that the above subspaces have the following dimensions: $\dim(\mathcal{B}) = 9$, $\dim(\mathcal{D}_1) = 18$ unless the set P has cardinality 1, in which case $\mathcal{D}_1 = \{0\}$, $\dim(\mathcal{D}_2) = 18$ unless the set C has cardinality 1, in which case $\mathcal{D}_2 = \{0\}$.

Proof. The indicated basis of $\mathcal{B}$ can be obtained from $iS_z \otimes S_z$ by taking Lie brackets with elements of the basis of $\mathcal{A}^C$ and $\mathcal{A}^P$ in (11). Now assume that the set P has cardinality strictly bigger than 1 and take the Lie bracket of the two elements in $\mathcal{B}$, $iS_x \otimes S_z$ and $iS_y \otimes S_z$, which is $[iS_x, iS_y] \otimes S_z^2 \vDash iS_z \otimes (\mathbf{1} + I_{zz})$. Since we know that $iS_z \otimes \mathbf{1}$ is in $\mathcal{G}$, as it belongs to $\mathcal{A}^C$, we have that $iS_z \otimes I_{zz}$ is in $\mathcal{G}$. By taking Lie brackets with $iS^C_x \otimes \mathbf{1}$ and $iS^C_y \otimes \mathbf{1}$ we obtain $iS_{(x,y,z)} \otimes I_{zz}$. By taking (possibly) repeated Lie brackets with $\mathbf{1} \otimes S^P_{(x,y,z)}$ (using possibly the fact that $iS_{(x,y,z)} \otimes I_{zz}$ belongs to $\mathcal{G}$) we obtain all other elements of the form $iS_{(x,y,z)} \otimes I_{(x,y,z)(x,y,z)}$. Analogously we obtain the elements in the indicated basis of $\mathcal{D}_2$. $\square$

3.2. Dynamical Lie algebra for $n_c = 1$

In the case $n_c = n_p = 1$, $\mathcal{A}^C \oplus \mathcal{A}^P \oplus \mathcal{B}$ is equal to $su(4)$, so that $\mathcal{G} = su(4)$. In this case the system is completely controllable and our analysis terminates here. We shall therefore assume that $n_p > 1$, and therefore $\mathcal{D}_1 \neq \{0\}$ in (13) while $\mathcal{D}_2 = \{0\}$.

Take B in $\mathcal{S}^P$ and D in $\mathcal{I}^P$. The Lie bracket of the matrices $S_z \otimes B := \sigma_z \otimes B \in \mathcal{B}$ and $S_z \otimes iD = \sigma_z \otimes iD \in \mathcal{D}_1$ gives

$$[S_z \otimes B, S_z \otimes iD] = S_z^2 \otimes [B, iD] = \mathbf{1} \otimes R, \tag{14}$$

for an arbitrary R in $i\mathcal{I}^P_0$ according to (5) of Lemma 2.2. We have therefore:

Lemma 3.4. *If $n_c = 1$, the dynamical Lie algebra $\mathcal{G}$ contains*

$$\mathcal{E}_1 := \mathbf{1} \otimes i\mathcal{I}_0^P. \tag{15}$$

Theorem 2. *If $n_c = 1$ and for any $n_p \geq 2$ the dynamical Lie algebra $\mathcal{G}$ is given by*

$$\mathcal{G} := \left((\mathtt{span}\{\sigma_{x,y,z}\}) \otimes \mathcal{L}^G\right) \oplus ((\mathtt{span}\{\mathbf{1}\}) \otimes \mathcal{L}) = \hat{\mathcal{L}}. \tag{16}$$

Proof. Using elements in $\mathcal{E}_1$ and elements of $\mathcal{A}^P$, since $\mathcal{L}$ is the Lie algebra generated by $i\mathcal{I}_0$ and $iS_{(x,y,z)}$ we obtain anything in $(\mathtt{span}\{\mathbf{1}\}) \otimes \mathcal{L}$. Now we know from Theorem 1 that iI_{zz}^P, $iS_{(x,y,z)}^P$ and $i\mathbf{1}$ generate all of $\mathcal{L}^G$. Therefore, basis elements of $\mathcal{L}^G \cap su(2^{n_p})$ are obtained by (repeated) Lie brackets of iI_{zz}^P and $iS_{(x,y,z)}^P$. Define the '*depth*' of a basis element K_1 as the number of Lie brackets to be performed to obtain K_1. In particular, the generators iI_{zz}, $iS_{(x,y,z)}$ are element of depth zero. We show by induction on the depth of the basis element K_1 that all elements of the form $\sigma_{(x,y,z)} \otimes K_1$ can be obtained. For depth zero, we already have $i\sigma_{(x,y,z)} \otimes S_{(x,y,z)} \in \mathcal{B}$ and $i\sigma_{(x,y,z)} \otimes I_{zz} \in \mathcal{D}_1$, from Proposition 3.2. For depth $d \geq 1$, assume by induction that we have all elements $i\sigma_{(x,y,z)} \otimes K_1$ for K_1 in the basis of $\mathcal{L}^G \cap su(2^{n_p})$, K_1 of depth $d-1$. If $K_2 = [K_1, iS_{(x,y,z)}]$, we can obtain

$$\begin{aligned}[\sigma_{(x,y,z)} \otimes K_1, i\mathbf{1} \otimes S_{x,y,z}] = &\quad \sigma_{(x,y,z)} \otimes [K_1, iS_{(x,y,z)}] \\ = &\quad \sigma_{(x,y,z)} \otimes K_2.\end{aligned}$$

If $K_2 := [K_1, iI_{zz}]$, write
$iI_{zz} = \frac{1}{3}i(I_{xx} - I_{yy}) - \frac{2}{3}i(I_{xx} - I_{zz}) + \frac{1}{3}iJ$, so that

$$\begin{aligned}K_2 = &\quad \left[K_1, \tfrac{1}{3}i(I_{xx} - I_{yy}) - \tfrac{2}{3}i(I_{xx} - I_{zz}) + \tfrac{1}{3}iJ\right] \\ = &\quad \left[K_1, \tfrac{1}{3}i(I_{xx} - I_{yy}) - \tfrac{2}{3}i(I_{xx} - I_{zz})\right].\end{aligned}$$

This is true because iJ commutes with $\mathcal{L}^G$ according to Corollary 2.5. This shows that $K_2 \in [K_1, \mathcal{L}]$ and since we have $\sigma_{(x,y,z)} \otimes K_1 \in \mathcal{G}$ (by inductive assumption) and $\mathbf{1} \otimes \mathcal{L} \in \mathcal{G}$ (because we showed it above), we have

$$\begin{aligned}\left[\sigma_{(x,y,z)} \otimes K_1, \mathbf{1} \otimes iI_{zz}\right] = &\quad \sigma_{(x,y,z)} \otimes [K_1, iI_{zz}] \\ \in &\quad \sigma_{(x,y,z)} \otimes [K_1, \mathcal{L}] \in \mathcal{G}.\end{aligned}$$

These arguments show that, in (16), the right hand side is included in the left hand side. We already know that $\mathcal{G} \subseteq \hat{\mathcal{L}}$ by Lemma 3.1, so the theorem is proved. □

The result of Theorem 2 (and Lemma 3.1) show that, in general, $\mathcal{G} \neq \mathcal{L}^{\hat{G}} \cap su(2^{1+n_p})$. In particular $\mathtt{span}\{i\mathbf{1} \otimes J\}$ belongs to $\mathcal{L}^{\hat{G}} \cap su(2^{1+n_p})$ but it is orthogonal to $\mathcal{G}$. This fact could also be proved without knowing Theorem 2 and using the general membership criteria of [5]. Consider for instance the case $n_p = 2$. The set $\mathcal{P} \cup i\mathbf{1} \otimes J$, where $\mathcal{P}$ is the set of generators for our model, has an Abelian commutant in $u(8)$ spanned by $i\mathbf{1}_2 \otimes \mathbf{1}_4$

and $i\mathbf{1}_2 \otimes J$. Using this basis, a direct calculation shows that rank condition B of Result 1 of [5] is not verified so that the Lie algebras generated by $\mathcal{P}$ and $\{\mathcal{P} \cup i\mathbf{1} \otimes J\}$ (which is included in $\mathcal{L}^{\hat{G}} \cap su(2^{1+n_p})$) are different.

3.3. Dynamical Lie algebra for $n_c = 2$

We start with some considerations for general $n_p \geq n_c = 2$. Then we will give separate results for the case $n_p = 2$ and $n_p > 2$.

Lemma 3.5. *If $n_c = 2$, $\mathcal{G}$ contains the spaces*

$$i\mathcal{I}_0^C \otimes \mathcal{I}_0^P, \quad (\mathbf{1} + \frac{1}{3}J) \otimes i\mathcal{I}_0^P. \tag{17}$$

Proof. Using (14), we have $S_z^2 \otimes [B, iD] = (\mathbf{1} + I_{zz}) \otimes R \in \mathcal{G}$, for each $R \in i\mathcal{I}_0^P$. Taking (repeated) Lie brackets with elements of the form $\mathcal{A}^C \subseteq \mathcal{G}$ and using formula (6) of Lemma 2.2 with $A := iI_{zz}^C$ we obtain the first one of (17). Repeating the calculation in (14) with S_z replaced by S_x or S_y, we obtain $i(\mathbf{1}+I_{xx}) \otimes \mathcal{I}_0^P \in \mathcal{G}$ and $i(\mathbf{1}+I_{yy}) \otimes \mathcal{I}_0^P \in \mathcal{G}$, which together with the corresponding one for x gives the second one in (17). □

Proposition 3.6. *If $n_c = 2$ and for all $n_p \geq 2$, it holds that:*

$$S_{x,y,z} \otimes A, \tag{18}$$

with $A \in \mathcal{L}^G$ belongs to $\mathcal{G}$.

Proof. The proof is by induction on the depth of A, with the generators $iS_{x,y,z}$, iI_{zz} and $i\mathbf{1}$ of $\mathcal{L}^G$. We know that the matrices:

$$iS_{x,y,z} \otimes \mathbf{1}, \quad iS_{x,y,z} \otimes S_{x,y,z}, \quad iS_{x,y,z} \otimes I_{zz},$$

are in $\mathcal{G}$, since the first type belongs to $\mathcal{A}^C$ in (11), the second type belongs to $\mathcal{B}$ and the third one to $\mathcal{D}_1$ in (13). Thus equation (18) holds for A of depth 0. Assume that it holds for all B of depth k. If A has depth $k+1$, then either $A = [B, S_{x,y,z}]$ or $A = [B, I_{zz}]$, and $S_{x,y,z} \otimes B \in \mathcal{G}$ by inductive assumption. In the first case, we have:

$$[S_x \otimes B, \mathbf{1} \otimes iS_{x,y,z}] = S_x \otimes A \in \mathcal{G}.$$

In the second case, we have:

$$A = [B, iI_{zz}] = [B, \frac{1}{3}J + \frac{1}{3}i(I_{xx} - I_{yy}) - \frac{2}{3}i(I_{xx} - I_{zz})] = [B, \frac{1}{3}i(I_{xx} - I_{yy}) - \frac{2}{3}i(I_{xx} - I_{zz})],$$

since J commutes with B because of Corollary 2.5. We also have

$$(\mathbf{1}+I_{xx})\otimes\frac{1}{3}i(I_{x,x}-I_{y,y})-\frac{2}{3}i(I_{x,x}-I_{z,z})\in\mathcal{G},$$

because of (17). Therefore we calculate

$$\begin{aligned}&[S_x\otimes B,(\mathbf{1}+I_{xx})\otimes\frac{1}{3}i(I_{x,x}-I_{y,y})-\frac{2}{3}i(I_{x,x}-I_{z,z})]=\\&=1/2\{S_x,(\mathbf{1}+I_{xx})\}\otimes[B,\frac{1}{3}i(I_{x,x}-I_{y,y})-\frac{2}{3}i(I_{x,x}-I_{z,z})]=\\&=(S_x+S_xI_{xx})\otimes[B,iI_{zz}]=(S_x+S_xI_{xx})\otimes A\in\mathcal{G}.\end{aligned}$$

Since for $n_c=2$, $S_xI_{xx}=S_x$, we have $S_x\otimes A\in\mathcal{G}$, and analogously for $S_y\otimes A$ and $S_z\otimes A$. □

Proposition 3.7. *If $n_c=2$, then all matrices of the type*

$$(I_{xx}-I_{zz})\otimes A,\quad \textit{and}\quad (I_{yy}-I_{zz})\otimes A \tag{19}$$

with $A\in\mathcal{L}$ belong to $\mathcal{G}$.

Proof. We will prove the statement by induction on the depth of the matrix A, by taking $iS_{x,y,z}$ and $i(I_{xx}-I_{zz})$ and $i(I_{yy}-I_{zz})$ as generators of $\mathcal{L}$ (by definition). By Lemma 3.2, we know that all matrices:

$$i(I_{xx}-I_{zz})\otimes S_{x,y,z},\quad i(I_{yy}-I_{zz})\otimes S_{x,y,z}$$

are in $\mathcal{G}$. Moreover from equation (17) we get also that the matrices:

$$i(I_{xx}-I_{zz})\otimes(I_{xx}-I_{zz}),\quad i(I_{xx}-I_{zz})\otimes(I_{yy}-I_{zz}),$$

and

$$i(I_{yy}-I_{zz})\otimes(I_{xx}-I_{zz}),\quad i(I_{yy}-I_{zz})\otimes(I_{yy}-I_{zz}),$$

are in $\mathcal{G}$. Thus the elements (19) are in $\mathcal{G}$, when A is of depth 0.

On the other hand, if the depth of $A\in\mathcal{L}$ is $k>0$, then either $A=[B,iS_{x,y,z}]$ or $A=[B,i(I_{xx}-I_{zz})]$ or $A=[B,i(I_{yy}-I_{zz})]$, for $B\in\mathcal{L}$ of depth $k-1$. In the first case, we have:

$$[(I_{xx}-I_{zz})\otimes B,\mathbf{1}\otimes S_{x,y,z}]=(I_{xx}-I_{zz})\otimes A\in\mathcal{G},$$

and similarly also $(I_{yy}-I_{zz})\otimes A\in\mathcal{G}$. For the second case, we know from Proposition 3.6 that $S_x\otimes B\in\mathcal{G}$, and from Lemma 3.2, $iS_x\otimes(I_{xx}-I_{zz})\in\mathcal{G}$. Thus

$$[S_x\otimes B,S_x\otimes i(I_{xx}-I_{zz})]=S_x^2\otimes[B,i(I_{xx}-I_{zz})]=2(\mathbf{1}+I_{xx})\otimes A\in\mathcal{G}.$$

Using S_z instead of S_x, we get also the matrix $2(\mathbf{1} + I_{zz}) \otimes A$ is in $\mathcal{G}$. Thus also $(I_{xx} - I_{zz}) \otimes A$ is in $\mathcal{G}$. Similarly, we prove that also $(I_{yy} - I_{zz}) \otimes A \in \mathcal{G}$, as desired. □

Proposition 3.8. $(\mathbf{1} + \frac{1}{3}J) \otimes \mathcal{L}$, *belongs to* $\mathcal{G}$.

Proof. Using the last ones of (11) and (13) we know that $\mathcal{G}$ contains $(1 + \frac{1}{3}J) \otimes iS_{x,y,z}$. Using the second one of (17) we also know that $\mathcal{G}$ contains $(1 + \frac{1}{3}J) \otimes \mathcal{I}_0$. Therefore, for every generator of $\mathcal{L}$, A, $(1 + \frac{1}{3}J) \otimes A$ belongs to $\mathcal{G}$. Now for two elements of $\mathcal{L}$, A and B, we have that

$$[(\mathbf{1} + \frac{1}{3}J) \otimes A, (\mathbf{1} + \frac{1}{3}J) \otimes B] = (\mathbf{1} + \frac{1}{3}J)^2 \otimes [A, B] = \frac{4}{3}(\mathbf{1} + \frac{1}{3}J) \otimes [A, B],$$

since a direct calculation gives $(\mathbf{1} + \frac{1}{3}J)^2 = \frac{4}{3}(\mathbf{1} + \frac{1}{3}J)$. Therefore $(1 + \frac{1}{3}J) \otimes A$ is in $\mathcal{G}$ whether A is a generator of $\mathcal{L}$ or it is a Lie bracket of two elements of $\mathcal{L}$. This implies that it is in $\mathcal{G}$ for any A in $\mathcal{L}$. □

The following theorem summarizes the spaces included in $\mathcal{G}$ which we have identified so far for $n_c = 2$.

Theorem 3. *Assume* $n_c = 2$. *Then the dynamical Lie algebra* $\mathcal{G}$ *contains the following subspaces:*

i)

$$i\mathcal{L} \otimes \mathcal{L} \tag{20}$$

ii)

$$(\mathbf{1} + \frac{1}{3}J) \otimes \mathcal{L} \tag{21}$$

iii)

$$\mathcal{L} \otimes \left(\mathbf{1} + \frac{2}{3n_p}J\right) \tag{22}$$

iv) $\mathcal{A}^C$ *and* $\mathcal{A}^P$ *from (11).*

Proof. The subspace in (20) comes from (18) of Proposition 3.6 and (19) of Proposition 3.7 by taking Lie brackets of the elements in (19) with $iS_{x,y,z} \otimes \mathbf{1}$ (which are in (18)) to obtain the rest of $\mathcal{I}_0 \otimes \mathcal{L}$. For the subspace in (22), recalling that in the case $n_c = 2$, $\mathcal{L} = \mathcal{S} \oplus i\mathcal{I}_0$, the part in (22) with $\mathcal{S}$ on the right comes from (18). The subspace $i\mathcal{I}_0 \otimes (1 + \frac{2}{3n_p}J)$ can be obtained as follows: By induction on n_p, we have

$$(S^P_{x,y,z})^2 = n_p\mathbf{1} + 2I^P_{xx,yy,zz}. \tag{23}$$

Take for instance S_x for $n_p = n$ which we denote by $S_{x,n}$. We have $S_{x,n} = S_{x,n-1} \otimes \mathbf{1} + \mathbf{1} \otimes \sigma_x$, and

$$S_{x,n}^2 = (S_{x,n-1} \otimes \mathbf{1} + \mathbf{1} \otimes \sigma_x)^2 = S_{x,n-1}^2 \otimes \mathbf{1} + S_{x,n-1} \otimes \sigma_x + S_{x,n-1} \otimes \sigma_x + \mathbf{1}.$$

Using the inductive assumption on the first term we have

$$S_{x,n}^2 = (n-1)\mathbf{1} + 2I_{x,x(n-1)} \otimes \mathbf{1} + 2S_{x,n-1} \otimes \sigma_x + \mathbf{1} = n\mathbf{1} + 2I_{x,x(n)},$$

since we have collected in $I_{x,x(n)}$ the terms containing pairs (σ_x, σ_x) in the first $n-1$ terms, which are in $I_{x,x(n-1)} \otimes \mathbf{1}$, and the terms displaying σ_x in the last factor, which are in $2S_{x,n-1} \otimes \sigma_x$. Summing (23) for x, y and z, we obtain

$$\frac{1}{3n_p}((S_x^P)^2 + (S_y^P)^2 + (S_z^P)^2) = \mathbf{1} + \frac{2}{3n_p}J. \tag{24}$$

Now using (18) and $\mathcal{D}_2$ in (13) with $A \in \mathcal{S}$ and $B \in i\mathcal{I}$, which are in $\mathcal{G}$, we have $[A \otimes S_x, B \otimes S_x] = [A, B] \otimes S_x^2$ and analogously for y and z. Summing them all and using (24), we have that in $\mathcal{G}$ we also have $[A, B] \otimes (\mathbf{1} + \frac{2}{3n_p}J)$, and using (5) of Lemma 2.2, we obtain the space $i\mathcal{I}_0 \otimes \left(\mathbf{1} + \frac{2}{3n_p}J\right)$ to complete (22). □

3.3.1. *Case $n_p = 2$*

Theorem 4. *Assume $n_c = 2$ and $n_p = 2$. Then the dynamical Lie algebra is the direct sum of the subspaces (20), (21), (22), $\mathcal{A}^C$, and $\mathcal{A}^P$, that is, of all subspaces listed in Theorem 3.*

Proof. For $n_p = 2$, the subspaces (20), (21), (22), $\mathcal{A}^C$, and $\mathcal{A}^P$, listed in Theorem 3, summarize as

$$i\mathcal{L} \otimes \mathcal{L}, \quad (\mathbf{1} + \frac{1}{3}J) \otimes \mathcal{L}, \quad \mathcal{L} \otimes (\mathbf{1} + \frac{1}{3}J), \quad \mathbf{1} \otimes \mathcal{S}, \quad \mathcal{S} \otimes \mathbf{1}. \tag{25}$$

Since these spaces contain the generators of the dynamical Lie algebra $\mathcal{L}$, it is enough to prove that their direct sum is closed under commutation. Denote the direct sum of the first three spaces in (25) as $\tilde{\mathcal{L}}$, so that we have to show that $\bar{\mathcal{L}} := \tilde{\mathcal{L}} \oplus \mathcal{A}^C \oplus \mathcal{A}^P$ is closed under commutation. It is obvious that $[\mathcal{A}^C, \mathcal{A}^C]$, $[\mathcal{A}^C, \mathcal{A}^P]$, $[\mathcal{A}^P, \mathcal{A}^P]$, $[\tilde{\mathcal{L}}, \mathcal{A}^C]$, and $[\tilde{\mathcal{L}}, \mathcal{A}^P]$ are all in $\bar{\mathcal{L}}$. Therefore, we only have to show that $[\tilde{\mathcal{L}}, \tilde{\mathcal{L}}] \subseteq \bar{\mathcal{L}}$. To this aim, it is useful to introduce the spaces $\mathcal{O}_1 := (\mathbf{1} - J) \otimes i\mathcal{I}_0$, $\mathcal{O}_2 := i\mathcal{I}_0 \otimes (\mathbf{1} - J)$, so that $\mathcal{O}_1 \oplus \mathcal{O}_2$, is the orthogonal complement of $\tilde{\mathcal{L}} \oplus \mathcal{A}^C \oplus \mathcal{A}^P$ in $\hat{\mathcal{L}}$. Using Lemma 2.6 and $J^2 = 3\mathbf{1}_4 - 2J$, one can verify that the first three subspaces in (25) commute with $\mathcal{O}_1$ and $\mathcal{O}_2$. Therefore, the commutator of any two elements, according to Lemma 2.1 is orthogonal to $\mathcal{O}_1$ and $\mathcal{O}_2$, and therefore it belongs to $\bar{\mathcal{L}}$. □

Notice that in this case $\mathcal{G}$ is a *proper* subalgebra of $\hat{\mathcal{L}}$.

3.3.2. Case $n_p > 2$

Theorem 5. *Assume $n_c = 2$ and $n_p > 2$ then*

$$\mathcal{G} = \left(i\mathcal{L} \otimes \mathcal{L}^G\right) \oplus \left(\left(\mathbf{1} + \frac{1}{3}J\right) \otimes \mathcal{L}\right) \oplus (\mathbf{1} \otimes \mathcal{S}) \tag{26}$$

Proof. First, we see that the right hand side is included in $\mathcal{G}$. The last two terms of the direct sum are in $\mathcal{A}^P$ in (11) and in (21). Moreover consider A, an arbitrary element of $\mathcal{S}$, and B, an arbitrary element of $i\mathcal{I}_0$. Then $A \otimes S_x \in i\mathcal{L} \otimes \mathcal{L} \subseteq \mathcal{G}$ because of (20) and $B \otimes I_{xxx} \in i\mathcal{L} \otimes \mathcal{L} \subseteq \mathcal{G}$ because of (20). We calculate

$$[A \otimes S_x, B \otimes I_{xxx}] = [A, B] \otimes S_x I_{xxx}.$$

$[A, B]$ can be an arbitrary element of $i\mathcal{I}_0$ according to (5) of Lemma 2.2, while $S_x I_{xxx}$ is a linear combination with nonzero coefficients of I_{xx} and (if $n_p \geq 4$) I_{xxxx}. Since $iI_{xxxx} \in \mathcal{L}$, $[A, B] \otimes I_{xxxx} \in i\mathcal{L} \otimes \mathcal{L}$ which is already in $\mathcal{G}$ because of (20). Therefore $[A, B] \otimes I_{xx} \in \mathcal{G}$. Repeating this calculation with x replaced by y or z and summing all the terms, we obtain that $i\mathcal{I}_0 \otimes J \in \mathcal{G}$. We also have $i\mathcal{I}_0 \otimes \mathbf{1}$ because of (22), $\mathcal{S} \otimes \mathbf{1}$ because of (11), $\mathcal{S} \otimes J$ because of (22) and $i\mathcal{L} \otimes \mathcal{L}$ because of (20). These together give $i\mathcal{L} \otimes \mathcal{L}^G$.

To show the fact that $\mathcal{G}$ is included in the right hand side we notice that all the generators of $\mathcal{G}$ are in the right hand side of (26). Moreover we can check the commutations of the subspaces in (26). We report only the checks that are not immediate. We have

$$[i\mathcal{L} \otimes \mathcal{L}^G, i\mathcal{L} \otimes \mathcal{L}^G] = [\mathcal{L}, \mathcal{L}] \otimes \{\mathcal{L}^G, \mathcal{L}^G\} + \{\mathcal{L}, \mathcal{L}\} \otimes [\mathcal{L}^G, \mathcal{L}^G] \subseteq i\mathcal{L} \otimes \mathcal{L}^G + \{\mathcal{L}, \mathcal{L}\} \otimes \mathcal{L}.$$

In the last term in the right hand side $\{\mathcal{L}, \mathcal{L}\}$ must be a linear combination of $\mathbf{1} + \frac{1}{3}J$ and elements in $i\mathcal{L}$ because it is in $i\mathcal{L}^G$ and orthogonal to $\mathbf{1} - J$ because of Lemma 2.6. In fact, for A and B in $\mathcal{L}$, we have $Tr((\mathbf{1} - J)(AB + BA)) = Tr(AB + BA - AB - BA) = 0$. Therefore these commutators are in the right hand side of (26).

$$[i\mathcal{L} \otimes \mathcal{L}^G, (\mathbf{1} + \frac{1}{3}J) \otimes \mathcal{L}] = \{\mathcal{L}, (\mathbf{1} + \frac{1}{3}J)\} \otimes [\mathcal{L}^G, \mathcal{L}] + [\mathcal{L}, (\mathbf{1} + \frac{1}{3}J)] \otimes \{\mathcal{L}^G, \mathcal{L}\}.$$

The last term is zero because of Lemma 2.4 while the first term is in $i\mathcal{L} \otimes \mathcal{L}$ because of Lemma 2.6. Moreover

$$\left[(\mathbf{1} + \frac{1}{3}J) \otimes \mathcal{L}, (\mathbf{1} + \frac{1}{3}J) \otimes \mathcal{L}\right] \subseteq (\mathbf{1} + \frac{1}{3}J)^2 \otimes \mathcal{L} = \frac{4}{3}(\mathbf{1} + \frac{1}{3}J) \otimes \mathcal{L}. \quad \square$$

We remark that $\mathcal{G}$ in (26) is always a *proper* subalgebra of $\hat{\mathcal{L}}$ in (12). In fact, if $\mathcal{C}$ is a subspace in $\mathcal{L}$ orthogonal to $\mathcal{S}$, the subspace in $\hat{\mathcal{L}}$, $(\mathbf{1} - J) \otimes \mathcal{C}$ belongs to $\hat{\mathcal{L}}$ but it is orthogonal to $\mathcal{G}$ in (26). Nevertheless, we will see in the next section that subspace controllability is verified in all cases considered in this paper.

4. Subspace controllability

In general terms, if a system of the form (1) admits a discrete group of symmetries $\hat{G}$, i.e., a group $\hat{G}$ such that $[A, P] = 0$, $[B_j, P] = 0$, $\forall P \in \hat{G}$, the maximal Lie subalgebra of $u(n)$ which commutes with $\hat{G}$, acts on certain invariant subspaces $\mathcal{H}_j$ of the Hilbert space $\mathcal{H}$ as $u(\dim(\mathcal{H}_j))$. Each of such subspaces is an irreducible representation of $\mathcal{L}^{\hat{G}}$ (cf., [8] Theorem 4). In an appropriate basis of $\mathcal{H}$, therefore, such a maximal Lie algebra can be written in block diagonal form, where each block can take values in $u(\dim(\mathcal{H}_j))$. The dynamical Lie algebra associated with a system having $\hat{G}$ as a group of symmetries also displays a block diagonal form in the same basis although not necessarily equal to the full maximal Lie algebra. In the preferred basis however one can study the action of the dynamical Lie algebra on each subspace and determine subspace controllability. This is the plan we follow here.

A method to find the desired basis was described in [8] and it uses the so-called *Generalized Young Symmetrizers (GYS)* where the word 'Generalized' refers to the fact that, in the case where the group $\hat{G}$ is the symmetry group, they reduce to the classical Young symmetrizers of group representation theory as described for instance in [17]. More precisely, consider the representation of $\hat{G}$ on $\mathcal{H}$ and the *group algebra* of $\hat{G}$ (i.e., the algebra over the complex field generated by a basis of $\hat{G}$), $C[\hat{G}]$. Then the GYS are elements of $C[\hat{G}]$, and operators on $\mathcal{H}$, Π_j satisfying C) (*Completeness*): $\sum_j \Pi_j = \mathbf{1}$; O) (*Orthogonality*): $\Pi_j \Pi_k = \delta_{j,k} \Pi_j$, where $\delta_{j,k}$ is the Kronecker delta; P) (*Primitivity*): $\Pi_j g \Pi_j = \lambda_g \Pi_j$, where λ_g is a scalar which depends only on g (and not on j) H) (*Hermiticity*): For every j, $\Pi_j^\dagger = \Pi_j$. If the GYS are known for a given group $\hat{G}$ on a Hilbert space $\mathcal{H}$, then the images of the various $\Pi_j : \mathcal{H} \to \mathcal{H}$ give the subspace decomposition of $\mathcal{H}$ which block diagonalizes the maximal Lie algebra in $u(n)$ commuting with $\hat{G}$. In the cases where $\hat{G}$ is the symmetric group S_n over n objects, the (generalized) Young symmetrizers can be found using the classical method of Young tableaux (see, e.g., [17]) modified in references [3] [13] to meet the Orthogonality and Hermiticity requirements. A method is given in [8] to compute the GYS in the case where $\hat{G}$ is Abelian. However, the calculation of GYS for general discrete groups is in general an open problem. We observe that if $\mathcal{H} : \mathcal{H}_C \otimes \mathcal{H}_P$ the tensor product of two Hilbert spaces $\mathcal{H}_C$, $\mathcal{H}_P$, as in bipartite quantum systems, and $\hat{G}$ is the product of two groups $\hat{G} := \hat{G}_C \otimes \hat{G}_P$, with $\hat{G}_{C(P)}$ acting on $\mathcal{H}_{C(P)}$, then the GYS can be found as tensor products of GYS on $\mathcal{H}_{C(P)}$ for $\hat{G}_{C(P)}$, $\Pi_j^C \otimes \Pi_k^P$. It is indeed readily verified that if $\{\Pi_j^C\}$ and $\{\Pi_k^P\}$ satisfy the requirements (C,O,P,H) above on $\mathcal{H}_C$ and $\mathcal{H}_P$, respectively, then $\{\Pi_j^C \otimes \Pi_k^P\}$ satisfy the same requirements (C,O,P,H) on $\mathcal{H}_C \otimes \mathcal{H}_P$. The invariant subspaces are $\mathcal{H}_{j,k} := (Im\, \Pi_j^C) \otimes (Im\, \Pi_k^C)$ and, in this basis, the (maximal) invariant Lie algebra takes the corresponding block diagonal form.

For the systems treated in this paper, the symmetry groups $\hat{G}_C$ and $\hat{G}_P$ are the symmetric groups on n_c and n_p objects, S_{n_c} and S_{n_p}, respectively. The decomposition is obtained using the GYS of [3], [13], [17]. Let G be now the symmetric group on n objects and, as we have done before, denote by $\mathcal{L}^G$ the maximal Lie subalgebra of $u(n)$ which commutes with G. Consider the matrix J defined in Lemma 2.4 and Corollary 2.5

in the basis determined by the GYS. In this basis, the elements of $\mathcal{L}^G$ are block diagonal and every block can be an *arbitrary matrix* in $u(m)$ for appropriate m (cf. Theorem 2 in [8]). Since each block of the matrices in $\mathcal{L}^G$ can be an arbitrary skew-Hermitian matrix of appropriate dimensions, iJ is also a block diagonal matrix, i.e.,

$$iJ := \begin{bmatrix} iJ_1 & & \\ & \ddots & \\ & & iJ_d \end{bmatrix},$$

with iJ_k, $k = 1, ..., d$ commuting with the corresponding block of the matrices in $\mathcal{L}^G$. Since such a block defines an *irreducible representation* of $u(m)$ for appropriate dimensions m, it follows from Schür's Lemma (see, e.g., [10]) that all iJ_k are scalar matrices. Consider now the matrices in $\mathcal{L}$ and $\mathcal{L}^G$ and their restrictions to one of the subspaces $Im\Pi_k$, of dimensions m_k. A basis for $\mathcal{L}^G$ restricted to $Im\Pi_k$ is given by a basis of $u(m_k)$ while a basis of $\mathcal{L}$ contains at least a basis of $su(m_k)$ since the restriction of $\mathcal{L}$ to $Im\Pi_k$ differs by $u(m_k)$ at most by multiples of the identity. This is due to Proposition 2.3, along with the fact, seen above, that iJ acts as a scalar matrix on $Im\Pi_k$.

We are now ready to conclude subspace controllability for all the situations treated in this paper. Consider first the **case** $n_c = 1$**, and** $n_p \geq 1$, for which we have proved in Theorem 2 that the dynamical Lie algebra is $\hat{\mathcal{L}}$ in (12). The only GYS on $\mathcal{H}^C$ is the identity, and the only invariant subspaces for the whole system are $\mathcal{H}^C \otimes \Pi_k\mathcal{H}^P$, where the Π_k are the GYS's for the system P. A basis of $\mathcal{G} = \hat{\mathcal{L}}$ is given by $\{\sigma_{x,y,z} \otimes \mathcal{B}^L, \sigma_{x,y,z} \otimes \{i\mathbf{1}, iJ\}, \mathbf{1} \otimes \mathcal{B}^L\}$, where with $\mathcal{B}^L$ we have denoted a basis of $\mathcal{L}$. Since, as we have seen above, $\mathcal{L}$ acts on $\Pi_k\mathcal{H}^P$ as $u(m_k)$, $m_k := \dim(\Pi_k\mathcal{H}^P)$, except possibly for multiples of the identity, a basis for the restriction of $\mathcal{G}$ to $\mathcal{H}^C \otimes \Pi_k\mathcal{H}^P$, contains $\sigma_{x,y,z} \otimes \mathcal{U}_k$, $i\sigma_{x,y,z} \otimes \mathbf{1}$ and $\mathbf{1} \otimes \mathcal{U}_k$, where $\mathcal{U}_k$ is a basis of $su(m_k)$. Therefore it contains a basis of $su(2m_k)$ and therefore controllability is verified. Consider now the **case** $n_c = 2$**,** $n_p = 2$, where the dynamical Lie algebra is described by Theorem 4. If $\mathcal{B}^L$ is a basis of $\mathcal{L}$, a basis for $\mathcal{G}$ is given by $i\mathcal{B}^L \otimes \mathcal{B}^L$, $(\mathbf{1} + \frac{1}{3}J) \otimes \mathcal{B}^L$, $\mathcal{B}^L \otimes (\mathbf{1} + \frac{1}{3}J)$, $\mathbf{1} \otimes i\sigma_{x,y,z}$, $i\sigma_{x,y,z} \otimes \mathbf{1}$. Consider two GYS, Π_j^C and Π_k^P, and the invariant space $\Pi_j^C\mathcal{H}^C \otimes \Pi_j^P\mathcal{H}^P$ with dimensions $m_j \times m_k$, $m_j := \dim(\Pi_j^C\mathcal{H}^C)$, $m_k := \dim(\Pi_k^C\mathcal{H}^k)$. A basis for the restriction of $\mathcal{G}$ to $\Pi_j^C\mathcal{H}^C \otimes \Pi_j^P\mathcal{H}^P$ contains $i\mathcal{U}_j \otimes \mathcal{U}_k$, $\mathbf{1} \otimes \mathcal{U}_k$, $\mathcal{U}_j \otimes \mathbf{1}$, and therefore it contains a basis of $su(m_j m_k)$. Analogously, consider the **case** $n_c = 2$**,** $n_p > 2$. A basis for the dynamical Lie algebra $\mathcal{G}$ described in Theorem 5 is, with the above notation, $\{i\mathcal{B}^L \otimes \mathcal{B}^L, \mathcal{B}^L \otimes \{1, J\}, (\mathbf{1} + \frac{1}{3}J) \otimes \mathcal{B}^L, \mathbf{1} \otimes iS_{x,y,z}\}$ whose restriction to $\Pi_j^C\mathcal{H}^C \otimes Pi_j^P\mathcal{H}^P$ contains $i\mathcal{U}_j \otimes \mathcal{U}_k$, $\mathcal{U}_j \otimes \mathbf{1}$, $\mathbf{1} \otimes \mathcal{U}_k$, and therefore $su(m_j m_k)$. We have therefore with the following theorem.

Theorem 6. *The system (1) with one or two central spins ($n_c = 1$ or $n_c = 2$) with any number $n_p \geq n_c$ of surrounding spins, simultaneously controlled, is subspace controllable.*

Example 4.1. To illustrate some of the concepts and procedures described above, we consider the system of one central spin $n_c = 1$ along with $n_p = 3$ surrounding spins.

The symmetric group on the central spin is trivial being made up of just the identity. There is a single GYS given by the identity. For the symmetric group S_3 on the P part of the space, we obtain the GYS using the method of [3], [13], [17], based on the Young tableaux. We refer to these references for details on the method. For $n = 3$ there are three possible *partitions* of n and therefore three possible *Young diagram* (also called *Young shapes*). Recall that a partition of an integer n is a sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$, with $\lambda_1 + \lambda_2 + \cdots + \lambda_d = n$ and the corresponding Young diagram is made up of boxes arranged in rows of length $\lambda_1, \lambda_2, ..., \lambda_d$. Therefore for $n = 3$, we have the partitions (3), $(2, 1)$, $(1, 1, 1)$ which correspond to the Young diagrams

$$\begin{array}{|c|c|c|}\hline & & \\ \hline\end{array}, \qquad \begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}, \qquad \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline\end{array}, \tag{27}$$

respectively. To each Young diagram, there corresponds a certain number of *Standard Young Tableaux* obtained by filling the boxes of the Young diagram with the numbers 1 through n (3 in this case) so that they appear in strictly increasing order in the rows and in the columns. The following are the possible standard Young tableaux corresponding to the Young diagrams in (27). In particular, the first one corresponds to the first diagram in (27), the second and third one correspond to the second one in (27) and the fourth one corresponds to the third one in (27)

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}, \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}, \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}, \qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}. \tag{28}$$

To each tableaux there corresponds a GYS whose image is an invariant subspace for the Lie algebra representation. We refer to [8] for a summary of the procedure to obtain such GYS's. In our case the GYS corresponding to the first diagram in (28) has 4-dimensional image, the ones corresponding to the second and third have two-dimensional images and the one corresponding to the last one has zero dimensional image. Therefore the invariant subspaces for the system with one central spin and $n_p = 3$ surrounding spin, simultaneously controlled, have dimensions 2×4, 2×2 and 2×2.

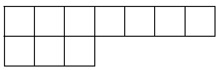
We conclude the section by discussing in general the dimension of the invariant (controllable) subspaces and how it increases with n_p. We recall (see, e.g., [8]) that there is an explicit general formula to obtain the dimension of the image of a GYS, Π_T, corresponding to a Young tableaux T. Such formula specializes to our case (where the dimension of the underlying subspace is 2) as

$$\dim(\mathtt{Im} P_T) = \frac{\prod_{l=1}^{r} \prod_{k=1}^{\lambda_l} (2 - l + k)}{\mathtt{Hook}(T)}. \tag{29}$$

Here r is the number of rows in the Young diagram associated with T, λ_l is the number of boxes in the l-th row, and $\mathtt{Hook}(T)$ is the *Hook length* of the Young diagram associated with T. It is calculated by considering, for each box, the number of boxes directly to the right + the number of boxes directly below + 1 and then taking the product of all the numbers obtained this way. Using formula (29) it is possible to derive, for each n_p, the dimensions of all invariant subspaces. Fix $n = n_p$. From formula (29), Young diagrams with more than two rows give zero dimensional spaces. So we have to consider only Young diagrams with one or two rows. There is only one diagram with one row, T_1, i.e., the diagram containing n_p boxes, and in (29) $r = 1$ and $\lambda_1 = n$. For this diagram, the Hook length is $n!$. We thus have:

$$\dim(\mathtt{Im}P_{T_1}) = \frac{\prod_{k=1}^{n}(1+k)}{n!} = n+1.$$

For diagrams with two rows, the possible partitions are of the type $\lambda_1 = n - k$ and $\lambda_2 = k$, with k integer and $k \leq \frac{n}{2}$. For example



is the Young diagram for the case $n = 10$ and $k = 3$. For the diagram corresponding to a given k, T_2^k, the Hook length is

$$\mathtt{Hook}(T_2^k) = (n+1-k)(n-k)\cdots(n-2k+2)\cdot(n-2k)!\cdot k!.$$

Thus we have

$$\dim(\mathtt{Im}P_{T_2^k}) = \frac{\prod_{j=1}^{n-k}(1+j)\prod_{j=1}^{k} j}{(n+1-k)(n-k)\cdots(n-2k+2)\cdot(n-2k)!\cdot k!} = n-2k+1.$$

So, for this central spin model, the dimension of the invariant subspaces grows linearly with n. The largest space has dimension $n + 1$. The dimensions of the full invariant subspaces of the model with 1 and 2 central spins are obtained by multiplying the dimensions obtained for $\mathcal{H}^P$ by the dimensions of the invariant subspaces of $\mathcal{H}^C$, which, with the same method of Young tableaux, can be shown to be 2 in the case $n_c = 1$ and 1 or 3 in the case $n_c = 2$. The largest possible dimension is therefore obtained for $n_c = 2$ and it is $3(n_p+1)$. This behavior is different from the one of the system considered in the paper [18], where the dimension of one of the invariant subspaces grows exponentially with the number of spins. This is essentially due to a much larger number of symmetries in our case.

5. Conclusions and generalizations

We have considered spin networks where the spins are arranged in two sets, a set P and a set C, and where the Ising interaction is exclusively between each spin of the set C and each spin of the set P. The model Hamiltonian is symmetric with respect to permutations on the spins in C and the spins in P. We now consider the possibility that the interaction between the spins in C and the spins in P is still symmetric but more general than Ising and-or that there are internal interactions within the set C and P. More specifically we replace the term $S_z^C \otimes S_z^P$ in (10) with the more general

$$H_+ := S_z \otimes S_z + aS_x \otimes S_x + bS_y \otimes S_y + H_C \otimes \mathbf{1} + \mathbf{1} \otimes H_P, \tag{30}$$

for real parameters a and b and H_C (H_P) represents the internal interactions of spins in C (P). We notice that the argument at the end of subsection 2.3 still holds. Therefore the Lie algebras $\mathcal{A}^C$ and $\mathcal{A}^P$ in (11) are still subalgebras of the dynamical Lie algebra. Moreover, a direct verification shows that

$$[[[[iH_+, iS_x \otimes \mathbf{1}], \mathbf{1} \otimes iS_y], \mathbf{1} \otimes iS_y], iS_x \otimes \mathbf{1}] \vDash iS_z \otimes S_z.$$

Therefore $iS_z \otimes S_z$ is still in the dynamical Lie algebra along with $\mathcal{A}^C$ and $\mathcal{A}^P$ and therefore the resulting Lie algebra includes the dynamical Lie algebra for the case considered in the above sections. It follows that subspace controllability is verified in these cases as well.

The assumption that all the coupling constants between elements in the set C and elements in the set P are equal is an idealization. However, more realistic systems where such couplings are nearly equal could be theoretically controllable [1] but require very high amplitude or long time control. Therefore they can in fact be considered uncontrollable for all practical purposes and satisfactorily approximated with the models we have considered. Small perturbations of the couplings appear to preserve the subspace controllability property we have proved and this generalizes the known fact that complete controllability is a property of quantum systems robust to small perturbations [4], [14]. For example, assume three spins in P and one in C. If there is in-homogeneity of the interaction strengths between one of the spins in P and the spin in C, then the Hamiltonian $\sigma_z \otimes S_z$ of (10) is replaced by

$$H_\epsilon = \sigma_z \otimes S_z + \epsilon \sigma_z \otimes \sigma_z \otimes \mathbf{1}_2 \otimes \mathbf{1}_2.$$

By taking the commutator $[iH_\epsilon, \mathbf{1} \otimes S_x]$, we obtain $i\sigma_z \otimes S_y + i\epsilon\sigma_z \otimes \sigma_y \otimes \mathbf{1} \otimes \mathbf{1}$, and calculating the commutator of this last one with iH_ϵ, we obtain $i\mathbf{1} \otimes \sigma_x \otimes \mathbf{1} \otimes \mathbf{1}$. Analogously we can obtain $i\mathbf{1} \otimes \sigma_y \otimes \mathbf{1} \otimes \mathbf{1}$, and therefore also $i\mathbf{1} \otimes \sigma_z \otimes \mathbf{1} \otimes \mathbf{1}$. By, repeated Lie brackets with iH_ϵ we can separate the term $\sigma_z \otimes \sigma_z \otimes \mathbf{1} \otimes \mathbf{1}$ from $\sigma_z \otimes \mathbf{1} \otimes (\sigma_z \otimes \mathbf{1} + \mathbf{1} \otimes \sigma_z)$. Therefore, once again, the dynamical Lie algebra contains the Lie algebra calculated in the fully symmetric Ising case and subspace controllability is preserved.

The calculation of the dynamical Lie algebra of a quantum system is the method of choice to study its controllability properties [7]. However such direct calculation might be difficult in cases of very large systems and in particular networks of spins where the dimension of the underlying full Hilbert space grows exponentially with the number of particles. For this reason, it is important to device methods to assess controllability from the topology of the network and its possible symmetries. Symmetries, in particular, prevent full controllability and determine a number of invariant subspaces on which the system evolves. In this paper we have considered a configuration of indistinguishable spins divided into sets interacting with each other. This is the first intermediate case between two extremes cases of all indistinguishable and all distinguishable spins previously treated in the literature. The full symmetric group acts on each set of spins without modifying the Hamiltonian which describes the dynamics. A common electromagnetic field is used for control. We have computed the dynamical Lie algebra and proved that such a system is subspace controllable, that is full controllability is verified on each invariant subsystem. Quantum evolution is a parallel of the evolution of various subsystems and we can use one of them to perform various tasks of, for instance, quantum computation and-or simulation.

Declaration of competing interest

The authors declare no competing interests.

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