How many randomly colored edges make a randomly colored dense graph rainbow hamiltonian or rainbow connected?

Michael Anastos* and Alan Frieze†

June 3, 2020

Abstract

In this paper we study the randomly edge colored graph that is obtained by adding randomly colored random edges to an arbitrary randomly edge colored dense graph. In particular we ask how many colors and how many random edges are needed so that the resultant graph contains a fixed number of edge disjoint rainbow Hamilton cycles w.h.p. We also ask when in the resultant graph every pair of vertices is connected by a rainbow path w.h.p.

1 Introduction

In this paper we study the following random graph model: We start with a graph $H = (V, E)$ and a set R of m edges chosen uniformly at random from $\binom{[n]}{2} \setminus E$, that is from the edges not found in H. We then add to H the edges in R to get the graph

$$G_{H,m} = (V, E \cup R).$$

After this, we color every edge of $G_{H,m}$ independently and uniformly at random with a color from $[r]$. We denote the resultant colored graph by

$$G^r_{H,m} = (V, E \cup R, c).$$

Here $c : E \cup R \mapsto [r]$ is the function that assigns to every edge in $E \cup R$ its color.

*Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh PA 15213, USA, Email: manastos@andrew.cmu.edu, Research supported in part by NSF grant DMS0753472

†Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh PA 15213, USA, Email: alan@random.math.cmu.edu, Research supported in part by NSF grant DMS0753472

The random graph model $G_{H,m}$ was first introduced by Bohman, Frieze and Martin in [2]. It can be considered as an extension of the Erdős-Rényi model which we can retrieve from $G_{H,m}$ by setting $H = \emptyset$. The main motivation for this model is the following: Let $\mathcal{G}$ be some class of graphs and $\mathcal{P}$ be a property, an increasing property in our case, that is satisfied by almost all the members of $\mathcal{G}$. The following question arises. For any graph $G \in \mathcal{G}$, suppose we perturb slightly its edge set at random, by adding a few random edges. How many are needed so that the result is a graph that satisfies property $\mathcal{P}$ w.h.p.[1] In [2] they study the case where $\mathcal{G}$ is the set of graphs of minimum degree δn, $\delta > 0$ and $\mathcal{P}$ is the property of a graph having a Hamilton cycle. They show that a linear number of random edges suffices in order to make any member of $\mathcal{G}$ Hamiltonian w.h.p. They also point out that for $\delta < 0.5$, complete bipartite graphs with bipartitions of sizes δn and $(1-\delta)n$ need a linear number of random edges in order to become Hamiltonian. Here it is worth mentioning that in the Erdős-Rényi random graph, $G(n,m)$, the threshold for Hamiltonicity is $(\log n + \log\log n)n$. $G_{H,m}$ has since been studied in a number of other contexts, see for example [1], [3], [9], [10] and [13].

In this paper we enhance this model by randomly coloring edges. We $[r]$-color the edges of $G_{H,m}$ independently and uniformly at random and we denote the resultant graph by $G^r_{H,m}$. We then ask about the existence of a rainbow Hamilton cycle in $G_{H,m}$ and whether $G^r_{H,m}$ is rainbow connected.

A Hamilton cycle is called rainbow if no color appears twice on its edges. It was shown by Frieze and Loh [8] and by Ferber and Krivelevich [5] that for $m \geq (1+o(1))(\log n+\log\log n)n$ if we color $G(n,m)$ randomly with $(1+o(1))n$ colors then $G(n,m)$ contains a rainbow Hamilton cycle. This implies that if we randomly $[(1+o(1))n]$-color a typical graph of minimum degree δn then w.h.p. the resultant graph will contain a rainbow Hamilton cycle. In comparison, as mentioned earlier, a graph of linear minimum degree needs only a linear number of edges in order to become Hamiltonian.

We say a graph is rainbow connected if every pair of its vertices are connected by a rainbow path. For a fixed graph G of minimum degree d is known that $\frac{20n}{d}$ colors are sufficient to color it such that the resultant graph is rainbow connected, see [12].

The class of graphs of interest in our case will be the graphs on n vertices, of minimum degree δn with $\delta > 0$ which we denote by $\mathcal{G}(n,\delta)$. For the rest of this paper we let $0 < \delta < 0.5$ and H be an arbitrary member of $\mathcal{G}(n,\delta)$. We also let

$$\theta = \theta(\delta) = -\log\delta \qquad \text{and} \qquad t_0 = t_0(\delta) = \min\left\{\frac{\delta n}{260}, \frac{n}{1000+200\theta}\right\}. \tag{1}$$

Our first theorem builds on the Hamiltonicity result of [2].

Theorem 1. *Let $1 \leq t \leq t_0$ and $m \geq \min\left(\frac{10(435+75\theta)tn}{9}, \left|\binom{[n]}{2} \setminus E(H)\right|\right)$ and $r \geq (120+20\theta)n$ then, w.h.p. $G^r_{H,m}$ contains t edge disjoint rainbow Hamilton cycles.*

When $t = 1$ the above theorem states that a linear number of edges and a linear number of colors suffice in order for $G^r_{H,m}$ to have rainbow-Hamilton cycle. In addition it says that

[1] We say that a sequence of events $\mathcal{E}_n$ holds *with high probability* if $\mathbf{P}\{\mathcal{E}_n\} \to 1$ as $n \to \infty$.

if you require a multiple number of edge disjoint rainbow Hamilton cycles then it suffices to multiply the number of random edges that are added appropriately.

The following Theorem concerns the rainbow connectivity of $G^r_{H,m}$.

Theorem 2. *For rainbow connectivity the following holds:*

(i) *If $r = 3$ and $m \geq 60\delta^{-2}\log n$ then w.h.p. $G^3_{H,m}$ is rainbow connected.*

(ii) *For $\delta \leq 0.1$ there exists $H \in \mathcal{G}(n,\delta)$ such that if $m \leq 0.5\log n$ then w.h.p. $G^4_{H,m}$ is not rainbow connected.*

(iii) *If $r \geq 7$ and $m = \omega(1)$ then w.h.p. $G^r_{H,m}$ is rainbow connected.*

The rest of the paper is divided as follows. In Section 2 we give the proof of Theorem 1 and in Section 3 we give the proof of Theorem 2. We close with Section 4.

2 Proof of Theorem 1

Let G be a graph of minimum degree at least k. A k-out random subgraph of G, denoted G_{k-out}, can be generated by adding independently at random k edges incident to vertex v for every $v \in V$.

We will often use the following version of the Chernoff bounds for the binomial distribution $B(n,p)$.

$$\mathbf{Pr}(|B(n,p) - np| \geq \epsilon np) \leq 2e^{-\epsilon^2 np/3}, \qquad \text{for } 0 \leq \epsilon \leq 1.$$

We partition H into two subgraphs H', H'' as follows. We include every edge of H into $E(H')$ independently with probability $p = \frac{1}{20}$. We then set $E(H'') = E(H) \setminus E(H')$. Since H has minimum degree δn and $p \cdot \delta n \gg \log n$, the Chernoff bounds imply that w.h.p. $E(H')$ has minimum degree $\delta(H') \geq \frac{\delta n}{21}$ and maximum degree $\Delta(H') \leq \frac{n}{19}$ and $\left(\frac{1}{20} + o(1)\right)|E(H)|$ edges. We partition H so that it will be easier to expose the random edges in stages.

To prove Theorem 1 we first prove a non-colored version of it. Namely we show

Theorem 3. *Let $Q \subset E(H'') \cup R$ be such that $|Q| = (81 + 15\theta)n$ and Q is distributed as a random subset of $\binom{[n]}{2} \setminus E(H')$ of the corresponding size. Then w.h.p. $H'_{6-out} \cup Q$ is Hamiltonian*

To get Theorem 1 from Theorem 3 we use a result of Ferber, Kronenberg, Mousset and Shikhelmam [6], stated as Theorem 6 below. We use it in order to extract t rainbow subgraphs from H' each of which has the properties of H_{6-out} needed in the proof of Theorem 3. And each of these subgraphs can become hamiltonian by adding to it $(81 + 15\theta)n$ random edges. We then argue that by suitably refining those edges the Hamilton cycles will be rainbow.

2.1 Proof of Theorem 3

We first randomly split Q into two sets Q_1, Q_2 of sizes $(45+15\theta)n$ and $36n$ respectively. We then show that $H'_{6-out} \cup Q_1$ is connected and has good expansion properties. Then we apply a standard Pósa rotation argument to show that $E(H') \cup Q_1 \cup Q_2$ spans a Hamilton cycle.

Lemma 4. *With probability $1 - o(n^{-2})$, the following hold:*

1. *$H'_{6-out} \cup Q_1$ is connected,*
2. *for every $S \subseteq V$ such that $|S| \leq n/5$ we have that $|N_{H'_{6-out} \cup Q_1}(S)| > 2|S|$.*

Proof. We start by examining the second property for "small" sets.

Claim 1: With probability $1 - o(n^{-2})$ every $S \subset V$ such that $|S| \leq \delta^2 n/200$ satisfies $|N_{H'_{6-out}}(S)| > 2|S|$.

Proof of Claim 1: Let $S, T \subset V$ be such that $|T| = 2|S|$ and $|S| \leq \delta^2 n/200$. In H' every vertex in S has at most $3|S| \leq 3\delta^2 n/200$ out neighbors in $S \cup T$. Furthermore observe that given H' every set of 6 edges adjacent to v in H is equaly likely to be chosen by v during the construction of H'_{6-out}. Thus

$$\mathbf{Pr}(N_{H'_{6-out}}(v) \subseteq S \cup T) \leq \binom{3|S|}{6} \Big/ \binom{\delta n}{6} \leq \left(\frac{3|S|}{\delta n}\right)^6.$$

Therefore, with $s = |S|$,

$$\begin{aligned}
\mathbf{Pr}(\text{Claim 1 is violated}) &\leq \sum_{s=1}^{\delta^2 n/200} \binom{n}{s}\binom{n}{2s}\left(\frac{3s}{\delta n}\right)^{6s} \\
&\leq \sum_{s=1}^{\delta^2 n/200} \left(\frac{en}{s}\right)^s \left(\frac{en}{2s}\right)^{2s} \left(\frac{3s}{\delta n}\right)^{6s} \\
&\leq \sum_{s=1}^{\delta^2 n/200} \left(\frac{e^3 \cdot 3^6 s^3}{\delta^6 n^3}\right)^s = o(n^{-2}).
\end{aligned}$$

End of proof of Claim 1.

Now we examine the second property for "large" sets. Here we are going to use the edges from Q_1. Let $\Delta(H')$ be the maximum degree of H'. Then $\Delta(H') \leq n/19$ w.h.p.

Claim 2: With probability $1 - o(n^{-2})$ every $S \subseteq V$ such that $\delta^2 n/200 < |S| \leq n/5$ satisfies $|N_{H'_{6-out} \cup Q_1}(S)| > 2|S|$.

Proof of Claim 2: For $\delta^2 n/200 \leq |S| \leq n/5$ by considering only the edges in Q_1 we have

$$\mathbf{Pr}(\text{Claim 2 is violated}) \leq o(n^{-2}) + (1+o(1)) \sum_{s=\frac{\delta^2 n}{200}}^{\frac{n}{5}} \binom{n}{s}\binom{n}{2s}\left(1 - \frac{s(n-3s) - sn/19}{\binom{n}{2}}\right)^{|Q_1|} \tag{2}$$

$$\leq o(n^{-2}) + (1+o(1)) \sum_{s=\frac{\delta^2 n}{200}}^{\frac{n}{5}} \left(\frac{en}{s}\right)^s \left(\frac{en}{2s}\right)^{2s} e^{-\frac{s(n-3s-n/19)|Q_1|}{\binom{n}{2}}}$$

$$\leq o(n^{-2}) + (1+o(1)) \sum_{s=\frac{\delta^2 n}{200}}^{\frac{n}{5}} \left(\frac{e^3 n^3}{4s^3} e^{-\frac{2|Q_1|}{5n}}\right)^s$$

$$\leq o(n^{-2}) + (1+o(1)) \sum_{s=\frac{\delta^2 n}{200}}^{\frac{n}{5}} \left(\frac{e^3 \cdot 200^3}{4\delta^6} e^{-(18+6\theta)}\right)^s = o(n^{-2}).$$

Explanation of (2): The $o(n^{-2})$ term accounts for the probability that $\Delta(H') > n/19$. The $(1+o(1))$ factor in front of the summation accounts for conditioning on this event. s denotes the size of S. We choose S and a set of size $2s$ that contains the neighborhood of S. We then estimate the probability p that an edge of Q_1 joins S to $V \setminus T$. Out of at most $\binom{n}{2}$ choices for an edge of Q_1 there are $s(n-3s)$ which join S to $V \setminus (S \cup T)$, of which at most $s\Delta(H')$ are in $E(H')$. The probability there is no S to $V \setminus (S \cup T)$ edge is bounded by $(1-p)^{|Q_1|}$.
End of proof of Claim 2.

Claims 1 and 2 imply the second property of our Lemma. At the same time Claim 1 implies that every connected component of H'_{6-out} has size at least $\frac{\delta^2 n}{200}$. In the event that any two of the at most $200\delta^{-2}$ components of H'_{6-out} are connected by an edge in Q_1 we have that $H_{6-out} \cup Q_1$ is connected. Observe for any two disjoint sets S_1, S_2 of size at least $\delta^2 n/200$, the Chernoff bounds imply that

$$\mathbf{Pr}(S_1 \cup S_2 \text{ span } \geq |S_1|\,||S_2|/10 \text{ edges in } H') = \exp\left\{-O(\delta^2 n^2)\right\}.$$

Since there are at most 2^n choices for each of S_1, S_2 we have that w.h.p. every pair of components S_1, S_2 of H'_{6-out} spans at most $|S_1||S_2|/10$ edges in $E(H')$, thus there are at least $9|S_1||S_2|/10$ edges in $\binom{[n]}{2} \setminus E(H')$ whose inclusion in Q_1 can connect S_1 to S_2. Therefore

$$\mathbf{Pr}(H'_{6-out} \cup Q_1 \text{ is not connected }) \leq \binom{200\delta^{-2}}{2}\left(1 - \frac{9}{10}\left(\frac{\delta^2 n}{200}\right)^2 \cdot \frac{1}{\binom{n}{2}}\right)^{|Q_1|} = o(n^{-2}).$$

□

Our next Lemma builds on Lemma 4 and completes the proof of Theorem 3. It is basically an adaptation of Pósa's argument to our setting.

Lemma 5.

$$\mathbf{Pr}(H'_{6-out} \cup Q_1 \cup Q_2 \text{ is not Hamiltonian}) = o(n^{-2}).$$

Proof. Let $Q_2 = \{e_1, ..., e_{|Q_2|}\}$ and set $G_i = (H'_{6-out} \cup Q_1) \cup \{e_1, ..., e_i\}$. Assume that G_i is not Hamiltonian and consider a longest path P_i in $G_i, i \geq 0$. Let x, y be the end-vertices of P_i. Given yv where v is an interior vertex of P_i we can obtain a new longest path $P'_i = x..vy..w$ where w is the neighbor of v on P_i between v and y. In such a case we say that P'_i is obtained from P_i by a rotation with the endpoint x being the fixed end-vertex.

Let $END_i(x; P_i)$ be the set of end-vertices of longest paths of G_i that can be obtained from P_i by a sequence of rotations that keep x as the fixed end-vertex. Thereafter, for $z \in END_i(x; P_i)$ let $P_i(x, z)$ be a path that has end-vertices x, z and can be obtain form P_i by a sequence of rotations that keep x as the fixed end-vertex. Observe that since G_i is connected but not Hamiltonian for $z \in END_i(x; P_i)$ and $z' \in END_i(z; P_i(x, z))$ neither xz nor zz' belong to G_i since otherwise we can close the path into a cycle that is not Hamiltonian and then use the connectivity of G_i to get a longer path than P. At the same time it follows from Pósa [14]

$$|N(END_i(x, P_i))| < 2|END_i(x, P_i)|.$$

Moreover for every $z \in END_i(x; P_i)$

$$|N(END_i(z, P_i(x, z)))| < 2|END_i(z, P_i(x, z))|.$$

As a consequence, since Lemma 4 states that $\forall S \subset V\ |S| \leq n/5$ we have $2|S| \geq N_{H_{6-out} \cup Q_1}(S)$, we get that $|END_i(x, P_i)| \geq \frac{n}{5}$.

Let $E_i = \{\{z, z'\} \notin H' : z \in END_i(x; P_i) \text{ and } z' \in END_i(z; P_i(x, z))\}$. H' has maximum degree at most $n/19$. Furthermore $|END(x, P_i)| \geq n/5$ and for every $z \in END_i(x, P_i)$ we have $|END_i(z; P_i(x, z))| \geq n/5$. Hence

$$|E_i| \geq \frac{1}{2} \cdot \frac{n}{5}\left(\frac{n}{5} - \frac{n}{19}\right) \geq \frac{n^2}{70}.$$

Now let Y_{i+1} be the indicator that $e_{i+1} \in E_i$ and $Z = \sum_{i=1}^{|Q_2|} Y_i$. Then $\mathbf{Pr}(Y_i = 1) \geq 1/35$. Therefore Z dominates the binomial $Bin(|Q_2|, 1/35)$. Thus since $|Q_2|/35 = 36n/35$, the Chernoff bound implies that $\mathbf{Pr}(Z \leq n) = e^{-\Omega(n)} = o(n^{-2})$. Hence $G_{|Q_2|} = H'_{6-out} \cup Q_1 \cup Q_2$ is Hamiltonian with the required probability. □

2.2 Partitioning H'

We are now ready to apply Theorem 6, given below and partition H' into $t+1$ edge disjoint subgraphs. t of them will be rainbow and will satisfy the property described in Theorem 3. The final step will be to show that for each $i \in [t]$ we can set aside a random subset $R_i \subseteq R$, of size $(81 + 15\theta)n$ such that $H_i \cup R_i$ is rainbow.

The following theorem is taken from Ferber et al [6]. It is a combination of Lemma 2.7 and Theorem 3.1 of that paper. From Theorem 3.1 we get the that each F_i is distributed as

a random graph F^*_{k-out}. Lemma 2.7 allows the substitution of F_{k-out} in place of F^*_{k-out} with the stated probability error $n^{-\omega(1)}$.

Theorem 6 ([6]). *Let $\epsilon > 0$ be a constant, $k \geq 2$ be an integer and $\mathcal{P}$ be a monotone increasing graph property. Let F be a graph on n vertices with minimum degree $\delta(F) = \omega(\log n)$ whose edges are colored independently and uniformly at random from $[kn]$. Then, w.h.p. F can be partitioned into $F = F_0 \cup F_1 \cup \cdots \cup F_r$ such that the following holds.*

1. *$F_0, F_1, \ldots, F_r$ are edge-disjoint subgraphs of F,*
2. *$r = (1-\varepsilon)\frac{\delta(F)}{2k}$,*
3. *For every $1 \leq i \leq r$, $E(F_i)$ is rainbow and of size at most kn, and*
4. *For every $1 \leq i \leq r$, $\Pr[F_{k\text{-}out} \text{ satisfies } \mathcal{P}] \leq \Pr[F_i \text{ satisfies } \mathcal{P}] + n^{-\omega(1)}$.*

□

By monotone, we mean here that if F_{k-out} satisfies $\mathcal{P}$ then adding edges to F_{k-out} gives a graph that also satisfies $\mathcal{P}$.

We apply the above Theorem with $F = H'$ and $r = t, k = 6$ and $\mathcal{P}$ the property that the addition of $(81+15\delta)n$ random edges from $\binom{[n]}{2} \setminus H'$ makes the graph Hamiltonian with probability at least $1 - n^{-2}$. This gives a partition of H' into $F_0 \cup F_1 \cup F_2 ... \cup F_t$ It follows from Theorem 6 and Lemma 5 that we can finish our proof by showing that w.h.p. we can pair each of the $F_i, i \in [t]$ with a random subset $Q_i \subseteq E(H'') \cup R$ of size $(81+15\delta)n$ such that $F_i \cup Q_i$ is rainbow.

2.3 Partitioning $E(H'') \cup R$

We can assume that $m = \min\{(435+75\theta)tn, \bar{m}_H\}, \bar{m}_H = \left|\binom{[n]}{2} \setminus E(H)\right|\}$. We start by extracting t disjoint sets, $Q'_1, ..., Q'_t$, from $E(H'') \cup R$, of size $m_1 = (435+75\theta)n$ such that Q'_i is uniformly distributed among all t element subsets of $\binom{[n]}{2} \setminus E(H')$. If $m = \bar{m}_H$ then $\binom{[n]}{2} \setminus E(H') = E(H'') \cup R$ and we can choose the sets directly by first randomly choosing a set Q' of size tm_1 from $E(H'') \cup R$. We can do this, because our choice of t_0 implies that $m_1 \leq 0.44n$ and $|E(H')| \leq n^2/19$ w.h.p. Otherwise $m = tm_1$ and $|E(H'') \cup R| \geq \bar{m}_H \geq tn$ and we proceed by generating two integers a, b whose distribution is the same as $|A \cap R|, |E(H'') \cap A|$ where A is a random subset of $\binom{[n]}{2}$ of size tm_1. We then generate Q' by adding to it a and b random elements of R and $E(H'')$ respectively. In both cases we proceed by randomly partitioning Q' into t pieces of the same size to get the desired sets $Q'_1, Q'_2, ..., Q'_{m_1}$.

From Lemma 5 any subset A_i of Q'_i that satisfies: (i) $A_i \cup F'_i$ is rainbow and (ii) $|A_i| = (81+15\theta)n$ satisfies the requirements for Q_i. In the case that at least $(87+15\theta)n$ colors appear in Q'_i, since at most $6n$ colors are used for the edges of F_i, such a set exists. Finally

the probability that fewer colors appear is bounded by

$$\binom{r}{(87+15\theta)n}\left(\frac{(87+15\theta)n}{r}\right)^{(435+75\theta)n}$$
$$\leq \left(\frac{er}{(87+15\theta)n}\right)^{(87+15\theta)n}\left(\frac{(87+15\theta)n}{r}\right)^{(435+75\theta)n}$$
$$= e^{(87+15\theta)n}\left(\frac{(87+15\theta)n}{(120+20\theta)n}\right)^{(348+60\theta)n} \leq \left[e\left(\frac{3}{4}\right)^4\right]^{(87+15\theta)n} = o\left(\frac{1}{n}\right).$$

This completes the proof of Theorem 1.

□

3 Proof of Theorem 2

3.1 Proof of Theorem 2 (i)

We will show that if R is large enough then w.h.p., for any pair of vertices $u, v \in V$ there are many edges in R between their neighborhoods $N(u), N(v)$. It will follow that w.h.p. there is a rainbow path of length 3 from u to v, which necessarily uses all of the three available colors.

Let $R = \{r_1, ..., r_m\}$. Let $\mathcal{C}$ be the event that G is rainbow connected. For $u, v \in V$ let $\mathcal{B}(u, v)$ be the event that there exist fewer than $10 \log n$ paths of the form (u, u_0, v_0, v) in G, where $\{u, u_0\}, \{v, v_0\} \in E(H)$ and $\{u_0, v_0\} \in R$. Then let $\mathcal{C}(u, v)$ be the event that there is a rainbow such path. Given $r_1, ..., r_{i-1}$ either there exist $10 \log n$ such paths or r_i creates such a path with probability at least $\delta n(\delta n - 10 \log n)/\binom{n}{2} \geq \delta^2$. The Chernoff bound implies that

$$\mathbf{Pr}(\mathcal{B}(u, v)) \leq \mathbf{Pr}(Binomial(60\delta^{-2} \log n, \delta^2) < 10 \log n) \leq \exp\left\{-\frac{1}{2} \cdot \frac{1}{9} \cdot 60 \log n\right\} \leq n^{-3}.$$

Observe that a path of length 3 in G is rainbow with probability $1 \cdot \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$ since we may assign any color to its first edge, then any of the other two colors to its second edge and finally the remaining color to its third edge. Thus,

$$\mathbf{Pr}(\neg\mathcal{C}) \leq \sum_{u,v \in V: u \neq v} \mathbf{Pr}(\neg\mathcal{C}(u, v)) \leq \sum_{u,v \in V: u \neq v} \mathbf{Pr}(\mathcal{B}(u, v)) + \mathbf{Pr}(\neg\mathcal{C}(u, v) | \neg\mathcal{B}(u, v))$$
$$\leq \sum_{u,v \in V: u \neq v} \left(n^{-3} + \left(1 - \frac{2}{9}\right)^{10 \log n}\right) \leq n^2\left(n^{-3} + n^{-20/9}\right) = o(1).$$

3.2 Proof of Theorem 2 (ii)

Our counterexample will consist of 2 disjoint copies of $G(0.5n, p)$ with $p = 0.11$. We will show that if $|R|$ is not sufficiently large then, in both copies there will be vertices that are in distance at least 2 from R.

Let $\delta \leq 0.1$. Partition V into 2 sets V_1, V_2 each of size $0.5n$. Then generate H by including in $E(H)$ every edge in $V_1 \times V_1$ or in $V_2 \times V_2$ independently with probability 0.22. Since $0.22 \cdot 0.5n = 0.11n$, the Chernoff bounds imply that for all $v \in V$ the degree of v, $d(v)$ satisfies $0.1n < d(v) < 0.12n$. In particular w.h.p. $H \in \mathcal{G}(n, 0.1)$.

In the case that $\exists v \in V_1$ and $u \in V_2$ such that no edge in R has an endpoint in each of $(\{v\} \cup N(v)) \times (\{u\} \cup N(u))$ then u and v are at distance at least 5 in G. Since any such path cannot be rainbow when is colored by four colors we have that G is not rainbow connected.

Observe that w.h.p. R covers sets $R_1 \subset V_1$ and $R_2 \subset V_2$ each of size at most $\log n$. Therefore a vertex in $V_1 \setminus R_1$ has at least one neighbor in R_1 independently with probability $1 - (1 - 0.22)^{|R_1|} \leq 1 - n^{-1/2}$. Therefore

$$\mathbf{Pr}(\exists v \in V_1 : (\{v\} \cup N(v)) \cap R = \emptyset) \geq 1 - (1 - n^{-1/2})^{0.5n} = 1 - o(1).$$

Similarly, $\mathbf{Pr}(\exists v \in V_2 : (\{v\} \cup N(v)) \cap R = \emptyset) = 1 - o(1)$. Hence w.h.p. G is not rainbow connected when $r = 4$.

3.3 Proof of Theorem 2 (iii)

We extract from V a small set of vertices S such that for every $v \in V$ there exists $s \in S$ that shares many neighbors with v in H (see Lemma 7). We then show that any two vertices in S are connected by a rainbow path of length 3. We extend these paths into many paths of length 7 to show that w.h.p. $G^7_{H,m}$ is rainbow connected.

Lemma 7. *Let $G \in \mathcal{G}(n, \delta)$. Then there exists $S \subset V$ satisfying the following conditions:*

1. $|S| \leq 2/\delta$.
2. $\forall v \in V \setminus S$, *there exists* $s \in S$ *such that* $|N(v) \cap N(s)| \geq \delta^2 n/4$.

Proof. Let S be a maximal subset of V such that for every $v, w \in S$ we have $|N(v) \cap N(w)| < \delta^2 n/4$. Then the maximality of S implies that S satisfies the second condition of our Lemma. Then either $|S| < 2/\delta$ or there exist $S_1 \subset S$ of size $\lceil 2/\delta \rceil$. In the latter case we have

$$n = |V| \geq \left| \bigcup_{s \in S_1} N(s) \right| \geq \sum_{s \in S_1} |N(s)| - \sum_{s_1 \neq s_2 \in S_1} |N(s_1) \cap N(s_2)|$$
$$\geq \delta n |S_1| - \frac{\delta^2 n}{4} \cdot \frac{|S_1|^2}{2} = \delta |S_1| n \left(1 - \frac{\delta |S_1|}{8}\right) > n.$$

Contradiction. □

Proof of Theorem 2 (iii). Assujme that $r = 7$. For larger r we can use the first 7 colors. Also, we will still need $m = \omega(1)$ for $r > 7, r = O(1)$. Let S be a set satisfying the conditions of Lemma 7. For $v \in V$ let $s_v \in S$ be such that $|N(v) \cap N(s)| \geq \delta^2 n/4$. Let $\mathcal{J}_S$ be the event that every pair of vertices $s_1, s_2 \in S$ are joined by three internally vertex disjoint rainbow paths of length 3. Since $|S| = O(1)$ and each vertex in S has $\Omega(n)$ neighbors and $m = \omega(1)$ and $r = 7$ we have

$$P(\mathcal{J}_S) \geq 1 - 3|S|^2 \cdot \left(1 - \frac{\omega}{n^2}\left(1 - \frac{7 \cdot 6 \cdot 5}{7^3}\right)\right)^{(\delta n/6)^2} = 1 - o(1).$$

Here we have partitioned the neighborhoods of two vertices $s_a, s_b \in S$ into 6 parts so that we can find 3 disjoint pairs of parts $A_i, B_i, i = 1, 2, 3$ and then have rainbow paths $a, A_i, B_i, b, i = 1, 2, 3$ w.h.p.

Given $\mathcal{J}_S$ occurring let $v_1, v_2 \notin S$. Then there is a rainbow path P_{v_1,v_2} of length 3 from s_{v_1} to s_{v_2} not containing v_1, v_2. Assume that v_1, v_2 share fewer than $\delta^2 n/8$ common neighbors. Let J_{v_1,v_2} be the event that P_{v_1,v_2} can be extended to a rainbow path from v_1 to v_2. Assume that P_{v_1,v_2} uses colors $5, 6, 7$. Then there will be a rainbow path from v_1 to v_2 if there is a vertex $w \in N(v_1) \cap N(s_{v_1})$ such that edge $\{v_1, w\}$ gets color 1 and edge $\{w, s_{v_1}\}$ gets color 2 and colors 3,4 are similarly used for some path of length 2 from s_{v_2} to v_2. It follows that

$$\Pr(J_{v_1,v_2} \text{ does not occur }) \leq 2\left(1 - \left(\frac{1}{7}\right)^2\right)^{\delta^2 n/4 - \delta^2 n/8} = o(n^{-3}).$$

The remaining cases for v_1, v_2 follow in a similar manner. Taking a union bound over v_1, v_2 gives us Theorem 2 (iii).

4 Conclusion

We have extended the notion of adding random edges to dense graphs and asking probabilistic questions to that of adding randomly colored edges. The most interesting question for us that is left open by the above analysis is as to whether 5 or 6 colors can be used to improve Theorem 2 (iii) .

References

[1] T. Bohman, A. Frieze, M. Krivelevich and R. Martin, Adding random edges to dense graphs, *Random Structures & Algorithms* **24** (2004), 105–117.

[2] T. Bohman, A. Frieze and R. Martin, How many random edges make a dense graph hamiltonian?, *Random Structures & Algorithms*, 22 (2003) 33-42.

[3] J. Böttcher, R. Montgomery, O. Parczyk and Y. Person, Embedding spanning bounded degree subgraphs in randomly perturbed graphs, *Electron. Notes Discrete Math.* **61** (2017), 155–161.

[4] G.A. Dirac, Some theorems on abstract graphs, *Proc. London Math. Soc.* **2** (1952), 69–81.

[5] A. Ferber and M. Krivelevich, Rainbow Hamilton cycles in random graphs and hyper-graphs, *Recent trends in combinatorics*, IMA Volumes in Mathematics and its applica-tions, A. Beveridge, J. R. Griggs, L. Hogben, G. Musiker and P. Tetali, Eds., Springer2016, 167-189

[6] A. Ferber, G. Kronenberg, F. Mousset and C. Shikhelman, Packing a randomly edge-colored random graph with rainbow k-outs, (2014).

[7] A. Freize and M. Karoński *Introduction to Random Graphs*, Cambridge University Press, 2016

[8] A. Frieze and P-S. Loh, Rainbow hamilton cycles in random graphs, *Random Structures and Algorithms*, 44 (2017) 328–354

[9] M. Krivelevich, M. Kwan and B. Sudakov, Bounded-degree spanning trees in randomly perturbed graphs, *SIAM J. Discrete Math.* **31** (2017), 155–171.

[10] M. Krivelevich, M. Kwan and B. Sudakov, Cycles and matchings in randomly perturbed digraphs and hypergraphs, *Combin. Probab. Comput.* **25** (2016), 909–927.

[11] M. Krivelevich, B. Sudakov and P. Tetali, On smoothed analysis in dense graphs and formulas, *Random Structures & Algorithms* **29** (2006), 180–193.

[12] M. Krivelevich and R. Yuster, The rainbow connection of a graph is (at most) reciprocal to its minimum degree. *Journal of Graph Theory* 63 (2009) 185-191.

[13] A. McDowell and R. Mycroft, Hamilton ℓ-cycles in randomly perturbed hypergraphs, in preparation.

[14] L. Pósa, Hamiltonian circuits in random graphs, *Discrete Mathematics* 14 (1976) 359-364.