Journal of Pure and Applied Algebra 224 (2020) 106347

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

Canonical bases for tensor products and super Kazhdan-Lusztig theory

Huanchen Bao [a], Weiqiang Wang [b], Hideya Watanabe [c,*]

[a] Department of Mathematics, National University of Singapore, Singapore 119076, Singapore
[b] Department of Mathematics, University of Virginia, Charlottesville, VA 22904, United States
[c] Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan

ARTICLE INFO

Article history:
Received 12 December 2019
Received in revised form 13 January 2020
Available online 24 February 2020
Communicated by A.D. Lauda

MSC:
Primary: 17B10

ABSTRACT

We generalize a construction in [5] by showing that, for a quantum symmetric pair $(\mathbf{U}, \mathbf{U}^\imath)$ of finite type, the tensor product of a based $\mathbf{U}^\imath$-module and a based $\mathbf{U}$-module is a based $\mathbf{U}^\imath$-module. This is then used to formulate a Kazhdan-Lusztig theory for an *arbitrary* parabolic BGG category $\mathcal{O}$ of the ortho-symplectic Lie superalgebras, extending a main result in [4].

1. Introduction

1.1. Following Lusztig [11], we shall refer to $(M, \mathbf{B})$, which consists of a $\mathbf{U}$-module M and its canonical basis $\mathbf{B}$, as a based $\mathbf{U}$-module, where $\mathbf{U}$ is a Drinfeld-Jimbo quantum group of finite type. Examples of such based $\mathbf{U}$-modules include any finite-dimensional simple $\mathbf{U}$-module or a tensor product of several such simple $\mathbf{U}$-modules. The canonical basis on a tensor product of several finite-dimensional simple $\mathbf{U}$-modules was constructed by Lusztig [10], and it has found applications to the Kazhdan-Lusztig theory for general linear Lie superalgebra $\mathfrak{gl}(m|n)$ of type A [2,7].

1.2. As a generalization of canonical bases for quantum groups, a theory of canonical basis arising from quantum symmetric pairs (QSP, for short) $(\mathbf{U}, \mathbf{U}^\imath)$ of finite type is systematically developed in [5]. For any finite-dimensional based $\mathbf{U}$-module $(M, \mathbf{B})$, a new bar involution $\psi_\imath$ on M was formulated and a $\psi_\imath$-invariant basis $\mathbf{B}^\imath$ of M (called an $\imath$-canonical basis for M) was constructed (see [5, Theorem 5.7]), which satisfies some specific properties when expanded with respect to $\mathbf{B}$; we shall call $(M, \mathbf{B}^\imath)$ a based $\mathbf{U}^\imath$-module.

* Corresponding author.
E-mail addresses: huanchen@nus.edu.sg (H. Bao), ww9c@virginia.edu (W. Wang), hideya@kurims.kyoto-u.ac.jp (H. Watanabe).

https://doi.org/10.1016/j.jpaa.2020.106347
0022-4049/

The first examples of based $\mathbf{U}^\imath$-modules were constructed in [4] for quasi-split QSP $(\mathbf{U}, \mathbf{U}^\imath)$ of type AIII. The $\imath$-canonical basis on a tensor product of $\mathbf{U}$-modules was used to formulate a Kazhdan-Lusztig theory on the full BGG category for ortho-symplectic Lie superalgebra of type B [4] (for type D see [1]); also see [6] on some parabolic BGG category via a super duality approach.

1.3. This paper is intended to supplement the two earlier papers [4,5] of the first two authors on canonical bases arising from quantum symmetric pairs and applications to super Kazhdan-Lusztig theory; it extends two principal results on $\imath$-canonical basis and super Kazhdan-Lusztig theory therein to full generalities.

1.4. By definition of a QSP $(\mathbf{U}, \mathbf{U}^\imath)$, $\mathbf{U}^\imath$ is a coideal subalgebra of $\mathbf{U}$ [9]; that is, the comultiplication Δ on $\mathbf{U}$ when restricting to $\mathbf{U}^\imath$ satisfies $\Delta : \mathbf{U}^\imath \to \mathbf{U}^\imath \otimes \mathbf{U}$. Hence $M \otimes N$ is a $\mathbf{U}^\imath$-module for any $\mathbf{U}^\imath$-module M and $\mathbf{U}$-module N. In the first main theorem (see Theorem 4) we show that, for a based $\mathbf{U}^\imath$-module $(M, \mathbf{B}^\imath)$ and a based $\mathbf{U}$-module $(N, \mathbf{B})$, there exists a $\psi_\imath$-invariant basis $\mathbf{B}^\imath \diamondsuit_\imath \mathbf{B}$ on the $\mathbf{U}^\imath$-module $M \otimes N$ such that $(M \otimes N, \mathbf{B}^\imath \diamondsuit_\imath \mathbf{B})$ is a based $\mathbf{U}^\imath$-module. This generalizes a main result in [5] on the $\imath$-canonical basis on a tensor product of $\mathbf{U}$-modules, since a based $\mathbf{U}^\imath$-module which is not a $\mathbf{U}$-module exists (cf. [4]).

The construction of the new bar involution $\psi_\imath$ on $M \otimes N$ above uses a certain element $\Theta^\imath$ in a completion of $\mathbf{U}^\imath \otimes \mathbf{U}^+$, which was due to [4] for quasi-split QSP of type AIII/IV and then established in Kolb [8] in full generality with an elegant new proof. We establish the integrality of $\Theta^\imath$ by using the integrality of the quasi-$\mathcal{R}$ matrix in [11] and the integrality of the quasi-$\mathcal{K}$ matrix in [5]. To construct the $\imath$-canonical basis on $M \otimes N$, we use crucially a partial order, which is different from and simpler than the old one used in [5] even when M is a $\mathbf{U}$-module; the old partial order does not make much sense in our new setting.

1.5. For the quasi-split QSP $(\mathbf{U}, \mathbf{U}^\imath)$ of type AIII/AIV, the quantum group $\mathbf{U}$ is of type A; we let $\mathbb{V}$ and $\mathbb{W}$ denote the natural representation of $\mathbf{U}$ and its dual. In this case, the $\imath$-canonical basis on a based $\mathbf{U}$-module was first constructed in [4] when a certain parameter $\kappa = 1$ (also see [1] with parameter $\kappa = 0$). The super Kazhdan-Lusztig theory for the *full* BGG category $\mathcal{O}_{\mathbf{b}}$ of an ortho-symplectic Lie superalgebra $\mathfrak{g}$ of type B in [4] (for type D see [1]) of integer or half-integer weights was formulated via the $\imath$-canonical basis on a mixed tensor $\mathbf{U}$-module with m copies of $\mathbb{V}$ and n copies of $\mathbb{W}$, where the order of the tensor product depends on the choice of a Borel subalgebra $\mathbf{b}$ in $\mathfrak{g}$.

As a consequence, a Kazhdan-Lusztig theory for parabolic categories $\mathcal{O}_{\mathbf{b}}^{\mathfrak{l}}$, where the Levi subalgebra $\mathfrak{l}$ in $\mathfrak{g}$ is a product of Lie algebras of type A, can be formulated and established via the $\imath$-canonical basis on a tensor product $\mathbf{U}$-module $\mathbb{T}$ of various exterior powers of $\mathbb{V}$ and of $\mathbb{W}$. Note however not all the parabolic categories of $\mathfrak{g}$-modules arise in this way; indeed a general Levi subalgebra of $\mathfrak{g}$ is isomorphic to a product of several Lie subalgebras of type A and a Lie subalgebra of type B.

Theorem 4, when specialized for the QSP $(\mathbf{U}, \mathbf{U}^\imath)$ of quasi-split type AIII/AIV, provides an $\imath$-canonical basis for a $\mathbf{U}^\imath$-module on the tensor product of the form $\wedge^a \mathbb{V}_- \otimes \mathbb{T}$. Here $\mathbb{T}$ is a tensor product $\mathbf{U}$-module of various exterior powers of $\mathbb{V}$ and of $\mathbb{W}$, while $\wedge^a \mathbb{V}_-$ (for $a > 0$) is a "type B" exterior power, which is a $\mathbf{U}^\imath$-module but not a $\mathbf{U}$-module. These new $\imath$-canonical bases are used to formulate the super Kazhdan-Lusztig theory for an *arbitrary* parabolic BGG category $\mathcal{O}$ of the ortho-symplectic Lie superalgebras of type B and D; see Theorem 10. The super Kazhdan-Lusztig polynomials $t_{gf}^{\mathbf{b}}(q)$ admit a positivity property; see Theorem 12.

1.6. This paper is organized as follows. Theorem 4 and its proof are presented in Section 2, and we shall follow notations in [5] throughout Section 2. The formulation of Theorem 10 is given in Section 3; its proof basically follows the proof for the Kazhdan-Lusztig theory for the full category $\mathcal{O}$ in [4, Part 2] once we have Theorem 4 available to us. We shall follow notations in [4] throughout Section 3. To avoid much repetition, we refer precisely and freely to the two earlier papers [4,5].

Acknowledgments

The research of WW is partially supported by an NSF grant DMS-1702254.

2. Tensor product modules as based $\mathbf{U}^\imath$-modules

2.1. We shall follow the notations in [5] throughout this section.

Let $\mathbf{U}$ denote a quantum group of finite type over the field $\mathbb{Q}(q)$ associated to a root datum of type $(\mathbb{I}, \cdot)$, and let Δ denote its comultiplication as in [11]. We denote the bar involution on $\mathbf{U}$ or its based module by ψ.

Let $\mathbf{U}^\imath \subset \mathbf{U}$ be a coideal subalgebra associated to a Satake diagram such that $(\mathbf{U}, \mathbf{U}^\imath)$ forms a quantum symmetric pair [9]. Let $\mathcal{A} := \mathbb{Z}[q, q^{-1}]$. Let $\dot{\mathbf{U}}^\imath$ be the modified version of $\mathbf{U}^\imath$ and let ${}_\mathcal{A}\dot{\mathbf{U}}^\imath$ be its $\mathcal{A}$-form, respectively; see [5, §3.7]. Let $\psi_\imath$ be the bar involution on $\mathbf{U}^\imath$, $\dot{\mathbf{U}}^\imath$ and ${}_\mathcal{A}\dot{\mathbf{U}}^\imath$. Let $X_\imath$ be the $\imath$-weight lattice [5, (3.3)]. A weight (i.e., $X_\imath$-weight) module of $\mathbf{U}^\imath$ can be naturally regarded as a $\dot{\mathbf{U}}^\imath$-module.

We introduce based $\mathbf{U}^\imath$-modules generalizing [11, §27.1.2]. Let $\mathbf{A} = \mathbb{Q}[[q^{-1}]] \cap \mathbb{Q}(q)$. We write $- \otimes - = - \otimes_{\mathbb{Q}(q)} -$ whenever the base ring is $\mathbb{Q}(q)$.

Definition 1. Let M be a finite-dimensional weight $\mathbf{U}^\imath$-module over $\mathbb{Q}(q)$ with a given $\mathbb{Q}(q)$-basis $\mathbf{B}^\imath$. The pair $(M, \mathbf{B}^\imath)$ is called a based $\mathbf{U}^\imath$-module if the following conditions are satisfied:

(1) $\mathbf{B}^\imath \cap M_\nu$ is a basis of M_ν, for any $\nu \in X_\imath$;
(2) The $\mathcal{A}$-submodule ${}_\mathcal{A}M$ generated by $\mathbf{B}^\imath$ is stable under ${}_\mathcal{A}\dot{\mathbf{U}}^\imath$;
(3) The $\mathbb{Q}$-linear involution $\psi_\imath : M \to M$ defined by $\psi_\imath(q) = q^{-1}, \psi_\imath(b) = b$ for all $b \in \mathbf{B}^\imath$ is compatible with the $\dot{\mathbf{U}}^\imath$-action, i.e., $\psi_\imath(um) = \psi_\imath(u)\psi_\imath(m)$, for all $u \in \dot{\mathbf{U}}^\imath, m \in M$;
(4) Let $L(M)$ be the $\mathbf{A}$-submodule of M generated by $\mathbf{B}^\imath$. Then the image of $\mathbf{B}^\imath$ in $L(M)/q^{-1}L(M)$ forms a $\mathbb{Q}$-basis in $L(M)/q^{-1}L(M)$.

We shall denote by $\mathcal{L}(M)$ the $\mathbb{Z}[q^{-1}]$-span of $\mathbf{B}^\imath$; then $\mathbf{B}^\imath$ forms a $\mathbb{Z}[q^{-1}]$-basis for $\mathcal{L}(M)$. (There are similar constructions for a based $\mathbf{U}$-module in similar notations.)

2.2. Let $\Upsilon = \sum_\mu \Upsilon_\mu$ (with $\Upsilon_0 = 1$ and $\Upsilon_\mu \in \mathbf{U}^+_\mu$) be the intertwiner (also called quasi-$\mathcal{K}$ matrix) of the quantum symmetric pair $(\mathbf{U}, \mathbf{U}^\imath)$ introduced in [4, Theorem 2.10]; for full generality see [3, Theorem 6.10], [5, Theorem 4.8, Remark 4.9]. It follows from [5, Theorem 5.7] (also cf. [4, Theorem 4.25]) that a based $\mathbf{U}$-module $(M, \mathbf{B})$ becomes a based $\mathbf{U}^\imath$-module with a new basis $\mathbf{B}^\imath$ (which is uni-triangular relative to $\mathbf{B}$) with respect to the involution $\psi_\imath := \Upsilon \circ \psi$.

Let $\widehat{\mathbf{U} \otimes \mathbf{U}}$ be the completion of the $\mathbb{Q}(q)$-vector space $\mathbf{U} \otimes \mathbf{U}$ with respect to the descending sequence of subspaces

$$\mathbf{U} \otimes \mathbf{U}^-\mathbf{U}^0\Big(\sum_{\mathrm{ht}(\mu)\geq N} \mathbf{U}^+_\mu\Big) + \mathbf{U}^+\mathbf{U}^0\Big(\sum_{\mathrm{ht}(\mu)\geq N} \mathbf{U}^-_\mu\Big) \otimes \mathbf{U}, \text{ for } N \geq 1, \mu \in \mathbb{Z}\mathbb{I}.$$

We have the obvious embedding of $\mathbf{U} \otimes \mathbf{U}$ into $\widehat{\mathbf{U} \otimes \mathbf{U}}$. By continuity the $\mathbb{Q}(q)$-algebra structure on $\mathbf{U} \otimes \mathbf{U}$ extends to a $\mathbb{Q}(q)$-algebra structure on $\widehat{\mathbf{U} \otimes \mathbf{U}}$. We know the quasi-$\mathcal{R}$ matrix Θ lies in $\widehat{\mathbf{U} \otimes \mathbf{U}}$ by [11, Theorem 4.1.2]. It follows from [5, Theorem 4.8] and [3, Theorem 6.10] that $\Upsilon^{-1} \otimes \mathrm{id}$ and $\Delta(\Upsilon)$ are both in $\widehat{\mathbf{U} \otimes \mathbf{U}}$.

We define

$$\Theta^\imath = \Delta(\Upsilon) \cdot \Theta \cdot (\Upsilon^{-1} \otimes \mathrm{id}) \in \widehat{\mathbf{U} \otimes \mathbf{U}}. \tag{2.1}$$

We can write

$$\Theta^\imath = \sum_{\mu \in \mathbb{N}\mathbb{I}} \Theta^\imath_\mu, \qquad \text{where } \Theta^\imath_\mu \in \mathbf{U} \otimes \mathbf{U}^+_\mu. \tag{2.2}$$

The following result first appeared in [4, Proposition 3.5] for the quantum symmetric pairs of (quasi-split) type AIII/AIV.

Lemma 2. *[8, Proposition 3.10] We have $\Theta^\imath_\mu \in \mathbf{U}^\imath \otimes \mathbf{U}^+_\mu$, for all μ. (The element $\Theta^\imath_\mu$ is denoted by R^θ_μ in [8].)*

Another basic ingredient which we shall need is the integrality property of $\Theta^\imath$.

Lemma 3. *We have $\Theta^\imath_\mu \in {}_\mathcal{A}\mathbf{U} \otimes_\mathcal{A} {}_\mathcal{A}\mathbf{U}^+_\mu$, for all μ.*

Proof. By a result of Lusztig [11, 24.1.6], we have $\Theta = \sum_{\nu \in \mathbb{N}\mathbb{I}} \Theta_\nu$ is integral, i.e., $\Theta_\nu \in {}_\mathcal{A}\mathbf{U}^-_\nu \otimes_\mathcal{A} {}_\mathcal{A}\mathbf{U}^+_\nu$. By [5, Theorem 5.3] we have $\Upsilon = \sum_{\mu \in \mathbb{N}\mathbb{I}} \Upsilon_\mu$ is integral, i.e., $\Upsilon_\mu \in {}_\mathcal{A}\mathbf{U}^+_\mu$ for each μ; it follows that $\Upsilon^{-1} = \psi(\Upsilon)$ is integral too thanks to [5, Corollary 4.11]. It is well known that the comultiplication Δ preserves the $\mathcal{A}$-form, i.e., $\Delta({}_\mathcal{A}\mathbf{U}) \subset {}_\mathcal{A}\mathbf{U} \otimes_\mathcal{A} {}_\mathcal{A}\mathbf{U}$. The lemma follows now by the definition of $\Theta^\imath$ in (2.1). □

2.3. Define a partial order $<$ on X by setting $\mu' < \mu$ if $\mu' - \mu \in \mathbb{N}\mathbb{I}$. Denote by $|b| = \mu$ if an element b in a $\mathbf{U}$-module is of weight μ. Now we are ready to prove the first main result of this paper.

Theorem 4. *Let $(M, \mathbf{B}^\imath)$ be a based $\mathbf{U}^\imath$-module and $(N, \mathbf{B})$ be a based $\mathbf{U}$-module.*

(1) *For $b_1 \in \mathbf{B}^\imath, b_2 \in \mathbf{B}$, there exists a unique element $b_1 \diamondsuit_\imath b_2$ which is $\psi_\imath$-invariant such that $b_1 \diamondsuit_\imath b_2 \in b_1 \otimes b_2 + q^{-1}\mathbb{Z}[q^{-1}]\mathbf{B}^\imath \otimes \mathbf{B}$.*
(2) *We have $b_1 \diamondsuit_\imath b_2 \in b_1 \otimes b_2 + \sum_{(b'_1, b'_2) \in \mathbf{B}^\imath \times \mathbf{B}, |b'_2| < |b_2|} q^{-1}\mathbb{Z}[q^{-1}]\, b'_1 \otimes b'_2$.*
(3) *$\mathbf{B}^\imath \diamondsuit_\imath \mathbf{B} := \{b_1 \diamondsuit_\imath b_2 \mid b_1 \in \mathbf{B}^\imath, b_2 \in \mathbf{B}\}$ forms a $\mathbb{Q}(q)$-basis for $M \otimes N$, an $\mathcal{A}$-basis for ${}_\mathcal{A}M \otimes_\mathcal{A} {}_\mathcal{A}N$, and a $\mathbb{Z}[q^{-1}]$-basis for $\mathcal{L}(M) \otimes_{\mathbb{Z}[q^{-1}]} \mathcal{L}(N)$. (This is called* the $\imath$-canonical basis *for $M \otimes N$.)*
(4) *$(M \otimes N, \mathbf{B}^\imath \diamondsuit_\imath \mathbf{B})$ is a based $\mathbf{U}^\imath$-module.*

Proof. It follows by Lemma 2 that the element $\Theta^\imath$ gives rise to a well-defined operator on the tensor product $M \otimes N$. Following [4, (3.17)], we define a new bar involution on $M \otimes N$ (still denoted by $\psi_\imath$) by letting

$$\psi_\imath := \Theta^\imath \circ (\psi_\imath \otimes \psi) : M \otimes N \longrightarrow M \otimes N.$$

Recall from [11] that $\Delta(E_i) = E_i \otimes 1 + \widetilde{K}_i \otimes E_i$. It follows that

$$\Delta(\Upsilon) \in \Upsilon \otimes 1 + \sum_{0 \neq \mu \in \mathbb{N}\mathbb{I}} \mathbf{U} \otimes \mathbf{U}^+_\mu.$$

Recalling (2.2), we have

$$\Theta^\imath_0 = (\Upsilon \otimes 1) \cdot (1 \otimes 1) \cdot (\Upsilon^{-1} \otimes 1) = 1 \otimes 1. \tag{2.3}$$

Let $b_1 \in \mathbf{B}^\imath$ and $b_2 \in \mathbf{B}$. By (2.3) and Lemma 3, we have

$$\psi_\imath(b_1 \otimes b_2) \in b_1 \otimes b_2 + \sum_{\substack{(b'_1, b'_2) \in \mathbf{B}^\imath \times \mathbf{B} \\ |b'_2| < |b_2|}} \mathcal{A}\, b'_1 \otimes b'_2. \tag{2.4}$$

Applying [11, Lemma 24.2.1], there exists a $\psi_\imath$-invariant element $(b_1 \otimes b_2)^\imath \in M \otimes N$ such that

$$b_1 \diamondsuit_\imath b_2 \in b_1 \otimes b_2 + \sum_{\substack{(b'_1, b'_2) \in \mathbf{B}^\imath \times \mathbf{B} \\ |b'_2| < |b_2|}} q^{-1}\mathbb{Z}[q^{-1}]\, b'_1 \otimes b'_2.$$

This proves (2), and part (3) follows immediately.

A by now standard argument shows the uniqueness of $b_1 \diamondsuit_\imath b_2$ as stated in (1); note a weaker condition than (2) is used in (1).

It remains to see that $(M, \mathbf{B}^\imath \diamondsuit_\imath \mathbf{B})$ is a based $\mathbf{U}^\imath$-module. The item (3) in the definition of a based $\mathbf{U}^\imath$-module is proved in the same way as for [4, Proposition 3.13], while the remaining items are clear.

This completes the proof. □

Remark 5.

(1) An elementary but key new ingredient in Theorem 4 above is the use of a (coarser) partial order $<$, which is different from the partial order $<_\imath$ used in [5, (5.2)].
(2) Assume that M is a based $\mathbf{U}$-module. Then the $\imath$-canonical basis for $M \otimes N$ in Theorem 4 coincides with the one in [5, Theorem 5.7], thanks to the uniqueness in Theorem 4(1).

Remark 6. Theorem 4 would be valid whenever we can establish the (weaker) integrality of $\Theta^\imath$ acting on $M \otimes N$. This might occur when we consider more general parameters for $\mathbf{U}^\imath$ than [5] or when we consider quantum symmetric pairs of Kac-Moody type in a forthcoming work of the first two authors.

3. Applications to super Kazhdan-Lusztig theory

3.1. In this section, we shall apply Theorem 4 to formulate and establish the (super) Kazhdan-Lusztig theory for an arbitrary parabolic category $\mathcal{O}$ of modules of integer or half-integer weights for ortho-symplectic Lie superalgebras, generalizing [4, Part 2] (also see [1]). We shall present only the details on an arbitrary parabolic category $\mathcal{O}$ consisting of modules of integer weights for Lie superalgebra $\mathfrak{osp}(2m+1|2n)$.

3.2. All relevant notations throughout this section shall be consistent with [4, Part 2]. In particular, we use a comultiplication for $\mathbf{U}$ different from [11]; this leads to a version of the intertwiner $\Upsilon = \sum_\mu \Upsilon_\mu$ with $\Upsilon_\mu \in \widehat{\mathbf{U}}^-$ (compare with §2.2), and a version of Theorem 4 in which the opposite partial order and the lattice $\mathbb{Z}[q]$ are used. To further match notations with [4] in this section, we denote the $\mathcal{A}$-form of any based $\mathbf{U}$ or $\mathbf{U}^\imath$-modules M, as $M_\mathcal{A}$ (instead of ${}_\mathcal{A}M$).

We consider the infinite-rank quantum symmetric pair $(\mathbf{U}, \mathbf{U}^\imath)$ as defined in [4, Section 8] (where the parameter is chosen to be $\kappa = 1$ in the notation of [5]). It is a direct limit of quantum symmetric pairs of type AIII, $(\mathbf{U}(\mathfrak{sl}_N), \mathbf{U}^\imath(\mathfrak{sl}_N))$, for N even. We denote by $\mathbb{V}$ the natural representation of $\mathbf{U}$, and by $\mathbb{W}$ the restricted dual of $\mathbb{V}$.

Associated to any given 0^m1^n-sequence $\mathbf{b} = (b_1, \dots, b_{m+n})$ starting with 0, we have a fundamental system of $\mathfrak{osp}(2m+1|2n)$, denoted by $\Pi_\mathbf{b} = \{-\epsilon_1^{b_1}, \epsilon_i^{b_i} - \epsilon_{i+1}^{b_{i+1}} \mid 1 \le i \le m+n-1\}$; here $\epsilon_i^0 = \epsilon_x$ for some $1 \le x \le m$ and $\epsilon_j^1 = \epsilon_{\bar{y}}$ for some $1 \le y \le n$ so that $\{\epsilon_i^{b_i} \mid 1 \le i \le m+n\}$ form a permutation of $\{\epsilon_a, \epsilon_b \mid 1 \le a \le m, 1 \le b \le n\}$.

3.3. Let W_{B_s} and $W_{A_{s-1}}$ be the Weyl group of type B_s and type A_{s-1} with unit e, respectively. We denote their corresponding Hecke algebras by $\mathcal{H}_{B_s}$ and $\mathcal{H}_{A_{s-1}}$, with Kazhdan-Lusztig bases by $\{\underline{H_w} | w \in W_{B_s}\}$ and $\{\underline{H_w} | w \in W_{A_{s-1}}\}$, respectively. Both algebras act naturally on the right on $\mathbb{V}^{\otimes s}$ and $\mathbb{W}^{\otimes s}$; cf. [4, Section 5]. We define

$$\wedge^s \mathbb{V}_- = \mathbb{V}^{\otimes s} \Big/ \sum_{e \neq w \in W_{B_s}} \mathbb{V}^{\otimes s} \cdot \underline{H_w},$$

$$\wedge^s \mathbb{V} = \mathbb{V}^{\otimes s} \Big/ \sum_{e \neq w \in W_{A_{s-1}}} \mathbb{V}^{\otimes s} \cdot \underline{H_w}.$$

We similarly define $\wedge^s \mathbb{W}_-$ and $\wedge^s \mathbb{W}$. Note $\wedge^s \mathbb{V}_-, \wedge^s \mathbb{V}, \wedge^s \mathbb{W}_-$ and $\wedge^s \mathbb{W}$ are all based $\mathbf{U}^\imath$-modules by [4, Theorem 5.8]. We shall denote

$$\mathbb{V}^c := \begin{cases} \mathbb{V} & \text{if } c = 0, \\ \mathbb{W} & \text{if } c = 1. \end{cases}$$

The following corollary is a direct consequence of Theorem 4.

Corollary 7. *Let $c_1, \dots, c_k \in \{0, 1\}$ and $a_0, a_1, \dots, a_k \in \mathbb{N}$. Then (a suitable completion of) the tensor product*

$$\mathbb{T}^{\mathbf{b},\mathfrak{l}} = \wedge^{a_0} \mathbb{V}_- \otimes \wedge^{a_1} \mathbb{V}^{c_1} \otimes \cdots \otimes \wedge^{a_k} \mathbb{V}^{c_k}$$

is a based $\mathbf{U}^\imath$-module.

The completion above arises since we deal with quantum symmetric pairs of infinite rank, and it is a straightforward generalization of the B-completion studied in [4, Section 9]. Note that the $\imath$-canonical basis lives in $\mathbb{T}^{\mathbf{b},\mathfrak{l}}$ (instead of its completion) by Theorem 12 below.

3.4. Associated to the fundamental system $\Pi_{\mathbf{b}}$ are the set of positive roots $\Phi^+_{\mathbf{b}}$ and the Borel subalgebra $\mathfrak{b}_{\mathbf{b}}$ of $\mathfrak{osp}(2m+1|2n)$. Let $\Pi_{\mathfrak{l}} \subset \Pi_{\mathbf{b}}$ be a subset of even simple roots. We introduce the corresponding Levi subalgebra $\mathfrak{l}$ and parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{osp}(2m+1|2n)$:

$$\mathfrak{l} = \mathfrak{h}_{m|n} \bigoplus \bigoplus_{\alpha \in \mathbb{Z}\Pi_{\mathfrak{l}} \cap \Phi_{\mathbf{b}}} \mathfrak{osp}(2m+1|2n)_\alpha, \qquad \mathfrak{p} = \mathfrak{l} + \mathfrak{b}_{\mathbf{b}}.$$

Recall [4, §7] the weight lattice $X(m|n) = \sum_{i=1}^m \mathbb{Z}\epsilon_i + \sum_{j=1}^n \mathbb{Z}\epsilon_{\overline{j}}$. We denote

$$X^{\mathfrak{l},+}_{\mathbf{b}} = \{\lambda \in X(m|n) \mid (\lambda|\alpha) \geq 0, \forall \alpha \in \Pi_{\mathfrak{l}}\}.$$

Let $L_0(\lambda)$ be the irreducible $\mathfrak{l}$-module with highest weight λ, which is extended trivially to a $\mathfrak{p}$-module. We form the parabolic Verma module

$$M^{\mathfrak{l}}_{\mathbf{b}}(\lambda) := \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{osp}(2m+1|2n)} L_0(\lambda).$$

Definition 8. Let $\mathcal{O}^{\mathfrak{l}}_{\mathbf{b}}$ be the category of $\mathfrak{osp}(2m+1|2n)$-modules M such that

(i) M admits a weight space decomposition $M = \bigoplus_{\mu \in X(m|n)} M_\mu$, and $\dim M_\mu < \infty$;
(ii) M decomposes over $\mathfrak{l}$ into a direct sum of $L_{\mathfrak{l}}(\lambda)$ for some $\lambda \in X^{\mathfrak{l},+}_{\mathbf{b}}$;
(iii) there exist finitely many weights ${}^1\lambda, {}^2\lambda, \dots, {}^k\lambda \in X^{\mathfrak{l},+}_{\mathbf{b}}$ (depending on M) such that if μ is a weight in M, then $\mu \in {}^i\lambda - \sum_{\alpha \in \Pi_{\mathbf{b}}} \mathbb{N}\alpha$, for some i.

The morphisms in $\mathcal{O}^{\mathfrak{l}}_{\mathbf{b}}$ are all (not necessarily even) homomorphisms of $\mathfrak{osp}(2m+1|2n)$-modules.

For $\lambda \in X_{\mathbf{b}}^{\mathfrak{l},+}$, we shall denote by $L_{\mathbf{b}}^{\mathfrak{l}}(\lambda)$ the simple quotient of the parabolic Verma module $M_{\mathbf{b}}^{\mathfrak{l}}(\lambda)$ in $\mathcal{O}_{\mathbf{b}}^{\mathfrak{l}}$ with highest weight λ. Following [4, Definition 7.4], we can define the tilting modules $T_{\mathbf{b}}^{\mathfrak{l}}(\lambda)$ in $\mathcal{O}_{\mathbf{b}}^{\mathfrak{l}}$, for $\lambda \in X_{\mathbf{b}}^{\mathfrak{l},+}$. We denote by $\mathcal{O}_{\mathbf{b}}^{\mathfrak{l},\Delta}$ the full subcategory of $\mathcal{O}_{\mathbf{b}}^{\mathfrak{l}}$ generated by all modules possessing finite parabolic Verma flags.

3.5. Recall the bijection $X(m|n) \leftrightarrow I^{m+n}$ [4, §8.4], where an element $f \in I^{m+n}$ is understood as a ρ-shifted weight. We consider the restriction $X_{\mathbf{b}}^{\mathfrak{l},+} \leftrightarrow I_{\mathfrak{l},+}^{m+n}$, where the index set $I_{\mathfrak{l},+}^{m+n}$ is defined as the image under the bijection.

Let $W_{\mathfrak{l}}$ be the Weyl group of $\mathfrak{l}$ with the corresponding Hecke algebra $\mathcal{H}_{\mathfrak{l}}$. Recall that $\Pi_{\mathfrak{l}} \subset \Pi_{\mathbf{b}}$ is a subset of even simple roots. Hence we have the natural right action of $\mathcal{H}_{\mathfrak{l}}$ on the $\mathcal{A}$-module $\mathbb{T}_{\mathcal{A}}^{\mathbf{b}} := \mathbb{V}_{\mathcal{A}}^{b_1} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \mathbb{V}_{\mathcal{A}}^{b_{m+n}}$ with a standard basis $M_f^{\mathbf{b}} \in \mathbb{T}_{\mathcal{A}}^{\mathbf{b}}$, for $f \in I_{\mathfrak{l}}^{m+n}$; cf. [4, §8.2]. We define

$$\mathbb{T}_{\mathcal{A}}^{\mathbf{b},\mathfrak{l}} = \mathbb{T}_{\mathcal{A}}^{\mathbf{b}} \Big/ \sum_{e \neq w \in W_{\mathfrak{l}}} \mathbb{T}_{\mathcal{A}}^{\mathbf{b}} \cdot \underline{H_w}.$$

The quotient space is an $\mathcal{A}$-form $\mathbb{T}_{\mathcal{A}}^{\mathbf{b},\mathfrak{l}}$ of the $\mathbb{Q}(q)$-space $\mathbb{T}^{\mathbf{b},\mathfrak{l}}$ appearing in Corollary 7:

$$\mathbb{T}_{\mathcal{A}}^{\mathbf{b},\mathfrak{l}} = \wedge^{a_0}\mathbb{V}_{-,\mathcal{A}} \otimes_{\mathcal{A}} \wedge^{a_1}\mathbb{V}_{\mathcal{A}}^{c_1} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \wedge^{a_k}\mathbb{V}_{\mathcal{A}}^{c_k}, \qquad \text{for } c_i \in \{0,1\}, \quad a_i \in \mathbb{N}, \tag{3.1}$$

where c_i and a_i are determined as follows. Let W' denote a subgroup of the Weyl group of $\mathfrak{osp}(2m+1|2n)$, $W' = W_{B_m} \times S_n = \langle s_0, s_1, \dots, s_{m-1}, s_{m+1}, \dots, s_{m+n-1}\rangle$, where $s_i = s_{\alpha_i}$, and $\alpha_0 = -\epsilon_1^{b_0}$, $\alpha_i = \epsilon_i^{b_i} - \epsilon_{i+1}^{b_{i+1}}$ for $1 \le i \le m+n-1$. Then, $W_{\mathfrak{l}}$ is the parabolic subgroup of W' generated by $\{s_i \mid \alpha_i \in \Pi_{\mathfrak{l}}\}$. Let us write $\{0,1,\dots,m+n\} \setminus \{i \mid \alpha_i \in \Pi_{\mathfrak{l}}\} = \{j_1 < j_2 < \cdots < j_{k+1}\}$. Then, $a_i = j_{i+1} - j_i$ and $c_{i+1} = b_{j_i}$, where it is understood that $j_0 = 0$.

For any standard basis element $M_f^{\mathbf{b}} \in \mathbb{T}_{\mathcal{A}}^{\mathbf{b}}$ with $f \in I_{\mathfrak{l},+}^{m+n}$, we denote by $M_f^{\mathbf{b},\mathfrak{l}}$ its image in $\mathbb{T}_{\mathcal{A}}^{\mathbf{b},\mathfrak{l}}$. Then $\{M_f^{\mathbf{b},\mathfrak{l}} | f \in I_{\mathfrak{l},+}^{m+n}\}$ forms an $\mathcal{A}$-basis of $\mathbb{T}_{\mathcal{A}}^{\mathbf{b},\mathfrak{l}}$. Let

$$\mathbb{T}_{\mathbb{Z}}^{\mathbf{b},\mathfrak{l}} = \mathbb{T}_{\mathcal{A}}^{\mathbf{b},\mathfrak{l}} \otimes_{\mathcal{A}} \mathbb{Z}$$

be the specialization of $\mathbb{T}_{\mathcal{A}}^{\mathbf{b},\mathfrak{l}}$ at $q=1$. Let $\widehat{\mathbb{T}}_{\mathbb{Z}}^{\mathbf{b},\mathfrak{l}}$ be the B-completion of $\mathbb{T}_{\mathbb{Z}}^{\mathbf{b},\mathfrak{l}}$ following [4, Section 9]. It follows from Corollary 7 the space $\widehat{\mathbb{T}}_{\mathbb{Z}}^{\mathbf{b},\mathfrak{l}}$ admits the $\imath$-canonical basis $\{T_f^{\mathbf{b},\mathfrak{l}} | f \in I_{\mathfrak{l},+}^{m+n}\}$. We can similarly define the dual $\imath$-canonical basis $\{L_f^{\mathbf{b},\mathfrak{l}} | f \in I_{\mathfrak{l},+}^{m+n}\}$ of $\widehat{\mathbb{T}}_{\mathbb{Z}}^{\mathbf{b},\mathfrak{l}}$ following [4, Theorem 9.9].

3.6. We denote by $[\mathcal{O}_{\mathbf{b}}^{\mathfrak{l},\Delta}]$ the Grothendieck group of the category $\mathcal{O}_{\mathbf{b}}^{\mathfrak{l},\Delta}$. We have the following isomorphism of $\mathbb{Z}$-modules:

$$\begin{aligned} \Psi : [\mathcal{O}_{\mathbf{b}}^{\mathfrak{l},\Delta}] &\longrightarrow \mathbb{T}_{\mathbb{Z}}^{\mathbf{b},\mathfrak{l}} \\ M_{\mathbf{b}}^{\mathfrak{l}}(\lambda) &\mapsto M_{f_\lambda^{\mathbf{b}}}^{\mathbf{b},\mathfrak{l}}(1), \qquad \text{for } \lambda \in X_{\mathbf{b}}^{\mathfrak{l},+}. \end{aligned}$$

We define $[[\mathcal{O}_{\mathbf{b}}^{\mathfrak{l},\Delta}]]$ as the completion of $[\mathcal{O}_{\mathbf{b}}^{\mathfrak{l},\Delta}]$ such that the extension of Ψ,

$$\Psi : [[\mathcal{O}_{\mathbf{b}}^{\mathfrak{l},\Delta}]] \longrightarrow \widehat{\mathbb{T}}_{\mathbb{Z}}^{\mathbf{b},\mathfrak{l}},$$

is an isomorphism of $\mathbb{Z}$-modules.

The following proposition is a reformulation of the Kazhdan-Lusztig theory for the parabolic category $\mathcal{O}$ of the Lie algebra $\mathfrak{so}(2m+1)$ (theorems of Brylinski-Kashiwara, Beilinson-Bernstein).

Proposition 9. *Let* $\mathbf{b} = (0^m)$ *(that is* $n = 0$*). The isomorphism* $\Psi : [[\mathcal{O}_{\mathbf{b}}^{\mathfrak{l},\Delta}]] \longrightarrow \mathbb{T}_{\mathbb{Z}}^{\mathbf{b},\mathfrak{l}}$ *sends*

$$\Psi([L_{\mathbf{b}}^{\mathfrak{l}}(\lambda)]) = L_{f_\lambda^{\mathbf{b}}}^{\mathbf{b},\mathfrak{l}}(1), \qquad \Psi([T_{\mathbf{b}}^{\mathfrak{l}}(\lambda)]) = T_{f_\lambda^{\mathbf{b}}}^{\mathbf{b},\mathfrak{l}}(1), \qquad \textit{for } \lambda \in X_{\mathbf{b}}^{\mathfrak{l},+}.$$

Note $\widehat{\mathbb{T}}_{\mathbb{Z}}^{\mathbf{b},\mathfrak{l}} = \mathbb{T}_{\mathbb{Z}}^{\mathbf{b},\mathfrak{l}}$ in this case, i.e. no completion is needed.

Proof. Thanks to [4, Theorem 5.8], the $\imath$-canonical basis on $\mathbb{T}^{\mathbf{b}}$ can be identified with the Kazhdan-Lusztig basis (of type B) on $\mathbb{T}^{\mathbf{b}}$. Note by [4, Theorem 5.4] that $\mathbb{S}^{\mathfrak{l}} := \sum_{e \neq w \in W_{\mathfrak{l}}} \mathbb{T}^{\mathbf{b}} \cdot \underline{H_w}$ is a $\mathbf{U}^\imath$-submodule of $\mathbb{T}^{\mathbf{b}}$, and it is actually a based $\mathbf{U}^\imath$-submodule of $\mathbb{T}^{\mathbf{b}}$ with its Kazhdan-Lusztig basis. Therefore the $\imath$-canonical basis on $\mathbb{T}_{\mathcal{A}}^{\mathbf{b},\mathfrak{l}}$ in Theorem 4 can be identified with the basis in the based quotient $\mathbb{T}^{\mathbf{b}}/\mathbb{S}^{\mathfrak{l}}$, which is exactly the parabolic Kazhdan-Lusztig basis. The proposition follows now from the classical Kazhdan-Lusztig theory (cf. [4]). □

Now we can formulate the super Kazhdan-Lusztig theory for $\mathcal{O}_{\mathbf{b}}^{\mathfrak{l}}$.

Theorem 10. *The isomorphism* $\Psi : [[\mathcal{O}_{\mathbf{b}}^{\mathfrak{l},\Delta}]] \longrightarrow \widehat{\mathbb{T}}_{\mathbb{Z}}^{\mathbf{b},\mathfrak{l}}$ *sends*

$$\Psi([L_{\mathbf{b}}^{\mathfrak{l}}(\lambda)]) = L_{f_\lambda^{\mathbf{b}}}^{\mathbf{b},\mathfrak{l}}(1), \qquad \Psi([T_{\mathbf{b}}^{\mathfrak{l}}(\lambda)]) = T_{f_\lambda^{\mathbf{b}}}^{\mathbf{b},\mathfrak{l}}(1), \qquad \textit{for } \lambda \in X_{\mathbf{b}}^{\mathfrak{l},+}.$$

Proof. Let us briefly explain the idea of the proof from [4]. The crucial new ingredient of this paper (cf. Remark 11 below) is the existence of the $\imath$-canonical basis and dual $\imath$-canonical basis on $\widehat{\mathbb{T}}^{\mathbf{b},\mathfrak{l}}$ thanks to Theorem 4. Here the dual $\imath$-canonical basis refers to a version of canonical basis where the lattice $\mathbb{Z}[q]$ is replaced by $\mathbb{Z}[q^{-1}]$; see [4].

We have already established the version of the theorem for the full category $\mathcal{O}$ of the Lie superalgebra $\mathfrak{osp}(2m+1|2n)$ in [4, Theorem 11.13]. We have the following commutative diagram of $\mathbb{Z}$-modules:

$$\begin{array}{ccc} [[\mathcal{O}_{\mathbf{b}}^{\mathfrak{l},\Delta}]] & \longrightarrow & \widehat{\mathbb{T}}_{\mathbb{Z}}^{\mathbf{b},\mathfrak{l}} \\ \downarrow & & \downarrow \\ [[\mathcal{O}_{\mathbf{b}}^{\Delta}]] & \longrightarrow & \widehat{\mathbb{T}}_{\mathbb{Z}}^{\mathbf{b}} \end{array}$$

(Note that the vertical arrow on the right is not a based embedding of $\mathbf{U}^\imath$-modules.) Then the theorem follows from comparison of characters entirely similar to [4, §11.2]. Note that this comparison uses only the classical Kazhdan-Lusztig theory. □

Remark 11. In the case of the full category $\mathcal{O}$ (i.e., $\mathfrak{l}$ is the Cartan subalgebra), the theorem goes back to [4, Theorem 11.13]. Following [4, Remark 11.16], the Kazhdan-Lusztig theory for the parabolic category $\mathcal{O}_{\mathbf{b}}^{\mathfrak{l}}$ with $\alpha_0 \neq \Pi_{\mathfrak{l}}$ was a direct consequence of [4, Theorem 11.13], via the $\imath$-canonical basis in [4, Theorem 4.25] in the ${}_{\mathcal{A}}\mathbf{U}$-module $\mathbb{T}_{\mathcal{A}}^{\mathbf{b},\mathfrak{l}}$ in (3.1) with $a_0 = 0$.

When $a_0 > 0$ (which corresponds to the condition $\alpha_0 \in \Pi_{\mathfrak{l}}$ on the Levi $\mathfrak{l}$), the space $\mathbb{T}_{\mathcal{A}}^{\mathbf{b},\mathfrak{l}}$ in (3.1) is a ${}_{\mathcal{A}}\mathbf{U}^\imath$-module but not a ${}_{\mathcal{A}}\mathbf{U}$-module, and hence Theorem 4 is needed.

Denote

$$T_f^{\mathbf{b},\mathfrak{l}} = M_f^{\mathbf{b},\mathfrak{l}} + \sum_g t_{gf}^{\mathbf{b}}(q) M_g^{\mathbf{b},\mathfrak{l}}, \qquad \text{for } t_{gf}^{\mathbf{b}}(q) \in \mathbb{Z}[q].$$

By Theorem 10, $t_{gf}^{\mathbf{b}}(q)$ plays the role of Kazhdan-Lusztig polynomials for $\mathcal{O}_{\mathbf{b}}^{\mathfrak{l}}$. The following positivity and finiteness results generalize [4, Theorem 9.11] and follow by the same proof.

Theorem 12.

(1) *We have* $t_{gf}^{\mathbf{b},\mathfrak{l}}(q) \in \mathbb{N}[q]$.
(2) *The sum* $T_f^{\mathbf{b},\mathfrak{l}} = M_f^{\mathbf{b},\mathfrak{l}} + \sum_g t_{gf}^{\mathbf{b}}(q) M_g^{\mathbf{b},\mathfrak{l}}$ *is finite, for all* f.

Remark 13. To formulate a super Kazhdan-Lusztig theory for the parabolic category $\mathcal{O}$ consisting of modules of half-integer weights for $\mathfrak{osp}(2m+1|2n)$, we use the quantum symmetric pair $(\mathbf{U}, \mathbf{U}^\imath)$ which is a direct limit of $(\mathbf{U}(\mathfrak{sl}_N), \mathbf{U}^\imath(\mathfrak{sl}_N))$ for N odd; cf. [4, Sections 6, 12]. Theorem 10 holds again in this setting.

Remark 14. Following [1], a simple conceptual modification allows us to formulate a super (type D) Kazhdan-Lusztig theory for the parabolic category $\mathcal{O}$ consisting of modules of integer (respectively, half-integer) weights for $\mathfrak{osp}(2m|2n)$. To that end, we use the $\imath$-canonical basis of the module (3.1) for the quantum symmetric pair $(\mathbf{U}, \mathbf{U}^\imath)$, where the parameter is now chosen to be $\kappa = 0$ in the notation of [5]. Theorem 10 holds again in this setting, where the cases new to this paper correspond to the cases $a_0 > 0$.

References

[1] H. Bao, Kazhdan-Lusztig theory of super type D and quantum symmetric pairs, Represent. Theory 21 (2017) 247–276, arXiv:1603.05105.
[2] J. Brundan, Kazhdan-Lusztig polynomials and character formulae for the Lie superalgebra $\mathfrak{gl}(m|n)$, J. Am. Math. Soc. 16 (2003) 185–231.
[3] M. Balagovic, S. Kolb, Universal K-matrix for quantum symmetric pairs, J. Reine Angew. Math. 747 (2019) 299–353, arXiv:1507.06276v2.
[4] H. Bao, W. Wang, A New Approach to Kazhdan-Lusztig Theory of Type B via Quantum Symmetric Pairs, Astérisque, vol. 402, 2018, vii+134 pp., arXiv:1310.0103.
[5] H. Bao, W. Wang, Canonical bases arising from quantum symmetric pairs, Invent. Math. 213 (2018) 1099–1177, arXiv:1610.09271v2.
[6] S.-J. Cheng, N. Lam, W. Wang, Super duality and irreducible characters of ortho-symplectic Lie superalgebras, Invent. Math. 183 (2011) 189–224.
[7] S.-J. Cheng, N. Lam, W. Wang, Brundan-Kazhdan-Lusztig conjecture for general linear Lie superalgebras, Duke Math. J. 164 (2015) 617–695.
[8] S. Kolb, Braided module categories via quantum symmetric pairs, arXiv:1705.04238v2.
[9] G. Letzter, Symmetric pairs for quantized enveloping algebras, J. Algebra 220 (1999) 729–767.
[10] G. Lusztig, Canonical bases in tensor products, Proc. Natl. Acad. Sci. 89 (1992) 8177–8179.
[11] G. Lusztig, Introduction to Quantum Groups, Modern Birkhäuser Classics, Birkhäuser, Boston, 2010 (reprint of the 1993 edition).