Fractals, Vol. 27, No. 6 (2019) 1950099 (22 pages)

DOI: 10.1142/S0218348X19500993

CONJECTURES ABOUT SIMPLE DYNAMICS FOR SOME REAL NEWTON MAPS ON $\mathbb{R}^2$

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Received December 1, 2018
Accepted May 28, 2019
Published October 11, 2019

*In memory of my father Giovanni (1942–2019),
that in the good old days encouraged me
to learn Basic on our Commodore 64.*

Abstract

We collect from several sources some of the most important results on the forward and backward limits of points under real and complex rational functions, and in particular real and complex Newton maps, in one variable and we provide numerical evidence that the dynamics of Newton maps N_f associated to real polynomial maps $f:\mathbb{R}^2 \to \mathbb{R}^2$ with no complex roots has a complexity comparable with that of complex Newton maps in one variable. In particular such a map N_f has no wandering domain, almost every point under N_f is asymptotic to a fixed point and there is some non-empty open set of points whose α-limit equals the set of non-regular points of the Julia set of N_f. The first two points were proved by B. Barna in the real one-dimensional case.

Keywords: Newton's Method; Barna's Theorem; Discrete Dynamical Systems; Attractors; Repellors; Iterated Function Systems.

1. INTRODUCTION

One of the most natural ways to understand the behavior of a continuous surjective map f of a compact manifold M into itself is studying the asymptotics of the forward and backward orbits of the points of M under f. Among the simplest things that can happen is that there is some finite number of attracting fixed points c_i such that the sequence of iterates $\{f^n(x)\}$ converges to one of them for almost all $x \in M$ (with respect to any measure equivalent to the Lebesgue measure on each chart) and that, again for some full measure set, the sets $f^{-n}(x)$ converge, in some suitable sense, to the set of points whose forward iterates do not converge to any c_i. In other words, the action of f on M is, asymptotically, to thicken points near the c_i while, at the same time, thinning them out near the boundaries of the basins of attraction of the c_i. We call functions with such behavior *plain*.

While the large diversity and complexity of behaviors of continuous maps of a manifold M into itself suggests that in the general case the situation is much more complicated, it was a surprising discovery that the same is true even in case of very elementary maps such as quadratic polynomials in one variable (e.g. the logistic map) or piecewise linear polynomials in one variable (e.g. the tent map) — see Refs. 1–4 and the references therein for a large panorama of the old and recent advances in this field.

Since being plain does not seem frequent among continuous functions, it is particularly important singling out properties that identify families of functions that behave so nicely under iteration. A large source of them is given by the rational maps coming from complex Newton's method. Consider, for instance, the case of the complex polynomial $f(z) = z^3 - 1$, whose Newton map $N_f : \mathbb{CP}^1 \to \mathbb{CP}^1$ is given by $N_f(z) = \frac{2z^3+1}{3z^2}$. It is well known that N_f has exactly three attractors, the cubic roots of the unity, and one repellor, namely the Julia set of N_f (see Fig. 1, top), which means that N_f is plain. Note that the situation can get more complicated even with different polynomials of same degree: as it was shown first numerically by Sullivan *et al.*,[5] in the space of all complex cubic polynomials there is a set of non-zero Lebesgue measure for which there exist attracting k-cycles, $k \geq 2$, and, for each of these polynomials, the basin of attraction of such attracting cycle has measure larger than zero (see Fig. 2, top).

While the dynamics of Newton maps on the complex line has been deeply and thoroughly studied over the last 40 years, especially in connection with the general problem of the dynamics of complex rational maps in one variable initiated exactly 100 years ago by Julia[6] and Fatou,[7–9] in comparison almost nothing has been done in the more general case of Newton maps on the real plane. The main aim of this paper is to attract the attention of the dynamical systems community to this topic by providing numerical evidence that Newton's maps on the real plane relative to generic polynomials with only real roots are (weakly) plain.

2. PRELIMINARIES

The following concepts are central for this paper.

Definition 1. Let (M, μ) be a compact manifold with a measure μ belonging to the *Lebesgue measure class*, namely a measure equivalent to the Lebesgue measure on any chart, and let f be a surjective continuous map of M into itself. The ω-limit of a point $x \in M$ under f is the (closed) set of the accumulation points of its forward orbit $\{x, f(x), f(f(x)), \ldots\}$, namely

$$\omega_f(x) = \bigcap_{n\geq 0} \overline{\bigcup_{m\geq n} \{f^m(x)\}},$$

while its α-limit is the (closed) set of the accumulation points of the sequence of preimages of x under f, namely

$$\alpha_f(x) = \bigcap_{n\geq 0} \overline{\bigcup_{m\geq n} \{f^{-m}(x)\}}.$$

The ω- and α-limits of a set are defined similarly. The *forward* (respectively, *backward*) *basin* $\mathcal{F}_f(C)$ (respectively, $\mathcal{B}_f(C)$) under f of a closed invariant subset $C \subset M$ is the set of all $x \in M$ such that $\omega_f(x) \subset C$ (respectively, $\alpha_f(x) \subset C$). Following Milnor,[10] we say that a closed subset $C \subset M$ is an *attractor* (respectively, *repellor*) for f if:

(1) $\mathcal{F}_f(C)$ (respectively, $\mathcal{B}_f(C)$) has strictly positive measure;
(2) there is no closed subset $C' \subset C$ such that $\mathcal{F}_f(C)$ (respectively, $\mathcal{B}_f(C)$) coincides with $\mathcal{F}_f(C')$ (respectively, $\mathcal{B}_f(C')$) up to a null set.

Finally, we say that f is *plain* if it has a finite number of attracting fixed points c_i, $i = 1, \ldots, N$, so that:

(i) $\bigcup_{i=1}^N \mathcal{F}_f(c_i) = M \backslash J$ is a full measure set;
(ii) the set of $x \in M$ such that $\alpha_f(x) = J$ is a full measure set.

If (ii) holds at least for a set of positive measure, then we say that f is *weakly plain*.

Remark 1. Conditions (i) and (ii) imply that a plain map cannot have any other attractor/repellor besides the c_i and J.

Remark 2. While ω-limits of discrete systems have been thoroughly studied, at least in one (real and complex) dimension, relatively very little has been done to date for α-limits (see Ref. 11 for a discussion on this topic).

Throughout this paper we will endow all manifolds M, as above, with a measure μ belonging to the Lebesgue measure class. Notice that all measures within the Lebesgue measure class of M have the very same null sets so that, since all our statements relative to measures of sets are about whether some set or its complement have zero or positive measure, our results do not depend on the particular measure used within this class. By a slight abuse of notation, we will refer to such a measure as *the Lebesgue measure on* M. The Hausdorff measures induced on the spheres $\mathbb{S}^n$ by their *round* metric, namely the angular distance between points, are an example of such measures.

Finite attractors and repellors play a major role in this theory.

Definition 2. A *periodic orbit* (or k-*cycle*) γ is a non-empty finite set of k points minimally invariant under f, namely that cannot be decomposed into the disjoint union of smaller invariant sets. A 1-cycle is also called a *fixed point*.

Example 1. Consider the map f of the Riemann sphere $\mathbb{CP}^1$ into itself given by $f([z:w]) = [2z:w]$. The only invariant proper sets of f are the two fixed points, the *south pole* $S = [0:1]$ (repellor) and the *north pole* $N = [1:0]$ (attractor). Clearly $\omega_f(x) = N$ for all x but S and $\alpha_f(x) = S$ for all x but N, so that $\mathcal{F}_f(N) = \mathbb{CP}^1\backslash\{S\}$ and $\mathcal{B}_f(S) = \mathbb{CP}^1\backslash\{N\}$. In particular, f is plain.

The following sets are of fundamental importance in the dynamics of a continuous map.

Definition 3. Given a compact manifold M and a continuous map $f: M \to M$, the *Fatou set* $F_f \subset M$ of f is the largest open set over which the family of iterates $\{f^n\}$ is *normal*, namely the largest open set over which there is a subsequence of the iterates of f that converges locally uniformly. The complement of F_f in M is the *Julia set* J_f of f. Finally, when f is differentiable we denote by Z_f the set of points $x \in M$ where its Jacobian $D_x f$ is degenerate.

In this paper, we focus on the case of rational functions so, from now on, we will restrict all our definitions and statements to this case.

Theorem 1 (Refs. 6 and 7). *Let* $f: \mathbb{CP}^1 \to \mathbb{CP}^1$ *be a rational map of degree larger than* 1. *Then*:

(1) F_f *contains all basins of attractions of* f;
(2) *Both* J_f *and* F_f *are forward and backward invariant and* J_f *is the smallest closed set with more than two points with such property*;
(3) J_f *is a perfect set*;
(4) J_f *has interior points if and only if* $F_f = \emptyset$;
(5) $J_f = J_{f^n}$ *for all* $n \in \mathbb{N}$;
(6) J_f *is the closure of all repelling cycles of* f;
(7) $\omega_f(z) = J_f$ *for a generic point* $z \in J_f$;
(8) $\alpha_f(z) = J_f$ *for every point* $z \in J_f$;
(9) $\partial\mathcal{F}_f(\gamma) = J_f$ *for every attracting periodic orbit* γ;
(10) *The dynamics of the restriction of* f *to its Julia set is highly sensitive to the initial conditions, namely* $f|_{J_f}$ *is* chaotic.

Remark 3. The possibility that $J_f = \mathbb{CP}^1$ in point (4) above does take place. Two well-known examples of functions with empty Fatou set are the Lattès example[12] $p(z) = \frac{(z^2+1)^2}{4z(z^2-1)}$, related to the theory of Elliptic functions (see Ref. 13), and $q(z) = \frac{(z-2)^2}{z^2}$ (see Ref. 13 and Corollary 6.2.4 in Ref. 14). In general, $F_f = \emptyset$ if and only if $\omega_f(z) = \mathbb{CP}^1$ for some z (see in Ref. 13, Theorem 4.3.2).

Remark 4. As soon as f has more than two attracting fixed points, J_f must have a fractal nature since, by point (9) above, all of its points belong to the boundary of more than two basins of attraction. In other words, in this case all basins of attraction have the Wada property[15] (see Ref. 16 and Sec. 4.1 of Ref. 17 for a series of examples and pictures of fractal Julia sets of polynomial, rational and transcendental complex maps).

Example 2. Consider the map $f(z) = z^2$ and denote by E, L and U respectively the *equator* $\{|z| = 1\}$, the *lower hemisphere* $\{|z| < 1\}$ and the *upper hemisphere* $\{|z| > 1\} \cup \{N\}$.

If $z \in L$ (respectively, U), then $\{f^n\}$ converges uniformly in some neighborhood of z to the constant map $z \mapsto S$ (respectively, $z \mapsto N$), so $L = \mathcal{F}_f(S) \subset F_f$ (respectively, $U = \mathcal{F}_f(N) \subset F_f$), namely f has exactly two attractors (the south and north poles) and F_f is the disjoint union of their basins. On the contrary, if $z \in E$, then for any neighborhood of z there will be some point converging to N and some point converging to S under $\{f^n\}$, so the family is not normal and $J_f = E$.

Notice that, as claimed by the theorem above, $\partial\mathcal{F}_f(N) = \partial\mathcal{F}_f(S) = J_f$ and that $f|_{J_f}$ is the doubling map on the circle, a classic example of chaotic map. Finally, notice that J_f is also the only repellor of f and that its basin is given by $\mathcal{B}_f(J_f) = \mathbb{CP}^1\backslash\{S, N\}$. In particular, f is plain.

Notice that the north and south poles in the example above both have a finite α-limit set: the only point in their preimages is themselves. This exceptional behavior can happen at most at two points for any complex rational map of degree two or more (see point (2) above and in Ref. 13, Theorem 4.2.2). The fact that the α-limit of every other point is J_f is a general result very useful in numerical exploration of Julia sets.

Theorem 2 (Ref. 8, see also in Ref. 18, Theorem 6.1 and Lemma 6.3). *Let $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ be a rational map of degree larger than 1. Then for all $z \in \mathbb{CP}^1$, with at most two exceptions, $\alpha_f(z) \supset J_f$. Moreover, $\alpha_f(z) = J_f$ if and only if z belongs to either J_f or to the basin of attraction of a root of f (except the root itself if it does not belong to J_f). More generally, if $E \subset \mathbb{CP}^1$ is a closed set disjoint from $\omega_f(F_f)$, then the sequence of sets $E_n = f^{-n}(E)$ converges uniformly to J_f.*

Corollary 1. *Let $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ be a rational map of degree larger than 1 whose only attractors are its fixed points and whose Julia set has measure zero. Then f is plain.*

Theorem 2 is reminiscent of what happens in case of hyperbolic Iterated Function Systems.

Definition 4. A Iterated Function System (IFS) $\mathcal{I}$ on a metric space (X, d) is a semigroup generated by some finite number of continuous functions $f_i : X \to X$, $i = 1, \ldots, n$. We say that $\mathcal{I}$ is *hyperbolic* when the f_i are all contractions. The Hutchinson operator associated to $\mathcal{I}$ is defined as $\mathcal{H}(A) = \bigcup_{i=1}^n f_i(A)$, $A \subset X$.

Theorem 3 (Refs. 19 and 20). *Let $\mathcal{I}$ be a hyperbolic IFS on X. Then there exists a unique non-empty compact set $K \subset X$ such that $\mathcal{H}(K) = K$. Moreover, $\lim_{n\to\infty} \mathcal{H}^n(A) = K$ for every non-empty compact set $A \subset X$.*

In the simplest cases, like Example 2, the naive idea is that ultimately the map f is a contraction close to its attractors while its inverses $\{w_1, \ldots, w_d\}$ (in case of complex rational maps, as many as their degree) are contractions close to its repellor (in the example above, the Julia set of f), which suggests that the Julia set can be found as the unique invariant compact set of the IFS defined by the $\boldsymbol{w_i}$. Indeed, in Sec. 7.3 of Ref. 21, Barnsley shows through an example how to apply these ideas to Julia sets of rational maps, namely how to write a Julia set as the invariant compact set of an IFS (notice that this important point of view is seldom mentioned in the literature about the dynamics of complex rational maps). In the real case the number of preimages, even taking into account multiplicity, is not the same for every point and it seems unlikely to be able to build in general an IFS out of them but, nevertheless, Barnsley's result shows that the invariant set for an open map f under mild conditions can be obtained as the limit of its inverse images.

A similar result, weaker but much more general, was stated by Barnsley (see Sec. 7.4 of Ref. 21) in the setting of continuous maps between metric spaces.

Theorem 4 (Ref. 21). *Let (Y, d) be a complete metric space and X a compact non-empty proper subset of Y. Denote by $\mathcal{K}(X)$ the set of the non-empty compact subsets of X endowed with the Hausdorff distance h (recall that h makes $\mathcal{K}(X)$ a complete metric space). Assume that one of the following conditions is satisfied:*

(1) *$f : X \to Y$ is an open map such that $f(X) \supset X$;*
(2) *$f : Y \to Y$ is an open map such that $f(X) \supset X$ and $f^{-1}(X) \subset X$.*

Then the map $F : \mathcal{K}(X) \to \mathcal{K}(X)$ defined by $F(K) = f^{-1}(K)$ is continuous, $\{F^n(K)\}$ is a Cauchy sequence, its limit $K_0 = \lim F^n(X) \in \mathcal{K}(X)$ is a repellor for f and it is equal to the set of points that never leave X under the action of f.

A useful algorithm based on these ideas was extracted by Hawkins and Taylor[22] from a Barnsley algorithm introduced in Ref. 21 for certain types of hyperbolic rational maps.

Definition 5. Let $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ be a rational map of degree $d > 1$. Given a point $z_0 \in \mathbb{CP}^1$, we call *backward orbit* of z_0 any sequence $\zeta_{z_0} = \{z_i\}_{i\in\mathbb{N}}$ such that $f(z_i) = z_{i-1}$ for all i. We endow the space of all backward orbits of z_0 with the equidistributed Bernoulli measure ν, namely the measure of the set of all sequences ζ_{z_0} with first k elements $\{z_0, z_1, \ldots, z_k\}$ is $d^{-(k+1)}$.

Based on a fundamental result of Freire, Lopes and Mañé,[23] Mañé[24] and, independently, Lyubich,[25] Hawkins and Taylor were able to prove the following.

Theorem 5 (Ref. 22). *Let* $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ *be a rational map of degree larger than* 1 *and* z_0 *a non-exceptional point. Then, for* ν*-almost all backward paths* ζ_{z_0}, *the set of limit points of* ζ_{z_0} *is equal to* J_f.

Going back to *forward* dynamics, since continuous function maps preserve connectedness, every complex rational map f induces a mapping of the connected components of F_f in themselves whose dynamics tells us what happens to the points of the Fatou set under iterations.

Theorem 6 (Ref. 26). *Let* $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ *be a rational map of degree larger than* 1. *Then every connected component of* F_f *ends up in a finite time inside a connected component* V *of one of the following types*:

(1) *an* attracting basin, *namely* V *contains an attracting periodic point* z_0 *of some period* $N \geq 1$ *such that* $\lim_{n\to\infty} f^{nN}(z) = z_0$ *for all* $z \in V$;
(2) *a* parabolic basin, *namely* ∂V *contains an attracting periodic point* z_0 *of some period* $N \geq 1$ *such that* $\lim_{n\to\infty} f^{nN}(z) = z_0$ *for all* $z \in V$;
(3) *a* Siegel disc, *namely* $f|_V$ *is conformally conjugate to an irrational rotation of the unit disc*;
(4) *a* Arnold-Herman ring, *namely* $f|_V$ *is conformally conjugate to an irrational rotation of an annulus of finite modulus.*

Note that, in case of general entire maps, there might be countably many disjoint connected components W_n of the Fatou set such that $f(W_n) \subset W_{n+1}$. Such sets are called *wandering domains* and the fact that they cannot arise for rational maps is one of the most important contents of Sullivan's theorem above, often called *Non Wandering Domain* Theorem.

The general picture in the real case is much more complicated. Even the simplest non-trivial case of *unimodal maps* on the interval, namely smooth maps from a closed interval into itself with a single critical point, whose rigorous study was started by Milnor and Thurston[27] after the seminal paper on the logistic map by biologist May,[28] has been fully understood only very recently thanks to fundamental contributions by Avila (see the survey by Lyubich[3] and the references therein).

We notice, first of all, that the characterization of the Julia and Fatou sets is weaker than in the complex case because real maps are not necessarily open.

Theorem 7 (see Chap. 5 of Ref. 29). *Let* $f : \mathbb{RP}^1 \to \mathbb{RP}^1$ *be a generic analytical function and denote by* γ_i *the attracting cycles of* f *and by* W_i *the set of wandering intervals of* f, *namely those intervals whose iterates are all disjoint and that do not converge to a cycle. Then*:

(1) F_f *is backward invariant*;
(2) J_f *is forward invariant*;
(3) $F_f = \sqcup_i \mathcal{F}(\gamma_i) \sqcup_j W_j$;
(4) $J_f = \alpha_f(Z_f)$;
(5) J_f contains *the closure of the set of repelling points of* f;
(6) $f|_{F_f}$ *is almost open in the sense that, if we denote by* U_i *the connected components of* F_f, *then* $f(U_i) \subset U_j$ *for some* j *if* $U_i \cap T_f = \emptyset$ *and* $f(U_i) \subset \overline{U}_j$ *for some* j *otherwise.*

One of the most general results on maps $\mathbb{RP}^1 \to \mathbb{RP}^1$ is the following generalization of Sullivan's Non-Wandering Domain theorem[4,29,30]:

Theorem 8 (Ref. 30). *Let* $f : \mathbb{RP}^1 \to \mathbb{RP}^1$ *be a generic non-invertible* C^2 *map. Then*:

(1) *every connected component of* F_f *falls, in a finite time, in a periodic component*;
(2) *there are only finitely many periodic components of* F_f.

Moreover, for almost all $x \in \mathbb{RP}^1$, *the set* $\omega_f(x)$ *is of the following three types*:

(i) *a periodic orbit*;
(ii) *a minimal Cantor set*;
(ii) *a finite union of intervals containing a critical point.*

Remark 5. Note that just in 2016 Astorg *et al.* showed[31] that this result is sharp in the sense that, both in the real and complex case, there are polynomial maps $\mathbb{KP}^2 \to \mathbb{KP}^2$, $K = \mathbb{R}$ or $\mathbb{C}$, which admit wandering domains.

In spite of an extraordinary number of articles and books devoted to the study of rational maps $\mathbb{CP}^1 \to \mathbb{CP}^1$ in the last forty years, very few have been dedicated to the general study of arguably the most natural generalization of them, namely rational maps $\mathbb{RP}^2 \to \mathbb{RP}^2$. Among the few exceptions are the study of Julia sets of dianalytic maps by Hawkins *et al.*[32–34] and of the dynamics of a particular family of birational maps by Bedford and Diller.[35–37] A similar situation holds in the subcase of Newton maps, that are the subject of this paper.

Definition 6. Let p be a polynomial in one variable over real or complex numbers. We call *Newton map* associated to p the rational map

$$N_p(z) = z - p(z)/p'(z).$$

The Newton's method for finding the root of a function (e.g. see Refs. 38 and 39), of paramount importance in the Numerical Analysis field, is based on the elementary facts that, for a generic function p, the following holds: (1) the set of the roots of p coincides with the set of (bounded) fixed points of N_p; (2) all of these fixed points are *attracting* (in fact, *super-attracting*). Hence iterations of N_p lead naturally to a root of p when the initial point is chosen close enough to it — see Refs. 39 and 40 for a very general classical proof of this fact in the context of Banach spaces and Ref. 41 for a clever algorithm to retrieve *all* roots of a complex polynomial based on strong general results of holomorphic dynamics.

Newton maps of complex polynomials are quite special rational functions: for instance, the point at infinity is always a fixed repelling point for them. The following theorem[42] provides a full characterization for them.

Theorem 9 (Ref. 42). *Every rational map $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ of degree d with d distinct*

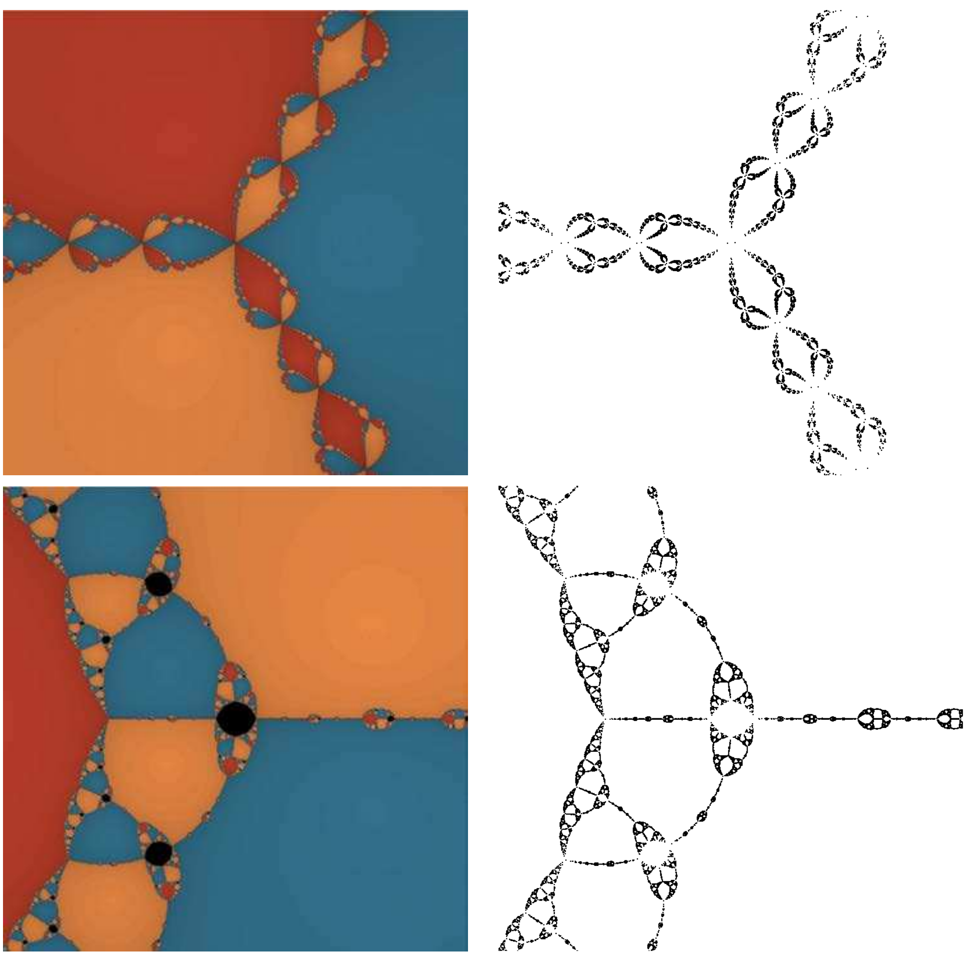

Fig. 1 Basins of attraction (left) and Julia set (right) of the Newton maps associated to $p(z) = z^3 - 1$ in the square $[-2, 2]^2$ (top row) and $q(z) = z^3 - 2z + 2$ in the square $[-1.5, 1.5]^2$ (bottom row). We assigned a color to each root so that all points belonging to a root's basin have been plotted with that same color. The interiors of the black islands that appear in case of q do not belong to J_q (see point (3) of Theorem 1) but rather correspond to Fatou components of points that are attracted to a non-trivial cycle rather than any of q's roots (equivalently, they are attracting basins of some root of f^k for some $k > 1$). The right column shows an approximation of the α-limit set of the point $z_0 = 5 + i$ at the 10th recursion level.

superattracting fixed points is conjugate, via a Mobius transformation, to the Newton's map N_p of a polynomial p of degree d. If ∞ is not superattracting for f and $f(\infty) = \infty$, then $f = N_p$.

Correspondingly, their Fatou and Julia sets have special properties (e.g. see Ref. 43).

Theorem 10. *Let p be a polynomial with complex coefficients. Then*:

(1) *J_{N_p} has empty interior*;
(2) *J_{N_p} is connected*;
(3) *All connected component of F_{N_p} are simply connected*;
(4) *F_{N_p} has no Arnold-Herman rings*;
(5) *$J_{N_p} = \alpha_{N_p}(Z_{N_p})$*;
(6) *All* immediate basins *B_i of the roots of p, namely the connected components of F_{N_p} containing those roots, are unbounded (i.e. $\infty \in \partial B_i$)*;
(7) *∞ is a repelling fixed point for N_p.*

Notice that it is enough to consider polynomials of degree three in order to find cases of Newton maps with parabolic basins and Siegel discs (e.g. see Sec. 3.2 of Ref. 43), although there seem to be no concrete example available in literature.

Corollary 2. *Let p be a generic complex polynomial of degree n with roots $R = \{c_1, \ldots, c_n\}$ and such that N_p has no Siegel domains or attracting k-cycles for $k \geq 2$ and J_{N_p} has Lebesgue measure zero. Then $\alpha_{N_p}(c_i) = J_{N_p} \cup \{c_i\}$, $i = 1, \ldots, n$, and $\alpha_{N_p}(z) = J_{N_p}$ for any other point. Equivalently, $\mathcal{B}(J_{N_p}) = \mathbb{CP}^1 \backslash R$ and $\mathcal{B}(J_{N_p} \cup R) = \mathbb{CP}^1$. In particular, N_p is a plain map.*

Example 3. This is the case of the Newton maps associated to the polynomials $p(z) = z^n - 1$. Consider again, for instance, the case of $p(z) = z^3 - 1$, so that $N_p(z) = \frac{2z^3+1}{3z^2}$ (see Fig. 1, top). N_p has exactly three attractors, the three roots of unity u_i, $i = 1, 2, 3$. There cannot be attracting cycles other than these fixed points because, by Theorem 11, if there were one then a critical point of N_p should converge to it, but for this map Z_{N_p} coincides with the set of zeros of N_p. Each forward basin $\mathcal{F}_p(u_i)$ is the disjoint union of countably many simply connected open sets and the boundary $\partial\mathcal{F}_p(u_i) = \overline{\mathcal{F}_p(u_i)} \backslash \mathcal{F}_p(u_i)$ of each of them is equal to the Julia set J_f.

The Julia set is the only repellor of N_p and $\mathcal{B}_p(J_f) = \mathbb{CP}^1 \backslash \{u_1, u_2, u_3\}$. In fact, the equation $N_p(w) = z$ has always three solutions (taking into account multiplicity) and this defines three meromorphic functions w_i so that $N_p^{-1}(z) = \{w_1(z), w_2(z), w_3(z)\}$. Following Barnsley (see Sec. 7.3 of Ref. 21), we can restrict the w_i to the complement of some open neighborhood of the roots of p, so that the Iterated Function System generated by these restrictions has a unique attractor, which coincides with J_{N_p}.

In this paper, we are mostly interested in the size of the set of points that do not converge to any root. Buff and Chéritat showed that there are complex quadratic polynomials whose Julia set has positive measure.[44,45] As a consequence of a deep study by Lei Tan on the dynamics of complex Newton maps coming from cubic polynomials,[46] we know that the Julia set of any quadratic polynomial can be found in the Julia set of the Newton map of a suitable cubic polynomial and therefore there are Newton maps on $\mathbb{CP}^1$ whose Julia set has measure larger than zero,[43] although no concrete example appears in literature so far. On the other side, the following theorem allows to find easily the existence of non-trivial attracting periodic cycles, whose presence also causes the set of non-converging points to be of non-zero measure.

Theorem 11 (Fatou). *If a rational map $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ has an attracting periodic cycle, then the orbit of at least one of its critical points will converge to it.*

When p has degree 1 or 2, the set of non-converging points has trivially measure zero: in the first case, $N_p^n(z)$ converges to the root for all $z \in \mathbb{C}$; in the second, J_{N_p} is diffeomorphic to a circle and the Fatou set is the disjoint union of two discs, each of which is the immediate basin of one of the two roots. In the first non-trivial case, when p has degree 3, it was found first numerically by Curry, Garnett and Sullivan[5] that there are such polynomials whose Newton map N_p has attracting orbits with period larger than 1 (see Fig. 2, top). This means that, even for such simple Newton maps, there is an open set of points (hence with measure larger than zero) that does not converge to any root. A simple example of such polynomials is $q(z) = z^3 - 2z + 2$ (see Fig. 1, bottom).

Of course the restriction of complex polynomials with real coefficients to the real line provides examples of dynamics of *real* Newton maps on $\mathbb{RP}^1$, so the example above shows that such behavior also

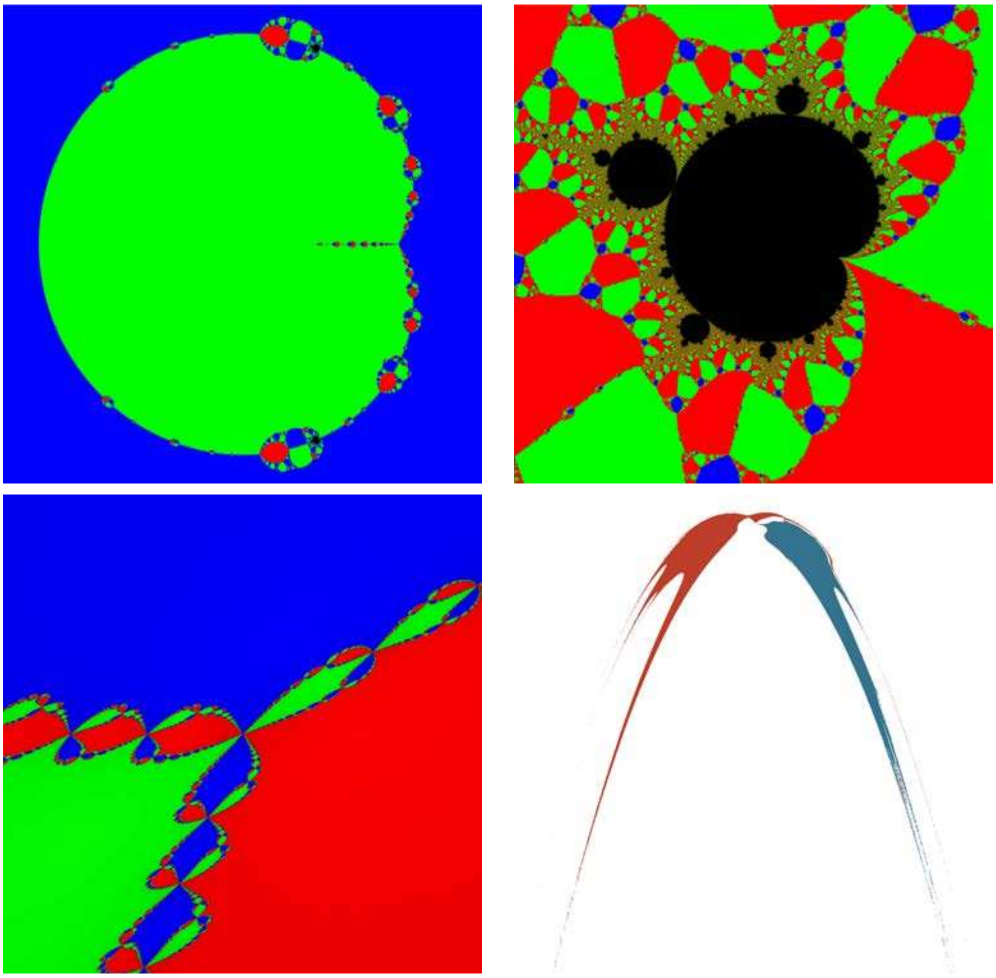

Fig. 2 Top: ω-limit of the origin under the Newton map of $f_A(z) = z^3 + (A-1)z - A$ for values of A in the square $[-2.3, 1.7] \times [-2, 2]$ of the complex plane (left). For most values of A, the origin converges to one of the three roots but there is a non-zero volume set of values (black points) for which the origin converges to a non-trivial attracting cycle. The zoom (right) shows that the connected components of this set are the celebrated Mandelbrot fractal. Bottom: Newton method applied to two intrinsically complex functions. On the left, f is the real version of the polynomial $p(z) = z^3 - 1$ but the complex structure has been modified so that the imaginary unit is represented by the vector $(1, 1)$ rather than $(0, 1)$. On the right, the function is $\psi^* f$, where $\psi(x, y) = (x, y + x^2)$ and f is the real version of the polynomial $p(z) = z^2 - 1$.

takes place in the real case (e.g. see Fig. 1, bottom). It is, therefore, non-trivial and particularly interesting the following result found by Barna,[47] way before the explosion of work on complex holomorphic dynamics.

Theorem 12 (Ref. 47). *Let p be a generic real polynomial of degree $n \geq 4$ without complex roots and denote by $c_1, \ldots, c_n$ its roots and by $N_p : \mathbb{RP}^1 \to \mathbb{RP}^1$ its Newton map. Then*:

(1) $F_{N_p} = \bigcup_{i=1}^n \mathcal{F}(c_i)$;
(2) F_{N_p} *has full Lebesgue measure*;
(3) N_p *has no attracting k-cycles with $k \geq 2$*;
(4) N_p *has repelling k-cycles of any order $k \geq 2$*;
(5) J_{N_p} *is equal, modulo a countable set, to a Cantor set $\mathcal{E}_{N_p}$ of Lebesgue measure zero.*

Remark 6. In fact, it is more generally true that the, for any complex polynomial p with all its roots $\{c_1, \ldots, c_n\}$ simple and real, $F_{N_p} = \bigcup_{i=1}^n \mathcal{F}(c_i)$ has full Lebesgue measure in $\mathbb{CP}^1$ (Theorem 1.27 of Ref. 48). Even more generally, $J_{N_p} \subset \mathbb{CP}^1$ is a set of Lebesgue measure zero if all critical points of N_p converge to attracting, repelling or neutral rational cycles (Theorem 1.26 of Ref. 48). For example, this last theorem covers all polynomials $p_n(z) = z^n - 1$, $n = 2, 3, \ldots$, showing that all the p_n are plain.

To our knowledge, the only result in literature about α-limits of maps on $\mathbb{RP}^1$ is that $J_f = \alpha_f(Z_f)$ (see Theorem 7). It is reasonable, though, to believe that this property extends to almost all points of $\mathbb{RP}^1$, leading to the following.

Conjecture 1. *Let p be a generic real polynomial of degree $n \geq 4$ without complex roots. Then its Newton map $N_p : \mathbb{RP}^1 \to \mathbb{RP}^1$ is plain.*

About 30 years later, Barna's work was revisited independently in the same year by Saari and Urenko,[49] Wong[50] and Hurley and Martin[51] leading, in particular, to the following important results.

Theorem 13 (Ref. 50). *A sufficient condition for Barna's theorem to hold is that the polynomial p has no complex root and at least four distinct real roots, possibly repeated.*

Theorem 14 (Ref. 49). *Let p be a generic real polynomial of degree $n \geq 3$, A_p the collection of all bounded intervals in $\mathbb{R}\backslash Z_p$ and $\mathcal{A}_p$ the set of all sequences of elements of A_p. Then the restriction of N_p to the Cantor set $\mathcal{E}_{N_p}$ is semi-conjugate to the one-sided shift map S on $\mathcal{A}_f$, namely there is a surjective homomorphism $h_p : \mathcal{E}_{N_p} \to A_p$ such that $T \circ h_p = h_p \circ N_p$.*

Theorem 15 (Ref. 51). *Let p be a generic real polynomial of degree $n \geq 3$. Then N_p has at least $(n-2)^k$ k-cycles for each $k \geq 1$ and its topological entropy is at least $\log(n-2)$.*

Remark 7. The theorem by Saari and Urenko actually holds for the much larger class of "polynomial-like" functions and similarly happens for the Hurley and Martin theorem (see Refs. 49 and 51 for details).

Despite the depth and interest of these results for Newton maps on the real line, no attempts to multidimensional generalizations of Barna's theorem appear in literature. In Sec. 3, we present numerical evidence showing that a similar statement might hold in higher dimension, or at least on the real plane.

3. NEWTON MAPS ON $\mathbb{R}^2$

The Newton method extends naturally to much more general settings than the real and complex lines, from finite-dimensional linear spaces[52] to Banach spaces[39,40] to Riemannian manifolds.[53] Moreover, since the map $f \mapsto N_f$ leaves invariant the subset of *complex* maps of $\mathbb{R}^{2n}$ into itself, in each setting one can consider separately the real and the complex case.

In this paper, we are only interested in the finite-dimensional real case.

Definition 7. Let $f : \mathbb{K}^n \to \mathbb{K}^n$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, be a polynomial map. We call *Newton map* associated to f the rational map $N_f : \mathbb{KP}^n \to \mathbb{KP}^n$ defined by

$$N_f(x) = x - D_x f^{-1}(f(x)).$$

In this paper, we will limit our discussion to the case $n = 2$. Notice that very little, compared to the one-dimensional case, is available in literature about Newton's method in $\mathbb{R}^2$ or $\mathbb{C}^2$. The complex case has been recently investigated in a few papers by Hubbard and Papadopol[41] and by Hubbard's pupil Roeder,[54,55] where they classify and study of the case of *quadratic polynomials*. As expected, technical difficulties are much more challenging than in dimension one. The real case was considered, to the author's knowledge, only by Peitgen, Prufer and Schmitt[16,56,57] about 30 years ago, mostly from the point of view of identifying the best definition of *Julia set* in the real multidimensional context, and about 20 years ago by Miller and Yorke,[58] that studied the size of attracting basins. In this work, we are rather interested to a somehow transversal point of view, namely we look for real polynomial maps of the plane into itself whose Newton maps are plain.

3.1. Points of Indeterminacy

To start, notice that, unlike the case $n = 1$, when $n \geq 2$ a Newton map N_f has a non-empty finite set I_f of points of indeterminacy, namely points where N_f is undefined and cannot be extended continuously to $\mathbb{KP}^n$. This happens even when the roots of f are simple (e.g. see Ref. 41). Because of this, in the study of such maps in principle one has to deal with the following situation: either restricting N_f to the complement of the set $\bigcup_{i=0}^{\infty} f^{-i}(I_f)$ of all points that fall eventually on the points of indeterminacy or using the blowup technique[59,60] to eliminate the singularities. Both points of view lead to non-trivial situations: in the first case, we leave the compact setting; in the second, we are led to an infinite series of blowups that make the space quite non-trivial (e.g. see Ref. 61 for a detailed construction of such set in case of the complexified Henon map).

Example 4. In case of the Newton map of the complex quadratic polynomial $p(z) = z^2 - 1$, namely $N_p(z) = (z^2 + 1)/(2z)$, it was shown already in late 19th century independently by Schröder[62,63] and

Cayley[64,65] that $J_{N_p} \subset \mathbb{CP}^1$ is the circle $\{\mathrm{Im}(z) = 0\} \cup \{\infty\}$ and there are exactly two basins of attraction, corresponding to the two roots ± 1 of p. Written in homogeneous coordinates, the Newton map of the real version of p, namely $f(x, y) = (x^2 - y^2 - 1, 2xy)$, reads

$$N_f([x : y : z]) = [x(x^2 + y^2 + z^2) : y(x^2 + y^2 - z^2) : 2z(x^2 + y^2)].$$

Corresponding to the fact that ∞ is a repelling fixed point for N_p, at infinity (namely for $z = 0$) N_f restricts to the identity map and a direct calculation in the chart $y = 1$, where the circle at infinity is the x axis, shows that the Jacobian of N_f at each infinity point is diagonal with eigenvalue 2, namely repelling, in the z-direction.

The problem here is the behavior of N_f at the origin: its complex version N_p sends 0 to ∞ but, passing from $\mathbb{CP}^1$ to $\mathbb{RP}^2$, the point at infinity is blown up into a whole circle and it is not clear *a priori* which point of that circle should be the image of the origin via N_f. In this elementary case, a single blowup is enough to resolve the singularity at p: we extend N_f to the Klein bottle K obtained by replacing a neighborhood U of the point p with a corresponding neighborhood of the graph of the map $U\backslash\{p\} \to \mathbb{RP}^1$ sending a point $q \in U$ into the unique straight line passing through p and q. In concrete, we switch to the coordinates $(x, u) = (x, y/x)$ nearby p. In this chart, N_f can be written as

$$(x, u) \to [x^2(1 + u^2) + 1 : u(x^2(1 + u^2) - 1) : 2x(1 + u^2)]$$

and so, at $x = 0$, it extends to the smooth rational map $u \to [1 : -u : 0]$. Note that the topological degree of this map is non-zero, corresponding to the fact that it is impossible to extend N_f to p continuously in $\mathbb{RP}^2$.

Denote by $\hat{N}_f : K \to \mathbb{RP}^2$ the above extension of N_f. While now it is clear what happens at p, we created the same problem at two other points, namely $f^{-1}(p) = \{[1 : 0 : 1], [0 : 1 : 1]\}$, since $\hat{N}_f$ would send them to the point p but we replaced it by a circle. Of course this can be fixed by a couple of blowups at those points but this would just move the problem at their preimages and so on. Hubbard *et al.* show in Ref. 61 how to proceed with such recursive construction and finally get a compact space on which N_f can be extended without singularities but this process is way beyond the scope of this paper. To us it is enough to notice that the projections on $\mathbb{RP}^2$ of the trajectories of points belonging to those blown-up circles under some globally regular extension $\tilde{N}_f$ coincide with the trajectory of the corresponding point on $\mathbb{RP}^2$ under N_f and so all those circles belong to the Julia set of $\tilde{N}_f$.

The example above shows that, in order to understand the dynamics of N_f, we do not really need to build a global regular extension but rather it is enough to study its regular extensions at the points of indeterminacy. In the case above, the only point of indeterminacy of N_f belongs to the Julia set because the projection of any orbit of the extension $\hat{N}_f$ passing over it ends up in the circle at infinity, whose points are all fixed and repelling (notice that the behavior of $\hat{N}_f$ is completely determined by N_f by continuity). Hence, the Julia set in $\mathbb{RP}^2$ of the real version of the complex Newton map of the polynomial $p(z) = z^2 - 1$ is the wedge sum of two circles: the x axis and the circle at infinity. In Example 5, we will show a case of point of indeterminacy belonging to the Fatou set.

In general, given a point of indeterminacy x_0 of a Newton map $N_f : \mathbb{RP}^2 \to \mathbb{RP}^2$ and an algebraic extension $\hat{N}_f : X \to \mathbb{RP}^2$ with projection $\pi : X \to \mathbb{RP}^2$ resolving the indeterminacy at x_0, we assign x_0 to F_{N_f} if $\hat{N}_f(\pi^{-1}(x_0)) \subset F_{N_f}$; otherwise, we assign it to J_{N_f}.

3.2. Intrinsically Complex Maps

Let us consider now the case of real Newton maps whose dynamics are essentially covered by the standard theory of the complex case. From a real point of view, complex differentiable maps $f : \mathbb{C} \to \mathbb{C}$ are just real differentiable maps $f_{\mathbb{R}} : \mathbb{R}^2 \to \mathbb{R}^2$ whose Jacobian $Df_{\mathbb{R}}$ commutes with the (imaginary unit) matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

or, equivalently, for which $f_{\mathbb{R}}$ we have that

$$Df_{\mathbb{R}} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

for some $\alpha, \beta \in C^0(\mathbb{R}^2)$. Clearly there is nothing special about the matrix J; that is, any matrix $\hat{J}$ with $\hat{J}^2 = -\mathbb{1}_2$ defines an equivalent complex structure on $\mathbb{R}^2$ where complex maps are differentiable maps $f_{\mathbb{R}}$ whose Jacobian commutes with $\hat{J}$.

Even more generally, recall that an almost complex structure on a smooth manifold M is given by a section $J: M \to T^{(1,1)}M$ of the bundle of vector bundle morphisms of TM into itself such that $J_x^2 = -\mathbb{1}_2$ for every $x \in M$. On a general manifold not all almost complex structures give rise to a global complex one but, for purely dimensional reasons, this is instead the case on the plane.

Definition 8. We say that $f:\mathbb{R}^2 \to \mathbb{R}^2$ is intrinsically complex if it is an almost complex map with respect to an almost complex structure J, namely if there exists an almost complex structure J such that $D_x f \circ J_{f(x)} = J_x \circ D_x f$ for every $x \in \mathbb{R}^2$.

Clearly all theorems of holomorphic dynamics apply to real intrinsically complex maps.

Proposition 1. *Let $f:\mathbb{R}^2 \to \mathbb{R}^2$ be a smooth intrinsically complex map. Then exactly one of the following holds*:

(1) *f enjoys all topological properties that hold in holomorphic dynamics (in particular, its Julia set is non-empty and it is equal to the boundary of any of the basins of its attracting cycles).*
(2) *f is conjugate via a diffeomorphism to a rotation about a fixed point.*
(3) *$\{f^n\}$ converges to a constant function uniformly on compact sets.*

Note that, in the last two cases, J_f is empty.

Proof. This is an immediate consequence of a few important results: the integrability of all almost complex structures in dimension 2,[66] the Uniformization Theorem and the Denjoy–Wolff Theorem.[17] By the first, every almost complex structure gives rise to a complex structure, so that every almost complex map, namely every map f that preserves the almost complex structure, is a complex map with respect to the corresponding complex structure. By the second, this complex structure must be diffeomorphic to one of the following two inequivalent ones: either the standard one on the plane or the standard one on the unit open disc. In the first case, f is smoothly conjugate to a standard complex function and therefore all topological results of holomorphic dynamics also apply to it. In the second case, f is conjugate to a holomorphic map of the unit disc in itself and, by the Denjoy–Wolff Theorem, this means that it is either conjugate to a (hyperbolic) rotation or its iterates converge, uniformly on compact sets, to a constant function. □

Newton maps are natural with respect to linear transformations, namely $N_{\psi^* f} = \psi^* N_f$ for every linear map $\psi:\mathbb{R}^2 \to \mathbb{R}^2$ (e.g. see Ref. 67), where $\psi^* f = \psi^{-1} f \psi$ and similarly for N_f. In other words, the Newton map N_f of every intrinsically complex map f that is complex with respect to a *constant* almost complex structure is the pull-back of a complex Newton map. In particular, those Newton maps have all properties of complex Newton maps in one variable, the only difference being that the point at infinity is replaced by a circle on which the Newton map is the identity. For instance, Fig. 2 (bottom, left) shows the Fatou components and Julia set of the real version of the polynomial $p(z) = z^3 - 1$ with respect to the complex structure where the imaginary identity is represented by the vector $(1,1)$ rather than the standard $(0,1)$ (compare with the one relative to the standard structure in Fig. 1, top), corresponding to the constant almost complex structure

$$J = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}.$$

For a general diffeomorphism ψ, though, $N_{\psi^* f} \neq \psi^* N_f$, namely the Newton map of an intrinsically complex map is not necessarily an intrinsically complex map, as the example below shows.

Example 5. Consider again the map $f(x,y) = (x^2 - y^2 - 1, 2xy)$ of Example 4. Under the diffeomorphism $\psi(x,y) = (x, y + x^2)$, f transforms into

$$\psi^* f(x,y) = \psi^{-1} f \psi(x,y) = (x^2 - (y+x^2)^2 \\ - 1, 2x(y+x^2) - (x^2 - (y+x^2)^2 - 1)^2),$$

which is a complex map with respect to the almost complex structure

$$J_{(x,y)} = \begin{pmatrix} -2x & -1 \\ 1 + 4x^2 & 2x \end{pmatrix}.$$

Clearly $\psi^* f$ and f enjoy the same dynamical and topological properties but, on the contrary, $\psi^* N_f$ and $N_{\psi^* f}$ turn out to be very different. Let us give a closer look to their Fatou sets. The restriction to $\mathbb{R}^2$ of the Fatou set of N_f is the union of two connected components, the left and right half-planes separated by the Julia set, the line $x = 0$. Since ψ leaves the line $x = 0$ invariant, the Fatou and Julia sets of $\psi^* N_f$ coincide with those of N_f, namely it is the wedge sum at $[0:1:0]$ of the circle $\{x = 0\}$ (the imaginary axis) with the circle at infinity $\{z = 0\}$.

Let us consider now $N_{\psi^* f}$. A direct calculation shows that

$$N_{\psi^* f}([x:y:z]) = [zp(x,y,z) : 2q(x,y,z) \\ : 2z^7(x^2z^2 + (x^2+yz)^2)],$$

where p and q are, respectively, homogeneous polynomials of order 10 and 11 with $p(x,y,0) = x^{10}$ and $q(x,y,0) = x^{11}$ and neither p nor q contain a term in z only. Hence $N_{\psi^* f}([x:y:0]) = [0:1:0]$ for every $[x:y:0] \neq [0:1:0]$ while it is undefined at $[0:1:0]$. Note that the third component of $N_{\psi^* f}([x:y:z])$ can be zero only when $z=0$ or $x=y=0$ so there are exactly two points of indeterminacy: the point at infinity above and the origin $[0:0:1]$.

We have already seen in Example 4 the reason behind the indeterminacy of the origin. The indeterminacy at $[0:1:0]$ is more interesting. Numerics suggest the presence of three basins of attraction: the ones shown in red and blue in Fig. 2 (bottom, right) are the basins relative to the two roots of $\psi^* f$, namely $(\pm 1, 1)$, while the points in the white basin converge at infinity to $[0:1:0]$. Of course an attracting point where the map is undefined is a quite unsatisfactory situation. A direct calculation in the projective chart $y=1$, where the circle at infinity is represented by the x axis and the point of indeterminacy by the origin, shows that we need two blowups to resolve this singularity: a first one passing from coordinates (x,z) to $(x, u = z/x)$ and a second one from (x,v) to $(x, v = u/x)$. The corresponding extension is well-defined at $(x,v) = (0,0)$, which is a fixed point. Moreover, its Jacobian at this fixed point is equal to the zero matrix, so this point at infinity is superattracting. Recall that infinity is, on the contrary, always repelling for complex Newton maps.

We can now reformulate Lyubich Theorem (Theorem 1.27) in Ref. 48 in our setting.

Theorem 16. *Let $f:\mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial map with n simple real roots $c_1,\ldots,c_n$ which, with respect to some constant almost complex structure, is a complex polynomial p of degree n. Then*:

(1) J_{N_f} *has Lebesgue measure zero*;
(2) $F_{N_f} = \bigcup_{i=1}^n \mathcal{F}(c_i)$;
(3) N_f *has no wandering domains*;
(4) F_{N_f} *has full Lebesgue measure*;
(5) N_f *has no attracting k-cycles with* $k \geq 2$;
(6) N_f *is a plain map.*

Note that a polynomial map f as in the hypotheses of the theorem above has real degree n^2 but can have no more than n real solutions, since every of its real solutions is a solution of the corresponding complex polynomial equation. The list of properties above suggests that having maximal number of real roots within a given family of Newton maps grants quite special properties. The main goal of this paper is to provide evidence that the same is true among the family of general polynomial maps of the plane into itself.

Note finally that the condition of having maximal number of real roots is not by any means necessary. As pointed out in Remark 6, the same result holds, for instance, for all real maps corresponding to the complex polynomials $p_n(z) = z^n - 1$. This is the case, for example, of the real map $f(x,y) = (x^3 - 6xy^2 + 4y^3 - 1, y(3x^2 - 6xy + 2y^2))$, whose basins are shown in Fig. 2 (bottom, left).

3.3. The Semilinear Case

As already pointed out by Yorke *et al.* in Ref. 68, complex maps (as well as intrinsically complex ones) are very special among real maps and it is not to be expected that their asymptotic behavior is shared by general real maps. For instance, complex polynomials of degree n, seen as real maps on the plane, have both components of degree n and have n roots (counted with multiplicity), while a real polynomial map does not need to have components of the same degree and the number of its roots can be as large as the product of the degrees of the components. From the dynamical point of view, as pointed out by Peitgen *et al.* in Ref. 57, unlike in the complex case, points at infinity can be attracting for real Newton maps on the plane, even in case of Newton maps of *intrinsically* complex ones (e.g. see Example 5). In particular, the circle at infinity is not necessarily contained inside the Julia set of a real Newton map.

In fact, it is expected that the dynamics of general *real* maps on the plane be more complicated and diverse than the one of holomorphic maps. Even in case of the logistic map $f_\mu = \mu x(1-x)$, $\mu \in [0,4]$, $x \in [0,1]$, arguably one of the most elementary nontrivial real discrete one-dimensional dynamical systems, the dynamics can be highly non-trivial. For instance, for a Cantor subset of parameters μ of positive Lebesgue measure, the (unique) attractor of f_μ is a cycle of intervals whose basin has full measure and on which the dynamics is chaotic[69,70] while for an open set of parameter values the attractor is a periodic orbit.[71] It is therefore natural to ask whether, besides intrinsically complex ones, there are other classes of real polynomial (and, more generally, rational) maps on the plane whose behavior is comparable, if not simpler, to the one of holomorphic maps on the complex line. In this section, we examine the case of a simple family of Newton maps on the plane whose behavior is essentially one-dimensional.

We call a map semilinear $f:\mathbb{R}^2 \to \mathbb{R}^2$ if it has a linear component. We can always choose coordinates (x, y) so that $f(x, y) = (p(x, y), y)$. Its Newton map is

$$N_f(x, y) = \left(\frac{x\partial_x p(x, y) + y\partial_y p(x, y) - p(x, y)}{\partial_x p(x, y)}, 0\right).$$

In particular, its image is one-dimensional and the map can be re-written as

$$N_f(x, y) = \left(N_{p_y}(x) + y\frac{\partial_y p(x, y)}{\partial_x p(x, y)}, 0\right),$$

where $p_y(x) = p(x, y)$, so in particular the action of N_f on the x-axis is given by

$$N_f(x, 0) = (N_{p_0}(x), 0).$$

Hence N_f is essentially a one-dimensional Newton map: a point (x_0, y_0) converges to a root $(c_x, 0)$ under N_f if and only if the x component of $N_f(x_0, y_0)$ converges to c_x under N_{p_0}. In turn, this means that $J_{N_f} = N_f^{-1}(J_{p_0})$, which makes possible to extend Barna's theorem to these maps.

Theorem 17. *Let $f:\mathbb{R}^2 \to \mathbb{R}^2$ be a generic semilinear polynomial map of degree $n \geq 4$ with n simple real roots $c_1, \ldots, c_n$. Then:*

(1) *J_{N_f} is the wedge sum of a Cantor set of circles of Lebesgue measure zero;*
(2) *$F_{N_f} = \bigcup_{i=1}^{n} \mathcal{F}(c_i)$;*
(3) *N_f has no wandering domains;*
(4) *F_{N_f} has full Lebesgue measure;*
(5) *N_f has no attracting k-cycles with $k \geq 2$;*
(6) *N_f has repelling k-cycles for all $k \geq 2$.*

Proof. Let $\mathrm{pr}_1:\mathbb{R}^2 \to \mathbb{R}$ be the projection on the first component. For a generic semilinear map f, the gradient of the map $\mathrm{pr}_1 \circ f$ is zero only in a finite number of points and so the preimages through N_f of null sets are null sets.[72] Since $J_{N_f} = N_f^{-1}(J_{p_0})$ and, by hypothesis, p_0 satisfies the conditions of Barna's theorem (Theorem 12), then J_{p_0} is a null set of $\mathbb{R}$ so that, for a generic f, J_{N_f} is a null set of $\mathbb{R}^2$. The preimage of each regular point of N_f is a finite number of circles and so J_{N_f} is a Cantor set of circles. The points where these circles meet are necessarily points of indeterminacy for N_f, since level sets of a function corresponding to different values cannot meet.

Since the image of every point under N_f lies on the circle $y = 0$ and from that moment on the dynamics coincides with the one of N_{p_0}, which satisfies Barna's theorem, the rest of properties follows trivially. □

Example 6. In Fig. 3 (top row), we show two pictures relative to the case of the cubic polynomial map $f(x, y) = (x^3 + 3xy - x, y)$, whose roots are roots $(0, 0)$, $(\pm 1, 0)$ and whose Newton map is

$$N_f(x, y) = \left(x\frac{2x^2 + 3y}{3x^2 - 1 + 3y}, 0\right).$$

In the left picture we show the basins of attraction of N_f in the square $[-4, 4]^2$; the three immediate basins of attraction are the largest visible basin for each of the three colors and are clearly unbounded. The extension of N_f on $\mathbb{RP}^2$, in homogeneous coordinates, is the map

$$[x : y : z] \mapsto [x(2x^2 + 3yz) : 0 : z(3x^2 - z^2 + 3yz)]$$

that restricts at infinity to the constant map $[x : y : 0] \mapsto [1 : 0 : 0]$ at all points except $[0 : 1 : 0]$, which is a point of indeterminacy. N_f has exactly other three points of indeterminacy in $\mathbb{R}^2$, namely the real roots of the system of sixth degree $x(2x^2 + 3y) = 0, 3x^2 - 1 + 3y = 0$, whose projective coordinates are $[0 : 1 : 3]$, $[-3 : -2 : 3]$ and $[3 : -2 : 3]$. These four points correspond exactly to the nodal points of J_f, three of which are visible in Fig. 3 (top, left).

The set of all preimages of the point $[3 : 0 : 2]$ up to the third recursion level is shown in Fig. 3 (top, right) and clearly suggests that Conjecture 1 holds for this type of maps. Notice that all smooth connected components of J_{N_f} are segments of cubic polynomials asymptotic to the vertical direction, corresponding to the fact that they all meet at the nodal point at infinity $[0 : 1 : 0]$.

In Fig. 3, we show also the Fatou and Julia sets of the quartic polynomial map $g(x, y) = (x^4 - 3x^2 + xy + 2, y)$ (bottom, left), that has four distinct real roots and five points of indeterminacy and behaves similarly to the previous example, and of the cubic polynomial map $h(x, y) =$

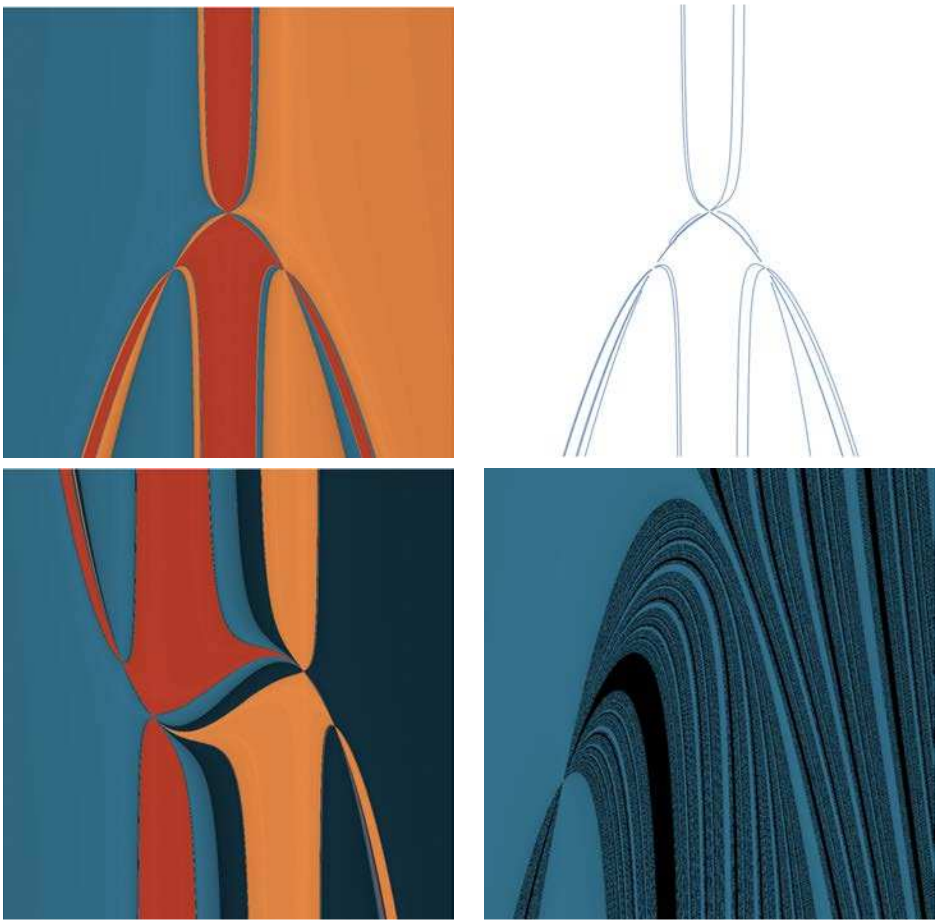

Fig. 3 (Top) (left) Basins of attraction of N_f, with $f(x,y) = (x^3 - x + 3xy, y)$, in the square $[-4,4]^2$ and (right) the corresponding set Z_{N_f} and a few of its preimages (right). Each of the three roots of f corresponds to a different color. The three visible nodes are the indeterminacy points of N_f. (Bottom, left) Basins of attraction of N_g, with $g(x,y) = (x^4 - 3x^2 + xy + 2, y)$. In this case there are four roots and four indeterminacy points, corresponding to the four colors and nodes in the picture. (Bottom, right) Basin of attraction of N_h, with $h(x,y) = (x^3 + xy - 2x + 2, y)$. This map has a single root and its basin, as the picture clearly suggests, is not of full measure because of the presence of an attracting 2-cycle.

$(x^3+xy-2x+2, y)$ (bottom, right), that instead has one real and two complex roots. In this last case, the Newton map has an attracting 2-cycle whose basin is shown in black.

Notice that the Newton map operator $f \mapsto N_f$ is invariant with respect to the action on f by $\mathrm{GL}_2(\mathbb{R})$ given by $g \to g \circ f$, namely $N_{g\circ f} = N_f$ for all $g \in \mathrm{GL}_2(\mathbb{R})$,[67] so the result above actually holds for all maps whose two components are linear combinations, via an invertible matrix, of a general polynomial with a linear one.

3.4. Numerical Results

In Figs. 4–8 we show several numerical results on the ω- and α-limits of points under iterations of real Newton maps associated to polynomial maps of various degrees in two variables. Every row (with the exception of Fig. 8 and the middle row in Fig. 4) shows, next to each other, the basins of attraction of a Newton map (left), with a different color associated to each attractor, and the α-limit of a suitable point under that Newton map (right) in black and white.

First, in Fig. 4, we consider two maps with quadratic components. The first two rows are relative to $f(x,y) = (y - x^2, x + 2 - (y-2)^2)$, whose four roots are all real: $c_1 = (0,0)$, $c_2 = (2,4)$, $c_3 \simeq (-1.62, 2.62)$ and $c_4 \simeq (0.62, 0.38)$. In the first row we compare the basins of attraction with the α-limit of a suitable point. Similarly to the complex case, the α-limit seem to be almost identical to J_f except for some arcs, most noticeably the arc at the boundary of the basin $\mathcal{F}(c_2)$, colored in cyan.

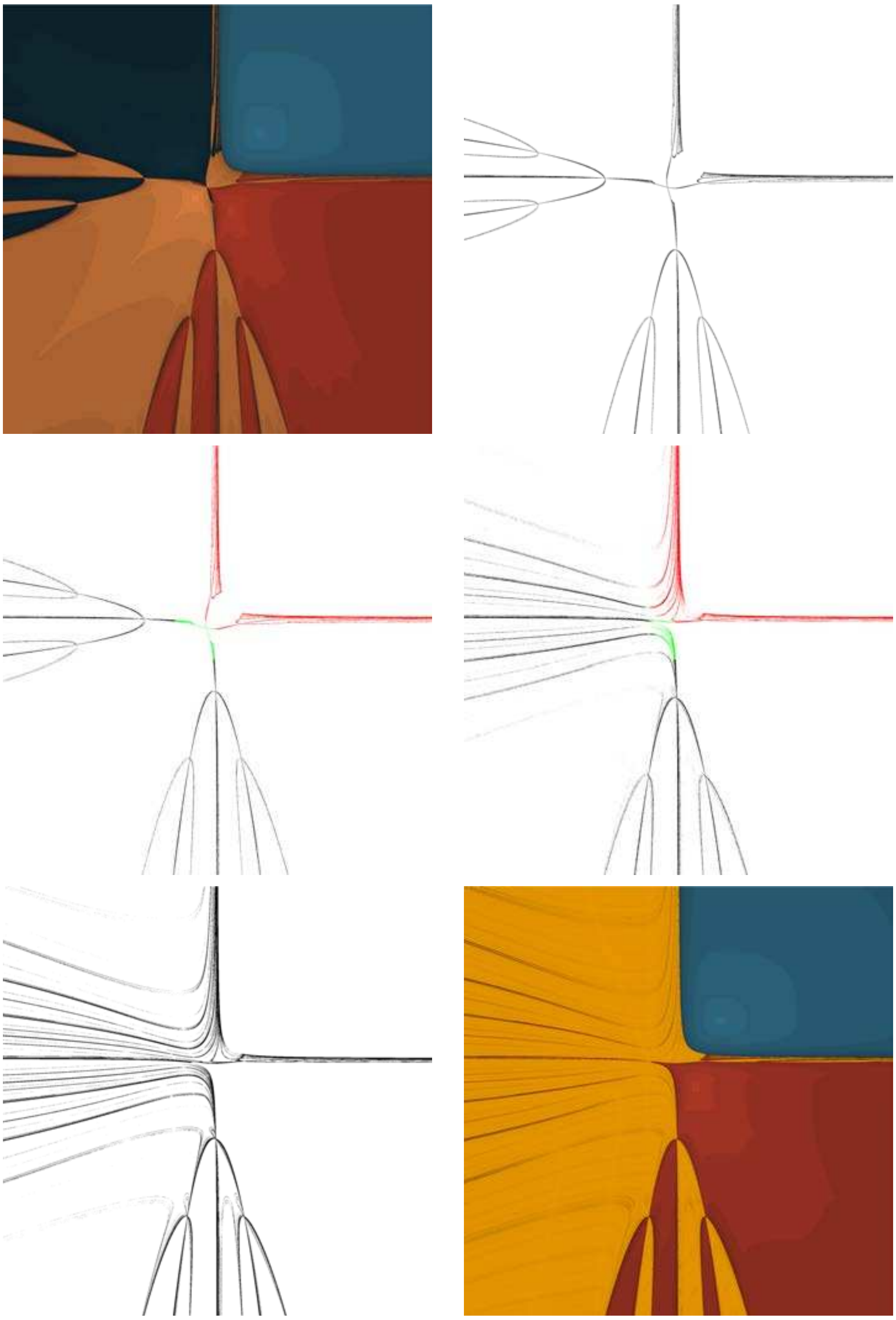

Fig. 4 (Color online) ω-limits (in color, each color corresponding to a different basin of attraction) and α-limits (in black and white) for the Newton maps of the polynomial maps $f(x, y) = (y - x^2, x + 2 - (y - 2)^2)$ (first row) and $g(x, y) = f(x, y) - (0, 1)$ (last row).

These points of J_f are characterized by the fact that they are *isolated*, in the sense that for each of these points $x \in J_f$ there is some neighborhood U_x such that $J_f \cap U_x$ is a single smooth curve. This pattern will repeat in all other examples below.

Definition 9. We say that a point p of the Julia set J_F of a rational map $F : \mathbb{R}^2 \to \mathbb{R}^2$ is *regular* if there is a neighborhood U of p such that $J_F \cap U$ is a connected 1-dimensional submanifold and U contains points from exactly two different basins.

Notice that, unlike the complex case, not all points have the same α-limit: e.g. the α-limit of a generic point in $\mathcal{F}(c_2)$ does not contain any bounded point. On the other side, the α-limit of

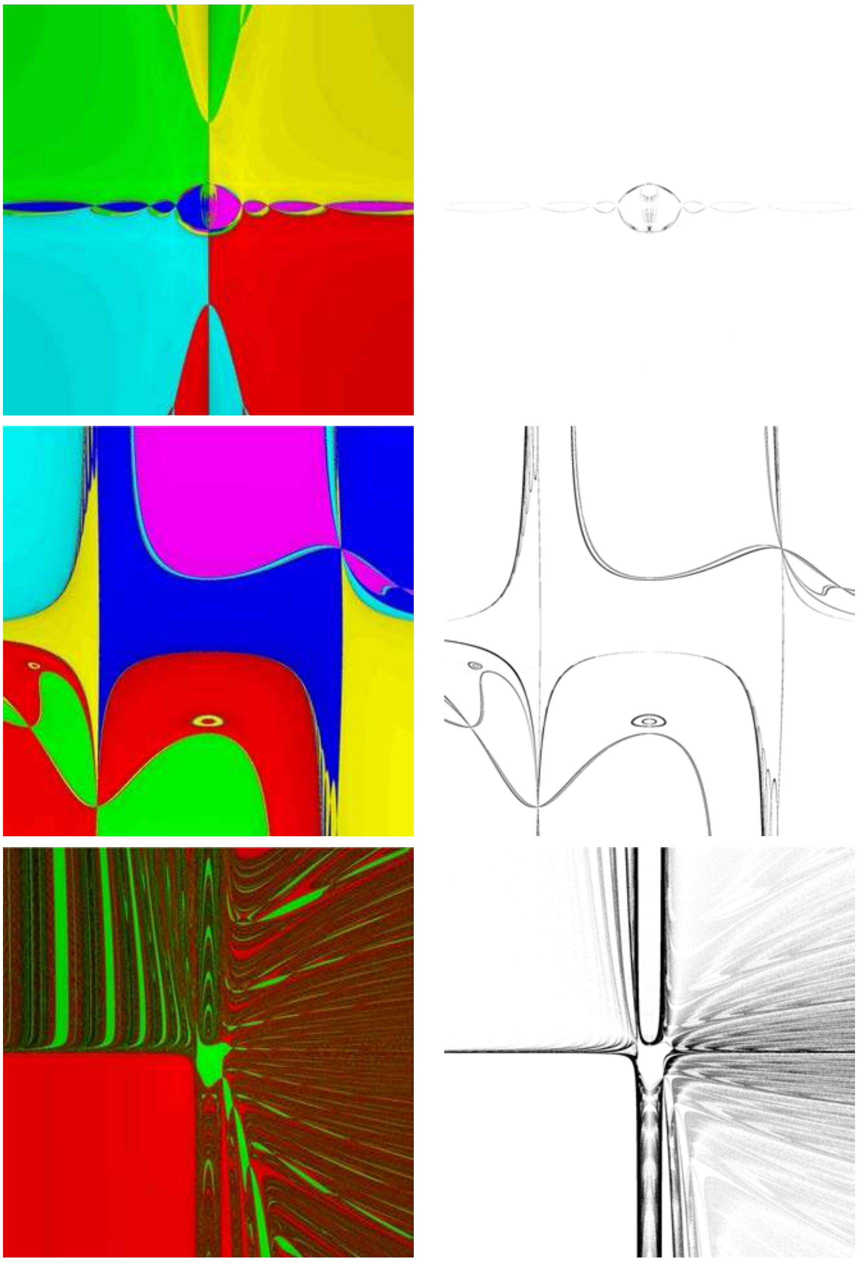

Fig. 5 ω-limits (in color, each color corresponding to a different basin of attraction) and α-limits (in black and white) for the Newton maps of the polynomial maps (from top to bottom) $f(x,y) = (6 - 9x^2 + 24y - 9x^2y + 9y^2 + y^3, x^2 + y^2 - 6)$, $g(x,y) = (5x(x^2-1) + y, y^2 + x - 2)$ and $h(x,y) = (5x(x^2-1) - 5y, 10(y^2+x) - 1)$.

a generic point below the line $x + y + 1 = 0$ seems to coincide with the one shown in Fig. 4, suggesting that in the real case one cannot expect Theorem 2 to hold in general for almost all points but rather only for some non-empty open set.

In the middle row we show the unique invariant set of the Iterated Function System $\mathcal{I}_f$ consisting in the free semigroup generated by three of the four branches of f^{-1}. Indeed, it turns out that the half-plane $D = \{x + y \geq -1\}$ is invariant under the action of $\mathcal{I}_f$ and its unique compact invariant set is shown in black in Fig. 4 (mid, left). To be precise, what the picture actually shows is the first $3 \cdot 10^5$ points of a backward orbit of a single point, suggesting that the result of Hawkins and Taylor (Theorem 5) holds even in the real setting. Notice that there is a small open set $E \subset D$, whose boundary contains c_1, c_3 and c_4, that is outside of the range of f. Whenever our algorithm generating a backward orbit fell on E, we marked that point in green and chose a different preimage to continue moving backwards. Those green points cover the

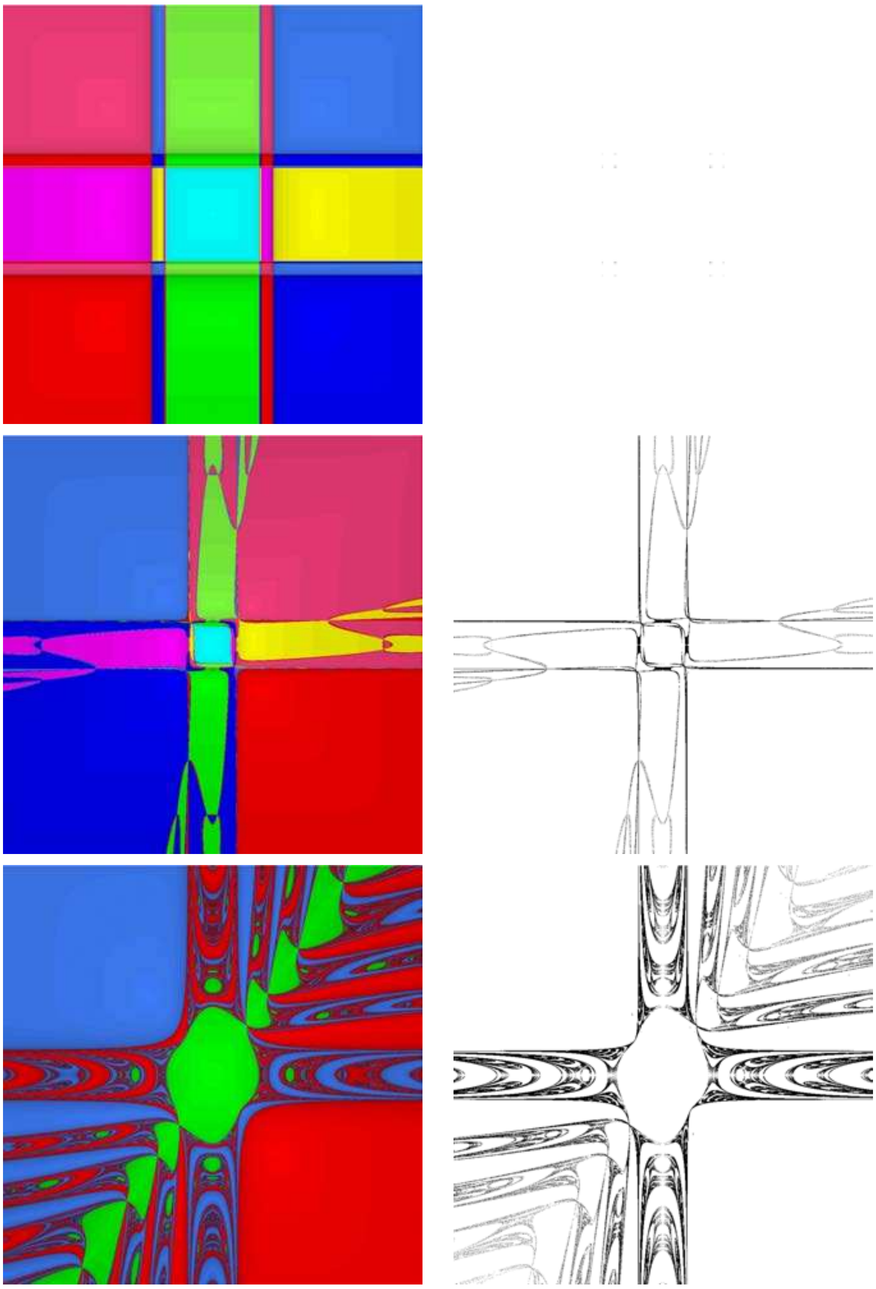

Fig. 6 ω-limits (in color, each color corresponding to a different basin of attraction) and α-limits (in black and white) for the Newton maps of the polynomial maps (from top to bottom) $f(x,y) = (x(x^2-1), y(y^2-1))$, $g(x,y) = (20x(x^2-1) + y, 20y(y^2-1) + x)$ and $h(x,y) = (x(x^2-1) + y, y(y^2-1) + 3x)$.

part of J_f lying in E, once again except for the regular points. Finally, the red points are obtained by applying to the black and green points the branch of f^{-1} that is not among the generators of $\mathcal{I}_f$. The union of the red, green and black points looks indistinguishable from the approximation of J_f obtained as the α-limit of a point, suggesting that Barnsley's idea that Julia sets can be obtained as invariant sets of IFS work also in the real setting (with the exception of the regular points).

In the bottom row and in the mid right one we show the corresponding pictures for the map $g(x,y) = f(x,y) - (0,1)$. A small change produces many qualitative differences here: g has only two roots, the points $c_1 \simeq (1.9, 3.7)$ and $c_2 \simeq (0.81, 0.65)$, whose basins are colored in cyan and red, respectively. As above, the Julia set can be obtained both as the α-limit of a suitable point and as the invariant set of an IFS. Unlike above, though, there is a third basin of attraction, colored in gold,

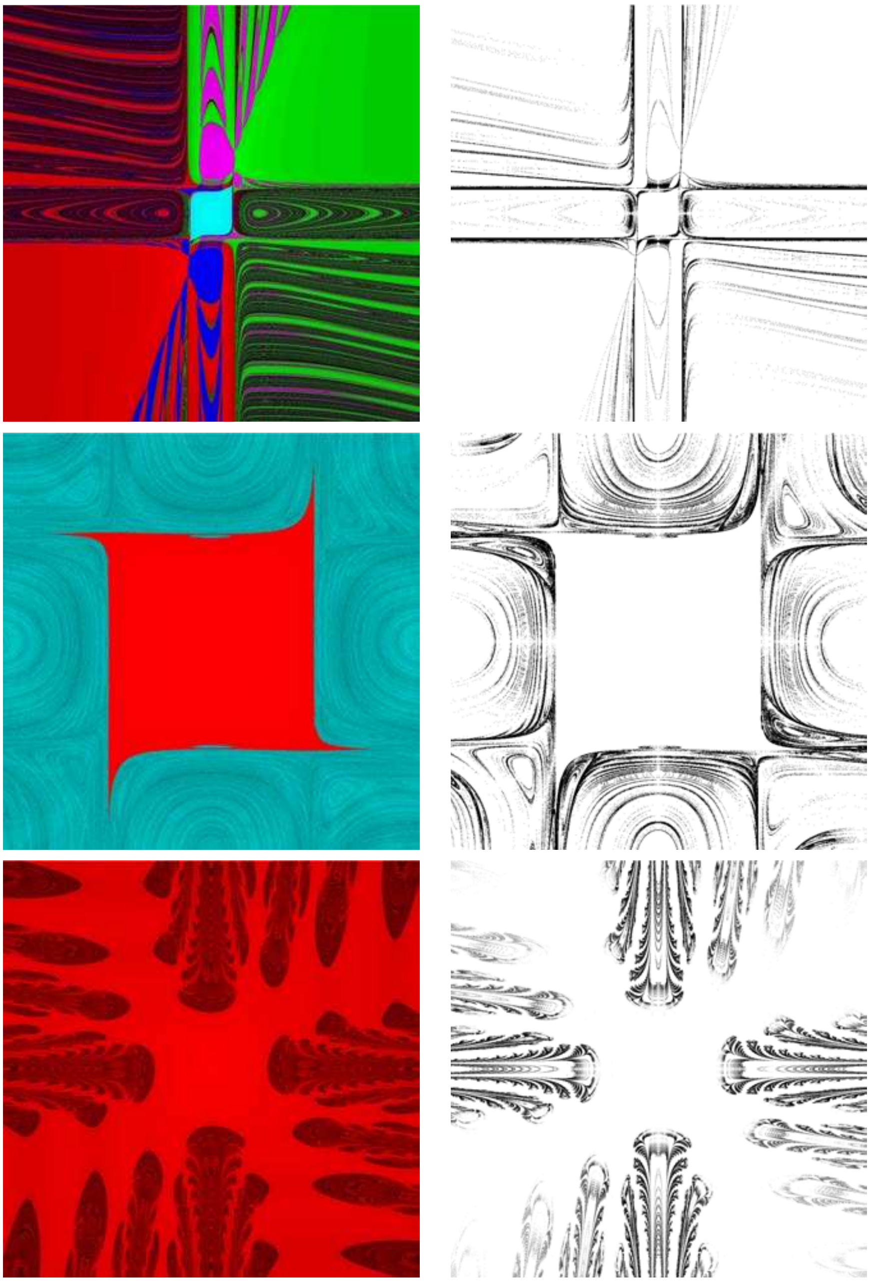

Fig. 7 ω-limits (in color, each color corresponding to a different basin of attraction) and α-limits (in black and white) for the Newton maps of the polynomial maps (from top to bottom) $f(x,y) = (10x(x^2-1)+3y, y(y^2-1)-x)$, $g(x,y) = (10x(x^2-1)+7y, y(y^2-1)-x)$ and $h(x,y) = (x(x^2-1)+60y, y(y^2-1)-60x)$.

corresponding to a chaotic attractor contained in an invariant line of N_g. In Fig. 8 (top, left) we show the main elements in the dynamics of N_g: the fixed points (red), the only bounded point of indeterminacy (blue), the invariant line passing through the two roots (light blue) and the one on which lie the third attractor (gold) together with its first preimage (purple), the set Z_{N_g} (light blue hyperbola passing through the indeterminacy point) with its first and second preimages (red and brown) and the first 500 points of the orbit of a generic point in the third basin (green). The orbits of points in the golden basin are essentially one-dimensional after the first few iterations and show high sensitivity to the initial conditions. In particular then this basin belongs to J_{N_g} rather than F_{N_g}. This suggests that, in the real case, the Julia set can have a non-empty interior without being necessarily the whole $\mathbb{RP}^2$ (as it happens, instead, for intrinsically holomorphic maps).

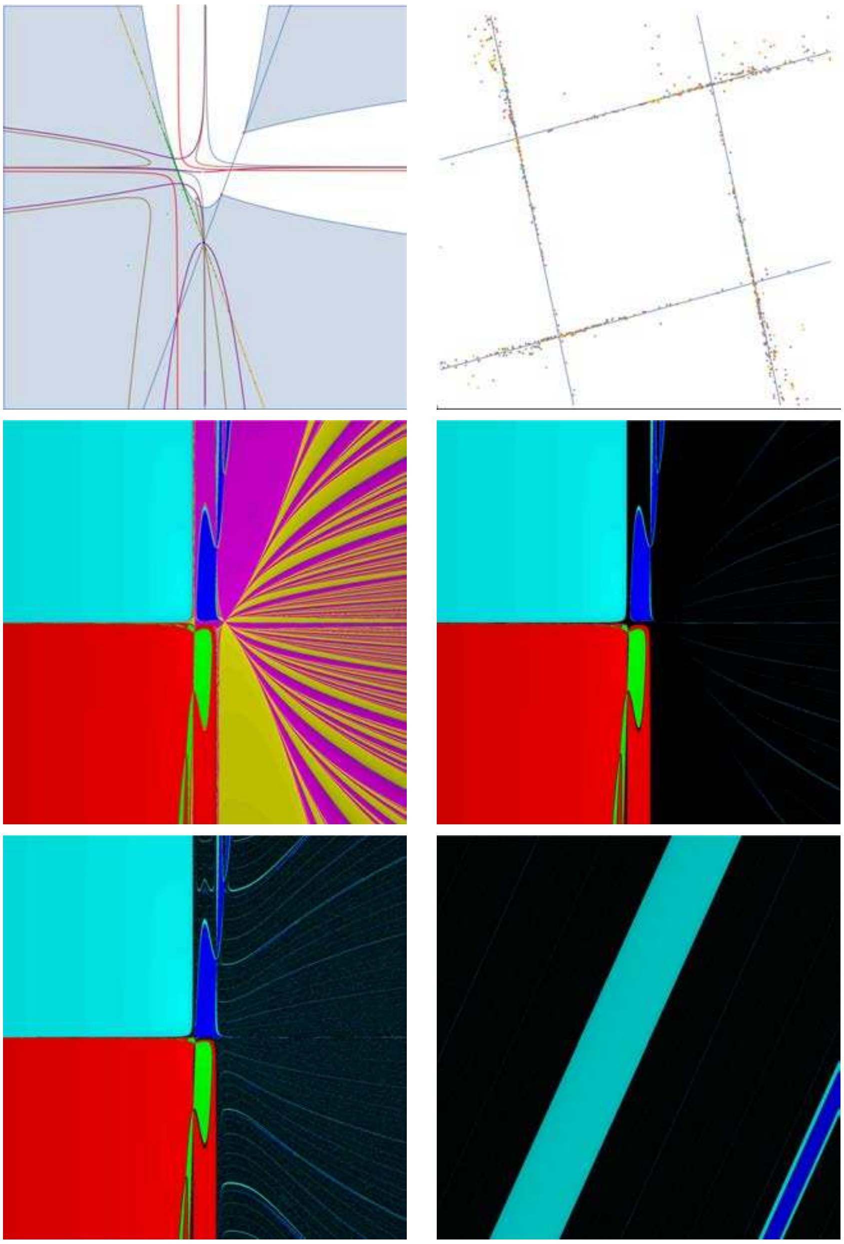

Fig. 8 (Top, left) Main elements of the dynamics of N_g in Fig. 4. (Top, right) A typical orbit of a cyan point in Fig. 7 (middle) and the four invariant lines of the map. (Middle and bottom) Basins of attraction of N_{f_α} for $f_\alpha(x,y) = (x^2(x-1)+y, x-\alpha-y^2)$ with $\alpha = -0.997462, -0.997461, -0.5$ in the square $[-10,10]^2$ and a detail in $[3.00000015, 3.00000025]\times[4, 4.0000001]$ for $\alpha = -0.995$.

We postpone to a separate paper a thorough numerical analysis of the dynamics of Newton maps of real polynomial maps with quadratic coefficients. The remaining pictures show that this qualitative behavior is not limited to such elementary maps and reveal some further element of difference with respect to the complex case. Figures 5 (top and middle) and 6 (top and middle) show the basins of attraction of Newton maps corresponding to real polynomial maps with maximal number of real solutions (respectively 6, 6, 9 and 9). These pictures strongly suggest the following facts for this kind of maps f:

(1) regular points of J_f cannot be reached via α-limits (Fig. 5 (top));
(2) every neighborhood of every point of J_f contains points from at least two basins — Fig. 7 (bottom) suggests that this is not the case when the number of real roots is not maximal;

(3) boundaries between basins of attraction are smooth, except at countably many nodal points;
(4) basins of attraction are not necessarily simply connected (Fig. 5, (middle));
(5) immediate basins arenot necessarily unbounded (Fig. 6, (top)).

Figures 5 (bottom), 6 (bottom) and 7 (all) show the basins of attraction of Newton maps corresponding to polynomials with fewer roots than maximal. Although we know from the intrinsically holomorphic case that, for some polynomials, basins of attraction can satisfy the same properties of those with maximal number of roots (e.g. see Fig. 6 (bottom)), numerics strongly suggest that this is not always the case.

The α-limit of points in some non-empty open set seems to be equal to the boundary of the Julia set, suggesting a more suitable split of $\mathbb{RP}^2$ in case of real maps: $\mathbb{RP}^2 = A_{N_f} \sqcup R_{N_f}$, where A_{N_f} is the union of the Fatou set with *all* basins of attraction and R_{N_f} its complement. Numerics suggest that, with this definition, the set R_{N_f}, as for J_{N_f} in the intrinsically holomorphic case, has no interior points, possibly even when f has no real roots at all, and, for some non-empty open set, the α-limits of points are equal to its non-regular points. Observe that, since in the holomorphic case Newton maps are always non-chaotic on their basins of attraction, this split coincides with the split in the Fatou and Julia sets in the intrinsically holomorphic case.

In Figs. 5 (bottom) and 7 (top), the basins of attraction are intertwined in such a way to suggest a *Cantor set of circles* structure with non-zero measure or Hausdorff dimension greater than 1 for the corresponding Julia set. Following numerically the evolution of the ω-limits in one-parametric families close to the bifurcation point where a couple of real roots disappear, we observed that usually the basins of attraction of the disappeared roots get replaced by the basin of attraction of a Cantor set lying in some neighborhood of an invariant line (see the middle and bottom rows of Fig. 8) and then the size of this basin usually decreases in favor of the basins of the remaining real roots.

In Fig. 7 (middle), we show the two basins of the Newton map of a polynomial map g of degree 9 with a single real root. In this case the basin of attraction of the only root (in red) is bounded and connected (but not simply connected) while the other one is the basin of attraction of a Cantor set (in cyan) in some neighborhood of the union of four invariant lines corresponding to the four pairs of mutually conjugate pairs of complex solutions. The α-limit seems to be equal to the difference between the Julia set and the basin of attraction of the Cantor set.

Finally, in Fig. 7 (bottom), we show the basin of attraction (in red) of the Newton map of a polynomial map h of degree 9 with a single root and its Julia set. Even in this case we can find points whose α-limit is equal to the Julia set.

We conclude the paper with two conjectures motivated by the several analytical and numerical results presented above.

Conjecture 2. *Let* $f:\mathbb{R}^2 \to \mathbb{R}^2$ *be a generic polynomial map of degree* $n \geq 3$. *Then there are non-empty open subsets* $V \subset U \subset f(\mathbb{RP}^2)$ *such that*:

(1) $\alpha_{N_f}(x)$ *is equal to the set of non-regular points of the boundary of* J_{N_f} *for all* $x \in U$;
(2) $V \cap J_f$ *is the unique attractor of an IFS.*

Conjecture 3. *Let* $f:\mathbb{R}^2 \to \mathbb{R}^2$ *be a polynomial map of degree* $n \geq 3$ *with* n *simple real roots* $\{c_1, \dots, c_n\}$. *Then*:

(1) $F_{N_f} = \bigcup_{i=1}^n \mathcal{F}_{N_f}(c_i)$. *In particular,* N_f *has no wandering domains or attracting* k*-cycles for* $k \geq 2$.
(2) J_{N_f} *is the countable union of wedge sums of countably many circles and of Cantor sets of circles of measure zero.*
(3) *Every neighborhood of any point of* J_{N_f} *intersects at least two distinct basins of attractions.*
(4) *Unlike the holomorphic case*:
    (a) *Basins of attractions are not necessarily simply connected.*
    (b) *Immediate basins of attraction are not necessarily unbounded.*
    (c) J_{N_f} *can have interior points without being equal to the whole* $\mathbb{RP}^2$.

Note that, as a corollary of these two conjectures, a Newton map N_f is weakly plain if f is a polynomial map of degree $n \geq 3$ with all simple real roots.

ACKNOWLEDGMENTS

I am in great debt with J. Yorke and J. Hawkins for many discussions that greatly helped the development of this paper and I am grateful to S. van Strien for some clarification on real one-dimensional dynamics. I am also grateful to the anonymous referee for comments that helped improving the overall quality of the manuscript. All pictures, except

the two in the top row of Fig. 8, were generated by code written by the author in Python and C/C++. All calculations were performed on the HPCC of the College of Arts and Sciences at Howard University.

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