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RIESZ MEANS OF FOURIER SERIES AND INTEGRALS: STRONG SUMMABILITY AT THE CRITICAL INDEX

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Abstract. We consider spherical Riesz means of multiple Fourier series and some generalizations. While almost everywhere convergence of Riesz means at the critical index $(d-1)/2$ may fail for functions in the Hardy space $h^1(\mathbb{T}^d)$, we prove sharp positive results for strong summability almost everywhere. For functions in $L^p(\mathbb{T}^d)$, $1 < p < 2$, we consider Riesz means at the critical index $d(1/p-1/2)-1/2$ and prove an almost sharp theorem on strong summability. The results follow via transference from corresponding results for Fourier integrals. We include an endpoint bound on maximal operators associated with generalized Riesz means on Hardy spaces $H^p(\mathbb{R}^d)$ for $0 < p < 1$.

1. Introduction

We consider multiple Fourier series of functions on $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. For $\ell \in \mathbb{Z}^d$, let $e_\ell(x) = e^{2\pi i \langle x,\ell\rangle}$, and define the Fourier coefficients of $f \in L^1(\mathbb{T}^d)$ by $\langle f, e_\ell\rangle = \int_{\mathbb{T}^d} f(y)e^{-2\pi i\langle y,\ell\rangle}dy$. We shall examine the pointwise behavior of (generalized) Riesz means of the Fourier series. Fix a homogeneous distance function ρ, continuous on $\mathbb{R}^d$, positive and C^∞ on $\mathbb{R}^d \setminus \{0\}$, and satisfying, for some $b > 0$, $\rho(t^b\xi) = t\rho(\xi)$ for all $\xi \in \mathbb{R}^d$. For $f \in L^1(\mathbb{T}^d)$, define the Riesz means of index λ with respect to ρ, by

$$\mathcal{R}_t^\lambda f = \sum_{\substack{\ell\in\mathbb{Z}^d:\\ \rho(\ell/t)\le 1}} \big(1-\rho(\ell/t)\big)^\lambda \langle f, e_\ell\rangle e_\ell. \tag{1.1}$$

The classical Riesz means are recovered for $\rho(\xi) = |\xi|$, and when in addition $\lambda = 1$ we obtain the Fejér means. The Bochner–Riesz means are covered with $b = 1/2$ by taking $\rho(\xi) = |\xi|^2$.

It is well known via classical results for Fourier integrals [34], [40], [31] and transference [25], [20], [1] that, for $\lambda > \frac{d-1}{2}$ and $f \in L^1(\mathbb{T}^d)$, we have $\lim_{t\to\infty}\mathcal{R}_t^\lambda f = f$, both with respect to the L^1 norm and almost everywhere. For the critical index $\lambda = \frac{d-1}{2}$, it is known that the Riesz means are of weak type $(1,1)$, and one has convergence in measure [8], [10] but Stein [35] showed early that a.e. convergence may fail (see also [40]). Indeed, extending the ideas of Bochner, he proved the existence of an $L^1(\mathbb{T}^d)$ function for which the Bochner–Riesz means at index $\frac{d-1}{2}$ diverge almost everywhere as $t \to \infty$. Stein's theorem can be seen as an analogue of the theorem by Kolmogorov [23] on the failure of a.e. convergence for Fourier series

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in $L^1(\mathbb{T})$; see [48, Ch. VIII-4]. Later, Stein [37] proved a stronger result showing that, even for some functions in the subspace $h^1(\mathbb{T}^d)$ (the local Hardy space), the Bochner–Riesz means at the critical index may diverge almost everywhere. It is then natural to ask what happens if we replace almost everywhere convergence with the weaker notion of strong convergence a.e. (also known as strong summability a.e.), which goes back to Hardy and Littlewood [18].

Definition 1.1. Let $0 < q < \infty$. Given a measurable function $g : (0,\infty) \to \mathbb{C}$, we say that $g(t)$ *converges q-strongly* to a, as $t \to \infty$, if

$$\lim_{T\to\infty} \Big(\frac{1}{T}\int_0^T |g(t)-a|^q\,dt\Big)^{1/q} = 0.$$

If $g(t)$ refers to the partial sum of a series, then one also says that the series is strongly H_q summable. Clearly if $\lim_{t\to\infty} g(t) = a$, then $g(t)$ converges q-strongly to a for all $q < \infty$. Conversely, if $g(t)$ converges q-strongly to a for some $q > 0$, then $g(t)$ is *almost convergent* to a as $t \to \infty$. That is, there is a (density 1) subset $E \subset [0,\infty)$ satisfying

$$\lim_{T\to\infty} \frac{|E\cap[0,T]|}{T} = 1 \quad \text{and} \quad \lim_{\substack{t\to\infty\\ t\in E}} g(t) = a. \tag{1.2}$$

See [48, Ch. XIII, (7.2)] and also Corollary 7.3.

For the classical case of a Fourier series of an $L^1(\mathbb{T})$ function, Zygmund [47] proved that the partial sum $\sum_{|l|\le t}\langle f, e_\ell\rangle e_\ell(x)$ converges q-strongly to $f(x)$ as $t\to\infty$ a.e. for all $q<\infty$, extending an earlier result by Marcinkiewicz [28] for $q = 2$. Zygmund used complex methods, but in more recent papers one can find alternative approaches with stronger results and some weaker extensions to rectangular partial sums of multiple Fourier series; see, e.g., [30], [46] and the references therein. See also [24] for an overview of recent developments on topics related to the convergence of Fourier series.

Regarding spherical partial sums of multiple Fourier series, q-strong convergence results have been available for $L^p(\mathbb{T}^d)$ functions for the Bochner–Riesz means of index $\lambda > \lambda(p)$ when $p \le 2$, $q = 2$, where $\lambda(p) = d(\frac{1}{p}-\frac{1}{2}) - \frac{1}{2}$ is the critical index (cf. [34], [42]). The question of strong convergence a.e. for the Bochner–Riesz means at the critical index $\lambda(1) = \frac{d-1}{2}$, for either $f \in L^1(\mathbb{T}^d)$ or $f \in h^1(\mathbb{T}^d)$, had been left open and was posed by Lu in the survey article [27]. We answer this question in the affirmative for $f \in h^1(\mathbb{T}^d)$ for generalized Riesz means with *any* distance function ρ under consideration.

Theorem 1.2. *Let $q < \infty$, and let $\lambda(1) = \frac{d-1}{2}$. Then, for all $f \in h^1(\mathbb{T}^d)$, the following statements hold.*

(i) *There is a constant C such that, for all $\alpha > 0$,*

$$\mathrm{meas}\Big(\Big\{x : \sup_{T>0}\Big(\frac{1}{T}\int_0^T |\mathcal{R}_t^{\lambda(1)} f(x)|^q dt\Big)^{1/q} > \alpha\Big\}\Big) \le C\alpha^{-1}\|f\|_{h^1}.$$

(ii)

$$\lim_{T\to\infty} \Big(\frac{1}{T}\int_0^T |\mathcal{R}_t^{\lambda(1)} f(x) - f(x)|^q dt\Big)^{1/q} = 0 \qquad \textit{for almost every } x \in \mathbb{T}^d.$$

We remark that for the classical Riesz means (or generalized Riesz means assuming finite type conditions on the cosphere $\Sigma_\rho = \{\xi : \rho(\xi) = 1\}$), Theorem 1.2 for the range $q \le 2$ could have been extracted from [32], although that result is not explicitly stated there. The full range $q < \infty$ obtained here seems to be new. Regarding the question posed for $f \in L^1(\mathbb{T}^d)$, in Section 7, we derive some weaker results including q-strong convergence up to passing to a subsequence.

We now address the question of strong convergence of Riesz means for $L^p(\mathbb{T}^d)$ functions at the critical index $\lambda = \lambda(p)$. In this case, q-strong convergence results may fail for large q. Our next result identifies a nearly sharp range of q for which $\mathcal{R}_t^{\lambda(p)} f(x)$ converges q-strongly to $f(x)$ almost everywhere for any $f \in L^p(\mathbb{T}^d)$. We denote by $p' = \frac{p}{p-1}$ the exponent dual to p.

Theorem 1.3. *Let $1 < p < 2$, let $q < p'$, and let $\lambda(p) = d(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}$. Then, for all $f \in L^p(\mathbb{T}^d)$, the following statements hold.*

(i) *There is a constant C such that, for all $\alpha > 0$,*

$$\operatorname{meas}\Big(\Big\{x \in \mathbb{T}^d : \sup_{T>0} \Big(\frac{1}{T}\int_0^T |\mathcal{R}_t^{\lambda(p)} f(x)|^q dt\Big)^{1/q} > \alpha\Big\}\Big) \le C\alpha^{-p}\|f\|_{L^p(\mathbb{T}^d)}.$$

(ii)

$$\lim_{T\to\infty} \Big(\frac{1}{T}\int_0^T |\mathcal{R}_t^{\lambda(p)} f(x) - f(x)|^q dt\Big)^{1/q} = 0 \text{ for almost every } x \in \mathbb{T}^d.$$

(iii) *For suitable $f \in L^p(\mathbb{T}^d)$, statements* (i) *and* (ii) *fail when $q > p'$.*

Part (ii)'s in both theorems follow by a standard argument from the respective part (i), using the fact that pointwise (in fact uniform) convergence holds for Schwartz functions. We note that Theorem 1.2 is sharp in view of the above mentioned example by Stein. Moreover, part (iii) of Theorem 1.3 shows that the result is essentially sharp for all $p \in (1, 2)$, but the case $q = p'$ remains open.

We state a special case of Theorem 1.3 for $\lambda(p) = 0$, i.e., for the case of generalized spherical partial sums of Fourier series, as a corollary.

Corollary 1.4. *Let $d \ge 2$, let $q < \frac{2d}{d-1}$, and let $f \in L^{\frac{2d}{d+1}}(\mathbb{T}^d)$. Then*

$$\lim_{T\to\infty} \Big(\frac{1}{T}\int_0^T \Big|\sum_{\rho(\ell/t)\le 1} \langle f, e_\ell\rangle e_\ell(x) - f(x)\Big|^q dt\Big)^{1/q} = 0 \qquad \text{for almost every } x \in \mathbb{T}^d.$$

In particular, for almost every $x \in \mathbb{T}^d$, the partial sums $\sum_{\rho(\ell/t)\le 1}\langle f, e_\ell\rangle e_\ell(x)$ are almost convergent to $f(x)$ as $t \to \infty$, in the sense of (1.2).

We remark that there are analogues of the above results for generalized Riesz means of Fourier integral in $\mathbb{R}^d$:

$$R_t^\lambda f(x) = \int_{\rho(\xi/t)\le 1} \big(1 - \rho(\xi/t)\big)^\lambda \widehat{f}(\xi)\, e^{2\pi i\langle \xi, x\rangle} d\xi. \tag{1.3}$$

See Section 2. Indeed, we derive Theorems 1.2 and 1.3 from corresponding theorems for Fourier integrals in $\mathbb{R}^d$ using transference arguments. Our proof uses somewhat technical arguments on atomic decomposition and Calderón–Zygmund theory. Unlike the proofs of the L^p boundedness of Bochner–Riesz means (such as in [38], [5] and the references therein), our proof does not rely on Fourier restriction theory thanks to the averaging over the dilation parameter t. In particular, the curvature of the cosphere $\Sigma_\rho = \{\xi : \rho(\xi) = 1\}$ does not play a role in the argument (cf. [10], [11]), which allows us to work with generalized Riesz means with respect to any smooth homogeneous distance function.

Notation. Given two quantities A, B, we use the notation $A \lesssim B$ to mean that there is a constant C such that $A \le CB$. We use $A \approx B$ if $A \lesssim B$ and $B \lesssim A$.

This paper. In Section 2, we formulate Theorems 2.1 and 2.2 on strong convergence for Riesz means of critical index in $\mathbb{R}^d$ and reduce their proof to the main weak type inequality stated in Theorem 2.3. Some preliminary estimates are contained in Section 3. The proof of the main result, Theorem 2.3, is given in Section 4. In Section 5, we use transference arguments to prove the positive results in Theorems 1.2 and 1.3. In Section 7, we discuss a weaker result for L^1 functions. In Section 6, we show the essential sharpness of our L^p results—namely, that Theorems 1.3 and 2.2 require the condition $q \le p'$ (the failure of the maximal theorems for h^1 already follows from Stein's example [37]). In Section 8, we include the proof of an extension of a theorem by Stein, Taibleson, and Weiss [39]—namely, an $H^p \to L^{p,\infty}$ estimate for the maximal function $\sup_{t>0} |R_t^{\lambda(p)} f(x)|$ associated with generalized Riesz means in Hardy spaces H^p with $p < 1$. Finally, we discuss some open problems in Section 9.

2. The main weak type estimate

We state results on $\mathbb{R}^d$ which are analogous to Theorems 1.2 and 1.3 and reduce them to a crucial inequality for a vector-valued operator stated in Theorem 2.3. Let ρ be as in the introduction. Recall the definition of Riesz means R_t^λ for Fourier integrals from (1.3).

Theorem 2.1. *Let $q < \infty$, and let $\lambda(1) = \frac{d-1}{2}$. Then, for all $f \in H^1(\mathbb{R}^d)$ and for all $\alpha > 0$,*

$$\operatorname{meas}\Big(\Big\{x \in \mathbb{R}^d : \sup_{T>0} \Big(\frac{1}{T}\int_0^T |R_t^{\lambda(1)} f(x)|^q dt\Big)^{1/q} > \alpha\Big\}\Big) \le C\alpha^{-1} \|f\|_{H^1(\mathbb{R}^d)}.$$

Theorem 2.2. *Let $1 < p < 2$, let $q < p'$, and let $\lambda(p) = d(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}$. Then, for all $f \in L^p(\mathbb{R}^d)$ and for all $\alpha > 0$,*

$$\operatorname{meas}\Big(\Big\{x \in \mathbb{R}^d : \sup_{T>0} \Big(\frac{1}{T}\int_0^T |R_t^{\lambda(p)} f(x)|^q dt\Big)^{1/q} > \alpha\Big\}\Big) \le C\alpha^{-p} \|f\|_{L^p(\mathbb{R}^d)}.$$

As a consequence of these estimates, we obtain

$$\lim_{T\to 0} \Big(\frac{1}{T}\int_0^T |R_t^{\lambda(p)} f(x) - f(x)|^q dt\Big)^{1/q} = 0$$

for almost every $x \in \mathbb{R}^d$, for every $f \in L^p(\mathbb{R}^d)$ when $1 < p < 2$ and $f \in h^1(\mathbb{R}^d)$ or $H^1(\mathbb{R}^d)$ when $p = 1$.

We start the reduction of Theorems 2.1 and 2.2 to Theorem 2.3 by replacing the multipliers for the Riesz means R_t^λ with similar multipliers supported away from the origin; see (2.2).

2.1. **Contribution near the origin.** Let $\upsilon_0 \in C^\infty(\mathbb{R})$ so that $\upsilon_0(\rho) = 1$ for $\rho \le 4/5$ and $\upsilon_0(\rho) = 0$ for $\rho \ge 9/10$. It is then standard that the maximal function $\sup_{t>0} |\mathcal{F}^{-1}[\upsilon_0(\rho(\cdot/t))(1-\rho(\cdot/t))_+^\lambda \widehat{f}]$ defines an operator of weak type $(1,1)$ and bounded on L^p for all $p > 1$. A small complication occurs if ρ is not sufficiently smooth at the origin. We address this complication as follows.

Define, for $N > 0$, the functions u, u_N with domain $(0,\infty)$ by $u(\tau) = \upsilon_0(\tau)(1-\tau)^\lambda$ and $u_N(s) = u(s^{1/N})$. It is then straightforward to check that, for all M,

$$\int_0^\infty s^M |u_N^{(M+1)}(s)| ds < \infty,$$

and we have the subordination formula [45]

$$u(\rho(\xi)) = u_N(\rho^N(\xi)) = \frac{(-1)^{M+1}}{M!} \int_0^\infty \left(1 - \frac{(\rho(\xi))^N}{s}\right)_+^M s^M u_N^{(M+1)}(s) ds, \tag{2.1}$$

which is proved by integration by parts. Given any $m > 0$, one has $|\mathcal{F}^{-1}[(1-\rho^N)_+^M](x)| \lesssim_m (1+|x|)^{-m}$ provided that M and N are large enough. This is used to show that $\sup_{t>0} |\mathcal{F}^{-1}[u \circ \rho(\cdot/t)\widehat{f}]$ is dominated by a constant times the Hardy–Littlewood maximal function of f (see also Lemma 8.2).

We can now replace the operator R_t^λ in Theorems 2.1 and 2.2 by S_t^λ defined by

$$\widehat{S_t^\lambda f}(\xi) = (1 - \upsilon_0(\rho(\xi/t)))(1-\rho(\xi/t))_+^\lambda \widehat{f}(\xi). \tag{2.2}$$

2.2. **Further decompositions.** We first recall standard dyadic decompositions on the frequency side. Let $\eta \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$ be such that η is nonnegative,

$$\eta(\xi) = 1 \qquad \text{on } \{\xi : \rho(\xi/t) \in [1/4, 4],\ 1/2 \le t \le 2\}. \tag{2.3}$$

Define $\mathcal{L}_k f$ by $\widehat{\mathcal{L}_k f}(\xi) = \eta(2^{-k}\xi)\widehat{f}(\xi)$.

We use the nontangential version of the Peetre maximal operators

$$\mathfrak{M}_k f(x) = \sup_{|h| \le 2^{-k+10}d} |\mathcal{L}_k f(x+h)|$$

and the associated square function

$$\mathfrak{S} f(x) = \Big(\sum_{k\in\mathbb{Z}} |\mathfrak{M}_k f(x)|^2\Big)^{1/2}. \tag{2.4}$$

Then

$$\|\mathfrak{S} f\|_{L^1} \le C\|f\|_{H^1} \tag{2.5a}$$

and

$$\|\mathfrak{S} f\|_{L^p} \le C_p \|f\|_{L^p}, \qquad 1 < p < \infty; \tag{2.5b}$$

see Peetre [29].

The inequalities in Theorems 2.1 and 2.2 follow from

$$\Big\| \sup_{T>0} \Big(\frac{1}{T}\int_0^T |S_t^{\lambda(p)} f|^q dt\Big)^{1/q} \Big\|_{L^{p,\infty}} \lesssim \|\mathfrak{S} f\|_p$$

for $1 \le p < 2$, $q < p'$. Here $L^{p,\infty}$ is the weak type Lorentz space and the expression $\|g\|_{L^{p,\infty}} = \sup_{\alpha>0} \alpha(\text{meas}(\{x : |g(x)| > \alpha\}))^{1/p}$ is the standard quasi norm on $L^{p,\infty}$. We may, by Hölder's inequality, assume that $2 \le q < p'$. We can then use

$$\sup_{T>0} \Big(\frac{1}{T}\int_0^T |S_t^{\lambda(p)} f(x)|^q\, dt\Big)^{1/q} \le 2^{1/q}\Big(\sum_{k\in\mathbb{Z}} 2^{-k}\int_{2^k}^{2^{k+1}} |S_t^{\lambda(p)} f(x)|^q dt\Big)^{1/q}. \tag{2.6}$$

We now use the standard idea to decompose the multiplier $(1 - \upsilon_0 \circ \rho)(1 - \rho)_+^\lambda$ into pieces supported where $\rho(\xi) \in [1 - 2^{-j}, 1 - 2^{-j-2}]$. Generalizing slightly, we assume that we are given C^∞ functions φ_j supported in $[1 - 2^{-j}, 1 - 2^{-j-2}]$ and satisfying

$$\|\partial^n \varphi_j\|_\infty \le C_n 2^{jn}$$

for $n = 0, 1, 2, \dots$. Let $I := [1, 2]$. For $t \in I$, $k \in \mathbb{Z}$, define

$$\widehat{T_j^k f}(\xi, t) = \varphi_j(\rho(2^{-k}t^{-1}\xi))\widehat{f}(\xi). \tag{2.7}$$

We may decompose $S^\lambda_{2^k t} f = \sum_{j\ge1} 2^{-j\lambda} T_j^k f(x, t)$, with T_k^j of the form in (2.7). The asserted estimates for $S_t^{\lambda(p)}$ follow now from weak type bounds for the expression on the right-hand side of (2.6). By (2.3), we have $\eta(2^{-k}\xi) = 1$ whenever $\rho(2^{-k}\xi/t) \in \text{supp}(\varphi_j)$ for any $t \in I$. Thus after changing variables, the desired estimate can be restated as

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_I \Big|\sum_{j=1}^\infty 2^{-j\lambda(p)} T_j^k \mathcal{L}_k f(\cdot, t)\Big|^q dt\Big)^{1/q}\Big\|_{L^{p,\infty}} \lesssim \|\mathfrak{S} f\|_p.$$

Since $\ell^2 \subset \ell^q$ for $q \ge 2$, this follows from the following stronger statement, our main estimate.

Theorem 2.3. *For $1 \le p < 2$, $\lambda(p) = d(1/p - 1/2) - 1/2$, $q < p'$,*

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\Big(\int_I \Big|\sum_{j=1}^\infty 2^{-j\lambda(p)} T_j^k \mathcal{L}_k f(\cdot, t)\Big|^q dt\Big)^{2/q}\Big)^{1/2}\Big\|_{L^{p,\infty}(\mathbb{R}^d)} \lesssim \|\mathfrak{S} f\|_{L^p(\mathbb{R}^d)}.$$

The theorem will be proved in Section 4. Some preparatory material is contained in Section 3.

3. Preliminary estimates

We gather elementary estimates for the operators T_j^k defined in (2.7).

Lemma 3.1. *For $2 \le q \le \infty$,*

$$\Big\|\Big(\int_1^2 |T_j^k f(\cdot, t)|^q dt\Big)^{1/q}\Big\|_2 \lesssim 2^{-j/q}\|f\|_2.$$

Proof. Use the convexity inequality, $\|\gamma\|_q \le \|\gamma\|_2^{2/q}\|\gamma\|_\infty^{1-2/q}$, for $\gamma \in L^q([1, 2])$, and for $\gamma \in C^1$ we have $\|\gamma\|_\infty \lesssim \|\gamma\|_2^{1/2}(\|\gamma\|_2 + \|\gamma'\|_2)^{1/2}$, and hence

$$\begin{aligned}\Big(\int_1^2 |\gamma(t)|^q dt\Big)^{1/q} &\lesssim \Big(\int_1^2 |\gamma(t)|^2 dt\Big)^{1/2} \\ &\quad + \Big(\int_1^2 |\gamma(t)|^2 dt\Big)^{\frac12(\frac12+\frac1q)}\Big(\int_1^2 |\gamma'(t)|^2 dt\Big)^{\frac12(\frac12-\frac1q)}.\end{aligned} \tag{3.1}$$

We obtain after some standard estimations

$$\Big\|\Big(\int_1^2 |T_j^k f(\cdot,t)|^2 dt\Big)^{1/2}\Big\|_2 + 2^{-j}\Big\|\Big(\int_1^2 |\frac{d}{dt}T_j^k f(\cdot,t)|^2 dt\Big)^{1/2}\Big\|_2 \lesssim 2^{-j/2}\|f\|_2,$$

and then the assertion of the lemma follows from (3.1) applied to $\gamma(t) = T_j^k f(x,t)$, followed by Hölder's inequality in x. □

To prove the L^1 estimate, we rely on a spherical decomposition introduced in [12]. For each fixed $j \geq 1$, we use a C^∞ partition of unity $\{\chi_{j,\nu}\}_{\nu\in\mathcal{Z}_j}$ for an index set $\mathcal{Z}_j$ with $\#\mathcal{Z}_j = O(2^{j(d-1)/2})$, which has the following properties: each $\chi_{j,\nu}$ is homogeneous of degree 0, the restriction of the support of $\chi_{j,\nu}$ to the sphere $\{\xi : |\xi| = 1\}$ is supported in a set of diameter $2^{-j/2}$, and each unit vector is contained in the supports of $\chi_{j,\nu}$ for $O(1)$ indices ν. We may choose the index set $\mathcal{Z}_j$ such that, for every ν, there is a unit vector $\xi_{j,\nu} \in \operatorname{supp}(\chi_{j,\nu})$ such that $\operatorname{dist}(\xi_{j,\nu}, \xi_{j,\nu'}) \geq c2^{-j/2}$ for $\nu \neq \nu'$. We assume that the $\chi_{j,\nu}$ satisfy the natural differential estimates, i.e., $\partial_\xi^\beta \chi_{j,\nu}(\xi) = O(2^{\frac{j}{2}(\beta_1+\cdots+\beta_d)})$. Define $T_{j,\nu}^k$ by

$$\widehat{T_{j,\nu}^k f}(\xi,t) = \chi_{j,\nu}(\xi)\varphi_j(\rho(2^{-k}t^{-1}\xi))\widehat{f}. \tag{3.2}$$

Let $K_j = \mathcal{F}^{-1}[\varphi_j(\rho(\cdot))]$, and let $K_{j,\nu} = \mathcal{F}^{-1}[\varphi_j(\rho(\cdot))\chi_{j,\nu}]$. Let $\Phi_0 \in C_c^\infty(\mathbb{R}^d)$ supported in $\{x : |x| \leq 1\}$ be such that $\Phi_0(x) = 1$ for $|x| \leq 1/2$, and, for $n \geq 1$, let $\Phi_n(x) = \Phi_0(2^{-n}x) - \Phi_0(2^{1-n}x)$. Define, for $n = 0, 1, 2, \ldots,$

$$K_j^n(x) = K_j(x)\Phi_n(2^{-j}x),$$
$$K_{j,\nu}^n(x) = K_{j,\nu}(x)\Phi_n(2^{-j}x)$$

and

$$T_j^{n,k} f(x,t) = (2^k t)^d K_j^n(2^k t\cdot) * f,$$
$$T_{j,\nu}^{n,k} f(x,t) = (2^k t)^d K_{j,\nu}^n(2^k t\cdot) * f.$$

Then

$$T_j^k f = \sum_{\nu\in\mathcal{Z}_j} T_{j,\nu}^k f = \sum_{n=0}^\infty T_j^{n,k} f = \sum_{n=0}^\infty \sum_{\nu\in\mathcal{Z}_j} T_{j,\nu}^{n,k} f. \tag{3.3}$$

Lemma 3.2. *Let* $\Sigma_\rho = \{\xi : \rho(\xi) = 1\}$. *Then*

$$|\widehat{K_j^n}(\xi)| \leq C_{M_0,M_1} 2^{-nM_0}(1 + 2^j \operatorname{dist}(\xi,\Sigma_\rho))^{-M_1}.$$

Sketch of proof. Let $\Psi(x) = \Phi_0(x/2) - \Phi_0(x)$. Then, for $n \geq 1$, we may use the fact that $\widehat{\Psi}$ has vanishing moments and write

$$\begin{aligned}\widehat{K_j^n}(\xi) &= \int \varphi_j(\rho(\xi - y))2^{(j+n)d}\widehat{\Psi}(2^{j+n}y)dy \\ &= \int_0^1 \frac{(1-s)^{N-1}}{(N-1)!}\int \langle y,\nabla\rangle^N [\varphi_j \circ \rho](\xi - sy)2^{(j+n)d}\widehat{\Psi}(2^{j+n}y)dy\,ds\end{aligned} \tag{3.4}$$

by Taylor's formula. The estimate is now straightforward. When $n = 0$, we just use the first line in (3.4), with Ψ replaced by Φ_0. □

For each ν, choose $\xi_{j,\nu}$ such that $\rho(\xi_{j,\nu}) = 1$ and $\xi_{j,\nu} \in \operatorname{supp}(\chi_{j,\nu})$. Take $e_{j,\nu} = \frac{\nabla\rho(\xi_{j,\nu})}{|\nabla\rho(\xi_{j,\nu})|}$, and let $P_{j,\nu}$ be the orthogonal projection to $e_{j,\nu}^\perp$, i.e.,

$$P_{j,\nu}h = h - \langle h, e_{j,\nu}\rangle e_{j,\nu}. \tag{3.5}$$

Lemma 3.3. *For every $M \ge 0$,*

$$\sup_{t\in I} |t^d K_{j,\nu}(tx)| \le C(M) \frac{2^{-j(d+1)/2}}{(1+2^{-j}|x|)^M (1+2^{-j/2}|P_{j,\nu}(x)|)^M}. \tag{3.6}$$

Proof. This is standard (and follows after integration by parts); see, e.g., [12], [11], or [31]. □

Lemma 3.4.

(i) *For $k \in \mathbb{Z}$,*

$$\big\| \sup_{t\in I} |T_j^{n,k} f(\cdot,t)| \big\|_1 \le C_N 2^{j\frac{d-1}{2}} 2^{-nN} \|f\|_1.$$

(ii) *For $1 < p \le 2$, $q \le p'$ and $k \in \mathbb{Z}$,*

$$\Big\| \Big(\int_I |T_j^{n,k} f(\cdot,t)|^q dt \Big)^{1/q} \Big\|_p \le C_N 2^{j(d(\frac1p - \frac12) - \frac12)} 2^{-nN} \|f\|_p.$$

(iii) *For $2 \le q \le \infty$,*

$$\Big\| \Big(\int_1^2 |T_j^{n,k} f(\cdot,t)|^q dt \Big)^{1/q} \Big\|_2 \lesssim 2^{-j/q} 2^{-nN} \|f\|_2.$$

Proof. Lemma 3.3 easily implies $\| \sup_{t\in I} |T_{j,\nu}^{n,k} f(\cdot,t)| \|_1 \le C_N 2^{-nN} \|f\|_1$, and part (i) follows after summing in ν. Using Lemma 3.2, we see that the proof of Lemma 3.1 also gives

$$\Big\| \Big(\int_1^2 |T_j^{n,k} f(\cdot,t)|^2 dt \Big)^{1/2} \Big\|_2 \lesssim_N 2^{-nN} 2^{-j/2} \|f\|_2.$$

Part (ii) now follows by complex interpolation.

Part (iii) for $q = 2$ is just the previous displayed inequality. For $q > 2$, it follows by the argument in Lemma 3.1 (cf. (3.1)) applied to $T_j^{n,k}$ in place of T_j^k, in conjunction with Lemma 3.2. □

4. Proof of Theorem 2.3

The proof combines ideas that were used in the proof of weak type inequalities for Bochner–Riesz means and other radial multipliers, and elsewhere [15], [8], [9], [10], [32]. It combines atomic decompositions with Calderón–Zygmund estimates using L^r bounds for $r > p$ in the complement of suitable exceptional sets together with analytic interpolation arguments inspired by [9].

In this section, we fix a Schwartz function f whose Fourier transform has compact support in $\mathbb{R}^d \setminus \{0\}$. Observe that then $\mathcal{L}_k f = 0$ for all but a finite number of indices k (depending on f). This assumption together with the Schwartz bounds can be used to justify the a priori finiteness of various expressions showing up in the arguments below, but they do not enter quantitatively in the estimates.

We need to prove the inequality

(4.1)
$$\operatorname{meas}\Big\{ x \in \mathbb{R}^d : \Big(\sum_k \Big[\int_I \Big| \sum_{j=1}^\infty 2^{-j\lambda(p)} T_j^k \mathcal{L}_k f(x,t) \Big|^q dt \Big]^{2/q} \Big)^{1/2} > \alpha \Big\} \lesssim \alpha^{-p} \|\mathfrak{S} f\|_p^p$$

for arbitrary but fixed $\alpha > 0$. The implicit constant does not depend on α or the choice of f.

4.1. **Preliminaries on atomic decompositions.** Let $\mathcal{R}_k$ be the set of dyadic cubes of side length 2^{-k} so that each $R \in \mathcal{R}_k$ is of the form $\prod_{i=1}^{d}[n_i 2^{-k}, (n_i+1)2^{-k})$ for some $n \in \mathbb{Z}^d$. For $\mu \in \mathbb{Z}$, let

$$\Omega_\mu = \{x : |\mathfrak{S}f(x)| > 2^\mu\},$$

and let $\mathcal{R}_k^\mu$ be the set of dyadic cubes of length 2^{-k} with the property that

$$|R \cap \Omega_\mu| \geq |R|/2$$

and

$$|R \cap \Omega_{\mu+1}| < |R|/2.$$

Clearly if $\mathfrak{S}f \in L^p$, then every dyadic cube in $\mathcal{R}_k$ belongs to exactly one of the sets $\mathcal{R}_k^\mu$. We then have [7]

$$\sum_{k\in\mathbb{Z}} \sum_{R\in\mathcal{R}_k^\mu} \int_R |\mathcal{L}_k f|^2 dx \lesssim 2^{2\mu}\text{meas}(\Omega_\mu). \tag{4.2}$$

For completeness, we give the argument. Observe that

$$|\mathcal{L}_k f(x)| \leq \mathfrak{M}_k f(z) \qquad \text{for } x, z \in R,\ R \in \mathcal{R}_k^\mu.$$

Let

$$\widetilde{\Omega}_\mu = \{x : M_{\text{HL}} \mathbb{1}_{\Omega_\mu} > 10^{-d}\},$$

where M_{HL} denotes the Hardy–Littlewood maximal operator. Then

$$\text{meas}(\widetilde{\Omega}_\mu) \lesssim \text{meas}(\Omega_\mu),$$

and we have $\bigcup_k \cup_{R\in\mathcal{R}_k^\mu} R \subset \widetilde{\Omega}_\mu$. Now

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \sum_{R\in\mathcal{R}_k^\mu} \|\mathbb{1}_R \mathcal{L}_k f\|_2^2 &\leq \sum_{k\in\mathbb{Z}} \sum_{R\in\mathcal{R}_k^\mu} 2 \int_{R\setminus\Omega_{\mu+1}} |\mathfrak{M}_k f(x)|^2 dx \\ &\leq 2 \int_{\widetilde{\Omega}_\mu \setminus \Omega_{\mu+1}} \sum_{k\in\mathbb{Z}} |\mathfrak{M}_k f(x)|^2 dx \leq 2^{2\mu+1}\text{meas}(\widetilde{\Omega}_\mu) \leq C2^{2\mu}\text{meas}(\Omega_\mu),\end{aligned}$$

which yields (4.2).

Next we work with a Whitney decomposition of the open set $\widetilde{\Omega}_\mu$, which is a disjoint union of dyadic cubes W, such that

$$\text{diam}(W) \leq \text{dist}(W, \widetilde{\Omega}_\mu^{\complement}) \leq 4\,\text{diam}(W).$$

See [36, Ch. VI.1]. We denote by $\mathfrak{W}^\mu$ the collection of these Whitney cubes. Each $R \in \mathcal{R}_k^\mu$ is contained in a unique $W(R) \in \mathfrak{W}^\mu$. For each W, define

$$\mathcal{R}_k^\mu(W) = \{R \in \mathcal{R}_k^\mu : R \subset W\} \tag{4.3}$$

and

$$\gamma_{W,\mu} = \Big(\frac{1}{|W|} \sum_k \sum_{R\in\mathcal{R}_k^\mu(W)} \int_R |\mathcal{L}_k f(y)|^2 dy\Big)^{1/2}.$$

Define

$$U(x) = \sum_\mu \sum_{W\in\mathfrak{W}^\mu} \gamma_{W,\mu}^p \mathbb{1}_W(x). \tag{4.4}$$

Observe that

$$
\begin{aligned}
\|U\|_1 &= \sum_\mu \sum_{W\in\mathfrak{W}^\mu} |W|\gamma_{W,\mu}^p = \sum_\mu \sum_{W\in\mathfrak{W}^\mu} |W|^{1-p/2}(|W|^{1/2}\gamma_{\mu,W})^p \\
&\le \sum_\mu \Big(\sum_{W\in\mathfrak{W}^\mu}|W|\Big)^{1-p/2}\Big(\sum_{W\in\mathfrak{W}^\mu}|W|\gamma_{\mu,W}^2\Big)^{p/2} \\
&\le \sum_\mu |\widetilde{\Omega}_\mu|^{1-p/2}\Big(\sum_k \sum_{W\in\mathfrak{W}^\mu}\sum_{R\in\mathcal{R}_k^\mu(W)} \|\mathbb{1}_R\mathcal{L}_k f\|_2^2\Big)^{p/2} \\
&\lesssim \sum_\mu |\Omega_\mu|^{1-p/2}(2^{2\mu}|\Omega_\mu|)^{p/2} \lesssim \sum_\mu 2^{\mu p}|\Omega_\mu|
\end{aligned}
\tag{4.5}
$$

by (4.2), and thus

$$
\sum_\mu \sum_{W\in\mathfrak{W}^\mu} |W|\gamma_{W,\mu}^p = \|U\|_1 \lesssim \|\mathfrak{S}f\|_p^p. \tag{4.6}
$$

For $\alpha>0$, let

$$
\mathcal{O}_\alpha = \{x : M_{\mathrm{HL}}U > \alpha^p\} \tag{4.7}
$$

and

$$
\widetilde{\mathcal{O}}_\alpha = \{x : M_{\mathrm{HL}}\mathbb{1}_{\mathcal{O}_\alpha}(x) > (10d)^{-d}\} \tag{4.8}
$$

so that $\mathcal{O}_\alpha \subset \widetilde{\mathcal{O}}_\alpha$ and

$$
\mathrm{meas}(\widetilde{\mathcal{O}}_\alpha) \lesssim \mathrm{meas}(\mathcal{O}_\alpha) \lesssim \alpha^{-p}\|\mathfrak{S}f\|_p^p. \tag{4.9}
$$

Let $\mathfrak{Q}_\alpha = \{Q\}$ be the collection of Whitney cubes for the set $\widetilde{\mathcal{O}}_\alpha$ (see [36, Ch. VI]) such that

$$
\mathrm{diam}(Q) \le \mathrm{dist}(Q, \widetilde{\mathcal{O}}_\alpha^{\complement}) \le 4\,\mathrm{diam}(Q).
$$

In analogy to the usual terminology of "good" and "bad" functions in Calderón–Zygmund theory we split, for fixed α, the collection $\mathfrak{W}^\mu$ into two subcollections $\mathfrak{W}^\mu_{\mathrm{good}} \equiv \mathfrak{W}^\mu_{\mathrm{good}}(\alpha)$ and $\mathfrak{W}^\mu_{\mathrm{bad}} \equiv \mathfrak{W}^\mu_{\mathrm{bad}}(\alpha)$ by setting

$$
\begin{aligned}
\mathfrak{W}^\mu_{\mathrm{bad}} &= \big\{W\in\mathfrak{W}^\mu : \gamma_{W,\mu} > \alpha\big\},\\
\mathfrak{W}^\mu_{\mathrm{good}} &= \big\{W\in\mathfrak{W}^\mu : \gamma_{W,\mu} \le \alpha\big\}.
\end{aligned}
\tag{4.10}
$$

We relate the collection $\mathfrak{W}^\mu_{\mathrm{bad}}$ with the collection of Whitney cubes $\mathfrak{Q}_\alpha$ for the set $\widetilde{\mathcal{O}}_\alpha$.

Lemma 4.1. *Let $W\in\mathfrak{W}^\mu_{\mathrm{bad}}$. Then $W\subset\mathcal{O}_\alpha$. Moreover, there is a unique cube $Q=Q(W)\in\mathfrak{Q}_\alpha$ containing W.*

Proof. For the first statement, assume otherwise that there is $x\in W\cap\mathcal{O}_\alpha^{\complement}$ for some $W\in\mathfrak{W}^\mu_{\mathrm{bad}}$. Then $U(x)\le\alpha^p$ and therefore $\gamma_{W,\mu}^p\le\alpha^p$, which is a contradiction.

For the second statement, we first claim that $W^*\subset\widetilde{\mathcal{O}}_\alpha$, where W^* is the $10d^{1/2}$-dilate of W (with the same center). The claim follows because, for all $y\in W^*$, we have $M_{\mathrm{HL}}\mathbb{1}_{\mathcal{O}_\alpha}(y)\ge|W|/|W^*|=(10\sqrt{d})^{-d}$ by the first statement. Let x_W be the center of W. Then, by the claim,

$$
\mathrm{dist}(x_W,(\widetilde{\mathcal{O}}_\alpha)^{\complement}) \ge \mathrm{dist}(x_W,(W^*)^{\complement}) = \frac{\mathrm{diam}(W^*)}{2\sqrt{d}} = 5\,\mathrm{diam}(W).
$$

Let $Q \in \mathfrak{Q}_\alpha$ be such that $x_W \in Q$. Then the last displayed inequality implies that

$$5\,\mathrm{diam}(W) \le \mathrm{dist}(x_W, (\widetilde{\mathcal{O}}_\alpha)^{\complement}) \le \mathrm{diam}(Q) + \mathrm{dist}(Q, (\widetilde{\mathcal{O}}_\alpha)^{\complement}) \le 5\,\mathrm{diam}(Q),$$

and hence $\mathrm{diam}(Q) \ge \mathrm{diam}(W)$. Since both W, Q are dyadic cubes containing x_W, this implies that $W \subset Q$. Uniqueness of Q follows since the cubes in $\mathfrak{Q}_\alpha$ have disjoint interior. □

In light of Lemma 4.1, we also set, for a dyadic cube $Q \in \mathfrak{Q}_\alpha$,

$$\mathfrak{W}^\mu(Q) = \{W \in \mathfrak{W}^\mu_{\mathrm{bad}} : W \subset Q\}. \tag{4.11}$$

Lemma 4.2. *Let $Q \in \mathfrak{Q}_\alpha$. Then*

$$\sum_\mu \sum_{W \in \mathfrak{W}^\mu(Q)} |W| \gamma^p_{W,\mu} \le 10^d \alpha^p |Q|.$$

Proof. Since Q is a Whitney cube for the set $\widetilde{\mathcal{O}}_\alpha$, there is an $\widetilde{x} \in \widetilde{\mathcal{O}}^{\complement}_\alpha \subset \mathcal{O}^{\complement}_\alpha$ such that $\mathrm{dist}(\widetilde{x}, Q) \le 4\,\mathrm{diam}(Q)$. If Q_* denotes the cube centered at $\widetilde{x}$ with diameter equal to $10\,\mathrm{diam}(Q)$, then $Q \subset Q_*$. Since $\widetilde{x} \in \mathcal{O}^{\complement}_\alpha$, we have $M_{\mathrm{HL}} U(\widetilde{x}) \le \alpha^p$. Hence $\int_Q U \le \int_{Q_*} U \le \alpha^p |Q_*| = 10^d \alpha^p |Q|$, and the assertion follows. □

4.2. **Outline of the proof of the weak type inequalities.** For $R \in \mathcal{R}_k$, let

$$e_R(x) = \mathbb{1}_R(x) \mathcal{L}_k f(x), \tag{4.12}$$

and, as in (4.10), split $\mathcal{L}_k f = g^k + b^k$, where

$$g^k = \sum_\mu \sum_{W \in \mathfrak{W}^\mu_{\mathrm{good}}} \sum_{R \in \mathcal{R}^\mu_k(W)} e_R, \tag{4.13}$$

$$b^k = \sum_\mu \sum_{W \in \mathfrak{W}^\mu_{\mathrm{bad}}} \sum_{R \in \mathcal{R}^\mu_k(W)} e_R. \tag{4.14}$$

In view of (4.9), it suffices to show that, for $2 \le q < \infty$,

$$\mathrm{meas}\Big\{x : \Big(\sum_k \Big(\int_I \Big|\sum_{j=1}^\infty 2^{-j\lambda(p)} T^k_j g^k(x,t)\Big|^q dt\Big)^{2/q}\Big)^{1/2} > \alpha/2\Big\} \lesssim \alpha^{-p} \|\mathfrak{S} f\|_p^p \tag{4.15}$$

and

$$\mathrm{meas}\Big\{x \in \widetilde{\mathcal{O}}^{\complement}_\alpha : \Big(\sum_k \Big(\int_I \Big|\sum_{j=1}^\infty 2^{-j\lambda(p)} T^k_j b^k(x,t)\Big|^q dt\Big)^{2/q}\Big)^{1/2} > \alpha/2\Big\} \lesssim \alpha^{-p} \|\mathfrak{S} f\|_p^p. \tag{4.16}$$

Since $\lambda(p) \ge 1/p - 1 > -1/q$, we can use Lemma 3.1 to bound

$$\begin{aligned}
&\Big\|\Big(\sum_k \Big(\int_I \Big|\sum_{j=1}^\infty 2^{-j\lambda(p)} T^k_j g^k(x)\Big|^q dt\Big)^{2/q}\Big)^{1/2}\Big\|_2 \\
&\lesssim \sum_{j=1}^\infty 2^{-j\lambda(p)} \Big\|\Big(\sum_k \Big(\int_I |T^k_j g^k(x)|^q dt\Big)^{2/q}\Big)^{1/2}\Big\|_2 \\
&\lesssim \sum_{j=1}^\infty 2^{-j(\lambda(p)+\frac{1}{q})} \Big(\sum_k \|g^k\|_2^2\Big)^{1/2} \lesssim \Big(\sum_k \|g^k\|_2^2\Big)^{1/2}.
\end{aligned}$$

Hence, by Tshebyshev's inequality, the left-hand side of (4.15) is bounded by

$$4\alpha^{-2}\Big\|\Big(\sum_k\Big(\int_I\Big|\sum_{j=1}^\infty 2^{-j\lambda(p)}T_j^k g^k(x)\Big|^q dt\Big)^{2/q}\Big)^{1/2}\Big\|_2^2 \lesssim \alpha^{-2}\sum_k \|g^k\|_2^2.$$

Now

$$\sum_k \|g^k\|_2^2 = \sum_k\Big\|\sum_\mu \sum_{W\in\mathfrak{W}^\mu_{\mathrm{good}}}\sum_{R\in\mathcal{R}_k^\mu(W)} e_R\Big\|_2^2 \le \sum_k\sum_\mu \sum_{W\in\mathfrak{W}^\mu_{\mathrm{good}}}\sum_{R\in\mathcal{R}_k^\mu(W)} \|e_R\|_2^2$$
$$= \sum_\mu \sum_{W\in\mathfrak{W}^\mu_{\mathrm{good}}} |W|\gamma_{W,\mu}^2 \lesssim \alpha^{2-p}\sum_\mu\sum_{W\in\mathfrak{W}^\mu}|W|\gamma_{W,\mu}^p \lesssim \alpha^{2-p}\|\mathfrak{S}f\|_p^p,$$

where we have used $\gamma_{W,\mu}\le\alpha$ for $W\in\mathfrak{W}^\mu_{\mathrm{good}}$. Equation (4.15) follows.

We turn to (4.16). We write $L(Q)=m$ if the side length of Q is 2^m. Define, for $m\ge -k$,

$$B_m^k = \sum_{\substack{Q\in\mathfrak{Q}_\alpha\\ L(Q)=m}}\sum_\mu\sum_{W\in\mathfrak{W}^\mu(Q)}\sum_{R\in\mathcal{R}_k^\mu(W)} e_R \tag{4.17}$$

so that $b^k=\sum_{m\ge -k}B_m^k$.

Note that, for $R\in\mathcal{R}_k^\mu(W)$, we have $L(W)\ge -k$. Then

$$b^k=\sum_{m\ge -k}B_m^k=\sum_{m\ge -k}\sum_{\sigma\ge 0}B^k_{m,\sigma},$$

where

$$B^k_{m,\sigma}=\sum_{\substack{Q\in\mathfrak{Q}_\alpha\\ L(Q)=m}}\sum_\mu\sum_{\substack{W\in\mathfrak{W}^\mu(Q)\\ L(W)=-k+\sigma}}\sum_{R\in\mathcal{R}_k^\mu(W)} e_R. \tag{4.18}$$

We handle the case of the contributions $T_j^kB^k_{m,\sigma}$ with $m\le j-k$ differently from those with $m>j-k$. Moreover, we distinguish the cases in which $|j-k-m|\ge\sigma$ and $|j-k-m|<\sigma$. If we use Tshebyshev's inequality and take into account (4.9), we see that, in order to establish (4.16), it suffices to show the following three inequalities, assuming that $2\le q<p'$ (and hence $p<q'\le 2$):

(4.19)
$$\Big\|\Big(\sum_k\Big[\int_I\Big|\sum_{j=1}^\infty 2^{-j\lambda(p)}\sum_{\substack{(m,\sigma):m\le j-k,\\ 0\le\sigma\le j-m-k}} T_j^kB^k_{m,\sigma}(\cdot,t)\Big|^q dt\Big]^{q'/q}\Big)^{1/q'}\Big\|^{q'}_{L^{q'}(\mathbb{R}^d)} \lesssim \alpha^{q'-p}\|\mathfrak{S}f\|_p^p,$$

(4.20)
$$\Big\|\Big(\sum_k\Big[\int_I\Big|\sum_{j=1}^\infty 2^{-j\lambda(p)}\sum_{\substack{(m,\sigma):m> j-k,\\ 0\le\sigma\le m+k-j}} T_j^kB^k_{m,\sigma}(\cdot,t)\Big|^q dt\Big]^{p/q}\Big)^{1/p}\Big\|^{p}_{L^{p}(\mathbb{R}^d\setminus\widetilde{\mathcal{O}}_\alpha)} \lesssim \|\mathfrak{S}f\|_p^p,$$

and

(4.21)
$$\Big\|\Big(\sum_k\Big[\int_I\Big|\sum_{j=1}^\infty 2^{-j\lambda(p)}\sum_{\substack{m,\sigma:\\ \sigma>|m+k-j|}} T_j^kB^k_{m,\sigma}(\cdot,t)\Big|^q dt\Big]^{2/q}\Big)^{1/2}\Big\|^{p}_{L^{p}(\mathbb{R}^d\setminus\widetilde{\mathcal{O}}_\alpha)} \lesssim \|\mathfrak{S}f\|_p^p.$$

The proofs will be given in Sections 4.4, 4.5, and 4.6. We shall handle the cases $p=1$, $2\le q<\infty$ and $1<p<2$, $2\le q<p'$ in a unified way, but we will need an additional analytic families interpolation argument for $1<p<2$.

4.3. **Analytic families.** Fix p, α, and consider for $0\le \mathrm{Re}\,(z)\le 1$ the family of functions

$$b^{k,z}_{Q,\sigma}=\sum_{\mu}\sum_{\substack{W\in\mathfrak{W}^\mu(Q)\\ L(W)=-k+\sigma}}\gamma_{W,\mu,z}\sum_{R\in\mathcal{R}^\mu_k(W)}e_R, \tag{4.22}$$

where for $W\in\mathfrak{W}^\mu_{\mathrm{bad}}$

$$\gamma_{W,\mu,z}=\gamma_{W,\mu}^{p(1-z/2)-1},$$

and Q belongs to $\mathfrak{Q}_\alpha$. Observe that $b^{k,z}_{Q,\sigma}$ is supported in Q. Notice that $z\mapsto\gamma_{W,\mu,z}$ is an entire function for $W\in\mathfrak{W}^\mu_{\mathrm{bad}}$. We also set

$$B^{k,z}_{m,\sigma}=\sum_{\substack{Q\in\mathfrak{Q}_\alpha\\ L(Q)=m}}b^{k,z}_{Q,\sigma} \tag{4.23}$$

and, for $0\le\mathrm{Re}\,(z)\le 1$, define p_z and $\lambda(p_z)$ by

$$\frac{1}{p_z}=1-z+\frac{z}{2},\qquad \lambda(p_z)=\frac{d(1-z)-1}{2}. \tag{4.24}$$

If $1<p<2$, then we set $\vartheta=2-2/p$ so that

$$p_\vartheta=p,\quad \lambda(p_\vartheta)=d(\frac1p-\frac12)-\frac12,\quad B^{k,\vartheta}_{m,\sigma}=B^k_{m,\sigma}.$$

For $\mathrm{Re}\,(z)=1$, we have the following.

Lemma 4.3. *For fixed k, $m\ge -k$, let $\mathcal{N}_{k,m}\subset\mathbb{Z}$. Then*

$$\sum_{k\in\mathbb{Z}}\sum_{m\ge -k}\Big\|\sum_{\sigma\in\mathcal{N}_{k,m}}B^{k,z}_{m,\sigma}\Big\|_2^2\lesssim\|\mathfrak{S}f\|_p^p,\qquad \mathrm{Re}\,(z)=1.$$

Proof. The left-hand side is equal to

$$\sum_k\sum_{m\ge -k}\Big\|\sum_{\substack{Q\in\mathfrak{Q}_\alpha\\ L(Q)=m}}\sum_\mu\sum_{\sigma\in\mathcal{N}_{k,m}}\sum_{\substack{W\in\mathfrak{W}^\mu(Q)\\ L(W)=-k+\sigma}}\gamma_{W,\mu}^{p(1-z/2)-1}\sum_{R\in\mathcal{R}^\mu_k(W)}e_R\Big\|_2^2.$$

Let, for each W, $Q(W)$ be the unique cube in $\mathfrak{Q}_\alpha$ such that $W\subset Q$. We use the fact that, for fixed k, the supports of the functions e_R, $R\in\mathcal{R}_k$, have disjoint interior and dominate for $\mathrm{Re}\,(z)=1$ the last display by

$$\begin{aligned}&\sum_k\sum_{m\ge -k}\sum_{\substack{Q\in\mathfrak{Q}_\alpha\\ L(Q)=m}}\sum_\mu\sum_{\substack{W\in\mathfrak{W}^\mu(Q):\\ L(W)+k\in\mathcal{N}_{k,m}}}\gamma_{W,\mu}^{(\frac p2-1)2}\sum_{R\in\mathcal{R}^\mu_k(W)}\|e_R\|_2^2\\ &\quad\lesssim\sum_\mu\sum_{W\in\mathfrak{W}^\mu_{\mathrm{bad}}}\gamma_{W,\mu}^{p-2}\sum_{\substack{k:L(W)+k\in\\ \mathcal{N}_{k,L(Q(W))}}}\sum_{R\in\mathcal{R}^\mu_k(W)}\|e_R\|_2^2\le\sum_\mu\sum_{W\in\mathfrak{W}^\mu}\gamma_{W,\mu}^p|W|\lesssim\|\mathfrak{S}f\|_p^p.\end{aligned}$$

□

For $\mathrm{Re}\,(z)=0$, we have the following.

Lemma 4.4. *There exists a universal constant C dependent only on the dimension such that, for every $Q \in \mathfrak{Q}_\alpha$ and every $\mathcal{N} \subset \mathbb{N} \cup \{0\}$,*

$$\int \Big(\sum_k \Big| \sum_{\sigma \in \mathcal{N}} b^{k,z}_{Q,\sigma}(x) \Big|^2 \Big)^{1/2} dx \le C \alpha^p |Q| \qquad \text{if } \operatorname{Re}(z) = 0.$$

Proof. For each $W \in \mathfrak{W}^\mu$, let W_* be its double. By Minkowski's inequality, the left-hand side is dominated by

$$\sum_\mu \sum_{\substack{W \in \mathfrak{W}^\mu \\ W \subset Q}} \gamma^{p-1}_{W,\mu} \int \Big(\sum_{\substack{k: \\ k+L(W) \in \mathcal{N}}} \Big| \sum_{R \in \mathcal{R}^\mu_k(W)} e_R(x) \Big|^2 \Big)^{1/2} dx$$
$$\lesssim \sum_\mu \sum_{\substack{W \in \mathfrak{W}^\mu \\ W \subset Q}} \gamma^{p-1}_{W,\mu} \int_{W_*} \Big(\sum_k \sum_{R \in \mathcal{R}^\mu_k(W)} |e_R(x)|^2 \Big)^{1/2} dx,$$

which by the Cauchy–Schwarz inequality can be estimated by

$$\sum_\mu \sum_{\substack{W \in \mathfrak{W}^\mu \\ W \subset Q}} \gamma^{p-1}_{W,\mu} \Big(\sum_k \sum_{R \in \mathcal{R}^\mu_k(W)} \|e_R\|^2 \Big)^{1/2} |W_*|^{1/2} \lesssim \sum_\mu \sum_{\substack{W \in \mathfrak{W}^\mu \\ W \subset Q}} |W^*| \gamma^p_{W,\mu} \lesssim \alpha^p |Q|.$$

Here we have used Lemma 4.2. □

4.4. **Proof of** (4.19)**.** Let $1 \le p < 2$, and let $2 \le q < p'$. The asserted inequality follows from

$$\text{(4.25a)} \quad \Big\| \Big(\sum_k \Big(\int_I \Big| \sum_{j \ge 2s} \sum_{0 \le \sigma \le s} 2^{-j\lambda(p)} T^k_j B^k_{j-k-s,\sigma}(\cdot, t) \Big|^q dt \Big)^{q'/q} \Big)^{1/q'} \Big\|_{q'}$$
$$\lesssim (1+s)^{1-\frac{2}{q}} 2^{-s(d-1)(\frac{1}{p}-\frac{1}{q'})} \alpha^{p(\frac{1}{p}-\frac{1}{q'})} \|\mathfrak{S} f\|_p^{p/q'}$$

and

$$\text{(4.25b)} \quad \Big\| \Big(\sum_k \Big(\int_I \Big| \sum_{0 \le \sigma \le s} 2^{-j\lambda(p)} T^k_j B^k_{j-k-s,\sigma}(\cdot, t) \Big|^q dt \Big)^{q'/q} \Big)^{1/q'} \Big\|_{q'}$$
$$\lesssim (1+j)^{1-\frac{2}{q}} 2^{-j\frac{d-1}{2}(\frac{1}{p}-\frac{1}{q'})} \alpha^{p(\frac{1}{p}-\frac{1}{q'})} \|\mathfrak{S} f\|_p^{p/q'}, \qquad \frac{j}{2} \le s \le j.$$

If in addition $p > 1$, we use a complex interpolation argument, embedding $B^k_{m,\sigma}$ in an analytic family of functions; see (4.23).

Define r by

$$\text{(4.26)} \qquad \frac{1}{r} = \Big(\frac{1}{p} - \frac{1}{q} \Big) \Big/ \Big(\frac{2}{p} - 1 \Big)$$

so that $1 < r \le 2$ and for $\vartheta = 2 - 2/p$ we have $(1-\vartheta)(1, \frac{1}{r}) + \vartheta(\frac{1}{2}, \frac{1}{2}) = (\frac{1}{p}, \frac{1}{q'})$. Then, by complex interpolation (i.e., the three line lemma and duality), we deduce (4.25a) and (4.25b) from

(4.27a)

$$\Big\| \Big(\sum_k \int_I \Big| \sum_{j \ge 2s} \sum_{0 \le \sigma \le s} 2^{-j\lambda(p_z)} T^k_j B^{k,z}_{j-k-s,\sigma}(\cdot, t) \Big|^2 dt \Big)^{1/2} \Big\|_2 \lesssim \|\mathfrak{S} f\|_p^{p/2}, \qquad \operatorname{Re}(z) = 1,$$

$$\Big\|\Big(\sum_k \int_I \Big| \sum_{0\le\sigma\le s} 2^{-j\lambda(p_z)} T_j^k B^{k,z}_{j-k-s,\sigma}(\cdot,t)\Big|^2 dt\Big)^{1/2}\Big\|_2 \lesssim \|\mathfrak{G}f\|_p^{p/2}, \qquad \frac{j}{2}\le s\le j, \mathrm{Re}\,(z)=1, \tag{4.27b}$$

and

$$\Big\|\Big(\sum_k \int_I \Big| \sum_{j\ge 2s}\sum_{0\le\sigma\le s} 2^{-j\lambda(p_z)} T_j^k B^{k,z}_{j-k-s,\sigma}(\cdot,t)\Big|^{r'} dt\Big)^{1/r'}\Big\|_r \lesssim (1+s)^{\frac{2}{r}-1} 2^{-s(d-1)(1-\frac{1}{r})} \alpha^{p(1-\frac{1}{r})} \|\mathfrak{G}f\|_p^{p/r}, \ \mathrm{Re}\,(z)=0, \tag{4.28a}$$

$$\Big\|\Big(\sum_k \int_I \Big| \sum_{0\le\sigma\le s} 2^{-j\lambda(p_z)} T_j^k B^{k,z}_{j-k-s,\sigma}(\cdot,t)\Big|^{r'} dt\Big)^{1/r'}\Big\|_r \lesssim (1+j)^{\frac{2}{r}-1} 2^{-j\frac{d-1}{2}(1-\frac{1}{r})} \alpha^{p(1-\frac{1}{r})} \|\mathfrak{G}f\|_p^{p/r}, \quad \frac{j}{2}\le s\le j, \quad \mathrm{Re}\,(z)=0. \tag{4.28b}$$

We note that, for the special case $p=1$, inequalities (4.28a), (4.28b) with $r=q'$ and $z=0$ imply inequalities (4.25a), (4.25b) with $p=1$.

The proof of (4.27a), (4.27b) is straightforward, using orthogonality, i.e., the fact that, for each k, t, ξ, there are at most five j for which $\varphi_j(\rho(2^{-k}t^{-1}\xi))\neq 0$. Therefore, we get for $\mathrm{Re}\,(z)=1$ (and thus $\mathrm{Re}\,(\lambda(p_z))=-1/2$)

$$\begin{aligned}
&\Big\|\Big(\sum_k \int_I \Big| \sum_{j\ge 2s}\sum_{0\le\sigma\le s} 2^{-j\lambda(p_z)} T_j^k B^{k,z}_{j-k-s,\sigma}(\cdot,t)\Big|^2 dt\Big)^{1/2}\Big\|_2^2 \\
&\lesssim \sum_k \int_I \sum_{j\ge 2s} 2^j \int |\phi_j(\rho(2^{-k}t^{-1}\xi))|^2 \Big|\sum_{0\le\sigma\le s} \widehat{B^{k,z}_{j-k-s,\sigma}}(\xi)\Big|^2 d\xi\, dt \\
&\lesssim \sum_{k\in\mathbb{Z}}\sum_{j\ge 2s} \Big\|\sum_{0\le\sigma\le s} B^{k,z}_{j-k-s,\sigma}\Big\|_2^2 = \sum_k \sum_{m\ge -k+s} \Big\|\sum_{0\le\sigma\le s} B^{k,z}_{m,\sigma}\Big\|_2^2 \lesssim \|\mathfrak{G}f\|_p^p
\end{aligned}$$

by Lemma 4.3. Similarly, for fixed j,

$$\begin{aligned}
&\Big\|\Big(\sum_k \int_I \Big| \sum_{0\le\sigma\le s} 2^{-j\lambda(p_z)} T_j^k B^{k,z}_{j-k-s,\sigma}(\cdot,t)\Big|^2 dt\Big)^{1/2}\Big\|_2^2 \\
&\lesssim \sum_k \Big\|\sum_{0\le\sigma\le s} B^{k,z}_{j-k-s,\sigma}\Big\|_2^2 \lesssim \|\mathfrak{G}f\|_p^p.
\end{aligned}$$

This concludes the proof of (4.27a) and (4.27b).

We now come to the main part of the proof—namely, the inequalities (4.28a), (4.28b) when $1\le p<2$ and $\mathrm{Re}\,(z)=0$. We fix z with $\mathrm{Re}\,(z)=0$ and then use another interpolation inequality based on

$$[L^1(\ell^1(L^\infty)), L^2(\ell^2(L^2))]_\theta = L^r(\ell^r(L^{r'})) \qquad \text{for } \theta = 2-\frac{2}{r},$$

where Calderón's complex interpolation method is applied to vector-valued L^p spaces (see [2, Theorem 5.1.2]. As a consequence, we have

$$\|\cdot\|_{L^r(\ell^r(L^{r'}))} \lesssim \|\cdot\|_{L^1(\ell^1(L^\infty))}^{\frac{2}{r}-1} \|\cdot\|_{L^2(\ell^2(L^2))}^{2-\frac{2}{r}}.$$

Assuming that $1 \le p < 2$, (4.28a), (4.28b) follow from

(4.29a)
$$\Big\|\Big(\sum_k \int_I \Big|\sum_{j\ge 2s}\sum_{0\le\sigma\le s} 2^{-j\lambda(p_z)} T_j^k B^{k,z}_{j-k-s,\sigma}(\cdot,t)\Big|^2 dt\Big)^{1/2}\Big\|_2 \lesssim 2^{-s\frac{d-1}{2}}\alpha^{p/2}\|\mathfrak{S}f\|_p^{p/2}, \qquad \mathrm{Re}\,(z)=0,$$

(4.29b)
$$\Big\|\Big(\sum_k \int_I \Big|\sum_{0\le\sigma\le s} 2^{-j\lambda(p_z)} T_j^k B^{k,z}_{j-k-s,\sigma}(\cdot,t)\Big|^2 dt\Big)^{1/2}\Big\|_2 \lesssim 2^{-j\frac{d-1}{4}}\alpha^{p/2}\|\mathfrak{S}f\|_p^{p/2}, \qquad \frac{j}{2}\le s\le j,\ \mathrm{Re}\,(z)=0,$$

and

(4.30a)
$$\Big\|\sum_k \sup_{t\in I}\Big|\sum_{j\ge 2s}\sum_{0\le\sigma\le s} 2^{-j\lambda(p_z)} T_j^k B^{k,z}_{j-k-s,\sigma}(\cdot,t)\Big|\Big\|_1 \lesssim (1+s)\|\mathfrak{S}f\|_p^p, \qquad \mathrm{Re}\,(z)=0,$$

(4.30b)
$$\Big\|\sum_k \sup_{t\in I}\Big|\sum_{0\le\sigma\le s} 2^{-j\lambda(p_z)} T_j^k B^{k,z}_{j-k-s,\sigma}(\cdot,t)\Big|\Big\|_1 \lesssim (1+j)\|\mathfrak{S}f\|_p^p, \qquad \frac{j}{2}\le s\le j,\ \mathrm{Re}\,(z)=0.$$

This proof of (4.29a), (4.29b) is inspired by the work of Christ and Sogge [10], [11]. We use the decomposition (3.3) and orthogonality, first in the j sum and then, for each j, also in the ν sums, where $\nu \in \mathcal{Z}_j$. We then see that

(4.31)
$$\begin{aligned}
&\Big\|\Big(\sum_k \int_I \Big|\sum_{j\ge 2s}\sum_{0\le\sigma\le s} 2^{-j\lambda(p_z)} T_j^k B^{k,z}_{j-k-s,\sigma}(\cdot,t)\Big|^2 dt\Big)^{1/2}\Big\|_2^2 \\
&\quad\lesssim \sum_k\sum_{j\ge 2s}\sum_{\nu\in\mathcal{Z}_j} 2^{-j(d-1)}\int_I \Big\|\sum_{0\le\sigma\le s} T^k_{j,\nu} B^{k,z}_{j-k-s,\sigma}(\cdot,t)\Big\|_2^2 dt \\
&\quad= \sum_m\sum_{k\ge 2s-m}\sum_{\nu\in\mathcal{Z}_{k+m}} 2^{-(k+m)(d-1)}\int_I \Big\|\sum_{0\le\sigma\le s} T^k_{k+m,\nu} B^{k,z}_{m-s,\sigma}(\cdot,t)\Big\|_2^2 dt.
\end{aligned}$$

We use
$$\int_I \|T^k_{j,\nu} g\|_2^2 dt = \int_I\iint (2^k t)^d h_{j,\nu}(2^k t(x-y)) g(y)\overline{g(x)}\,dy\,dx\,dt,$$
where $h_{j,\nu}(x) = \mathcal{F}^{-1}[|\chi_{j,\nu}\varphi_j(\rho(\cdot))|^2](x)$. The kernel $h_{j,\nu}$ satisfies kernel estimates which are analogous to the right-hand side of (3.6), i.e.,
$$\sup_{t\in I}|t^d h_{j,\nu}(tx)| \lesssim_N \frac{2^{-j\frac{d+1}{2}}}{(1+2^{-j}|x|)^N(1+2^{-j/2}|P_{j,\nu}(x)|)^N}.$$

Using $j=k+m$, we can then estimate, for $t\in I$,
$$2^{-(k+m)(d-1)}\Big\|\sum_{0\le\sigma\le s} T^k_{k+m,\nu} B^{k,z}_{m-s,\sigma}(\cdot,t)\Big\|_2^2 \le C_N \times \iint 2^{-(k+m)\frac{d-1}{2}} \frac{2^{-md}}{(1+2^{-m}|x-y|)^N}\frac{1}{(1+2^{-m+\frac{k+m}{2}}|P_{k+m,\nu}(x-y)|)^N}|\beta^{k,z}_{m,s}(y)|\,dy\,|\beta^{k,z}_{m,s}(x)|\,dx,$$

with

$$\beta^{k,z}_{m,s} := \sum_{0\le\sigma\le s} B^{k,z}_{m-s,\sigma}. \tag{4.32}$$

Consider a maximal set Z^s of $c2^{-s}$ separated unit vectors η_ς, and let P^s_ς be the orthogonal projection to the orthogonal complement of $\nabla\rho(\eta_\varsigma)$. Notice that, for each ς, there are $\approx 2^{(d-1)(\frac{j}{2}-s)}$ of the vectors ξ_ν with $\nu \in \mathcal{Z}_j$ which are of distance $\le C2^{-s}$ to η_ς. For those ν, we then have $|\frac{\nabla\rho(\xi_\nu)}{|\nabla\rho(\xi_\nu)|} - \frac{\nabla\rho(\eta_\varsigma)}{|\nabla\rho(\eta_\varsigma)|}| = O(2^{-s})$. Consequently, for those ν and $j = k+m \ge 2s$,

$$\begin{aligned}&\frac{2^{-md}}{(1+2^{-m}|x-y|)^N}\frac{1}{(1+2^{-m+\frac{k+m}{2}}|P_{k+m,\nu}(x-y)|)^N}\\ &\qquad\lesssim_N \frac{2^{-md}}{(1+2^{-m}|x-y|)^N}\frac{1}{(1+2^{-m+s}|P^s_\varsigma(x-y)|)^N},\end{aligned}$$

and there are $O(2^{(d-1)(\frac{k+m}{2}-s)})$ indices $\nu \in \mathcal{Z}_{k+m}$ for which we may use this inequality. Then, setting

$$A_{k,m,\varsigma}(x) = \int 2^{-s(d-1)}\frac{2^{-md}}{(1+2^{-m}|x-y|)^N}\frac{1}{(1+2^{-m+s}|P^s_\varsigma(x-y)|)^N}|\beta^{k,z}_{m,s}(y)|\,dy, \tag{4.33}$$

we get by the above considerations

$$\begin{aligned}(4.31) &\lesssim \sum_{\varsigma\in Z^s}\sum_m\sum_{k\ge 2s-m}\int A_{k,m,\varsigma}(x)|\beta^{k,z}_{m,s}(x)|dx\\ &\lesssim \sum_{\varsigma\in Z^s}\sum_m\int\Big(\sum_{k\ge 2s-m}[A_{k,m,\varsigma}(x)]^2\Big)^{1/2}\Big(\sum_k|\beta^{k,z}_{m,s}(x)|^2\Big)^{1/2}dx.\end{aligned} \tag{4.34}$$

We first establish that

$$\sup_m \sum_{\varsigma\in Z^s}\Big\|\Big(\sum_{k\ge 2s-m}|A_{k,m,\varsigma}|^2\Big)^{1/2}\Big\|_\infty \lesssim \alpha^p 2^{-s(d-1)}. \tag{4.35}$$

For each dyadic cube Q, let y_Q be the center of Q. Using (4.33), we estimate for fixed $x\in\mathbb{R}^d$

$$\begin{aligned}\Big(\sum_{k\ge 2s-m}|A_{k,m,\varsigma}(x)|^2\Big)^{1/2} &\lesssim 2^{-s(d-1)}\\ &\times \sum_{\substack{Q:\\ L(Q)=m-s}}\frac{2^{-md}}{(1+2^{-m}|x-y_Q|)^N}\frac{1}{(1+2^{-m+s}|P^s_\varsigma(x-y_Q)|)^N}\\ &\times\int\Big(\sum_k\Big|\sum_{0\le\sigma\le s}b^{k,z}_{Q,\sigma}(y)\Big|^2\Big)^{1/2}dy,\end{aligned}$$

and using Lemma 4.4, we bound this expression by

$$2^{-s(d-1)} \sum_{\substack{Q:\\ L(Q)=m-s}} \frac{2^{-md}}{(1+2^{-m}|x-y_Q|)^N} \frac{1}{(1+2^{-m+s}|P^s_\varsigma(x-y_Q)|)^N} \alpha^p |Q|$$
$$\lesssim \alpha^p 2^{-s(d-1)} \int \frac{2^{-md}}{(1+2^{-m}|x-w|)^N} \frac{1}{(1+2^{-m+s}|P^s_\varsigma(x-w)|)^N} dw$$
$$\lesssim \alpha^p 2^{-2s(d-1)}.$$

We sum over $\varsigma \in Z^s$ and use $\#Z^s = O(2^{s(d-1)})$ to obtain (4.35).

Combining (4.35) and (4.34), we obtain

$$(4.31) \lesssim 2^{-s(d-1)} \alpha^p \sum_m \sum_{\substack{Q\in\mathfrak{Q}_\alpha:\\ L(Q)=m-s}} \Big\| \Big(\sum_k \Big| \sum_{0\le\sigma\le s} b^{k,z}_{Q,\sigma} \Big|^2 \Big)^{1/2} \Big\|_1 .$$

Finally, by Lemma 4.4 again,

$$\sum_m \sum_{\substack{Q\in\mathfrak{Q}_\alpha:\\ L(Q)=m-s}} \Big\| \Big(\sum_k \Big| \sum_{0\le\sigma\le s} b^{k,z}_{Q,\sigma} \Big|^2 \Big)^{1/2} \Big\|_1 \lesssim \sum_{Q\in\mathfrak{Q}_\alpha} |Q|\alpha^p \lesssim \alpha^p |\widetilde{\mathcal{O}}_\alpha| \lesssim \|\mathfrak{S} f\|_p^p$$

by (4.9). This finishes the proof of (4.29a).

The proof of (4.29b) uses the same idea. We estimate, for fixed $j \in [s/2, s]$, $\mathrm{Re}\,(z) = 0$,

$$(4.36)\quad \Big\| \Big(\sum_k \int_I \Big| \sum_{0\le\sigma\le s} 2^{-j\lambda(p_z)} T^k_j B^{k,z}_{j-k-s,\sigma}(\cdot,t) \Big|^2 dt \Big)^{1/2} \Big\|_2^2$$
$$\lesssim 2^{-j(d-1)} \sum_k \int_I \big\| T^k_j \beta^{k,z}_{j-k,s}(\cdot,t) \big\|_2^2 \lesssim 2^{-j(d-1)} \sum_{\nu\in\mathcal{Z}_j} \sum_k \int_I \big\| T^k_{j,\nu} \beta^{k,z}_{j-k,s}(\cdot,t) \big\|_2^2 dt$$
$$\lesssim 2^{-j(d-1)} \sum_{\nu\in\mathcal{Z}_j} \sum_k \int \mathcal{A}_{k,j,\nu}(x) \big|\beta^{k,z}_{j-k,s}(x)\big|\, dx,$$

where again $\beta^{k,z}_{m,s}$ is as in (4.32) and

$$\mathcal{A}_{k,j,\nu}(x) := \int \frac{2^{kd} 2^{-j\frac{d+1}{2}}}{(1+2^{k-j}|x-y|)^N (1+2^{k-\frac{j}{2}}|P_{j,\nu}(x-y)|)^N} |\beta^{k,z}_{j-k,s}(y)|\, dy.$$

Now $\mathcal{A}_{k,j,\nu}(x) \lesssim$

$$\int \frac{2^{kd} 2^{-j\frac{d+1}{2}}}{(1+2^{k-j}|x-w|)^N (1+2^{k-\frac{j}{2}}|P_{j,\nu}(x-w)|)^N} dw \sup_{\substack{Q\in\mathfrak{Q}_\alpha\\ L(Q)=j-k-s}} \frac{1}{|Q|} \int \Big| \sum_{0\le\sigma\le s} b^{k,z}_{Q,\sigma}(y) \Big| dy,$$

which is bounded by $C\alpha^p$. Consequently,

$$(4.36) \lesssim 2^{-j(d-1)} \sum_{\nu\in\mathcal{Z}_j} \alpha^p \sum_k \sum_{\substack{Q\in\mathfrak{Q}_\alpha\\ L(Q)=j-k-s}} \Big\| \sum_{0\le\sigma\le s} b^{k,z}_{Q,\sigma}\Big\|_1$$
$$\lesssim 2^{-j\frac{d-1}{2}} \alpha^p \sum_{Q\in\mathfrak{Q}_\alpha} \Big\| \sum_{0\le\sigma\le s} b^{j-s-L(Q),z}_{Q,\sigma}\Big\|_1$$
$$\lesssim 2^{-j\frac{d-1}{2}} \alpha^p \sum_{Q\in\mathfrak{Q}_\alpha} \Big\| \Big(\sum_k \Big| \sum_{0\le\sigma\le s} b^{k,z}_{Q,\sigma}\Big|^2\Big)^{1/2}\Big\|_1 \lesssim 2^{-j\frac{d-1}{2}}\alpha^p \|\mathfrak{S}f\|_p^p$$

by Lemma 4.4.

We now turn to the proof of (4.30a), (4.30b), here still $\mathrm{Re}\,(z)=0$. We estimate the left-hand side of (4.30a) using Lemma 3.4 by

$$\sum_k \sum_{j\ge 2s} 2^{-j\frac{d-1}{2}} \sum_{0\le\sigma\le s} \big\| \sup_{t\in I} |T^k_j B^{k,z}_{j-k-s,\sigma}(\cdot,t)|\big\|_1 \lesssim \sum_k \sum_{j\ge 2s} \sum_{0\le\sigma\le s} \|B^{k,z}_{j-k-s,\sigma}\|_1,$$

and the right-hand side of the last display is dominated by

$$\sum_{0\le\sigma\le s}\sum_k\sum_{j\ge 2s}\sum_\mu \sum_{\substack{Q\in\mathfrak{Q}_\alpha:\\ L(Q)=j-k-s}} \sum_{\substack{W\in\mathfrak{W}^\mu(Q)\\ L(W)=-k+\sigma}} \gamma^{p-1}_{W,\mu} \Big\| \sum_{R\in\mathcal{R}^\mu_k(W)} e_R\Big\|_1$$
$$\lesssim \sum_{0\le\sigma\le s}\sum_k\sum_{j\ge 2s}\sum_\mu \sum_{\substack{Q\in\mathfrak{Q}_\alpha:\\ L(Q)=j-k-s}} \sum_{\substack{W\in\mathfrak{W}^\mu(Q)\\ L(W)=-k+\sigma}} \gamma^{p-1}_{W,\mu} |W|^{1/2} \Big(\sum_{R\in\mathcal{R}^\mu_k(W)} \|e_R\|_2^2\Big)^{1/2}$$
$$\lesssim \sum_{0\le\sigma\le s}\sum_\mu \sum_{W\in\mathfrak{W}^\mu} \gamma^p_{W,\mu}|W| \lesssim (1+s)\|\mathfrak{S}f\|_p^p.$$

The left-hand side of (4.30b) is estimated for fixed $j\in[s,2s]$ by

$$2^{-j\frac{d-1}{2}} \sum_k \sum_{0\le\sigma\le s} \Big\| \sup_{t\in I} \Big|T^k_j B^{k,z}_{j-k-s,\sigma}(\cdot,t)\Big|\Big\|_1 \lesssim \sum_{0\le\sigma\le s}\sum_k \|B^{k,z}_{j-k-s,\sigma}\|_1,$$

and the subsequent estimation is as for (4.30a). This concludes the proof of (4.19). □

4.5. **Proof of** (4.20)**.** It suffices to show, assuming that $1\le p<2$, $q=p'$, that, for some $a(p,q)>0$ and $s\ge 0$,

$$\Big\|\Big(\sum_k \Big[\int_I \Big|\sum_{j=1}^\infty 2^{-j\lambda(p)} \sum_{0\le\sigma\le s} T^k_j B^k_{j-k+s,\sigma}(\cdot,t)\Big|^q dt\Big]^{p/q}\Big)^{1/p}\Big\|_{L^p(\mathbb{R}^d\setminus\widetilde{\mathcal{O}}_\alpha)}$$
$$\lesssim 2^{-a(p,q)s}\|\mathfrak{S}f\|_p.$$

When $p>1$, we use the analytic family of functions in (4.23). It suffices to prove the inequalities

$$(4.37)\qquad \Big\|\Big(\sum_k \int_I \Big|\sum_{j=1}^\infty 2^{-j\lambda(p_z)} \sum_{0\le\sigma\le s} T^k_j B^{k,z}_{j-k+s,\sigma}(\cdot,t)\Big|^2 dt\Big)^{1/2}\Big\|_{L^2(\mathbb{R}^d\setminus\widetilde{\mathcal{O}}_\alpha)}$$
$$\lesssim \|\mathfrak{S}f\|_p^{p/2}, \qquad \mathrm{Re}\,(z)=1,$$

and

$$\Big\| \sum_k \sup_{t\in I} \Big| \sum_{j=1}^{\infty} 2^{-j\lambda(p_z)} \sum_{0\le\sigma\le s} T_j^k B^{k,z}_{j-k+s,\sigma}(\cdot,t) \Big| \Big\|_{L^1(\mathbb{R}^d\setminus\widetilde{\mathcal{O}}_\alpha)} \lesssim 2^{-\varepsilon s}\|\mathfrak{S}f\|_p^p, \quad \mathrm{Re}\,(z)=0, \tag{4.38}$$

for some $\varepsilon > 0$.

To show (4.37), we replace the $L^2(\mathbb{R}^d \setminus \widetilde{\mathcal{O}}_\alpha)$ norm with the $L^2(\mathbb{R}^d)$ norm and argue exactly as in the proof of (4.27a), using Lemma 4.3.

To show (4.38), it suffices to prove, after Minkowski's inequality for the σ-summation (involving $O(1+s)$ terms),

$$\Big\| \sum_k \sup_{t\in I} \Big| \sum_{j=1}^{\infty} 2^{-j\lambda(p_z)} T_j^k B^{k,z}_{j-k+s,\sigma}(\cdot,t) \Big| \Big\|_{L^1(\mathbb{R}^d\setminus\widetilde{\mathcal{O}}_\alpha)} \lesssim 2^{-\varepsilon s}\|\mathfrak{S}f\|_p^p, \qquad \mathrm{Re}\,(z)=0,\ 0\le\sigma\le s. \tag{4.39}$$

For the proof, observe that, for $t\in I$, $T_j^{n,k} B^{k,z}_{j-k+s,\sigma}(\cdot,t)$ is supported in $\widetilde{\mathcal{O}}_\alpha$ when $n \le s$ and thus does not contribute to the $L^1(\mathbb{R}^d \setminus \widetilde{\mathcal{O}}_\alpha)$ norm. We then use the simple L^1 estimate in Lemma 3.4(i) for $n > s$ and $\mathrm{Re}\,(\lambda(p_z)) = (d-1)/2$ to estimate the left-hand side of (4.39) by a constant times

$$2^{-sN} \sum_k \sum_j \|B^{k,z}_{j-k+s,\sigma}\|_1 \lesssim 2^{-sN} \sum_k \sum_j \sum_{\substack{Q\in\mathfrak{Q}_\alpha \\ L(Q)=j-k+s}} \sum_\mu \sum_{\substack{W\in\mathfrak{W}^\mu(Q) \\ L(W)=-k+\sigma}} \gamma_{W,\mu}^{p-1} \Big\| \sum_{R\in\mathcal{R}_k^\mu(W)} e_R \Big\|_1.$$

We interchange the sums and note that each W is contained in a unique cube $Q \in \mathfrak{Q}_\alpha$, and thus, because of the disjointness of the cubes in $\mathfrak{Q}_\alpha$, the (j,Q) sums corresponding to a fixed W collapse to a single term. Hence we can bound the previous expression by C_N times

$$2^{-sN} \sum_k \sum_\mu \sum_{\substack{W\in\mathfrak{W}^\mu \\ L(W)=-k+\sigma}} \gamma_{W,\mu}^{p-1} |W|^{1/2} \Big(\sum_{R\in\mathcal{R}_k^\mu(W)} \|e_R\|_2^2 \Big)^{1/2} \lesssim 2^{-sN} \sum_\mu \sum_{W\in\mathfrak{W}^\mu} \gamma_{W,\mu}^p |W| \lesssim 2^{-sN} \|\mathfrak{S}f\|_p^p.$$

This completes the proof of (4.20). □

4.6. **Proof of** (4.21)**.** The estimate follows from the inequalities

$$\Big\| \Big(\sum_k \Big[\int_I \Big| \sum_{j\ge1} 2^{-j\lambda(p)} \sum_{\substack{m,\sigma: \\ \sigma>|m+k-j|, \\ \sigma\ge j}} T_j^k B^k_{m,\sigma}(\cdot,t) \Big|^q dt \Big]^{2/q} \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^d\setminus\widetilde{\mathcal{O}}_\alpha)} \lesssim \|\mathfrak{S}f\|_p \tag{4.40}$$

and

(4.41)

$$\Big\|\Big(\sum_k\Big[\int_I\Big|\sum_{j\ge1}^{\infty}2^{-j\lambda(p)}\sum_{\substack{m,\sigma:\\ \sigma>|m+k-j|\\ \sigma<j}}T_j^kB^k_{m,\sigma}(\cdot,t)\Big|^q dt\Big]^{p/q}\Big)^{1/p}\Big\|_{L^p(\mathbb{R}^d)}\lesssim\|\mathfrak{S}f\|_p.$$

4.6.1. *Proof of* (4.40). We use the decomposition $T_j^k=\sum_{n>0}T_j^{n,k}$ and use Minkowski's inequality for the j and n sums. When $j+n\le\sigma$, the support of $T_j^{k,n}B^k_{m,\sigma}(\cdot,t)$ is contained in $\widetilde{\mathcal{O}}_\alpha$ for all $t\in I$. Thus in (4.40) we need to consider only the terms with $|m+k-j|<\sigma$ and $j\le\sigma\le j+n$. Since $\lambda(p)+1/q>0$, it suffices to show, for fixed $j\ge1$, that

(4.42)

$$\Big\|\Big(\sum_k\Big[\int_I\Big|\sum_{\substack{m,\sigma:\\ \sigma>|m+k-j|\\ j\le\sigma\le j+n}}T_j^{n,k}B^k_{m,\sigma}(\cdot,t)\Big|^q dt\Big]^{2/q}\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^d)}\lesssim2^{-n}2^{-j/q}\|\mathfrak{S}f\|_p.$$

This follows from

$$\Big\|\Big(\sum_k\Big[\int_I\Big|\sum_{\substack{m,\sigma:\\ \sigma>|m+k-j|\\ j\le\sigma\le j+n}}T_j^{n,k}B^{k,z}_{m,\sigma}(\cdot,t)\Big|^q dt\Big]^{2/q}\Big)^{1/2}\Big\|_{L^2(\mathbb{R}^d)}\lesssim2^{-n}2^{-j/q}\;\|\mathfrak{S}f\|_p^{p/2},\qquad \mathrm{Re}\,(z)=1,\tag{4.43}$$

and

$$\Big\|\Big(\sum_k\Big[\int_I\Big|\sum_{\substack{m,\sigma:\\ \sigma>|m+k-j|\\ j\le\sigma\le j+n}}T_j^{n,k}B^{k,z}_{m,\sigma}(\cdot,t)\Big|^q dt\Big]^{2/q}\Big)^{1/2}\Big\|_{L^1(\mathbb{R}^d)}\lesssim2^{-n}2^{-j/q}\|\mathfrak{S}f\|_p^{p},\qquad \mathrm{Re}\,(z)=0.\tag{4.44}$$

By Lemma 3.4(iii), the left-hand side of (4.43) is

$$\Big(\sum_k\Big\|\Big(\int_I\Big|\sum_{\substack{m,\sigma:\\ \sigma>|m+k-j|\\ j\le\sigma\le j+n}}T_j^{n,k}B^{k,z}_{m,\sigma}(\cdot,t)\Big|^q dt\Big)^{1/q}\Big\|_2^2\Big)^{1/2}$$
$$\lesssim2^{-n-j/q}\Big(\sum_k\Big\|\sum_{\substack{m,\sigma:\\ \sigma>|m+k-j|\\ j\le\sigma\le j+n}}B^{k,z}_{m,\sigma}\Big\|_2^2\Big)^{1/2}.$$

Recall that

$$\mathrm{supp}(B^{k,z}_{m,\sigma})\subset\bigcup_{\substack{Q\in\mathfrak{Q}_\alpha\\ L(Q)=m}}Q.$$

Therefore, for $\mathrm{Re}\,(z)=1$, we have

$$\Big(\sum_k\Big\|\sum_{\substack{m,\sigma:\\ \sigma>|m+k-j|\\ j\le\sigma\le j+n}}B^{k,z}_{m,\sigma}\Big\|_2^2\Big)^{1/2}=\Big(\sum_k\sum_{m\ge-k}\Big\|\sum_{\substack{\sigma:\\ \sigma>|m+k-j|\\ j\le\sigma\le j+n}}B^{k,z}_{m,\sigma}\Big\|_2^2\Big)^{1/2}\lesssim\|\mathfrak{S}f\|_p^p$$

by Lemma 4.3. Hence (4.43) follows.

We now turn to the proof of (4.44), where $\mathrm{Re}\,(z)=0$. For $W\in\mathfrak{W}^\mu_{\rm bad}$, let $Q(W)$ be the unique cube in $\mathfrak{Q}_\alpha$ containing W. We can split

$$B^{k,z}_{m,\sigma}=\sum_{\mu\in\mathbb{Z}}\sum_{\substack{W\in\mathfrak{W}^\mu_{\rm bad}\\ L(W)=-k+\sigma}}\widetilde{B}^{k,z}_{m,\sigma,\mu,W},$$

where

$$\widetilde{B}^{k,z}_{m,\sigma,\mu,W}=\begin{cases}\gamma_{W,\mu,z}\sum_{R\in\mathcal{R}^\mu_k(W)}e_R & \text{if } L(Q(W))=m \text{ and } L(W)=-k+\sigma,\\ 0 & \text{if either } L(Q(W))\neq m \text{ or } L(W)\neq -k+\sigma.\end{cases}$$

Observe that, for $j\le\sigma$, $L(W)=-k+\sigma$, the function $T^{n,k}_j\widetilde{B}^{k,z}_{m,\sigma,\mu,W}$ is supported in a 2^{n+3}-dilate of W (with respect to its center). Hence, by the Minkowski and Cauchy–Schwarz inequalities, we estimate for fixed j,n

$$\begin{aligned}&\Big\|\Big(\sum_k\Big[\int_I\Big|\sum_{\substack{m,\sigma:\\ \sigma>|m+k-j|\\ \sigma\le j+n}}T^{n,k}_jB^{k,z}_{m,\sigma}(\cdot,t)\Big|^q dt\Big]^{2/q}\Big)^{1/2}\Big\|_{L^1(\mathbb{R}^d)}\\
&\lesssim\sum_\mu\sum_{W\in\mathfrak{W}^\mu_{\rm bad}}2^{nd/2}|W|^{1/2}\\
&\quad\Big\|\Big(\sum_{\substack{k:|L(Q(W))+k-j|\\ <L(W)+k\le j+n}}\Big[\int_I\Big|T^{n,k}_j\widetilde{B}^{k,z}_{L(Q(W)),L(W)+k,\mu,W}(\cdot,t)\Big|^q dt\Big]^{2/q}\Big)^{1/2}\Big\|_2,\end{aligned}$$

which by an application of Lemma 3.4 is bounded by

$$\begin{aligned}&C_N2^{-n(N-d/2)}2^{-j/q}\sum_\mu\sum_{W\in\mathfrak{W}^\mu_{\rm bad}}|W|^{1/2}\\
&\qquad\Big(\sum_{\substack{k:\\ |L(Q(W))+k-j|<L(W)+k\\ L(W)+k\le j+n}}\Big\|\gamma_{W,\mu,z}\sum_{R\in\mathcal{R}^\mu_k(W)}e_R\Big\|_2^2\Big)^{1/2}\\
&\qquad\lesssim 2^{-n-j/q}\sum_\mu\sum_{W\in\mathfrak{W}^\mu}|W|^{1/2}\gamma^{p-1}_{W,\mu}\Big(\sum_k\sum_{R\in\mathcal{R}^\mu_k(W)}\|e_R\|_2^2\Big)^{1/2}\\
&\qquad\lesssim 2^{-n-j/q}\sum_\mu\sum_{W\in\mathfrak{W}^\mu}|W|\gamma^p_{W,\mu}\lesssim 2^{-n-j/q}\|\mathfrak{S}f\|_p^p.\end{aligned}$$

4.6.2. *Proof of* (4.41). By Minkowski's inequality, (4.41) follows if we can prove, for fixed $\sigma>0$,

$$\begin{aligned}&\Big\|\Big(\sum_k\Big[\int_I\Big|\sum_{j>\sigma}2^{-j\lambda(p)}\sum_{\substack{m:\\ \sigma>|m+k-j|}}T^k_jB^k_{m,\sigma}(\cdot,t)\Big|^q dt\Big]^{p/q}\Big)^{1/p}\Big\|_{L^p(\mathbb{R}^d)}\\
&\qquad\lesssim(1+\sigma)^{1/p}2^{-\sigma d(\frac1q-\frac1{p'})}\|\mathfrak{S}f\|_p.\end{aligned}\tag{4.45}$$

When $p>1$, we use complex interpolation to deduce this from

$$\begin{aligned}&\Big\|\Big(\sum_k\int_I\Big|\sum_{j>\sigma}2^{-j\lambda(p_z)}\sum_{\substack{m:\\ \sigma>|m+k-j|}}T^k_jB^{k,z}_{m,\sigma}(\cdot,t)\Big|^2 dt\Big)^{1/2}\Big\|_{L^2(\mathbb{R}^d)}\\
&\qquad\lesssim(1+\sigma)^{1/2}\|\mathfrak{S}f\|_p^{p/2},\qquad \mathrm{Re}\,(z)=1,\end{aligned}\tag{4.46}$$

and, with $\frac{1}{q_0}=(\frac{1}{p}-\frac{1}{q'})/(\frac{2}{p}-1)$,

$$\Big\|\sum_k\Big(\int_I\Big|\sum_{j>\sigma}2^{-j\lambda(p_z)}\sum_{\substack{m:\\ \sigma>|m+k-j|}}T_j^kB^{k,z}_{m,\sigma}(\cdot,t)\Big|^{q_0}dt\Big)^{1/q_0}\Big\|_{L^1(\mathbb{R}^d)} \lesssim(1+\sigma)2^{-\sigma d/q_0}\|\mathfrak{S}f\|_p^p,\qquad \mathrm{Re}\,(z)=0. \tag{4.47}$$

Note that $1/q_0=1-1/r$, where r is as in (4.26), and we have $(1-\vartheta)(1,\frac{1}{q_0})+\vartheta(\frac12,\frac12)=(\frac1p,\frac1q)$ for $\vartheta=2/p'$.

We first consider the inequality for $\mathrm{Re}\,(z)=1$. We can use the orthogonality of the functions $\varphi_j(\rho(\cdot/t))$ to estimate

$$\begin{aligned}&\Big\|\Big(\sum_k\int_I\Big|\sum_{j>\sigma}2^{-j\lambda(p_z)}\sum_{\substack{m:\\ \sigma>|m+k-j|}}T_j^kB^{k,z}_{m,\sigma}(\cdot,t)\Big|^2dt\Big)^{1/2}\Big\|_2\\ &\quad\lesssim\Big(\sum_k\int_I\sum_{j>\sigma}2^j\Big\|\sum_{\substack{m:\\ \sigma>|m+k-j|}}T_j^kB^{k,z}_{m,\sigma}(\cdot,t)\Big\|_2^2dt\Big)^{1/2}\\ &\quad\lesssim\Big(\sum_k\sum_{j>\sigma}\Big\|\sum_{\substack{m:\\ \sigma>|m+k-j|}}B^{k,z}_{m,\sigma}\Big\|_2^2\Big)^{1/2}.\end{aligned}$$

We use the disjointness of the cubes in $\mathfrak{Q}_\alpha$ and then interchange the m,j summations. Using the fact that, for fixed m,k, there are $O(1+\sigma)$ terms in the j summation, we bound the last expression by

$$\begin{aligned}\Big(\sum_k\sum_{j>\sigma}\sum_{\substack{m\geq -k:\\ \sigma>|m+k-j|}}\|B^{k,z}_{m,\sigma}\|_2^2\Big)^{1/2}&\lesssim(1+\sigma)^{1/2}\Big(\sum_k\sum_{m\geq-k}\|B^{k,z}_{m,\sigma}\|_2^2\Big)^{1/2}\\ &\lesssim(1+\sigma)^{1/2}\|\mathfrak{S}f\|_p^{p/2},\end{aligned}$$

where in the last line we have applied Lemma 4.3 to conclude (4.46).

We now turn to (4.47). We split $T_j^k=\sum_{n=0}^\infty T_j^{n,k}$, set

$$b^k_{W,\mu,z}=\gamma_{W,\mu,z}\sum_{R\in\mathcal{R}_k^\mu(W)}e_R,$$

and estimate the left-hand side of (4.47) by

$$\sum_k\sum_{n\geq0}\sum_{j>\sigma}2^{-j\frac{d-1}{2}} \sum_{\substack{m:\\ \sigma>|m+k-j|}}\sum_{\substack{Q\in\mathfrak{Q}_\alpha\\ L(Q)=m}}\sum_\mu\sum_{\substack{W\in\mathfrak{W}^\mu(Q)\\ L(W)=-k+\sigma}}\Big\|\Big(\int_I|T_j^{n,k}b^k_{W,\mu,z}(\cdot,t)|^{q_0}dt\Big)^{1/q_0}\Big\|_1. \tag{4.48}$$

We bound for fixed W, with $L(W)=-k+\sigma$,

$$\begin{aligned}&\Big\|\Big(\int_I|T_j^{n,k}b^k_{W,\mu,z}(\cdot,t)|^{q_0}dt\Big)^{1/q_0}\Big\|_1\\ &\quad\lesssim 2^{(-k+j+n)d/q_0}\Big\|\Big(\int_I|T_j^{n,k}b^k_{W,\mu,z}(\cdot,t)|^{q_0}dt\Big)^{1/q_0}\Big\|_{q_0'}\\ &\quad\lesssim 2^{(-k+j+n)d/q_0}2^{j(d(\frac{1}{q_0'}-\frac12)-\frac12)}2^{-nN}\|b^k_{W,\mu,z}\|_{q_0'}\end{aligned}$$

by Lemma 3.4(ii). Hence, after summing in n,

$$(4.48) \lesssim \sum_k \sum_{j>\sigma} \sum_{\substack{m:\\ \sigma>|m+k-j|}} \sum_{\substack{Q\in\mathfrak{Q}_\alpha\\ L(Q)=m}} \sum_\mu \sum_{\substack{W\in\mathfrak{W}^\mu(Q)\\ L(W)=-k+\sigma}} 2^{-kd/q_0}\|b^k_{W,\mu,z}\|_{q_0'}.$$

Observe that, for $L(W)+k=\sigma$,

$$\begin{aligned}2^{-kd/q_0}\|b^k_{W,\mu,z}\|_{q_0'} &\le 2^{-kd/q_0}|W|^{1/q_0'-1/2}\|b^k_{W,\mu,z}\|_2\\ &\lesssim 2^{-\sigma d/q_0}|W|^{1/2}\gamma^{p-1}_{W,\mu}\Big(\sum_{R\in\mathcal{R}^\mu_k(W)}\|e_R\|_2^2\Big)^{1/2} \le 2^{-\sigma d/q_0}|W|\gamma^p_{W,\mu}.\end{aligned}$$

We interchange summations and use, for fixed $W\in\mathfrak{W}^\mu_{\mathrm{bad}}$,

$$\#\{j\ge\sigma : |L(Q(W))+\sigma-L(W)-j|<\sigma\} = O(1+\sigma).$$

We then obtain

$$(4.48) \lesssim 2^{-\sigma d/q_0}(1+\sigma)\sum_\mu \sum_{W\in\mathfrak{W}^\mu}|W|\gamma^p_{W,\mu} \lesssim 2^{-\sigma d/q_0}(1+\sigma)\|\mathfrak{S}f\|_p^p.$$

This completes the proof of (4.41), then (4.21), and finally the proof of Theorem 2.3.

5. Proofs of Theorems 1.2 and 1.3

In this section, we use Theorems 2.1 and 2.2 proved in $\mathbb{R}^d$ and transference argument to establish the corresponding versions for periodic functions. Such transference arguments go back to de Leeuw [25]. See also [20] for transference of maximal operators and [26], [14] inequalities in Hardy spaces on $\mathbb{T}^d$. In our presentation, we rely on the method in [14].

5.1. **The $h^1(\mathbb{T}^d)\to L^{1,\infty}(\mathbb{T}^d)$ bound.** We identify functions f on $\mathbb{T}^d$ with functions on $\mathbb{R}^d$ satisfying $f(x+n)=f(x)$ for all $n\in\mathbb{Z}^d$. Let $Q^0=[-\frac12,\frac12]^d$.

Let

$$h_\lambda(s) = (1-\upsilon_0(s))(1-s)_+^\lambda$$

and $\mathcal{S}^\lambda_t f = \sum_{\ell\in\mathbb{Z}^d} h(\rho(\ell/t))\langle f, e_\ell\rangle e_\ell$. Let $\lambda(1)=\frac{d-1}{2}$. After a reduction analogous to the one in Section 2.1, we need to prove the bound

$$\Big\|\Big(\sum_{k>0}\int_I |\mathcal{S}^{\lambda(1)}_{2^k t} f|^q dt\Big)^{1/q}\Big\|_{L^{1,\infty}(\mathbb{T}^d)} \lesssim \|f\|_{h^1(\mathbb{T}^d)}.$$

By normalizing, we may assume that $\|f\|_{h^1(\mathbb{T}^d)}=1$.

By the atomic decomposition for periodic functions [17], [14], we may assume that

$$f = f_0 + \sum_{Q\in\mathcal{Q}} c_Q a_Q,$$

where $f_0\in L^2$, $\|f_0\|_2\lesssim 1$, where $\mathcal{Q}$ is a collection of cubes of side length at most 1/4 which intersect the fundamental cube Q^0 and where a_Q is periodic and supported in $Q+\mathbb{Z}^d$, satisfying $\|a_Q\|_{L^2(Q^0)}\le |Q|^{-1/2}$ and

$$(5.1)\qquad \int_Q a_Q(x)P(x)dx = 0$$

for all polynomials of degree at most $2d$. Moreover,

$$(5.2)\qquad \|f\|_{h^1} \approx \|f_0\|_2 + \sum_{Q\in\mathcal{Q}}|c_Q| \approx 1.$$

The contribution acting on f_0 is taken care of by standard L^2 estimates.

Now let $\gamma = (\gamma_1, \dots, \gamma_d) \in \{-\frac{1}{2}, 0, \frac{1}{2}\}^d =: \Gamma$, and let $Q^\gamma = \gamma + Q^0$. We can then split the family of cubes $\mathcal{Q}$ into 3^d disjoint families $\mathcal{Q}_\gamma$ so that each cube $Q \in \mathcal{Q}_\gamma$ has the property that its double is contained in the cube Q^γ. By periodicity and the monotone convergence theorem, it suffices to prove for each finite subset $\mathcal{N}$ of $\mathbb{N}$, and for each $\gamma \in \Gamma$,

$$\sup_{\alpha>0} \alpha \text{ meas}\Big(\Big\{x \in Q^\gamma : \Big(\sum_{k\in\mathcal{N}} \int_I \big|\mathcal{S}^{\lambda(1)}_{2^k t}[\sum_{Q\in\mathcal{Q}^\gamma} c_Q a_Q]\big|^q dt\Big)^{1/q} > \alpha\Big\}\Big) \lesssim 1. \tag{5.3}$$

It suffices to show for every finite subset $\mathcal{F}^\gamma$ of $\mathcal{Q}^\gamma$

(5.4)
$$\sup_{\alpha>0} \alpha \text{ meas}\Big(\Big\{x \in Q^\gamma : \Big(\sum_{k\in\mathcal{N}} \int_I \big|\mathcal{S}^{\lambda(1)}_{2^k t}[\sum_{Q\in\mathcal{F}^\gamma} c_Q a_Q]\big|^q dt\Big)^{1/q} > \alpha\Big\}\Big) \lesssim \sum_{Q\in\mathcal{F}^\gamma} |c_Q|,$$

where the implicit constant is independent of $\mathcal{F}^\gamma$. To see the reduction, we split $\mathcal{Q}_\gamma = \bigcup_{n=0}^\infty \mathcal{F}^{\gamma,n}$, where $\mathcal{F}^{\gamma,n}$ is finite and $\sum_{Q\in\mathcal{F}^{\gamma,n}} |c_Q| \le 2^{-n}$. By using the result of Stein and Weiss on adding $L^{1,\infty}$ functions [41, Lemma 2.3], the left-hand side in (5.3) is bounded by $C\sum_{n=0}^\infty (1+n)2^{-n} \lesssim 1$, as claimed.

In order to prove (5.4), we can renormalize again, replacing c_Q with $c_Q(\sum_{Q'\in\mathcal{F}^\gamma} |c_{Q'}|)^{-1}$ and α with $\alpha(\sum_{Q'\in\mathcal{F}^\gamma} |c_{Q'}|)^{-1}$. It therefore remains to prove for every finite subset $\mathcal{F}^\gamma$ of $\mathcal{Q}^\gamma$, and for $\sum_{Q\in\mathcal{F}^\gamma} |c_Q| = 1$, that

$$\sup_{\alpha>0} \alpha \text{ meas}\Big(\Big\{x \in Q^\gamma : \Big(\sum_{k\in\mathcal{N}} \int_I \big|\mathcal{S}^{\lambda(1)}_{2^k t}[\sum_{Q\in\mathcal{F}^\gamma} c_Q a_Q]\big|^q dt\Big)^{1/q} > \alpha\Big\}\Big) \lesssim 1, \tag{5.5}$$

where the implicit constant is independent of $\mathcal{F}^\gamma$.

Now fix $\alpha > 0$. Let $\phi \in C^\infty$ be supported in $\{x : |x| \le 1\}$ such that $\int \phi(x)dx = 1$, and let $\phi_\varepsilon = \varepsilon^{-d}\phi(\varepsilon^{-1}\cdot)$. Choose ε_Q to be small, less than one tenth of the side length of Q, so that in addition $\|\phi_{\varepsilon(Q)} * a_Q - a_Q\|_2 < \alpha^{1/2}$. Let $\widetilde{a}_Q = \phi_{\varepsilon(Q)} * a_Q$. Then, by Tshebyshev's inequality and standard L^2 estimates (such as in Section 3),

$$\begin{aligned}
&\text{meas}\Big(\Big\{x \in Q^\gamma : \Big(\sum_{k\in\mathcal{N}} \int_I \Big|\mathcal{S}^{\lambda(1)}_{2^k t}[\sum_{Q\in\mathcal{F}^\gamma} c_Q(a_Q - \widetilde{a}_Q)]\Big|^q dt\Big)^{1/q} > \alpha\Big\}\Big) \\
&\quad\lesssim \alpha^{-2}\Big\|\Big(\sum_{k\in\mathcal{N}} \int_I \Big|\mathcal{S}^{\lambda(1)}_{2^k t}[\sum_{Q\in\mathcal{F}^\gamma} c_Q(a_Q - \widetilde{a}_Q)]\Big|^q dt\Big)^{1/q}\Big\|_2^2 \\
&\quad\lesssim \alpha^{-2}\Big(\sum_{Q\in\mathcal{F}^\gamma} |c_Q|\|a_Q - \widetilde{a}_Q\|_2\Big)^2 \lesssim \alpha^{-1}\Big(\sum_{Q\in\mathcal{F}^\gamma} |c_Q|\Big)^2 \lesssim \alpha^{-1};
\end{aligned}$$

here we have used the normalization $\sum_Q |c_Q| \le 1$.

It suffices to show that

$$\text{meas}\Big(\Big\{x \in Q^\gamma : \Big(\sum_{k\in\mathcal{N}} \int_I \Big|\mathcal{S}^{\lambda(1)}_{2^k t}[\sum_{Q\in\mathcal{F}^\gamma} c_Q \widetilde{a}_Q]\Big|^q dt\Big)^{1/q} > \alpha\Big\}\Big) \lesssim \alpha^{-1}. \tag{5.6}$$

We shall now follow the argument in [14] and set

$$\Psi(x) = \prod_{i=1}^d (1 - x_i^2/4)_+, \qquad \Psi_N^\gamma(x) = \Psi(N^{-1}(x-\gamma)). \tag{5.7}$$

As in [14] we use the following formula, which is valid at least for g in the Schwartz space of $\mathbb{T}^d$, for all $x\in\mathbb{R}^d$:

(5.8)
$$\begin{aligned}&\Psi_N^\gamma(x)\mathcal{S}^\lambda_{2^kt}g(x)-S^\lambda_{2^kt}[\Psi_N^\gamma g](x)\\&\quad=\sum_{\ell\in\mathbb{Z}^d}\langle g,e_\ell\rangle e_\ell(x)\int\Big[h_\lambda(\rho(\frac{\ell}{2^kt}))-h_\lambda(\rho(\frac{\ell+N^{-1}\xi}{2^kt}))\Big]\widehat{\Psi}(\xi)e^{2\pi i\langle x-\gamma,N^{-1}\xi\rangle}d\xi.\end{aligned}$$

As the Fourier coefficients $\langle g,e_\ell\rangle$ decay rapidly, $\widehat{\Psi}\in L^1$, and h_λ is Hölder continuous for $\lambda>0$, this implies that

$$\lim_{N\to\infty}\sup_{t\in I}\sup_{x\in\mathbb{R}^d}\big|\Psi_N^\gamma(x)\mathcal{S}^{\lambda(1)}_{2^kt}g(x)-S^{\lambda(1)}_{2^kt}[\Psi_N^\gamma g](x)\big|=0 \tag{5.9}$$

for $k\in\mathcal{N}$.

Next we observe that $\Psi_N^\gamma(x)\ge(3/4)^d$, for all $x\in m+Q^\gamma$, when $-N\le m_i\le N$ for $i=1,\dots,d$. Using periodicity, we see that the left-hand side of (5.6) is equal to

$$\begin{aligned}&(2N+1)^{-d}\\&\sum_{\substack{-N\le m_i\le N\\ i=1,\dots,d}}\text{meas}\Big(\Big\{x\in m+Q^\gamma:\Big(\sum_{k\in\mathcal{N}}\int_I\Big|\mathcal{S}^{\lambda(1)}_{2^kt}\big[\sum_{Q\in\mathfrak{F}^\gamma}c_Q\widetilde{a}_Q\big](x)\Big|^qdt\Big)^{1/q}>\alpha\Big\}\Big)\\&\le(2N+1)^{-d}\text{meas}\\&\quad\Big(\Big\{x\in\mathbb{R}^d:\Big(\sum_{k\in\mathcal{N}}\int_I\Big|\Psi_N^\gamma(x)\mathcal{S}^{\lambda(1)}_{2^kt}\big[\sum_{Q\in\mathfrak{F}^\gamma}c_Q\widetilde{a}_Q\big](x)\Big|^qdt\Big)^{1/q}>(3/4)^d\alpha\Big\}\Big).\end{aligned}$$

Consider the periodic C^∞ function $g=\sum_{Q\in\mathfrak{F}^\gamma}c_Q\widetilde{a}_Q$, and apply (5.9). Hence there is an $N_0=N_0(g,\alpha,\mathcal{N})$ such that, for every $x\in\mathbb{R}^d$ and $N>N_0$,

$$\begin{aligned}&\Big(\sum_{k\in\mathcal{N}}\int_I\Big|\Psi_N^\gamma(x)\mathcal{S}^{\lambda(1)}_{2^kt}\big[\sum_{Q\in\mathfrak{F}^\gamma}c_Q\widetilde{a}_Q\big](x)-S^{\lambda(1)}_{2^kt}\big[\Psi_N^\gamma\sum_{Q\in\mathfrak{F}^\gamma}c_Q\widetilde{a}_Q\big](x)\Big|^qdt\Big)^{1/q}\\&\qquad<(3/4)^d\alpha/2.\end{aligned}$$

Assuming that $N>N_0$, in what follows, we see that it suffices to bound

$$\begin{aligned}&(2N+1)^{-d}\text{meas}\Big(\Big\{x\in\mathbb{R}^d:\Big(\sum_{k\in\mathcal{N}}\int_I\Big|S^{\lambda(1)}_{2^kt}\big[\Psi_N^\gamma\sum_{Q\in\mathfrak{F}^\gamma}c_Q\widetilde{a}_Q\big](x)\Big|^qdt\Big)^{1/q}\\&\qquad\qquad>(3/4)^d\alpha/2\Big\}\Big).\end{aligned}\tag{5.10}$$

Define, for $Q\in\mathfrak{F}^\gamma$, $m\in\mathbb{Z}^d$,

$$a_{Q,m}(y)=\mathbb{1}_{m+Q^\gamma}(y)\Psi_N^\gamma(y)\widetilde{a}_Q(y).$$

Then the support of $a_{Q,m}$ is in the interior of $m+Q^\gamma$ and Ψ_N^γ coincides on the support of $a_{Q,m}$ with a bounded polynomial of degree $2d$. Hence $a_{Q,m}$ is an L^2 function supported on the double of Q such that $\int a_{Q,m}(y)dy=0$ and such that $\|a_{Q,m}\|_2\lesssim|Q|^{-1/2}$. Moreover, $a_{Q,m}$ is nontrivial only when $|m_i|\le 2N$ for $i=1,\dots,d$. This implies that

$$\Big\|\Psi_N^\gamma\sum_{Q\in\mathfrak{F}^\gamma}c_Q\widetilde{a}_Q\Big\|_{H^1(\mathbb{R}^d)}\le\sum_{\substack{-2N\le m_i\le 2N\\ i=1,\dots,d}}\sum_{Q\in\mathfrak{F}^\gamma}|c_Q|\|a_{Q,m}\|_{H^1(\mathbb{R}^d)}\lesssim(4N+1)^d.$$

We now apply Theorem 2.1 to see that the left-hand side of (5.10) is bounded by

$$C\alpha^{-1}(2N+1)^{-d}\Big\|\Psi_N^\gamma \sum_{Q\in\mathcal{F}^\gamma} c_Q\widetilde{a}_Q\Big\|_{H^1(\mathbb{R}^d)} \lesssim (2N+1)^{-d}(4N+1)^d\alpha^{-1} \lesssim \alpha^{-1},$$

which finishes the proof of the theorem. □

5.2. **The $L^p \to L^{p,\infty}$ bound.** The proof is similar (but more straightforward); therefore, we will be brief. Now $\lambda(p)$ can be negative, but we have $\lambda > -1/q$. The limiting relation (5.9) is now replaced by

$$\lim_{N\to\infty} \sup_{x\in\mathbb{R}^d} \Big(\int_I \big|\Psi_N^\gamma(x)\mathcal{S}_{2^kt}^{\lambda(p)}g(x) - S_{2^kt}^{\lambda(p)}[\Psi_N^\gamma g](x)\big|^q dt\Big)^{1/q} = 0, \qquad k\in\mathcal{N}. \tag{5.11}$$

Here we consider $g\in\mathcal{S}(\mathbb{T}^d)$. We sketch a proof of (5.11) based on (5.8).

We start by observing that

$$\int_I |h_\lambda(\rho(\zeta/t))|^q dt \le C, \tag{5.12}$$

uniformly in $\zeta\in\mathbb{R}^d$. To see this, note that $\rho(\zeta/t) = \rho(\zeta)t^{-1/b}$, and we may assume that $\rho(\zeta)\sim 1$ due to the support of h_λ. Therefore, (5.12) follows by a change of variable. From this observation, we may reduce (5.11) to

$$\lim_{N\to\infty} \Big(\int_I \big|h_\lambda(\rho(\frac{\ell}{2^kt})) - h_\lambda(\rho(\frac{\ell+N^{-1}\xi}{2^kt}))\big|^q dt\Big)^{1/q} = 0 \tag{5.13}$$

for fixed l,k,ξ using (5.8), Minkowski's inequality, and the dominated convergence theorem.

For (5.13), we argue as follows. Let $h\in L^q(J)$ for a compact subinterval J of $(0,\infty)$. Then, for any $a>0$,

$$\lim_{\delta\to 0}\Big(\int_J |h(as)-h((a+\delta)s)|^q ds\Big)^{1/q} = 0,$$

and the limit is uniform if a is taken from a compact subset of $(0,\infty)$. This is easily seen for smooth h and follows for general $h\in L^q(J)$ by an approximation argument. Changing variables $s=t^{-1/b}$, we obtain that, for any compact subinterval $I\subset(0,\infty)$,

$$\lim_{\delta\to 0}\Big(\int_I |h(at^{-1/b}) - h((a+\delta)t^{-1/b})|^q dt\Big)^{1/q} = 0. \tag{5.14}$$

Then (5.13) follows from (5.14), with $h=h_\lambda$, $\delta = \rho((\ell+N^{-1}\xi)/2^k) - \rho(\ell/2^k)$, and $a=\rho(\ell/2^k)$, using the homogeneity and continuity of ρ.

Finally, using (5.11), we get, for sufficiently large N,

$$\begin{aligned}
&\mathrm{meas}\Big(\Big\{x\in Q^0 : \Big(\sum_{k\in\mathcal{N}}\int_I \big|\mathcal{S}_{2^kt}^{\lambda(p)}g\big|^q dt\Big)^{1/q} > \alpha\Big\}\Big)\\
&\quad\lesssim (2N+1)^{-d}\mathrm{meas}\Big(\Big\{x\in\mathbb{R}^d : \Big(\sum_{k\in\mathcal{N}}\int_I \big|S_{2^kt}^{\lambda(p)}[\Psi_N^0 g](x)\big|^q dt\Big)^{1/q} > (3/4)^d\alpha/2\Big\}\Big),
\end{aligned}$$

and by Theorem 2.2 we bound the right-hand side by

$$C(2N+1)^{-d}\alpha^{-p}\|\Psi_N^0 g\|^p_{L^p(\mathbb{R}^d)} \lesssim \alpha^{-p}\|g\|^p_{L^p(\mathbb{T}^d)}.$$

Remark 5.1. It is also possible to build a proof of Theorem 1.3 from Theorem 2.2 using modifications of a duality argument by de Leeuw [25]; see also [40], [20].

6. Sharpness

In this section, we show that Theorems 1.3 and 2.2 fail for $q > p'$. We shall first reduce the argument for Fourier series to the one for Fourier integrals by a familiar transplantation method and then modify an argument that was used by Tao to obtain necessary conditions for the Bochner–Riesz maximal operator (see [43, Sect. 5]), and also the work by Carbery and Soria [6] where a related argument appears in the context of localization results for Fourier series. Note that the almost everywhere convergence assertion in part (ii) of Theorem 1.3 also fails for $q > p'$, by Stein–Nikishin theory [35].

6.1. **Fourier series.** We have for $f \in L^p(\mathbb{T}^d)$

$$\Big\| \sup_{T>0} \Big(\frac{1}{T} \int_0^T |\mathcal{R}_t^\lambda f|^q dt \Big)^{1/q} \Big\|_{L^{p,\infty}(\mathbb{T}^d)} \geq \sup_{T>0} \Big\| \Big(\frac{1}{T} \int_0^T |\mathcal{R}_t^\lambda f|^q dt \Big)^{1/q} \Big\|_{L^{p,\infty}(\mathbb{T}^d)}, \tag{6.1}$$

and our necessary condition will follow from Proposition 6.2 and the following result.

Lemma 6.1. *Let $1 \leq p \leq 2$. Suppose that, for some $C > 0$,*

$$\sup_{\|f\|_{L^p(\mathbb{T}^d)}=1} \sup_{T>0} \Big\| \Big(\frac{1}{T} \int_0^T |\mathcal{R}_t^\lambda f|^q dt \Big)^{1/q} \Big\|_{L^{p,\infty}(\mathbb{T}^d)} \leq C. \tag{6.2}$$

Then also

$$\sup_{\|f\|_{L^p(\mathbb{R}^d)}=1} \sup_{T>0} \Big\| \Big(\frac{1}{T} \int_0^T |R_t^\lambda f|^q dt \Big)^{1/q} \Big\|_{L^{p,\infty}(\mathbb{R}^d)} \leq C. \tag{6.3}$$

Proof. By scaling, density of C_c^∞ functions in L^p, and the monotone convergence theorem it suffices to show, for all $f \in C_c^\infty(\mathbb{R}^d)$, all compact sets K, all $\delta \in (0,1)$, all $\varepsilon \in (0,1)$, and all $\alpha > 0$,

$$\operatorname{meas}\Big(\Big\{ x \in K : \Big(\int_\delta^1 |R_t^\lambda f(x)|^q dt \Big)^{1/q} > \alpha \Big\} \Big) \leq C^p (1-\varepsilon)^{-p} \alpha^p \|f\|_p^p.$$

Fix such f, α, δ, ε, and K. For large $L \in \mathbb{N}$, define

$$V_{L,t}^\lambda f(x) = \sum_{\ell \in \mathbb{Z}^d} L^{-d} \widehat{f}(L^{-1}\ell)(1 - \rho(t^{-1}L^{-1}\ell))_+^\lambda e^{2\pi i L^{-1}\langle x, \ell \rangle}.$$

Then $V_{L,t}^\lambda f(x)$ is a Riemann sum for the integral representing $R_t^\lambda f(x)$. Hence we have

$$\lim_{L \to \infty} V_{L,t}^\lambda f(x) = R_t^\lambda f(x)$$

with the limit uniform in $t \in [\delta, 1]$, $x \in K$. We may therefore choose L such that

$$\operatorname{supp}(f(L\cdot)) \subset \{x : |x| < 1/4\}$$

and

$$K \subset LQ^0$$

with $Q^0 = [-1, 2, 1/2]^d$, and

$$\sup_{\delta \leq t \leq 1} \sup_{x \in K} |R_t^\lambda f(x) - V_{L,t}^\lambda f(x)| < \alpha\varepsilon.$$

It remains to show that

$$\operatorname{meas}\Big(\Big\{ x \in K : \Big(\int_\delta^1 |V_{L,t}^\lambda f(x)|^q dt \Big)^{1/q} > \alpha(1-\varepsilon) \Big\} \Big) \leq C^p (1-\varepsilon)^{-p} \alpha^p \|f\|_p^p. \tag{6.4}$$

Observe that, for $w \in Q^0$,

$$\Big(\int_\delta^1 |V^\lambda_{L,t} f(Lw)|^q dt\Big)^{1/q} = \Big(\frac{1}{L}\int_{\delta L}^L \Big|\sum_{\ell\in\mathbb{Z}^d} L^{-d}\widehat{f}(L^{-1}\ell)(1-\rho(\ell/s))_+^\lambda e^{2\pi i\langle w,\ell\rangle}\Big|^q ds\Big)^{1/q}.$$

Let $f_L^{\mathrm{per}}(w) = \sum_{\kappa\in\mathbb{Z}^d} f(L(w+\kappa))$. Then from the Poisson summation formula the Fourier coefficients of the periodic function f_L^{per} are given by $\langle f_L^{\mathrm{per}}, e_\ell\rangle = L^{-d}\widehat{f}(L^{-1}\ell)$. Hence the expression on the right-hand side of the last display is equal to $(L^{-1}\int_{\delta L}^L |\mathcal{R}_t^\lambda f_L^{\mathrm{per}}|^q dt)^{1/q}$. Replacing K with the larger set LQ_0 and then changing variables $x = Lw$, we see that the expression on the left-hand side of (6.4) is dominated by

$$\begin{aligned} &L^d \mathrm{meas}\Big(\Big\{w \in Q^0 : \Big(\frac{1}{L}\int_{\delta L}^L |\mathcal{R}_t^\lambda f_L^{\mathrm{per}}(w)|^q dt\Big)^{1/q} > \alpha(1-\varepsilon)\Big\}\Big)\\ &\qquad \le L^d C^p \alpha^{-p}(1-\varepsilon)^{-p}\int_{Q_0} |f_L^{\mathrm{per}}(w)|^p dw,\end{aligned}$$

where the last inequality follows from assumption (6.2). Since the support of $f(L\cdot)$ is contained in Q_0, one immediately gets

$$\|f_L^{\mathrm{per}}\|^p_{L^p(Q^0)} = \|f(L\cdot)\|^p_{L^p(\mathbb{R}^d)} = L^{-d}\|f\|^p_{L^p(\mathbb{R}^d)}.$$

This shows (6.4) and concludes the proof. □

6.2. **Fourier integrals.** Using the $\mathbb{R}^d$ analogue of (6.1), we reduce the sharpness claim in Theorem 2.2 to the following proposition.

Proposition 6.2. *Let $1 < p \le 2$, and let $\lambda > -1/2$. Assume that there is a constant $C > 0$ such that*

$$\sup_{T>1}\Big\|\Big(\frac{1}{T}\int_0^T |R_t^\lambda f|^q dt\Big)^{1/q}\Big\|_{L^{p,\infty}(\mathbb{R}^d)} \le C\|f\|_{L^p(\mathbb{R}^d)} \tag{6.5}$$

for all Schwartz functions f. Then

$$\lambda \ge \lambda(p) + \frac{1}{2}\Big(\frac{1}{p'} - \frac{1}{q}\Big).$$

In particular, if (6.5) holds for $\lambda = \lambda(p)$, then $q \le p'$.

Proof. We note that the inequality with a given ρ is equivalent to the inequality with $\rho\circ A$, where A is any rotation.

Let $\xi^\circ \in \Sigma_\rho$ be such that $|\xi^\circ|$ is maximal. Then the Gaussian curvature does not vanish at ξ^0. Choose small neighborhoods U_1, U_0 of ξ^0 in Σ_ρ such that $\overline{U}_1 \subset U_0$, the Gauss map is injective in a neighborhood of $\overline{U}_0$, and the curvature is bounded below on U_0. Let γ be homogeneous of degree 0, and let $\gamma(\xi) \neq 0$ for $\xi \in U_1$, with γ supported on the closure of the cone generated by U_0. Let $n(\xi_0) = \frac{\nabla\rho(\xi^\circ)}{|\nabla\rho(\xi^\circ)|}$ the outer normal at ξ_0, and let $\Gamma_\varepsilon = \{x \in \mathbb{R}^d : \big|\frac{x}{|x|} - n(\xi_0)\big| \le 2\varepsilon\}$, with ε so small that Γ_ε is contained in the cone generated by the normal vectors $\nabla\rho(\xi)$ with $\xi \in U_1$. Let, for $R \gg 1$, $\Gamma_{\varepsilon,R} = \{x \in \Gamma_\varepsilon : |x| \ge R\}$. By the choice of ε there is, for each $x \in \Gamma_\varepsilon$, a unique $\Xi(x) \in \Sigma_\rho$ such that $\gamma(\Xi(x)) \neq 0$ and such that x is normal to Σ_ρ at $\Xi(x)$. Clearly $x \mapsto \Xi(x)$ is homogeneous of degree 0 on Γ, smooth away from the origin. By a rotation, we may assume that

$$n(\xi^\circ) = (0,\dots,0,1). \tag{6.6}$$

By Section 2.1, inequality (6.5) also implies the similar inequality where $R_t^\lambda f$ is replaced with $S_t^\lambda f$ and S_t^λ is as in (2.2). Let $h_\lambda(s) = (1-v_0(s))(1-s)_+^\lambda$ and

$$K_{\lambda,t}(x) = t^d \mathcal{F}^{-1}[\gamma\, h_\lambda \circ \rho](tx).$$

Observe that $K_{\lambda,t} * f = S_t^\lambda f_\gamma$ with $\widehat{f_\gamma} = \gamma \widehat{f}$. By the Hörmander multiplier theorem, γ is a Fourier multiplier of L^p, and we see that (6.5) implies that

$$\sup_{T>0} \Big\| \Big(\frac{1}{T} \int_0^T |K_{\lambda,t} * f|^q dt \Big)^{1/q} \Big\|_{L^{p,\infty}(\mathbb{R}^d)} \le C \|f\|_{L^p(\mathbb{R}^d)}. \tag{6.7}$$

We now derive an asymptotic expansion for $K_{\lambda,1}(x)$ when $x \in \Gamma_{\varepsilon,R}$. Recall that ρ is homogeneous of degree $1/b$; i.e., $\rho(t^b\xi) = t\rho(\xi)$. We use generalized polar coordinates $\xi = \rho^b \xi(\omega)$ where $\omega \to \xi(\omega)$ is a parametrization of Σ_ρ in a neighborhood of U_0. Then

$$\begin{aligned} d\xi &= b\rho^{db-1} d\rho \, \langle \xi(\omega), n(\xi(\omega))\rangle \big(\det(\frac{\partial \xi}{\partial \omega})^\intercal \frac{\partial \xi}{\partial \omega}\big)^{1/2} d\omega \\ &= \rho^{db-1} d\rho \, |\nabla \rho(\xi')|^{-1} d\sigma(\xi'), \qquad \xi' = \xi(\omega). \end{aligned}$$

Here we have used Euler's homogeneity relation $b\langle \xi, \nabla\rho(\xi)\rangle = \rho(\xi)$ for vectors on Σ_ρ. Then

$$K_{\lambda,1}(x) = \int_0^\infty h_\lambda(\rho)\rho^{bd-1} \int_{\Sigma_\rho} \gamma(\xi') e^{2\pi i \rho \langle \xi', x\rangle} \frac{d\sigma(\xi')}{|\nabla\rho(\xi')|}\, d\rho. \tag{6.8}$$

We use the method of stationary phase and get, for $x \in \Gamma_{\varepsilon,R}$,

$$K_{\lambda,1}(x) = I(x) + \sum_{j=1}^N II_j(x) + III(x), \tag{6.9}$$

where

$$I(x) = c \int_0^\infty h_\lambda(\rho)\rho^{bd-1-\frac{d-1}{2}} e^{2\pi i \rho\langle \Xi(x), x\rangle} d\rho \; \frac{\gamma(\Xi(x))|\nabla\rho(\Xi(x))|^{-1}}{(\langle \Xi(x), x\rangle)^{\frac{d-1}{2}} |\text{curv}(\Xi(x))|^{1/2}},$$

where $\text{curv}(\Xi(x))$ is the Gaussian curvature at $\Xi(x)$ and $c \neq 0$, and

$$II_j(x) = c_j \int_0^\infty h_\lambda(\rho)\rho^{bd-1-\frac{d-1}{2}-j} e^{2\pi i \rho\langle \Xi(x), x\rangle} d\rho \; \frac{\gamma_j(\Xi(x))}{(\langle \Xi(x), x\rangle)^{\frac{d-1}{2}+j} |\text{curv}(\Xi(x))|^{1/2}},$$

where γ_j is smooth. For the remainder term, we get

$$|III(x)| \lesssim_N \|h\|_1 |x|^{-N}, \qquad x \in \Gamma_{\varepsilon,R}.$$

In the resulting ρ integrals, we use asymptotics for the one-dimensional Fourier transform of h_λ (see [13, Section 2.8]) and see that, for $x \in \Gamma_{\varepsilon,R}$,

$$\int_0^\infty h_\lambda(\rho)\rho^{bd-\frac{d+1}{2}} e^{2\pi i \rho\langle \Xi(x), x\rangle} d\rho \; = C_\lambda \langle \Xi(x), x\rangle^{-\lambda-1} e^{2\pi i \langle \Xi(x), x\rangle} + O(\langle \Xi(x), x\rangle)^{-\lambda-2},$$

with similar asymptotics for the ρ-integrals in the terms II_j.

Now set, for $x \in \Gamma_\varepsilon$, $H(x) = \langle \Xi(x), x\rangle$, and use Euler's homogeneity relation to write

$$H(x) = |x| \langle \Xi(x), \frac{\nabla\rho(\Xi(x))}{|\nabla\rho(\Xi(x))|}\rangle = |x| \frac{\rho(\Xi(x))}{b|\nabla\rho(\Xi(x))|} = \frac{|x|}{b|\nabla\rho(\Xi(x))|}.$$

If ε is small, we then have, for $t|x| \gg R$,

$$K_{\lambda,t}(x) = A_\lambda(x,t) + B_\lambda(x,t), \qquad |x'| \le \varepsilon^2 |x_d|,$$

where

$$A_\lambda(x,t) = C(\lambda) t^{d-\frac{d+1}{2}-\lambda} G(x) e^{2\pi i t H(x)},$$
$$\text{where } G(x) = H(x)^{-\frac{d+1}{2}-\lambda} \frac{\gamma(\Xi(x))|\nabla\rho(\Xi(x))|^{-1}}{|\mathrm{curv}(\Xi(x))|^{1/2}}, \tag{6.10}$$

and

$$B_\lambda(x,t) \lesssim t^{d-\frac{d+3}{2}-\lambda} H(x)^{-\frac{d+3}{2}-\lambda}.$$

Recall (6.6) and split $y = (y', y_d)$. We now let

$$P(T,\varepsilon) = \{y : |y'| \le T^{-1}\varepsilon,\ |y_d| \le T^{-1/2}\varepsilon\}$$

and define

$$f_T(y) = \mathbb{1}_{P(T,\varepsilon)}(y) e^{2\pi i \varepsilon T y_d}.$$

Then

$$\|f_T\|_p \lesssim T^{\frac{1}{p}(\frac{1}{2}-d)}. \tag{6.11}$$

We examine the integrals $K_{\lambda,t} * f_T(x)$ for $|x| \approx 1$ and $R \ll t \approx \varepsilon T$. We may obtain a lower bound for the absolute value of this integral if we can choose t for a given x such that

$$2\pi(\varepsilon T y_d + tH(x-y) - tH(x)) \in (-\frac{\pi}{4}, \frac{\pi}{4}) \qquad \text{for all } y \in \mathrm{supp}(f_T). \tag{6.12}$$

As the Gauss map is invertible near ξ°, we observe that H is smooth and homogeneous of degree 1. We have $\nabla H(x) = \xi^\circ + O(\varepsilon)$, and thus $\partial_{x_d} H(x) \ge c > 0$. Now

$$\begin{aligned} &\varepsilon T y_d + tH(x-y) - tH(x) \\ &\quad = -t \sum_{i=1}^{d-1} y_i \partial_{x_i} H(x) + y_d(\varepsilon T - t\partial_{x_d} H(x)) \\ &\quad\quad + t \sum_{i,j=1}^{d} y_i y_j \int_0^1 (1-s) \partial^2_{x_i x_j} H(x - sy) ds. \end{aligned} \tag{6.13}$$

The first and the third term on the right-hand side are $O(\varepsilon)$ when $y \in \mathrm{supp}(f_T)$. We choose t in the interval

$$I_{x,T} = \Big[\frac{\varepsilon T}{\partial_{x_d} H(x)} - \varepsilon T^{1/2}, \frac{\varepsilon T}{\partial_{x_d} H(x)} + \varepsilon T^{1/2}\Big]. \tag{6.14}$$

We assume that ε is chosen so small that $I_{x,T} \subset [0,T]$. If $t \in I_{x,T}$, the second term on the right-hand side of (6.13) will be $O(\varepsilon)$ as well so that (6.12) is satisfied.

We now split

$$K_{\lambda,t} * f_T(x) = \mathfrak{I}_1(x,t) + \mathfrak{I}_2(x,t) + \mathfrak{I}_3(x,t)$$

with

$$\mathfrak{I}_1(x,t) = C(\lambda) G(x) e^{2\pi i t H(x)} t^{\frac{d-1}{2}-\lambda} \int e^{2\pi i (T\varepsilon y_d + tH(x-y) - tH(x))} \mathbb{1}_{P(T,\varepsilon)}(y)\, dy,$$

$$\mathfrak{I}_2(x,t) = C(\lambda) t^{\frac{d-1}{2}-\lambda} \int e^{2\pi i (T\varepsilon y_d + tH(x-y))} (G(x-y) - G(x)) \mathbb{1}_{P(T,\varepsilon)}(y)\, dy,$$

$$\mathfrak{I}_3(x,t) = \int B_\lambda(x-y,t) f_T(y)\, dy.$$

We estimate these terms for

$$x \in \Omega := \{x : |x'| \le \varepsilon^2 |x_d|,\ 1/2 \le |x_d| \le 1\}, \qquad t \in I_{x,T}. \tag{6.15}$$

Then, by (6.12), the real part of the integrand in the definition of $\mathcal{J}_1(x,t)$ is bounded below by $2^{-1/2}\mathbb{1}_{P(T,\varepsilon)}(y)$, and therefore, for $x \in \Omega$,

$$\begin{aligned} |\mathcal{J}_1(x,t)| &\ge CG(x) t^{\frac{d-1}{2}-\lambda} \int \mathbb{1}_{P(T,\varepsilon)}(y)\, dy \\ &\ge c t^{\frac{d-1}{2}-\lambda} T^{\frac{1}{2}-d}. \end{aligned}$$

Moreover,

$$\begin{aligned} |\mathcal{J}_2(x,t)| &\lesssim t^{\frac{d-1}{2}-\lambda} \varepsilon T^{-1/2} T^{\frac{1}{2}-d}, \\ |\mathcal{J}_3(x,t)| &\lesssim t^{\frac{d-3}{2}-\lambda} T^{\frac{1}{2}-d}. \end{aligned}$$

Hence, for small ε and $t|x| \gg R$, $t \in I_{x,T}$, the term $|\mathcal{J}_1(x,t)|$ is significantly larger than the terms $|\mathcal{J}_2(x,t)|$ and $|\mathcal{J}_3(x,t)|$. Consequently, by $|I_{x,T}| \ge \varepsilon T^{1/2}$ and assuming (6.15), we get

$$\begin{aligned} \Big(\frac{1}{T}\int_0^T |K_{\lambda,t} * f_T(x)|^q dt\Big)^{1/q} &\ge \Big(\frac{1}{T}\int_{I_{x,T}} |K_{\lambda,t} * f_T(x)|^q dt\Big)^{1/q} \\ &\ge c\varepsilon^{1/q} T^{-1/2q} (\varepsilon T)^{\frac{d-1}{2}-\lambda} T^{1/2-d} = c_\varepsilon T^{-\frac{d}{2}-\lambda-\frac{1}{2q}}, \end{aligned}$$

and thus

$$\Big\| \Big(\frac{1}{T}\int_0^T |K_{\lambda,t} * f_T|^q dt\Big)^{1/q} \Big\|_{L^{p,\infty}} \gtrsim_\varepsilon T^{-\frac{d}{2}-\lambda-\frac{1}{2q}} T^{\frac{d}{p}-\frac{1}{2p}} \|f_T\|_p,$$

which for $T \to \infty$ implies that $\lambda \ge \lambda(p) + \frac{1}{2}(1 - \frac{1}{p} - \frac{1}{q})$. □

7. An L^1 result

We currently do not have an analogue of Theorem 1.2 for general functions in $L^1(\mathbb{T}^d)$. We formulate a weaker result which is essentially a consequence of Theorem 1.2.

Theorem 7.1.

(i) *Let $f \in L^1(\mathbb{T}^d)$. Then, for all $q < \infty$ and for $\lambda(1) = \frac{d-1}{2}$,*

$$\lim_{T\to\infty} \Big\| \Big(\frac{1}{T}\int_0^T |\mathcal{R}_t^{\lambda(1)} f - f|^q dt\Big)^{1/q} \Big\|_{L^{1,\infty}(\mathbb{T}^d)} = 0.$$

(ii) *The analogous statement holds on $L^1(\mathbb{R}^d)$ with $R_t^{\lambda(1)} f$ in place of $\mathcal{R}_t^{\lambda(1)} f$.*

Proof. Since the convergence holds for Schwartz functions, one can by a standard approximation argument reduce the proof of (ii) to the inequality

$$\sup_{T>0} \Big\| \Big(\frac{1}{T}\int_0^T |R_t^{\lambda(1)} f|^q dt\Big)^{1/q} \Big\|_{L^{1,\infty}(\mathbb{R}^d)} \lesssim \|f\|_{L^1(\mathbb{R}^d)}. \tag{7.1}$$

Similarly, the proof of (i) is reduced to a corresponding inequality on $\mathbb{T}^d$, with the supremum in T extended over $T \ge 1$. The weak type $(1,1)$ inequality in the $\mathbb{T}^d$ case follows from the $\mathbb{R}^d$ case by the transference arguments of Section 5. Therefore, it suffices to show (7.1).

By the maximal estimate in Section 2.1, it remains to prove that

$$\Big\|\Big(\frac{1}{T}\int_0^T |S_t^{\lambda(1)} f|^q dt\Big)^{1/q}\Big\|_{L^{1,\infty}(\mathbb{R}^d)} \lesssim \|f\|_{L^1(\mathbb{R}^d)}, \tag{7.2}$$

where $S_t^{\lambda(1)}$ is as in (2.2). We may assume that $q \geq 2$. Now

$$\Big(\frac{1}{T}\int_0^T |S_t^{\lambda(1)} f(x)|^q dt\Big)^{1/q} \leq \sum_{l=0} 2^{-l/q}\Big(\frac{1}{T2^{-l}}\int_{T2^{-l}}^{T2^{-l+1}} |S_t^{\lambda(1)} f(x)|^q dt\Big)^{1/q},$$

and we claim the inequality

$$\sup_{A>0}\Big\|\Big(\frac{1}{A}\int_A^{2A} |S_t^{\lambda(1)} f|^q dt\Big)^{1/q}\Big\|_{L^{1,\infty}} \leq C_q\|f\|_1. \tag{7.3}$$

Assuming that (7.3) is verified, we can deduce that the left-hand side of (7.2) is bounded by $C_q\tilde{C}\sum_{l>0}(1+l)2^{-l/q}\|f\|_1 \lesssim_q \|f\|_1$, by the theorem of Stein and Weiss [41, Lemma 2.3] on summing $L^{1,\infty}$ functions.

Let η be as in (2.3). Then our main result, Theorem 2.3, yields for all $A > 0$

$$\Big\|\Big(\frac{1}{A}\int_A^{2A} |S_t^{\lambda(1)} f|^q dt\Big)^{1/q}\Big\|_{L^{1,\infty}(\mathbb{R}^d)} \leq C_q\|\mathcal{F}^{-1}[\eta(A^{-1}\cdot)] * f\|_{H^1(\mathbb{R}^d)}.$$

Since η is C^∞ and is compactly supported away from the origin, we have

$$\|\mathcal{F}^{-1}[\eta(A^{-1}\cdot)] * f\|_{H^1(\mathbb{R}^d)} \lesssim \|f\|_{L^1(\mathbb{R}^d)},$$

uniformly in A. This yields (7.3) and concludes the proof of (7.2). □

As an immediate consequence of Theorem 7.1, we get the following corollary.

Corollary 7.2. *Let $f \in L^1(\mathbb{T}^d)$. There is a subsequence $T_j \to \infty$ such that*

$$\lim_{j\to\infty}\Big(\frac{1}{T_j}\int_0^{T_j} |\mathcal{R}_t^{\lambda(1)} f(x) - f(x)|^q dt\Big)^{1/q} = 0 \ a.e. \tag{7.4}$$

Arguing as in [48, Ch. XIII.7] or [46, Section 4], we get the following corollary.

Corollary 7.3. *Let $f \in L^1(\mathbb{T}^d)$. For almost every $x \in \mathbb{T}^d$, there is a measurable set $E = E(f,x)$ of upper density 1, i.e., satisfying*

$$\limsup_{T\to\infty}\frac{|E\cap[0,T]|}{T} = 1, \tag{7.5}$$

such that

$$\lim_{\substack{t\to\infty\\ t\in E}} \mathcal{R}_t^{\lambda(1)} f(x) = f(x).$$

For the convenience of the reader, we give a proof.

Proof. Fix x such that (7.4) in Corollary 7.2 holds, and let $g(t) = |\mathcal{R}_t^{\lambda(1)} f(x) - f(x)|$. We may assume that T_j is increasing in j. For $m = 1, 2, \dots$, let $E_m = \{t : g(t) \leq 1/m\}$. By Tshebyshev's inequality, we have

$$\frac{|E_m^{\complement}\cap[0,T_j]|}{T_j} \leq m^q \frac{1}{T_j}\int_0^{T_j} g(t)^q dt,$$

which by assumption tends to 0 as $j\to\infty$. Hence $\lim_{j\to\infty} T_j^{-1}|E_m\cap[0,T_j]| = 1$. Thus we may choose a strictly increasing sequence j_m of positive integers such that

$T_j^{-1}|E_m \cap [0,T_j]| > 1-m^{-1}$ for $j \geq j_m$. Let $E = [0,T_{j_1}] \cup \bigcup_{m=1}^{\infty}(E_m \cap [T_{j_m}, T_{j_{m+1}}])$. Since the sets E_m are decreasing, we have

$$|E \cap [0,T_{j_{m+1}}]| \geq |E_m \cap [0,T_{j_{m+1}}]| \geq (1-m^{-1})T_{j_{m+1}},$$

and hence $\limsup_{T\to\infty} T^{-1}|E \cap [0,T]| = 1$. Now $E \cap [T_{j_m}, \infty] \subset E_m$, and thus $g(t) \leq m^{-1}$ on this set. It follows that $g(t) \to 0$ as $t \to \infty$ within E. □

It would be desirable to replace the lim sup in (7.5) with the lim inf. The proof of the corollary shows that this would require the existence a.e. of the limit in (7.4) for *all* sequences $T_j \to \infty$. We can currently prove this only for functions in h^1.

8. Maximal functions on $H^p(\mathbb{R}^d)$ for $p < 1$

We now consider the maximal operator associated with the generalized Riesz means when they act on functions or distributions in the Hardy space $H^p(\mathbb{R}^d)$ for $p < 1$. The following result generalizes one by Stein, Taibleson, and Weiss [39] for the standard Bochner–Riesz means. Other generalizations for specific rough ρ were considered in [19] and the references therein.

Let R_t^λ be as in (1.3).

Theorem 8.1. *For $0 < p < 1$, $\lambda(p) = d(1/p - 1/2) - 1/2$, we have for all $f \in H^p(\mathbb{R}^d)$*

$$\big\| \sup_{t>0} |R_t^{\lambda(p)} f| \big\|_{L^{p,\infty}(\mathbb{R}^d)} \lesssim \|f\|_{H^p(\mathbb{R}^d)}.$$

We use the same reductions as in Section 2. Write, for $t > 0$,

$$R_t^\lambda f(x) = \mathcal{F}^{-1}[u(\rho(\cdot/t))\widehat{f}](x) + \sum_{j=1}^{\infty} 2^{-j\lambda} T_j f(x,t),$$

where u is as in Section 2.1 and $\widehat{T_j f}(\xi,t) = \varphi_j(\rho(\xi/t))\widehat{f}(\xi)$, with ϕ_j as in Section 2.2. This is similar to (2.7) (except that now t ranges over $(0,\infty)$). The functions u, φ_j depend on λ but satisfy uniform estimates, as λ is taken over a compact subset of $\mathbb{R}$. Let

$$\mathcal{M}_0 f(x) = \sup_{t>0} |\mathcal{F}^{-1}[u(\rho(\cdot/t))\widehat{f}](x),$$

and for $j \geq 1$,

$$\mathcal{M}_j f(x) = \sup_{t>0} |T_j f(x,t)|.$$

We then have

$$\sup_{t>0} |\mathcal{R}_t^{\lambda(p)} f(x)| \leq \mathcal{M}_0 f(x) + \sum_{j\geq 1} 2^{-j\lambda(p)} \mathcal{M}_j f(x), \tag{8.1}$$

and we shall derive a weak type inequality on H^p for the right-hand side in (8.1). The ingredients are $H^p \to L^p$ bounds for the maximal operators $\mathcal{M}_0$ and $\mathcal{M}_j$.

Let M be a nonnegative integer. We recall that a function a supported on a ball B is a (p,M) atom if $\|a\|_\infty \leq \mathrm{vol}(B)^{-1/p}$ and $\int a(x)P(x)dx = 0$ for all polynomials of degree at most M. By the atomic decomposition, it suffices to check the $H^p \to L^p$ bounds on (p,M) atoms for every nonnegative integer $M > d(p^{-1}-1)-1$. The bound for $\mathcal{M}_0 a$ is straightforward.

Lemma 8.2. *Let $M+1 > d(p^{-1}-1)$, and let a be a (p,M)-atom. For $0 < p \leq 1$, we have*

$$\|\mathcal{M}_0 a\|_p \lesssim 1.$$

Proof. This follows by a variant of the argument in Section 2.1. Define

$$\mathfrak{M}_{N_1,N_2} f = \sup_{\tau>0} |\mathcal{F}^{-1}[(1-\rho(\cdot/\tau)^{N_1})_+^{N_2} \widehat{f}]|.$$

Let N_1, N_2 be large so that $\mathfrak{M}_{N_1,N_2}$ maps H^p to L^p. By the subordination formula (2.1), we have

$$\sup_{t>0} |\mathcal{F}^{-1}[u(\rho(\cdot/t))\widehat{a}](x)| \lesssim |\mathfrak{M}_{N_1,N_2} a(x)| \frac{1}{N_2!} \int_0^\infty s^{N_2} |u_{N_1}^{(N_2+1)}(s)| ds, \tag{8.2}$$

and the integral is finite. Hence we get the desired L^p bound for $\mathcal{M}_0 a$. □

8.1. **The main $H^p \to L^p$ bound.**

Proposition 8.3. *Let $0 < p \le 1$, let $j \ge 1$, and let $\nu \in \mathcal{Z}_j$. Let $M+1 > d(p^{-1}-1)$, and let a be a (p, M)-atom. Then*

$$\|\mathcal{M}_j a\|_p \lesssim 2^{j(d(\frac{1}{p}-\frac{1}{2})-\frac{1}{2})}.$$

We further decompose $T_j f(x,t) = \sum_{\nu\in\mathcal{Z}_j} T_{j,\nu} f(x,t)$, where we use the homogeneous partition of unity as in (3.2). Let, for $\nu \in \mathcal{Z}_j$,

$$\mathcal{M}_{j,\nu} f(x) = \sup_{t>0} |T_{j,\nu} f(x,t)|.$$

Then $\mathcal{M}_j f(x) \le \sum_{\nu\in\mathcal{Z}_j} \mathcal{M}_{j,\nu} f(x)$. Since $\#\mathcal{Z}_j = O(2^{j(d-1)/2})$, we can use the triangle inequality in L^p, $p \le 1$, to see that the proposition follows from

$$\|\mathcal{M}_{j,\nu} a\|_p \lesssim 2^{j\frac{d+1}{2}(\frac{1}{p}-1)}. \tag{8.3}$$

We proceed with the proof of (8.3).

By translation and scaling, we may assume that a is supported in the ball B of radius 1 centered at the origin, that $\|a\|_\infty \le 1$, and that $\int a(x)P(x)dx = 0$ for all polynomials of degree $\le M$. By a rotation, we may also assume that $\nabla\rho(\xi_{j,\nu})$ is parallel to $(1, 0, \dots, 0)$, and thus, writing $x = (x_1, x')$, we have

$$|\partial^\alpha K_{j,\nu}(x)| \le C_{N_1,N_2,\alpha} \frac{2^{-j(d+1)/2}}{(1 + 2^{-j}|x_1|)^{N_1} (1 + 2^{-j/2}|x'|)^{N_2}} \tag{8.4}$$

for all multi-indices $\alpha \in \mathbb{N}_0^d$ and all $N_1, N_2 \ge 0$; cf. [11] or [31]. Let

$$\mathcal{D} = \{(x_1, x') \in \mathbb{R}^d : |x_1| \le 5 \cdot 2^j, |x'| \le 5 \cdot 2^{j/2}\}.$$

In the following subsections, we estimate the L^p quasi norm of $\mathcal{M}_{j,\nu} a(x)$ over $\mathcal{D}$ and $\mathcal{D}^\complement$, respectively, using the cancellation condition for the atom when $x \in \mathcal{D}^\complement$.

8.1.1. *Estimation over $\mathcal{D}$.* Let

$$\mathcal{D}_0 = \{(x_1, x') \in \mathbb{R}^d : |x_1| \le 5, |x'| \le 5\},$$
$$\mathcal{D}_1 = \{(x_1, x') \in \mathbb{R}^d : |x_1| \le 5 \cdot 2^{j/2}, |x'| \le 5\},$$

and

$$E = \{(x_1, x') \in \mathbb{R}^d : |x'| \ge 2^{-j/2}|x_1|\}. \tag{8.5}$$

We derive the following pointwise estimates:

$$\mathcal{M}_{j,\nu}a(x) \lesssim \begin{cases} 1 & \text{if } x \in \mathcal{D}_0, \\ 2^{\epsilon j/2}|x_1|^{-(1+\epsilon)} & \text{if } x \in \mathcal{D}_1 \setminus \mathcal{D}_0, \\ 2^{-j/2}|x'|^{-d} & \text{if } x \in (\mathcal{D} \setminus \mathcal{D}_1) \cap E, \\ 2^{j\frac{d-1}{2}}|x_1|^{-d} & \text{if } x \in (\mathcal{D} \setminus \mathcal{D}_1) \cap E^{\complement}. \end{cases} \tag{8.6}$$

If we use this for $0 < \varepsilon < \frac{1}{p} - 1$, then straightforward integrations give the desired bound,

$$\|\mathcal{M}_{j,\nu}a\|_{L^p(\mathcal{D})} \lesssim 2^{j\frac{d+1}{2}(\frac{1}{p}-1)}. \tag{8.7}$$

To verify (8.6), first observe the pointwise bound $\mathcal{M}_{j,\nu}a(x) \le \sup_{t>0} \|t^d K_{j,\nu}(t\cdot)\|_1 \|a\|_\infty \lesssim 1$. This gives (8.6) for $x \in \mathcal{D}_0$. Secondly, for any $x \in \mathcal{D}_1 \setminus \mathcal{D}_0$ and $y \in B_1(0)$, we have $|x_1 - y_1| \gtrsim |x_1|$. Using (8.4) with $N_1 = 1+\epsilon$ and $N_2 = d-1-\epsilon$, we have

$$\begin{aligned} |t^d K_{j,\nu}(t\cdot) * a(x)| &\lesssim t^d 2^{-j(d+1)/2}(2^{-j}t|x_1|)^{-(1+\epsilon)} \int_{|y'|\le 1} (2^{-j/2}t|x'-y'|)^{-(d-1-\epsilon)} dy' \\ &\lesssim 2^{\epsilon j/2}|x_1|^{-(1+\epsilon)} \end{aligned}$$

for all $x \in \mathcal{D}_1 \setminus \mathcal{D}_0$.

Assume that $x \in (\mathcal{D} \setminus \mathcal{D}_1) \cap E$. Then $|x'| \ge 5$, and thus $|x' - y'| \ge c|x'|$ for some $c > 0$ for all $|y'| \le 1$. Using (8.4) with $N_1 = 0$ and $N_2 = d$, we have

$$|t^d K_{j,\nu}(t\cdot) * a(x)| \lesssim t^d 2^{-j(d+1)/2}(2^{-j/2}t|x'|)^{-d} = 2^{-j/2}|x'|^{-d}.$$

Finally, when $x \in (\mathcal{D} \setminus \mathcal{D}_1) \cap E^{\complement}$, we have $|x_1 - y_1| \ge c|x_1|$, and necessarily $|x_1| \ge 2^{-j/2}$. If we put $N_1 = d$, $N_2 = 0$ in (8.4), we get

$$|t^d K_{j,\nu}(t\cdot) * a(x)| \lesssim t^d 2^{-j(d+1)/2}(2^{-j}t|x_1|)^{-d} = 2^{j(d-1)/2}|x_1|^{-d}.$$

This concludes the proof of the pointwise estimate (8.6), which implies (8.7).

8.1.2. *Estimation over* $\mathcal{D}^{\complement}$. When $x \in \mathcal{D}^{\complement}$, we use the cancellation of the atom and Taylor's formula to write

$$\begin{aligned} t^d K_{j,\nu}(t\cdot) * a(x) &= t^d \int \Big(K_{j,\nu}(tx - ty) - \sum_{n=1}^{M} \frac{\langle -ty, \nabla\rangle^n K_{j,\nu}(tx)}{n!} \Big) a(y)\, dy \\ &= \frac{(-1)^{M+1}}{M!} t^{d+M+1} \int_0^1 (1-s)^M \int \langle y, \nabla\rangle^{M+1} K_{j,\nu}(tx - sty) a(y) dy\, ds. \end{aligned}$$

We now use (8.4) for the derivatives of order $M+1$. Also notice that if E is as in (8.5), we have $|x'| \ge 5 \cdot 2^{j/2}$ for $x \in \mathcal{D}^{\complement} \cap E$, and $|x_1| \ge 5 \cdot 2^j$ for $x \in \mathcal{D}^{\complement} \cap E^{\complement}$. We obtain

$$\mathcal{M}_{j,\nu}a(x) \lesssim \begin{cases} 2^{jM/2}|x'|^{-d-1-M} & \text{if } x \in \mathcal{D}^{\complement} \cap E, \\ 2^{j(M+\frac{d+1}{2})}|x_1|^{-d-M-1} & \text{if } x \in \mathcal{D}^{\complement} \cap E^{\complement}, \end{cases}$$

where for $x \in \mathcal{D}^{\complement} \cap E$ we took $N_1 = 0$, $N_2 = d+M+1$ in (8.4), and for $x \in \mathcal{D}^{\complement} \cap E^{\complement}$ we took $N_1 = d+M+1$ and $N_2 = 0$. Hence

$$\|\mathcal{M}_{j,\nu}a\|_{L^p(\mathcal{D}^{\complement}\cap E)} \lesssim 2^{jM/2}\Big(\int_{|x'|\gtrsim 2^{j/2}} |x'|^{-(d+1+M)p} \int_{|x_1|\lesssim 2^{j/2}|x'|} dx_1 dx' \Big)^{1/p}$$

and

$$\|\mathcal{M}_{j,\nu}a\|_{L^p(\mathfrak{D}^\complement\cap E^\complement)} \lesssim 2^{j(M+\frac{d+1}{2})}\Big(\int_{|x_1|\gtrsim 2^j}|x_1|^{-(d+1+M)p}\int_{|x'|\lesssim 2^{-j/2}|x_1|}dx'dx_1\Big)^{1/p}.$$

Both integrals are $\lesssim 2^{j\frac{d+1}{2}(\frac{1}{p}-1)}$ provided that $p>\frac{d}{d+1+M}$, which is the hypothesis on p and M. This concludes the proof of (8.3). □

8.2. **Proof of Theorem 8.1: Conclusion.** As a crucial ingredient, we shall use the generalized triangle inequality for $L^{p,\infty}$—namely,

$$\Big\|\sum_l f_l\Big\|_{L^{p,\infty}} \lesssim A_p\Big(\sum_l \|f_l\|_{L^{p,\infty}}^p\Big)^{1/p}, \tag{8.8}$$

which holds with $A_p=O((1-p)^{-1/p})$ for $0<p<1$. See either the paper by Kalton [21] or the paper by Stein, Taibleson, and Weiss [39]. By Lemma 8.2, it suffices to prove that

$$\Big\|\sum_{j\ge 1}2^{-j\lambda(p)}\mathcal{M}_jf\Big\|_{L^{p,\infty}} \lesssim \|f\|_{H^p}, \tag{8.9}$$

and by (8.8) and the atomic decomposition, we may assume that f is a (p,M)-atom a, with $M+1>d(p^{-1}-1)$. By dilation and translation invariance, we may assume that a is a function supported in $\{x:|x|\le 1\}$ such that $\|a\|_\infty\le 1$ and $\int a(x)P(x)dx=0$ for all polynomials of degree $\le M$. Because of this normalization, we notice that (up to a harmless constant) the function a is also a (p_1,M)- and a (p_0,M)-atom where $p_1<p<p_0<1$, and we pick p_1 sufficiently close to p that $M+1>d(p_1^{-1}-1)$.

We need to verify, for all $\alpha>0$, that

$$\text{meas}\big(\{x:\sum_{j\ge1}2^{-j\lambda(p)}\mathcal{M}_ja>\alpha\}\big)\lesssim \alpha^{-p}. \tag{8.10}$$

By Proposition 8.3, we have for every $j\ge 1$

$$\Big\|2^{-j\lambda(p)}\mathcal{M}_ja\Big\|_{p_i}\lesssim 2^{j(\lambda(p_i)-\lambda(p))}. \tag{8.11}$$

We employ a variant of an interpolation argument in [4] to estimate

$$\text{meas}\big(\{x:\sum_{j\ge1}2^{-j\lambda(p)}\mathcal{M}_ja>\alpha\}\big)\le I+II,$$

where I is the measure of the set on which $\sum_{2^j\le\alpha^{-p/d}}2^{-j\lambda(p)}\mathcal{M}_ja>\alpha/2$, and II is the measure of the set on which $\sum_{2^j>\alpha^{-p/d}}2^{-j\lambda(p)}\mathcal{M}_ja>\alpha/2$. By Tshebyshev's inequality,

$$I\le(2/\alpha)^{p_1}\Big\|\sum_{\substack{j\in\mathbb{N}:\\2^j\le\alpha^{-p/d}}}2^{-j\lambda(p)}\mathcal{M}_ja\Big\|_{p_1}^{p_1},$$

$$II\le(2/\alpha)^{p_0}\Big\|\sum_{\substack{j\in\mathbb{N}:\\2^j>\alpha^{-p/d}}}2^{-j\lambda(p)}\mathcal{M}_ja\Big\|_{p_0}^{p_0}.$$

Apply (8.11) to obtain

$$I + II \lesssim \alpha^{-p_1} \sum_{2^j \le \alpha^{-p/d}} 2^{j(\lambda(p_1)-\lambda(p))p_1} + \alpha^{-p_0} \sum_{2^j > \alpha^{-p/d}} 2^{j(\lambda(p_0)-\lambda(p))p_0}$$
$$= \alpha^{-p_1} \sum_{2^{jd} \le \alpha^{-p}} 2^{jd(1-\frac{p_1}{p})} + \alpha^{-p_0} \sum_{2^{jd} > \alpha^{-p}} 2^{jd(1-\frac{p_0}{p})} \lesssim \alpha^{-p}.$$

This yields (8.10) and concludes the proof. □

Remark 8.4. Versions of the Fan–Wu transference argument in Section 5.1 for maximal functions and h^p for $p < 1$ can be used to prove a theorem for Riesz means of Fourier series analogous to Theorem 8.1; i.e., the maximal function $\sup_{t>0} |\mathcal{R}_t^{d(1/p-1/2)-1/2} f|$ defines an operator that maps $h^p(\mathbb{T}^d)$ to $L^{p,\infty}(\mathbb{T}^d)$ when $p < 1$.

9. Open problems

9.1. **Spaces near L^1.** For $f \in L^1(\mathbb{T}^d)$, it remains open whether the Riesz means $\mathcal{R}_t^{\lambda(p)} f(x)$ converge q-strongly a.e. for any $q < \infty$. In particular, can one upgrade in Corollary 7.3 the conclusion of upper density 1 of $E(f, x)$ to density 1?

It may also be interesting to investigate strong convergence a.e. for spaces intermediate between L^1 and $L \log L$.

9.2. **The case $q = p'$.** For $f \in L^p(\mathbb{T}^d)$, $1 < p < 2$, prove or disprove that $\mathcal{R}_t^{\lambda(p)} f(x)$ converges q-strongly a.e. when $q = p'$. For $f \in h^1(\mathbb{T}^d)$, is there a version of Rodin's theorem [30], in one dimension, that applies to Riesz means at the critical index $\lambda(1) = \frac{d-1}{2}$ where the L^q-average norm in t-variable is replaced by a BMO-average?

9.3. **Problems involving nonisotropic dilations.** One can ask the same questions for quasi-radial Riesz means when the isotropic dilation group is replaced by a nonisotropic dilation group t^P, where P is a matrix with positive eigenvalues and ρ satisfies $\rho(t^P \xi) = t\rho(\xi)$. It turns out that the results depend on the geometry of the surface in relation to the eigenvectors of P. In the case in which $\Sigma_\rho = \{\xi : \rho(\xi) = 1\}$ has nonvanishing curvature everywhere, one has almost everywhere convergence for $\lambda > \frac{d-1}{2}$, but there are other examples where a.e. convergence fails for $\lambda < d/2$; see [22] for details. Even in the case of nonvanishing curvature, we have currently no endpoint results for strong convergence of $\mathcal{R}_t^\lambda f$ for the critical $\lambda = \lambda(p)$ when the dilations are nonisotropic.

9.4. **Almost everywhere convergence.** For $1 < p < 2$, the problem of a.e. convergence, and the critical q for strong summability for $\lambda > \lambda(p)$, is wide open. Optimal results for the maximal operators are currently known only for the subspace L^p_{rad} of radial L^p functions; see [16]. For general L^p functions, results that improve on Stein's classical theorem for a.e. convergence of Riesz means of index $> (d-1)(1/p - 1/2)$ are currently known only in two dimensions; see Tao's paper [44].

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References

[1] Nakhlé Asmar, Earl Berkson, and Jean Bourgain, *Restrictions from* $\mathbf{R}^n$ *to* $\mathbf{Z}^n$ *of weak type* $(1,1)$ *multipliers*, Studia Math. **108** (1994), no. 3, 291–299, DOI 10.4064/sm-108-3-291-299. MR1259281

[2] Jöran Bergh and Jörgen Löfström, *Interpolation spaces. An introduction*, Grundlehren der Mathematischen Wissenschaften, No. 223, Springer-Verlag, Berlin-New York, 1976. MR0482275

[3] Salomon Bochner, *Summation of multiple Fourier series by spherical means*, Trans. Amer. Math. Soc. **40** (1936), no. 2, 175–207, DOI 10.2307/1989864. MR1501870

[4] Jean Bourgain, *Estimations de certaines fonctions maximales* (French, with English summary), C. R. Acad. Sci. Paris Sér. I Math. **301** (1985), no. 10, 499–502. MR812567

[5] Jean Bourgain and Larry Guth, *Bounds on oscillatory integral operators based on multilinear estimates*, Geom. Funct. Anal. **21** (2011), no. 6, 1239–1295, DOI 10.1007/s00039-011-0140-9. MR2860188

[6] Anthony Carbery and Fernando Soria, *Pointwise Fourier inversion and localisation in* $\mathbf{R}^n$, Proceedings of the Conference Dedicated to Professor Miguel de Guzmán (El Escorial, 1996), J. Fourier Anal. Appl. **3** (1997), no. Special Issue, 847–858, DOI 10.1007/BF02656490. MR1600203

[7] Sun-Yung A. Chang and Robert Fefferman, *A continuous version of duality of* H^1 *with BMO on the bidisc*, Ann. of Math. (2) **112** (1980), no. 1, 179–201, DOI 10.2307/1971324. MR584078

[8] Michael Christ, *Weak type* $(1,1)$ *bounds for rough operators*, Ann. of Math. (2) **128** (1988), no. 1, 19–42, DOI 10.2307/1971461. MR951506

[9] Michael Christ, *Weak type endpoint bounds for Bochner-Riesz multipliers*, Rev. Mat. Iberoamericana **3** (1987), no. 1, 25–31, DOI 10.4171/RMI/44. MR1008443

[10] F. M. Christ and C. D. Sogge, *The weak type* L^1 *convergence of eigenfunction expansions for pseudodifferential operators*, Invent. Math. **94** (1988), no. 2, 421–453, DOI 10.1007/BF01394331. MR958838

[11] Michael Christ and Christopher D. Sogge, *On the* L^1 *behavior of eigenfunction expansions and singular integral operators*, Miniconferences on harmonic analysis and operator algebras (Canberra, 1987), Proc. Centre Math. Anal. Austral. Nat. Univ., vol. 16, Austral. Nat. Univ., Canberra, 1988, pp. 29–50. MR953981

[12] A. Córdoba, *A note on Bochner-Riesz operators*, Duke Math. J. **46** (1979), no. 3, 505–511. MR544242

[13] A. Erdélyi, *Asymptotic expansions*, Dover Publications, Inc., New York, 1956. MR0078494

[14] Da Shan Fan and Zhi Jian Wu, *Transference of maximal multipliers on Hardy spaces*, Proc. Amer. Math. Soc. **123** (1995), no. 10, 3169–3174, DOI 10.2307/2160677. MR1273491

[15] Charles Fefferman, *Inequalities for strongly singular convolution operators*, Acta Math. **124** (1970), 9–36, DOI 10.1007/BF02394567. MR0257819

[16] Gustavo Garrigós and Andreas Seeger, *A note on maximal operators associated with Hankel multipliers*, Rev. Un. Mat. Argentina **50** (2009), no. 2, 137–148. MR2656531

[17] David Goldberg, *A local version of real Hardy spaces*, Duke Math. J. **46** (1979), no. 1, 27–42. MR523600

[18] G. H. Hardy and J. E. Littlewood, *Sur la séries de Fourier d'une fonction à carré sommable*, Comptes Rendus (Paris) **156** (1913), 1307–1309.

[19] Sunggeum Hong, Joonil Kim, and Chan Woo Yang, *Riesz means associated with certain product type convex domain*, J. Math. Anal. Appl. **380** (2011), no. 2, 585–606, DOI 10.1016/j.jmaa.2011.02.070. MR2794416

[20] Carlos E. Kenig and Peter A. Tomas, *Maximal operators defined by Fourier multipliers*, Studia Math. **68** (1980), no. 1, 79–83, DOI 10.4064/sm-68-1-79-83. MR583403

[21] N. J. Kalton, *Linear operators on* L_p *for* $0 < p < 1$, Trans. Amer. Math. Soc. **259** (1980), no. 2, 319–355, DOI 10.2307/1998234. MR567084

[22] Y. Kim and A. Seeger, *A note on pointwise convergence of quasiradial Riesz means*, Acta Sci. Math. (Szeged) **62** (1996), no. 1-2, 187–199. MR1412927

[23] A. N. Kolmogorov, *Une série de Fourier-Lebesgue divergente presque partout*, Fund. Math. **4** (1923), 324–328 (French).

[24] Sergey V. Konyagin, *Almost everywhere convergence and divergence of Fourier series*, International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, 2006, pp. 1393–1403. MR2275651

[25] Karel de Leeuw, *On L_p multipliers*, Ann. of Math. (2) **81** (1965), 364–379, DOI 10.2307/1970621. MR0174937

[26] Zhi Xin Liu and Shan Zhen Lu, *Transference and restriction of maximal multiplier operators on Hardy spaces*, Studia Math. **105** (1993), no. 2, 121–134, DOI 10.4064/sm-105-2-121-134. MR1226622

[27] Shanzhen Lu, *Conjectures and problems on Bochner-Riesz means*, Front. Math. China **8** (2013), no. 6, 1237–1251, DOI 10.1007/s11464-013-0323-9. MR3122308

[28] J. Marcinkiewicz, *Sur la sommabilité forte de séries de Fourier* (French), J. London Math. Soc. **14** (1939), 162–168, DOI 10.1112/jlms/s1-14.3.162. MR0000066

[29] Jaak Peetre, *On spaces of Triebel-Lizorkin type*, Ark. Mat. **13** (1975), 123–130, DOI 10.1007/BF02386201. MR0380394

[30] V. A. Rodin, *The space BMO and strong means of Fourier series* (English, with Russian summary), Anal. Math. **16** (1990), no. 4, 291–302, DOI 10.1007/BF02630362. MR1094184

[31] Andreas Seeger, *Estimates near L^1 for Fourier multipliers and maximal functions*, Arch. Math. (Basel) **53** (1989), no. 2, 188–193, DOI 10.1007/BF01198570. MR1004277

[32] Andreas Seeger, *Endpoint estimates for multiplier transformations on compact manifolds*, Indiana Univ. Math. J. **40** (1991), no. 2, 471–533, DOI 10.1512/iumj.1991.40.40024. MR1119186

[33] Andreas Seeger, *Endpoint inequalities for Bochner-Riesz multipliers in the plane*, Pacific J. Math. **174** (1996), no. 2, 543–553. MR1405600

[34] Elias M. Stein, *Localization and summability of multiple Fourier series*, Acta Math. **100** (1958), 93–147, DOI 10.1007/BF02559603. MR0105592

[35] E. M. Stein, *On limits of sequences of operators*, Ann. of Math. (2) **74** (1961), 140–170, DOI 10.2307/1970308. MR0125392

[36] Elias M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, NJ, 1970. MR0290095

[37] E. M. Stein, *An H^1 function with nonsummable Fourier expansion*, Harmonic analysis (Cortona, 1982), Lecture Notes in Math., vol. 992, Springer, Berlin, 1983, pp. 193–200, DOI 10.1007/BFb0069159. MR729354

[38] E. M. Stein, *Oscillatory integrals in Fourier analysis*, Beijing lectures in harmonic analysis (Beijing, 1984), Ann. of Math. Stud., vol. 112, Princeton Univ. Press, Princeton, NJ, 1986, pp. 307–355. MR864375

[39] Elias M. Stein, Mitchell H. Taibleson, and Guido Weiss, *Weak type estimates for maximal operators on certain H^p classes*, Proceedings of the Seminar on Harmonic Analysis (Pisa, 1980), Rend. Circ. Mat. Palermo (2) **suppl. 1** (1981), 81–97. MR639468

[40] Elias M. Stein and Guido Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Mathematical Series, vol. 32, Princeton University Press, Princeton, NJ, 1971. MR0304972

[41] E. M. Stein and N. J. Weiss, *On the convergence of Poisson integrals*, Trans. Amer. Math. Soc. **140** (1969), 35–54, DOI 10.2307/1995121. MR0241685

[42] Gen-ichirô Sunouchi, *On the Littlewood-Paley function g^* of multiple Fourier integrals and Hankel multiplier transformations*, Tôhoku Math. J. (2) **19** (1967), 496–511, DOI 10.2748/tmj/1178243254. MR0225097

[43] Terence Tao, *The weak-type endpoint Bochner-Riesz conjecture and related topics*, Indiana Univ. Math. J. **47** (1998), no. 3, 1097–1124, DOI 10.1512/iumj.1998.47.1544. MR1665753

[44] Terence Tao, *On the maximal Bochner-Riesz conjecture in the plane for $p < 2$*, Trans. Amer. Math. Soc. **354** (2002), no. 5, 1947–1959, DOI 10.1090/S0002-9947-02-02942-2. MR1881025

[45] Walter Trebels, *Some Fourier multiplier criteria and the spherical Bochner-Riesz kernel*, Rev. Roumaine Math. Pures Appl. **20** (1975), no. 10, 1173–1185. MR0404967

[46] Bobby Wilson, *On almost everywhere convergence of strong arithmetic means of Fourier series*, Trans. Amer. Math. Soc. **367** (2015), no. 2, 1467–1500, DOI 10.1090/S0002-9947-2014-06297-1. MR3280051

[47] A. Zygmund, *On the convergence and summability of power series on the circle of convergence. II*, Proc. London Math. Soc. (2) **47** (1942), 326–350, DOI 10.1112/plms/s2-47.1.326. MR0007042

[48] A. Zygmund, *Trigonometric series: Vols. I, II*, 2nd ed., Cambridge University Press, London–New York, 1968. MR0236587

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