

SUBSPACE CONTROLLABILITY OF QUANTUM ISING SPIN NETWORKS WITH A CENTRAL SPIN

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Abstract—We consider a class of spin networks where each spin in a certain set interacts via Ising coupling with a single central spin. This is a common situation for instance in NV centers in diamonds. Due to the permutation symmetries of the network, the system is not globally controllable but it displays invariant subspaces of the underlying Hilbert space. The system is said to be *subspace controllable* if it is controllable on each of these subspaces. We characterize the given invariant subspaces and the dynamical Lie algebra of this class of systems and prove subspace controllability.

Keywords: Controllability of Quantum Systems, Spin Networks, Symmetry Groups, Dynamical Decomposition, Subspace Controllability.

I. INTRODUCTION

Controllability of finite dimensional quantum systems, described by a Schrödinger equation of the form

$$|\dot{\psi}\rangle = (A + \sum_j B_j u_j(t))|\psi\rangle, \quad (1)$$

is usually assessed by computing the Lie algebra $\mathcal{G}$ generated by the matrices in $u(N)$, A and B_j (see, e.g., [5], [11], [15]). The Lie algebra $\mathcal{G}$ is called the *dynamical Lie algebra*. Here $u_j = u_j(t)$ are the (semiclassical) control electromagnetic fields and $|\psi\rangle$ is the quantum mechanical state varying in a Hilbert space $\mathcal{H}$. If $e^{\mathcal{G}}$ denotes the connected component containing the identity of the Lie group associated with $\mathcal{G}$, then the set of states reachable from $|\psi_0\rangle$ by changing the control fields is (dense in) $\{|\psi\rangle := X|\psi_0\rangle \in \mathcal{H} \mid X \in e^{\mathcal{G}}\}$. In particular if $\mathcal{G} = u(N)$ or $\mathcal{G} = su(N)$ the system is said to be (*completely*) *controllable* and every unitary operation, or special unitary operation in the $su(n)$ case, can be performed on the quantum state. This is important in quantum information processing [14] when we want to ensure that every quantum operation can be obtained for a certain physical experiment (universal quantum computation). Although controllability is a generic property (see, e.g., [13]), often symmetries of the physical system prevent it and the dynamical Lie algebra $\mathcal{G}$ is a *proper* Lie subalgebra of $su(N)$. In this case the given representation of the Lie algebra $\mathcal{G}$, splits into its irreducible components which act on invariant subspaces of the full Hilbert space $\mathcal{H}$ on

which the system state $|\psi\rangle$ is defined. It is therefore of interest to study whether, on each subspace, controllability is verified, so that, in particular, one can perform universal quantum computation and/or generate interesting entangled states on a smaller portion of the Hilbert space (see, e.g., [8], [10]). This situation has recently been studied in detail for networks of particles with spin in the papers [18], [19]. In particular in [19] various topologies of the spin network were considered for various possible interactions among the spins and results were proven concerning the controllability of the *first excitation space*, that is, the invariant subspace of the network of states of the form $\sum_j a_j |000 \cdots 00100 \cdots 000\rangle$, i.e., superpositions of states where only one spin is in the excited state. In [18], only *chains* with next neighbor interactions were considered (instead of general networks) but comprehensive controllability results were given on all the invariant subspaces of this type of systems. In both these papers, the control affects *only one* of the spins in the network, which may be placed in various places in the network. One interesting achievement of the paper [18] is to show that the dimension of one of the invariant subspaces of the system still grows *exponentially* with the number of spins. Given that subspace controllability is proven on this subspace, this opens the door to achieving universal quantum computation with the given set-up, in particular with physical control on a single spin.

The present paper is motivated by experimental situations where control on a single spin particle is not possible and all the spins of the network are controlled simultaneously. We want to study the structure of the dynamical Lie algebra and subspace controllability in this situation. We shall consider the case where the spins of the network are arranged around a *central spin* C and interact in the same (Ising) way with such a spin C but do not interact with each other. This type of arrangement has been considered recently for instance in $N - V$ centers in diamonds [7] [16], where a central (electronic) spin (of type C), interacts (via Ising interaction) with a bath of surrounding (nuclear) spins as in Figure 1. Since the electrons are easier to manipulate than the nuclei (which however have longer coherence time) such an experiment is used in [16] to indirectly control the nuclei via the interaction with the central spin. Such a situation can be in principle extended to the case where there are more than one central spins and some of our results concern the more general situation where the network is composed of two types of spins, C and P , with identical spins within the set C and within the set P . The situation however becomes more complicated as soon as we consider two central spins

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and, because of space limitation, we restrict ourselves to the case of one central spin. The more general case is presented in [2]. Systems of this type admit *symmetries*. In particular,

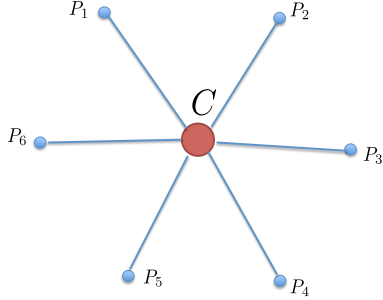


Fig. 1. Schematic representation of a spin network with a central spin.

by permuting the spins in the set C and/or the spins in the group P , the Hamiltonians describing the dynamics of the system as in (1) are left unchanged (see next section for details). Then, if n_c is the cardinality of the set C and n_p is the cardinality of the set P , the group of symmetries is the product between the symmetry group on n_c elements, S_{n_c} , and the symmetry group on n_p elements, S_{n_p} . In this context, the results of this paper are the first step towards developing a theory for controllability of spin networks where symmetries are ‘localized’ within certain subsets of the network.

In general if a discrete group G of symmetries is present for a quantum mechanical system, the dynamical Lie algebra $\mathcal{G}$ associated with the system will be a subalgebra of $\mathcal{L}^G$, the largest subalgebra of $u(N)$ (N being the dimension of the system) which commute with G . If $\mathcal{G}$ is equal to $\mathcal{L}^G$, subspace controllability is satisfied for each of the invariant subspaces of the system [6]. However $\mathcal{G}$ might be a *proper* Lie subalgebra of $\mathcal{L}^G$ and subspace controllability will not be satisfied. For the systems we consider in this paper where there is only one central spin (in C), we will see that $\mathcal{G}$ is not exactly equal to $\mathcal{L}^G$. However, this does not affect the subspace controllability of the system for each of its invariant subspaces.

The controllability of spin networks where one can permute the spins arbitrarily (*completely symmetric spin networks*) was studied in [1] expanding upon a study that was started in [4] motivated by [8], [10]. In [6] it was shown how to use *Generalized Young Symmetrizers* for the group G to characterize $\mathcal{L}^G$ in every case, extending some of the results of [1] to higher dimensions. We shall use some of the results of these papers in the following.

The paper is organized as follows. In the next section we set up the notations and the basic definitions so that we can

describe the model we want to treat and the problem we want to consider precisely. The main results are given in section III where we describe the dynamical Lie algebra for Ising networks of spins with a central spin under global control. Subspace controllability will come as a consequence of this in section IV. Some concluding remarks on the given results will be given in section V.

For space reasons we omit some of the more technical proofs, that can be found in [2].

II. PRELIMINARIES

A. Notations, Basic Definitions and Properties

In the following, we will have to compute a basis for a Lie algebra generated by a given set of matrices. In these calculations, it is not important if we obtain a matrix A or a matrix kA with $k \neq 0$. Therefore we shall use the notation $[A, B] \models D$ to indicate that the commutator of A and B ($[A, B] := AB - BA$) is kD for some $k \neq 0$ and therefore D belongs to the Lie algebra that contains A and B . We shall also often use the formula

$$[A \otimes B, C \otimes D] = \frac{1}{2}\{A, B\} \otimes [B, D] + \frac{1}{2}[A, C] \otimes \{B, D\},$$

where $\{A, B\}$ denotes the *anti-commutator* of A and B , i.e., $\{A, B\} := AB + BA$. We will do this routinely without explicitly referring to this formula.

In $u(n)$ we shall use the *inner product* $\langle A, B \rangle := \text{Tr}(AB^\dagger)$. One property of this inner product which will be useful is given by the following:

Lemma 2.1: If A is orthogonal to B and C and commutes with B and C , then it is also orthogonal and commutes with $[B, C]$.

Proof: Commutativity follows from the Jacobi identity. Moreover, $\text{Tr}(A[B, C]) = \text{Tr}(ABC - ACB) = \text{Tr}(BAC - CAB) = \text{Tr}(BAC - BCA) = \text{Tr}(B[A, C]) = 0$. ■

The *Pauli matrices* $\sigma_{(x,y,z)}$ are defined as

$$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y := \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2)$$

If $\mathbf{1}$ denotes the two dimensional identity matrix, the Pauli matrices satisfy

$$\begin{aligned} \sigma_x \sigma_x &= \sigma_y \sigma_y = \sigma_z \sigma_z = \mathbf{1}, \\ \sigma_x \sigma_y &= -i\sigma_z, \quad \sigma_y \sigma_z = -i\sigma_x, \quad \sigma_z \sigma_x = -i\sigma_y, \\ \sigma_y \sigma_x &= i\sigma_z, \quad \sigma_z \sigma_y = i\sigma_x, \quad \sigma_x \sigma_z = i\sigma_y \end{aligned} \quad (3)$$

which give the commutation relations

$$[i\sigma_x, i\sigma_y] = 2i\sigma_z, \quad [i\sigma_y, i\sigma_z] = 2i\sigma_x, \quad [i\sigma_z, i\sigma_x] = 2i\sigma_y. \quad (4)$$

In the most general setting, our model consists of $n = n_c + n_p$ spin $\frac{1}{2}$ particles, with n_c of a type C (for example n_c nuclei) and n_p of the type P (for example n_p electrons). In our conventions, the first n_c positions in a tensor product refer to operators on the spins in the set C , while the following n_p refer to operators on the set P . Our main results on the characterization of the dynamical Lie algebra and subspace controllability will concern the special case where there is a

single central spin in the set C , i.e., $n_c = 1$, but we prefer to start with the more general situation where n_c is arbitrary.

We denote by $S_{(x,y,z)}^{C(P)}$ the sum of $n_{c(p)}$ tensor products $\sum_{j=1}^{n_{c(p)}} \mathbf{1} \otimes \cdots \otimes \sigma_{(x,y,z)} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}$ where the Pauli matrix $\sigma_{(x,y,z)}$ varies among all the possible $n_{c(p)}$ positions. For example, if $n_c = 2$, $S_x^C := \sigma_x \otimes \mathbf{1} + \mathbf{1} \otimes \sigma_x$. When it is not important or it is clear whether we refer to the set C or the set P , we shall simply denote this type of matrices as $S_{(x,y,z)}$. In particular matrices on the left (right) of a tensor product always refer to operators on the set C (P). We notice that $S_{(x,y,z)}$ satisfy the same commutation relations as $\sigma_{(x,y,z)}$ and therefore $iS_{(x,y,z)}$ give a representation of $su(2)$ in the appropriate dimensions. We shall denote the 3-dimensional Lie algebra spanned by $iS_{(x,y,z)}$ with $\mathcal{S}$.

We denote by $I_{(x,y,z)(x,y,z)}^{C(P)}$ matrices which are sum of the tensor products of 2×2 identities, $\mathbf{1}$, except in all possible pairs of positions which are occupied by $\sigma_{(x,y,z)}$ and $\sigma_{(x,y,z)}$. For example, if $n_c = 3$, we have

$$\begin{aligned} I_{xx}^C &:= \sigma_x \otimes \sigma_x \otimes \mathbf{1} + \sigma_x \otimes \mathbf{1} \otimes \sigma_x + \mathbf{1} \otimes \sigma_x \otimes \sigma_x, \\ I_{xy}^C &:= \sigma_x \otimes \sigma_y \otimes \mathbf{1} + \sigma_y \otimes \sigma_x \otimes \mathbf{1} + \sigma_x \otimes \mathbf{1} \otimes \sigma_y + \\ &\quad + \sigma_y \otimes \mathbf{1} \otimes \sigma_x + \mathbf{1} \otimes \sigma_x \otimes \sigma_y + \mathbf{1} \otimes \sigma_y \otimes \sigma_x. \end{aligned}$$

As before, when it is not important or it is clear in the given context, whether we refer to the set C or P , we omit the superscript C or P . $\mathcal{I}^{C(P)}$ denotes the 6-dimensional span of $I_{(x,y,z)(x,y,z)}^{C(P)}$, while $\mathcal{I}_0^{C(P)}$ denotes the 5-dimensional subspace of $\mathcal{I}^{C(P)}$ spanned by $\{I_{xy}^{C(P)}, I_{xz}^{C(P)}, I_{yz}^{C(P)}, I_{xx}^{C(P)} - I_{yy}^{C(P)}, I_{yy}^{C(P)} - I_{zz}^{C(P)}\}$.

Lemma 2.2:

$$[\mathcal{S}, i\mathcal{I}] = [\mathcal{S}, i\mathcal{I}_0] = i\mathcal{I}_0. \quad (5)$$

Furthermore, if $A := iI_{zz}$ or iI_{xx} or iI_{yy} ,

$$[\mathcal{S}, \text{span}\{A\}] \oplus [\mathcal{S}, [\mathcal{S}, \text{span}\{A\}]] = i\mathcal{I}_0. \quad (6)$$

For the proof see [2].

With $\mathcal{L}^G$ we denote the full Lie algebra of matrices in $u(\hat{n})$, which commute with the symmetric group $S_{\hat{n}}$. The dimension of $\mathcal{L}^G$ was calculated in [1] and it is given by $\binom{\hat{n}+3}{\hat{n}}$. With $\mathcal{L}$, we shall denote the Lie algebra generated by $i\{S_x, S_y, S_z, I_{xx} - I_{yy}, I_{yy} - I_{zz}\}$. The following fact was one of the main results of [1].

Theorem 1: Let $\hat{n}$ a certain number of spin $\frac{1}{2}$ particles and I_{zz} , $S_{(x,y,z)}$ matrices of the corresponding dimension $2^{\hat{n}}$. Then $iI_{z,z}$, $iS_{(x,y,z)}$, generate all $\mathcal{L}^G \cap su(2^{\hat{n}})$.

In Theorem 1 I_{zz} models the *Ising interaction* among each pair of spins in a network, while $S_{(x,y,z)}$ models the interaction with the external control magnetic field in the (x, y, z) direction, respectively.

The matrix $J := I_{xx} + I_{yy} + I_{zz}$, which models an *Heisenberg interaction* for each pair of spins, will be important in our description of the dynamical Lie algebra for the systems studied here. The Lie algebra $\mathcal{L}$ above defined is the same as $\mathcal{L}^G \cap su(2^{\hat{n}})$ except for iJ . More precisely, we have the following results (for the proof see [2]):

Proposition 2.3:

$$\mathcal{L}^G \cap su(2^{\hat{n}}) = \mathcal{L} \oplus \text{span}\{iJ\}. \quad (7)$$

We shall also use the following property of the matrix J , for the proof see [2].

Lemma 2.4: The matrix iJ commutes with $\mathcal{L}$.

Using Proposition 2.3, we have

Corollary 2.5: The matrix iJ commutes with $\mathcal{L}^G$.

B. The model

We consider a network of spin $\frac{1}{2}$ particles divided into two sets, C and P . Each spin in the set C interact via Ising interaction with each spin in the set P but there is no (significant) interaction within spins in the set C (P). The system is controlled by a common electro-magnetic field which is arbitrary in the x and y direction. Up to a proportionality factor, the quantum mechanical Hamiltonian of the system can be written as

$$\begin{aligned} H = & S_z^C \otimes S_z^P + u_x(\gamma_C S_x^C \otimes \mathbf{1} + \gamma_P \mathbf{1} \otimes S_x^P) + \\ & + u_y(\gamma_C S_y^C \otimes \mathbf{1} + \gamma_P \mathbf{1} \otimes S_y^P). \end{aligned} \quad (8)$$

Here the term $S_z^C \otimes S_z^P$ models the Ising interaction of each spin of the set C with each spin of the set P . This should not be confused with a term of the form I_{zz} which models Ising interaction between *any pair* of spin in a network. The terms $u_x := u_x(t)$ and $u_y := u_y(t)$ are control electromagnetic fields in the x and y directions. The parameters γ_C and γ_P are (proportional to) the *gyromagnetic ratios* of the spins in set C and set P , respectively. The dimensions of the identity matrices $\mathbf{1}$ in (8) are either 2^{n_c} or 2^{n_p} . The Schrödinger equation for the system takes the form (1) where $A + \sum_j B_j u_j = -iH$ with H in (8).

C. Dynamical Lie algebra and subspace controllability

We want to describe the possible evolutions that can be obtained by changing the controls in (8) and therefore we want to describe the dynamical Lie algebra $\mathcal{G}$ generated by

$$\{iS_z^C \otimes S_z^P, i(\gamma_C S_x^C \otimes \mathbf{1} + \gamma_P \mathbf{1} \otimes S_x^P), i(\gamma_C S_y^C \otimes \mathbf{1} + \gamma_P \mathbf{1} \otimes S_y^P)\}.$$

Once $\mathcal{G}$ is described, its elements will take, in appropriate coordinates, a block diagonal form which describes the *sub-representations* of $\mathcal{G}$. The Hilbert space $\mathcal{H}$ for the quantum state is accordingly decomposed into invariant subspaces. Subspace controllability is verified if, on each subspace, $\mathcal{G}$ acts as $u(m)$ or $su(m)$ where m is the dimension of the given subspace. Our problem is to determine the Lie algebra $\mathcal{G}$ and then find all its sub-representations and prove subspace controllability.

As a preliminary step, we remark that, letting

$$W = [i\gamma_C S_x^C \otimes \mathbf{1} + i\gamma_P \mathbf{1} \otimes S_x^P, i\gamma_C S_y^C \otimes \mathbf{1} + i\gamma_P \mathbf{1} \otimes S_y^P],$$

then

$$[i\gamma_C S_x^C \otimes \mathbf{1} + i\gamma_P \mathbf{1} \otimes S_x^P, W] \models i\gamma_C^3 S_y^C \otimes \mathbf{1} + i\gamma_P^3 \mathbf{1} \otimes S_y^P.$$

Therefore, since the Lie algebra contains $i\gamma_C S_y^C \otimes \mathbf{1} + i\gamma_P \mathbf{1} \otimes S_y^P$ also, assuming $|\gamma_C| \neq |\gamma_P|$, we have that $iS_y^C \otimes \mathbf{1}$ and $i\mathbf{1} \otimes S_y^P$ belong to $\mathcal{G}$. Taking the Lie brackets of $i\gamma_C S_x^C \otimes$

$1 + i\gamma_P \mathbf{1} \otimes S_x^P$ with $iS_y^C \otimes \mathbf{1}$ and $i\mathbf{1} \otimes S_y^P$ we obtain that $iS_z^C \otimes \mathbf{1}$ and $i\mathbf{1} \otimes S_z^P$ are in $\mathcal{G}$, and taking the Lie bracket between $iS_y^C \otimes \mathbf{1}$ ($i\mathbf{1} \otimes S_y^P$) and $iS_z^C \otimes \mathbf{1}$ ($i\mathbf{1} \otimes S_y^P$) we obtain $iS_x^C \otimes \mathbf{1}$ ($i\mathbf{1} \otimes S_x^P$). Therefore $\mathcal{G}$ contains the 3-dimensional subspaces

$$\mathcal{A}^C := \text{span}\{iS_{(x,y,z)}^C \otimes \mathbf{1}\}, \mathcal{A}^P := \text{span}\{i\mathbf{1} \otimes S_{(x,y,z)}^P\}, \quad (9)$$

under the assumption that $|\gamma_C| \neq |\gamma_P|$. We shall assume this to be the case in the following. Therefore the dynamical Lie algebra $\mathcal{G}$ is the Lie algebra generated by $\mathcal{A}^C$, $\mathcal{A}^P$ and $iS_z^C \otimes S_z^P$.

III. DESCRIPTION OF THE DYNAMICAL LIE ALGEBRA

A. Results for general $n_c \geq 1$

Without loss of generality we assume $n_c \leq n_p$. Consider the group $\hat{G}$, $\hat{G} := S_{n_c} \otimes S_{n_p}$, where S_{n_c} is the group of permutation matrices (symmetric group) on the first n_c positions, corresponding to spins of the type C and S_{n_p} is the group of permutation matrices (symmetric group) on the second n_p positions, corresponding to spins of the type P . This is a *group of symmetries* for the system described by the Hamiltonian (8) since for every element $Q_C \otimes Q_P \in S_{n_c} \otimes S_{n_p}$, we have

$$\begin{aligned} [iS_z^C \otimes S_z^P, Q_C \otimes Q_P] &= 0, \\ [i(\gamma_C S_x^C \otimes \mathbf{1} + \gamma_P \mathbf{1} \otimes S_x^P), Q_C \otimes Q_P] &= 0, \\ [i(\gamma_C S_y^C \otimes \mathbf{1} + \gamma_P \mathbf{1} \otimes S_y^P), Q_C \otimes Q_P] &= 0. \end{aligned}$$

The generators of $\mathcal{G}$ all commute with $\hat{G}$ and therefore all of $\mathcal{G}$ commutes with $\hat{G}$. This implies that the dynamical Lie algebra $\mathcal{G}$ must be a Lie subalgebra of the maximal subalgebra $\mathcal{L}^{\hat{G}}$ of $u(2^n)$ which commutes with $\hat{G}$. We have $\mathcal{L}^{\hat{G}} = i\mathcal{L}^{GC} \otimes \mathcal{L}^{GP}$. Here $\mathcal{L}^{GC}$ ($\mathcal{L}^{GP}$) is the Lie subalgebra of $u(2^{n_c})$ ($u(2^{n_p})$) invariant under S_{n_c} (S_{n_p}). Therefore a basis of $\mathcal{L}^{\hat{G}}$ can be obtained by taking tensor products of a basis of $\mathcal{L}^{GC}$ with a basis of $\mathcal{L}^{GP}$ and the dimension of $\mathcal{L}^{\hat{G}}$ is $M(n_c)M(n_p)$, where $M(n) := \binom{n+3}{n}$ (from [1]). In fact, $\mathcal{G}$ is a Lie subalgebra of a slightly smaller Lie algebra. For the proof see [2].

Lemma 3.1: The Lie algebra

$$\hat{\mathcal{L}} = (i\mathcal{L} \otimes \mathcal{L}^G) + (i\mathcal{L}^G \otimes \mathcal{L}), \quad (10)$$

is a super Lie algebra of $\mathcal{G}$.

We shall see that in the case $n_c = 1$, $\mathcal{G} = \hat{\mathcal{L}}$. We now identify certain subspaces of $\hat{\mathcal{L}}$ which belongs to the dynamical Lie algebra $\mathcal{G}$ (for the proof see [2]).

Proposition 3.2: The following vector spaces belong to $\mathcal{G}$:

$$\begin{aligned} \mathcal{B} &:= \text{span}\{iS_{x,y,z}^C \otimes S_{x,y,z}^P\}, \\ \mathcal{D}_1 &:= \text{span}\{iS_{(x,y,z)}^C \otimes I_{(x,y,z)(x,y,z)}^P\}, \\ \mathcal{D}_2 &:= \text{span}\{I_{(x,y,z)(x,y,z)}^C \otimes S_{(x,y,z)}^P\}. \end{aligned} \quad (11)$$

Remark 3.3: Notice that the above subspaces have the dimensions $\dim(\mathcal{B}) = 9$, $\dim(\mathcal{D}_1) = 18$ unless the set P has cardinality 1, in which case $\mathcal{D}_1 = \{0\}$, $\dim(\mathcal{D}_2) = 18$ unless the set C has cardinality 1, in which case $\mathcal{D}_2 = \{0\}$.

B. Dynamical Lie algebra for $n_c = 1$

In the case $n_c = n_p = 1$, $\mathcal{A}^C \oplus \mathcal{A}^P \oplus \mathcal{B}$ which is equal to $su(4)$ is the dynamical Lie algebra $\mathcal{G}$. In this case the system is completely controllable and our analysis terminates here. We shall therefore assume that $n_p > 1$, and therefore $\mathcal{D}_1 \neq \{0\}$ in (11) while $\mathcal{D}_2 = \{0\}$.

Take B in S^P and D in $\mathcal{I}^P$. The Lie bracket of the matrices $S_z \otimes B := \sigma_z \otimes B \in \mathcal{B}$ and $S_z \otimes iD = \sigma_z \otimes iD \in \mathcal{D}_1$ gives

$$[S_z \otimes B, S_z \otimes iD] = S_z^2 \otimes [B, iD] = \mathbf{1} \otimes R, \quad (12)$$

for an arbitrary R in $i\mathcal{I}_0^P$ according to (5) of Lemma 2.2. We have therefore:

Lemma 3.4: If $n_c = 1$, the dynamical Lie algebra $\mathcal{G}$ contains

$$\mathcal{E}_1 := \mathbf{1} \otimes i\mathcal{I}_0^P. \quad (13)$$

We are now ready to characterize the dynamical Lie algebra $\mathcal{G}$ in the case $n_c = 1$.

Theorem 2: If $n_c = 1$ and for any $n_p \geq 2$ the dynamical Lie algebra $\mathcal{G}$ is given by

$$\mathcal{G} := ((\text{span}\{\sigma_{x,y,z}\}) \otimes \mathcal{L}^G) \oplus ((\text{span}\{\mathbf{1}\}) \otimes \mathcal{L}) = \hat{\mathcal{L}}. \quad (14)$$

Proof: Using elements in $\mathcal{E}_1$ and elements of $\mathcal{A}^P$, since $\mathcal{L}$ is, by definition, the Lie algebra generated by $i\mathcal{I}_0$ and $iS_{(x,y,z)}$ we obtain anything in $(\text{span}\{\mathbf{1}\}) \otimes \mathcal{L}$. Now we know from Theorem 1 that iI_{zz}^P , $iS_{(x,y,z)}^P$ and $i\mathbf{1}$ generate all of $\mathcal{L}^{GP}$. Therefore, basis elements of $\mathcal{L}^{GP} \cap su(2^{n_p})$ are obtained by (repeated) Lie brackets of iI_{zz}^P and $iS_{(x,y,z)}^P$. Define the ‘depth’ of a basis element K_1 as the number of Lie brackets to be performed to obtain K_1 . In particular, the generators iI_{zz} , $iS_{(x,y,z)}$ are element of depth zero. We show by induction on the depth of the basis element K_1 that all elements of the form $\sigma_{(x,y,z)} \otimes K_1$ can be obtained. For depth zero, we already have $i\sigma_{(x,y,z)} \otimes S_{(x,y,z)} \in \mathcal{B}$ and $i\sigma_{(x,y,z)} \otimes I_{zz} \in \mathcal{D}_1$, from Proposition 3.2. For depth $d \geq 1$, assume by induction that we have all elements $i\sigma_{(x,y,z)} \otimes K_1$ for K_1 in the basis of $\mathcal{L}^G \cap su(2^{n_p})$, K_1 of depth $d-1$. If $K_2 = [K_1, iS_{(x,y,z)}]$, we can obtain

$$\begin{aligned} [\sigma_{(x,y,z)} \otimes K_1, i\mathbf{1} \otimes S_{x,y,z}] &= \sigma_{(x,y,z)} \otimes [K_1, iS_{(x,y,z)}] \\ &= \sigma_{(x,y,z)} \otimes K_2. \end{aligned}$$

If $K_2 := [K_1, iI_{zz}]$, write

$$iI_{zz} = \frac{1}{3}i(I_{xx} - I_{yy}) - \frac{2}{3}i(I_{xx} - I_{zz}) + \frac{1}{3}iJ, \text{ so that}$$

$$\begin{aligned} K_2 &= \left[K_1, \frac{1}{3}i(I_{xx} - I_{yy}) - \frac{2}{3}i(I_{xx} - I_{zz}) + \frac{1}{3}iJ \right] \\ &= \left[K_1, \frac{1}{3}i(I_{xx} - I_{yy}) - \frac{2}{3}i(I_{xx} - I_{zz}) \right]. \end{aligned}$$

This is true because iJ commutes with $\mathcal{L}^G$ according to Corollary 2.5. This shows that $K_2 \in [K_1, \mathcal{L}]$ and since we have $\sigma_{(x,y,z)} \otimes K_1 \in \mathcal{G}$ (by inductive assumption) and $\mathbf{1} \otimes \mathcal{L} \in \mathcal{G}$ (because we showed it above), we have

$$\begin{aligned} [\sigma_{(x,y,z)} \otimes K_1, \mathbf{1} \otimes iI_{zz}] &= \sigma_{(x,y,z)} \otimes [K_1, iI_{zz}] \\ &\in \sigma_{(x,y,z)} \otimes [K_1, \mathcal{L}] \in \mathcal{G}. \end{aligned}$$

These arguments show that, in (14), the right hand side is included in the left hand side. We already know that $\mathcal{G} \subseteq \hat{\mathcal{L}}$ by Lemma 3.1, so the theorem is proved. ■

IV. SUBSPACE CONTROLLABILITY

In general, if a system of the form (1) admits a discrete group of symmetries $\hat{G}$, i.e., a group $\hat{G}$ such that $[A, P] = 0$, $[B_j, P] = 0$, $\forall P \in \hat{G}$, the maximal Lie subalgebra of $u(\hat{n})$ which commutes with $\hat{G}$, $\mathcal{L}^{\hat{G}}$, acts on subspaces $\mathcal{H}_j$ of the Hilbert space $\mathcal{H}$ as $u(\dim(\mathcal{H}_j))$. Each of such subspaces is an irreducible representation of $\mathcal{L}^{\hat{G}}$ [6]. In an appropriate basis of $\mathcal{H}$, therefore, $\mathcal{L}^{\hat{G}}$ can be written in block diagonal form, where each block can take values in $u(\dim(\mathcal{H}_j))$. The dynamical Lie algebra associated with a system having $\hat{G}$ as a group of symmetries also displays a block diagonal form in the same basis although not necessarily equal to the full $\mathcal{L}^{\hat{G}}$. In the preferred basis however one can study the action of the dynamical Lie algebra on each subspace and determine subspace controllability. This is the plan we follow here.

A method to find the desired basis was described in [6] and is based on the so-called *Generalized Young Symmetrizers (GYS)* where the word ‘Generalized’ refer to the fact that in the case where the group $\hat{G}$ is the symmetry group they reduce to the classical Young symmetrizers of group representation theory as described for instance in [17]. More precisely consider the representation of $\hat{G}$ on $\mathcal{H}$ and the *group algebra* of $\hat{G}$ (i.e., the algebra over the complex field generated by a basis of $\hat{G}$), $C[\hat{G}]$. Then the GYS are elements of $C[\hat{G}]$, and operators on $\mathcal{H}$, Π_j satisfying C) (*Completeness*): $\sum_j \Pi_j = 1$; O) (*Orthogonality*): $\Pi_j \Pi_k = \delta_{j,k} \Pi_j$, where $\delta_{j,k}$ is the Kronecker delta; P) (*Primitivity*): $\Pi_j g \Pi_j = \lambda_g \Pi_j$, where λ_g is a scalar which depends only on g (and not on j) H) (*Hermiticity*): For every j , $\Pi_j^\dagger = \Pi_j$. If the GYS are known for a given group $\hat{G}$ on a Hilbert space $\mathcal{H}$ then the images of the various $\Pi_j : \mathcal{H} \rightarrow \mathcal{H}$ give the subspace decomposition of $\mathcal{H}$ which block diagonalizes the Lie algebra $\mathcal{L}^{\hat{G}}$. In the cases where $\hat{G}$ is the symmetric group $S_{\hat{n}}$ over $\hat{n}$ objects, the (generalized) Young symmetrizers can be found using the classical method of Young tableaux (see, e.g., [17]) modified in references [3] [12] to meet the Orthogonality and Hermiticity requirements.

A method is given in [6] to compute the GYS in the case where $\hat{G}$ is Abelian, but the calculation of GYS for general discrete groups is an open problem. We observe that if $\mathcal{H} : \mathcal{H}_C \otimes \mathcal{H}_P$ the tensor product of two Hilbert spaces $\mathcal{H}_C$, $\mathcal{H}_P$, as in bipartite quantum systems, and $\hat{G}$ is the product of two groups $\hat{G} := \hat{G}_C \otimes \hat{G}_P$, with $\hat{G}_{C(P)}$ acting on $\mathcal{H}_{C(P)}$, then the GYS can be found as tensor products of GYS on $\mathcal{H}_{C(P)}$ for $\hat{G}_{C(P)}$, $\Pi_j^C \otimes \Pi_k^P$. It is indeed readily verified that if $\{\Pi_j^C\}$ and $\{\Pi_k^P\}$ satisfy the requirements (C,O,P,H) above on $\mathcal{H}_C$ and $\mathcal{H}_P$, then $\{\Pi_j^C \otimes \Pi_k^P\}$ satisfy the same requirements (C,O,P,H) on $\mathcal{H}_C \otimes \mathcal{H}_P$. The invariant subspaces are $\mathcal{H}_{j,k} := (Im \Pi_j^C) \otimes (Im \Pi_k^P)$ and in this basis the (maximal) invariant Lie algebra $\mathcal{L}^{\hat{G}_C} \otimes \mathcal{L}^{\hat{G}_P}$ takes a block diagonal form.

For the systems treated in this paper the symmetry groups $\hat{G}_C$ and $\hat{G}_P$ are the symmetric group on n_c and n_p objects, respectively. The decomposition is obtained using the GYS of [17], [3], [12]. Let G be now the symmetric group and consider the matrix J defined in Lemma 2.4 and Corollary 2.5 in the basis determined by the GYS. In this basis, the

elements of $\mathcal{L}^G$ are block diagonal and every block is an *arbitrary matrix* in $u(m)$ for appropriate m (cf. Theorem 2 in [6]).¹ Since each block of the matrices in $\mathcal{L}^G$ can be an arbitrary skew-Hermitian matrix of appropriate dimensions, iJ is also a block diagonal matrix, i.e.,

$$iJ := \begin{bmatrix} iJ_1 & & \\ & \ddots & \\ & & iJ_d \end{bmatrix},$$

with iJ_k , $k = 1, \dots, d$ commuting with the corresponding block of the matrices in $\mathcal{L}^G$. Since such a block defines an *irreducible representation* of $u(m_k)$ for appropriate dimensions m_k , it follows from Schür’s Lemma (see, e.g., [9]) that all iJ_k are scalar matrices. Consider now the matrices in $\mathcal{L}$ and $\mathcal{L}^G$ and their restrictions to one of the subspaces $Im \Pi_k$, of dimensions m_k . A basis for $\mathcal{L}^G$ restricted to $Im \Pi_k$ is given by a basis of $u(m_k)$ while a basis of $\mathcal{L}$ is given by a basis of $su(m_k)$ since the restriction of $\mathcal{L}$ to $Im \Pi_k$ is $su(m_k)$ for what we have seen above and Proposition 2.3.

Consider now the Lie algebra $\hat{\mathcal{L}}$ defined in (10) on the invariant space $\Pi_j \mathcal{H}_C \otimes \Pi_k \mathcal{H}_P$ for two generalized Young symmetrizers Π_j and Π_k for system C and P . Then a basis of $\hat{\mathcal{L}}$ restricted to $\Pi_j \mathcal{H}_C \otimes \Pi_k \mathcal{H}_P$ is given by all tensor products of bases of $u(m_j)$ with bases of $u(m_k)$, where $m_j = \dim \Pi_j \mathcal{H}_C$ and $m_k = \dim \Pi_k \mathcal{H}_P$, except for the product of scalar matrices. Therefore the restriction of $\hat{\mathcal{L}}$ on any invariant subspace (of dimension $m_j m_k$) is given by $su(m_j m_k)$ and the system is subspace controllable.

Since $\mathcal{G} = \hat{\mathcal{L}}$ when we have a single central spin as we have proved in Theorem 2, we have:

Theorem 3: The system (1) of a single central spin ($n_c = 1$) with any number $n_p \geq 1$ of surrounding spins simultaneously controlled is subspace controllable.

A. Example

To illustrate some of the concepts and procedures described above, we consider the system of a central spin along with $n_p = 3$ surrounding spins. The symmetric group on the central spin is trivial being made up of just the identity. There is a single GYS given by the identity. For the symmetric group S_3 on the P spins part of the space we obtain the GYS using the method of [17], [3], [12], based on the Young tableaux. We refer to these references and [6] for details on the method. For $n = 3$ there are three possible *partitions* of n and therefore three possible *Young diagram* (also called *Young shapes*). Recall that a partition of an integer n is a sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_d$, with $\lambda_1 + \lambda_2 + \dots + \lambda_d = n$ and the corresponding Young diagram is made up of boxes arranged in rows of length $\lambda_1, \lambda_2, \dots, \lambda_d$. Therefore for $n = 3$, we have the partitions (3), (2, 1), (1, 1, 1) which correspond to the Young diagrams

$$\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}, \quad \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}, \quad (15)$$

¹Formulas are available to determine the dimension of each block [6].

respectively. To each Young diagram corresponds a certain number of *Standard Young Tableaux* obtained by filling the boxes of the Young diagram with the numbers 1 through n (3 in this case) so that they appear in strictly increasing order in the rows and in the columns. The following are the possible standard Young tableaux corresponding to the Young diagrams in (15). In particular, the first one corresponds to the first diagram in (15), the second and third correspond to the second one in (15) and the fourth one corresponds to the third one in (15)

$$\begin{array}{|c|c|c|}, \\ \hline 1 & 2 & 3 \\ \hline \end{array}, \quad \begin{array}{|c|c|}, \\ \hline 1 & 2 \\ \hline 3 \\ \hline \end{array}, \quad \begin{array}{|c|c|}, \\ \hline 1 & 3 \\ \hline 2 \\ \hline \end{array}, \quad \begin{array}{|c|}, \\ \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}. \quad (16)$$

To each tableaux there corresponds a GYS whose image is an invariant subspace for the Lie algebra representation. We refer to [6] for a summary of the procedure to obtain such GYS. We limit ourselves to remark that there is an explicit formula to obtain the dimension of the image of a GYS, Π_T , corresponding to a Young tableaux T . Such formula specializes to our case (where the dimension of the underlying subspace is 2) as

$$\dim(\text{Im}P_T) = \frac{\prod_{l=1}^r \prod_{k=1}^{\lambda_l} (2 - l + k)}{\text{Hook}(T)}. \quad (17)$$

Here, r is the number of rows in the tableaux, λ_l is the length of the l -th row and $\text{Hook}(T)$ is the *Hook length* of the Young diagram associated with T . It is calculated by considering, for each box of the Young diagram the number of boxes directly to the right + the number of boxes directly below + 1 and then taking the product of all the numbers obtained. Applying this formula, we obtain that the subspace corresponding to the fourth tableaux in (16) is zero dimensional. On the other hand the subspace corresponding to the first tableaux has dimension 4 while the two subspaces corresponding to the second and third tableaux have dimension 2. By tensoring with the two dimensional space corresponding to the central spin C we obtain invariant subspaces of dimensions 8, 4 and 4. From the above analysis the system is controllable on each of these subspaces and therefore subspace controllable (cf. Theorem 3).

V. CONCLUSIONS

The calculation of the dynamical Lie algebra of a quantum system is the method of choice to study its controllability properties [5]. However such direct calculation might be difficult in cases of very large systems and in particular networks of spins where the dimension grows exponentially with the number of particles. For this reason it is important to devise methods to assess controllability from the topology of the network and its possible symmetries. Symmetries, in particular, prevent full controllability and determine a number of invariant subspaces on which the system evolves. Such invariant subspaces are obtained as images of Generalized Young Symmetrizers. Full controllability on each of these subspaces is then possible.

In this paper we have taken the first steps in understanding such dynamical decomposition and subspace controllability

for *multipartite* systems where different groups of symmetries act on different subspaces. Motivated by common experimental situations with N-V centers in diamonds, we have considered a configuration of a central spin surrounded by a number of spins which can be arbitrarily exchanged without modifying the Hamiltonian which describes the dynamics. We have computed the dynamical Lie algebra and proved that such a system is subspace controllable. More complicated configurations and symmetry groups different from the full symmetric group will be the object of future studies.

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